ANALYSIS OF A GSVD APPROACH
TO FULL-STATE FEEDBACK CONTROL DESIGN USING
SINGULAR VALUE LOCALIZATION OF EIGENVALUES.

A Thesis Presented to
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Master of Science

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CHAPTER 1

INTRODUCTION

Figure 1-1. Schematic representation of a closed-loop system

Most control systems are closed-loop systems, consisting of: (1) the plant, which is the original, unalterable system to be controlled; (2) one or more sensors, which give information about the plant; and (3) the controller, the central processing unit of the control system, which compares the measured values of the states to the desired values and adjusts the input values (to the plant) accordingly.

Feedback has several attractive properties. Since the actual operation is continuously compared to the desired operation, the performance of feedback
control systems is less sensitive to adverse conditions such as disturbances that act upon the system, or variations in plant properties. This feedback feature also makes it possible to stabilize a system if it is initially unstable or to improve its performance if its unforced phenomena do not die out sufficiently fast. The unforced properties mean the response of the control system to nonzero initial conditions of the plant.

At present, there are two conventional ways of improving the stability and unforced properties of linear systems by state feedback. One is that of pole assignment, where under a mildly restrictive condition (namely, the system must be completely controllable), all closed-loop poles (characteristic roots) can be arbitrarily placed by choosing $F$ suitably, where $F$ is the constant feedback gain matrix. However the pole assignment theorem gives no guidance as to where in the left-half complex (for continuous-time systems) plane the closed-loop poles should be located with respect to achieving a certain unforced response. Even more, uncertainties occur in multi-input systems where many solutions for the feedback gain matrix exist for the same closed-loop pole configurations.

Because the closed-loop poles can be located anywhere in the complex plane, they can be chosen to the far left in the complex plane so that the convergence to the zero state can be made arbitrarily fast. This feat, however, requires large input amplitudes and adequate information about the system dynamics. This is quite unlikely since in any practical problem, large inputs are
unrealistic and the plant parameters may not be known precisely. As a result, the
distance over which the closed-loop poles can be moved to the left is restricted.
These considerations lead to the formulation of an optimization problem, namely
the linear regulator problem, where both the speed of reducing the controlled
variable to zero and the input amplitudes are taken into account by choosing
suitable weighting matrices to minimize the integrated square factors.

The deterministic linear optimal regulator problem gives us the means to
develop asymptotically stable, linear feedback systems and at the same time
ensure the convergence of any nonzero initial state to zero in an "optimal"
fashion. These features are indeed attractive, but the regulator problem does have
several drawbacks. One is that, although the solution of the regulator problem
provides us with a rational prescription for placing these poles, it gives no
indication as to which combination of weighting indices gives us the desired
performance. Another drawback is that observers used in the implementation of
multivariable linear-quadratic optimal regulators have no guaranteed robustness
properties. In addition, optimal regulator systems can have "hidden modes" which
appear in the state and the input but not in the controlled variable.

This thesis is an effort to investigate an alternate method to pole placement
and the optimal linear regulator approach. The proposed design method
synthesizes a feedback system using damping as the performance criterion while
imposing an indirect constraint on the control effort required.
The concept proposed is based on the direct relationship between the moduli of the eigenvalues of the discrete-time, state-space system and the system damping. This innovation employs the Generalized Singular Value Decomposition (GSVD) to localize, but not completely specify a priori, system eigenvalue locations, thus enabling the damping for all individual modes of the closed-loop, discrete-time system to be specified, while simultaneously constraining the input. The objectives of this thesis are:

(1) Developing a new concept of synthesizing the full-state feedback gain,

(2) Developing design curves relating damping bounds to some parameters of $F$, the feedback gain matrix, and

(3) Investigating the feasibility of the proposed scheme through simulation.

This thesis is organized as follows. In Chapter 2, different formulations of the Generalized Singular Value Decomposition are reviewed and applied to the closed-loop, characteristic matrix $[A - BF]$. Detailed algebraic manipulations and analyses are presented to demonstrate how application of the GSVD leads to a compact and meaningful expression for the 2-norm of $[A - BF]$, and how this reduced form is related to $F$.

The material in Chapter 3 consists of detailed application of the proposed theory on four significant problems. These examples consist of two unstable
open-loop systems and two stable open-loop systems. Each class of examples will be made up of a single-input / single-output (SISO) and a multi-input / multi-output (MIMO) problem. Each worked example will be followed by its respective results and a step by step analysis that leads to reported observations and relevant implications.

Chapter 5, the final chapter, presents the conclusion that the proposed scheme is able to improve performance and guarantee stability only if the original plant has poles in the stable region (radius less than 1 in discrete-time, z-plane). In the unfortunate situation where the original system possesses unstable poles, the design method is capable of moving the poles by a small amount, enough to improve "performance" but not enough to stabilize the system. Lastly, the chapter concludes with suggestions for future studies.
CHAPTER 2
THEORY

Linear, time-invariant, discrete-time systems are described by difference equations of the form

\[
\begin{align*}
\mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k) \\
\mathbf{y}(k) &= C\mathbf{x}(k) + D\mathbf{u}(k)
\end{align*}
\] (2.1)

where the system behavior is observed at a sequence of equally spaced instants of time \( t_k, k = 0, 1, 2, \ldots \). Assuming that all states can be measured, it is possible to implement a linear, time-invariant, full state feedback control law of the form

\[
\mathbf{u}(k) = -F\mathbf{x}(k).
\] (2.2)

As with most feedback control systems, it is of interest to improve performance, i.e., it is desired to improve the speed with which the response of a control system converges to the zero state. For the discrete-time representation above, time-response characteristics are determined by location of the constant damping loci in the z-plane. Response characteristics are easier to determine by examining the location of poles in the s-plane for the continuous time, therefore,
mapping of the constant damping loci in the s-plane into the z-plane is considered.

Figure 2-1. Mapping constant damping loci into the z-plane

Since time-response characteristics in the continuous time are dictated by constant damping loci specified by $\sigma_i$ (which shall be referred to as the damping factor), the next step then is to determine how $\sigma$ is related to the parameters in the z-plane.

$Z$-plane poles occur at a $z = e^{sT}$ where $s = \sigma \pm j\omega$. Substituting for $s$, 
\[ z = e^{(\sigma \pm j\omega)T} \]
\[ = e^{\sigma T} e^{\pm j\omega T} \]  
(2.3)

which may be represented as

\[ z = e^{\sigma T} \angle \pm \omega T \]  
(2.4)

where \( e^{\sigma T} \) is the distance from the origin to the location of the pole, and \( \omega T \) is the angle. For the discrete-time representation, the distance from the origin to each pole location is simply the magnitude of the individual eigenvalue. That is,

\[ z = |\lambda_i| \angle \pm \Theta \]  
(2.5)

where \( \lambda_i \) is an eigenvalue of \( A \). Therefore, a relationship between the individual damping factor \( \sigma_i \) and the eigenvalues of \( A \) exists, and is given by

\[ |\lambda_i| = e^{\sigma_i T} \]  
(2.6)

which, solving for \( \sigma_i \) gives
\[ \sigma_i = \ln \left( \frac{|\lambda_i|}{T_s} \right) \quad (2.7) \]

Now that the relationship between the eigenvalues of \( A \) and the damping factor \( \sigma_i \) has been established, it is of interest to know how \( \sigma \) is related to the singular values of \( A \).

In terms of the singular value decomposition of \( A \),

\[ A = U \Sigma V^T \quad (2.8) \]

where \( U \) and \( V \) are orthogonal (unitary) and

\[ \Sigma = \text{diag} \left( \alpha_1, \alpha_2, \ldots, \alpha_n \right). \quad (2.9) \]

The diagonal elements of \( \Sigma \) are ordered such that

\[ \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq 0 \quad (2.10) \]
where $\alpha_1$ and $\alpha_n$ are, respectively, the largest and smallest singular values of $A$ and are related to $\lambda$, the eigenvalues of $A$, as shown by Kailath [7]:

$$
\alpha_n \leq \min_i |\lambda_i| \leq \max_i |\lambda_i| \leq \alpha_1,
$$

(2.11)

which can be further extended to a more general form,

$$
\alpha_n \leq |\lambda_i| \leq \alpha_1.
$$

(2.12)

With the knowledge that

$$
\| A \|_2 = \alpha_1 \geq \max_i |\lambda_i|,
$$

(2.13)

it is easy to see that the upper bound on the maximum achievable damping is simply the 2-norm of $A$.

Since the objective of feedback is to move the eigenvalues to the origin (in the digital case), naturally the next step should be that of finding a feedback gain, $F$, such that $\| A - BF \|_2$ is well within some specified bound. It is known that if
the system is controllable, an $F$ (with no constraints) can be found such that the closed-loop system matrix $[A - BF]$ satisfies the damping constraint

$$\|A - BF\|_2 \leq \sigma \quad (2.14)$$

for an arbitrary positive $\sigma$.

However, it is recalled that input amplitudes need to be constrained (large inputs are impractical) and robustness maintained. With this in mind, the goal then is not only to achieve damping but to achieve it with $Bu(k)$ of minimum norm. This is the same as requiring that $\|BF\|_2$ be minimum since

$$\|BF\|_2 = \max_{\|x(k)\|_2 = 1} \|BFx(k)\|_2 \quad (2.15)$$

If $B$ is square and invertible, the problem can be easily solved via the singular value decomposition (SVD) of $A$ and $[A - BF]$ as follows. Let

$$A' = U\Sigma'V^T \quad (2.16)$$

By setting
\[ \Sigma' = \text{diag} \left( \gamma_1, \gamma_2, \ldots, \gamma_n \right) \quad (2.17) \]

such that

\[ \gamma_i = \begin{cases} \alpha_i, & \alpha_i \leq \sigma \\ \sigma, & \alpha_i > \sigma \end{cases} \quad (2.18) \]

where \( \alpha_i \) is a singular value of \( A \) and \( \sigma \) is the damping constraint on \( \| A - BF \|_2 \).

the feedback gain matrix \( F \) can be found by requiring

\[ A - BF = A' \quad (2.19) \]

which, after substituting for \( A \), \( B \) and \( A' \), gives

\[ F = B^{-1} U \left( \Sigma - \Sigma' \right) V^T \quad (2.20) \]

Since \( B \) is singular (therefore noninvertible) most of the time, a solution for \( F \) cannot be obtained. Thus the problem must be reconfigured such that an \( F \) (with \( \| BF \|_2 \) minimum) can be found that fulfills

\[ [A - BF] = Q_c \Sigma_c U_c^T \quad (2.21) \]
where $U_c, Q_c$ are orthogonal, and $\Sigma_c$ is specified (partially) and parameterized via the elements of $F$. As will be seen next, it turns out that a diagonalization of this form plays an important role in producing a minimum $\|BF\|_2$.

Golub and Van Loan [6] have shown that for the constrained least squares problem

\[
\begin{align*}
\text{minimize} & \quad \|\hat{A}y - b\|_2 \\
\text{subject to} & \quad \|\hat{E}y - d\|_2 = \zeta
\end{align*}
\]  

(2.22a)

(2.22b)

where $\hat{A}$ is m\!\times\!n, $b$ is a vector of dimension m, $\hat{E}$ is p\!\times\!n, $d$ is a p-length vector, and $\zeta \geq 0$, a solution for $y$ exists if and only if $\min \|\hat{E}y - d\|_2 \leq \zeta$. If a solution exists, then the solution that minimizes $\|\hat{A}y - b\|_2$ also satisfies $\|\hat{E}y - d\|_2 = \zeta$.

By the same token, the least squares problem can be recast, in terms of control system dynamics, as

\[
\begin{align*}
\text{minimize} & \quad \|D_B^T D_F\|_2 \\
\text{subject to} & \quad \|D_A - D_B^T D_F\|_2 = \sigma
\end{align*}
\]

(2.23a)

(2.23b)

where $\sigma$ is some arbitrary positive value, and $D_A, D_B$ and $D_F$ are, respectively, the
diagonal elements of the singular value decomposition of A, B and F, as will be revealed in greater detail later. Since the goal of achieving damping and imposing an approximate constraint on the input can be restated as

\[
\begin{align*}
\text{minimize} & \quad \| BF \|_2 \\
\text{subject to} & \quad \| A - BF \|_2 = \sigma
\end{align*}
\] (2.24a, 2.24b)

Equations (2.24a) and (2.24b) can be reconfigured as Equations (2.23a) and (2.23b) if there exist \( D_A, D_B \) and \( D_F \) such that

\[
\| BF \|_2 \leq f\{ \| D_B^T D_F \|_2 \}
\] (2.25)

and

\[
\| A - BF \|_2 \leq f\{ \| D_A - D_B^T D_F \|_2 \}
\] (2.26)

where \( f(.) \) denotes function.

In light of this, it is thus desirable to cast \([ A - BF ]\) in the form given by Equation (2.21). To obtain \([ A - BF ]\) in that form, the generalized singular value decomposition (GSVD) is proposed. At present, there are a few approaches to the GSVD problem, two of which will be explored in the next two topics.
2.1 Application of the GSVD that utilizes the CS decomposition

Van Loan \[ 7 \] claims that, for some given matrices \( A ( n_1 \times p ) \) and \( B ( n_2 \times p ) \), a simultaneous diagonalization is possible if some subset of \( AQ \) is a well-conditioned matrix that has nearby orthogonal columns, whereby it can be safely diagonalized by the QR factorization. According to him, there exist orthogonal matrices \( Q ( p \times p ) \), \( U ( n_1 \times n_1 ) \) and \( V ( n_2 \times n_2 ) \) such that

\[
U^T A Q = D_A \quad \text{(2.27)}
\]

and

\[
V^T B Q = D_B \quad . \quad \text{(2.28)}
\]

However, in most control problems the realistic case is that of which \( A \) is \((p \times p)\) and \( B \) is \((p \times n_2)\), thus resulting in \( D_A (p \times p) \) being square and \( D_B (n_2 \times p) \) rectangular. This means \( D_A \) and \( D_B \) are of the form

\[
D_A = \begin{bmatrix}
c_p & \cdots & 0 \\
. & \ddots & . \\
. & . & . \\
0 & \cdots & c_1
\end{bmatrix} \quad \text{(2.29)}
\]

and
\[ D_B = \begin{bmatrix} 0 & 0 & \cdots & 0 & \mid & s_p & \cdots & 0 \\ \vdots & \vdots & \ddots & \hdots & \mid & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \mid & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \mid & 0 & \cdots & s_l \end{bmatrix} \begin{bmatrix} q \\ p-q \\ q \end{bmatrix} \] (2.30)

where \( q = \min \{ p, n2 \} \) and \( c_i \) and \( s_i \) are ordered as follows:

\[
0 \leq c_2 \leq \ldots \leq c_q \leq c_{q+1} = c_p = 1
\]
\[
1 \geq s_1 \geq \ldots \geq s_q \geq 0
\] (2.31)

Naturally the next step now is to obtain some desired final form of \([ A - BF ]\) (specifically, that of Equation (2.21)). For this purpose, much liberty is taken to "mold" A and B into suitable forms prior to the GSVD process. Employing the GSVD on \( A^{-1} \) and \( B^T \),

\[
A^{-1} = UD_A Q^T \] (2.32)

and

\[
B^T = VD_B Q^T \] (2.33)

Taking the inverse of both sides of Equation (2.32) and the transpose of both sides of Equation (2.33), yields
\[ A = QD_A^{-1}U^T \]  \hspace{1cm} (2.34)

and

\[ B = QD_B^T V^T \]  \hspace{1cm} (2.35)

Making the appropriate substitutions in A and B,

\[ A - BF = QD_A^{-1}U^T - QD_B^T V^T F \]  \hspace{1cm} (2.36)

F is chosen such that

\[ F = VD_F U^T \]  \hspace{1cm} (2.37)

where \( VD_F U^T \) is some SVD of F. Substituting Equation (2.37) in Equation (2.36) yields

\[ A - BF = QD_A^{-1}U^T - QD_B^T D_F U^T \]

\[ = Q \left( D_A^{-1} - D_B^T D_F \right) U^T \]  \hspace{1cm} (2.38)

Since Q and V are orthogonal, Equation (2.38) has the form

\[ A - BF = Q \Sigma U^T \]  \hspace{1cm} (2.39)
where \( \Sigma \) is diagonal and \( Q \) and \( U^T \) are orthogonal. This is the same as requiring

\[
D_{A^{-1}} - D_B^T D_F = \Sigma = \text{diag} (\gamma_1, \ldots, \gamma_p) \tag{2.40}
\]

Since \( D_A, D_B, D_F \) are all diagonal matrices, it follows that a \( D_F \) can always be found such that Equation (2.40) holds true. Consequently,

\[
\| A - BF \|_2 = \| D_{A^{-1}} - D_B^T D_F \|_2 . \tag{2.41}
\]

Indeed this is a most convenient form since an approximation of the maximum achievable damping and the proximity of the closed-loop poles to the unit circle can be obtained just by inspecting the generalized singular values of \( A^{-1} \) and \( B \).

However, attractive as it may seem, Van Loan's GSVD algorithm is not universal in that it exclusively relies on chance that \( AQ \) is diagonalizable via the QR factorization. Since the odds of \( AQ \) being well-conditioned, in Van Loan's sense, are unpredictable, it would be a risky venture to apply this GSVD algorithm to the proposed design procedure. Thus another alternative will be explored: a GSVD algorithm that results in a simultaneous triangularization of \( A \) and \( B \) will be the subject of the next topic.
2.2 Application of the GSVD that utilizes the Direct Algorithm

Bai [1] has shown that the given matrices \( A \ (p \times p) \) and \( B \ (n_2 \times p) \) can be written

\[
U^T A Q = D_A R \tag{2.42}
\]

and \( V^T B Q = D_B R \tag{2.43} \)

where

\[
D_A = \begin{bmatrix} I & C \\ p - n_2 & n_2 \\ \end{bmatrix} p - n_2 \tag{2.44}
\]

and \( D_B = \begin{bmatrix} Q & S \\ p - n_2 & n_2 \end{bmatrix} n_2 \tag{2.45} \)

\( R \ (p \times p) \) is non-singular and upper triangular, \( U \ (p \times p) \), \( V \ (n_2 \times n_2) \) and \( Q \ (p \times p) \) are orthogonal, and
\[ C = \text{diag} \left( \alpha_1, \ldots, \alpha_{n_2} \right), \quad (2.46) \]

\[ S = \text{diag} \left( \beta_1, \ldots, \beta_{n_2} \right) \quad (2.47) \]

such that

\[ 1 \geq \alpha_1 \geq \ldots \geq \alpha_p > 0 \quad (2.48) \]

\[ 0 < \beta_1 \leq \ldots \leq \beta_q < 1. \]

The next step then is to achieve the form of Equation (2.21). To begin with, the GSVD is applied on \( A^T \) and \( B^T \), which yields

\[ A^T = U D_A R Q^T \quad (2.49) \]

and

\[ B^T = V D_B R Q^T. \quad (2.50) \]

Taking the transpose of both sides of Equations (2.49) and (2.50) respectively,

\[ A = Q R^T D_A U^T \quad (2.51) \]

and

\[ B = Q R^T D_B T V^T \quad (2.52) \]
\( \mathbf{D}_A \) is rewritten as \( \mathbf{D}_A = \mathbf{D}_A^{-1} \mathbf{D}_A^2 \) thus resulting in

\[
\mathbf{A} = \mathbf{Q} \mathbf{R}^T \mathbf{D}_A^{-1} \mathbf{D}_A^2 \mathbf{U}^T
\]  (2.53)

Substituting for \( \mathbf{A} \) and \( \mathbf{B} \) gives

\[
\mathbf{A} - \mathbf{B} \mathbf{F} = \mathbf{Q} \mathbf{R}^T \mathbf{D}_A^{-1} \mathbf{D}_A \mathbf{D}_A^2 \mathbf{U}^T - \mathbf{Q} \mathbf{R}^T \mathbf{D}_B^T \mathbf{V}^T \mathbf{F} .
\]  (2.54)

Let

\[
\mathbf{F} = \mathbf{V} ( \mathbf{D}_F \mathbf{D}_A^2 ) \mathbf{U}^T ,
\]  (2.55)

whose 2-norm can be written as

\[
\| \mathbf{F} \|_2 = \| \mathbf{D}_F \mathbf{D}_A^2 \|_2.
\]  (2.56)

After appropriate substitutions

\[
\mathbf{A} - \mathbf{B} \mathbf{F} = \mathbf{Q} \mathbf{R}^T \mathbf{D}_A^{-1} \mathbf{D}_A^2 \mathbf{U}^T - \mathbf{Q} \mathbf{R}^T \mathbf{D}_B^T \mathbf{D}_F \mathbf{D}_A^2 \mathbf{U}^T
\]  (2.57)

which may be factored as
\[ A - BF = Q \left( R^T D_A^{-1} D_A^2 - R^T D_B^T D_F D_A^2 \right) U^T. \quad (2.58) \]

It follows then that

\[ \| A - BF \|_2 = \| R^T D_A^{-1} D_A^2 - R^T D_B^T D_F D_A^2 \|_2 \]

\[ = \| R^T \left( D_A^{-1} - D_B^T D_F \right) D_A^2 \|_2 \quad (2.59) \]

which can be rewritten

\[ \| A - BF \|_2 \leq \| R^T \|_2 \| D_A^{-1} - D_B^T D_F \|_2 \| D_A^2 \|_2. \quad (2.60) \]

This configuration has several attractive features:

1. By setting the elements of \( D_F \) to their maximum allowable (realistic) values, the maximal achievable damping can be estimated.

2. This bound is parameterized via some element of \( F \) that we specify.

3. An approximate minimum norm constraint is imposed on \( BF \) as shown in Equations (2.23) through (2.26).

4. More importantly, the bound of Equation (2.60) serves as a guideline for choosing the elements of \( D_F \) as will be shown next.
To illustrate how the elements of $D_F$ are to be chosen, a fictitious four-state, two-input system is considered. This means that the resulting $D_A$ is 4x4, $D_B$ is 4x2, and $D_F$ is 2x4. The diagonal matrices, $D_A$, $D_B^T$ and $D_F$, of the GSVD of $A$ and $B$ that satisfy Equations (2.42), (2.43) and (2.55) are selected such that

$$D_A = \begin{bmatrix}
a_1 & 0 & 0 & 0 \\
0 & a_2 & 0 & 0 \\
0 & 0 & a_3 & 0 \\
0 & 0 & 0 & a_4
\end{bmatrix} \quad (2.61)$$

where

$$a_1 \geq a_2 \geq a_3 \geq a_4 > 0, \quad (2.62)$$

$$D_B^T = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
b_1 & 0 \\
0 & b_2
\end{bmatrix}, \quad (2.63)$$

and

$$D_F = \begin{bmatrix}
0 & 0 & f_1 & 0 \\
0 & 0 & 0 & f_2
\end{bmatrix}. \quad (2.64)$$

Multiplying $D_B^T$ and $D_F$ gives
\[
D_B^T D_F = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & b_1 f_1 & 0 \\
0 & 0 & 0 & b_2 f_2 \\
\end{bmatrix}.
\]

(2.65)

Since the singular value decomposition (SVD) of any matrix yields a diagonal matrix that has real, positive entries, \( D_A, D_B \) and \( D_F \) all have positive entries.

In terms of the 2-norm of \( [D_A^{-1} - D_B^T D_F] \),

\[
\| D_A^{-1} - D_B^T D_F \|_2 = \max \{ 1/a_1, 1/a_2, 1/a_3 - b_1 f_1, 1/a_4 - b_2 f_2 \} \quad (2.66)
\]

where

\[
1/a_4 \geq 1/a_3 \geq 1/a_2 \geq 1/a_1 > 0.
\]

(2.67)

The diagonal elements of \( D_F \) are initially let to be zero. Then \( f_2 \) is incremented until \( 1/a_4 - b_2 f_2 \) is less than or equal to \( 1/a_3 \). When that happens, \( f_1 \) is incremented along with \( f_2 \). Both are incremented until the \( \max \{ 1/a_3 - b_1 f_1, 1/a_4 - b_2 f_2 \} \) is less than or equal to \( 1/a_2 \).

Naturally, when \( \| D_A^{-1} - D_B^T D_F \|_2 \) reaches \( 1/a_2 \), it can go no lower since there is no other element of \( D_F \) to be deducted from \( 1/a_2 \). Hence the range of \( \| D_A^{-1} - D_B^T D_F \|_2 \) is limited by the degrees of freedom in \( D_F \), which in turn
is directly dependent on the number of inputs.

It is stressed that this method of choosing the elements of $D_F$ (hence $F$) is a guideline for computing $[D_{A^{-1}} - D_B T D_F]$ which is then used to compute $[R T (D_{A^{-1}} - D_B T D_F) D_A^2]$. Once $[R T (D_{A^{-1}} - D_B T D_F) D_A^2]$ is computed, the singular values of $[A - BF]$ can be calculated since

$$SVD(A - BF) = SVD(R^T [D_{A^{-1}} D_A^2 - D_B T D_F] D_A^2).$$ (2.68)

If the singular value decomposition of $[A - BF]$ yields $\Sigma_c$ such that

$$\Sigma_c = \text{diag}(\gamma_{\text{max}}, \ldots, \gamma_{\text{min}})$$ (2.69)

and $\lambda_i$ is an eigenvalue of $[A - BF]$, it follows from Equation (2.11) that

$$\gamma_{\text{min}} \leq |\lambda_i| \leq \gamma_{\text{max}}.$$ (2.70)

Since

$$\sigma_i = (\ln |\lambda_i|) / T_s$$ (2.71)
it is easy to see that

\[
\frac{\ln \gamma_{\min}}{T_s} \leq \sigma_i \leq \frac{\ln \gamma_{\max}}{T_s}.
\] (2.72)

Thus given \([A - BF]\) in the form of Equation \((2.58)\), design curves can be developed to depict the required \(\|F\|_2\) for a given improvement in the closed-loop singular values or the damping factor. Investigating the feasibility of these design curves will be the subject of the next chapter.
CHAPTER 3
EXAMPLES

This chapter presents the analysis of four problems, two of which are stable systems, and the other two, unstable systems. To analyze the results presented here, the highlights of Chapter 2 are recalled:

1. The matrices that describe a system, A and B, are simultaneously decomposed to obtain U (orthogonal), V (orthogonal), Q (orthogonal), R (triangular), \( D_A \) (diagonal), and \( D_B \) (diagonal) that satisfy Equations (2.53) and (2.54).

2. \([A - BF]\) is related to the elements of the simultaneous decomposition of A and B such that

\[
A - BF = Q \left[ R^T \left( D_{A^{-1}} - D_B^T D_F \right) D_A^2 \right] U^T
\]

where \( D_F \) is chosen and is related to \( F \) and \( \| F \|_2 \) such that

\[
F = V D_F D_A^2 U^T
\]

and
\[ \| F \|_2 = \| D_F D_A^2 \|_2. \]

If \( \{ \gamma_{\text{max}}, \ldots, \gamma_{\text{min}} \} \) are the singular values of \( [ R^T (D_A^{-1} - D_B^T D_F) D_A^{-2} ] \),

then \( \{ \gamma_{\text{max}}, \ldots, \gamma_{\text{min}} \} \) are also the singular values of \( [ A - BF ] \).

(3) It was shown that

\[
\gamma_{\text{min}} \leq |\lambda_i| \leq \gamma_{\text{max}},
\]

where \( \gamma_{\text{min}} \) and \( \gamma_{\text{max}} \) are, respectively, the smallest and largest singular values of \( [ A - BF ] \), \( \lambda_i \) denotes an eigenvalue of \( [ A - BF ] \), and \( \sigma_i \) is the closed-loop damping factor. Since \( \sigma_i = (1/n |\lambda|)/T_s \), it follows that

\[
\sigma_{\text{min}} \leq \sigma_i \leq \sigma_{\text{max}}
\]

where \( \sigma_{\text{min}} = (1/n \gamma_{\text{min}})/T_s \), \( \sigma_{\text{max}} = (1/n \gamma_{\text{max}})/T_s \).
and $T_s$ is the sampling interval. Clearly, each eigenvalue is logarithmically related to its individual damping, thus $\sigma_{\text{min}}$ constitutes the upper bound on the maximum possible damping, and $\sigma_{\text{max}}$ constitutes the lower bound on the minimum possible damping in the structure. Graphically, this is depicted as follows.

![Graph showing $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ vs $\| F \|_2$.]

**Figure 3-1.** Plot of $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ vs $\| F \|_2$
EXAMPLE 1: ANALYSIS OF AN UNSTABLE, 2-STATE, SISO SYSTEM

As a first example, consider the counter example [4] used by Doyle to demonstrate that there are no guaranteed margins for LQG regulators. The example consists of an unstable system that possesses two real and repeated roots at \( s = 1 \). In state-space form, the system is represented as:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u
\]

\[
y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

Discretizing the system at a sampling period, \( T_s = 0.1s \),

\[
A = \begin{bmatrix} 1.1052 & 0.1105 \\ 0.0000 & 1.1052 \end{bmatrix},
\]

and

\[
B = \begin{bmatrix} 0.0053 \\ 0.1052 \end{bmatrix}
\]
Using the GSVD to decompose A and B, the elements of their decomposition are

\[ U = \begin{bmatrix} 9.987721021971819E-001 & 4.954076980247573E-002 \\ 4.954076980247573E-002 & -9.987721021971819E-001 \end{bmatrix}, \]

\[ V = \begin{bmatrix} -1.000000000000 \end{bmatrix}, \]

\[ Q = \begin{bmatrix} -9.987841691096406E-001 & 4.929689174750004E-002 \\ -4.929689174750004E-002 & -9.987841691096406E-001 \end{bmatrix}, \]

\[ D_A = \begin{bmatrix} 1.000000000000 & 0.000000000000 \\ 0.000000000000 & 9.95489333564303E-01 \end{bmatrix}, \]

\[ D_B = \begin{bmatrix} 0.000000000000 & 9.487353252393453E-02 \end{bmatrix}, \]

and

\[ R = \begin{bmatrix} -1.110667566347584 & 0.000000000000 \\ 1.099602691873832E-01 & 1.099759349250449 \end{bmatrix}. \]
where \( U, V, Q \) are orthogonal, \( R \) is lower triangular, and \( D_A \) and \( D_B \) are diagonal.

Computing the SVD of \( [ R^T ( D_A^{-1} - D_B^T D_F ) D_A^2 ] \) for each increment in \( \| F \|_2 \), we obtain a maximum singular value and a minimum singular value for each \( \| F \|_2 \). The results are shown in Figures 3-2a and 3-2b.

![Figure 3-2a](image)

**Figure 3-2a.** Plot of \( \gamma_{\text{max}} \) and \( \gamma_{\text{min}} \) vs \( \| F \|_2 \) (different scale)
**Figure 3-2b.** Plot of $\gamma_{\max}$ and $\gamma_{\min}$ vs $\| F \|_2$ (same scale)

Figures 3-2a – 3-2b show that the maximum singular value decreases very slowly as $\| F \|_2$ increases while the minimum singular value decreases at a much faster rate. This may possibly indicate that poles located close to the unit circle can be moved inward very little whereas poles further away from the unit circle can be shifted toward the origin by a greater amount.

In terms of the damping factor, $\sigma_i = (\ln \alpha_i) / T_s$. The $\sigma_i$'s that correspond to the singular values above are shown in Figures 3-2c and 3-2d.
**Figure 3-2c.** Plot of $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ vs $\|F\|_2$ (different scale)

**Figure 3-2d.** Plot of $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ vs $\|F\|_2$ (same scale)
As observed in Figures 3-2a – 3-2d, both the maximum singular value and the maximum damping factor stay above the stability boundary of $|z| = 1$ and $\text{Re}(s) = 0$, respectively, for all values of $\|F\|_2$, thus indicating that there may be some closed-loop poles lying outside the stability region.

Hence this example of the design procedure seems to be unsuccessful since performance is "improved" at the expense of guaranteed stability.
EXAMPLE 2: ANALYSIS OF AN UNSTABLE, 8-STATE, MIMO SYSTEM

This next example is an unstable, two-axis pointing system whose parameters are very sensitive to plant variations and disturbances. It is used by Mitchell [12] to test robustness of flexible, large space structures.

![Block Diagram Of A Two-Axis Pointing System]

**Figure 3-3.** Block Diagram Of A Two-Axis Pointing System

For the system above the transfer functions are given as

\[ G_{11}(s) = \frac{Y_1}{U_1} \]
Figure 3-4 Reduced-order block diagram of system in Example 2
\[
G_{12}(s) = \frac{0.1}{s^2} + \frac{0.05}{s^2 + 0.01s + 1}
\]

\[
G_{21}(s) = \frac{Y_1}{U_2} = \frac{0.15}{s^2} + \frac{-1.0}{s^2 + 0.02s + 4}
\]

\[
G_{22}(s) = \frac{Y_2}{U_2} = \frac{0.8}{s^2} - \frac{0.04}{s^2 + 0.01s + 1}
\]

Model reduction yields the irreducible (thereby controllable) block diagram form illustrated in Figure 3-4. The coefficients $V_{ij}$ and $K_{ij}$ are found using signal flow graph technique as follows.
Figure 3-5. Signal Flow Graph corresponding to Figure 3-4

For convenience, the following notation will be used.

\[ G_1 (s) = \frac{1}{s^2} \quad G_2 (s) = \frac{1}{s^2 + 0.01s + 1} \]

and

\[ G_3 (s) = \frac{4}{s^2 + 0.02s + 4} \]
From the signal flow graph of Figure 3-5,

\[
G_{11} = \frac{Y_1}{U_1} = V_{11}K_{11}G_1 + V_{12}K_{12}G_2 + 4V_{13}K_{13}G_3
= 0.1G_1 + 0.05G_2
\]

\[
G_{12} = \frac{Y_2}{U_1} = V_{11}K_{21}G_1 + V_{12}K_{22}G_2 + 4V_{13}K_{23}G_3
= 0.08G_1 - 0.04G_2
\]

\[
G_{21} = \frac{Y_1}{U_2} = V_{21}K_{14}G_1 + V_{22}K_{12}G_2 + 4V_{23}K_{13}G_3
= 0.15G_1 - 1.0G_3
\]

\[
G_{22} = \frac{Y_2}{U_2} = V_{21}K_{24}G_1 + V_{22}K_{22}G_2 + 4V_{23}K_{23}G_3
= 0.2G_1 - 0.05G_3
\]

Equating like terms,

\[
V_{11}K_{11} = 0.1, \quad V_{12}K_{12} = 0.05, \quad 4V_{13}K_{13} = 0
\]

\[
V_{11}K_{21} = 0.08, \quad V_{12}K_{22} = -0.04, \quad 4V_{13}K_{23} = 0
\]

\[
V_{21}K_{14} = 0.15, \quad 4V_{23}K_{13} = -1.0, \quad V_{22}K_{12} = 0
\]

\[
V_{21}K_{24} = 0.2, \quad 4V_{23}K_{23} = -0.05, \quad V_{22}K_{22} = 0
\]
Setting $V_{13} = V_{22} = 0$, $V_{11} = V_{12} = V_{21} = V_{23} = 1$, and solving for $K_{ij}$,

$$
\begin{align*}
K_{11} &= 0.1 & K_{12} &= 0.05 \\
K_{21} &= 0.08 & K_{22} &= -0.04 \\
K_{14} &= 0.15 & K_{13} &= -0.25 \\
K_{24} &= 0.2 & K_{23} &= -0.0125
\end{align*}
$$

Having found the necessary coefficients, and substituting them in the block diagram of Figure 3-4, the resulting continuous-time state-space form is found, by inspection, to be:

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5 \\
\dot{x}_6 \\
\dot{x}_7 \\
\dot{x}_8 
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -0.01 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -4 & -0.02 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
x_8 
\end{bmatrix} + 
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
$$

$$
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} = 
\begin{bmatrix}
0.1 & 0 & 0.05 & 0 & 0.15 & 0 & -0.25 & 0 \\
0.08 & 0 & -0.04 & 0 & 0.2 & 0 & -0.0125 & 0
\end{bmatrix}
\begin{bmatrix}
x
\end{bmatrix}
$$
To obtain the discrete-time, state-space realization of the form

\[ x(k+1) = A x(k) + B u(k) \]
\[ y(k) = C x(k) \]

the system is discretized at a sampling frequency of 25 hertz. The resulting discretized A and B are as given below.

\[
A = \begin{bmatrix}
1.0000D+0 & 4.0000D-02 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 \\
0.0000D-01 & 1.0000D+0 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 \\
0.0000D-01 & 0.0000D-01 & 9.9920D-01 & 4.0000D-02 & 0.0000D-01 \\
0.0000D-01 & 0.0000D-01 & -4.0000D-02 & 9.9880D-01 & 0.0000D-01 \\
0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 1.0000D+00 \\
0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 \\
0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 \\
0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01
\end{bmatrix}
\]
Columns 6 through 8

\[
\begin{array}{ccc}
0.0000D-01 & 0.0000D-01 & 0.0000D-01 \\
0.0000D-01 & 0.0000D-01 & 0.0000D-01 \\
0.0000D-01 & 0.0000D-01 & 0.0000D-01 \\
4.0000D-02 & 0.0000D-01 & 0.0000D-01 \\
1.0000D+00 & 0.0000D-01 & 0.0000D-01 \\
0.0000D-01 & 9.9680D-01 & 3.9900D-02 \\
0.0000D-01 & -1.5980D-01 & 9.9600D-01 \\
\end{array}
\]

\[
B = \begin{bmatrix}
8.0000D-04 & 0.0000D-01 \\
4.0000D-02 & 0.0000D-01 \\
8.0000D-04 & 0.0000D-01 \\
4.0000D-02 & 0.0000D-01 \\
0.0000D-01 & 8.0000D-04 \\
0.0000D-01 & 4.0000D-02 \\
0.0000D-01 & 3.2000D-03 \\
0.0000D-01 & 1.5980D-01 \\
\end{bmatrix}
\]

Applying the GSVD on A and B, the orthogonal matrices U, V and Q, diagonal matrices \( D_A \) and \( D_B \), and a triangular matrix R, are
\[ U = \text{Columns 1 through 5} \]

\[
\begin{array}{cccc}
0.0000D-01 & 1.9986D-04 & 9.9900D-01 & 6.3073D-19 \\
0.0000D-01 & -9.9930D-03 & 3.9960D-02 & -6.7277D-18 \\
0.0000D-01 & 9.9950D-01 & 7.9888D-04 & -2.1233D-19 \\
0.0000D-01 & 3.0007D-02 & -1.9956D-02 & 1.8100D-18 \\
-7.9935D-04 & -3.7068D-17 & 2.7451D-17 & 3.8782D-02 \\
4.9822D-03 & -6.3677D-18 & -3.1854D-16 & -2.4172D-01 \\
\end{array}
\]

Columns 6 through 8

\[
\begin{array}{cccc}
2.4488D-18 & -1.4136D-02 & 1.7177D-19 \\
-1.7363D-16 & 7.0682D-01 & -2.1860D-17 \\
-1.0846D-17 & -1.4159D-02 & -1.4186D-18 \\
9.2226D-18 & 7.0711D-01 & 7.9100D-19 \\
-6.5875D-04 & -2.2082D-16 & 4.8518D-03 \\
3.2937D-02 & -3.1561D-19 & -2.4259D-01 \\
-9.9906D-01 & -4.9154D-18 & 1.9355D-02 \\
-2.8178D-02 & -5.8149D-18 & -9.6992D-01 \\
\end{array}
\]

\[ V = \begin{bmatrix} 1. & 0. \\ 0. & -1. \end{bmatrix} , \]
\[ Q = \begin{array}{cccccc}
-2.7668D-36 & -1.0001D-02 & 3.9952D-06 & -1.0727D-21 & -7.0718D-01 \\
1.6966D-35 & -1.0001D-02 & -1.9992D-02 & 2.3567D-18 & 7.0689D-01 \\
3.1404D-19 & -1.5484D-33 & 1.9390D-33 & -3.1027D-18 & -4.7151D-33 \\
\end{array} \]

\[ Columns \ 6 \ through \ 8 \]
\[
\begin{array}{cccc}
3.7082D-21 & 1.4139D-02 & 5.5304D-34 \\
1.1237D-19 & 7.0697D-01 & 3.7599D-18 \\
3.7082D-21 & 1.4139D-02 & -1.5960D-35 \\
1.8541D-19 & 7.0697D-01 & -5.0612D-34 \\
9.4319D-05 & 1.4915D-20 & -4.8555D-03 \\
4.7160D-03 & -9.7420D-19 & -2.4277D-01 \\
-9.9981D-01 & 2.6221D-19 & -1.9422D-02 \\
1.8840D-02 & 2.9792D-18 & -9.6988D-01 \\
\end{array} \]

\[ D_A = \begin{array}{cccccc}
1.0000D+00 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 \\
0.0000D-01 & 1.0000D+00 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 \\
0.0000D-01 & 0.0000D-01 & 1.0000D+00 & 0.0000D-01 & 0.0000D-01 \\
0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 1.0000D+00 & 0.0000D-01 \\
0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 1.0000D+00 \\
0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 \\
0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 \\
0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 \\
0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 & 0.0000D-01 \\
\end{array} \]
<table>
<thead>
<tr>
<th>Columns 6 through 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0000D-01  0.0000D-01  0.0000D-01</td>
</tr>
<tr>
<td>0.0000D-01  0.0000D-01  0.0000D-01</td>
</tr>
<tr>
<td>0.0000D-01  0.0000D-01  0.0000D-01</td>
</tr>
<tr>
<td>0.0000D-01  0.0000D-01  0.0000D-01</td>
</tr>
<tr>
<td>1.0000D+00  0.0000D-01  0.0000D-01</td>
</tr>
<tr>
<td>0.0000D-01  9.9840D-01  0.0000D-01</td>
</tr>
<tr>
<td>0.0000D-01  0.0000D-01  9.8668D-01</td>
</tr>
</tbody>
</table>

\[ D_B = \begin{align*}
\text{Columns 1 through 5} \\
& 0.0000D-01  0.0000D-01  0.0000D-01  0.0000D-01  0.0000D-01 \\
& 0.0000D-01  0.0000D-01  0.0000D-01  0.0000D-01  0.0000D-01 \\
\end{align*} \]

<table>
<thead>
<tr>
<th>Columns 6 through 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0000D-01  5.6501D-02  0.0000D-01</td>
</tr>
<tr>
<td>0.0000D-01  0.0000D-01  1.6269D-01</td>
</tr>
</tbody>
</table>

\[ R = \begin{align*}
\text{Columns 1 through 5} \\
& 0.0000D-01  1.0000D+00  -3.9948D-04  1.0726D-19  -1.6968D-05 \\
& 0.0000D-01  0.0000D-01  1.0008D+00  -3.4003D-19  -2.8225D-02 \\
& 0.0000D-01  0.0000D-01  0.0000D-01  9.9972D-01  8.9635D-18 \\
& 0.0000D-01  0.0000D-01  0.0000D-01  0.0000D-01  9.9990D-01 \\
& 0.0000D-01  0.0000D-01  0.0000D-01  0.0000D-01  0.0000D-01 \\
& 0.0000D-01  0.0000D-01  0.0000D-01  0.0000D-01  0.0000D-01 \\
& 0.0000D-01  0.0000D-01  0.0000D-01  0.0000D-01  0.0000D-01 \\
\end{align*} \]
Columns 6 through 8

\[
\begin{array}{ccc}
4.1091D-04 & 3.9017D-20 & 9.6372D-03 \\
1.1774D-17 & -5.6529D-06 & 2.0192D-19 \\
-2.8639D-17 & 2.8284D-02 & -3.2103D-19 \\
-2.9092D-02 & -1.2288D-17 & 3.7681D-03 \\
-1.8409D-17 & 3.9964D-04 & -2.5966D-18 \\
0.0000D-01 & 9.9980D-01 & 3.0711D-18 \\
0.0000D-01 & 0.0000D-01 & 9.9924D-01 \\
\end{array}
\]

Then using the equation \[ A - BF \] = \[ R^T ( D_A^{-1} - D_B D_F ) D_A^2 \], the SVD of \[ A - BF \] is computed for a suitable range of \( \| F \|_2 \). Given the known relationship between the singular values and the damping factor, the damping factors corresponding to the maximum and minimum singular values of \[ A - BF \] are also calculated. The results are as shown in Figures 3-6a, 3-6b, 3-6c and 3-6d.
Figure 3-6a. Plot of $\gamma_{\text{max}}$ and $\gamma_{\text{min}}$ vs $\|F\|_2$ (different scale)

Figure 3-6b. Plot of $\gamma_{\text{max}}$ and $\gamma_{\text{min}}$ vs $\|F\|_2$ (same scale)
Figure 3-6c. Plot of $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ vs $\|F\|_2$ (different scale)

Figure 3-6d. Plot of $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ vs $\|F\|_2$ (same scale)
Figures 3-6a and 3-6b show the maximum and the minimum singular value of \([ A - BF ]\) decreasing as \(\| F \|_2\) increases. In comparison, the maximum singular value decreases faster than the minimum singular value. This may be an indication that the closer the open-loop poles are to the stability boundary, the more difficult it is to move them into the region of stability.

In addition, it is observed from Figures 3-6a through 3-6d that both the maximum singular value and the maximum damping factor of \([ A - BF ]\) are always outside the stability boundary, thus implying that some of the closed-loop poles might be in the unstable region. This, as we know, is undesirable.

Hence it is concluded that for this example of an eight-state, MIMO, unstable system, the consequences of using the proposed design method are unfavorable since the closed-loop system may be asymptotically unstable for any improvement in the damping.
EXAMPLE 3: ANALYSIS OF A STABLE, 2-STATE, SISO SYSTEM

Now consideration is given to an arbitrary discrete-time, stable system with poles chosen such that there is a pole close to the unit circle and one close to the origin. The hypothetical system is described by the linearized, discrete-time, difference equation of the form

\[
\begin{bmatrix}
    x_1(k+1) \\
    x_2(k+1) \\
    y(k)
\end{bmatrix} = \begin{bmatrix}
    0.8858 & 0.3625 \\
    0.000 & 0.1546 \\
    1.000 & 0.000
\end{bmatrix} \begin{bmatrix}
    x_1(k) \\
    x_2(k)
\end{bmatrix} + \begin{bmatrix}
    0.5786 \\
    0.2339
\end{bmatrix} u
\]

Applying the GSVD on the given matrices A and B where

\[
A = \begin{bmatrix}
    0.8858 & 0.3625 \\
    0.0000 & 0.1546
\end{bmatrix}
\]

and

\[
B = \begin{bmatrix}
    0.5786 \\
    0.2339
\end{bmatrix},
\]
the following results are obtained.

\[
U = \begin{bmatrix}
9.997468553730108E-001 & -2.249944825493706E-002 \\
-2.249944825493702E-002 & -9.997468553730107E-001
\end{bmatrix},
\]

\[
V = \begin{bmatrix}
-1.000000000000000
\end{bmatrix},
\]

\[
Q = \begin{bmatrix}
-9.999921419855260E-001 & -3.964336918021340E-003 \\
3.964336918021340E-003 & -9.999921419855260E-001
\end{bmatrix},
\]

\[
D_A = \begin{bmatrix}
1.000000 & 0.000000000000 \\
0.000000 & 5.513064088185218E-001
\end{bmatrix},
\]

\[
D_B = \begin{bmatrix}
0.000000000000E+000 & 8.343028488478419E-001
\end{bmatrix},
\]

and
\[ R = \begin{bmatrix} -8.774266093279941E-01 & 0.00000000000E+00 \\ 3.817225105788611E-01 & 1.560753669242873E-01 \end{bmatrix}, \]

U, V, Q are orthogonal, \(D_A\) and \(D_B\) are diagonal, and \(R\) is lower triangular.

Using the above matrices the maximum and minimum singular values of \([ A - BF ]\) as well as their corresponding damping factors are computed for some range of \(\| F \|_2\). The results are as plotted in Figures 3-7a, 3-7b, 3-7c and 3-7d.

**Figure 3-7a.** Plot of \(\gamma_{\text{max}}\) and \(\gamma_{\text{min}}\) vs \(\| F \|_2\) (different scale)
Figure 3-7b. Plot of $\gamma_{\text{max}}$ and $\gamma_{\text{min}}$ vs $\|F\|_2$ (same scale)

Figure 3-7c. Plot of $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ vs $\|F\|_2$ (different scale)
Figure 3-7d. Plot of $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ vs $\| F \|_2$ (same scale)

Figures 3-7a and 3-7b show that the maximum singular value decreases at a very slow rate while the minimum singular value decreases at a much faster rate as $\| F \|_2$ increases. This may possibly indicate that poles located close to the unit circle can be moved very little whereas poles further away from the unit circle can be shifted toward the origin by a greater amount.

It is also seen that the maximum singular value and the maximum damping factor are never above the stability boundary of $|z| = 1$ and $\Re(s) = 0$, respectively, for all values of $\| F \|_2$, indicating that all closed-loop poles lie within the stability
region.

Hence it is concluded that for this example of a two-state, SISO, stable system, the proposed design method is successful in determining the feedback gain matrix for some given improvement in the damping bound without sacrificing stability.
EXAMPLE 4: ANALYSIS OF A STABLE, 4-STATE, MIMO SYSTEM

As a further example of a stable system, consider a hypothetical two-input/two-output system that has continuous-time poles located close to the origin at \( s = -1, -2, -3 \), having the following transfer function:

\[
H_{11} = \frac{Y_1}{U_1} = \frac{1}{s + 1}
\]

\[
H_{12} = \frac{Y_2}{U_1} = \frac{1}{(s + 1)(s + 3)}
\]

\[
H_{21} = \frac{Y_1}{U_2} = \frac{2}{(s + 1)(s + 2)}
\]

\[
H_{22} = \frac{Y_2}{U_2} = \frac{1}{s + 3}
\]

Notice that \( H_{12} \) and \( H_{21} \) may be rewritten, using partial-fraction expansion, as

\[
H_{12} = \frac{1/2}{s + 1} - \frac{1/2}{s + 3}
\]

\[
H_{21} = \frac{2}{s + 1} - \frac{2}{s + 2}
\]
Model reduction yields the irreducible block diagram as shown in Figure 3-8.

![Block Diagram](image)

**Figure 3-8.** Reduced-order block diagram of the system in Example 4

From the block diagram of Figure 3-8, the state-space form for the system can be directly derived, resulting in the continuous state-space form

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -2 & 0 \\
0 & 0 & 0 & -3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} +
\begin{bmatrix}
1 & 2 \\
1/2 & 0 \\
0 & -2 \\
-1/2 & 1
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]
\[
\begin{bmatrix}
  y_1 \\
y_2
\end{bmatrix}
= \begin{bmatrix}
  1 & 0 & 1 & 0 \\
  0 & 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
\]

To obtain the discrete-time, state-space realization of the form

\[
\begin{align*}
x(k+1) &= A x(k) + B u(k) \\
y(k) &= C x(k)
\end{align*}
\]

the system is discretized at a sampling frequency of 25 hertz. The resulting discretized

A and B are as given below.

\[
A = \begin{bmatrix}
  0.9608 & 0.0000 & 0.0000 & 0.0000 \\
  0.0000 & 0.9608 & 0.0000 & 0.0000 \\
  0.0000 & 0.0000 & 0.9231 & 0.0000 \\
  0.0000 & 0.0000 & 0.0000 & 0.8869
\end{bmatrix}
\]

and

\[
B = \begin{bmatrix}
  0.0392 & 0.0784 \\
  0.0196 & 0.0000 \\
  0.0000 & -0.0769 \\
 -0.0188 & 0.0377
\end{bmatrix}
\]
The generalized singular value decomposition (via the direct algorithm) of A and B yields diagonal matrices $D_A$ and $D_B$, an upper triangular matrix $R$, and orthogonal matrices $U$, $V$ and $Q$, such that

$$U = \text{Columns 1 through 3}$$

\[
\begin{array}{ccc}
4.124490167072410\text{E-01} & 1.848927499299579\text{E-01} & -6.958017235211479\text{E-01} \\
-2.579575144171500\text{E-01} & -8.600363771939690\text{E-01} & -2.906525599521619\text{E-02} \\
6.823930188890360\text{E-01} & -5.963662611502636\text{E-02} & 6.510155036182553\text{E-01} \\
5.456037913582197\text{E-01} & -4.718003537006177\text{E-01} & -3.019834208820119\text{E-01} \\
\end{array}
\]

Column 4

\[
\begin{array}{c}
-5.581759948130774\text{E-01} \\
-4.392613818135300\text{E-01} \\
-3.270505383664637\text{E-01} \\
6.233192942753386\text{E-01} \\
\end{array}
\]

$$V = \begin{bmatrix}
-1.793645069782939\text{E-01} & -9.837826861845220\text{E-01} \\
-9.837826861845220\text{E-01} & 1.793645069782940\text{E-01}
\end{bmatrix},$$

$$Q = \text{Columns 1 through 3}$$

\[
\begin{array}{ccc}
-3.949661894760025\text{E-01} & 1.994384301487895\text{E-01} & -6.861176073719678\text{E-01} \\
2.470232498780838\text{E-01} & -8.557974025854798\text{E-01} & -8.480631093861974\text{E-03} \\
-6.801558943970972\text{E-01} & -3.020739572148947\text{E-02} & 6.563536089881167\text{E-01} \\
-5.660116452047262\text{E-01} & -4.763639696193007\text{E-01} & -3.136409535100456\text{E-01} \\
\end{array}
\]
Column 4
-5.774674454934746E-001
-4.544429542240323E-001
-3.250775484847953E-001
5.952625792223886E-001

\[ R = \begin{array}{cccc}
-9.200737533044071E-01 & 4.160049443711986E-02 & -8.114805227540746E-03 \\
0.000000000000000E+00 & 9.435868130466051E-01 & -1.977039372641436E-02 \\
0.000000000000000E+00 & 9.435868130466051E-01 & 9.373639489742065E-01 \\
0.000000000000000E+00 & 0.000000000000000E+00 & 0.000000000000000E+00
\end{array} \]

Column 4
-3.293439824227185E-002
4.174665429005967E-002
4.395812482642156E-002
9.287025615064992E-001

\[ D_A = \begin{bmatrix}
1.00 & 0.00 & 0.000000000000000 & 0.000000000000000 \\
0.00 & 1.00 & 0.000000000000000 & 0.000000000000000 \\
0.00 & 0.00 & 9.9216902190E-01 & 0.000000000000000 \\
0.00 & 0.00 & 0.000000000000000 & 9.9895794842E-01
\end{bmatrix} \]

and
\[ D_B = \begin{bmatrix} 0.00 & 0.00 & 1.2490248983E-01 & 0.000000000000000 \\ 0.00 & 0.00 & 0.000000000000000 & 4.564008403E-02 \end{bmatrix} \]

The maximum and minimum singular values of \([ A - BF ]\) and their corresponding damping factors are computed for some range of \(\| F \|_2\). The results are shown in Figures 3-9a, 3-9b, 3-9c and 3-9d.

**Figure 3-9a.** Plot of \(\gamma_{\text{max}}\) and \(\gamma_{\text{min}}\) vs \(\| F \|_2\) (different scale)
Figure 3-9b. Plot of $\gamma_{\max}$ and $\gamma_{\min}$ vs $\| F \|_2$ (same scale)

Figure 3-9c. Plot of $\sigma_{\max}$ and $\sigma_{\min}$ vs $\| F \|_2$ (different scale)
Figure 3-9d. Plot of $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ vs $\|F\|_2$  

Figures 3-9a and 3-9b show the maximum and the minimum singular value of $[A - BF]$ decreasing as $\|F\|_2$ increases. In comparison, the maximum singular value decreases faster than the minimum singular value. This may be an indication that the closer the open-loop poles are to the stability boundary, the more difficult it is to move them into the origin.

In addition, it is observed from Figures 3-9a through 3-9d that both the maximum singular value and the maximum damping factor of $[A - BF]$ are always well within the stability boundary, thus indicating that all the poles of the closed-loop are in the stable region.
Here it is observed that the feedback gain can be determined using damping as the performance criterion. More importantly, an improvement in damping can be achieved without making the closed-loop system unstable.
CHAPTER 4
CONCLUSIONS

In the preceding chapters it has been extensively discussed how linear state feedback control systems may be designed using the GSVD approach. The chapter on examples is devoted to addressing the question of whether such systems exhibit desired characteristics namely, guaranteed stability and favorable response from initial conditions to a reference value.

The results of the previous chapter, which concern the stability properties and response characteristics of worked problems, give considerable insight into the applicability of the proposed scheme.

As seen in the analysis of the four examples presented in the preceding chapter, the GSVD can be applied successfully to devise the full-state feedback gain using damping as the performance criterion. As a consequence, it is possible to develop design curves that map damping factor to the 2-norm of the feedback gain matrix $F$ which, in turn, is related to the elements of the singular value decomposition of $F$. However, the extent to which these design curves are meaningful in the implementation of a fast-response, stable, closed-loop full state feedback control system is restricted.
In examples 3 and 4, it is observed that for open-loop stable systems, use of the GSVD approach to obtain the damping bound for some specified diagonal elements of the feedback gain matrix results in stable closed-loop systems that exhibit an improvement in "performance". However, this is not the case with asymptotically unstable systems. Results of examples 1 and 2 show that for asymptotically unstable systems, the proposed theory is not sufficient to guarantee closed-loop stability even though it promises enhanced "performance".

Hence, in essence, the control law devised relocates open-loop poles to a region that guarantees stability and improved response only if the open-loop poles themselves are stable. Therefore, an unfavorable situation may arise, particularly when a system possesses unstable open-loop poles.

It is also observed that all four examples depict a general trend in that the largest singular value (hence the maximum damping factor) decreases at a very slow rate, while the smallest singular value (hence the minimum damping factor) decreases at a faster pace. This serves to show that poles located near the stability boundary (unit circle, \( z = 1 \)) are more difficult to shift (inward) compared to poles located near the origin. This may be advantageous since it is found in practice that small margins in performance enhancement lead to robust systems.

A drawback of the design method suggested is that the guideline for choosing the diagonal factor of \( F \) gets numerically complicated as the number of
inputs (to a system) increases. This is due to the fact that each additional input results in an added degree of freedom in the elements of the diagonal factor. Lastly, as with most feedback control systems, it is often impossible or too costly to measure all components of the state, thus application of the GSVD approach is, at this stage, restricted to the case where it is assumed everything needed to be known about "A" is known.

**Future Studies**

Since the estimator/observer design is the dual to the full-state feedback control problem, the GSVD approach can be applied to the separate observer/estimator design. The difference now is that instead of having a BF term, the observer has a perturbation term, LC, where C is the parameter that describes the output of the system and L is some chosen feedback gain of the observer.

A possible approach would be to let \( \| A - LC \|_2 = \| A - BF \|_2 \), that is requiring the state estimates of the observer to decay at the same minimum rate as the plant state variables with minimum perturbation of the system matrix. Lastly, the robustness and stability of the closed-loop system would have to be investigated by analysing the eigenvalues of \( [ A - BF + LC ] \) which describes the closed-loop,
compensated system.

This thesis is merely a preliminary effort in the investigation of the feasibility of an alternate method to conventional feedback control design techniques. In terms of rendering a solution for practical linear full-state feedback control problems, it is modestly prospective, even favorable at times. To truly assess its worth as a likely successor to existing control theory, subsequent effort will have to be devoted to the investigation of observer design and robustness properties as proposed.
REFERENCES


