MODELING AND OPTIMAL SHAPE CONTROL OF A LAMINATED COMPOSITE THIN PLATE WITH PIEZOELECTRIC ACTUATORS SURFACE EMBEDDED OR BONDED

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CHAPTER 1

INTRODUCTION

Piezoelectric materials exhibit both direct piezoelectric effect and converse piezoelectric effect,\textsuperscript{1-3} which are to develop a strain field from an applied electric field and to produce an electric field in response to an applied strained field respectively. Since the first recorded findings of the two effects (1880, 1881), the application of piezo materials has passed two generations and is now the third generation.\textsuperscript{3} From the early 1980s, piezoelectric ceramics have been attracting significant attention for active control and for sensor of space structures, aircraft structures, satellite structures, and the like.\textsuperscript{6-26}

1-1 Recent Research on Piezoelectric Structures

Large space structures, owing to low weight requirements, will be very flexible and therefore they will need some type of active control system for vibration suppression and to maintain shape specifications.\textsuperscript{4-6} Some surface-bonded beam-like structures,\textsuperscript{7-11} plates,\textsuperscript{10-19, 26} and shells\textsuperscript{20,21} have been previously studied, usually to explore the use of piezo actuator technology for vibration control of space structures. An active control system using distributed piezoelectric sensors and actuators has been proven effective in controlling the vibration of distributed elastic systems.

The integration of composite structural design with the “smart structures” concept could potentially result in significant improvement in the performance of aircraft and
space structures. The feasibility of such integrated smart structures has been demonstrated by some examples of cantilever beams and several analyses and numerical models have also been developed to analyze such integrated structures.\textsuperscript{10-16,20} Most of them are based on analytic approaches and the Ritz method.\textsuperscript{11,12,14} Finite element methods (FEM) for composite structures with integrated piezoelectric materials are described in Ref. 24, 25. Finite difference analysis, which is the early stage of this research, is also applied in Ref. 20. An overview of smart structure technology is presented in Ref. 27. But, still, few problems attacked have extended to practical designs.\textsuperscript{22,23}

In all applications, the determination of optimal shape of such structures is of much importance,\textsuperscript{6} especially in space applications where weight restrictions are stringent. The effectiveness of the control system strongly depends on the active element locations.\textsuperscript{9,10}

In those 3-D models, when applied for a thin plate, some special techniques are required, thus making the problems large and complex. For example, a plate with thin sensors and actuators are modeled with the isoparametric hexahedron solid element,\textsuperscript{23} requiring reduce the total degrees of freedom by using Guyan reduction. Another example is, when an 8-node, 32-DOF brick element was applied to a thin plate example,\textsuperscript{24} three-dimensional incompatible modes was necessary for predicting the deflection. In the former example, when the thickness of a plate is very thin, there are problems of excessive shear strain energies and higher stiffness coefficients in the thickness direction.
Although finite element methods have been developed for piezoelectric structures, the use of finite element techniques analyzing the integrated piezo-composite structures has not been fully established. To advance this technology further for more complicated and large-scaled structures, a through and comprehensive development in theory and numerical methods is critically important and essentially needed.

1-2 The Goals of the Research

The objectives of this research work are:

(a) to develop a simpler 2-D (relative to 3-D model and refer to independent variables \( x \) and \( y \) and \( z \)) mathematical model for the deflection of a thin plate composed of laminate composite materials with piezo actuators surface embedded/bonded using Kirchhoff thin plate theory,

(b) to study the validity and the efficiency of applying finite element method by using a 4-node, 12-degrees-of-freedom thin plate DKQ (discrete Kirchoff quadrilateral) bending element, with one electrical degree of freedom, to solve normal deflection of a large-size plate composed of laminated composite materials and piezoelectric actuators,

(c) to develop a mathematical model for finding out the optimal actuation voltages applied to the actuators and their optimal layout as well as the optimal number of those actuators to match the deflection of a plate to certain desired deflection through dividing the entire plate domain into a discrete set of possible locations and

(d) to develop an algorithm/algorithms to solve the problems of (c).
1-3 About Piezoelectric Behavior

The general linear relationships between the mechanical and electrical variables, the so called piezoelectric constitutive relations, and the piezoceramic constitutive matrix, which specifies the special case of a piezoceramic, are reviewed in this section.

1-3.1 Piezoelectric Constitutive Relations

There are two mechanical and two electrical quantities, electric field $E$ and electric displacement $D$ and stress $\sigma$ and strain $\varepsilon$, involved in describing piezoceramic behavior. When there is electromechanical coupling, the following constitutive relations\(^1\,^2\)

$$D_i = \varepsilon_{ij} E_j + d_{ijk} \sigma_{kl}$$

(1.1)

$$\varepsilon_{ij} = d_{kij} E_k + S_{ijkl} \sigma_{kl}$$

(1.2)

couple the mechanical and electrical problems $D_i = \varepsilon_{ij} E_j$ and $\varepsilon_{ij} = S_{ijkl} \sigma_{ij}$.

In the above tensorial expressions, $\varepsilon_{ij}$ are permittivities or dielectricities, and $S_{ijkl}$ are elastic compliances, a fourth order tensor. The superscripts $\sigma$ and $E$ denote that those quantities are for a constant elastic stress field and a constant electric field, respectively. The coupling terms $d_{ijk}$ are the piezoelectric strain constants which relate strain to applied electric field. Their physical meaning is clear: the higher the $d_{ijk}$ coefficient the larger the amount of strain per field.
According to the definition of the strain tensor, for small deformation, there is
\[ \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \]  
where the tensorial components \( \varepsilon_{ij} \) should not be confused as the engineering strain components**. The electric field \( E_i \) is related to the electric potential \( V \) by
\[ E_i = -V_{,i} \]  
Now if a set of equilibrium equations, i.e., the equation of motion for mechanical behavior and the Maxwell’s equation for electrical behavior, are added to the above Eqs.(1.1-1.4), the system describing piezo electromechanical behavior will be complete.

1.3.2 Piezoceramic Constitutive Equations

For the special case of a piezoceramic plate, the full matrix form for the degraded constitutive Eqs.(1.1, 1.2) is shown as below:\(^1\,^2\,^10\):

\[
\begin{bmatrix}
D_1 & \varepsilon^\sigma_1 & 0 & 0 & 0 & 0 & d_{15} & 0 \\
D_2 & 0 & \varepsilon^\sigma_1 & 0 & 0 & 0 & 0 & d_{15} & 0 \\
D_3 & 0 & 0 & \varepsilon^\sigma_2 & d_{31} & d_{31} & d_{33} & 0 & 0 \\
\varepsilon_1 & 0 & 0 & d_{31} & S_{11}^E & S_{12}^E & S_{13}^E & 0 & 0 \\
\varepsilon_2 & 0 & 0 & d_{31} & S_{12}^E & S_{11}^E & S_{13}^E & 0 & 0 \\
\varepsilon_3 & 0 & 0 & d_{33} & S_{13}^E & S_{13}^E & S_{33}^E & 0 & 0 \\
\varepsilon_4 & 0 & d_{15} & 0 & 0 & 0 & 0 & S_{55}^E & 0 \\
\varepsilon_5 & d_{15} & 0 & 0 & 0 & 0 & 0 & 0 & S_{55}^E \\
\varepsilon_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & S_{66}^E \\
\end{bmatrix}
\begin{bmatrix}
E_1 \\
E_2 \\
E_3 \\
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6 \\
\end{bmatrix}
\]  

* The components of the strain tensor are defined as \( \varepsilon_{ij} = \frac{1}{2} (u_{ij} + u_{ji}) \). For small deformation, the higher order terms are neglected. Thus one has Eq.(1.3).
** From Eq.(1.3), the tensorial strain components, in relating to the engineering strain components, are
\[ \varepsilon_{11} = \varepsilon_x, \; \varepsilon_{22} = \varepsilon_y, \; \varepsilon_{33} = \varepsilon_z, \; \varepsilon_{12} = \frac{1}{2} \gamma_{xy}, \; \varepsilon_{13} = \frac{1}{2} \gamma_{xz}, \; \varepsilon_{23} = \frac{1}{2} \gamma_{yz}. \]
where, $S_{66}^E$ is the compliance matrix, the inverse of the stiffness matrix $Q_{ij}$ (Chapter 2 section 2.2), and the strains $\varepsilon_i$ are engineering components. The condensed notation for stress, strain and compliance, with subscripts 1-6 is used.

To understand the information given in the $9\times9$ constitutive block matrix in Eq.(1.7), take a piezoceramic and set the coordinate system as such that $x_3$ parallels the direction of the initial polarization of the piezoceramic with $x_1$ and $x_2$ chosen arbitrarily in the plane perpendicular to $x_3$ direction. In an actual specimen, the $x_1$ and $x_2$ directions are usually defined so they align with the principal or physical axes of symmetry of the manufactured ceramic. Fig.(1.1) illustrates this coordinate system.

The subscripts 1, 2 and 3 correspond to the properties along the $x_1$, $x_2$ and $x_3$ axes and the subscripts 4, 5 and 6 refer to the shear properties in the $x_2$-$x_3$, $x_3$-$x_1$ and $x_1$-$x_2$ planes respectively. Such shear coupling appears in the mechanical part of the problem. The diagonal form of dielectric matrix implies that there is no such analogue in the dielectrical part of the problem. The shear strain $\varepsilon_6$ in the $x_1$-$x_2$ plane can not be expected by the electric field $E_3$ applied in the poling direction $x_3$.

---

* The subscripts 1 through 6 may assume any order of six engineering strain components. Here choose

$\varepsilon_1=\varepsilon_{11}=\varepsilon_{xx}, \varepsilon_2=\varepsilon_{22}=\varepsilon_{yy}, \varepsilon_3=\varepsilon_{33}=\varepsilon_{zz}, \varepsilon_4=2\varepsilon_{12}=\gamma_{yx}, \varepsilon_5=2\varepsilon_{13}=\gamma_{xz}, \varepsilon_6=2\varepsilon_{13}=\gamma_{xy},$

$\sigma_1=\sigma_{11}=\sigma_x, \sigma_2=\sigma_{22}=\sigma_y, \sigma_3=\sigma_{33}=\sigma_z, \sigma_4=\sigma_{12}=\tau_{yx}, \sigma_5=\sigma_{13}=\tau_{xz}, \sigma_6=\sigma_{12}=\tau_{xy}.$
In the piezoelectric matrices, the subscripts are written in such a conventional way that the first subscript is for electric variable while the second for the mechanical. So the strain in \( x_i \) due to the electric field in \( x_3 \) is expressed as \( \varepsilon_i \propto d_{31}E_3 \), rather than \( \varepsilon_i \propto d_{13}E_3 \). The subscripts 2 and 4 are substituted by 1 and 5, because, ideally, piezoceramic plates are transversely isotropic with respect to electrical, mechanical, and piezoelectric properties. Also, \( S_{66}^E \) is not an independent coefficient. It can be expressed as

\[
S_{66}^E = 2(S_{11}^E - S_{12}^E) .
\]

Therefore, in the piezoceramic matrices there are only 10 independent material property coefficients (2 electrical, 5 elastic, and 3 piezoelectric).

![Poling direction](image)

(a) A piezoceramic with no electric field applied

![Illustrations](image)

(b) for \( d_{33} \) and \( d_{31} \)

(c) for \( d_{15} \)

Fig.(1.2) Explanations of Piezoelectric Strain Coefficients

The illustrations for understanding the physical meaning of the piezoelectric strain coefficients \( d_{31}, d_{33}, d_{15} \) are shown in the Fig.(1.2). For \( x_3 \), the strain is
\[ \frac{\Delta t}{t} = d_{33} \frac{V_3}{t} \quad \text{or} \quad \varepsilon_3 = d_{33} E_3, \quad (1.7a) \]

and for \( x_1 \) (or \( x_2 \)), the strain is

\[ \frac{-\Delta w}{w} = d_{31} \frac{V_3}{t} \quad \text{or} \quad \varepsilon_1 = d_{31} E_3, \quad (1.7b) \]

and for \( x_3-x_1 \) (or \( x_2-x_3 \)) the shear strain is

\[ \frac{dx}{t} = d_{15} \frac{V_1}{t} \quad \text{or} \quad \varepsilon_5 = d_{15} E_1, \quad (1.7c) \]

where \( \Delta \) indicates the changes in \( t \) and \( w \) after the actuation voltages are applied on the actuators.
CHAPTER 2
ACTUATOR/PLATE SYSTEM MODEL

On obtaining mathematical description for the behavior of an actuator/plate system, stress resultants, stress couples and shear resultants are needed. On deriving those resultants, strain-displacement and stress-strain relationships have to be determined. And on getting those relationships, certain assumptions for the behavior of the actuator/plate system are the basic.

In this chapter a displacement $U-V-W$ model and a stress-function-normal-displacement $\Phi-W$ model, are going to be presented. Both of the two models are based on Kirchhoff's thin plate hypotheses.

2-1 Physical System and Basic Assumptions

The physical actuator/plate system under study is a homogeneous, anisotropic and elastic laminated composite thin plate or isotropic thin plate, which has piezoelectric actuators embedded or bonded on its top and bottom surfaces in a bimorph arrangement, i.e., to every actuator centered at $(x, y)$ on the bottom surface of the plate, there is an actuator on the top surface centered at
(x, y) as shown in Fig.(2.1). The piezo actuators are modeled as homogeneous, transversely isotropic in mechanical and piezoelectric behavior and also elastic.

The system coordinates x-y-z respectively relate to the laminae principle material coordinates x₁-x₂-x₃ which has been described in Section 1-3.2 as shown in Fig.(2.1). In the laminae, axis 1 is parallel to the fibers while axis 2 is normal to them, and the orientation θ can be arbitrarily chosen.

Cross sectional structures of the substrate and the plate with actuators surface embedded and bonded are shown in Fig.(2.2). The actuator and substrate plates are given the subscripts a and s respectively. The actuator plate is tₐ and the total thickness of the plate is t. The poling directions of all actuator plates are the same as the z direction. The applied electric fields on the top and bottom actuators of each bimorph have a same electric potential with one in the poling direction while the other against it, thus inducing extension in one actuator but contraction in the other. Such pairing is favorable for the Kirchhoff's assumptions for a thin plate system under bending which will be given later.

The actuator plates are perfectly bonded in the pockets dug out on the top and bottom surfaces of the substrate plate or perfectly bonded on both surfaces. Perfect here means that the deformations in the actuators and substrates at a point on a bonding
surface between the actuators and substrates are consistent. Such a consistent plate model is based on the assumption that the interlaminar bonding layer between the actuators and the substrates is sufficiently thin that neglecting the shear layer will not introduce any significant errors into the model\textsuperscript{13}.

The Kirchhoff’s hypotheses\textsuperscript{29} used in the mathematical modeling says:

(1) Plane sections initially normal to the midsurface remain plane and normal after the bending. This results to $\gamma_{xz}=\gamma_{zx}=0$ and $\varepsilon_{x}=0$, or $\varepsilon_{4}=\varepsilon_{5}=\varepsilon_{3}=0$ in Eq.(1.5).

(2) The stress normal to the midplane is small compared with the other stress components and may be neglected. Therefore $\sigma_{z}=0$, or $\sigma_{3}=0$ in Eq.(1.5).

2-2 Basic Relations

Based on the Kirchhoff’s assumptions, strain-displacement and stress-strain relationships can be compactly expressed as equations(2.1, 2.5) respectively.

2-2.1 Strain-Displacement Relationship

For convenience of the studies a compacted basic strain-displacement relationship

\[ \varepsilon = \varepsilon_{0} + z\kappa_{0} \]  

(2.1)

with $\varepsilon$, $\varepsilon_{0}$ and $\kappa_{0}$ given as in Eqs.(2.3, 2.4a,b) respectively is reviewed in this section.

Let $u$, $v$, $w$ denote respectively the $X$, $Y$, $Z$ displacements of a point $(x, y, z)$ on the plate and $U$, $V$, $W$ denote respectively the $X$, $Y$, $Z$ displacements of the point $(x, y)$ on the midplane of the plate. Based on the first assumption in Section 2-1, the displacements $u$ and $v$ are related to the mid-plate displacements $U$, $V$ and $W$ as
According to the classical small deflection theory, the normal strains $\varepsilon_x$ and $\varepsilon_y$ and the shear strain $\gamma_{xy}$ at the point $(x, y)$, in a vector form, are

$$
\varepsilon = \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial y}
\end{bmatrix} \quad \text{(2.3)}
$$

Substituting $u$ and $v$ in Eq.(2.3) results for the equation (2.1) with

$$
\varepsilon_0 = \begin{bmatrix}
\varepsilon_{0x} \\
\varepsilon_{0y} \\
\gamma_{0xy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial U}{\partial x} \\
\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \\
\frac{\partial U}{\partial y}
\end{bmatrix} \quad \text{and} \quad \kappa_0 = \begin{bmatrix}
\kappa_{0x} \\
\kappa_{0y} \\
2\kappa_{0xy}
\end{bmatrix} = \begin{bmatrix}
-\frac{\partial^2 W}{\partial x^2} \\
-\frac{\partial^2 W}{\partial y^2} \\
-2\frac{\partial^2 W}{\partial x\partial y}
\end{bmatrix} \quad \text{(2.4a, b)}
$$

being the mid-plane strain and the mid-plane curvature corresponding to the point $(x, y)$ which are the functions of midplane displacements $U, V$ and $W$ only.

**2-2.2 Stress-Strain Relationship**

In this section, it will be shown that the stresses can generally be expressed in terms of mid-plane deflection via $\varepsilon_0$ and $\kappa_0$, i.e.,

$$
\sigma = Q(\varepsilon - \varepsilon_a) = Q\varepsilon_0 + Q\varepsilon_a
$$

where $\varepsilon_a$ is given in Eq.(2.9) and $Q$ is $Q_a$ in Eq.(2.11) within a domain of actuators, and $\varepsilon_a$ is zero and $Q$ is $\overline{Q}$, in Eq.(2.18) within a domain of substrate.
2-2.2.1 In Actuators

For a piezoceramic plate, based on the Kirchhoff hypotheses, the constitutive Equation (1.5) can be written as, by crossing out the rows of $\varepsilon_3$, $\varepsilon_4$, $\varepsilon_5$ and the relative 6, 7, 8 columns, two sets of direct and converse equations of linear piezoelectric couplings between the elastic field and the electric field for a plate-shape sensor and an actuator:

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_{z1} & d_{z1} & 0 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} + \begin{bmatrix} \xi_{s1} & 0 & 0 \\ 0 & \xi_{s1} & 0 \\ 0 & 0 & \xi_{s3} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$  \hfill (2.6)

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} S_{11}^E & S_{12}^E & 0 \\ S_{12}^E & S_{11}^E & 0 \\ 0 & 0 & S_{66}^E \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} + \begin{bmatrix} 0 & 0 & d_{z1} \\ 0 & 0 & d_{z1} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix},$$  \hfill (2.7)

where subscript 3 refers to the poling axis which is $z$, 1 and 2 to $x$ and $y$ respectively, and 1 in $d_{z1}$ especially indicates the behavior in the $x$-$y$ plane.

The in-plane actuation strain can be derived from the converse piezoelectric equation Eq.(2.7). Since no stress field is applied to the actuator layer and also the electric field $E$ (which is $\begin{bmatrix} 0 & 0 & -V / t_a \end{bmatrix}^T$, recalling Eqs.(1.4, 1.7b) with the subscript omitted) is applied only on $z$ axis which corresponds to the axis of initial polarization of the piezoceramic, the strain $\varepsilon$ developed by the electric field on the actuator layer can be expressed as a linear combination of strains due to the local stresses $\begin{bmatrix} \sigma_x & \sigma_y & \sigma_z \end{bmatrix}^T$ and the actuation strains $\begin{bmatrix} \varepsilon_{ax} & \varepsilon_{ay} & \varepsilon_{az} \end{bmatrix}^T$, i.e.,
\[ \varepsilon = S_a \sigma + \varepsilon_a \]  

(2.8)

where \( S_a \) is the stiffness matrix of piezoceramic actuators in Eq.(2.7) with the superscript \( E \) omitted and the actuation strain \( \varepsilon_a \) is

\[
\varepsilon_a = \begin{bmatrix}
  d_{z1} E_z \\
  d_{z1} E_z \\
  0
\end{bmatrix} = \begin{bmatrix}
  -d_{z1} / t_a \\
  -d_{z1} / t_a \\
  0
\end{bmatrix} V .
\]

(2.9)

Therefore \( \sigma \) can be solved as

\[
\sigma = Q_a \varepsilon_0 + Q_a z \kappa_0 - Q_a \varepsilon_a
\]

(2.10)

where Eq.(2.1) has been employed and the matrix \( Q_a \), which is the inverse of \( S_a \), is

\[
Q_a = S_a^{-1} = \frac{E_{1a}}{1 - \nu_{1a}^2} \begin{bmatrix}
  1 & \nu_{1a} & 0 \\
  \nu_{1a} & 1 & 0 \\
  0 & 0 & (1 - \nu_{1a}) / 2
\end{bmatrix} .
\]

(2.11)

Here, the subscript 1 are, again, associated with the behavior in the \( x-y \) plane. The initial strains \( \varepsilon_0 \) caused by thermal effect could be simply added on to the right hand side of Eq.(2.8), but are not included in the studies of the system behavior for simplicity. It will be directly introduced in the section of numerical implementation (Chapter 3 & 4) later.

2-2.2.2 In Substrate

For a laminae of the substrate composite plate, according to Kirchhoff hypotheses, the generalized Hoke's law\(^{29,34}\) is reduced to

\[ \varepsilon' = S'_z \sigma' , \]

(2.12)
where \( \varepsilon' = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_6 \end{bmatrix}^T \) and \( \sigma' = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_6 \end{bmatrix}^T \). If the strains and stresses under the physical system coordinates are denoted as \( \varepsilon = \begin{bmatrix} \varepsilon_x & \varepsilon_y & \gamma_{xy} \end{bmatrix}^T \) and \( \sigma = \begin{bmatrix} \sigma_x & \sigma_y & \tau_{xy} \end{bmatrix}^T \), then the transformation from the physical system coordinates to the principle material coordinates for strains and stresses can be expressed as

\[
\varepsilon' = T^T \varepsilon \quad \text{and} \quad \sigma' = T^{-1} \sigma \tag{2.13a, b}
\]

where

\[
T = \begin{bmatrix}
m^2 & n^2 & -2mn \\
n^2 & m^2 & 2mn \\
mn & -mn & m^2 - n^2
\end{bmatrix} \tag{2.14}
\]

is the strain transformation matrix\(^{39} \) and \( m \) and \( n \) are

\[
m = \cos \theta \quad \text{and} \quad n = \sin \theta \tag{2.15}
\]

with \( \theta \) positive if the orientation of the fibers from physical system axis \( x \) to principle material axis \( 1 \) is counterclockwise as shown in Fig.(2.1).

Therefore, by substituting Eqs.(2.13a, b) in Eq.(2.12), the stress-strain relationship under the system coordinates is obtained:

\[
\sigma = \bar{Q}_s \varepsilon \tag{2.16}
\]

where

\[
\bar{Q}_s = TQ_s T^T \tag{2.17}
\]
\[ Q_s = S_s^{-1} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \] (2.18)

assuming the material properties of each ply of the substrate as generally anisotropic.

Noted that \( \bar{Q}_s \) must be symmetric like \( Q_s \), the explicit forms of each component of \( \bar{Q}_s \) are obtained by performing matrix products in Eq.(2.17) as

\[
\begin{align*}
\bar{Q}_{11} &= m^4 Q_{11} + 2m^2 n^2 (Q_{12} + 2Q_{66}) + n^4 Q_{22} - 2m^2 n Q_{16} - 2mn^3 Q_{26} \\
\bar{Q}_{12} &= m^4 n^2 (Q_{11} + Q_{22} - 4Q_{66}) + (m^4 + n^4) Q_{12} + 2mn(m^2 - n^2)(Q_{16} - Q_{26}) \\
\bar{Q}_{16} &= m^4 n(Q_{11} - Q_{12} - 2Q_{66}) + mn^3 (Q_{12} - Q_{22} + 2Q_{66}) + m^2 (m^2 - 3n^2) Q_{16} + n^2 (3m^2 - n^2) Q_{26} \\
\bar{Q}_{22} &= m^4 Q_{22} + 2m^2 n^2 (Q_{12} + 2Q_{66}) + n^4 Q_{12} + 4mn^3 Q_{16} + 4m^3 n Q_{26} \\
\bar{Q}_{26} &= m^4 n(Q_{12} - Q_{22} + 2Q_{66}) + mn^3 (Q_{11} - Q_{12} - 2Q_{66}) + n^2 (3m^2 - n^2) Q_{16} + m^2 (m^2 - 3n^2) Q_{26} \\
\bar{Q}_{66} &= m^2 n^2 (Q_{11} - 2Q_{12} + Q_{22}) + (m^2 - n^2) Q_{66} + 2mn(m^2 - n^2)(Q_{16} - Q_{26}) 
\end{align*}
\] (2.19)

Normally, the material properties of most fibers in laminated composite plate are orthotropic which results in \( Q_{16}=Q_{26}=0 \). The stiffness matrix takes the form of

\[
\bar{Q}_s = \frac{E_2}{1 - \nu_{21}^2} \begin{bmatrix} n & n\nu_{21} & 0 \\ n\nu_{21} & 1 & 0 \\ 0 & 0 & m(1 - \nu_{21}^2) \end{bmatrix}
\] (2.20)

with \( n = E_1 / E_2 = \nu_{12} / \nu_{21} \) and \( m = G_{12} / E_2 \). Hence there are \( \bar{Q}_{16} = \bar{Q}_{26} = 0 \). If the substrate material is isotropic, then the matrix simply becomes

\[
\bar{Q}_s = Q_s = \frac{E_a}{1 - \nu_a^2} \begin{bmatrix} 1 & \nu_a & 0 \\ \nu_a & 1 & 0 \\ 0 & 0 & (1 - \nu_a)/2 \end{bmatrix}
\] (2.21)
Recalling Eq.(2.1) and substituting it in Eq.(2.16), the final compact expression of
the stresses in terms of mid-plane deflections, vie $\varepsilon_0$ and $\kappa_0$ in Eqs.(2.4a,b), is obtained as
\[
\sigma = \bar{Q}_s \varepsilon_0 + \bar{Q}_s \varepsilon_0.
\]  
(2.22)

Now with the stresses in Eq.(2.5) determined, stress resultants and stress couples
can be obtained.

2-2.3 Stress Resultants, Stress Couples and Shear Resultants

In determining those resultants, it is found that by arranging the actuators mid-
plane symmetric and applying the opposite electric potential fields to the actuators in a
bimorph arrangement, the piezo actuation (characterized by the input voltages $V$) only
changes the stress couples through providing bending moments $M_\lambda$ as
\[
M_\lambda = \begin{bmatrix} M_{\lambda x} & M_{\lambda y} & 0 \end{bmatrix}^T = LV
\]  
(2.23)
where $L$ is given as in Eq.(2.33) for a location with actuation unit and zero for without
actuation unit. Piezo actuation makes no contribution to stress resultants, nor to shear
resultants.

2-2.3.1 Stress Resultants and Stress Couples

After integrating the stresses $\sigma$ in Eq.(2.5) across the overall plate thickness $t$, the
stress resultants and the stress couples, denoted by $N$ and $M$ respectively, turn out to be
\[
N = \int_0^t \sigma dz = A\varepsilon_0 + b\kappa_0 - N_\lambda
\]  
(2.24)
and
\[ M = \int_{i} \sigma z dz = B \varepsilon_{o} + D \kappa_{o} - M_{\lambda}, \quad (2.25) \]

where \( N = \begin{pmatrix} N_{x} & N_{y} & N_{xy} \end{pmatrix}^{T} \) and \( M = \begin{pmatrix} M_{x} & M_{y} & M_{xy} \end{pmatrix}^{T} \), \( \varepsilon_{o} \) and \( \kappa_{o} \) are given in Eqs.(2.4a,b), and \( A, B, D, N_{\lambda} \) and \( M_{\lambda} \) are different for where there are actuators and no actuators. They are given as follows:

For a laminated substrate of composite material, at a location with no actuators, \( A, B \) and \( D \) are

\[ A = \int_{i} Q dz = \sum_{k=1}^{n} \Omega_{s_{k}} (h_{k} - h_{k-1}) \quad (2.26a) \]

\[ B = \int_{i} Q z dz = \frac{1}{2} \sum_{k=1}^{n} \Omega_{s_{k}} (h_{k}^{2} - h_{k-1}^{2}) \quad (2.27a) \]

\[ D = \int_{i} Q z^{2} dz = \frac{1}{3} \sum_{k=1}^{n} \Omega_{s_{k}} (h_{k}^{3} - h_{k-1}^{3}) \quad , \quad (2.28a) \]

and \( N_{\lambda} \) and \( M_{\lambda} \) are

\[ N_{\lambda} = \int_{i} Q \varepsilon_{a} dz = 0 \quad (2.29a) \]

\[ M_{\lambda} = \int_{i} Q \varepsilon_{a} z dz = 0 \quad (2.30a) \]

due to zero \( \varepsilon_{a} \) outside the domain of actuators, where \( h_{k} \) is the \( z \) coordinate of the top surface of each substrate ply as shown in Fig.(2.2) and the bold zeros indicate matrices.

At a location with actuators surface embedded, \( A', B' \) and \( D' \) are
\[ A' = \sum_{k=j+1}^{n-j} Q_{sk}(h_k - h_{k-1}) + Q_{s_j}(h_j + h_a) + Q_{s_{n-j+1}}(h_a - h_{n-j}) + 2Q_a t_a \]  
(2.26b)

\[ B' = \frac{1}{2} \sum_{k=j+1}^{n-j} Q_{sk}(h_k^2 - h_{k-1}^2) + Q_{s_j}(h_j^2 - h_a^2) + Q_{s_{n-j+1}}(h_a^2 - h_{n-j}^2) \]  
(2.27b)

\[ D' = \frac{1}{3} \sum_{k=j+1}^{n-j} Q_{sk}(h_k^3 - h_{k-1}^3) + \frac{1}{3} Q_{s_j}(h_j^3 + h_a^3) + \frac{1}{3} Q_{s_{n-j+1}}(h_a^3 - h_{n-j}^3) + \frac{2}{3} Q_a (h_a^3 - h_a^3) \]  
(2.28b)

with

\[ h_a = h_n - t_a = \frac{t}{2} - t_a \]  
(2.31)

where \( j \) is the layer number of the beginning substrate ply which contains no piezo actuator (see Fig.(2.2)), and the ' sign is used to differentiate \( A, B, D, N_\lambda \) and \( M_\lambda \) from those of no actuators. For where there are actuators surface bonded

\[ A' = \sum_{k=1}^{n} Q_{sk}(h_k - h_{k-1}) + 2Q_a t_a \]  
(2.26c)

\[ B' = \frac{1}{2} \sum_{k=1}^{n} Q_{sk}(h_k^2 - h_{k-1}^2) \]  
(2.27c)

\[ D' = \frac{1}{3} \sum_{k=1}^{n} Q_{sk}(h_k^3 - h_{k-1}^3) + \frac{2}{3} Q_a [(h_n + t_a)^3 - h_n^3] \]  
(2.28c)

For both bonding types of actuators, \( N'_\lambda \) and \( M'_\lambda \) are

\[ N'_\lambda = \int_{-t/2}^{-t/2 + t_a} Q_a e_a^- dz + \int_{t/2}^{t/2} Q_a e_a^+ dz = 0 \]  
(2.29b)
\[ M'_{\lambda} = \int_{-t/2}^{t/2} Q_a \varepsilon_a^- zdz + \int_{t/2}^{t/2-t_a} Q_a \varepsilon_a^+ zdz = Q_a \varepsilon_a (t - t_a), \quad (2.30b) \]

where \( \varepsilon_a^- \) and \( \varepsilon_a^+ \) are respectively the actuation strains in the bottom and top actuators of a bimorph and \( \varepsilon_a^- = -\varepsilon_a = \varepsilon_a \). Further, if one substitutes Eqs.(2.9, 2.11) in Eq.(2.30b) \( M'_{\lambda} \) can be expressed in term of the input voltage \( V \) explicitly as

\[ M'_{\lambda} = LV \quad (2.32) \]

where

\[ L = -\frac{E_{1a}}{1 - \nu_{1a}} (t - t_a) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad (2.33) \]

For isotropic substrate material, the above matrices take simpler expressions. Since the substrate is also mid-plane symmetric, so there is \( B = 0 \), and there are

\[ A = Q_s t \quad (2.26Is) \]
\[ D = Q_s \frac{t^3}{12} \quad (2.28Is) \]

for a location with no actuators and

\[ A' = Q_s (t - 2t_a) + 2Q_a t_a \quad (2.26Ia) \]
\[ D' = Q_s \frac{(t - 2t_a)^3}{12} + Q_a \frac{t^3 - (t - 2t_a)^3}{12} \quad (2.28Ia) \]

for a location with actuators surface embedded. For the surface bonded case, simply substitute \( t - 2t_a \) for \( t \) to compute \( A \) and \( D \), while \( A' \) and \( D' \) remain the same as above.
2-2.3.2 Shear Resultants

Although the effect of the shear strain components $\gamma_{yz}$ and $\gamma_{zx}$ on bending is omitted, the vertical forces $Q_x$ and $Q_y$ are not negligible. In fact they are of the same order of the magnitude as surface loads and moments and are, therefore, included in the following derivation of governing equations and boundary conditions.

Since $\tau_{yz}$ and $\tau_{zx}$ cannot be obtained by using generalized Hookes's law, i.e., $\gamma_{23} = \tau_{23}/G_{23}$ and $\gamma_{13} = \tau_{13}/G_{13}$, the shear resultants per unit length in the $x$ and $y$ directions, denoted by $Q_x$ and $Q_y$, are determined by a set of moment equilibrium equations (2.38, 2.39) and the stress couples in Eq. (2.25). Due to the uniform distribution of the actuation moment $M_A$ within the plane domain of a bimorph, $M_A$ does not affect $Q_x$ and $Q_y$. Hence the shear resultants are

\begin{align}
Q_x &= \frac{\partial}{\partial x} (Be_0 + D\kappa_0)_x + \frac{\partial}{\partial y} (Be_0 + D\kappa_0)_{xy} + \frac{h}{2} (\tau_{1x} + \tau_{2x}) \tag{2.34a} \\
Q_y &= \frac{\partial}{\partial y} (Be_0 + D\kappa_0)_y + \frac{\partial}{\partial x} (Be_0 + D\kappa_0)_{xy} + \frac{h}{2} (\tau_{1y} + \tau_{2y}) \tag{2.34b}
\end{align}

where subscripts $x$, $y$ and $xy$ indicate the relevant components of the vector $(\bullet)$, and $B$ and $D$, and $e_0$ and $\kappa_0$ are given as in Eqs. (2.27, 2.28) and Eqs. (2.4a, b) respectively.

2-3 Mathematical Model

Two mathematical models, a displacement $U-V-W$ model and a stress-function-normal-displacement $\Phi-W$ model, with their corresponding expressions of boundary conditions, are presented in this section.
In deriving the governing equations, it is observed that piezo actuation (characterized by the input voltages) produces bending moments at the edges of the actuators but the input voltage is not present in the governing equations.

2-3.1 Governing Equations

Making a force and moment balance, neglecting, for simplicity, the body force term over a differential plate element of dimension $dx$, $dy$, $dz$, integrating term by term across each ply, and summing across the plies of the plate results in five equilibrium equations:

\[
\frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} + \tau_1 = \tau_2 = 0 \tag{2.35}
\]

\[
\frac{\partial N_y}{\partial y} + \frac{\partial N_x}{\partial x} + \tau_1 = \tau_2 = 0 \tag{2.36}
\]

\[
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = 0 \tag{2.37}
\]

and

\[
\frac{\partial M_x}{\partial x} - \frac{\partial M_y}{\partial y} - Q_x + \frac{h}{2}(\tau_{1x} + \tau_{2x}) = 0 \tag{2.38}
\]

\[
\frac{\partial M_y}{\partial y} + \frac{\partial M_x}{\partial x} - Q_y + \frac{h}{2}(\tau_{1y} - \tau_{2y}) = 0 \tag{2.39}
\]

where $M_{xy} = M_{yx}$ and $\tau_{1x}$, $\tau_{2x}$, $\tau_{1y}$ and $\tau_{2y}$ are respectively the surface shear stresses or the surface traction terms on the top and bottom surfaces of a $dx\,dy$ dimension.

On substituting $Q_x$ and $Q_y$ obtained from Eqs.(2.38, 2.39) in Eq.(2.37), the result
and Eqs.(2.35, 36) form a new set of equilibrium equations.

2-3.1.1 Displacement Model

The governing equations for the displacements $U$, $V$, $W$ of the plate are obtained by direct substitution of the expressions of all those resultants obtained in previous sections in the above new set of equilibrium equations. As a summary of the previous stage of the research work, these governing equations are also presented in this paper, as well as in the section deriving boundary conditions. They are

$$
\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \frac{h}{2} \left[ \frac{\partial (\tau_{1x} + \tau_{2x})}{\partial x} + \frac{\partial (\tau_{1y} + \tau_{2y})}{\partial y} \right] = 0 \tag{2.40}
$$

and

$$
\frac{\partial^3 U}{\partial x^2 \partial y} + 2A_{16} \frac{\partial^2 U}{\partial y^2} + A_{66} \frac{\partial^2 U}{\partial y^2} + A_{16} \frac{\partial^2 V}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 V}{\partial y^2} + A_{26} \frac{\partial^2 V}{\partial y^2} \tag{2.41}
$$

$$
- B_{11} \frac{\partial^3 W}{\partial x^3} - 3B_{16} \frac{\partial^3 W}{\partial x^2 \partial y} - (B_{12} + 2B_{66}) \frac{\partial^3 W}{\partial x \partial y^2} - B_{26} \frac{\partial^3 W}{\partial y^3} + \xi_x = 0
$$

$$
A_{16} \frac{\partial^2 U}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 U}{\partial x \partial y} + A_{26} \frac{\partial^2 V}{\partial x^2} + A_{66} \frac{\partial^2 V}{\partial x^2} + 2A_{66} \frac{\partial^2 V}{\partial x \partial y} + A_{22} \frac{\partial^2 V}{\partial y^2} \tag{2.42}
$$

$$
- B_{16} \frac{\partial^3 W}{\partial x^3} - (B_{12} + 2B_{66}) \frac{\partial^3 W}{\partial x^2 \partial y} - 3B_{26} \frac{\partial^3 W}{\partial x \partial y^2} - B_{22} \frac{\partial^3 W}{\partial y^3} + \xi_y = 0
$$

$$
B_{11} \frac{\partial^3 U}{\partial x^3} + 3B_{16} \frac{\partial^3 U}{\partial x^2 \partial y} + (B_{12} + 2B_{66}) \frac{\partial^3 U}{\partial x \partial y^2} + B_{26} \frac{\partial^3 U}{\partial y^3} \tag{2.43}
$$

$$
+ B_{16} \frac{\partial^3 V}{\partial x^3} + (B_{12} + 2B_{66}) \frac{\partial^3 V}{\partial x^2 \partial y} + 3B_{26} \frac{\partial^3 V}{\partial x \partial y^2} + B_{22} \frac{\partial^3 V}{\partial y^3}
$$
\[- D_{11} \frac{\partial^4 W}{\partial x^4} - 4D_{16} \frac{\partial^4 W}{\partial x^3 \partial y} - 2(D_{12} + 2D_{66}) \frac{\partial^4 W}{\partial x^2 \partial y^2} - 4D_{26} \frac{\partial^4 W}{\partial x \partial y^3} - D_{22} \frac{\partial^4 W}{\partial y^4} \]

\[+ \frac{h}{2} \left( \frac{\partial \xi_x}{\partial x} + \frac{\partial \xi_y}{\partial y} \right) = 0 \quad (2.43)\]

where

\[\xi_x = \tau_{1x} - \tau_{2x}, \quad \xi_y = \tau_{1y} - \tau_{2y}, \quad (2.44a, b)\]

\[\zeta_x = \tau_{1x} + \tau_{2x}, \quad \text{and} \quad \zeta_y = \tau_{1y} + \tau_{2y}. \quad (2.45a, b)\]

For orthotropic substrate material, i.e., \((\cdot)_{16}=0\), \(U\) and \(V\) are still coupled to \(W\). For isotropic substrate material, i.e., \(B=0\) also, \(U\) and \(V\) are uncoupled from \(W\), and therefore on simplifying Eq.(2.43) it becomes

\[\nabla^4 W + \frac{h}{2} \left( \frac{\partial \zeta_x}{\partial x} + \frac{\partial \zeta_y}{\partial y} \right) = 0 \quad (2.46)\]

for both locations with actuator pair and without actuator pair, where

\[\nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}. \]

That is right the governing equation obtained in previous FDM analysis.\(^{20}\) Actually, from Eq.(2.27a) the uncoupling occurs even if the substrate material is mid-plane symmetric, that is, \(B=0\) in Eqs.(2.41-2.43).

In the above governing equations, \(A_y\), \(B_y\) and \(D_y\) are given as in Eqs.(2.26-2.28) for a location with actuators and without actuators when the substrate is anisotropic, orthotropic or isotropic, respectively.
It is observed that the voltages applied on the actuators do not appear in the governing equations. This is because that the piezo actuation strain is not a function of the coordinates \((x, y)\) but a function of the input voltages only.

**2-3.1.2 Stress-Function-Normal-Displacement Model**

The three independent variables, displacements \(U\) and \(V\) and \(W\), can be further reduced to two dependent variables. Noticing that in Eq.(2.3) the in-plane displacements \(U\) and \(V\) are separated from the normal displacement \(W\) and also in Eq.(2.4a) the in-plane shear strain \(\gamma_{xy}\) is related to the normal strains \(\varepsilon_{xx}\) and \(\varepsilon_{yy}\) via \(U\) and \(V\), by introducing an Airy stress function \(\Phi\) such that\(^{35,39}\)

\[
N_x = \frac{\partial^2 \Phi}{\partial y^2} - \eta_x, \quad (2.47a)
\]

\[
N_y = \frac{\partial^2 \Phi}{\partial x^2} - \eta_y, \quad (2.47b)
\]

and

\[
N_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} \quad (2.47c)
\]

where

\[
\eta_x = \int (\tau_{1x} - \tau_{2x}) \, dx \quad \text{and} \quad \eta_y = \int (\tau_{1y} - \tau_{2y}) \, dy, \quad (2.48a, b)
\]

the in-plane displacements \(U\) and \(V\) can be combined in the dependent variable \(\Phi\). If Eqs.(2.47a,b,c) are substituted in Eqs.(2.35,2.36), it can be found that the equilibrium equations are automatically satisfied.
The in-plane strains $\varepsilon_\theta$ in Eq.(2.4a) and moments $M$ can be solved from Eqs.(2.24, 2.25) in terms of $N$ and $\kappa_\theta$ by applying $N_\lambda=0$ as

$$
\varepsilon_\theta = aN - b\kappa_\theta \quad (2.49)
$$

$$
M = b^T N - d\kappa_\theta - M_\lambda , \quad (2.50)
$$

where $a$, $b$ and $d$ are matrices defined as

$$
a = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{12} & a_{22} & a_{23} \\
a_{13} & a_{23} & a_{33}
\end{bmatrix} = A^{-1} \quad (2.51)
$$

$$
b = \begin{bmatrix}
b_{11} & b_{12} & b_{13} \\
b_{21} & b_{22} & b_{23} \\
b_{31} & b_{32} & b_{33}
\end{bmatrix} = A^{-1} B = aB \quad (2.52)
$$

$$
d = \begin{bmatrix}
d_{11} & d_{12} & d_{13} \\
d_{12} & d_{22} & d_{23} \\
d_{13} & d_{23} & d_{33}
\end{bmatrix} = BA^{-1} - D = b^T B - D . \quad (2.53)
$$
Here since both $A$ and $B$ are symmetric matrices, $a$ and $d$ must be symmetric but not necessarily $b$. In obtaining Eq.(2.50) the fact that $BA^{-1} = (A^{-1}B)^T = b^T$ has been used. By substituting all the explicit forms of $a$, $b$, $d$, $N$, $N'$ and $\kappa_0$, that is, Eqs.(2.51, 2.52, 2.53, 2.47, 2.23, 2.4b) respectively, in Eqs.(2.49, 2.50) and solving for $\varepsilon_0$ and $M$ one gets the explicit matrix forms

$$
\varepsilon_0 = \begin{bmatrix}
\frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}
\end{bmatrix}
= \begin{bmatrix}
a_{11} & a_{12} & a_{13}
a_{12} & a_{22} & a_{23}
a_{13} & a_{23} & a_{33}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial^2 \Phi}{\partial y^2} & \frac{\partial^2 \Phi}{\partial x \partial y} & \frac{\partial^2 W}{\partial x^2} & -\frac{\partial^2 W}{\partial x \partial y} & -2 \frac{\partial^2 W}{\partial x \partial y}
\end{bmatrix}
- \begin{bmatrix}
b_{11} & b_{12} & b_{13} & 0
b_{21} & b_{22} & b_{23} & 0
b_{31} & b_{32} & b_{33} & 0
\end{bmatrix}
\begin{bmatrix}
\eta_x
\eta_y
0
\end{bmatrix}
$$

(2.54)

and

$$
M = \begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix}
= \begin{bmatrix}
\begin{bmatrix}
b_{11} & b_{12} & b_{13} & 0
b_{12} & b_{22} & b_{23} & 0
b_{13} & b_{23} & b_{33} & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial^2 \Phi}{\partial y^2} & \frac{\partial^2 \Phi}{\partial x \partial y} & \frac{\partial^2 W}{\partial x^2} & -\frac{\partial^2 W}{\partial x \partial y} & -2 \frac{\partial^2 W}{\partial x \partial y}
\end{bmatrix}
- \begin{bmatrix}
b_{11} & b_{12} & b_{13} & 0
b_{21} & b_{22} & b_{23} & 0
b_{31} & b_{32} & b_{33} & 0
\end{bmatrix}
\begin{bmatrix}
\eta_x
\eta_y
0
\end{bmatrix}
\end{bmatrix}
$$

(2.55)

For now only the last one in the new equilibrium equation set Eqs.(2.35, 2.36, 2.40) is not used, the compatibility equation of the in-plane strains, i.e.,
\[
\frac{\partial^2 \varepsilon_{0x}}{\partial y^2} - \frac{\partial^2 \varepsilon_{0y}}{\partial x \partial y} + \frac{\partial^2 \varepsilon_{0y}}{\partial x^2} = 0
\]  
(2.56)

is added here to reduce Eqs.(2.54, 2.55) to the final expressions of the governing equations.

On substituting Eq.(2.54) in Eq.(2.56) and Eq.(2.55) in Eq.(2.40), the final expressions of the plate governing equations turn out to be

\[
\begin{align*}
& a_{22} \frac{\partial^4 \Phi}{\partial x^4} - 2a_{23} \frac{\partial^4 \Phi}{\partial x^3 \partial y} + (2a_{12} + a_{33}) \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} - 2a_{13} \frac{\partial^4 \Phi}{\partial x \partial y^3} + a_{11} \frac{\partial^4 \Phi}{\partial y^4} \\
& + b_{21} \frac{\partial^4 W}{\partial x^4} + (2b_{23} - b_{31}) \frac{\partial^4 W}{\partial x^3 \partial y} + (b_{11} + b_{22} - b_{33}) \frac{\partial^4 W}{\partial x^2 \partial y^2} + (2b_{13} - b_{32}) \frac{\partial^4 W}{\partial x \partial y^3} + b_{12} \frac{\partial^4 W}{\partial y^4} \\
& - a_{22} \frac{\partial^2 \eta_y}{\partial x^2} - a_{11} \frac{\partial^2 \eta_x}{\partial y^2} - a_{12} \left( \frac{\partial \varepsilon_y}{\partial x} + \frac{\partial \varepsilon_x}{\partial y} \right) + a_{23} \frac{\partial \varepsilon_y}{\partial x} + a_{13} \frac{\partial \varepsilon_x}{\partial y} = 0 \\
& + b_{21} \frac{\partial^4 \Phi}{\partial x^4} + (2b_{23} - b_{31}) \frac{\partial^4 \Phi}{\partial x^3 \partial y} + (b_{11} + b_{22} - b_{33}) \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + (2b_{13} - b_{32}) \frac{\partial^4 \Phi}{\partial x \partial y^3} + b_{12} \frac{\partial^4 \Phi}{\partial y^4} \\
& + d_{11} \frac{\partial^4 W}{\partial x^4} + 4d_{13} \frac{\partial^4 W}{\partial x^3 \partial y} + (2d_{12} + 2d_{33}) \frac{\partial^4 W}{\partial x^2 \partial y^2} + 4d_{23} \frac{\partial^4 W}{\partial x \partial y^3} + d_{22} \frac{\partial^4 W}{\partial y^4} \\
& - b_{21} \frac{\partial^2 \eta_y}{\partial x^2} - b_{12} \frac{\partial^2 \eta_x}{\partial y^2} - b_{11} \frac{\partial \varepsilon_y}{\partial x} - b_{22} \frac{\partial \varepsilon_y}{\partial y} - 2b_{23} \frac{\partial \varepsilon_y}{\partial x} - 2b_{13} \frac{\partial \varepsilon_x}{\partial y} + \frac{h}{2} \left( \frac{\partial \varepsilon_x}{\partial x} + \frac{\partial \varepsilon_y}{\partial y} \right) = 0.
\end{align*}
\]
(2.57)

For orthotropic substrate plies, from Eqs.(2.51-2.53) and \( A_{16} = A_{26} \), there are \((\bullet)_{13} = (\bullet)_{23} = b_{31} = b_{32} = 0\) for the matrices \( a, b \) and \( d \). For isotropic substrate material there
are \((\bullet)_{13}=(\bullet)_{23}=0\) and \(b=0\) and \(d=-D\). If the substrate material is mid-plane symmetric, i.e., \(b=0\) and \(d=-D\), \(\Phi\) is uncoupled from \(W\) and the equations become

\[
\begin{align*}
2a_{12} \frac{\partial^4 \Phi}{\partial x \partial y^3} + 2a_{13} \frac{\partial^4 \Phi}{\partial x^2 \partial y} + 2a_{13} \frac{\partial^4 \Phi}{\partial x \partial y^3} + a_{11} \frac{\partial^4 \Phi}{\partial y^4} = 0 \\
2a_{12} \frac{\partial^2 \eta_y}{\partial x^2} + 2a_{11} \frac{\partial^2 \eta_x}{\partial y^2} + 2a_{12} \left( \frac{\partial \xi_y}{\partial x} + \frac{\partial \xi_x}{\partial y} \right) + 2a_{23} \frac{\partial \xi_y}{\partial x} + 2a_{13} \frac{\partial \xi_x}{\partial y} = 0 \\
D_{11} \frac{\partial^4 W}{\partial x^4} + 4D_{16} \frac{\partial^4 W}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 W}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 W}{\partial x \partial y^3} + D_{22} \frac{\partial^4 W}{\partial y^4} = 0
\end{align*}
\] (2.57S)

Although the \(\Phi-W\) model looks long with those additions, it is still simpler than the \(U-V-W\) model refer to variables, and thus also simpler for numerical discrete.

2-3.2 Boundary Conditions

To complete the mathematical description of the plate system behavior under study where a fourth order differential equation is involved, one needs certain four boundary conditions corresponding to the edge constraints for each edge of the plate.

2-3.2.1 In-Plane Forces and Moments and In-Plane Displacement

For the \(U-V-W\) model, i.e., Eqs.(2.41-2.43), on substituting Eqs.(2.4a,b) in the stress resultants Eq.(2.24) and the stress couples Eq.(2.25) respectively, the in-plane forces \(N\) and the moments \(M\) on a boundary can finally be expressed in terms of the mid-plane displacements \(U, V\) and \(W\). For the moments, for example, they are
where $M_{xx}$ and $M_{xy}$ are given as in Eq.(2.23). For orthotropic material, i.e., $(\bullet)_{16}=(\bullet)_{26}=0$, $U$ and $V$ are still involved. For mid-plane symmetric substrate material, i.e., $B=0$, or isotropic substrate material, i.e., $(\bullet)_{16}=(\bullet)_{26}=0$ additionally, the moments are functions of $W$ only.

For the $\Phi$-$W$ model, i.e., Eqs.(2.57, 2.58), the expressions of the moments on boundary are given as in Eq.(2.55) if the matrix products are performed. The in-plane forces are defined as in Eq.(2.47). The in-plane displacements $U$ and $V$ can be expressed in terms of the dependent variables $\Phi$ and $W$ by integrating the first two components of the in-plane strains $\varepsilon_0$ in Eq.(54) as
where \( f_1(y) \) and \( f_2(x) \) are arbitrary functions provided that they can be evaluated by substituting \( \partial U / \partial y \) and \( \partial V / \partial x \) in the third equation of Eq. (2.54).

It is found that the input voltages only affect the normal deflection of the plate through moment boundary condition. In the bimorph arrangement, piezo actuation produces bending moments along the edges of actuators.

Now those \( N, M, U, V, \) i.e., Eqs. (2.47, 2.55, 2.60a,b) respectively, together with \( W \) can be used to describe those constrained boundary conditions.

### 2-3-2.2 Effective Transverse Forces

For a free edge of a plate, take \( y=0 \) for example, most generally the in-plane normal force \( N_y \) and shear force \( N_{xy} \), the transverse shear resultant \( Q_y \), the bending moment \( M_y \) and the twist moment \( M_{xy} \) are all zero. Also for a thin plate whose in-plane deflection can be ignored, governing equation for the normal deflection \( W \) is the same as Eq. (2.58S) for the case of mid-plane symmetric substrate material. It is obvious that two boundary conditions are sufficient for each edge of the plate. Of course, for both cases, there are too many boundary conditions for solving the normal deflection. Kirchhoff’s
free edge boundary conditions\textsuperscript{29-31} are then introduced: in addition to zero in-plane forces and bending moment, the effective transverse shear force is zero also on a free edge. This is based on that a distribution of $M_{xy}$ along an edge is statically equivalent to a distribution of vertical shear forces.

According to the definition, the effective transverse shear forces per unit length in $x$ and $y$ directions are

\begin{align}
V_x &= Q_x + \frac{\partial M_{xy}}{\partial y} = \frac{\partial}{\partial x} (B\varepsilon_0 + Dx_0)_x + 2 \frac{\partial}{\partial y} (B\varepsilon_0 + Dx_0)_{xy} + \frac{h}{2} \zeta_x \tag{2.61a} \\
V_y &= Q_y + \frac{\partial M_{xy}}{\partial x} = \frac{\partial}{\partial y} (B\varepsilon_0 + Dx_0)_y + 2 \frac{\partial}{\partial x} (B\varepsilon_0 + Dx_0)_{xy} + \frac{h}{2} \zeta_y . \tag{2.61b}
\end{align}

For $U-V-W$ model, on substituting Eqs.(2.4a,b) in the above equations, the effective transverse shear forces can be written in terms of the mid-plane displacements as

\begin{align}
V_x &= B_{11} \frac{\partial^2 U}{\partial x^2} + 3B_{16} \frac{\partial^2 U}{\partial x \partial y} + 2B_{66} \frac{\partial^2 U}{\partial y^2} + B_{16} \frac{\partial^3 V}{\partial x^2} + (B_{12} + 2B_{16}) \frac{\partial^3 V}{\partial x \partial y} + 2B_{26} \frac{\partial^3 V}{\partial y^2} \\
&\quad - D_{11} \frac{\partial^3 W}{\partial x^3} - 4D_{16} \frac{\partial^3 W}{\partial x^2 \partial y} - (D_{12} + 4D_{66}) \frac{\partial^3 W}{\partial x \partial y^2} - 2D_{26} \frac{\partial^3 W}{\partial y^3} \tag{2.62a} \\
V_y &= 2B_{16} \frac{\partial^2 U}{\partial x^2} + (B_{12} + 2B_{66}) \frac{\partial^2 U}{\partial x \partial y} + 2B_{26} \frac{\partial^2 U}{\partial y^2} + 2B_{66} \frac{\partial^2 V}{\partial x^2} + 3B_{26} \frac{\partial^2 V}{\partial x \partial y} + 2B_{22} \frac{\partial^2 V}{\partial y^2} \\
&\quad - 2D_{16} \frac{\partial^3 W}{\partial x^3} - (D_{12} + 4D_{66}) \frac{\partial^3 W}{\partial x^2 \partial y} - 4D_{26} \frac{\partial^3 W}{\partial x \partial y^2} - D_{22} \frac{\partial^3 W}{\partial y^3} . \tag{2.62b}
\end{align}

For $\Phi-W$ model, they are
For orthotropic substrate material, \((\bullet)_16\) and \((\bullet)_26\) or \((\bullet)_13\) and \((\bullet)_23\) are zero. For mid-plane symmetric substrate material and isotropic substrate since there is \(B=0\) or \(b=0\) also, thus the effective transverse shear forces are only functions of the normal deflection on a free edge.

### 2-3.2.3 Boundary Conditions

Now one can write down those boundary conditions for different edge constraints (still taking the \(x\) axis for example, i.e., at \(y=0\)). For a clamped edge

\[
\overline{V}_x = b_{11} \frac{\partial^3 \Phi}{\partial x^3} + (2b_{23} - b_{31}) \frac{\partial^3 \Phi}{\partial x^2 \partial y} + (b_{12} - 2b_{33}) \frac{\partial^3 \Phi}{\partial x \partial y^2} + 2b_{13} \frac{\partial^3 \Phi}{\partial y^3} + d_{11} \frac{\partial^3 W}{\partial x^3} + 4d_{13} \frac{\partial^3 W}{\partial x^2 \partial y} + (d_{12} + 4d_{33}) \frac{\partial^3 W}{\partial x \partial y^2} + 2d_{23} \frac{\partial^3 W}{\partial y^3} - b_{11} \frac{\partial \eta_x}{\partial x} - 2b_{13} \frac{\partial \eta_y}{\partial y} - b_{12} \frac{\partial \eta_x}{\partial x} - 2b_{23} \frac{\partial \eta_y}{\partial y} + \frac{h}{2} \zeta_y.
\]

(2.63a)

\[
\overline{V}_y = 2b_{21} \frac{\partial^3 \Phi}{\partial x^3} + (b_{22} - 2b_{33}) \frac{\partial^3 \Phi}{\partial x^2 \partial y} + (2b_{13} - b_{23}) \frac{\partial^3 \Phi}{\partial x \partial y^2} + b_{12} \frac{\partial^3 \Phi}{\partial y^3} + 2d_{13} \frac{\partial^3 W}{\partial x^3} + (d_{12} + 4d_{33}) \frac{\partial^3 W}{\partial x^2 \partial y} + 4d_{23} \frac{\partial^3 W}{\partial x \partial y^2} + d_{22} \frac{\partial^3 W}{\partial y^3} - 2b_{13} \frac{\partial \eta_x}{\partial x} - b_{12} \frac{\partial \eta_y}{\partial y} - 2b_{23} \frac{\partial \eta_x}{\partial x} - b_{22} \frac{\partial \eta_x}{\partial y} + \frac{h}{2} \zeta_y.
\]

(2.63b)

For orthotropic substrate material, \((\bullet)_16\) and \((\bullet)_26\) or \((\bullet)_13\) and \((\bullet)_23\) are zero. For mid-plane symmetric substrate material and isotropic substrate since there is \(B=0\) or \(b=0\) also, thus the effective transverse shear forces are only functions of the normal deflection on a free edge.

### 2-3.2.3 Boundary Conditions

Now one can write down those boundary conditions for different edge constraints (still taking the \(x\) axis for example, i.e., at \(y=0\)). For a clamped edge

\[
U = V = W = \frac{\partial W}{\partial y} = 0.
\]

(2.64)

For a simply supported edge
\[ U = V = W = M_y = 0 \]  
(2.65)

if the edge is pinned, or

\[ N_y = N_{xy} = W = M_y = 0 \]  
(2.66)

if the edge is on the rollers. And for a free edge

\[ N_y = N_{xy} = M_y = \bar{V}_y = 0 \]  
(2.67)

When the shape control of the plate or the response of the flectural vibration is considered, usually only normal displacement is of interest. If in plane displacements can be ignored, only two boundary conditions are required on each edge of a plate. They can be transverse deflection and slope, or force and moment, or some combination else\textsuperscript{31}. In the above discussed boundary conditions, if drop the in-plane deflection \( U \) and \( V \) and the in-plane normal forces \( N_x \) and \( N_y \), then the remaining expressions are the boundary conditions for solving for normal deflection \( W \) only.

Up to now, the mathematical model for solving for the deflection of the actuator/plate system described in the beginning of this chapter is complete.
CHAPTER 3

NUMERICAL FEM APPROACH

A FEM model for solving the system governing Eqs.(2.41-2.43) or Eqs.(2.57, 2.58) with a set of specified boundary conditions, like those given in the last section of Chapter 2, may be formed by using certain general, complete field variational principles for elasticity such as The Theorem of Minimum Potential Energy, The Theorem of Complementary Energy and Reissner's Variational Theorem.\textsuperscript{33,39} The conceptual introduction of using complementary energy function to solve Airy stress function model can be found in Ref. 39. Also finite difference method (FDM) can be used.\textsuperscript{20,35}

In addition to the assumptions stated in Section 2-1 of Chapter 2, it is further assumed that the in-plane displacements $U$ and $V$ are small and negligible. Therefore the governing equation to solve is exactly the same as Eq.(2.58S). For convenience it is still written out here

\[
D_{11} \frac{\partial^4 W}{\partial x^4} + 4 D_{16} \frac{\partial^4 W}{\partial x^3 \partial y} + 2(D_{12} + 2 D_{66}) \frac{\partial^4 W}{\partial x^2 \partial y^2} + 4 D_{26} \frac{\partial^4 W}{\partial x \partial y^3} + D_{22} \frac{\partial^4 W}{\partial y^4}
\]

\[
-\frac{h}{2} \left( \frac{\partial \zeta_x}{\partial x} + \frac{\partial \zeta_y}{\partial y} \right) = 0
\]

(3.1)

In this chapter, a simpler 2-D FEM model for solving the normal deflection $W$ through using Discrete Kirchhoff Quadrilateral (DKQ) thin plate bending element is studied. The Theorem of Minimum Potential Energy is adopted in the FEM formulation.
3-1. Total Potential Energy

The Theorem of Minimum Potential Energy can be stated as:

"Of all of the displacements satisfying compatibility and the prescribed boundary conditions, those which satisfy the equilibrium equations make the potential energy a minimum."

Its mathematical expression is

$$\delta \Pi = 0 \quad (3.2)$$

where $\Pi$ is the total potential energy and $\delta$ is the variational operator.

The total potential energy of any generalized elastic body can be written as follows

$$\Pi = \overline{U}(\varepsilon) + \overline{W}(u) \quad (3.3)$$

where $\varepsilon$ is a vector of strains, $u$ a vector of displacements, $\overline{U}(\varepsilon)$ the strain energy and $\overline{W}(u)$ the work done by the surface tractions and the body forces. For the elastic, linear material described by Eq.(2.5), $\overline{U}$ and $\overline{W}$ are

$$\overline{U} = \frac{1}{2} \int_{\Omega} \varepsilon^T Q \varepsilon \, d\Omega - \int_{\Omega} \varepsilon^T Q \varepsilon \, d\Omega - \int_{\Gamma} \varepsilon^T Q \varepsilon \, d\Omega \quad (3.4)$$

$$\overline{W} = -\int_{\Omega} u^T \bar{b} \, d\Omega - \int_{\Gamma} u^T \bar{t} \, d\Gamma \quad (3.5)$$

where $\bar{b}$ is the body force, $\bar{t}$ the surface traction, $\Omega$ the volume of the elastic body and $\Gamma_I$ the portion of the body surface over which tractions are prescribed. Notice that the initial strain $\varepsilon_T$ caused by the thermal effect has been introduced in Eq.(3.4), that is, Eq.(2.5) has been modified as $\sigma = Q(\varepsilon - \varepsilon_0 - \varepsilon_T)$. 
3-2 Problem Discretization

The formulation of the DKQ element is based on the discretization of the strain energy when the transverse shear strain energy is neglected and that Eq.(3.3) can be divided into a finite number of element domains $\Omega^e$ within the total domain $\Omega$ in such a way that

$$\Pi = \sum_{i=1}^{n_e} \overline{U}_i^e + \sum_{i=1}^{n_e} \overline{W}_i^e \quad (3.6)$$

with

$$\overline{U}_i^e = \frac{1}{2} \int_{\Omega^e} \varepsilon^T Q \varepsilon d\Omega - \int_{\Omega^e} \varepsilon^T \alpha e \varepsilon d\Omega - \int_{\Omega^e} \varepsilon^T Q \alpha \tau d\Omega \quad (3.7)$$

$$\overline{W}_i^e = - \int_{\Gamma_i^e} \mathbf{u}^T \gamma d\Gamma \quad , \quad (3.8)$$

where the body force $\mathbf{b}$ is neglected, and the subscript $e$ indicates relevant to the elements and $n_e$ is the number of the elements divided. After Eq.(2.1) is substituted in Eq.(3.7), with $\varepsilon_0$ term neglected, the strain energy becomes

$$\overline{U}_i^e = \frac{1}{2} \int_{\Gamma_i^e} \kappa_0^T Q \varepsilon^2 \kappa_0 dz d\Gamma - \int_{\Gamma_i^e} \kappa_0^T Q \varepsilon_0 \varepsilon_0 dz d\Gamma - \int_{\Gamma_i^e} \kappa_0^T Q \varepsilon_0 \gamma d\Gamma + \int_{\Gamma_i^e} \kappa_0^T Q \mu T dz d\Gamma \quad (3.9)$$

or

$$\overline{U}_i^e = \frac{1}{2} \int_{\Gamma_i^e} \kappa_0^T D \kappa_0 dz d\Gamma - \int_{\Gamma_i^e} \kappa_0^T M_a d\Gamma - \int_{\Gamma_i^e} \kappa_0^T M_\gamma d\Gamma \quad (3.10)$$

where the element surface $\Gamma_i^e$ and the plate thickness $t$ form the element domain $\Omega_i^e$, and matrix $D$ and actuation moments $M_a$ are determined from Eq.(2.28) and Eq.(2.30)
respectively for a domain with no actuators and a domain with actuators surface embedded or bonded, and $M_T$ is

$$M_T = \int_Q z \mu T \, dz = \sum_{k=1}^{n} \int_Q z_k T_k \, dz$$

(3.11)

with $\mu$ the coefficients of thermal expansion and $T$ the difference between the operating and laminating temperatures.

The total potential energy $\Pi$ of a laminated composite plate embedded or bonded with piezo actuators, after substituting Eqs.(3.8, 3.10) in Eq.(3.6), becomes

$$\Pi = \sum_{i=1}^{n} \frac{1}{2} \int_{\Gamma_i} \begin{pmatrix} \kappa_0 & \kappa_0^T \\ \kappa_0^T & M_\lambda \end{pmatrix} d\Gamma - \sum_{j=1}^{n} \int_{\Gamma_j} \begin{pmatrix} \kappa_0 & M_\gamma \end{pmatrix} d\Gamma - \sum_{i=1}^{n} \int \mathbf{u}^T \mathbf{t} d\Gamma .$$

(3.12)

For substrate material mid-plane symmetric, i.e., $B=0$, $W$ is uncoupled from $U$ and $V$, or $\varepsilon_0 = \varepsilon_0(U, V)$ and $\kappa_0 = \kappa_0(W)$. So the bending potential energy $\Pi_b$ in Eq.(3.13), which is exactly the same as the right hand side in Eq.(3.12), is of the form $\Pi_b = \Pi_b(\kappa_0)$, that is, independent of other terms, even if $\varepsilon_0$ is present in Eq.(2.1) when Eqs.(3.7,3.8) are substituted in Eq.(3.6):

$$\Pi = \frac{1}{2} \sum_{i=1}^{n} \int_{\Gamma_i} \begin{pmatrix} \varepsilon_0 & \varepsilon_0^T \\ \varepsilon_0^T & N_\lambda \end{pmatrix} - \sum_{j=1}^{n} \int_{\Gamma_j} \begin{pmatrix} \varepsilon_0 & \kappa_0^T \\ \kappa_0 & M_\gamma \end{pmatrix} d\Gamma - \sum_{i=1}^{n} \int \mathbf{u}^T \mathbf{t} d\Gamma$$

(3.13)

where $N_T$ is given as
Thus to minimize the total potential energy must minimize the bending potential energy.
So theoretically the model is more oriented for solving uncoupled bending deformation.

3-3 Element Equations

Following the standard finite element procedures\textsuperscript{37}, the equilibrium equations of
Eq.(3.2) of each element can be expressed in terms of the element nodal displacements as

\[ K^e q = F^e \]  \hspace{1cm} (3.15)

where \( q \) is a vector of the nodal variables in Eq.(3.16), \( K^e \) the element stiffness matrix
given in Eq.(3.35), and \( F^e \) the force vector in Eq.(3.31) together with Eqs.(3.36-3.38).

3-3.1 Nodal Variables and Interpolation Function

The four-node, 12-degrees-of-freedom quadrilateral DKQ element is shown in
Fig.(3.1). It relates the
rotations of the normal to that of the undeformed midplane,
\[ \beta_x = -\partial W/\partial x \text{ and } \beta_y = -\partial W/\partial y, \]
in such a way that
(a) the nodal variables must be the transverse displacement \( W \)
and its derivatives \( \theta_x = W_y \) and
\[ \theta_y = -W_{,x} \text{ at the four corner nodes (with } \varphi)\text{,}\]

(b) the Kirchhoff assumptions are satisfied along the boundary of the element in order to satisfy the Kirchhoff boundary conditions.

The explicit expressions for the rotations \( \beta_x \) and \( \beta_y \) of a general quadrilateral of the final DKQ nodal variables

\[ q^e = \begin{bmatrix} W_1 \theta_{x1} \theta_{y1} W_2 \theta_{x2} \theta_{y2} W_3 \theta_{x3} \theta_{y3} W_4 \theta_{x4} \theta_{y4} \end{bmatrix}^T \quad (3.16) \]

are

\[ \beta_x = H^x(\xi, \eta)q^e \quad (3.17a) \]

\[ \beta_y = H^y(\xi, \eta)q^e, \quad (3.17b) \]

where \( H^x(\xi, \eta) \) and \( H^y(\xi, \eta) \) are the interpolation functions for \( \beta_x \) and \( \beta_y \) respectively with

\[ H^x = \begin{bmatrix} H^x_1 & \cdots & H^x_{12} \end{bmatrix} \quad (3.18a) \]

\[ H^y = \begin{bmatrix} H^y_1 & \cdots & H^y_{12} \end{bmatrix}. \quad (3.18b) \]

Those \( H_i^x \) and \( H_i^y \) \((i=1, \ldots, 12)\) are

\[ H_1^x = -a_8 N_8 + a_5 N_5 \quad H_1^y = -d_8 N_8 + d_5 N_5 \]
\[ H_2^x = b_8 N_8 + b_5 N_5 \quad H_2^y = -N_1 + e_8 N_8 + e_5 N_5 \]
\[ H_3^x = N_1 - c_8 N_8 - c_5 N_5 \quad H_3^y = -b_8 N_8 - b_5 N_5 = -H_2^x \]
\[ H_4^x = -a_5 N_5 + a_6 N_6 \quad H_4^y = -d_5 N_5 + d_6 N_6 \]
\[ H_5^x = b_5 N_5 + b_6 N_6 \quad H_5^y = -N_2 + e_5 N_5 + e_6 N_6 \]
\[ H_6^x = N_2 - c_5 N_5 - c_6 N_6 \quad H_6^y = -b_5 N_5 - b_6 N_6 = -H_5^x \quad (3.19) \]
where

\[ N_i = \begin{cases} 
\frac{1}{4} (1 + \xi_0) (1 + \eta_0) (\xi_0 + \eta_0 - 1) & \xi_i / \eta_i = -1/-1, 1/1, -1/1, +1/-1
\\ 
\frac{1}{2} (1 - \xi^2) (1 + \eta_0) & \xi_i / \eta_i = 0/0, -1/1
\\ 
\frac{1}{2} (1 + \xi_0) (1 - \eta^2) & \xi_i / \eta_i = +1/0, -1/0
\end{cases} \quad i = 1, 2, 3, 4
\]

with

\[ \xi_0 = \xi_\xi , \quad \eta_0 = \eta \eta \]

and

\[ a_k = -\frac{3}{2} \frac{x_{ij}}{l_{ij}^2} , \quad b_k = \frac{3}{4} \frac{x_{ij} y_{ij}}{l_{ij}^2} , \]

\[ c_k = \frac{x_{ij}^2 - y_{ij}^2}{4 l_{ij}^2} , \quad d_k = -\frac{3}{2} \frac{y_{ij}}{l_{ij}^2} , \]

\[ e_k = -\frac{x_{ij}^2 + y_{ij}^2}{4 l_{ij}^2} \]

with \( x_{ij} = x_i - x_j, y_{ij} = y_i - y_j, l_{ij}^2 = x_{ij}^2 - y_{ij}^2 \) and \( k = 5, 6, 7, 8 \) for the sides \( ij = 12, 23, 34, 41 \).

Since \( \kappa_0 \) can be expressed as
therefore \( \kappa_0 \) can, by substituting Eqs.(3.17a,b) in Eq.(3.23), be expressed in terms of the nodal variables as

\[
\kappa_0 = Bq^c
\]

with

\[
B = \begin{bmatrix}
H^x_{x,x} & j_{11}H^x_{x,\eta} + j_{12}H^x_{\eta,\eta} \\
H^y_{y,y} & j_{21}H^y_{y,\xi} + j_{22}H^y_{\xi,\xi} \\
H^x_{x,y} + H^y_{y,x} & j_{11}H^y_{y,\xi} + j_{12}H^y_{\xi,\eta} + j_{21}H^x_{x,\eta} + j_{22}H^x_{\eta,\eta}
\end{bmatrix}
\]

(3.25)

where \( j_{11}, j_{12}, j_{21}, \text{ and } j_{22} \) are the components of the inverse of the Jacobian matrix \( J \) of the transformation between the parent and the actual elements. The Jacobian matrix \( J \) and \( j_{kl} \) are given as follows

\[
J = \frac{1}{4} \begin{bmatrix}
x_{21} + x_{34} + \eta(x_{12} + x_{34}) & y_{21} + y_{34} + \eta(y_{12} + y_{34}) \\
x_{32} + x_{41} + \xi(x_{12} + x_{34}) & y_{32} + y_{41} + \xi(y_{12} + y_{34})
\end{bmatrix}
= \begin{bmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{bmatrix}
\]

(3.26)

\[
j_{11} = \frac{1}{|J|} J_{22}, \quad j_{12} = -\frac{1}{|J|} J_{12},
\]

\[
j_{21} = -\frac{1}{|J|} J_{21}, \quad j_{22} = \frac{1}{|J|} J_{11}
\]

(3.27)

with

\[
|J| = \frac{1}{4} \begin{bmatrix}
x_{21} + x_{34} + \eta(x_{12} + x_{34}) & y_{21} + y_{34} + \eta(y_{12} + y_{34}) \\
x_{32} + x_{41} + \xi(x_{12} + x_{34}) & y_{32} + y_{41} + \xi(y_{12} + y_{34})
\end{bmatrix}
\]
3.3.2 Stiffness Matrix and Force Vector

On substituting Eq.(3.24) in Eq.(3.12) where \( u \) is \( q \) and then in Eq.(3.2), there is

\[
\sum_{i=1}^{n_e} \delta \left( \frac{1}{2} \int_{\Gamma_e} (q^e)^T B^T D B q^e \, d\Gamma - \int_{\Gamma_e} (q^e)^T B^T M \, d\Gamma - \int_{\Gamma_e} (q^e)^T B^T M_T \, d\Gamma - \int_{\Gamma_e} (q^e)^T B^T \tilde{f} \, d\Gamma \right) = 0. 
\]

(3.28)

Taking the variation with respect to the nodal variables \( q \), since the variation is arbitrary one has

\[
\int_{\Gamma_e} B^T D B d\Gamma \, q^e = \int_{\Gamma_e} B^T M \, d\Gamma + \int_{\Gamma_e} B^T M_T \, d\Gamma + \int_{\Gamma_e} B^T \tilde{f} \, d\Gamma
\]

(3.29)

for each element. On comparing Eq.(3.29) to Eq.(3.15), the expressions of \( K^e \) and \( F^e \) are obtained as

\[
K^e = \int_{\Gamma_e} B^T D B d\Gamma
\]

(3.30)

\[
F^e = F_{Tr}^e + F_T^e + F_V^e,
\]

(3.31)

where \( F_{Tr}^e \), \( F_T^e \) and \( F_V^e \) are element surface traction force, thermal force and piezo actuation force respectively with

\[
F_{Tr}^e = \int_{\Gamma_e} B^T \tilde{t} d\Gamma
\]

(3.32)

\[
F_T^e = \int_{\Gamma_e} B^T M_T d\Gamma
\]

(3.33)

\[
F_V^e = \int_{\Gamma_e} B^T M_V d\Gamma.
\]

(3.34)
For those elements with no actuators, $F_v^e$ is zero.

The element stiffness matrix $K^e$ and the force vectors are calculated by using a 2×2 Gauss numerical integration:

$$K^e = \int_{-1}^{+1} \int_{-1}^{+1} B^T DB |J| d\xi d\eta$$  \hspace{1cm} (3.35)

$$F_{7r}^e = \int_{-1}^{+1} \int_{-1}^{+1} B^T \bar{r} |J| d\xi d\eta$$  \hspace{1cm} (3.36)

$$F_T^e = \int_{-1}^{+1} \int_{-1}^{+1} B^T M_T |J| d\xi d\eta$$  \hspace{1cm} (3.37)

$$F_V^e = \int_{-1}^{+1} \int_{-1}^{+1} B^T M_A |J| d\xi d\eta$$  \hspace{1cm} (3.38)

where $J$ is given in Eq.(3.26), $B$ in Eq.(3.25), $D$ in Eq.(2.28), $M$ in Eq.(3.11), $M_A$ in Eq.(2.30) and $\bar{r}$ is known. Since the matrix $B$ is of dimensions 3×12, so the element stiffness matrix $K^e$ is a 12×12 matrix and it is a symmetric matrix.

### 3-4 System Equations

With the element equations written as Eq.(3.15), on substituting Eq.(3.6) in Eq.(3.2) the global system equation can be formulated as

$$K q = F$$  \hspace{1cm} (3.39)

where $K$ and $F$ can be determined through standard FEM procedures.

---

* According to the Ref. 36, a standard numerical integration scheme using 2×2 Gauss integration points is sufficient for the integration.
According to these standard procedures, a three dimensional array LM storing the relations among the three numbers, i.e., global degrees of freedom number, local node number and element number, is formed. The details are given in the explanations before each printout of code (see Appendix III). With the information stored in the array LM, a mapping matrix $P^e$ which relates the contribution of the element stiffness matrix to the global stiffness matrix is formed for each element. The global stiffness matrix $K$ is assembled as

$$K = \sum_{i=1}^{n_e} (P_i^e)^T K_i^e P_i^e . \quad (3.40)$$

For the force vector $F^e$, it is assembled as

$$F = \sum_{i=1}^{n_e} (P_i^e)^T F_i^e = \sum_{i=1}^{n_e} (P_i^e)^T (F_{\tau i}^e + F_{\tau j}^e + F_{\psi i}^e) , \quad (3.41)$$

where $K^e$ is given as in Eq.(3.35) and $F_{\tau i}^e$ and $F_{\psi i}^e$ are given as in Eqs.(3.37, 3.38) respectively.

Up to now, the numerical discrete of global linear system for solving the deflection of the piezo composite thin plate is complete.

3-5 Numerical Implementation

In this section it will be shown that the 2-D DKQ FEM approach is very efficient: The predictions of the plate deformation are accurate and the model is simple.

Numerical implementation is applied to two cases of an actuator/plate system, whose experiment results are presented by E.H. Anderson and E.F. Crawely in Ref. 13,
for verifying the validity of the mathematical models presented in Section 2-3 of Chapter 2 and the numerical 2-D DKQ FEM approach presented in the previous sections of this Chapter 3. This actuator/plate system is more representative of practical induced strain actuator/substrate system than those pseudo two dimensional cantilever actuator/beam systems which have only one actuator along x axis.\textsuperscript{7-11, 16, 20, 27}

The tested structures were cantilever laminated composite thin plates with 15 piezo actuators bonded to each surface of the substrates in a bimorph arrangement. The two substrates whose stacking sequences are $[0/\pm 45]_s$ and $[+30_z/0]_s$ were constructed from Hercules AS4/3501 Graphite/Exposy and were designed for the laminated composite substrates to increase transverse bending and to produce twist through

![Fig.(3.2) Dimensions and Structure of the Plate for Verification](image-url)
bending/twist coupling respectively. The dimensions and structure of the actuator/plate system are illustrated in Fig.(3.2). The material properties of the substrate and the piezoceramics G1195 are given in Appendix I.

3-5.1 The Validity of the 2-D Model

The predictions of 2-D model for three non-dimensional characteristic magnitudes, \( W_1, W_2, \) and \( W_3 \), which represent respectively the longitudinal bending \( W_1 = M_2 / C \), the twist in radians \( W_2 = (M_3 - M_1) / C \) and the fractional transverse chamber \( W_3 = [M_2 - (M_3 - M_1) / 2] / C \) of an actuator/plate system, are investigated. Here \( M_1 \) and \( M_3 \) refer to the deflection measurement positions at the outer transverse edges and \( M_2 \) at the centerline of the plate respectively and \( C \) is the width of the plate as shown in Fig.(3.2).

The validity of the model can be seen from the comparisons of those non-dimensional deflections in Figs.(3.3-3.8). The results calculated from the simplified thin plate model show a very nice agreement with the data obtained from the experiments as well as the closeness of the predictions between the 2-D and 3-D models for both the \([0/±45]_s\) substrate and \([+30_2/0]_s\) substrate. For both \([0/±45]_s\) and \([+30_2/0]_s\) plate cases the predictions for longitudinal bending, \( W_1 \) in Figs.(3.3,3.6), are very close to the experiment data, like 3-D’s predictions. The predictions for the lateral twisting \( W_2 \) shown in Figs.(3.4,3.7) also give a nice agreement with the experiment data, closer than 3-D’s predictions. Figures(3.5,3.8) show that the predictions for transverse bending \( W_3 \) are in good agreement with the experiment data within the length of the first half plate but a lower convergence at the opposite edge far enough from the clamped edge. But the
**Fig. (3.3) Comparison of the W1 Predictions, [0/±45°]s Plate**

**Fig. (3.4) Comparison of the W2 Predictions: [0/±45°]s Plate**
Fig.(3.5) Comparison of the W3 Predictions: [0/±45°]s Plate

Fig.(3.6) Comparison of the W1 Predictions: [(+30°)2/0]s Plate
Fig. (3.7) Comparison of the W2 Predictions: \((+30^\circ)2/0\) Plate

Fig. (3.8) Comparison of the W3 Predictions: \((+30^\circ)2/0\) Plate
**Fig. (3.9) Simulation of the Plate Deformation: [0/±45°]s Plate**

**Fig. (3.10) Plate Deformation Simulation: [(+30°)2/0]s Plate**
predictions still provide reasonable accuracy. The deviation of the prediction of 2-D model for $W_3$, like 3-D model, can be attributed to the scattering of the experiment data and the sensitivity of the deformation to both the actuator/substrate material properties such as the piezoelectric strain coefficients and their prediction as well as the geometry of the structures.

The numerical simulations of the plate deformation are performed by a computer code OPSC (see Section 4-5 of Chapter 4) using Matlab on SUN workstation. The simulation of the results are shown in Fig.(3.8) and Fig.(3.9).

3-5.2 The Simplicity of the 2-D FEM Approach

The efficiency of using 2-D FEM model is obvious. Since the plate model downgrades to 2-D, the number of degrees of freedoms of each element is lowered from 3-D model's 24 of mechanical and 8 of electrical to 2-D's 12 and 1 respectively. Therefore the computation work is greatly reduced. Such efficiency will be the key factor for real-time vibration control and optimal design of large-scale intelligent piezo sensor/actuator structures. It makes solving optimal design problem much more efficient than that of using 3-D model. Such efficiency of the 2-D model is shown in Table(3.1) by comparing with 3-D model for both element basis and problem basis.

<table>
<thead>
<tr>
<th>Model</th>
<th>Number of nodes</th>
<th>Number of MDOF*</th>
<th>Number of EDOF*</th>
<th>Order of K</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present 2-D DKQ</td>
<td>4(187)</td>
<td>12(561)</td>
<td>1(15)</td>
<td>12(561)</td>
</tr>
<tr>
<td>Ha's 3-D incompatible</td>
<td>8(374)</td>
<td>24(1122)</td>
<td>4(374)</td>
<td>32(1496)</td>
</tr>
<tr>
<td>Improvement (1-2-D/3-D)</td>
<td>50 % (50 %)</td>
<td>50 % (50 %)</td>
<td>75 % (96 %)</td>
<td>62.5 % (62.5 %)</td>
</tr>
</tbody>
</table>

* MDOF stands for mechanic degrees of freedom and EDOF for electric degrees of freedom.
CHAPTER 4

OPTIMAL SHAPE CONTROL

From an engineering standpoint, the shape control of the plate has useful applications, especially in design.

The deflection of a piezo composite plate is a function of, refer to the piezo actuators, the geometry of actuators, the material property of actuators, the voltage applied to the actuators, the layout of the piezo actuators on the plate, and the number of actuators, i.e.,

\[ W = f(\text{geometry, property, voltage, layout, number}) \]

This paper works on finding out the optimal input voltages to actuators and the optimal layout of actuators with finite number of actuator pairs to be deployed on the plate. Based on that, the problem of determination of optimal actuator number is therefore formulated.

4-1 General Statement of Optimal Shape Control Problem

Let \( \chi \) represents the design variables, that is, the number of the piezo actuator pairs \( p \), the layout of those actuator pairs \( l \) and the input voltages to the actuators \( V \), where \( l \) can be a vector composed of the coordinates of the centers of the actuator pairs and \( V \) is a column vector composed of the input voltages. A general statement for shape control optimization problem (GP), therefore, can be qualitatively stated as follows:
Seek a design \((\chi^*)\) which minimizes some measure of actuator/plate system performance subject to the condition that all of the appropriate measures of the system behavior and all of the design variables remain within prescribed bounds.

The foregoing \((\text{GP})\) statement can be quantified by formulating it as a general linear or nonlinear mathematical programming of the form

\[
\begin{align*}
\text{(GP)} & \quad \begin{cases}
\text{Minimize} & f(\chi) \\
\text{subject to} & g_i(\chi) \leq 0 \quad i = 1, 2, \ldots, n_g \\
& \chi_j \leq \chi_j \leq \bar{\chi}_j \quad j = 1, 2, \ldots, n_x
\end{cases}
\end{align*}
\]

(4.1)

where subscripts \(i\) and \(j\) indicate components, \(\chi = \langle p \ l \ V \rangle^T\), and \(n_g\) the number of constraints and \(n_x\) the number of the variables. It should be remarked that the form \((\text{GP})\) covers the case of equality constraints as well.

**4-2 Problem of Optimal Input Voltages to Actuators (VP)**

For a specified layout of piezo actuators, the input voltages to the actuators are the only means of active control of the shape. The objective of the shape control problem is to find out a set of optimal input voltages which can most closely produce the desired deflection.

**4-2.1 Deflection-Voltage Relationship**

Let the dimensions of the input voltages \(V\) and the nodal deflections \(W\) are \(p\) and \(m\) respectively where \(p < m\). Physically, this means that the number of actuators on the plate is less than the number of nodes which are of interest.
It is found that the map between the vector $V$ of input voltages and the vector $W$ of output deflections is linear for a certain layout of actuators, i.e.,

$$ W = W_T + W_T + CV $$

(4.2)

where $V$ and $C$ are

$$ V = \begin{pmatrix} V_1 & V_2 & \cdots & V_p \end{pmatrix}^T $$

(4.3)

$$ C = \begin{bmatrix} C_1 & C_2 & \cdots & C_p \end{bmatrix}, $$

(4.4)

and $W_T$ is the deflection due to the surface traction and $C$ is a matrix whose column vectors $C_j$ is the output nodal deflections when a unit voltage is applied to the $j$th actuator with the voltages of all other actuators set to zero. The output vector $(W - W_T - W_T)$ due to the input voltages is a span of the column vectors $C_1, C_2, \ldots, C_p$. $W_T$ and $C$ are obtained through the following derivation.

On substituting Eq.(3.34) with Eqs.(2.23) in Eq.(3.41), the force vector becomes:

$$ F = \sum_{i=1}^{n_e} \left( P_i^e \right)^T \left( F_T^e + F_T^e + L_i^e V_i^e \right) $$

(4.5)

where $L_i^e$ is

$$ L_i^e = \int_{\Gamma^e} B_i^T L_i d\Gamma $$

(4.6)

with $L_i$ given as in Eq.(2.33) for a domain with actuators and zero for a domain with no actuators, and $V_i^e$ is a vector of intermediate variables of element voltages

$$ V^e = \begin{pmatrix} V_1^e & V_2^e & \cdots & V_p^e \end{pmatrix}^T. $$

(4.7)
Further, on writing the vector of forces (Eq.4.5) in terms of a vector of input voltages to actuators, it becomes

\[ F = F_{Tr} + F_T + L' P_{ae} V \]  

(4.8)

where \( F_{Tr}, F_T \) and \( L' \) are

\[
F_{Tr} = \sum_{i=1}^{n_e} (P_i^e)^T F_{Tr_i} 
\]

(4.9)

\[
F_T = \sum_{i=1}^{n_e} (P_i^e)^T F_{T_i} 
\]

(4.10)

\[
L' = \begin{bmatrix} (P_1^e)^T L_1' & (P_2^e)^T L_2' & \cdots & (P_{n_e}^e)^T L_{n_e}' \end{bmatrix}
\]

(4.11)

and \( P_{ae} \) is a matrix which maps the voltages of elements to the voltages of actuators

\[
V' = P_{ae} V.
\]

(4.12)

Matrix \( P_{ae} \) is formed through an array LMAC which stores the actuator number. Details are in Appendix III.

With Eq.(4.8) one can write \( q \) in Eq.(3.39) in terms of vector of input voltages

\[
q = K^{-1} F_{Tr} + K^{-1} F_T + K^{-1} L' P_{ae} V.
\]

(4.13)

The transverse deflections \( W \) of nodal points can be stretched out from the nodal variables \( q \) of the linear system Eq.(3.39) through defining a selecting matrix \( S \) such that

\[
W = Sq = SK^{-1} F_{Tr} + SK^{-1} F_T + SK^{-1} L' P_{ae} V,
\]

(4.14)

since for each nodal point the nodal variables are of a form \( \begin{bmatrix} W & \theta_x & \theta_y \end{bmatrix}^T \). Comparing with the form in Eq.(4.2) one get
\[ W_{tr} = SK^{-1}F_{tr}, \quad (4.15) \]

\[ W_T = SK^{-1}F_T, \quad (4.16) \]

and

\[ C = SK^{-1}L^e P_{ae} \quad (4.17) \]

with \( L_e \) given in Eq.(4.11).

**4-2.2 Model (VP) and Its Solution**

Since the number of the actuators is less than the number of nodes, in general a desired deflection \( W_d \) can not be achieved exactly. In some special cases, if \((W_d-W_T-W_{tr})\) lies in the span of \( C_1, C_2, \ldots, C_p \) with \( V \) within limits, exact solution of \( W_d \) can be achieved. In all other cases, an approximate shape \( W^* \) close to the desired shape \( W_d \) can be obtained.

Now let \( \chi \) in the statement (GP) be the voltages \( V \) and the vector of desired deflections \( W_d \). Define a quadric cost function \( f(W(V)) \) of the nodal displacements \( W \) as

\[ e^TQe \]

where \( e \) is the error \((W-W_d)\) in deflection and \( Q \) is a positive diagonal matrix. The \( Q \) matrix assigns relative weighs to the elements of the vector of error \( e \). If the \( i \)th diagonal entry is larger than the \( j \)th, the optimization procedure will make the deflection of the \( i \)th point closer to the desired deflection as compared to the \( j \)th point. The matrix \( Q \) can thus be used to attach relative importance to the errors at different nodal points on the plate. The constraints \( g \) are the governing equations and boundary conditions. Since the transverse nodal deflections \( W \) have been solved in terms of input voltages \( V \), i.e.,
$W = W_r + W_T + CV$, the equality constraints $g = 0$ (refer to $W$) are merged into the expression of the cost function $f(W(V))$. The upper and lower bonds of the voltages on $p$ actuators, denoted as $\overline{V}$ and $\underline{V}$, can be expressed as $2p$ linear inequality constraints on $V$. These constraints can be written in a matrix form as $KV - R \leq 0$ where $K$ is a $2p \times p$ matrix and $R$ is a vector of constants composed of the actuator voltage bonds.

Now mathematically, the voltage-shape optimization problem can be stated as:

$$\begin{align*}
\text{(VP)} \\
\text{Minimize } & f(V) = (W_r + W_T + CV - W_d)^T Q(W_r + W_T + CV - W_d) \\
\text{subject to } & KV - R \leq 0
\end{align*}$$

with $Q$, $K$ and $R$ given as above.

The identity of the voltage solution $V^*$, relative to $W^*$, is given by the Kuhn-Tucker necessary condition\textsuperscript{42,43} (Kuhn and Tucker, 1951) which says:

For $V^*$ to be the local optima of the statement of (VP), there exists Kuhn and Tucker multipliers $\lambda \geq 0$ such that

$$\begin{align*}
\text{(KT)} \\
\nabla f(V^*) + \lambda^T \nabla (KV^* - R) = 0 \\
\text{and} \\
\lambda_i (KV - R)_i = 0 \\
i = 1, 2, \ldots, 2p
\end{align*}$$

where $\lambda$ is a $2p \times 1$ vector and $\nabla$ represents the gradient.

This results in solving the following equation

$$2C^T QC V^* + K^T \lambda = 2C^T Q(W_d - W_r - W_T)$$

subject to the constraints $\lambda \geq 0$ and $\lambda_i (KV - R)_i = 0$, where $i$ indicates the $i$th elements of $\lambda$. 
and \((KV^-R)\), not a Kronecker summation sign.

From the choice of the matrix \(Q\) as a positive definite diagonal matrix, the objective function \(f(V)\) is strictly convex. The search space of the input voltages \(V\) is a strictly convex set. Hence, the solution of \((VP)\) is global minimum and it must be unique.

4-3 Problem of Optimal Layout of Actuators (LP)

More generally, the location of the actuators on the plate may also need to be determined. The actuators are best placed in the regions of high average strain and away from areas of zero strain which are called "strain nodes". If the actuators are placed at the strain nodes the desired deflection of a plate can hardly be reached even with high input voltages to the actuators. So, in a design, the actuation location greatly affects the effectiveness of the vibration and shape control.

Finding an optimal layout of the actuators for a plate to reach a desired shape by taking the center coordinates of all actuators as continuous variables is very complicated. The difficulty is at that to solve the deflection Eq.(3.39) of the plate the coordinates of the actuators are required, i.e., \(K(l)q=F(l)\) in a standard FEM expression with \(l\) being a vector of the center coordinates of the actuators which needs to be determined also. Here \(K\) and \(F\) are the stiffness matrix and force vector respectively.

But in design further relaxation is allowed. Some possible locations for a set of actuator pairs going to be deployed on the plate can be preassigned. Through introducing a vector of binary variables \(\alpha\) whose element \(\alpha_i\) is \((0, 1)\) which indicates the presence/activation \((1)\) of an actuation unit or the absent/inactivation \((0)\) of an actuation unit at a
given location, a hybrid function $\vec{F}_i$ in a domain of the $i$th element on the plate can be formed as

$$\vec{F}_i = (1 - \alpha_i) \vec{F}_{s_i} + \alpha_i \vec{F}_{a_i} \quad (4.21)$$

with the subscripts $s$ for the domain of substrate only or no actuation and $a$ for the domain with actuation. Generally, function $\vec{F}_i$ can be any z-direction-integrated functions. Here they are the element stiffness matrix $K^e$ and the force vector $F^e$. Then for the $i$th element Eq.(3.15) can be written as

$$[(1 - \alpha_i^e) K_{s_i}^e + \alpha_i^e K_{a_i}^e] q_i^e = (1 - \alpha_i^e) F_{s_i}^e + \alpha_i^e F_{a_i}^e = F_{Tr_i}^e + F_{T_i}^e + \alpha_i^e F_{v_i}^e \quad (4.22)$$

On substituting Eq.(4.22) in Eq.(3.39) and following the assembling procedure in Section 3-4 of Chapter 3 and the similar procedure for determining matrix $C$ in Section 4-2.1 in this Chapter 4, $K$ and $F$ can be expressed as functions of the layout vector $\alpha = \langle \alpha_1, \alpha_2, \ldots, \alpha_p \rangle^T$, i.e.,

$$K(\alpha)q = F_{Tr} + F_{T} + L(\alpha)V \quad (4.23)$$

Hence the transverse nodal deflections $W$ are

$$W = W_{Tr} + W_{T} + C(\alpha)V \quad (4.24)$$

In the above equations, $F_{Tr}, F_{T}, W_{Tr},$ and $W_{T}$ are the same as in Eqs.(4.9, 4.10, 4.15, 4.16), and $K(\alpha), L(\alpha)$ and $C(\alpha)$ are

$$C(\alpha) = SK^{-1}(\alpha)L(\alpha) \quad (4.25)$$

$$K(\alpha) = K_s + \left[ K_1 P_{pe} \alpha \quad K_2 P_{pe} \alpha \quad \cdots \quad K_n P_{pe} \alpha \right] \quad (4.26)$$
with

\[ K_s = \sum_{i=1}^{n_p} (P_i^e)^T K_i^e P_i^e \]  \hspace{1cm} (4.27)

\[ L(\alpha) = L^e \text{Diag}(P_{pe} \alpha) P_{ae} \]  \hspace{1cm} (4.28)

where the subscript \( s \) indicates the plate global stiffness matrix \( K \) composed of substrate material only, and \( \overline{K_i} \) is a matrix formed by all the \( i \)th columns of the transformed element stiffness matrices \( (P_i^e)^T (K_{i}^e - K_{i}^s) P_i^e \) with those columns arranged in the sequence as \( i \) (from 1 to \( n_e \)), \( P_{pe} \) a matrix formed from the array LMAC (like \( P_{ae} \), see Appendix III) mapping the possible locations to the elements, \( L^e \) as given in Eq.(4.10), and \( \text{Diag}(P_{pe} \alpha) \) a diagonal matrix whose elements on the diagonal are the elements of \( \alpha^e \) vector (= \( P_{pe} \alpha \)) of the intermediate layout variables.

Now the quantified layout-shape control optimization problem can be written as a quadratic, nonlinear, integer mixed mathematical programming

\[
\begin{align*}
\text{Minimize} \quad f(V, \alpha) &= (W_T + W_r + C(\alpha)V - W_d)^T Q(W_T + W_r + C(\alpha)V - W_d) \\
\text{subject to} \quad &KV - R \leq 0 \\
\alpha_i &= 0 \text{ or } 1 \\
\sum_{i=1}^{n_p} \alpha_i &= p
\end{align*}
\]  \hspace{1cm} (4.29)

where \( \alpha_i \) is the element of \( \alpha \), \( p \) is the number of the actuator pairs to be deployed on the plate and \( n_p \) is the number of possible locations preassigned.
The identity of the solution, \( \alpha \sim V' \), of (LP) under certain layout is also given by the Kuhn-Tucker necessary condition (KT) and the equation to solve is the same as Eq.(4.20). So \( \alpha \) solution searching appears to be an outer loop optimization problem. When the \( \alpha \sim V' \) solution is identified both the layout of actuators and the voltages applied on the actuators are optima.

The design optimization problem represented by the mathematic statement (LP) is highly nonlinear. The methods to solve (LP) depends on the size of the problem, or more precisely, the number of the actuator pairs and their possible locations as well as the constraints. The problem statement (4.29) is necessary for solving mid-to-large-size optimal design problems since it makes it possible for certain integer mixed optimization programming such as Branch-and-Bond technique\(^{40,41} \) to be applied to size down the feasible solution searching space.

4-4 Problem of Optimal Number of Actuators

By changing the constraint \( \sum_{i=1}^{n_p} \alpha_i = p \) in the Statement (4.29) to \( \underline{p} \leq \sum_{i=1}^{n_p} \alpha_i \leq \overline{p} \)

where \( \underline{p} \) and \( \overline{p} \) are the upper and lower bonds for \( p \), the quantitative statement of the problem of finding optimal number of actuator pairs can be obtained. Obviously, the problem of the optimal number of actuation units becomes a new outer loop and the \( \alpha \sim V' \) solution searching will be more complicated.
4-5 Numerical Implementation

In this section an algorithm for solving either (VP) or (LP) is constructed. A corresponding computer code OPSC (stands for optimal shape control) is developed. Numerical implementations are performed on two thermal-deformation-correction cases of an actuator/plate system exposed to certain thermal environments to see the efficiency of voltage optimization design and layout-voltage optimization design with different actuator bonding styles, i.e., surface embedded and bonded, prescribed input voltage constraints, as well as various boundary conditions.

4-5.1 Algorithm and Computer Code OPSC

The solution searching algorithm is aimed at determining the optimal actuator layout and input voltages which satisfy the Kuhn-Tucker optimality conditions Eq.(4.17). The implementation of this algorithm consists of the following steps:

(1) Assign $p$ actuator pairs to any $p$ of those pre-assigned $n_p$ possible locations for bonding or embedding actuator pairs on the plate. Create two arrays, $L$ and $V$, to store the layout variables $\alpha$ and the input voltages $V$ respectively, and a $F$ to store the value of the cost function $f$.

(2) Start the layout $\alpha$ optimal solution searching loop with moving one of the $p$ actuator pairs each time.

(3) Start the voltage $V$ optimal solution searching by selecting a set of $\lambda_i$ as zeros and making the voltage constraints $(KV-R)_i=0$ active for those $\lambda_i \neq 0$. Here one must be careful in selecting those $\lambda_i$, which are not zeros, to avoid making such constraints active that
they will result in an inconsistent solution space, that is, one must avoid making an actuator simultaneously take both of its lower and upper bonds.

(4) Compute the actuator input voltages $V$ using Eq.(4.20) and check whether the voltages obtained satisfy the inequality constraints. If the (KT) conditions are satisfied the current set of the Kuhn and Tucker multipliers results in the optimal input voltages. Stop searching and move onto step (5). If not satisfied, go back to step (3).

(5) Compute the cost function $f$. If the current value of $f$ is smaller than or equal to the one stored in $F$ renew $F$ with the current $f$ and meanwhile renew $L$ and $V$ with the current $\alpha$ and $V$. Otherwise, keep $L$, $V$ and $F$ unchanged.

(6) If the set of feasible layout solution space is not all searched go back to step (2) until the set of layout solution space is all searched. The layout variables $\alpha$ and the input voltages $V$ in the arrays $L$ and $V$ form the final design variables.

The illustration of this algorithm is given in Fig.4.1.

Based on the above algorithm, a computer code called OPSC is developed in a general way for analysis during this investigation. The code OPSC uses Matlab to simulate the actuator/plate system deformation on SUN workstation. The control information such as whether the problem is (LP) or (VP) and therefore input voltages and number of actuator pairs, the material properties of substrate and piezoceramics, and the geometry of the actuator/plate system and actuation units as well as their possible locations are defined in different input files as IN_CONT, IN_MATE and IN_ELEM respectively. The file MAIN performs the optimal solution calculation as well as the
deformation simulation. More details are given in the statements in the beginning of each file in the Appendix III.

Although OPSC is an optimal shape control design tool, it is believed that due to the use of FEM for the plate behavior the code can be further developed for analyzing the design and control of a thin plate composed of large-scale piezo actuator/sensor laminated structures of various geometry, as well as dynamic analysis. For dynamic vibration control analysis, simply add the time derivative term to Eq.(3.15).

4-5.2 Numerical Results

Two examples are presented in this section. The first one shows (VP) design while the second one is for (LP) design.

![Dimensions and Structure of the Plate for Implementation](image)
The actuator/plate system in first case was presented by Sung Kyu Ha, Charles Keilers and Fu-Kuo Chang in Ref. 26. Its dimensions and the system structure are illustrated in Fig.(4.2).

The substrate material is T300/976 Graphite/Epoxy with a stacking sequence of $[0/\pm45^\circ]$. The piezo ceramics are PZT G1195N. The substrate and actuator material properties are given as in the Appendix II. Two edges of the plate are simply supported with $y=0$ pinned and $y=372$ on the rollers while the rest two edges free. The plate were exposed to an elevated thermal environment with a temperature increase of 50°F (relative to the zero strain temperature) on the top and a drop of 50°F on the bottom. The temperature distribution reached thermal stationary.

Both the optimal input voltages determined by the OPSC and the corrected plate deformation simulated by the OPSC are shown in Fig.(4.3). Its three characteristic non-dimensional deflections, i.e. $W_1$, $W_2$, and $W_3$, are shown in Fig.(4.5). For comparison, the results of applying uniform input voltage to all actuators, which were obtained when the central deflection of the plate reached certain tolerance$^{26}$, are also computed by the code OPSC and shown in Fig.(4.4). Its characteristic non-dimensional deflections are also illustrated in Fig.(4.6). The effects of different corrections are indicated by the values of the cost function as well as the non-dimensional deflection tolerance shown in $W_1-W_2-W_3$ graphs (Fig.(4.6)). Those differences can also be told from the deformation-simulation graphs.
Fig. (4.3) Plate-Deformation Simulation: Optimal Input Voltages

Optimal Input Voltages (V)

<table>
<thead>
<tr>
<th>Voltage Values</th>
<th>85.0</th>
<th>-36.2</th>
<th>29.7</th>
<th>-5.3</th>
<th>50.7</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>140.0</td>
<td>59.3</td>
<td>113.7</td>
<td>59.3</td>
<td>140.0</td>
</tr>
<tr>
<td></td>
<td>50.7</td>
<td>-5.3</td>
<td>29.7</td>
<td>-36.2</td>
<td>85.0</td>
</tr>
</tbody>
</table>

Fig. (4.4) Plate-Deformation Simulation: Uniform Input Voltages

Uniform Input Voltage of 58(V)
Two Edges Simply Supported
Substrate: T300/976 Graphite/Epoxy, [0°/±45°]s
Actuator: PZT G1195N, Surface Bonded
Fig. (4.5) Nondimensional Deflections W1, W2, W3: Optimal Input Voltages

Fig. (4.6) Nondimensional Deflections W1, W2, W3: Uniform Input Voltage
It can be seen from Fig.(4.6) that the uniform-input-voltage correction sacrifices the fractional transverse chamber $W_3$ and controls the longitudinal bending $W_1$. But the optimal-input-voltage correction says contrarily: The control of $W_3$ has to be tightened instead of $W_1$ because $W_3$ is more significant than $W_1$. And the maximum magnitudes of the non-dimensional deflection tolerance of the plate for two different correction styles show that the optimal-input-voltage correction improves the effect from close to $12e^{-4}$ to $6e^{-4}$ — about 50%. The uniform-input-voltage correction leaves large deflection along the two free edges of the plate. Its solution is subjected to not only the iteration step but also the deflection tolerance and the desired shape. Such method works only for few special cases and the solution is not optimal.

Obviously, it is better to use more actuator pairs than just few to reach the desired shape. But using too many actuators is not practical nor economic.

Therefore the second example is designed for understanding the importance of layout, i.e., (LP) design. The example is to solve for optimal layout of the given total nine actuators to correct the thermal deformation of a laminated composite plate of the same dimensions and material properties and also exposed to the same elevated thermal environment as in the previous example, but with the actuators surface embedded instead of surface bonded and also the four corners simply supported in stead of the two edges simply supported, i.e., one pinned at $(0, 0)$ and the rest three on the rollers.

In the search of the optimal layout-voltage designing variables, voltage constraints turn out to be important. Before moving on to solving this nine-actuator-pairs problem,
the optimal input voltages for fifteen actuator pairs are solved for checking the probability of unfeasible input voltages since solving the optimal input voltage problem is more economical than solving the optimal layout-voltage problem. The results are given in Table (4.1), arranged in the same sequence as the locales of the actuators on the plate.

<table>
<thead>
<tr>
<th>Table (4.1)</th>
<th>Optimal Input Voltages (Electric Fields) for 15 Actuator Pairs without Voltage Constraints</th>
<th>V (V/mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>85.0(656.9)</td>
<td>-36.2(-278.5)</td>
<td>29.7(228.5)</td>
</tr>
<tr>
<td>140.0(1076.9)</td>
<td>59.3(456.2)</td>
<td>113.7(874.6)</td>
</tr>
<tr>
<td>50.7(390.0)</td>
<td>-5.2(-40.0)</td>
<td>29.7(228.5)</td>
</tr>
</tbody>
</table>

It is found that some of the fields (like 1076.9 V/mm) obtained under no voltage constraints reaches almost 90% of, very close to, the 1200 V/mm coercive field (at which the piezo ceramics will depolarize) of the G1195 piezo ceramics. With all fifteen actuator pairs present, since some of the fields are so close to the limit, it is very much likely that with nine actuator pairs activated some of the input voltages will cause the input electric fields exceed the coercive field. Actually, the optimal layout-voltage solution under no voltage constraints has been searched for this nine-actuator-pairs plate problem. The results of the optimal input voltages with relative electric fields as well as the layout of the actuator pairs are shown in Table(4.2). It can be seen that three out of nine exceed the coercive limit.
Optimal Input Voltages (V)

120.0 ------ -14.7 ------ 104.5
120.0 ------ 120.0 120.0 120.0
120.0 ------ ------ ------ 91.8

Four Corners Simply Supported
Substrate: T300/976 Graphite/Epoxy, [0/±45°]s
Actuator: PZT G1195N, Surface Embedded

Fig. (4.7) Plate-Deformation Simulation: 9 Actuator Pairs under Optimal Layout and Input Voltages with Vi<=120 (V)

Thermal Deformation Only

Four Corners Simply Supported
Substrate: T300/976 Graphite/Epoxy, [0/±45°]s

Fig. (4.8) Plate-Deformation Simulation: No Actuator Pairs
Table (4.2) Optimal Layout and Input Voltages (and Electric Fields) for 9 Actuator Pairs without Voltage Constraints \( V(V/mm) \)

<table>
<thead>
<tr>
<th>Voltage (V/mm)</th>
<th>Optimal Layout</th>
<th>Input Voltage</th>
<th>Electric Field</th>
<th></th>
<th>Voltage (V/mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>53.4(41.8)</td>
<td>----</td>
<td>-55.1(-423.8)</td>
<td>-</td>
<td>56.5(434.6)</td>
<td></td>
</tr>
<tr>
<td>223.2(1716.9)</td>
<td>----</td>
<td>243.2(1870.8)</td>
<td>-</td>
<td>232.3(1786.9)</td>
<td></td>
</tr>
<tr>
<td>53.9(414.6)</td>
<td>----</td>
<td>----</td>
<td>-73.9(-568.5)</td>
<td>75.0(576.9)</td>
<td></td>
</tr>
</tbody>
</table>

Therefore the design of optimal layout and input voltages have to be determined under certain voltage constraints. Figure (4.7) is for optimal layout-voltage solution under a voltage constraint \( V \leq 120 \) (V) which is 76.9% of the coercive field for each piezo actuator. Fig. (4.8) shows the thermal deformation of the substrate plate exposed to such a temperature environment with no actuators embedded/bonded on its surfaces.

Now the design of layout of the actuator pairs and the input voltages are both improved since the shape is more close to the desired shape, reflected by the smaller value of the cost function.

If the design problem is to determine the optimal activation of any certain number among all the actuators embedded/bonded on the plate, the code OPSC can also find the solution by taking the inactivated actuators as extra layers of laminae of different material properties.
CHAPTER 5

CONCLUSIONS

Two complete analytical models, the displacement $U-V-W$ model and stress-function-displacement $\Phi-W$ model, for describing the behavior of a thin plate composed of multi-directional laminated substrate with piezoceramic actuators surface embedded or bonded in a bimorph arrangement have been developed for an actuator/plate system with complicated stiffness couplings, various boundary conditions, and external loads including thermal loads. The introducing of the Kirchoff’s thin plate hypotheses in the modeling made the mathematic descriptions simpler.

In numerical FEM approach for the displacement model, the plate total potential energy formulation was derived using the consistent plate model. The 2-D FEM formulation for solving such an actuator/plate system, which correspondingly adopted DKQ element, has been developed and proved accurate in predicting deformation of the plate and meanwhile more efficient and simpler than 3-D models in saving computation work. This efficiency and simplicity along with the FEM capability made it possible to solve a control problem of the plate system with various number of piezo actuation units which could be of various shapes.

A generalized mathematical statement for optimal shape control of the plate, which takes the input voltages to actuators, the layout of those actuators and the number
of actuators as design variables, has been presented. With this statement, mathematical formulations for the optimal input voltage problem (VP) and the optimal actuator layout-voltage problem (LP) even the problem of optimal number-layout-voltage of actuation units to be allocated on the substrate plate to best reach the desired shape have been developed by introducing a binary design variable $\alpha$ for the optimal locations of actuation units. The (VP) design is a linear mathematical programming but the (LP) design turns out to be an integer involved highly non-linear mathematical programming.

Corresponding to the generalized problem statement, an algorithm for solving such a problem has been presented and a corresponding computer code OPSC using Matlab has also been developed as a design tool for designing large-scale laminated structures containing distributed piezo actuator bimorph units. Compared to the previous research work adopting FDM, it was found that the efficiency and the simplicity of the 2-D DKQ FEM model was critical for solving large-scale shape design and shape optimization problems.

It is believed that the 2-D DKQ FEM model can also be expended to dynamic analysis of the actuator/plate system under study without major technical difficulties, and same is the code OPSC. For future research work, a better binary tree searching algorithm for non-linear optimization problem should be developed. The potential of using Branch and Bond technique should especially be investigated. This will be critical to expending research work to dynamic analysis and certain large-scale plate systems. To establish an experiment system for this stage of research work will be necessary.
REFERENCES


APPENDIX I  Material Properties for the Substrates and Piezoceramics
of the Examples in Chapter 3

<table>
<thead>
<tr>
<th>Material Properties (^{13})</th>
<th>AS4/3501 Graphite/Epoxy</th>
<th>PZT G1195</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Moduli (Gpa)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E_1 = E_L )</td>
<td>143.0</td>
<td>63.0</td>
</tr>
<tr>
<td>( E_2 = E_T )</td>
<td>9.7</td>
<td>63.0</td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \nu_{12} = \nu_{LT} )</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Shear Moduli (Gpa)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( G_{12} = G_{LT} )</td>
<td>6.0</td>
<td></td>
</tr>
<tr>
<td>Density (Kg/m(^3))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \rho )</td>
<td></td>
<td>7650</td>
</tr>
<tr>
<td>Piezoelectric Strain Coefficients (pm/V) (^*)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( d_{Z_1} = d_{31} )</td>
<td>0</td>
<td>258</td>
</tr>
<tr>
<td>( d_{Z_2} = d_{32} )</td>
<td>0</td>
<td>354</td>
</tr>
<tr>
<td>Electrical Permittivity (nf/m)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \xi_{11}, \xi_{22}, \xi_{33} )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Thermal Expansion Coefficient ((\mu m/\text{m}^\circ\text{C}))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \alpha )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\(^*\) Predicted by Crawley, et. al., in Ref. 13


APPENDIX II  Material Properties for the Substrates and Piezoceramics

of the Examples in Chapter 4

<table>
<thead>
<tr>
<th>Material Properties</th>
<th>T300/976 Graphite/Epoxy</th>
<th>PZT G1195N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young's Moduli (Gpa)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_1 = E_{xx}$</td>
<td>150.0</td>
<td>63.0</td>
</tr>
<tr>
<td>$E_2 = E_{yy}$</td>
<td>9.0</td>
<td>63.0</td>
</tr>
<tr>
<td>Poisson's Ratio</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\nu_{12} = \nu_{xy}$</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Shear Moduli (Gpa)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_{12} = G_{xy} = G_{zx}$</td>
<td>7.1</td>
<td>24.2</td>
</tr>
<tr>
<td></td>
<td>$G_{yz}$</td>
<td>2.5</td>
</tr>
<tr>
<td>Density (Kg/m$^3$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td>1600</td>
<td>7600</td>
</tr>
<tr>
<td>Piezoelectric Strain Coefficients (pm/V)$^\dagger$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d_{z1} = d_{z2}$</td>
<td>0</td>
<td>254</td>
</tr>
<tr>
<td>$d_{y4}$</td>
<td>0</td>
<td>584</td>
</tr>
<tr>
<td>$d_{z3}$</td>
<td>0</td>
<td>374</td>
</tr>
<tr>
<td>Electrical Permittivity (nf/m)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\varepsilon_1 = \varepsilon_2 = \varepsilon_{xx} = \varepsilon_{yy}$</td>
<td>0</td>
<td>15.3</td>
</tr>
<tr>
<td>$\varepsilon_3 = \varepsilon_{zz}$</td>
<td>0</td>
<td>15.0</td>
</tr>
<tr>
<td>Thermal Expansion Coefficient (µm/m$^\circ$C)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_1 = \alpha_x$</td>
<td>1.1</td>
<td>0.9</td>
</tr>
<tr>
<td>$\beta_2 = \alpha_y = \alpha_z$</td>
<td>25.2</td>
<td>0.9</td>
</tr>
</tbody>
</table>
function Main

% Preprocessing phase
COOR=in_COOR;
[ID,ELEM]=in_ELEM;
MATE=in_MATE;
CONT=in_CONT;
nelxy=CONT(2);
nelx=nelxy(1,1);
nely=nelxy(2,1);

% Solution phase
flag=CONT(2);
if flag==1
    [L,V,W,F]=s1_W(COOR,ID,ELEM,MATE,CONT);
elseif flag==2
    [L,V,W,F]=s2_lagr(COOR,ID,ELEM,MATE,CONT);
elseif flag==3
    [L,V,W,F]=s3_srch(COOR,ID,ELEM,MATE,CONT);
else
    L=[]; V=[]; W=[]; F=[];
end
save g_data L V W F; % Save results in g_data file for different cases
Lc=fliplr(reshape(L,3,5)');
V=zeros(9,1);
V(find(L~=0))=V;
Vc=fliplr(reshape(V,3,5)');
Wc=fliplr(reshape(W,nelx+1,nely+1)*1E3);
Fc=F;
fprintf('result','%e',Lc,Vc,Wc,Fc);

% Postprocessing phase
W_graph(COOR,ID,ELEM,MATE,CONT,L,V,W,F);
pause(10);
M_graph(COOR,ID,ELEM,MATE,CONT,L,V,W,F);
function CONT=in_CONT

% example: Ha's plate of size nelx=10 nely=16 and flag=2

% flag--- for defining the type of the problem to solve:
% 4 3 2 1
% unknowns: number of layout, voltage, deflection
% actuators, voltage, deflection
% layout, deflection
% voltage, deflection
%
% unknowns: number of layout, voltage, deflection
% actuators, voltage, deflection
% layout, deflection
% voltage, deflection
%
% knowns: possible number of number of number of
% location actuators, actuators, actuators,
% possible location location location,
% location voltage
%
% knowns: possible number of number of number of
% location actuators, actuators, actuators,
% possible location location location,
% location voltage
%
% nelxy---------number of elements in x and y directions
% npl----------number of possible locations of actuators
% nac----------the number of actuators expected, given
% alpha---------(0,1) variable indicating whether there is actuation on the
% possible location: 1 for actuation, 0 for no actuation
% V----------the voltage input onto the actuators
% Vup,Vlow---the up and low bond of the voltage on the actuators
% Wd----------the desired deflection of the nodal points
% *Note: Be sure form the CONT vector in the specified sequence.

flag=2;
nelx=10;
nely=16;
nelxy=[nelx;nely];
npl=nelx*nely;
nac=15;
alpha=ones(15,1);
V=0.*ones(15,1);
a=[1 1 1 0 1 0 0 1 0 0 1 0 1 1 1]';
Vup=100.0*a;
Vlow=-100.0*(ones(15,1)-a);
Vbond=[Vup;Vlow];
Wd=zeros(187,1);
W0=zeros(187,1);

CONT=[le50; flag; le50;
     2e50; nelxy; 2e50;
     3e50; npl; 3e50;
     4e50; nac; 4e50;
     5e50; alpha; 5e50;
     6e50; V; 6e50;
     7e50; Vbond; 7e50;
     8e50; Wd; 8e50;
     9e50; W0; 9e50];

function CONT=in_CONT

% example: Crawley's plate of size nelx=9 nely=16 and flag=1
% Stacking sequence [+45/-45/0]s
flag=1;
nelx=10;
nely=16;
nelxy=[nelx;nely];
npl=nelx*nely;
nac=15;
alpha=ones(15,1);
V=100.0*ones(15,1); % For the case of [+30/+30/0]s, V=394.0*ones(15,1)
Vup=zeros(15,1);
Vlow=-Vup;
Vbond=[Vup;Vlow];
Wd=zeros(187,1);
W0=zeros(187,1);

CONT=[1e50; flag; 1e50;
     2e50; nelxy; 2e50;
     3e50; npl; 3e50;
     4e50; nac; 4e50;
     5e50; alpha; 5e50;
     6e50; V; 6e50; 
function [ID,ELEM]=in_ELEM

% example: Ha's plate nelx=10 nely=16 (See Ref.26), four corners simply supported

% alpha---Binary variables: 0 for no actuator presented domain
% 1 for actuator presented domain
% ID------Destination array:
% ID(i,A)=P if A is in the domain excluding the prescribed boundary
% =0 if A is on the prescribed boundary
% i----degrees of freedom number
% P---global equation number
% IEN----Element nodes array:
% IEN(a,e)=A
% a---local node number
% e---element number
% A---global node number
% LM----Location matrix:
% LM(i,a,e)=ID(i,A) or sometimes
% LM(p,e)=LM(1,a,e), p=ned*(a-1)+i
% p---local equation number
% ned---number of element degrees of freedom/node
% The degree of freedom of an element, or the number of element equations neq, is 12.
% The size of the global stiffness matrix is 144.
% The nodes with prescribed deflections have been removed from the unknowns of the linear system through the help of ID,IEN and LM arrays.
% See reference: Thomas J. R. Hughes, The Finite Element Method
% linear static and dynamic finite element analysis,
% Nodes are numbered from 1 to 170 started from y=0 edge. The mesh size is
% 10x17. Totally there are 144 elements. Local nodes are numbered in a sequence of (1 2 3 4) respectively for
% (x(i)-y(i) x(i+1)-y(i) x(i+1)-y(i+1) x(i)-y(i+1))
% for each element. The degree of freedoms are numbered in a sequence of w,
% thetax, thetay for each node.

ID=[ ];
for i=1:10
    ID=[ID [0;1;2]+(i-1)*3];
end
ID=[ID [0;30;31]];
for i=12:176
    ID=[ID [32;33;34]+(i-12)*3];
end
ID=[ID [0;527;528]];
for i=178:186
    ID=[ID [529;530;531]+(i-178)*3];
end
ID=[ID [0;556;557]];

IEN=[ ];
for j=1:16
end

LM=[ ];
for i=1:160
    LMi=[ID(:,IEN(1,i));ID(:,IEN(2,i));ID(:,IEN(3,i));ID(:,IEN(4,i))];
    LM=[LM LMi];
end

a=zeros(1,10);
b1=[0 1 1 0 2 2 0 3 3 0];
b2=[0 4 4 0 5 5 0 6 6 0];
b3=[0 7 7 0 8 8 0 9 9 0];
b4=[0 10 10 0 11 11 0 12 12 0];
b5=[0 13 13 0 14 14 0 15 15 0];
LMAC=[a b1 b1 a b2 b2 a b3 b3 a b4 b4 a b5 b5 a];

ELEM=[IEN;LM;LMAC];

%%%%%%%%%%%%%%%%%%%%%
function [ID,ELEM]=in_ELEM

% example: Ha's plate nelx=10 nely=16, two edges y=0 and y=292 clamped

ID(1:2:3,:)=zeros(2,11);
ID(2,:)=[1:11];
for i=12:176
    ID=[ID [12;13;14]+(i-12)*3];
end
ID=[ID [zeros(1,11);507:517;zeros(1,11)]];

IEN=[];
for j=1:16
end

LM=[];
for i=1:160
    LMi=[ID(:,IEN(1,i));ID(:,IEN(2,i));ID(:,IEN(3,i));ID(:,IEN(4,i))];
    LM=[LM LMi];
end

da=zeros(1,10);
b1=[0 1 1 0 2 2 0 3 3 0];
b2=[0 4 4 0 5 5 0 6 6 0];
b3=[0 7 7 0 8 8 0 9 9 0];
b4=[0 10 10 0 11 11 0 12 12 0];
b5=[0 13 13 0 14 14 0 15 15 0];
LMAC=[a b1 b1 a b2 b2 a b3 b3 a b4 b4 a b5 b5 a];
ELEM=[IEN;LM;LMAC];

function [ID,ELEM]=in_ELEM

% example: Crawley's plate nelx=9 nely=16
ID(1:3,1:2)=zeros(3,2);
for i=3:4
    ID=[ID [1;2;3]+(i-3)*3];
end

IEN=[1;2;4;3];

LMi=[ID(:,IEN(1,i));ID(:,IEN(2,i));ID(:,IEN(3,i));ID(:,IEN(4,i))];

ELEM=[IEN;LM];

function COOR=in_COOR

% The nodal coordinates for the example whose deflection has been
% calculated by S.K. Ha, C. Keilers, and F.K. Chang (Ref.26)

a=60.0/2;
b=12.0;
x(1)=0.0;
x(2)=b;
x(3)=x(2)+a;
x(4)=x(3)+a;
x(5)=x(4)+b;
x(6)=228.0/2;
x(7)=228.0-x(5);
x(8)=228.0-x(4);
x(9)=228.0-x(3);
x(10)=228.0-x(2);
x(11)=228.0;
y(1)=0.0;
y(2)=b;
y(3)=y(2)+a;
y(4)=y(3)+a;
y(5)=y(4)+b;
y(6)=y(5)+a;
y(7)=y(6)+a;
y(8)=y(7)+b;
y(9)=y(8)+a;
y(10)=372.0-y(8);
y(11)=372.0-y(7);
y(12)=372.0-y(6);
y(13)=372.0-y(5);
y(14)=372.0-y(4);
y(15)=372.0-y(3);
y(16)=372.0-y(2);
y(17)=372.0;
COOR=[];
for i=1:17
    COOR=[COOR;[x' ones(11,1)*y(i)]*1.E-3];
end;

function COOR=in_COOR

  a=51.0/2;
  b=25.0;
  cx=(152.0-4.0*a-b)/4.0;
  cy=(292.0-10.0*a)/6.0;

  x(1)=0.0;
  x(2)=cx;
  x(3)=x(2)+a;
  x(4)=x(3)+a;
  x(5)=x(4)+cx;
  x(6)=152.0/2;
  x(7)=152.0-x(5);
  x(8)=152.0-x(4);
  x(9)=152.0-x(3);
  x(10)=152.0-x(2);
  x(11)=152.0;

  y(1)=0.0;
  y(2)=cy;
  y(3)=y(2)+a;
  y(4)=y(3)+a;
  y(5)=y(4)+cy;
\[ y(6) = y(5) + a; \]
\[ y(7) = y(6) + a; \]
\[ y(8) = y(7) + c y; \]
\[ y(9) = y(8) + a; \]
\[ y(10) = 292.0 - y(8); \]
\[ y(11) = 292.0 - y(7); \]
\[ y(12) = 292.0 - y(6); \]
\[ y(13) = 292.0 - y(5); \]
\[ y(14) = 292.0 - y(4); \]
\[ y(15) = 292.0 - y(3); \]
\[ y(16) = 292.0 - y(2); \]
\[ y(17) = 292.0; \]

\[
\text{COOR} = []; \\
\text{for } i = 1: 17 \\
\quad \text{COOR} = [\text{COOR}; [x' \text{ ones}(11, 1) * y(i)] * 1.0E-3]; \\
\text{end;}
\]

\[
\text{function MATE = in_MATE} \\
\text{example: Ha's plate of size nelx=10, nely=16, and flag=2} \\
\text{% surface bonded actuators (See Ref.26)}
\]
\[
\text{Es1} = 150.0E9; \\
\text{Es2} = 9.0E9; \\
\text{nus12} = 0.3; \\
\text{nus21} = (\text{nus12} / \text{Es1}) * \text{Es2}; \\
\text{Gs12} = 7.1E9; \\
\text{p} = \text{Es1} / \text{Es2}; \\
\text{q} = \text{Gs12} / \text{Es2}; \\
\text{Ea} = 63.0E9; \\
\text{nua} = 0.3; \\
\text{ta} = 0.13E-3; \\
\text{ts} = 0.75E-3; \\
\text{tt} = 2 * \text{ta} + \text{ts}; \\
\text{dz1} = 254.0E-12; \\
\text{dz2} = \text{dz1}; \\
\text{nply} = 6; \\
\text{theta1} = \pi / 2 + 0.0;
\]
\[ \theta_2 = \pi/2 + \pi/4; \]
\[ \theta_3 = \pi/2 - \pi/4; \]
\[ \theta_4 = \theta_3; \]
\[ \theta_5 = \theta_2; \]
\[ \theta_6 = \theta_1; \]
\[ \theta = [\theta_1 \ \theta_2 \ \theta_3 \ \theta_4 \ \theta_5 \ \theta_6]; \]
\[ dT = 100.0 \times (5/9); \]
\[ \beta_{a1} = 0.9 \times 10^{-6}; \]
\[ \beta_{a2} = 0.9 \times 10^{-6}; \]
\[ \beta_{a} = [\beta_{a1} \ \beta_{a2} \ 0.0]; \]
\[ \beta_{s1} = 1.1 \times 10^{-6}; \]
\[ \beta_{s2} = 25.2 \times 10^{-6}; \]
\[ \beta_{s} = [\beta_{s1} \ \beta_{s2} \ 0.0]; \]
\[ \text{MATE} = [\text{nus21} \ \text{nua} \ \text{Es2} \ \text{Ea} \ \text{tt} \ \text{ts} \ \text{ta} \ \text{dz1} \ \text{dz2} \ \text{p} \ \text{q} \ dT \ \beta_{a} \ \beta_{s} \ \text{nply} \ \theta]; \]

function CONTi = CONTi(CONT, i)

% Picking out the parameters CONTi in the i-th section (denoted by i*1e50 at % both the beginning and end) in CONT

if i ~= 7
    addr = find(CONT == i*1e50);
    CONTi = CONT(addr(1)+1:addr(2)-1);
else
    addr = find(CONT == 7e50);
    CONTi = CONT(addr(1)+1:addr(2)-1);
end

function B = mtrx_Bel(J, xyel, xieta)

% \text{Chi} = B \times \text{q} 
% \text{Chi} --- \text{curvatures} 
% B ----- interpolation matrix 
% q ----- nodal variables or degrees of freedom 
% J ----- Jacobian transformation matrix 
% See reference: Jean-Louis Batoz & Mabrouk Ben Tahar, 
% Evaluation Of A New Quadrilateral Thin Plate Bending Bending Element,
\begin{verbatim}

j = inv(J);
Hp = shpfc_Hp(xyel, xieta);
B = [j(1,:) 0 0;
     0 0 j(2,:);
     j(2,:) j(1,:)]*Hp;

function [WTh, C] = mtrx_C(COOR, ID, ELEM, MATE, CONT)
% W = WTh + C*V
% From K*q = FTh + L*V, q can be solved as q = inv(K)*(FTh + L*V), where V is based
% on possible locations. With a selecting matrix S by which W = S*q, W can be
% written as W = S*inv(K)*(FTh + L*V). So WTh = S*inv(K)*FTh and C = S*inv(K)*L.
% Since S is a sparse matrix, it is implicitly included in the calculation
% of the above matrix products S*inv(K)*L for saving the computation time.
[nee, nnp] = size(ID);
npn = find(ID(1,:)) = 0;
[n, nel] = size(ELEM);
neq = max(max(ID));
nac = CONT(1, 4);
alp = CONT(1, 5);
[K, L, FTh] = mtrx_KgI(COOR, ID, ELEM, MATE, CONT);
SinvK = zeros(nnp, neq); % computing the product of S*inv(K)
invK = inv(K);
SinvK(npn, :) = invK(ID(1, npn), :);
WTh = SinvK*FTh; % computing the thermal deflection WTh
C = SinvK*L; % computing the product of S*inv(K)*L

function [Ds, Da, L, MTs, MTa] = mtrx_Del(MATE)
% Qs, Qa --- stiffness matrices for the substrate ply and actuator relatively
% Ds, Da --- elasticity matrices for the substrate and actuator relatively,
% integrated in the thickness
% p, q --- p = E11/E22, q = G12/E22
% T --- the strain transformation matrix from the material principle coordinates
\end{verbatim}
to the physical system coordinates
theta-----the orientation from x-1 axis to x axis, positive if counterclockwise
nply-----number of plies of substrate
L---------element actuation moment under unit area and unit input voltage
The following program is for even thickness plies.

nus21=MATE(1);
nua=MATE(2);
Es2=MATE(3);
Ea=MATE(4);
tt=MATE(5);
ts=MATE(6);
ta=MATE(7);
dz1=MATE(8);
dz2=MATE(9);
p=MATE(10);
q=MATE(11);
dT=MATE(12);
betaa=MATE(13:15)';
betas=MATE(16:18)';
nply=MATE(19);
theta(1:nply)=MATE(20:19+nply);

Qs=(Es2/(1-p*nus21^2))* [ p  p*nus21  0;
                        p*nus21  1  0;
                        0   0   q*(1-p*nus21^2)];

Qa=(Ea/(1-nua^2))* [ 1  nua  0;
                     nua  1  0;
                     0   0   (1-nua)/2];

% for determining Ds and MTs
if tt==ts
    j=ceil(ta/(tt/nply));
    ha=tt/2-ta;
    hj=tt/2+j*(tt/nply);
end
Ds=zeros(3,3);
Dsa=zeros(3,3);
MTs=zeros(3,1);
MTa=zeros(3,1);
MTsa=zeros(3,1);
for k=1:nply
    Tk=mtrx_T(theta(k));
    hk1=-ts/2+(k-1)*ts/nply;
    hk2=hk1+ts/nply;
    Ds=Ds+(1/3)*Tk*Qs*Tk'*((hk2^3-hk1^3);
    MTs=MTs+(1/3)*Tk*Qs*betas*(dT/ts)*((hk2^3-hk1^3);
    if tt==ts & k>=j+1 & k<=nply-j
        Dsa=Dsa+(1/3)*Tk*Qs*Tk'*((hk2^3-hk1^3);
        MTsa=MTsa+(1/3)*Tk*Qs*betas*(dT/tt)*((hk2^3-hk1^3);
    end
end
% for determining Da and MTa
if tt==ts
    Da=Ds+(2/3)*Qa*((ts/2+ta)^3-(ts/2)^3);
    MTa=(ts/tt)*MTs+(2/3)*Qa*betaa*(dT/tt)*((ts/2+ta)^3-(ts/2)^3);
else
    Tj=mtrx_T(theta(j));
    Tnj1=mtrx_T(theta(nply-j+1));
    Dsaj=(1/3)*((Tj*Qs*Tj'+Tnj1*Qs*Tnj1')*(ha^3+hj^3);
    Da=Dsa+Dsaj+(2/3)*Qa*(tt^3/8-ha^3);
    MTsa=(1/3)*((Tj+Tnj1)*Qs*betas*(dT/tt)*(ha^3+hj^3);
    MTa=MTsa+MTsaj+(2/3)*Qa*betaa*(dT/tt)*(tt^3/8-ha^3);
end
% for determining L
L=-Qa*[dz1;dz2;0]*(tt-ta);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%% function J=matrix_Jcb(xyel,xieta) %
% J---Jacobian matrix of the transformation between the parent and the
% actual element
% See reference: Jean-Louis Batoz & Mabrouk Ben Tahar,
% Evaluation Of A New Quadrilateral Thin Plate Bending Bending Element,
a=[-1 1 1 -1;
    -1 -1 1 1]
b=[1 -1 1 -1];
J=(1/4)*[a+flipud(xieta')]b*xyel;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [Kel, Lel, FTel] = mtrx_Kel(xy, D, L, MT)

% Kel*q = Frel + FThel + FVel
% Kel ---- element stiffness matrix
% Lel ---- actuation force vector under unit input voltage
% FTel ---- thermal force vector
% xy ---- element nodal coordinates
% GIP ---- the local coordinate Gauss integration points: (xi, eta).
% A standard numerical integration scheme using 2x2 Gauss integration
% points is used. See FEM 3rd. ed., p.198, O.C. Zienkiewicz.
% H = 1.000000000000000, a = (+)-0.577350269189626

a = 0.577350269189626;
GIP = [-1 -1; 1 -1; 1 1; -1 1] * a;
Kel = zeros(12, 12);
Lel = zeros(12, 1);

J1 = mtrx_Icb(xy, GIP(:,1));
B1 = mtrx_Bel(J1, xy, GIP(:,1));
J2 = mtrx_Icb(xy, GIP(:,2));
B2 = mtrx_Bel(J2, xy, GIP(:,2));
J3 = mtrx_Icb(xy, GIP(:,3));
B3 = mtrx_Bel(J3, xy, GIP(:,3));
J4 = mtrx_Icb(xy, GIP(:,4));
B4 = mtrx_Bel(J4, xy, GIP(:,4));

Kel = B1' * D * B1 * det(J1) + B2' * D * B2 * det(J2) + B3' * D * B3 * det(J3) + B4' * D * B4 * det(J4);
BtdetJ = B1' * det(J1) + B2' * det(J2) + B3' * det(J3) + B4' * det(J4);
Lel = BtdetJ * L;
FTel = BtdetJ * MT;

function [Kgl, Lgl, FTgl] = mtrx_Kgl(COOR, ID, ELEM, MATE, CONT)

% Kgl*q = FThgl + FVgl = WTr + WTh + C*V
% Kgl ---- global stiffness matrix
% Lgl ---- actuation force matrix, Lgl*V = FVgl
% FTgl ---- thermal force vector
% FVgl ---- actuation force vector
% neq----the size of the global stiffness matrix Kg

IEN=ELEM(1:4,:);
LM=ELEM(5:16,:);
LMAC=ELEM(17,:);
[nee,nel]=size(LM);
neq=max(max(ID));
nac=CONTi(CONT,4);
alpha=CONTi(CONT,5);
lac=find(alpha==0);

[Ds,Da,L,MTs,MTa]=mtrx_Del(MATE);
UKgl=zeros(neq,neq);  %computing matrices Kg and Lg, and vector FTg
Lg=zeros(neq,nac);
FTg=zeros(neq,1);

for j=1:nel
  addr=find(LMAC==lac);
  acn=sum(alpha(1:LMAC));
  if addr==[]
    [Kel,Lel,FTel]=mtrx_Kel(COOR(IEN(:,j),:),Da,L,MTa);
  else
    [Kel,Lel,FTel]=mtrx_Kel(COOR(IEN(:,j),:),Ds,[0;0;0],MTs);
  end
  [DOF,dof]=sort(LM(:,j));  %assembling Kg, Lg, and FTg
  for i=1:12
    if DOF(i)==0
      for k=i:12
        if DOF(k)==0 & Kel(dof(i),dof(k))==0
          UKgl(DOF(i),DOF(k))=UKgl(DOF(i),DOF(k))+Kel(dof(i),dof(k));
        end
      end
    end
    if addr==[]
      Lg(DOF(i),acn)=Lg(DOF(i),acn)+Lel(dof(i));
    end
    FTg(DOF(i))=FTg(DOF(i))+FTel(dof(i));
  end
end

Kg=UKgl+UKgl'-diag(diag(UKgl));
function \[K,R\]=mtrx_KR(CONT)

% \(K \times V - R \leq 0\) --- constrains of the voltages
% \(K\) --- sign matrix of \(V\)
% \(R\) --- vectors consist of lower and upper bonds of the actuator voltages

nac=CONTi(CONT,4);
Vbond=CONTi(CONT,7);
Vup=Vbond(1:length(Vbond)/2);
Vlow=Vbond((length(Vbond)/2)+1:length(Vbond));
addr1=find(Vup==0);
addr2=find(Vlow==0);
K=zeros(2*nac,nac);
K(2*addr1-1,addr1)=eye(length(addr1));
K(2*addr2,addr2)=-eye(length(addr2));
R=kron(Vup,[1;0])+kron(Vlow,[0;-1]);

function T=mtrx_T(theta);

% Transformation matrix: from principle material coordinates to physical
% system coordinates
% \(\theta\)---the orientation from \(x_1\) axis to \(x\) axis, positive if counter-clockwise

m=cos(theta);
n=sin(theta);

T=[m^2  n^2  -2*m*n;
   n^2  m^2  2*m*n;
   m*n -m*n m^2-n^2];

function [L,V,W,F]=sl_W(COOR,ID,ELEM,MATE,CONT)

% \(W = W_{Th} + C \times V\)
nac=CONTi(CONT,4);
L=CONTi(CONT,5);
V=CONTi(CONT,6);
Wd=CONTi(CONT,8);
W0=CONTi(CONT,9);
IEN=ELEM(1:4,:);
LM=ELEM(5:16,:);
[ndof,nnp]=size(ID);
[nee,nel]=size(LM);
neq=max(max(ID));
[WTh,C]=mtrx_C(COOR,ID,ELEM,MATE,CONT);
W=WTh+C*V;
Q=eye(nnp);
F=(WO-W)'*Q*(WO-W);

function [L,V,W,F]=s2_lagr(COOR,ID,ELEM,MATE,CONT)

% 2*C'*Q*C*V+K'*mu=2*C'*Q*(Wd-WTh)--lagrange optimization system
% [a11 a12][V]=[b1] a11--2*C'*Q*C a12--K' b1--2*C'*Q*Wd
% [a21 a22][mu] [b2] a21---0 or Ki a22---0 or 1 b2---0 or Ri
% V---optimal voltages
% mu---Lagrange multipliers
% Wd---desired deflection
% WTh---thermal deflection
% Algorithm: Solve the linear system through relaxing several constrains
% (i.e., let the relevant mus be zeros) in each trial set of mu.
% Search for the solution by checking whether it satisfies the
% Kuhn-Tucker optimality conditions:
% mu>=0 and mui*(K*V-R)i=0,
% starting from setting all mus zero. If some mus are set non zero, then the
% relevant constrains, K*V-R=0, are put into the linear system forming a new
% expended system with those mus which are zeros. Sets making activate both
% up and low bonds of one actuator have to be excluded. Because the search
% space is strictly convex, the solution is unique.

nac=CONTi(CONT,4);
L=CONTi(CONT,5);
Wd=CONTi(CONT,8);
[ndof,nnp]=size(ID);
Q=eye(nnp);
[WTh,C]=mtrx_C(COOR,ID,ELEM,MATE,CONT);
[K,R]=mtrx_KR(CONT);
[Y,I]=find(K==0);
nmu=length(Y);

a11=2*C'*Q*C;
a12=K';
b1=2*C'*Q*(Wd-WTh);

F=1e50;
mu=zeros(2*nac,1);

for i=0:2^nmu-1
    a21=zeros(2*nac,nac);
    a22=zeros(2*nac,2*nac);
    b2=zeros(2*nac,1);
    m=i;
    for j=1:2*nac
        %create a binary mu searching for all combination
        if find(2*nac-j+1==Y)==[]
            mu(2*nac-j+1)=rem(m,2);
            m=fix(m/2);
        end
    end
    if mu(1:2:2*nac) .*mu(2:2:2*nac) == 0  %exclude taking both bonds together
        if sum(mu)==0
            V=a11 \ b1;
            mu=zeros(2*nac,1);
        else
            addr1=find(mu==0);
            addr2=find((mu-1)==0);
            a21(addr1,:)=K(addr1,:);
            a22(addr2,addr2)=eye(length(addr2));
            b2(addr1,1)=R(addr1,1);
            Vmu=[a11 a12;a21 a22][b1;b2];
            V=Vmu(1:nac,1);
            mu=Vmu(nac+1:3*nac,1);
        end
    end
    if mu>=0 & K*V-R<=0  %check Kuhn-Tucker optimality conditions
        W=WTh+C*V;
        F=(Wd-W)*Q*(Wd-W);
        mu=i
        break
function [L,V,W,F]=s3_srch(COOR,ID,ELEM,MATE,CONT)

% Enumerating all the space of alpha, comparing the cost F
% L------optimal layout
% V------optimal voltages
% W------deflection
% F------cost function
% Q------positive definite matrix
% Wd-----desired deflection

npl=CONT(3);
nac=CONT(4);
acfix=CONT(5);
a5=find(CONT==5e50);

addrfix=find(acfix==0);
addrplc=find(acfix==0);
nacfix=length(addrfix);

n1=2^(nac-nacfix)-1;
n2=2^npl-2^(npl-(nac-nacfix));

F=1e50;
alpha=acfix;
p=1;

for i=n1:n2
    m=i;
    for j=1:npl
        alpha1(npl-j+1)=rem(m,2);
        m=fix(m/2);
    end
    if sum(alpha1)==nac-nacfix
        binary=i-n1+1
        alpha(addrplc)=alpha1';
        % start searching for alpha
        % create an alpha vector, npl---number of bits
        % m---the decimal number to be changed
symmdeci(p) = sum(2.^(flipud(find(alpha==0))-1));
deci = sum(2.^(length(alpha)-find(alpha==0))); if find(deci==symmdeci)==[]
    CONT(a5(1)+1:a5(2)-1)=alpha; %change alpha vector in CONT
    [Li,Vi, Wi, Fi]=s2_lagr(COOR, ID, ELEM, MATE, CONT); %Lagrange multiplier
    if Fi-F<=0 %compare cost and pick out the better one
        L=Li;
        V=Vi;
        W=Wi;
        F=Fi;
    end
end
p=p+1;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function Hp=shpfc_Hp(xyel, xieta)

% For the DKQ plate bending element, see reference:
% Jean-Louis Batoz & Mabrouk Ben Tahar,
% Evaluation Of A New Quadrilateral Thin Plate Bending Bending Element,
% Note: The signs in the paper is incorrect.

shift=[xyel(2:4,:);xyel(1,:)];
X=xyel(:,1)-shift(:,1);
Y=xyel(:,2)-shift(:,2);
LL=X.^2+Y.^2;
a=zeros(1,8);
b=zeros(1,8);
c=[ones(1,4) zeros(1,4)];
d=zeros(1,8);
e=[ones(1,4) zeros(1,4)];
a(5:8)=-(3/2)*X./LL;
b(5:8)=(3/4)*X.*Y./LL;
c(5:8)=(1/4)*X.^2-(1/2)*Y.^2)./LL;
d(5:8)=-(3/2)*Y./LL;
e(5:8)=-(1/2)*X.^2+(1/4)*Y.^2)./LL;
\[ r_1 = [0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ -1]; \]
\[ r_2 = [0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1]; \]
\[ r_3 = [1 \ 0 \ 0 \ 0 \ -1 \ 0 \ 0 \ -1]; \]
\[ r_4 = [0 \ 0 \ 0 \ 0 \ -1 \ 1 \ 0 \ 0]; \]
\[ r_5 = [0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0]; \]
\[ r_6 = [0 \ 1 \ 0 \ 0 \ -1 \ 1 \ 0 \ 0]; \]
\[ r_7 = [0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0]; \]
\[ r_8 = [0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0]; \]
\[ r_9 = [0 \ 0 \ 1 \ 0 \ 0 \ -1 \ 1 \ 0 \ 0]; \]
\[ r_{10} = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ -1 \ 1]; \]
\[ r_{11} = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1]; \]
\[ r_{12} = [0 \ 0 \ 0 \ 1 \ 0 \ 0 \ -1 \ 1 \ 1]; \]

\[ A_x = [r_1.*a; \ r_2.*b; \ r_3.*c; \ r_4.*a; \ r_5.*b; \ r_6.*c; \ r_7.*a; \ r_8.*b; \ r_9.*c; \ r_{10}.*a; \ r_{11}.*b; \ r_{12}.*c]; \]

\[ A_y = [r_1.*d; \ -r_3.*e; \ -r_2.*b; \ r_4.*d; \ -r_6.*e; \ -r_5.*b; \ r_7.*d; \ -r_9.*e; \ -r_8.*b; \ r_{10}.*d; \ -r_{12}.*e; \ -r_{11}.*b]; \]

\[ N_p = \text{shpfc}_N_p(xi, eta); \]
\[ H_p = [N_p*A_x'; \ N_p*A_y']; \]

---

function \( N_p = \text{shpfc}_N_p(xi, eta) \)

% The shape functions \( N_j \) are given in the reference:
% The derivatives of \( N_j \) are as follows
% for corner nodes:
% \( N_j, xi = \frac{1}{4}(x_{ij})(1 + (eta)(eta)(eta)[1+(eta)(eta)]] \)
% \( N_j, eta = \frac{1}{4}(eta)(1+xi(xi)[1+(xi)(xi)][x_{ij}+2(eta)(eta)]) \)
% for mid-side nodes:
% \( x_{ij} = 0, N_j, xi = \frac{1}{2}(x_i)(1+(eta)(eta)) \)
% \( N_j, eta = \frac{1}{2}(eta)(1-(xi)(xi)) \)
% \( eta = 0, N_j, xi = \frac{1}{2}(x_i)(1-(eta)(eta)) \)
% \( N_j, eta = \frac{1}{2}(xi)(xi)(eta) \)
% Element nodal coordinates: P(xi, eta)

\[ P = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \]
\[ xi = P(:,1); \]
\[ eta = P(:,2); \]
\[ xio = xieta(1) \times xi; \]
\[ etao = xieta(2) \times eta; \]
\[ Npxi(1:4) = (1/4) \times xi(1:4) \times (2 \times xio(1:4) + etao(1:4)) \times (1 + etao(1:4)); \]
\[ Npxi(5:2:7) = -xieta(1) \times (1 + etao(5:2:7)); \]
\[ Npxi(6:2:8) = (1/2) \times xi(6:2:8) \times (1 - xieta(2)^2); \]
\[ Npeta(1:4) = (1/4) \times eta(1:4) \times (2 \times etao(1:4) + xio(1:4)) \times (1 + xio(1:4)); \]
\[ Npeta(5:2:7) = (1/2) \times eta(5:2:7) \times (1 - xieta(1)^2); \]
\[ Npeta(6:2:8) = -(1 + xio(6:2:8)) \times xieta(2); \]
\[ Np = [Npxi; Npeta]; \]

function W_graph(COOR, ID, ELEM, MATE, CONT, L, V, W, F)

% Plot the deflection of the plate for different examples and save in a file.

load g_data;

COOR = in_COOR;
[ID, ELEM] = in_ELEM;
MATE = in_MATE;
CONT = in_CONT;

IEN = ELEM(1:4,:);
LM = ELEM(5:16,:);
LMAC = ELEM(17,:);
Wd = CONTi(CONT,8);
W0 = CONTi(CONT,9);
nnp = length(W);
a = max(max(COOR(:,1))) \times 1E3;
b = max(max(COOR(:,2))) \times 1E3;
c1 = min(W) \times 1E3;
c0 = W((nnp+1)/2) \times 1E3;
c2 = max(W) \times 1E3;
ax = [0 b];
by = [0 a];
\begin{verbatim}
cz=[0 5];
[ned nel]=size(LM);
adtra=find(L==0);
Wmax=max(W)*1E3
Wmin=min(W)*1E3
Wctr=W((nnp+1)/2)*1E3
addrax=find(Wmax==W*1E3);
addrmin=find(Wmin==W*1E3);
Xmax=COOR(addrax,1)*1E3
Ymax=COOR(addrax,2)*1E3
Xmin=COOR(addrmin,1)*1E3
Ymin=COOR(addrmin,2)*1E3
Xctr=COOR((nnp+1)/2,1)*1E3
Yctr=COOR((nnp+1)/2,2)*1E3;

c1g;
orient tall;
colormap('gray');
subplot(211);
set(gca,'fontsize',8,'fontangle','oblique','xtick','ax','ytick','by','ztick',cz,...
    'defaulttextfontsize',8,'defaulttextfontangle','oblique',...
    'position',[0.15 0.55 0.70 0.37])
text(30,200,20,...
    'Fig. 4.7 Simulation of the Plate Deformation Under the Optimal Layout and',...
    'fontsize',10);
text(30,200,18.5,...
    'Input Voltages of 9 Sets of Actuators with Constraints Vi<=120 (V)',...
    'fontsize',10);
text(30,200,17,...
    '(T300/976, [0/±45°]s, four corners simply supported)',...
    'fontsize',10);
text(250,0,15,'Optimal Input Voltages (V)');
text(230,0,12,'120.0 ----- -14.7 ----- 104.5');
text(230,0,10.5,'120.0 ----- 120.0 120.0 120.0');
text(230,0,9,'120.0 ----- ----- ----- 91.8');
text([50;50],[220;220],[2;3],['+50°F';'-50°F']);
text(372+5,115,-1,'x (mm)');
text(186.228+22,-1,'y (mm)');
text(-10,228.15,-1,'z (mm)');
text(Ymax,Xmax,Wmax+0.5,'Wmax=+0.648');
text(Ymin,Xmin,abs(Wmin)+0.5,'Wmin=-0.728');
\end{verbatim}
text(Yctr,Xctr,abs(Wctr)+0.5,'Wctr=-0.724');
hold on;
for i== 1:nel
    lnd=IEN(:,i);
    X=COOR(lnd,1)*1E3;
    x=[X;X(1)];
    Y=COOR(lnd,2)*1E3;
    y=[Y;Y(1)];
    Z=W(lnd)*1E3;
    z=[Z;Z(1)];
    Zd=Wd(lnd)*1E3;
    zd=[Zd;Zd(1)];
    Z0=W0(lnd)*1E3;
    z0=[Z0;Z0(1)];
    if LMAC(i)-addra==0
        %line(y,x,z);
        fill3(y,x,z,[0.9 0.9 0.9]);
        %line(y,x,zd);fill3(y,x,zd,'y');
        %line(y,x,z0);fill3(y,x,z0,'c');
        line(y,x);
    else
        %line(y,x,z);
        fill3(y,x,z,[0.8 0.8 0.8]);
        %line(y,x,zd);fill3(y,x,zd,'b');
        %line(y,x,z0);fill3(y,x,z0,'r');
        line(y,x);
    end
end
hold off;
axis('ij');
axis([0 b (-b+a) a 0 15]);
view(90-37.5,50);
%print -dmeta fig50.wmf

load g_data;

COOR=in_COOR;
[ID,ELEM]=in_ELEM;
MATE=in_MATE;
CONT=in_CONT;
IEN=ELEM(1:4,:);
LM=ELEM(5:16,:);
LMAC=ELEM(17,:);
Wd=CONTi(CONT,8);
W0=CONTi(CONT,9);
nnp=length(W);
a=max(max(COOR(:,1)))*1E3;
b=max(max(COOR(:,2)))*1E3;
c1=min(W)*1E3;
c0=W((nnp+1)/2)*1E3;
c2=max(W)*1E3;
ax=[0 b];
by=[0 a];
cz=[0 5];
[ned nel]=size(LM);
addra=find(L==0);
Wmax=max(W)*1E3
Wmin=min(W)*1E3
Wctr=W((nnp+1)/2)*1E3
addrmax=find(Wmax==W*1E3);
addrmin=find(Wmin==W*1E3);
Xmax=COOR(addrmax,1)*1E3;
Ymax=COOR(addrmax,2)*1E3;
Xmin=COOR(addrmin,1)*1E3;
Ymin=COOR(addrmin,2)*1E3;
Xctr=COOR((nnp+1)/2,1)*1E3;
Yctr=COOR((nnp+1)/2,2)*1E3;

%clg;
subplot(212);
set(gca,'fontsize',8,'fontangle','oblique','xtick',ax,'ytick',by,'ztick',cz,...
'defaulttextfontsize',8,'defaulttextfontangle','oblique',...
'position',[0.15 0.15 0.7 0.37]);

text(30,200,20,...
   'Fig. 4.7 Simulation of the Plate Deformation Under the Optimal Layout and',...
   'fontsize',10);

text(30,200,18.5,...
   'Input Voltages of 9 Sets of Actuators with Constraints Vi<=120 (V)',...
   'fontsize',10);

text(30,200,17,...
   'Input Voltages of 9 Sets of Actuators with Constraints Vi<=120 (V)',...
(T300/976, [0/±45°]s, four corners simply supported),

'fontsize',10); text(250,0,15,'Optimal Input Voltages (V)');
text(230,0,12, '120.0 ----- -14.7 ----- 104.5');
text(230,0,10.5,'120.0 ----- 120.0 120.0 120.0');
text(230,0,9,'120.0 ----- ----- 91.8');
text([50;50],[220;220],[2:-3],['+50°F;' ['-50°F']);
text(372+5,115,-1,'x (mm)');
text(186,228+22,-1,'y (mm)');
text(-10,228+15,'z (mm)');
text(Ymax,Xmax,Wmax+0.5,'Wmax=+0.648');
text(Ymin,Xmin,abs(Wmin)+0.5,'Wmin=-0.728');
text(Yctr,Xctr,abs(Wctr)+0.5,'Wctr=-0.724');

hold on;
for i=1:nel
    Ind=IEN(:,i);
    X=COOR(Ind,1)*1E3;
    x=[X;X(I)];
    Y=COOR(Ind,2)*1E3;
    y=[Y;Y(I)];
    Z=W(Ind)*1E3;
    z=[Z;Z(1)];
    Zd=Wd(Ind)*1E3;
    zd=[Zd;Zd(I)];
    Z0=W0(Ind)*1E3;
    z0=[Z0;Z0(1)];
    if LMAC(i)-addra--=0
        line(y,x,z);
        fill3(y,x,z,[0.9 0.9 0.9]);
        line(y,x,zd);fill3(y,x,zd,'y');
        line(y,x,z0);fill3(y,x,z0,'c');
        line(y,x);
    else
        line(y,x,z);
        fill3(y,x,z,[0.8 0.8 0.8]);
        line(y,x,zd);fill3(y,x,zd,'b');
        line(y,x,z0);fill3(y,x,z0,'r');
        line(y,x);
    end
end
hold off;
axis([0 b (-b+a) a 0 15]);
axis('ij');
view(90-37.5,50);
%print -dmeta fig50.wmf

function W1_graph
% For Fig.(3.5): Crawley's plate, [0+45-45]s

load g_data;

COOR=in_COR32;
a1=[1 1+[1:16]*11];
a2=a1+5;
a3=a2+5;
W12D=(W(a2)*1E3/152.0);  % for both g_data71 and g_data72
xL=COOR(a2,2)*1E3/292.0;
%W22D=(W(a3)-W(a1))*1E3/152.0;
%W32D=-(W(a2)-(W(a3)+ W(a1))/2)* 1E3/152.0;
d1=[0.1280 0.2134 0.3045 0.3863 0.4729 0.5752];
d2=[0.6503 0.7389 0.8245 0.9210 1.0000];
xLEXP=[d1 d2]';

% for g_data71
b711=[0.000 0.000 0.069 0.897 1.379 3.138 5.276 6.207 9.103]*1E-3;
b712=[12.69 13.72 18.00 23.10 24.17 29.72 36.21 37.66]*1E-3;
W13D=[b711 b712];
c711=[0.862 1.931 3.879 5.931 8.000 12.05]*1E-3;
c712=[16.86 21.03 26.81 33.86 43.00]*1E-3;
W1EXP=[c711 c712]';

% for g_data72
b721=[0.000 0.000 0.001 1.448 1.779 4.221 7.490 8.483 12.41]*1E-3;
b722=[17.50 18.74 24.79 31.70 33.35 40.68 48.91 51.06]*1E-3;
%W13D=[b721 b722];
c721=[0.0009 0.0019 0.0040 0.0076 0.0121 0.0168];
c722=[0.0224 0.0285 0.0369 0.0428 0.0543];
%W1EXP=[c721 c722]';
\%b21 = \[0.000 0.034 6.034 12.93 14.662 6.034 35.17 38.79 49.31\] \* 1E-4;
\%b22 = \[61.21 63.45 75.17 88.79 90.52 103.5 114.7 117.93\] \* 1E-4;
\%W2D = \[b21 b22\];
\%c21 = \[25.52 34.31 42.59 79.48 89.14 109.8\] \* 1E-4;
\%c22 = \[120.1 130.1 140.5 161.5 180.3\] \* 1E-4;
\%W2EXP = \[c21 c22\];
\%b31 = \[0.000 0.066 8.110 15.48 16.88 22.14 26.65 27.89 30.29\] \* 1E-4;
\%b32 = \[32.65 33.35 34.12 34.76 34.97 34.88 34.76 34.59\] \* 1E-4;
\%W3D = \[b31 b32\];
\%c31 = \[8.896 17.75 20.15 20.15 20.15 29.30\] \* 1E-4;
\%c32 = \[31.78 37.24 44.44 51.89 19.24\] \* 1E-4;
\%W3EXP = \[c31 c32\];

clc;
hold on;
plot(xLEXP, W1EXP, 'o');
plot(xL, W1D, '-');
plot(xL, W13D, '--');
plot(xLEXP, W2EXP, '*');
plot(xL, W22D, '-');
plot(xL, W23D, '--');
plot(xLEXP, W3EXP, 'x');
plot(xL, W32D, '-');
plot(xL, W33D, '--');
xlabel('Distance, x/L');
ylabel('Displacement, W1');
	ext{(0.1,0.047, ' Applied Field = 394 V/mm');
	ext{(0.1,0.043, ' Experiments, Ref. 13');
	ext{(0.1,0.037, '--- 3-D Model, Ref. 26');
	ext{(0.1,0.033, ' 2-D Model');

% text(0.1,0.057, ' Applied Field = 472 V/mm');
% text(0.1,0.053, ' Experiments, Ref. 13');
% text(0.1,0.047, '--- 3-D Model, Ref. 26');
% text(0.1,0.043, ' 2-D Model');
grd;
function W2_graph

load g_data;

COOR=in_COOR;
al=[1 1+1:16]*11;
a2=a1+5;
a3=a2+5;
    % for both g_data71 and g_data72
W22D=(W(a3)- W(a1 ))* 1E3/152.0;
xL=COOR(a2,2)* 1E3/292.0;
d1=[0.1280 0.2134 0.3045 0.3863 0.4729 0.5752];
d2=[0.6503 0.7389 0.8245 0.9210 1.0000];
xLEXP=[d1 d2];
    % for g_data71
b711=[0.000 0.000 12.41 24.83 29.97 39.42 48.35 62.27 70.35 79.03 96.75 105.7];
b712=[13.91 23.74 41.51 69.22 98.14 126.7];
W23D=[b711 b712];
c711=[1.903 0.000 39.73 79.24 171.3 262.8];
c712=[263.2 434.7 485.4 537.7 544.6];
W2EXP=[c711 c712];
    % for g_data72
b721=[0.000 0.003 6.034 12.93 14.66 24.83 35.17 38.79 49.31];
b722=[61.21 63.45 75.17 88.79 90.52 103.5 114.7 126.8];
W23D=[b721 b722];
c721=[25.52 34.31 42.59 79.48 89.14 109.8];
c722=[120.1 130.1 140.5 161.5 180.3];
W2EXP=[c721 c722];

clg;
hold on;
plot(xLEXP,W2EXP,'o');
plot(xL,W22D,'-');
plot(xL, W23D,'--');
xlabel('Distance, x/L');
ylabel('Displacement, W2');
    % for g_data71
text(0.12,0.0057,' Applied Field == 394 V/mm');
text(0.12,0.0053,' o Experiments, Ref. 13');
function W3_graph
load g_data;
COOR=in_COOR;
a1=[1 1+[1:16]*11];
a2=a1+5;
a3=a2+5;
% for both g_data71 and g_data72
W32D=-(W(a2)-(W(a3)+W(a1))/2)*1E3/152.0;
xL=COOR(a2,2)*1E3/292.0;
d1=[0.1280 0.2134 0.3045 0.3863 0.4729 0.5752];
d2=[0.6503 0.7389 0.8245 0.9210 1.0000];
xLEXP=[d1 d2];
% for g_data71
b711=[0.000 8.234 66.21 126.2 140.7 184.6 221.4 234.6 260.7]*1E-5;
b712=[286.3 290.9 306.2 315.7 319.9 323.2 327.3 327.3]*1E-5;
W33D=[b711 b712];
c711=[71.79 120.2 229.7 226.3 261.3 282.4]*1E-5;
c712=[299.6 386.9 391.0 482.5 331.0]*1E-5;
W3EXP=[c711 c712];
% for g_data72
b721=[0.000 0.066 8.110 15.48 16.88 22.14 26.65 27.89 30.29]*1E-4;
b722=[32.65 33.35 34.12 34.76 34.97 34.88 34.76 34.59]*1E-4;
%W33D=[b721 b722];
c721=[8.896 17.75 20.15 20.15 20.15 29.30]*1E-4;
c722=[31.78 37.24 44.44 51.89 19.24]*1E-4;
%W3EXP=[c721 c722];
clg;
hold on;
plot(xLEXP,W3EXP,'o');
plot(xL,W32D,'-');
plot(xL,W33D,'--');
xlabel('Distance, x/L');
ylabel('Displacement, W3');

% for g_data71
text(0.1,0.0047,' Applied Field = 394 V/mm');
text(0.1,0.0043,' o Experiments, Ref. 13');
text(0.1,0.0037,'--- 3-D Model, Ref. 26');
text(0.1,0.0033,'_ 2-D Model');

% for g_data72
%text(0.1,0.0057,' Applied Field = 472 V/mm');
%text(0.1,0.0053,' o Experiments, Ref. 13');
%text(0.1,0.0047,'--- 3-D Model, Ref. 26');
%text(0.1,0.0043,'_ 2-D Model');
grid;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
ABSTRACT

Future technologies in microsensing, microactuation, active changes in shapes of aerofoils, turbine blades justify, and shape and vibration control of some light-weight and flexible large space structures necessitate a comprehensive study of piezo actuators for shape control of structures. Piezo materials have the property to develop strain on applying voltage and voltage on applying strain, hence, these materials are useful both as sensors and actuators. If a piezo actuator is bonded to a substrate plate, on applying voltages, the resulting strain in the actuator induces a deflection in the plate. This deflection is a function of geometry, material properties of actuators and substrate, number of actuators chosen and their layout, and voltage inputs to the actuators. The objectives of this paper are (a) to develop a simpler 2-D (relative to 3-D models and refer to independent variables x-y-z) mathematical model for deflection of a multi-directional laminated composite thin plate surface embedded or bonded with piezoceramic actuators using Kirchhoff’s elastic thin plate theory, (b) to compute plate deflection by solving the mathematical model using finite element method (FEM) with correspondingly adopting the discrete Kirchhoff quadrilateral (DKQ) plate bending element, (c) to develop a mathematical programming statement for the problems of finding optimal actuation voltages to actuators and optimal layout of those actuators and even optimal number of those actuators to match the deflection of the plate to a desired shape and (d) to construct