ANALYSIS OF A PLANAR SPIRAL DISPLACER SPRING
FOR USE IN FREE-PISTON STIRLING ENGINES

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>List of Figures</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chapter 1: Rationale</td>
<td></td>
</tr>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>Alternate Spring Designs</td>
<td>1</td>
</tr>
<tr>
<td>Planar Spiral Springs</td>
<td>2</td>
</tr>
<tr>
<td>Purpose of Study</td>
<td>2</td>
</tr>
<tr>
<td>Closure</td>
<td>5</td>
</tr>
<tr>
<td>Chapter 2: Review of Literature</td>
<td>7</td>
</tr>
<tr>
<td>Definition and Purpose of Mechanical Springs</td>
<td>7</td>
</tr>
<tr>
<td>Spring Design</td>
<td>7</td>
</tr>
<tr>
<td>Force-Deflection Curves and Spring Rate</td>
<td>8</td>
</tr>
<tr>
<td>Stress and Fatigue</td>
<td>9</td>
</tr>
<tr>
<td>Energy Storage</td>
<td>15</td>
</tr>
<tr>
<td>Planar Spiral Springs</td>
<td>15</td>
</tr>
<tr>
<td>Comparisons with Conical Springs</td>
<td>16</td>
</tr>
<tr>
<td>Background of Spring Design Theory</td>
<td>21</td>
</tr>
<tr>
<td>Problems in Using Conical Spring Design Theory with Planar Spiral Springs</td>
<td>23</td>
</tr>
<tr>
<td>Hypotheses</td>
<td>25</td>
</tr>
<tr>
<td>Closure</td>
<td>25</td>
</tr>
<tr>
<td>Chapter 3: Methods</td>
<td>27</td>
</tr>
<tr>
<td>Selection of Analysis Method</td>
<td>27</td>
</tr>
<tr>
<td>Equipment and Software</td>
<td>28</td>
</tr>
<tr>
<td>Finite Element Model</td>
<td>28</td>
</tr>
<tr>
<td>Analysis Approach</td>
<td>33</td>
</tr>
</tbody>
</table>
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 a</td>
<td>Face of a Planar Spiral Spring</td>
<td>3</td>
</tr>
<tr>
<td>1 b</td>
<td>Rotated View of a Planar Spiral Spring</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>(\sigma)-N Diagram</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>Goodman Diagram</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>Modified Goodman Diagram</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>Deformed Planar Spiral Spring</td>
<td>17</td>
</tr>
<tr>
<td>6</td>
<td>Conical Spring</td>
<td>18</td>
</tr>
<tr>
<td>7 a</td>
<td>Conical Spring Section View</td>
<td>19</td>
</tr>
<tr>
<td>7 b</td>
<td>Planar Spiral Spring Section View</td>
<td>19</td>
</tr>
<tr>
<td>8</td>
<td>Force-Deflection Curve for Conical Springs</td>
<td>20</td>
</tr>
<tr>
<td>9</td>
<td>Planar Spiral Spring Testing Rig</td>
<td>29</td>
</tr>
<tr>
<td>10</td>
<td>Transition Mesh</td>
<td>32</td>
</tr>
<tr>
<td>11</td>
<td>Deformed Transition Mesh</td>
<td>32</td>
</tr>
<tr>
<td>12 to 16</td>
<td>Force-Deflection Curves for Laboratory Results</td>
<td>39-41</td>
</tr>
<tr>
<td>17 a</td>
<td>Finite Element Mesh</td>
<td>42</td>
</tr>
<tr>
<td>17 b</td>
<td>Finite Element Mesh Closeup</td>
<td>42</td>
</tr>
<tr>
<td>18 a</td>
<td>Finite Element Mesh</td>
<td>44</td>
</tr>
<tr>
<td>18 b</td>
<td>Finite Element Mesh Closeup</td>
<td>44</td>
</tr>
<tr>
<td>19 to 23</td>
<td>Force-Deflection Curves</td>
<td>45-47</td>
</tr>
<tr>
<td>24 to 26</td>
<td>Convergence Studies for Deflection</td>
<td>48-49</td>
</tr>
<tr>
<td>27 to 30</td>
<td>Force-Deflection Curves</td>
<td>50-51</td>
</tr>
<tr>
<td>31</td>
<td>Deflection vs. Thickness</td>
<td>52</td>
</tr>
<tr>
<td>32</td>
<td>Deflection vs. Width of Material Cut</td>
<td>54</td>
</tr>
</tbody>
</table>
33 to 35 Convergence Studies for Maximum Von Mises Stress

36 to 44 Force vs. Maximum Von Mises Stress

45 Maximum Von Mises Stress vs. Thickness

46 Maximum Von Mises Stress vs. Width of Material Cut

47 Von Mises Stress Contours

48 to 51 Position of Primary and Secondary Stress Peaks

52 to 63 Von Mises Stress Contours

64 Planar Spiral Spring Testing Rig

Table

Compendium of Finite Element Results
CHAPTER 1: Rationale

Introduction

Certain designs of Stirling engines incorporate a mechanical spring to help provide the restoring force necessary to maintain cyclic operation of the reciprocating elements (26). The spring may act in combination with a gas spring to tune the displacer so that it will resonate at a frequency near the engine's design operating frequency (4).

In many instances the displacer spring is a helical type of spring attached to the displacer rod. However, there are applications of Stirling technology where a premium is put on compactness and a helical spring will not work. This study is an analysis of a spring design intended for use in Stirling engines where helical types of displacer springs cannot be used.

Alternate Spring Designs

An example of a Stirling engine where a helical displacer spring is replaced by an alternate spring design is the Thermo-mechanical Generator (TMG) described by Cooke-Yarborough (3). The TMG is a special type of free-piston Stirling engine where the piston is actually a vibrating diaphragm (25). Instead of a helical spring, Cooke-Yarborough described a displacer spring for a TMG made from a flat disk of stainless steel which consisted of four circular arcs (3, 4, 15). Further development lead to a design consisting of a single ring with four lugs as attachment points (4). Though these designs were successful in TMG's, there are problems in using either of them in other Stirling applications. First, the four arc springs are difficult to
manufacture (4) and secondly, the spring with lugs has a length of stroke that is somewhat limited.

**Planar Spiral Springs**

There are applications of other types of free-piston Stirling engines that also impose space limitations so strict that a helical spring cannot be used on the displacer. However, these applications call not only for a compact spring but one that has a stroke length or operating length greater than that of a TMG. Figure 1 shows the face and profile of a design idea intended to meet these criteria.

The coils of the spring are the result of removing two spirals of material from a round flat disk of spring steel. In this particular design example, spirals the shape of involutes have been removed by cutting or stamping. The two resulting coils are also in the shape of involutes. A spring of this type could have more than two coils (resulting from an equal number of material cuts) and the coils could be in a shape other than involutes. The scope of this study, however, is limited to the basic design of Figure 1.

As it appears that a common name for such a spring does not exist, the author has chosen to use the descriptive term of planar spiral spring for this study.

**Purpose of Study**

Recognizing that any design must be analyzed to determine whether performance will comply with specification (22), the goal of this study is to determine the relationships between the geometric parameters of a planar spiral spring and the resulting performance characteristics. With regard to Stirling engine applications there are three design criteria for the planar spiral spring: the spring must
Figure 1a: Face of a Planar Spiral Spring

Figure 1b: Rotated View of a Planar Spiral Spring
have a predictable, preferably constant, spring rate, the spring must have infinite fatigue life and the spring must be easily manufacturable.

The spring rate is important in Stirling engine applications because of its effect on the resonant frequency of the system (26). The natural frequency of a simple spring mass system is a function of the spring rate (26). As springs are essential for providing the restoring force necessary to maintain cyclic operation in free-piston Stirling engines (26), the spring rate of a planar spiral spring must be known if the spring is to be used to tune the displacer's natural frequency to the engine's operating frequency.

In order for a spring to meet the second design criteria of infinite fatigue life the maximum working stress must be kept below the endurance limit of the material (21). The endurance limit of the material is the stress level below which fatigue failure is not expected to occur regardless of the number of loading cycles (14). Consequently, the maximum working stress in a planar spiral spring must be known if comparison is to be made with the material endurance limit to determine the expected operating life of the spring.

As fatigue cracks generally originate at the surface of a part, the surface condition has a significant influence on the fatigue life of a spring (25). A quality surface finish will improve a spring's resistance to fatigue. Typically the highest stresses in a part occur at the surface (5). Consequently, having a good surface finish at the location of the maximum working stress will help ensure that a planar spiral spring has infinite life. Conversely, knowing where the
maximum stress occurs is useful to the extent that efforts can be made to ensure a good surface finish at least at that location.

Economics is reason enough to look for an easily manufacturable design. Appearance suggests that planar spiral springs could meet this last design criteria. In the case of sizeable production runs the design lends itself to a stamping process thus obviating the need for any winding or grinding steps. If determined necessary, finishing processes such as heat treatment, shot peening or polishing could still be considered.

By evaluating the spring rate, maximum working stress and the location of the maximum working stress, it can be determined whether the design criteria for a compact displacer spring in Stirling engines can be met by planar spiral springs. These baseline analysis results may then be used by the spring designer as a general guide in selecting the geometry of a planar spiral spring such that the desired spring rate and working stress levels are achieved.

Closure

Some types of Stirling engines use a mechanical spring to help tune the displacer so that it resonates at a frequency at or near the engine's operating frequency. Some of these engines require compactness, thus alternative designs to the typical helical displacer spring must be used. Two designs of flat springs for TMG's have been described but they do not fulfill the needs for compact springs in all Stirling engine applications.

A new design of displacer spring, called a planar spiral spring, has been proposed to help fill this need. To be used, the spring must have a predictable spring rate, an infinite fatigue life and be easily
manufacturable. An analysis is necessary in order to determine whether these criteria can be met by planar spiral springs.
CHAPTER 2: Review of Literature

Definition and Purpose of Mechanical Springs

A mechanical spring may be defined as an elastic body whose primary function is to deflect or distort under load and which returns to its original shape when the load is removed (25). Gross (11) gives a definition in a narrower sense in that any device which is specially made for converting mechanical work into potential energy and re-converting it into mechanical work, by virtue of the elastic deformation of material, is a mechanical spring. It is the ability to convert and store energy that makes mechanical springs useful and suitable for exerting force, providing flexibility and reducing shock in machine design applications.

In free-piston Stirling engines springs are essential to provide the restoring force necessary to maintain cyclic operation of the reciprocating components (26). In many designs of Stirling engines the restoring force is provided by a gas spring. Some designs, however, require that a mechanical spring be used to tune the displacer so that it will resonate at a frequency at or near the operating frequency of the engine (4).

Spring Design

Gross notes that the whole purpose of doing spring design is to find the most suitable spring to fulfill a specific purpose (11). This work is centered on finding the three main aspects of spring performance: the force-deflection curve, the maximum stress intensity and the energy storage capacity.
Force-Deflection Curves and Spring Rate

The deflection as a function of applied force is an essential characteristic of any mechanical spring due to the fact that the force-deflection curve is needed to obtain the stiffness. When coupled to the displacer in a Stirling engine a spring will alter the natural frequency of the displacer by an amount which is a function of the spring’s rate. Treated as a simple spring-mass system the natural frequency of the displacer is determined by

\[ f_n = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \]  

(1)

where \( f_n \) is the natural frequency, \( k \) is the spring rate and \( m \) is the mass of the spring-mass system (25).

Provided that working stresses do not exceed the elastic limit, most springs will have a linear force-deflection characteristic. That is the deflection of the spring will be proportional to the load applied to it; doubling the load will double the deflection (25). This relation holds for springs designed for torque or moment loads as well providing angular deflection is used in place of linear deflection (22).

There are also a number of types of springs whose force-deflection curves are not linear. These springs may exhibit nonlinear stiffening or nonlinear softening or some other characteristic whose curve shape is not a straight line.

In general, the relationship between force, \( F \), and deflection, \( \delta \), can be given as (22):

\[ F = F(\delta) \]  

(2)

The spring rate is defined as (22):

\[ k(\delta) = \lim (\Delta F/\Delta \delta) = dF/d\delta \]  

(3)
where $\delta$ is measured in the direction of, and at the point of application of $F$. For springs where the force-deflection curves are linear, $k$ is a constant, known as the spring constant and equation (2) becomes (22):

$$k = \frac{F}{\delta} \quad (4)$$

**Stress and Fatigue**

In addition to the required force-deflection characteristics of a spring from which the spring rate is obtained, the maximum allowable stress will also influence the design of a spring. This is complicated by the fact that the maximum allowable stress depends on the material strength and the type of loading of the spring. For convenience, the types of loading can be divided into static loading and cyclic loading. In static loading a spring is subject to load just once or only a few times. In cyclic loading, the number of load cycles the spring must be able to withstand may be only a few hundred or may be millions of cycles as is the case with Stirling engines. Consequently, springs loaded in this way are subject to fatigue.

In general, fatigue loading may be in the form of fully reversed stresses or loading may be either tension or compression. However, regardless of the type of dynamic loads applied, a mechanical part's ability to withstand fatigue loading is dependent upon the properties of the material and the level of working stresses.

The fatigue strength of a material is the level of stress above which fatigue failure can be expected to occur for a specified number of loading cycles (14). The characteristic behavior of a material under uniaxial fatigue loading is typically shown using a stress-
number of cycles diagram (σ-N diagram). Figure 2 is an example of such a diagram.

For a material subjected to low-cycle fatigue, the fatigue-strength may be at or near the ultimate-strength of the material. For very high-cycle loadings, the σ-N curve forms a knee and the fatigue-strength levels off at a level known as the endurance limit (σe) or fatigue limit. The endurance limit is interpreted as the maximum stress which can be applied repeatedly for any number of cycles without causing failure (14).

Springs are rarely subjected to both positive and negative stresses. Typically springs are subjected to fluctuating stresses which never pass through zero. In other words, the stresses, though cyclic, are either always compressive or always tensile. The life of a spring, that is to say, the number of load cycles which it can withstand before failure, depends on the mean of the fluctuating stress but even more particularly on the amplitude of the fluctuation. At a given mean stress, life increases as the stress amplitude decreases (11).

The fatigue strength can also be determined using the values for mean stress and stress amplitude along with either a Goodman diagram or a Modified Goodman diagram (21, 22, 23, 25). Figure 3 shows an example of a Goodman diagram and Figure 4 shows an example of a Modified Goodman diagram.

The use of a Modified Goodman diagram is especially useful when the cyclic loading causes combined stress instead of uniaxial stress. When combined stresses, due to some combination of torsion,
Figure 2: $\sigma$-N Diagram

- $\sigma_u$ - ultimate strength
- $\sigma_e$ - endurance limit

Number of Cycles (Log Scale)
Figure 3: Goodman Diagram

- $\sigma_u$ - ultimate stress
- $\sigma_y$ - yield stress
- $\sigma_e$ - endurance limit
- $\sigma_a$ - alternating stress
- $\sigma_r$ - stress range
- $\sigma_m$ - mean stress
Figure 4: Modified Goodman Diagram

\( \sigma_y \) - yield stress
\( \sigma_e \) - endurance limit
\( \sigma_a \) - alternating stress
\( \sigma_m \) - mean stress
\( \sigma_{yc} \) - compressive yield stress
\( \sigma_{ut} \) - ultimate tensile stress
\( \sigma_{yt} \) - tensile yield stress
bending and shear, from the fluctuating stresses a part is subjected to, the distortion-energy failure theory is used along with a Modified Goodman diagram to find fatigue strength. This is done by first determining the mean Von Mises stress and the alternating Von Mises stress amplitude and then applying them to a Modified Goodman diagram. The justification for using this approach is that all available experimental evidence shows it to be conservative (20, 21).

Application of this approach to planar spiral springs is actually simpler than usual because, contrary to most spring applications, the spring is intended to be subjected to fully and equally reversed loading cycles. This results in a mean stress of zero, thus only the alternating stress amplitude need be considered.

To ensure that a spring will survive the cycles of variable loading expected during its service life, the spring must be designed such that the working stresses never exceed the endurance limit of the material. In order to do this a spring designer needs an understanding of how working stresses are affected by geometric design parameters.

As springs are primarily subject to some combination of shear forces, bending moments or torsional moments, the maximum stress will occur somewhere on the surface. This is because the maximum stress in any part caused by any one of these forces occurs at the surface (5). Consequently, fatigue cracks, which initiate at points of high stress, tend to start on the surface of a spring (14). The tendency for a fatigue crack to start is increased if there are stress risers on the surface such as scratches, notches, scale or corrosion. A spring's resistance to fatigue will be improved if the surface is of
high quality (11, 14, 25), especially in the vicinity of the maximum working stress. Consequently, determination of the maximum stress location is essential.

**Energy Storage**

All mechanical springs store energy when they are deformed. Springs whose purpose is to bring moving masses to a stop, such as buffer springs, are designed with energy storage capacity in mind. The energy stored in a spring is simply

\[ U = \int_{0}^{\delta} F(\tau) d\tau \]

where \( \tau \) is the integration variable. This equation reduces to

\[ U = \frac{1}{2} F \delta = \frac{1}{2} k \delta^2 \]

when the spring rate is constant (13)

**Planar Spiral Springs**

In Figure 1, the shape of the cut out spiral is that of an involute of a circle. The result of using two involute shaped cuts, 180° apart, is that the coils are also involutes having constant width as they progress out from the center.

The material cuts in planar spiral springs do not need to be in the shape of involutes necessarily. Cuts of various spiral shapes would result in planar spiral springs but of different geometries. Springs of several coils can be made by introduction of nonintersecting material cuts. Two cuts, resulting in two coils, is the minimum, where a coil is defined as the arm or spline of material that connects the outer ring to the center.
Comparisons with Conical Springs

Figure 5 shows a planar spiral spring with an applied load. The outer edge of the spring is fixed in position and the load, applied at the center causes the spring to deflect out of its plane. In this distorted state the two coils appear to be similar to two conical springs acting in parallel.

The conical spring, shown in Figure 6, is a type of helical spring where the mean diameter of the coils is not constant.

When the conical spring is designed so that each coil nests wholly within an adjacent coil the spring can be compressed to a solid height of one coil. Comparatively, planar spiral springs have a height or thickness of one coil in the unstressed state because all of the coils lie in a single plane. When loaded and deformed, a planar spiral spring takes on more of a conical shape. Hence, planar spiral springs can be thought of as being more like "conical extension springs" rather than as compression springs as conical springs are typically used.

Conical springs are usually wound with wire having a circular cross-section. Conical springs made with material having a rectangular cross-section with the long side of the cross-section parallel to the spring axis are called volute springs (1, 25). When the long side of the cross-section is normal to the spring axis the orientation is similar to that of the planar spiral spring, see Figure 7.

Regardless of the shape of the wire cross-section, conical springs have a force-deflection curve that is not fully linear. An example curve is shown in Figure 8. As a conical spring is compressed it initially exhibits a constant spring rate. But as
Figure 5: Deformed Planar Spiral Spring
Figure 6: Conical Spring
Figure 7a: Conical Spring Section View

Figure 7b: Planar Spiral Spring Section View
"bottoming out" of active coils.

Figure 8: Force-Deflection Curve for Conical Springs
compression is continued, a transition point is reached where the active coils begin to "bottom out" and the spring rate begins to increase. In contrast, as planar spiral springs are loaded in tension all active coils remain active so the "bottoming out" phenomena is not possible.

Another advantage of planar spiral springs is that they are potentially inexpensive to make in large numbers as it may be possible to stamp them out of sheet steel as opposed to the wire winding process used to make most types of springs. Also, as conical springs have increased lateral stability compared to cylindrical helical springs (13), planar spiral springs may be quite stable too. Finally, by simulating two or more conical springs in parallel with the coils evenly spaced, planar spiral springs may exhibit less load eccentricity than do many sorts of helical springs.

Planar spiral springs are not without certain disadvantages. For example, the coil orientation with the long side of the cross-section normal to the spring axis results in an unfavorable stress distribution. This is especially significant for infinite life requirements for conical springs with similar coil cross-section orientations so it may also be a concern for planar spiral springs (25).

Another disadvantage of planar spiral springs is the fact that very little fatigue data is available on any springs with rectangular coil cross-sections (25) and to the best of the author's knowledge, none has been published for planar spiral springs.

**Background of Spring Design Theory**

A number of authoritative publications on the design and analysis of various types of springs are in existence. These include
helical springs in tension or compression, torsion springs, leaf springs, flat springs, conical springs, spiral springs, and special designs such as Belleville springs. In the literature are methodologies for designing many types of springs for a needed force-deflection response, for determining working stress levels and for determining energy storage in a given spring type (1, 11, 13, 23, 25).

Computational procedures for determining the necessary geometry of a cylindrical helical spring for a required spring rate and maximum working stress level have been developed and are reasonably accurate (11, 25). The equations for spring rate can be applied to conical springs if the spring is treated as a series of cylindrical springs. The rate for each coil or fraction of a coil if the wire has a circular cross-section is computed using (13):

$$k = \frac{P}{\delta} = \frac{Gd^4}{8D^3N_a}$$  \hspace{1cm} (7)

where

- \( P \) - load, N
- \( \delta \) - deflection, mm
- \( G \) - shear modulus, MPa
- \( d \) - wire diameter, mm
- \( D \) - mean spring diameter, mm
- \( N_a \) - number of active coils

If the wire is rectangular the equation to use is (13):
where \( b \) - width, mm
\( t \) - thickness, mm
\( K_2 \) - correction factor based on the ratio \( b/t \)

Note that rectangular is the cross-sectional shape after winding. A keystone cross-section wire will deform to a rectangular shape when wound (13, 25).

The rate for the complete spring is found by combining these individual rates according to the series relationship (13):

\[
k = \frac{1}{1/k_1 + 1/k_2 + \ldots + 1/k_n}
\]

(9)

where \( k_1, k_2, \ldots k_n \) are found using equations (7) or (8).

To calculate the maximum stress at a given load, the mean diameter of the largest active coil is used. The effect of coil curvature must also be accounted for when computing stress because neglecting curvature in rectangular coil springs may result in errors of 15 percent or more (25).

Problems in Using Conical Spring Design Theory with Planar Spiral Springs

The computational procedures for the design of conical springs can be tedious as they involve an iterative process to find the best geometry for a given application. Using these procedures in the design of planar spiral springs would be no less tedious. Also, by
looking at the equations it is not immediately clear what the relationships are between geometric parameters such as thickness and coil width and response characteristics such as spring rate and stress distribution.

It should be noted that inspite of the similarities between conical springs and planar spiral springs there are significant differences. The equations have been developed for conical springs loaded between two flat plates. This allows the ends of the springs to rotate freely. Planar spiral springs, on the other hand, have their coiled ends fixed at the outer ring and symmetry at the center controls rotations there.

Furthermore, as conical springs are usually wound using round wire the maximum stress is found at the inside of the largest coil. A planar spiral spring has a coil whose cross-section is that of a flat rectangle. That is, the long sides of the rectangle are perpendicular to the spring axis. A spring coiled in this way may have the maximum working stress occur at the inside of the coil or at some point along the sides (25). Knowing where the peak stresses will occur can be helpful when specifying manufacturing processes to ensure that a quality surface finish, which improves fatigue life, will result at that location. The usual design equations do not give the location of the maximum stress for a rectangular cross-section coil and an exact analysis which also takes into account the curvature effects of the coil involves elasticity theory and is quite complicated (25).
Hence, the design theory for conical springs may prove insightful as to how planar spiral springs will behave but it cannot meet all of one's needs.

**Hypotheses**

In order to understand the relationship between the geometric parameters of a planar spiral spring and its performance characteristics some form of analysis is necessary. The most easily controllable geometric parameters of greatest interest are thickness and coil width (which is controlled by the width of the removed material). With the results of a careful analysis it can be determined whether planar spiral springs can meet the design criteria for use as displacer springs in Stirling engines. For this analysis two hypotheses are put forth: (1) planar spiral springs are a viable design idea for application as displacer springs in free-piston Stirling engines, and (2) the working envelope for a specific type of planar spiral spring can be established with reasonable confidence without an extensive test program.

**Closure**

Mechanical springs can be defined by their ability to store and release energy. The process of spring design centers on finding the three main aspects of spring performance: the force-deflection curve, the maximum stress intensity and the energy storage capacity.

A spring's rate, which is important in applying vibration theory to spring-mass systems, is obtained from the force-deflection curve. The maximum stress intensity along with the number of loading cycles will determine the fatigue life of a spring. The surface finish of a spring will also influence its fatigue life. The energy storage
capacity of a spring can be calculated using the force-deflection curve and the maximum deflection information.

In appearance, planar spiral springs resemble conical springs in parallel. Inspite of the similarities there are enough differences such that using the design equations for conical springs on planar spiral springs may not be judicious. These equations, should they apply, can be quite tedious to use.

In order to get an understanding of the relationships between the geometric parameters of a planar spiral spring and its performance characteristics an analysis of the design is necessary. Chapter 3 will discuss the method used to make such an analysis.
CHAPTER 3: Methods

Selection of Analysis Method

As discussed in Chapter 2, even though planar spiral springs share similarities with conical springs, it is not clear that the spring design equations for conical springs can be used to design a planar spiral spring. Regardless of the applicability of these equations to planar spiral springs, they tend to get quite tedious, especially as the number of approximating arcs to the actual coil shape increases. Also, by looking at the equations it is not clear what the relationships are between geometric parameters and spring performance.

A rigorous and thorough test program on a statistically significant number of planar spiral springs would generate data for checking whether planar spiral springs will meet the design criteria for displacer springs in Stirling engines. This data could also be used to make graphs that would demonstrate relationships between design parameters and spring behavior. Such a test program, however, would be time consuming and expensive as the commonly encountered large variation in results when testing for fatigue (5) would require a large number of tests. Also, the testing would probably need to include tests such as photoelastic coating tests to determine maximum stress locations.

Finite element analysis makes a more attractive, time efficient initial approach to this design analysis problem. Using finite element models, a number of design parameters can be evaluated much more quickly than can be done through testing alone. Finite element analysis does not supercede the testing program, it still should be
done, but the viability of planar spiral springs as a design concept can be largely substantiated using finite element analysis.

**Equipment and Software**

The PATRAN software package by PDA Engineering (versions 2.4 and 2.5) was used to generate the finite element model for the spring and the finite element code ABAQUS by Hibbitt, Karlsson and Sorensen, Inc. (releases 4.8) was used for analysis. Results from ABAQUS were output as data reports and were also post-processed using PATRAN. PATRAN was run on a Micro-VAX II host processor through an Intergraph Corporation Interpro 220 workstation. ABAQUS was run on a VAX 751 computer for small models and on a CRAY Y-MP8/864 for larger models. With the exception of the CRAY, all of the software and equipment used are either owned or licensed by Ohio University. The Ohio Supercomputing Center's CRAY was accessed remotely from the Ohio University campus.

Data for generating force-deflection curves for actual springs was obtained using a testing rig designed and built by Sunpower Incorporated, Athens, Ohio. Figure 9 shows a schematic of the testing rig. The Appendix contains a picture and a complete description of the testing rig.

**Finite Element Model**

Selection of the proper element is the important first step in finite element modeling (8). Because the coil radii increase from the center outward and because the ends are fixed, bending as well as torsion and shear forces are present in the coils. Shell theory based elements are generally capable of modeling these combined forces.
Figure 9: Planar Spiral Spring Testing Rig
and shell elements are well suited for the geometry of planar spiral springs in both the undeformed and deformed (i.e. unloaded and loaded) states.

The material cuts in a planar spiral spring and the small center hole unavoidably introduce geometric discontinuities. Since any discontinuity in a machine part alters the stress distribution in the vicinity of the discontinuity (22), stress concentrations were expected. Stress concentrations, having highly localized effects (22), can cause steep stress gradients in these local areas. If constant-stress elements such as four node quadrilaterals were used, an exceptionally fine mesh would be required for an accurate solution to such stress fields (8). Consequently, higher order quadratic elements were used which gave linear instead of constant stress distributions across elements. In particular, the shear flexible S8R5 eight node quadrilateral shell element in the ABAQUS element library was chosen. For these elements ABAQUS computes stresses at the upper and lower surfaces of the shell (12).

As the finite element models were refined, regions of low or constant stress were discretized using relatively large elements resulting in a coarse mesh. In regions having stress concentrations, relatively small elements were used resulting in a fine mesh. Between these regions of coarse and fine meshes a transition in mesh refinement naturally had to exist.

The transitions were accomplished by the use of multi-point constraints (MPC's). MPC's are a technique whereby the position of a node can be restrained to a line defined by two or more nodes. The application of MPC's to the models used in this study are best
illustrated with an example. In Figure 10 are shown three elements that form a transition.

Elements 2 and 3 are clearly topologically incompatible with element 1. An MPC can be used to ensure that nodes a and b, the mid-side nodes of elements 2 and 3, remain on the line or curve defined by nodes A, B, and C. Nodes a and b are allowed to "slide" along the ABC curve but they must remain on that curve. This constraint avoids what would be a compatibility problem at the boundary of elements 1, 2, and 3 as illustrated in Figure 11. A transition of this type without the use of MPC's could result in inter-element gaps or overlaps which could affect the computed solutions (12, 18).

The displacer rod in a Stirling engine is attached to the central hole of a planar spiral spring. The force transmitted by the displacer rod to a planar spiral spring was modeled as lumped forces equally distributed among the nodes on the edge of the center hole. This ad hoc lumping results in mid-side nodes and corner nodes carrying equal loads which will result in less accurate solutions for coarse meshes. But with mesh refinement there will be convergence toward the correct solution (2).

A planar spiral spring is intended to operate with the outside edge of the spring fixed in place. This boundary condition was modeled by constraining all of the element corner nodes on the outside edge of the spring in all six degrees of freedom. The element mid-side nodes on the outside edge were not constrained. The intent of this approach was to give a slight amount of flexibility for deformation to the element edges forming the spring outside edge.
Figure 10: Transition Mesh

Figure 11: Deformed Transition Mesh
It was felt that this would better represent the actual boundary conditions of the spring than would a fully-fixed outside edge.

Young's modulus and Poisson's ratio for steel are, to a certain degree, dependent upon the alloy considered. As there does not appear to be any reason that planar spiral springs must be limited to any particular alloy of steel, the widely used "mid-range" values of 200 GPa for Young's modulus and 0.30 for Poisson's ratio were used in this study.

**Analysis Approach**

Before any confidence can be placed in the solutions resulting from a finite element analysis, the validity of the finite element model must be proven (19). There are various techniques for doing this. One is to compare the finite element solution with the analytical results available for a comparable problem. Perhaps the best validation technique is comparison of the finite element solution with results derived empirically (10). Another technique is to do a convergence study, sometimes called a mesh refinement study, by comparing a number of trial solutions each with a different number of degrees of freedom in the model. If these trial solutions are plotted, a converging curve can be made for a model that is sufficiently and properly refined (10, 19). In this study the latter two validation techniques were used.

A set of planar spiral springs comprising five different geometries were manufactured by Sunpower, Incorporated of Athens, Ohio. In the laboratory facilities of Sunpower and using a testing rig designed and built by Sunpower (see Appendix), each spring was subjected to a series of incremental loads and the
resulting deflections were recorded. This data was used to develop the force-deflection curve for each spring geometry in the set.

PATRAN geometric models of these lab tested springs were developed and finite element models were generated using the geometric models. The finite element models were then translated into ABAQUS input files. Using at least three different load cases per model, a linear static analysis of each load case was run using ABAQUS. The computed deflections were then compared to the lab results to verify the finite element model solutions. When the computed results deviated from the lab results the finite element model was refined and the load cases were rerun.

For three of the five experimental spring geometries, finite element model convergence studies for both deflection and maximum Von Mises stress were conducted. In refining the meshes for these convergence studies, care was taken to ensure that all of the coarser meshes were included in the more refined meshes of each model. This along with using the same order of elements are necessary in order for the sub-division process to converge to the exact solution (6, 10).

Since the variation of stress within an element is of one order less than displacement, a model predicting an accurate displacement solution may indicate an inaccurate stress solution (8). Consequently, a second test to ensure accurate stress solutions was used.

For S8R5 eight node quadrilateral reduced integration shell elements, ABAQUS computes the stresses at the four Gaussian integration points. The stresses at the nodes of an element are then computed by extrapolating out from the integration points. If a node
is shared by two elements then there will be two computed stresses at that node, one from the extrapolation in the first element and another from the extrapolation in the second element. The computed stresses at a node shared between two elements are, in general, not the same. The difference in the computed stresses for a node is the inter-element stress jump at the given node.

In developing finite element models in this study, a mesh was refined until the inter-element Von Mises stress jump at any shared node between elements was less than some $\Delta$ computed as:

$$\Delta = \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{\sigma_{\text{avg}}}$$

A value of 4% was chosen for this $\Delta$ as small enough such that when the nodal stresses computed for each element sharing a given node were averaged, the resulting nodal stress average would be within $1/2 \, \Delta$ or 2% of the exact solution. This would seem to be sufficiently accurate for design engineering purposes considering other variations inherent in making planar spiral springs such as the variation in material thickness, warpage or distortion due to heat treatment and imperfections in geometry from the cutting or stamping process. Also, the inherent scatter in fatigue test results mandates a sizeable factor of safety if the goal is infinite life (22, 25). Thus, 2% error is not out of proportion in relation to other design considerations involved in designing and making planar spiral springs.

When modeling the lab tested springs the actual measured material cut widths were used. On the other hand, the measured thicknesses were compared to the thicknesses for various standard
gages of sheet steel (16) and the thickness of the closest matching gage steel was used. This was done because standard gage sheet steel had been used to make the springs and the measured thicknesses were suspect to error due to machining burrs and warpage.

With the modeling approach for planar spiral springs verified, a number of standard gage thicknesses and material cut widths were analyzed. Solutions for deflection, maximum Von Mises stress and the distribution of Von Mises stress within a planar spiral spring were computed and recorded.

Closure

There are several possible methods that may be used to determine whether planar spiral springs will meet the design criteria for displacer springs in Stirling engines. Design equations for conical springs are not fully suitable for planar spiral springs. A rigorous testing program of a statistically significant number of planar spiral springs would be time consuming and expensive. Finite element analysis is an attractive approach because a number of parameters can be tested in a timely fashion.

Because of the anticipated forces in a planar spiral spring under load, shell elements were used in the finite element model. As geometric discontinuities existed, steep stress gradients were also anticipated hence quadratic order elements were used which gave linear element stress distributions.

Model validation is very important in order to justify confidence in any finite element solution. Both convergence studies and comparison with experimental data were used to validate the
models. After validating various models, analysis runs were made for several load cases for each model to determine the force-deflection characteristics, maximum Von Mises stress levels and the Von Mises stress distributions in planar spiral springs.

With a plan for model creation and validation in hand, the test work was scheduled and the geometric and finite element modeling began.
Experimental Results

Figures 12 through 16 show the force-deflection curves for the five spring geometries tested. Two springs were used to generate Figure 12, six springs for Figure 13, one spring for Figure 14, ten springs for Figure 15 and three springs for Figure 16.

Each of the five figures shows a straight line fitted to the data. The computed slope of each line, which corresponds to the spring constant, k, of each spring is also shown. These spring constants range from 1.90 N/mm to 5.95 N/mm.

Finite Element Solutions for Deflection

The meshes in the finite element models used to model the experimental springs varied in the number of elements and nodes depending upon the spring being modeled. As few as 1872 nodes and 518 elements to as many 5164 nodes and 1570 elements were used. In all of the finite element analysis runs, the node with the maximum computed deflection was not one of the loaded nodes around the central hole but was at a node on the outer edge of where the spring coil connects to the central disk. The difference between the maximum nodal displacement and the displacement at the center hole was always less than 1%. Since deflections in the lab were measured relative to the edge of the small center hole, finite element results for nodes at this same position were used in place of the maximum nodal deflections.

Figures 17a and 17b show the finite element mesh for a model of a spring with a material cut width of 1.71 mm. Figure 17a shows
Figure 12

Force-Deflection Curve
Laboratory Results

$k = 4.19 \text{ N/mm}$

$W(\text{cut}) = 1.71 \text{ mm}$
$W(\text{coil}) = 4.35 \text{ mm}$
$t = 1.062 \text{ mm (19 gage)}$

Figure 13

Force-Deflection Curve
Laboratory Results

$k = 1.90 \text{ N/mm}$

$W(\text{cut}) = 1.84 \text{ mm}$
$W(\text{coil}) = 4.32 \text{ mm}$
$t = 0.836 \text{ mm (21 gage)}$
Figure 14

Figure 15
Force-Deflection Curve
Laboratory Results

\[ k = 2.75 \text{ N/mm} \]

\[ W(\text{cut}) = 3.09 \text{ mm} \]
\[ W(\text{coil}) = 3.16 \text{ mm} \]
\[ t = 1.062 \text{ mm (19 gage)} \]

Figure 16
17a: Finite Element Mesh

17b: Finite Element Mesh Closeup
a fully meshed spring and Figure 17b shows a close-up of the refined mesh area in the lower right of Figure 17a. Figures 18a and 18b show the mesh for a spring with a material cut width of 3.09 mm for comparison.

Figures 19 through 23 show the force-deflection curves of the five sets of lab tested springs with the results from the finite element analysis runs also plotted. The spring constant, determined from the finite element results are shown along with the spring constant derived from the lab results. Figures 19, 20 and 23 show that the finite element results predict a higher spring stiffness than that determined experimentally. For the spring of Figure 20 the finite element stiffness is 15% greater than the lab determined stiffness. Figures 24, 25 and 26 show the results of convergence studies for deflection for three different springs. All three studies were done with a force of 5 Newtons and the results for lab tests at a force level of 5 Newtons are also shown.

Figures 27 through 30 show the force-deflection curves for four spring geometries not tested in the lab. The computed slopes of straight lines fitted to the data are shown. Just as in Figures 12 through 16, these line slopes also represent the spring constants.

Figure 31 shows the relationship between spring thickness and deflection for four springs with different widths of material cut. All four curves were generated using data from finite element analysis runs using a force of 5 Newtons.
18a: Finite Element Mesh

18b: Finite Element Mesh Closeup
Force-Deflection Curve

\[ k_{lab} = 4.19 \text{ N/mm} \]
\[ k_{FEA} = 4.53 \text{ N/mm} \]
\[ W(\text{cut}) = 1.71 \text{ mm} \]
\[ W(\text{coil}) = 4.35 \text{ mm} \]
\[ t = 1.062 \text{ mm (19 gage)} \]
Mesh SK1D

Figure 19

Force-Deflection Curve

\[ k_{lab} = 1.90 \text{ N/mm} \]
\[ k_{FEA} = 2.19 \text{ N/mm} \]
\[ W(\text{cut}) = 1.84 \text{ mm} \]
\[ W(\text{coil}) = 4.32 \text{ mm} \]
\[ t = 0.836 \text{ mm (21 gage)} \]
Mesh SL21

Figure 20
Force-Deflection Curve

$k_{lab} = 5.95 \text{ N/mm}$
$k_{FEA} = 5.81 \text{ N/mm}$

$W(\text{cut}) = 2.18 \text{ mm}$
$W(\text{spline}) = 4.03 \text{ mm}$
$t = 1.214 \text{ mm (18 gage)}$

Mesh SJ1E

Figure 21

Force-Deflection Curve

$k_{lab} = 3.92 \text{ N/mm}$
$k_{FEA} = 3.95 \text{ N/mm}$

$W(\text{cut}) = 2.20 \text{ mm}$
$W(\text{coil}) = 4.00 \text{ mm}$
$t = 1.062 \text{ mm (19 gage)}$

Mesh SG1D

Figure 22
Force-Deflection Curve

\[ k_{\text{lab}} = 2.75 \text{ N/mm} \]
\[ k_{\text{FEA}} = 2.89 \text{ N/mm} \]

\[ W(\text{cut}) = 3.09 \text{ mm} \]
\[ W(\text{coil}) = 3.16 \text{ mm} \]
\[ t = 1.062 \text{ mm (19 gage)} \]
Mesh SH7D

Figure 23
Convergence Study for Deflection

Figure 24

Convergence Study for Deflection

Figure 25
Figure 26

Convergence Study for Deflection

W_{(cut)} = 3.09 \text{ mm} \\
t = 1.062 \text{ mm (19 gage)} \\
Force = 5 \text{ N} \\
Meshes SHxD

$W_{(CU)} = 3.09 \text{ mm}$ \\
t = 1.062 \text{ mm (19 gage)} \\
Force = 5 \text{ N} \\
Meshes SHxD
Figure 27

Force-Deflection Curve

\[ k = 4.21 \text{ N/mm} \]

\[ W(\text{cut}) = 1.984 \text{ mm} \]

\[ W(\text{coil}) = 4.29 \text{ mm} \]

\[ t = 1.062 \text{ mm (19 gage)} \]

Mesh SB1D

Figure 28

Force-Deflection Curve

\[ k = 3.76 \text{ N/mm} \]

\[ W(\text{cut}) = 2.381 \text{ mm} \]

\[ W(\text{coil}) = 3.878 \text{ mm} \]

\[ t = 1.062 \text{ mm (19 gage)} \]

Mesh SC8D
**Force-Deflection Curve**

**FEA Results**

![Diagram](image)

- \( k = 3.27 \text{ N/mm} \)
- \( W(\text{cut}) = 2.778 \text{ mm} \)
- \( W(\text{coil}) = 3.46 \text{ mm} \)
- \( t = 1.062 \text{ mm (19 gage)} \)
- Mesh SD2D

**Figure 29**

![Diagram](image)

- \( k = 1.76 \text{ N/mm} \)
- \( W(\text{cut}) = 3.969 \text{ mm} \)
- \( W(\text{coil}) = 2.14 \text{ mm} \)
- \( t = 1.062 \text{ mm (19 gage)} \)
- Mesh SF2D

**Figure 30**
Deflection vs. Thickness

- $W(cut) = 3.09 \, mm$
- $W(cut) = 2.20 \, mm$
- $W(cut) = 1.71 \, mm$

Force = 5 N
Meshes SK1, SG1, SH7, SF2

Figure 31
Figure 32 shows the relationship between the width of material cut and deflection. The relationship is for a material thickness of 1.062 mm and a load of 5 Newtons.

**Finite Element Solutions for Stress**

Figures 33, 34 and 35 show the results of convergence studies for maximum Von Mises stress for three spring geometries. These three geometries are the same as those used in the convergence study for deflection shown in Figures 24, 25 and 26. The convergence studies for maximum Von Mises stress were also done with a force of 5 Newtons on each spring. Note that these and all other computed stresses are at the surface of the shell elements.

The Table shows a compendium of all of the analysis runs made using fully refined finite element meshes with a load level of 5 Newtons. The Table includes the number of nodes in each model, the maximum average nodal Von Mises stress and the percent Von Mises stress jump at the node where the maximum was computed.

Figures 36 through 44 show the relationship between applied force and the maximum Von Mises stress produced in a spring. Figures 36, 37, 39, 40 and 43 correspond to the spring geometries tested in the lab. Figures 38, 41, 42 and 44 are the results for additional geometries.

Figure 45 shows the relationship between spring thickness and the resulting maximum Von Mises stress for a force of 5 Newtons. The four widths of cut of material are the same in these figures as Figure 31.
Deflection vs. Width of Material Cut

$t = 1.062$ mm (19 gage)

Force = 5 N

Meshes K,B,G,D,H,F

Figure 32
Convergence Study for Maximum Von Mises Stress

Figure 33

Convergence Study for Maximum Von Mises Stress

Figure 34
Convergence Study for Maximum Von Mises Stress

Max. Von Mises Stress, MPa

Number of Nodes

W(cut) = 3.09 mm

t = 1.062 mm (19 gage)

Force = 5 N

Meshes SHxD

Figure 35
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<th># nodes</th>
<th>$\sigma_{max}$ (MPa) (Von Mises)</th>
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Table: Compendium of Finite Element Results
Force vs. Max. Von Mises Stress

Figure 36

W(cut) = 1.71 mm

slope = 0.0512 N/MPa

t = 1.062 mm (19 gage)

Mesh SK1D

Force vs. Max. Von Mises Stress

Figure 37

W(cut) = 1.84 mm

slope = 0.0296 N/MPa

t = 0.836 mm (21 gage)

Mesh SI21
Force vs. Max. Von Mises Stress

Figure 38

Force vs. Max. Von Mises Stress

Figure 39
Force vs. Max. Von Mises Stress

- Figure 40: Slope = 0.0492 N/MPa
  - $W(cut) = 2.20 \text{ mm}$
  - $t = 1.062 \text{ mm (19 gage)}$
  - Mesh SG1D

Force vs. Max. Von Mises Stress

- Figure 41: Slope = 0.0495 N/MPa
  - $W(cut) = 2.381 \text{ mm}$
  - $t = 1.062 \text{ mm (19 gage)}$
  - Mesh SC8D
Force vs. Max. Von Mises Stress

Figure 42

Force vs. Max. Von Mises Stress

Figure 43
Force vs. Max. Von Mises Stress

\[ \text{slope} = 0.0286 \text{ N/MPa} \]

\[ W(\text{cut}) = 3.969 \text{ mm} \]
\[ t = 1.062 \text{ mm (19 gage)} \]
\[ \text{Mesh SF2D} \]

Figure 44
Max. Von Mises Stress vs. Thickness

- W(cut) = 3.969 mm
- W(cut) = 3.09 mm
- W(cut) = 2.20 mm
- W(cut) = 1.71 mm

Force = 5 N
Meshes SK1, SG1, SH7, SF2

Figure 45
Figure 46 shows the relationship between the width of material cut and maximum Von Mises stress. The results are for a spring with a thickness of 1.062 mm and a load of 5 Newtons.

Figure 47 shows the Von Mises stress contours for a spring with a material cut width of 1.71 mm, a thickness of 1.062 mm and a load of 5 Newtons.

Figures 48 through 51 show closeup views of meshes in the area of stress concentration (see Figures 17 and 18) of four geometries. The material width cuts were 1.71 mm, 2.20 mm, 3.09 mm and 3.969 mm, respectively. For a material thickness of 1.062 mm and a load of 5 Newtons the maximum Von Mises stress was computed for each and the node associated with the primary stress peak indicated. The secondary stress peak, when it occurred, is also indicated.

Figures 52 through 57 show the Von Mises stress contours for a spring with a material cut width of 1.71 mm and loaded with a 5 Newton force. Each figure is for a different thickness of spring. Figures 58 through 63 show the same sequence of material thicknesses but for a spring with a material cut width of 3.09 mm and a 5 Newton force.

**Closure**

Presented are the results from lab tests of five geometries of springs. Finite element model convergence study results are also presented along with finite element analysis results for deflection and maximum Von Mises stress. The relationships between force, material thickness and the material cut width and deflection and Von Mises stress are shown graphically and in contour plots.
Max. Von Mises Stress vs. Width of Material Cut

$t = 1.062 \text{ mm (19 gage)}$

Force = 5 N

Meshes K, B, G, D, H, F

Figure 46
Figure 47: Von Mises Stress Contours

MPa

96.0 = A
89.4 = B
82.7 = C
76.1 = D
69.4 = E
62.7 = F
56.1 = G
49.4 = H
42.8 = I
36.1 = J
29.4 = K
22.8 = L
16.1 = M
9.46 = N
2.80 = 0

W(cut) = 1.71 mm

Force = 5 N
Figure 48: Position of Primary Stress Peaks

Primary 97.65 MPa

W(cut) = 1.71 mm
\( t = 1.062 \text{ mm} \)
Force = 5 N
Figure 50: Position of Primary and Secondary Stress Peaks

- Primary Stress: 113.6 MPa
- Secondary Stress: 111.2 MPa

- $W_{\text{cut}} = 3.09$ mm
- $t = 1.062$
- Force $= 5$ N
Figure 51: Position of Primary and Secondary Stress Peaks

Primary 174.7 MPa
Secondary 131.5 MPa

W(cut) = 3.969 mm
t = 1.062 mm
Force = 5 N
Figure 52: Von Mises Stress Contours

MPa

288 = A
268 = B
248 = C
228 = D
208 = E
188 = F
168 = G
148 = H
128 = I
108 = J
87.6 = K
67.6 = L
47.6 = M
27.6 = N

W(cut) = 1.71 mm
\( t = 0.607 \text{ mm} \)
Force = 5 N
7.57 = 0
Figure 53: Von Mises Stress Contours

W(cut) = 1.71 mm
\[ t = 0.759 \text{ mm} \]
\[ \text{Force} = 5 \text{ N} \]
Figure 55: Von Mises Stress Contours

MPa

129. = A
120. = B
111. = C
102. = D
93.0 = E
84.1 = F
75.1 = G
66.2 = H
57.2 = I
48.3 = J
39.4 = K
30.4 = L
21.5 = M
12.5 = N

W(cut) = 1.71 mm
\( t = 0.912 \text{ mm} \)
Force = 5 N
3.60 = O
Figure 59: Von Mises Stress Contours

\[ W(\text{cut}) = 3.09 \text{ mm} \]
\[ t = 0.759 \text{ mm} \]
\[ \text{Force} = 5 \text{ N} \]
Figure 60: Von Mises Stress Contours

- Force = 5 N
- t = 0.836 mm
- W(cut) = 3.09 mm

MPa

172. = A
160. = B
148. = C
136. = D
124. = E
112. = F
100. = G
88.4 = H
76.5 = I
64.6 = J
52.6 = K
40.7 = L
28.7 = M
16.8 = N
4.82 = 0
Figure 62: Von Mises Stress Contours

MPa

110. = A
103. = B
95.1 = C
87.5 = D
79.8 = E
72.2 = F
64.5 = G
56.8 = H
49.2 = I
41.5 = J
33.8 = K
15.3 = L
18.5 = M
10.9 = N

W(cut) = 3.09 mm

t = 1.062 mm
Force = 5 N

3.20 = O
Experimental and Finite Element Force-Deflection Results

The spring constants of the experimental springs ranged from a minimum of 1.90 N/mm for a spring with a material cut width of 1.84 mm and a thickness of 0.836 mm, to a maximum of 5.92 N/mm for a spring with a material cut width of 2.18 mm and a thickness of 1.214 mm. The experimental results shown in Figures 12 through 16 show a linear force-deflection curve for planar spiral springs. This is reasonable because all coils remain active through the operating deflection distance.

The force-deflection curves resulting from the finite element analysis runs are consistent in showing a linear deflection response to an applied force for all five geometries of springs tested. The computed spring constants differed from the lab results by +8.1%, +15.3%, -2.3%, +0.8% and +5.1% as shown in Figures 19 through 23, respectively. The springs where the computed spring constant and the experimental spring constant vary by -2.3% and +0.8% indicate a good correlation between the finite element model and the actual spring in terms of gross deflection.

For the three spring geometries where the finite element spring constant deviates from the associated experimental spring constant by more than 5%, the finite element results are consistently higher. That is, the spring of the finite element model is stiffer than the actual spring. This is a characteristic of results using finite element code which is based on displacement functions. Because the number of degrees of freedom, which is related to the number of
nodes, are fewer in a finite element model than actually exist, the finite element model is always stiffer than the actual part being modeled (6, 7).

The deviations of 8.1% and 15.3%, though sizeable, do not necessarily indicate finite element models of marginal quality and/or of marginal use. The geometry of the finite element models, and in particular the width of the material cuts, were measured by hand using a caliper. The measurements were only taken at one place on each spring. Since all of the springs had been handled and many of them tested or used to one degree or another they may have been slightly deformed. The result could have been that the measured material cut widths do not reflect the true cut width of the actual spring. This would be important because the geometric model used to make the finite element model was based on the measured dimension. If the measured material cut width was smaller than the true width, the resulting spring coil would be wider and hence a stiffer spring model than the actual spring.

Consequently, the two finite element models that deviate by 8.1% and 15.3% from the experimental results may actually be giving good results for the geometries modeled. However, the geometries modeled may, in fact, not match the geometries of the actual experimental springs. Because of this possibility it is necessary to use convergence studies for deflection to help substantiate the validity of the finite element models and their results.

The convergence studies shown in Figures 24, 25 and 26 were done using finite element models of experimental springs all having
the same material thickness, 1.062 mm. In each of the three analysis runs on each spring the applied load was the same, 5 Newtons.

In all three convergence studies the finite element results show good stability. The percent change from the coarsest meshes to the most refined meshes were 5.0%, 3.7% and 0.0%, respectively. This stability indicates that the results will not improve appreciably regardless of how much more mesh refinement is done. Because of this stability and because the spring constant results of three of the models were within 5% of the experimental results, it is concluded that the general modeling technique used is viable for determining deflections in planar spiral springs.

The force-deflection curves of Figures 27 through 30 are the results of finite element analysis runs for additional spring geometries. All four of these springs were modeled with a thickness of 1.062 mm, only the material width cuts varied.

The curves in Figure 31 show that there is a nonlinear relationship between deflection and spring thickness. As the thickness increases the stiffness of the spring increases thus reducing deflection for a given load. This nonlinear relationship is reasonable as the stiffness of any part in either bending or torsion is related in a non-linear way to geometry. That is, stiffness in bending is related to the second moment of area or moment of inertia, I, which for a beam of rectangular cross-section is given as 1/12(bh^3). For a rectangular cross-section in torsion the stiffness is given as βbt^3(G/L) where β is a coefficient related to the ratio of width to thickness in a nonlinear fashion.
The possible material cut widths range from zero, or no cut, to a maximum of when the adjacent material cut paths just intersect. Figure 32 shows a nonlinear relationship between the width of the material cut and deflection for a given material thickness and load. As the width of cut increases, the width of the resulting spring coil decreases along with the stiffness of the spring. As with the change in thickness, the change in material cut width which controls the resulting coil width, is related to stiffness in a nonlinear way and hence the resulting curve.

For a given desired spring constant there may be several geometries of planar spiral spring that will result in the necessary force-deflection response. Figures 19 to 23 and 27 to 32 can be used to identify the spring geometries which will satisfy the spring constant requirement. The best geometry for a given design situation can then be identified by considering load induced stress which is discussed presently.

Finite Element Stress Results

No experimental tests were conducted to determine stresses in a planar spiral spring. In order to validate the finite element models for stress, convergence studies were done for three different geometries. Because combined loading necessitated the use of Von Mises stresses, the convergence studies were done using the maximum computed Von Mises stress.

In Figures 33, 34 and 35 the stability of the finite element solutions is clear. Using a spring thickness of 1.062 mm and a force of 5 Newtons in all three studies, refinement of the mesh did not appreciably change the computed maximum stress values. The
percent change from the medium refined mesh to the finest mesh were -0.6%, 1.5% and 0.3%, respectively. This indicates that the finite element stress solutions would not change any appreciable amount regardless of the amount of further mesh refinement.

The Table shows the results of another type of study used to help validate the finite element models for computed stresses. For each analysis run made, the node with the highest computed Von Mises stress contributed by any single element was identified. Then, the largest difference in computed Von Mises stress between any two elements sharing that node was determined. This difference, divided by the average of the Von Mises stresses of the node contributed by each element sharing the node, gave the percent maximum inter-element stress jump at the node. The values of these computed stress jumps are shown in column six of the Table.

All of the inter-element stress jumps were less than 4%. In fact, with exception to the SG1 series of models (width of cut of of 2.20 mm), all of the stress jumps were only 1.12% or less. Such small stress jumps in the area of greatest interest, that is, in the area of greatest stress, indicates a well refined mesh.

The node with the highest average Von Mises stress was also identified. Without exception, this node was the same node that had the highest stress from any single element. The maximum average Von Mises nodal stress for each model loaded with a 5 Newton force is shown in column 5.

To put these stress values into perspective, an AISI 1095 heat treated steel, quenched in oil and tempered, has a yield strength of 813 MPa (21). Since the Von Mises failure criteria without regard to
fatigue is when the maximum Von Mises stress exceeds $\sqrt[2]{\beta(\sigma_y)}$, where $\sigma_y$ is the material yield strength (22, 24), failure is expected in this steel when the maximum Von Mises stress exceeds 383 MPa.

Column 5 of the Table shows that the computed maximum Von Mises stress for model SF2A1 of 486.3 MPa is the only value in excess of the failure limit. As this steel has a rather high strength, such a thin planar spiral spring with such a wide material cut probably cannot be made to work in Stirling applications under loadings of 5 Newtons or more. Especially since fatigue loading reduces the allowable working stress in the spring.

The other spring geometries may or may not be feasible for Stirling engines depending upon the factor of safety desired with regard to the endurance limit. A modified Goodman diagram can be constructed for this material and, with a factor of safety in mind, the viable geometries for a 5 Newton load can be identified. For other loads, Figures 36 to 46 can be used to get the approximate maximum Von Mises stress for various geometries.

The relationships between the maximum Von Mises stress, force and geometry are shown graphically in Figures 36 to 44. In each case the relationship is linear. This is reasonable as shear stress, bending stress and shear stress due to torsion are related to load in a linear fashion. That is, shear stress is determined by the equation $\tau=F/A$, which is linear in $F$. Bending stress is determined by $\sigma=Mc/I$ where $M=Fx$ which is linear in $F$. Shear stress due to torsion is determined by $\tau=Tr/J$ where $T=Fx$ which is also linear in $F$.

The curves in Figure 45 show a nonlinear relationship between the maximum Von Mises stress and spring thickness. In general, the
wider the material cut, the steeper the slope of the resulting curve for a given thickness. This nonlinear relationship is reasonable as energy, which increases nonlinearly with thickness because the spring rate is nonlinear with respect to thickness (see Figure 31), is stored in the form of stress.

Note how similar the curves are for widths of cut of 1.71 mm and 2.20 mm. Inspite of a 29% increase in material cut width, the maximum Von Mises stress changes by less than 1% for $t=0.607$ mm and by less than 5.0% for $t=1.214$ mm.

Figure 46 shows a nonlinear relationship between the width of the material cut and maximum Von Mises stress for a given material thickness and load. As with the change in thickness, the energy, which increases nonlinearly with the width of material cut because the spring rate is nonlinear with respect to the width of material cut (see Figure 32), is stored in the form of stress.

The Von Mises stress contours of the active coils in Figure 47 show that in general the stress at a point on the inside edge of a coil is higher than the point directly across from it on the outside edge of the coil. The figure also shows that the stresses increase as the coil spirals from the center out. This is probably due to the fact that as the coil spirals out from the center, the moment arm from the center to the coil continually increases. Hence, the torque, and the accompanying shear stress due to torsion, increase.

In Figures 48 to 51 can be seen what the effect of changing the material cut width has on the position of the maximum Von Mises stress. For a narrow width cut as shown in Figure 48, the maximum Von Mises stress or stress peak is on the edge of the semicircular end
of the tool cut path. In Figure 49 for a wider cut, the highest or primary stress peak is also on the edge but a secondary peak which is 91% in magnitude of the primary peak has formed to the interior face of the coil. The same is true for Figure 50 but the secondary stress peak is 98% of the primary stress peak. Figure 51 shows that for a wide material cut the primary stress peak has shifted to the interior face of the coil and the secondary stress peak, which is 75% of the primary stress peak, is now at the cut edge.

A possible reason for this stress peak shift may have to do with the degree of abruptness of the change in section. That is, the narrow cut width of Figure 48 causes a sudden change in geometry which forms a severe stress concentration. As the material cut width is increased the radius of the cut edge increases and the change in geometry becomes more gradual. This reduces the severity of the stress concentration until the stress peak due to the combined stresses on the rectangular cross-section coil exceeds the stress peak due to stress concentration.

Figures 52 through 57 and 58 through 63 show the effect of material thickness on the Von Mises stress distribution in general, and the position of the maximum Von Mises stress in particular. For the narrow material cut width of Figures 52 through 57, as thickness is increased the maximum stress declines but the position of the stress peak remains unchanged; the stress peak stays on the edge of the cut. Though the magnitude of Von Mises stress associated with each stress contour changes as the thickness changes, the overall pattern formed by the contours is virtually constant.
For the wider material cut shown in Figures 58 through 63, changes in material thickness have a much more pronounced effect. The magnitude of the Von Mises stress peak declines as thickness is increased but the position of the stress peak shifts as well. In Figure 58 the region of highest stress is on the interior face of the coil and a secondary stress peak exists on the edge of the cut. Figure 59 shows that with an increase in thickness the relative difference in magnitude between the primary and secondary stress regions decreases. Figures 60 and 61 show a continuation of this trend as material thickness increases. In Figures 62 and 63 the position of the stress peak has shifted to the edge of the cut and the secondary stress region is now on the interior face of the coil.

This shift in position of the primary stress peak is important because it may influence the machining and/or the heat treatment process used to make the spring. That is, different processes may be necessary in order to ensure a quality surface finish at the different primary stress areas.

**Closure**

Both the experimental planar spiral springs and the finite element models of such springs had linear force-deflection curves which means that the spring rate for these springs is a constant. A variety of thicknesses and material width cuts were analyzed and the results are presented here graphically. These graphs may be used either directly or by interpolation to determine the approximate geometry necessary for a desired spring constant.

The maximum Von Mises stresses induced in planar spiral springs are linear with respect to induced loads. The magnitude of
the stresses are dependent upon material thickness and the material cut width. The results of finite element analysis runs for a variety of thicknesses and cut widths are presented in both graphical and tabular form. The graphs or tabulated results may be used either directly or by interpolation to determine stress levels induced in planar spiral springs of various geometries.

Stress contour plots indicate the areas where stress peaks generally occur in planar spiral springs. Attention during manufacturing to ensure a quality surface finish in these general areas will improve fatigue life.
CHAPTER 6: Conclusions and Recommendations

Conclusions

1. The finite element modeling technique used in this study is acceptable for determining deflections in planar spiral springs.

2. The finite element modeling technique used in this study is also acceptable for determining Von Mises stresses in planar spiral springs.

3. Planar spiral springs are a viable design idea for application as displacer springs in free-piston Stirling engines.

4. The response characteristics of various geometries of planar spiral springs can be identified using the finite element analysis results presented.

Note that conclusions 3 and 4 satisfy the two original hypotheses of: (1) planar spiral springs are a viable design idea for applications in Stirling engines, and (2) the working envelope for a specific type of planar spiral spring can be established with reasonable confidence without an extensive test program.

Recommendations

This study does not complete the necessary analysis work to be done on planar spiral springs. This study was limited to planar spiral springs having only two coils both in the shape of involutes. Other numbers and shapes of coils remain to be studied. The effect of the length of the coil on response characteristics is still unknown. Also, for a given coil length, what effect do the radii of the starting and
Stoping points of the material cut tool path have on spring rate and stress?

Changes in material properties will have an effect on results as well. Thus, reanalysis using the modulus of elasticity and poisson's ratio of other potential spring materials is in order.

As some Stirling engines operate at frequencies of as much as 60 Hz., a modal analysis of planar spiral springs is called for in order to understand their behavior under such rapid cyclic loading.

Certainly more test work is needed as well. Test work to further substantiate the results of this and further studies and to prove the durability and reliability of planar spiral springs.
REFERENCES


APPENDIX

The planar spiral testing rig was designed and developed by Sunpower Incorporated. It included a Sunpower manufactured variable inductor type fast linear displacement transducer (FLDT) to measure displacement. An OMEGA LCL-20-20 load cell with a full active temperature compensated bridge was used to measure load. Two FLUKE 3 1/2 digit multimeters provided digital output. The signal conditioner and filtered differential amplifier were designed and built by Sunpower. See Figure 64.

Figure 64: Planar Spiral Spring Testing Rig