DYNAMICS OF CIRCULAR FOOTING ON ELASTIC FOUNDATION

A Thesis Presented to the Faculty of The College of Engineering and Technology Ohio University

In Partial Fulfillment of the Requirements for the Degree of Master of Science in Civil Engineering

by
Zheng Sheng
June, 1986
A two-parameter elastic foundation is more accurate than a one-parameter (Winkler) foundation. In this paper a circular plate resting on a single layer elastic foundation is analyzed by using Vlasov's two parameter elastic foundation model to determine its natural frequencies under an axisymmetrical loading condition.

This model is used to study the effects of various foundation parameters on the natural frequencies of the plate foundation system by the application of Bessel functions and modified wave equations.
ACKNOWLEDGMENTS

I would like to express my sincere thanks to Dr. S. M. Sargand and Dr. Y. C. Das for their constant support and guidance through the course of this research paper, without which I could not have completed my investigation in this period of time.

I also wish to express my deep appreciation to Marietta Concrete Company and Marietta Structures Corporation, for their encouragement and support in my graduate studies at Ohio University.

I would also like to thank the faculty of the Department of Civil Engineering and Mechanical Engineering. I would especially like to thank Dr. Gene Adams, for his guidance in the field of mathematics and dynamics which have provided me the essential background for this investigation and research.

Finally, most of all, the encouragement and patience of my wife Mimi has been indispensable, and has made this time of writing pleasant and enjoyable in spite of the pressures of the project.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ABSTRACT</td>
<td>iii</td>
</tr>
<tr>
<td></td>
<td>ACKNOWLEDGMENTS</td>
<td>iv</td>
</tr>
<tr>
<td></td>
<td>TABLE OF CONTENTS</td>
<td>v</td>
</tr>
<tr>
<td>I</td>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II</td>
<td>BASIC EQUATIONS</td>
<td>7</td>
</tr>
<tr>
<td>III</td>
<td>SOLUTION OF THE FUNDAMENTAL DIFFERENTIAL EQUATION</td>
<td>14</td>
</tr>
<tr>
<td>IV</td>
<td>BOUNDARY CONDITIONS</td>
<td>29</td>
</tr>
<tr>
<td>V</td>
<td>NUMERICAL RESULTS</td>
<td>36</td>
</tr>
<tr>
<td>VI</td>
<td>CONCLUSION</td>
<td>39</td>
</tr>
<tr>
<td></td>
<td>NOTATION</td>
<td>45</td>
</tr>
<tr>
<td></td>
<td>REFERENCES</td>
<td>46</td>
</tr>
</tbody>
</table>
CHAPTER I

Introduction

The analysis of beams and plates on elastic foundations has wide applications in the field of engineering. Generally, the analysis is based on the Winkler hypothesis [1] which considered only the normal interaction forces in the foundation. The foundation can be treated as if it consisted of many closely spaced linear springs. The reaction forces are linearly proportional to the plate deflection at any point. Although the Winkler hypothesis is mathematically simple, it neglects the interactions between springs and does not accurately represent the true characteristics of many foundations because of this simplification.

Two-parameter elastic foundations have been suggested to improve the Winkler hypothesis. Filonenko-Borodich [2] introduced the second foundation parameter by assuming an elastic membrane connected to the top ends of the springs. This elastic membrane is further assumed to be stretched by a constant tension force. Pasternak [3] assumed that the shear interactions exist between the springs. Vlasov and
Leont'ev [4] also considered the shear interactions in their analysis of an elastic foundation. They treated the foundation as a semi-infinite elastic medium, and formulated their elastic foundation model through the variational principle.

The Vlasov foundation model has been adopted by many authors to analyze foundation problems. T. Y. Yang [5] used Vlasov's foundation model to analyze a plate on elastic foundation by the application of the finite element method. Y. C. Das and S. M. Sargand [6] used the finite element method to determine the natural frequencies of a beam on an elastic foundation under dynamic conditions.

The dynamics of a footing on an elastic foundation is a very popular problem in the field of engineering. Vlasov and Leont'ev also solved this type of problem. They analyzed a plate with an infinite size in order to simplify the complicated boundary conditions. For most of the problems in practical engineering, the plate sizes are considerably smaller than the supporting elastic foundation and the free edge condition is the most common. In this investigation, a circular plate with its radius equal to $R$ (Figure 1) resting on a single-layer elastic foundation is
analyzed by using Vlasov's two-parameter elastic foundation model to determine its natural frequencies under an axisymmetrical condition. Based on conventional stress-strain relationships, the generalized equilibrium condition is derived by considering the virtual displacement of the foundation through the variational principle.

The general solutions in the region of the plate are represented in terms of Bessel functions. Since the plate is freely supported on the elastic foundation, the region outside the plate also needs to be considered in order to account for the shear interaction forces in the foundation. The modified wave equations are used to predict the harmonic motion of the elastic foundation. Because the region outside the plate is infinite, there is no reflected wave produced. Therefore, the wave which moves towards the plate is not considered in the analysis. For some values of foundation parameters, the solution corresponds to spatially varying but non-propagating waves. The waves in that solution which cause the deflection to be very large at locations far from the plate are also not considered.

Finally, along the edge of the plate, the continuity of the deformed surface of the elastic foundation requires
that the vertical displacement and shearing forces should have the same values from both sides of the boundary. (i.e. the vertical deflection of the plate should be identical to the vertical deflection of the elastic foundation along the edge). Since the plate is freely supported on the elastic foundation the bending moment equals zero along the plate edge. The natural frequencies of the system can be determined by solving the characteristic equations of the above boundary conditions.
Dynamics of Circular Footing
On Elastic Foundation

FIGURE 1
FIGURE 2
CHAPTER II

Basic Equations

In this chapter, starting with the basic stress-strain relationship, a variational method is used in order to develop a two-parameter elastic foundation model. The equations for the normal and shearing stresses in an elastic foundation for the three dimensional case is shown below:\[7\]

\[
\begin{align*}
\sigma_x &= 2G\varepsilon_x + \lambda e, \\
\sigma_y &= 2G\varepsilon_y + \lambda e, \\
\sigma_z &= 2G\varepsilon_z + \lambda e, \\
\tau_{xy} &= G_S\gamma_{xy} \\
\tau_{yz} &= G_S\gamma_{yz} \\
\tau_{xz} &= G_S\gamma_{xz}
\end{align*}
\]

where

\[
\begin{align*}
&\varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z \\
&\lambda = \frac{E_S}{(1+\mu_s)(1-2\mu_s)} \\
&G_S = \frac{E_s}{2(1+\mu_s)}
\end{align*}
\]

\(E_S\) and \(\mu_s\) are the modulus of elasticity and Poisson's ratio for the elastic foundation, respectively.

Two assumptions were made in order to develop Vlasov's
elastic foundation model. First, the horizontal displacements were assumed to be negligible. Second, the vertical displacement was represented by the products of two independent functions:

\[
\begin{align*}
U(x,y,z) &= 0 \\
V(x,y,z) &= 0 \\
W(x,y,z) &= W(x,y) \Phi(z)
\end{align*}
\]

where function \(\Phi(z)\) in equation (2) determines the variation with respect to the height of the vertical displacement. Function \(W(x,y)\) equals the vertical deflection of the elastic foundation surface. The following expressions can be obtained:

\[
\begin{align*}
\epsilon_x &= \frac{\partial U}{\partial x} = 0, & \epsilon_y &= \frac{\partial V}{\partial y} = 0, \\
\epsilon_z &= \frac{\partial W}{\partial z} = W(x,y) \frac{\partial \Phi(z)}{\partial z}, \\
\gamma_{xy} &= \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = 0, \\
\gamma_{xz} &= \frac{\partial V}{\partial z} + \frac{\partial W}{\partial y} = \frac{\partial W(x,y)}{\partial y} \Phi(z) \\
\gamma_{yz} &= \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} = \frac{\partial W(x,y)}{\partial x} \Phi(z)
\end{align*}
\]
Let
\[ E_s = \frac{E_0}{1-\mu_s}, \quad \mu_s = \frac{\mu_0}{1+\mu_0} \] (4)

It can be shown:
\[ \frac{E_s}{1+\mu_s} = \frac{E_0}{1+\mu_0}, \quad 1+\frac{1}{1-2\mu_s} = \frac{1}{1-\mu_0} \]

Substituting these expressions into equation (1), the following equations can be derived:

\[
\begin{align*}
\sigma_z &= \frac{E_s \varepsilon_z}{(1+\mu_s)(1+\frac{1}{1-2\mu_s})} - \frac{E_s \varepsilon_z}{(1+\mu_0)(1-\mu_0)} \\
&= \frac{E_0}{1-\mu_0} W(x,y) \frac{\partial \phi(z)}{\partial z} \\
\tau_{zy} &= \tau_{yz} = \frac{E_0 \phi(z)}{2(1+\mu_0)} \frac{\partial W(x,y)}{\partial y} \\
\tau_{zx} &= \tau_{xz} = \frac{E_0 \phi(z)}{2(1+\mu_0)} \frac{\partial W(x,y)}{\partial x}
\end{align*}
\] (5)

In the case of the vibration analysis, the elastic foundation is considered to be in the static equilibrium condition by introducing a inertia force \((-\tilde{m}_0 \frac{\partial^2 W}{\partial t^2})\) in addition to the external loading.

Now, assume there is an elementary column in the elastic foundation with its height equal to H, and sides dx and dy equal to 1. The external distributed load acting on
the surface of the elastic foundation is given by \( q(x,y,t) \).
The virtual displacements of the foundation for the condition of equilibrium is given by the equation:[4]

\[
\int_0^H \frac{\partial^2 w}{\partial x^2} \varnothing dz - \int_0^H \frac{\partial^2 w}{\partial x \partial y} \partial dz + \int_0^H \frac{\partial^2 w}{\partial t^2} \varnothing dz + m_0 \frac{\partial^2 w}{\partial t^2} \varnothing^2 dz + q = 0 \quad (6)
\]

where, \( m_0 \) equals mass per unit volume of elastic foundation, and \( H \) is thickness of elastic foundation. Substitution of equation (5) into equation (6) yields:

\[
-c \nabla^2 W(x,y,t) + k W(x,y,t) + m_0 \frac{\partial^2 W(x,y,t)}{\partial t^2} = q(x,y,t) \quad (7)
\]

where

\[
\begin{align*}
\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \\
k &= \frac{E_\theta}{1-\mu^2} \int_0^H \varnothing^2(z) \, dz \\
c &= \frac{E_\theta}{2(1+\mu) \theta} \int_0^H \varnothing^2(z) \, dz \\
m_0 &= \bar{m}_0 \int_0^H \varnothing^2(z) \, dz
\end{align*}
\]

Similarly, a plate resting on an elastic foundation can be analyzed by assuming that the plate elements are being acted upon by an inertia force \(-m_0 \frac{\partial^2 W}{\partial t^2}\) in addition to
a distributed load \( Q(x,y,t) \). The assumptions usually made in the theory of bending of thin plates will be applied to this case. For this analysis, the friction and cohesion between the plate and surface of the foundation are not considered.

The differential equation of plate vibration can thus be obtained in the following form:[10]

\[
D \frac{\partial^2 \eta}{\partial t^2} w(x,y,t) = Q(x,y,t) - \frac{m^2}{3} \frac{\partial^2 w(x,y,t)}{\partial t^2}
\]

where

- \( m \) = mass per unit plate area
- \( D \) = flexural rigidity of plate
  \[ D = \frac{Eh^3}{12(1-\mu^2)} \]
- \( h \) = thickness of the plate
- \( Q(x,y,t) = p(x,y,t) - q(x,y,t) \)
- \( p(x,y,t) \) = given external load per unit plate area

\( E \) and \( \mu \) are the modulus of elasticity and poison's ratio for the plate, respectively.

Combining equation (7) and equation (9) and eliminating the foundation reaction \( q(x,y,t) \), the following equation is obtained:
In the case of free vibration, where no external load acts, equation (10) reduces to:

\[ \nabla^2 \nabla^2 W(x, y, t) - c \nabla^2 W(x, y, t) + kW(x, y, t) = p(x, y, t) - m_1 \frac{\partial^2 W}{\partial t^2} \quad (11) \]

and equation (7), the governing equation for the region outside the plate becomes:

\[ \nabla^2 W_f(x, y, t) + kW_f(x, y, t) = -m_0 \frac{\partial^2 W_f(x, y, t)}{\partial t^2} \quad (12) \]

where \( W_f(x, y, t) \) designates the vertical displacement at the surface of the elastic foundation outside the plate edge.

For the analysis of a circular plate, it is convenient to express the governing equations in the polar coordinate system. These equations can be transformed by a coordinate transformation. Since the loading condition in this analysis is radially symmetric with respect to the origin of the polar coordinate system, the deflected plate will depend on
only the space coordinate $r$ and time $t$. Therefore, the Laplace operator $\nabla^2$, in terms of polar coordinates, becomes:

\[
\begin{align*}
\nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \\
\nabla^2 \nabla^2 &= \frac{\partial^4}{\partial r^4} + 2 \frac{\partial^3}{\partial r^3} \frac{1}{r^2} \frac{\partial^2}{\partial r^2} + \frac{1}{r^3} \frac{\partial}{\partial r} \nonumber
\end{align*}
\]

(13)
CHAPTER III

Solution of the Fundamental Differential Equation

1) The governing equation in the region of plate \((0 \leq r \leq R)\) is:

where \(R = \) radius of the plate

\[
D \frac{\partial^2 W(r,t)}{\partial r^2} - c \frac{\partial^2 W(r,t)}{\partial t^2} + k W(r,t) = -m_t \frac{\partial^2 W(r,t)}{\partial t^2}
\] (14)

let

\[
\left\{
\begin{array}{l}
2C = c/D \\
K = k/D \\
M = m_t/D
\end{array}
\right.
\] (15)

equation (14) becomes

\[
\frac{\partial^2 W(r,t)}{\partial r^2} - 2C \frac{\partial^2 W(r,t)}{\partial t^2} + k W(r,t) = -M \frac{\partial^2 W(r,t)}{\partial t^2}
\] (16)

In an attempt to separate variables in this partial differential equation and reduce the problem to the solution of two ordinary differential equations, the product shown below can be assumed as a tentative solution:

\[
W(r,t) = W(r) \exp(i \omega t)
\] (17)
where $W(r) =$ function of $r$ only
and $\exp(i\omega t) =$ function of $t$ only

substitution of eq. (17) into eq. (16) yields

$$(\nabla^2 \nabla^2 W(r) - 2C \nabla^2 W(r) + K W(r)) \exp(i\omega t) = M \omega^2 \exp(i\omega t)$$

or by eliminating the term $\exp(i\omega t)$ from both sides of the equation, the equation can be expressed as:

$$\nabla^2 \nabla^2 W(r) - 2C \nabla^2 W(r) = (M \omega^2 - K) W(r)$$  

(18)

This is a fourth order ordinary differential equation. The solution of this equation determines the axisymmetrical modes of the free vibration plate. By using the following technique, it can be further reduced to two equivalent second order differential equations.

It is assumed that the solution can be expressed as:

$$\nabla^2 W(r) = n W(r)$$  

(19)

where $n$ is a constant which needs to be determined. By applying the Laplace Operator to equation (19), it can be
shown:

\[
\begin{align*}
\nabla^2 W(r) &= nW(r) \\
\nabla^2 \nabla^2 W(r) &= n^2 W(r)
\end{align*}
\]  \hspace{1cm} (20)

Substitution of eq.(20) into eq.(18), the following equation can be obtained:

\[
n^2 W(r) - 2W(r)(M^2 - K)W(r) = 0
\]

Solving this quadratic equation, the constant n can be written as:

\[
n = \pm \sqrt{C^2 + M\omega^2 - K} + C
\]  \hspace{1cm} (21)

It is seen from eq.(20) and eq.(21) that the fourth order differential equation (18) is equivalent to the two second order differential equations.

\[
\begin{align*}
\nabla^2 W(r) + (\sqrt{C^2 + M\omega^2 - K} - C)W(r) &= 0 \\
\nabla^2 W(r) - (\sqrt{C^2 + M\omega^2 - K} + C)W(r) &= 0
\end{align*}
\]  \hspace{1cm} (22)

let \[a = \sqrt{C^2 + M\omega^2 - K} - C\] \hspace{1cm} (23)

\[
\begin{align*}
b &= \sqrt{C^2 + M\omega^2 - K} + C
\end{align*}
\]
equation (22) becomes
\[ \nabla^2 \bar{W}(r) + a \bar{W}(r) = 0 \]
\[ \nabla^2 \bar{W}(r) - b \bar{W}(r) = 0 \]

or

\[
\begin{align*}
    r^2 \frac{d^2 \bar{W}}{dr^2} + r \frac{d \bar{W}}{dr} + a r^2 \bar{W} &= 0 \\
    r^2 \frac{d^2 \bar{W}}{dr^2} + r \frac{d \bar{W}}{dr} - b r^2 \bar{W} &= 0
\end{align*}
\]

(24)

For different values of \( a \) and \( b \), different solutions of equation (24) will result.

CASE 1

\[ M \omega^2 - K > 0 \]
then \( a > 0, \quad b > 0 \)

let \( A = \sqrt{a}, \quad \hat{a} = \sqrt{b} \)

by introducing the new variables:

\[ L = Ar, \quad S = \hat{a} r \]

Equation (24) can be transformed into two zero order Bessel equations:

\[
\begin{align*}
    \frac{d^2 W}{dL^2} + \frac{1}{L} \frac{dW}{dL} + W &= 0 \\
    \frac{d^2 W}{ds^2} + \frac{1}{S} \frac{dW}{ds} - W &= 0
\end{align*}
\]

(25)
The solution of eq.(25) can be represented in the following forms:

\[
W = A_1 J_0(L) + A_2 Y_0(L)
\]

\[
W = A_3 I_0(S) + A_4 K_0(S)
\]
or

\[
W = A_1 J_0(\bar{r}) + A_2 Y_0(\bar{r})
\]

\[
W = A_3 I_0(\bar{r}) + A_4 K_0(\bar{r})
\]

where \( J_0 \) and \( Y_0 \) are Bessel functions of the first and second kind of order zero, \( I_0 \) and \( K_0 \) are modified Bessel functions of the first and second kind of order zero, respectively. From equation (26), the solution of equation (18) can be written into the final form:

\[
W(r) = A_1 J_0(\bar{r}) + A_2 Y_0(\bar{r}) + A_3 I_0(\bar{r}) + A_4 K_0(\bar{r})
\]

Since when \( r = 0 \), the functions \( Y_0 \) and \( K_0 \) approach infinity, the constants \( A_2 \) and \( A_4 \) must set to equal zero and equation (27) can thus be reduced to:

\[
W(r) = A_1 J_0(\bar{r}) + A_3 I_0(\bar{r})
\]
CASE 2

\[ M_0^2-k=0 \]

then \[ a=0, \quad b=2C \]

let \[ \hat{a}=\sqrt{2C} \]

equation (24) becomes:

\[ r^2 \frac{d^2W}{dr^2} + r \frac{dW}{dr} = 0 \] (29)

\[ r^2 \frac{d^2W}{dr^2} + r \frac{dW}{dr} - \hat{a}^2 r^2 = 0 \] (30)

equation (29) is the second order Euler-Chauchy equation.

Under the transformation:

\[ r = \exp(L) \]

or \[ L = \ln(r) \]

by a straightforward application of the chain rule, it can be shown:

\[ \frac{dW}{dr} = \frac{dW}{dL} \frac{dL}{dr} = \frac{1}{r} \frac{dW}{dL} \]

\[ \frac{d^2W}{dr^2} = -\frac{1}{r} \frac{dW}{dL} \left( \frac{1}{r} \frac{dL}{dr} \right) = \frac{1}{r^2} \left( \frac{d^2W}{dL^2} \frac{dL}{dL} \right) \]
Substituting these expressions into equation (29) will simplify the equation to:

\[
\frac{d^2W}{dL^2} - \frac{dW}{dL} + \frac{dW}{dL} = 0
\]

or

\[
\frac{d^2W}{dL^2} = 0
\]

The complete solution can then be obtained immediately:

\[
W = A_1 + A_2 L
\]

or

\[
W(r) = A_1 + A_2 \ln(r)
\]  \hspace{1cm} (31)

While the solution of equation (30) is still the Bessel function, the final solution form can be written as:

\[
W(r) = A_1 + A_2 \ln(r) + A_3 I_0(\bar{\alpha}r) + A_4 K_0(\bar{\alpha}r)
\]

Since when \( r=0 \), both functions \( \ln(r) \) and \( K_0 \) approach infinity, the constants \( A_2 \) and \( A_4 \) must be set to equal zero. Thus, the equation above can be simplified to:

\[
W(r) = A_1 + A_3 I_0(\bar{\alpha}r)
\]  \hspace{1cm} (32)
CASE 3

\[ M\omega^2 - k < 0 \]

But \[ C^2 + M\omega^2 - K > 0 \]

let

\[ A = \sqrt{-a}, \quad \dot{A} = \sqrt{b} \]

Similarly, the solution of equation (22) can be written in the following form:

\[ W(r) = A_1 I_0(\dot{A}r) + A_2 K_0(\dot{A}r) + A_3 I_0(\dot{A}r) + A_4 K_0(\dot{A}r) \]

Since when \( r=0 \), \( K_0 \) approaches infinity, again the constants \( A_2 \) and \( A_4 \) can be set equal to zero. Thus the equation above can rewritten as:

\[ W(r) = A_1 I_0(\dot{A}r) + A_3 I_0(\dot{A}r) \quad (33) \]

CASE 4

\[ C^2 + M\omega^2 - K < 0 \]

let

\[ \dot{A} = \sqrt{K - C^2 - M\omega^2} \]

substituting into equation (22), it can be shown that the following two independent second order differential
equations correspond to two conjugate complex numbers:

\[
\begin{align*}
\nabla^2 W(r) - (C - Ai) W(r) &= 0 \\
\nabla^2 W(r) - (C + Ai) W(r) &= 0
\end{align*}
\]

For the solution of the practical problems it is convenient to write the equation to the following form:

\[
\begin{align*}
\frac{r^2 d^2 W}{dr^2} + \frac{r dW}{dr} - r^2 \rho^2 (\cos \phi - i \sin \phi)^2 W &= 0 \\
\frac{r^2 d^2 W}{dr^2} + \frac{r dW}{dr} - r^2 \rho^2 (\cos \phi + i \sin \phi)^2 W &= 0
\end{align*}
\]

where

\[
\rho = (A^2 + C^2)^{1/4} \\
\phi = 1/2 \tan(A/C)
\]

Similar to case 3, the solution of the following form can be obtained:

\[
W(r) = B_1 I_\phi (r\rho(\cos \phi - i \sin \phi)) + B_2 K_\phi (r\rho(\cos \phi - i \sin \phi))
\]

\[
B_3 I_\phi (r\rho(\cos \phi + i \sin \phi)) + B_4 K_\phi (r\rho(\cos \phi + i \sin \phi))
\]

Since when \( r = 0 \), \( k_\phi \) approaches infinity, the constants \( B_2 \) and \( B_4 \) can be set equal to zero. So, the final solution becomes:
\[ W(r) = B_1 I_0(\rho r (\cos \varnothing - i \sin \varnothing)) + B_3 I_0(\rho r (\cos \varnothing + i \sin \varnothing)) \quad (37) \]

Since the function \( I_0(\rho r (\cos \varnothing + i \sin \varnothing)) \) is complex while the plate deflections are real, the constants \( B_1 \) and \( B_3 \) must be complex numbers. In order to express the solution through the real functions, the above Bessel functions can be expressed in its original series forms:

\[
W(r) = B_1 \sum_{n=0}^{\infty} (\frac{1}{2} \rho r)^2 n (\cos(2n\varnothing) + i \sin(2n\varnothing)) \left( \frac{1}{(n!)^2} \right) \\
+ B_3 \sum_{n=0}^{\infty} (\frac{1}{2} \rho r)^2 n (\cos(2n\varnothing) - i \sin(2n\varnothing)) \left( \frac{1}{(n!)^2} \right)
\]

let

\[
A_1 = \frac{1}{2}(B_1 + B_3) \\
A_3 = \frac{1}{2}(B_1 - B_3)i
\]

then by substitution of the equations above:

\[
W(r) = A_1 \sum_{n=0}^{\infty} (\frac{1}{2} \rho r)^2 n \cos(2n\varnothing) \frac{1}{(n!)^2} + A_3 \sum_{n=0}^{\infty} (\frac{1}{2} \rho r)^2 n \sin(2n\varnothing) \frac{1}{(n!)^2} \quad (38)
\]

2) The governing equation in the region outside the plate \((r > R)\) is:

\[
-c \nabla^2 W_f + kW_f = -m_0 \frac{\partial^2 W_f}{\partial t^2} \quad \text{or} \quad c \frac{\partial^2 W_f}{\partial r^2} - \frac{c}{r} \frac{\partial W_f}{\partial r} + kW_f = -m_0 \frac{\partial^2 W_f}{\partial t^2} \quad (39)
\]
\[
\int_{0}^{2\pi} iy \sin \phi \exp(iyr \sin \phi) \, d\phi
\]
\[
= \int_{0}^{2\pi} -iy \exp(iyr \sin \phi) \, d(\cos \phi)
\]
\[
= \int_{0}^{2\pi} -y \cos \phi \exp(iyr \sin \phi) \, d\phi
\]
\[
= -\int_{0}^{2\pi} y^2 \cos^2 \phi \exp(iyr \sin \phi) \, d\phi
\]

Substituting of these expressions into eq. (39), yields:

\[
\frac{c}{2\pi} \int_{0}^{2\pi} y^2 \sin^2 \phi \exp(iyr \sin \phi) \, d\phi - \frac{c}{2\pi} \int_{0}^{2\pi} iyr \sin \phi \exp(iyr \sin \phi) \, d\phi
\]
\[
+ \frac{k}{2\pi} \int_{0}^{2\pi} \exp(iyr \sin \phi) \, d\phi - \frac{m_0 \omega^2}{2\pi} \int_{0}^{2\pi} \exp(iyr \sin \phi) \, d\phi = 0
\]
\[
\frac{c}{2\pi} \int_{0}^{2\pi} y^2 \sin^2 \phi \exp(iyr \sin \phi) \, d\phi + \frac{c}{2\pi} \int_{0}^{2\pi} y^2 \cos^2 \exp(iyr \sin \phi) \, d\phi
\]
\[
+ \frac{k}{2\pi} \int_{0}^{2\pi} \exp(iyr \sin \phi) \, d\phi - \frac{m_0 \omega^2}{2\pi} \int_{0}^{2\pi} \exp(iyr \sin \phi) \, d\phi = 0
\]

or

\[
y^2 = \frac{m_0 \omega^2 - k}{c}
\]

(41)

CASE A

\[
y^2 > 0, \quad y = \pm \left(\frac{m_0 \omega^2 - k}{c}\right)^{1/2}
\]

\[
W_I = A \frac{1}{2\pi} \int_{0}^{2\pi} \exp(iyr \sin \phi + i\omega t) \, d\phi + A \frac{1}{2\pi} \int_{0}^{2\pi} \exp(-iyr \sin \phi + i\omega t) \, d\phi
\]

Two real roots yields inwards and outwards propagating waves depending which sign is selected. Since in this analysis,
Because of normal and shearing interaction forces in the foundation, the simple wave solution, propagating without distortion is not likely satisfied in this solution. It is logical to expect some type of pulse distortion to occur in a system governed by equation (39). A first step in assessing this effect is to determine the necessary conditions for the propagation of harmonic waves. Since we have used Bessel functions in the region inside the plate, it is logical to assume the solutions will be expressed in a form similar to the Bessel functions. Thus, the question remaining is, under what condition will the solution shown below satisfy equation (39):

\[
W_f = \frac{1}{2\pi} \int_{0}^{2\pi} \exp(iyr \sin \omega + i\omega t) \, d\omega
\]  

(40)

where \( y \) is a constant which needs to be determined. The derivatives of the solution can be expressed as:

\[
\frac{\partial W_f}{\partial r} = \frac{1}{2\pi} \int_{0}^{2\pi} iy \sin \omega \exp(iyr \sin \omega + i\omega t) \, d\omega
\]

\[
\frac{\partial^2 W_f}{\partial r^2} = \frac{1}{2\pi} \int_{0}^{2\pi} -y^2 \sin^2 \omega \exp(iyr \sin \omega + i\omega t) \, d\omega
\]

\[
\frac{\partial^2 W_f}{\partial t^2} = -\omega^2 \frac{1}{2\pi} \int_{0}^{2\pi} \exp(iyr \sin \omega + i\omega t) \, d\omega
\]

Also the following terms can be simplified to:
the region outside the plate is infinity, there is no reflecting wave produced, therefore, the wave moving towards the plate is not considered in the solution. Thus $A_3=0$

$$W_f = A \frac{1}{2 \pi} \int_0^{2\pi} \exp(-iyr \sin \omega + i \omega t) \, d\omega$$

The function $\int_0^{2\pi} \exp(-iyr \sin \omega) \, d\omega$ can be further simplified to another form by the following method.

Let $\theta = 2\pi - \omega$.

$$d\omega = -d\theta, \quad \text{and} \quad \sin \omega = -\sin \theta$$

$$\int_0^{2\pi} \exp(-iyr \sin \omega) \, d\omega = \int_0^\pi \exp(-iyr \sin \omega) \, d\omega + \int_\pi^{2\pi} \exp(-iyr \sin \omega) \, d\omega$$

$$= \int_0^\pi \exp(-iyr \sin \omega) \, d\omega + \int_0^\pi -\exp(iyr \sin \theta) \, d\theta$$

$$= \int_0^\pi \exp(-iyr \sin \omega) \, d\omega + \int_0^\pi \exp(iyr \sin \theta) \, d\theta$$

By changing the dummy variable $\theta$ back to $\omega$ it can be shown:

$$\int_0^\pi \exp(-iyr \sin \omega) \, d\omega + \int_0^\pi \exp(iyr \sin \omega) \, d\omega$$

$$= \int_0^\pi 2 \cos(yr \sin \omega) \, d\omega$$
Thus, the final solution becomes:

\[
W_f = A_6 \frac{1}{2\pi} \int_0^\pi \cos(yr\sin\phi) \exp(i\omega t) \, d\phi
\]

(CASE B)

\[y^2 < 0\]

let \[Y = (k-m_0\omega^2)\frac{1}{c},\]

\[y = \pm iY\]

\[
W_f = B_5 \frac{1}{2\pi} \int_0^{2\pi} \exp(Yr\sin\phi + i\omega t) \, d\phi + B_6 \frac{1}{2\pi} \int_0^{2\pi} \exp(-Yr\sin\phi + i\omega t) \, d\phi
\]

Since \(\int_0^{2\pi} \exp(-Yr\sin\phi) \, d\phi\)

\[
= \int_0^\pi \exp(-Yr\sin\phi) \, d\phi + \int_\pi^{2\pi} \exp(-Yr\sin\phi) \, d\phi
\]

\[
= \int_0^\pi \exp(-Yr\sin\phi) \, d\phi + \int_0^\pi \exp(Yr\sin\phi) \, d\phi
\]

By using this relationship, the equation can be arranged in a different way, and new integration constants can be assigned.

\[
W_f = A_5 \frac{1}{2\pi} \int_0^\pi \exp(Yr\sin\phi + i\omega t) \, d\phi + A_6 \frac{1}{2\pi} \int_0^\pi \exp(-Yr\sin\phi + i\omega t) \, d\phi
\]

Since when \(r\) becomes very large, the function \(\int_0^{2\pi} \exp(Yr\sin\phi) \, d\phi\) approaches infinity and \(A_5\) must equal zero. So the equation can be expressed as:
\[ \dot{W}_f = A_6 \frac{1}{2\pi} \int_0^\pi \exp(-Y\sin \phi + i\omega t) \, d\phi \]  

(43)

This solution corresponds to a spatially varying but non-propagating disturbance of wave motion.

**CASE C**

Finally, the special case of \( y^2 = 0 \) needs to be considered. The waves in this case represent the transition from propagation to non-propagation.

Since \( m_0 \omega^2 - k = 0 \)

The final solution of equation (40) becomes

\[ \dot{W}_f = A_6 \exp(i\omega t) \]  

(44)
CHAPTER IV

Boundary Conditions

Since the load acting on the circular plate is symmetrically distributed about the axis perpendicular to the plate through its center, the deflection and bending moment along the same radius are constant. The variations between angles are not considered. The equations for the general radial moment and shearing force for the plate are expressed in the following forms:[4]

\[ M_r = -D \left( \frac{d^2W}{dr^2} + \frac{1}{r} \frac{dW}{dr} \right) \]

(45)

\[ Q_r = -D \frac{d}{dr} \left( \frac{d^2W}{dr^2} + \frac{1}{r} \frac{dW}{dr} \right) + c \frac{dW}{dr} \]

(46)

where, \( c \frac{dW}{dr} \) represents the shearing forces from the elastic foundation.

Similarly, the generalized shearing force \( Q_f \) in the foundation beyond the plate edge ( \( r > R \) ) is defined by [4]

\[ Q_f = c \frac{dW_f}{dr} \]

(47)

Three basic boundary conditions (at \( r = R \)) can be
drawn in order to find the natural frequencies of the system.

\[ M_r(R) = 0 \]  
\[ W(R) = W_f(R) \]  
\[ Q_r(R) = Q_f(R) \]

where \( W \) and \( W_f \) are the vertical displacements of plate and surface of elastic foundation at \( r < R \) and \( r > R \) respectively. Equation (48) indicates the bending moments equal zero along the edge of the plate. Equations (49) and (50) describe the continuity requirement at the surface of the elastic foundation. In order to determine the natural frequencies of the plate-foundation system, these three basic boundary conditions will be applied to all cases previously discussed.

Before trying to find the final results of all these conditions, the derivatives of the following Bessel functions must be determined.

\[ J_0'(r) = -J_1(r) \]
\[ J_0'(ar) = -aJ_1(ar) \]
\[ J_0''(ar) = -a^2(1/2 J_0(ar) - 1/2 J_2(ar)) \]
\[ J''_0(ar) = -1/2 a^3(-1.5J_1(ar) + 1/2 J_3(ar)) \]
Now, all these boundary conditions can be applied to different cases and their combinations.

**CASE 1 and CASE A**

\[ M \omega^2 - K > 0 \quad \text{and} \quad m_0 \omega^2 - k > 0 \]

\[ W(R) = W_f(R) \]

Using equation (28) and equation (42), it follows:

\[ A_1 J_0(\alpha R) + A_3 I_0(\alpha R) = A \frac{1}{2\pi} \int_0^{\pi} \cos(yr \sin \phi) d\phi \tag{52} \]

\[ M_f(R) = 0 \]

\[ A_1 D(-J_0'\alpha R - \frac{h}{r} J_0(\alpha R)) + A_3 D(-I_0'\alpha R - \frac{h}{r} I_0(\alpha R)) = 0 \tag{53} \]

Combining equation (46) together with equations (28) and (42), the following expressions result:
\[ Q_r = -D \frac{d}{dr} \left( \frac{d^2 W}{dr^2} + \frac{1}{r} \frac{dW}{dr} \right) + c \frac{dW}{dr} \]  
\[ = -D \frac{d^3 W}{dr^3} + \frac{D}{r^2} \frac{dW}{dr} - \frac{D}{r} \frac{d^2 W}{dr^2} + c \frac{dW}{dr} \]  
\[ Q_r(0) - Q_f(0) = 0 \]  
\[ A_1 D(-J_0'(AR) - \frac{1}{R} J_0'(AR) + \frac{1}{R^2} + \frac{C}{D}) J_0'(AR) + \frac{1}{R} I_0'(AR) + \frac{1}{R^2} + \frac{C}{D} I_0'(AR) \]

\[ + A_3 D(-I_0'(AR) - \frac{1}{R} I_0'(AR) + \frac{1}{R^2} + \frac{C}{D} I_0'(AR)) \]

\[ + A_6 \frac{1}{2 \pi} \int_0^\pi (-y \sin \phi \sin(yR \sin \phi)) d\phi = 0 \]

Finally, in order to find the non-trivial solution of the above system of equations, the characteristic values of the equations must be determined. These characteristic values can be found by setting the determinant of the equations equal to zero, and the corresponding values of \( \omega \) is the natural frequency of the system.

\[
\begin{vmatrix}
J_0'(AR) & I_0'(AR) & -\frac{1}{2 \pi} \int_0^\pi \cos(yR \sin \phi) d\phi \\
D(-J_0'(AR) - \frac{1}{R} J_0'(AR)) & D(-I_0'(AR) - \frac{1}{R} I_0'(AR)) & 0 \\
X_1, & X_2, & X_3 \\
\end{vmatrix} = 0
\]
where

\[ X_1 = D(-J'^\prime (\dot{\alpha} R) - \frac{1}{R} J^\prime (\dot{\alpha} R) + \left( \frac{1}{R^2} + \frac{C}{D} \right) J (\dot{\alpha} R)) \]

\[ X_2 = D(-I'^\prime (\dot{\alpha} R) - \frac{1}{R} I^\prime (\dot{\alpha} R) + \left( \frac{1}{R^2} + \frac{C}{D} \right) I (\dot{\alpha} R)) \]

\[ X_3 = \frac{1}{2\pi} \int_0^\pi (y \sin \phi \sin(y R \sin \phi)) \, d\phi \]

Similarly, equations (48), (49), and (50) can be applied to each individual case. For example, the combinations of case 1, 2, 3, and 4 with cases A, B, and C give a total of twelve possibilities. The following equations are the results from all the different cases after applying the boundary conditions to them.

**CASE 2**

\[ M \omega^2 - K = 0 \]

\[ W(R) = A_1 + A_3 I_0 (\dot{\alpha} R) \]  \hspace{1cm} (32)

\[ M_r(R) = A_3 D(-I^\prime (\dot{\alpha} R) - \frac{1}{R} I (\dot{\alpha} R)) = 0 \]  \hspace{1cm} (54)

\[ Q_0(R) = A_3 D(-I^\prime (\dot{\alpha} R) - \frac{1}{R} I^\prime (\dot{\alpha} R) + \left( \frac{1}{R^2} + \frac{C}{D} \right) I (\dot{\alpha} R)) \]  \hspace{1cm} (55)

**CASE 3**

\[ M \omega^2 - K < 0 \]

but \[ C^2 + M \omega^2 - K > 0 \]
\[ W(R) = A_1 I_0(\rho R) + A_3 I_0(\rho R) \quad (33) \]

\[ M_R(R) = 0 \]
\[ M_R(R) = A_1 D(-I''_0(\rho R) - \frac{1}{R} I'_0(\rho R)) + A_3 D(-I''_0(\rho R) - \frac{1}{R} I'_0(\rho R)) = 0 \quad (56) \]
\[ Q(R) = A_1 D(-I''(\rho R) - \frac{1}{R} I'_0(\rho R)) + (\frac{1}{R^2 + C^2}) I_0(\rho R) \]
\[ + A_3 D(-I''(\rho R) - \frac{1}{R} I'_0(\rho R)) + (\frac{1}{R^2 + C^2}) I_0(\rho R) \]

**CASE 4**

\[ M \omega^2 - K + C^2 < 0 \]
\[ W(R) = \sum_{n=0}^{\infty} (\frac{1}{2} \rho R)^2 n(2n-1)(\frac{1}{2} \rho R)^{2n-1} \cos(2n\omega) \frac{1}{(n!)^2} \]
\[ M_R(R) = 0 \]
\[ M_R(R) = A_1 D \sum_{n=0}^{\infty} (\frac{1}{2} \rho R)^2 n(2n-1)(\frac{1}{2} \rho R)^{2n-1} \cos(2n\omega) \frac{1}{(n!)^2} \]
\[ + A_3 D \sum_{n=0}^{\infty} (\frac{1}{2} \rho R)^2 n(2n-1)(\frac{1}{2} \rho R)^{2n-1} \sin(2n\omega) \frac{1}{(n!)^2} \quad (58) \]
\[ Q(R) = A_1 D \sum_{n=0}^{\infty} (\frac{1}{2} \rho R)^3 n(2n-1)(2n-2) \frac{1}{2} \rho R)^{2n-3} + (\frac{1}{R^2 + C^2}) \rho n(\frac{1}{2} \rho R)^{2n-1} \]
\[ - \frac{1}{2R} \rho^2 n(2n-1)(\frac{1}{2} \rho R)^{2n-2} \cos(2n\omega) \frac{1}{(n!)^2} \]
\[ + A_3 D \sum_{n=0}^{\infty} (\frac{1}{2} \rho R)^3 n(2n-1)(2n-2) \frac{1}{2} \rho R)^{2n-3} + (\frac{1}{R^2 + C^2}) \rho n(\frac{1}{2} \rho R)^{2n-1} \]
\[ - \frac{1}{2R} \rho^2 n(2n-1)(\frac{1}{2} \rho R)^{2n-2} \sin(2n\omega) \frac{1}{(n!)^2} \quad (59) \]
CASE A

\[ y^2 > 0 \]

\[ W_f(R) = A_6 \frac{1}{2\pi} \int_0^\pi \cos(yR\sin\phi) \, d\phi \]  
\[ Q_f(R) = A_6 \frac{1}{2\pi} \int_0^\pi (-y\sin\phi \sin(yR\sin\phi)) \, d\phi \]

CASE B

\[ y^2 < 0 \]

\[ W_f(R) = A_6 \frac{1}{2\pi} \int_0^\pi \exp(-yR\sin\phi) \, d\phi \]  
\[ Q_f(R) = A_6 \frac{1}{2\pi} \int_0^\pi -y\sin\phi \exp(-yR\sin\phi) \, d\phi \]

CASE C

\[ y^2 = 0 \]

\[ W_f(R) = A_6 \]  
\[ Q_f(R) = 0 \]
Numerical Results

To calculate the values of foundation parameter in equation (8), the value of the function $\phi(z)$ must be selected in order to account for the variation with the height of the elastic foundation. Vlasov and Leon'tev suggested selecting the following expression for $\phi(z)$:[4]

$$\phi(z) = \frac{\sinh \frac{g(H-z)}{R}}{\sinh \frac{gH}{R}}$$

where $R$ is the radius of plate in this analysis and $g$ is a constant determining the rate of the displacement with respect to the depth. It can be shown that the value of $\phi(0)$ equals one and as $z$ increases, $\phi(z)$ decreases at a rate best defined mathematically by a hyperbolic function.

Substituting $\phi(z)$ into equation (8), the following equations can be obtained: 4

$$k = \frac{E_0}{1 - \mu^2} \int_0^H \phi'(z) \, dz$$

$$= \frac{E_0}{2R(1 - \mu^2)} \frac{\sinh(gh/R) \cosh(gh/R) + gh/R}{\sinh^2 gh/R}$$
\[ c = \frac{E_0}{2(1+\mu_0)} \int_0^H \Omega^2(z) \, dz \]

\[ m_0 = \hat{m}_0 \int_0^H \Omega^2(z) \, dZ \]

As a numerical example, a circular plate with Young's modulus \( E \) equal to 2.5 \( \times \) 10\(^{10}\) pa. and mass density equal to 2,405 kg/cu. m. is considered. The size of plate varies, in order to find the effects due to the radius of plate. For various foundations, their Young's modulus and mass density are varied. Also, in the analysis, the foundation parameter \( C \) and \( K \), regardless of their physical meaning, are varied independently with other foundation parameter values in order to determine their direct effects on the natural frequencies of the plate foundation system. In all of the computations, a value of 1.5 is assigned to the decay rate \( g \).

A computer program was written in order to determine the natural frequencies of the system. Based on the analysis discussed above, the system of equations' characteristic
value was found by using iterative methods. The changes of the signs of the determinant values indicated the characteristic values of the equations.

It can be seen from (figure 3) that the natural frequencies of the plate foundation system is affected by the second elastic foundation parameter C (shear deformation). The ratio of frequencies $\omega_1 / \omega_0$ equals 1.08 and $\omega_2 / \omega_0$ equals 1.35 were found, where $\omega_0$ is the natural frequency of a plate without the effects of shear deformation.

Figure 4 represents the effects of the first foundation parameter K (normal deformation). Figure 5 shows the effects of foundation density on the natural frequencies, in which the frequencies decrease with the increase of foundation density. Figure 6 shows that the natural frequencies decrease with the increase of the plate size. Finally, from Figure 7, it can be observed that effects of an increase in Young's modulus for the foundation results in an increase in the natural frequencies.
CHAPTER VI

Conclusion

The analysis shown above indicated that the dynamics of a circular footing on an elastic foundation can be solved by introducing the propagating wave equations. The directions of the wave motion can be used as an additional boundary condition. The solutions obtained from the computer analysis were given reasonable results.

The application of the axisymmetrical natural frequency of a circular footing on an elastic foundation can be used in many areas, especially, in some places where heavily dynamic loadings are involved. The design criteria in these areas are usually very conservatively based on the static analysis. The dynamic analysis provides the designer with a better understanding of the structure's behavior, thus resulting in an improved design.

Since the above investigation is based only on a mathematical solution, further studies including extensive experimental research is recommended.
Figure 3. Effects of foundation parameter (shear interaction) on frequency

h=0.5 meter    R=10.0 meter
E=2.5x10^{10} pa.    D=2.173x10^8 N-M
\mu=0.2    H=40.0 meter
E_s=1.8x10^7 pa.
Figure 4. Effects of foundation parameter (normal interaction) on frequency
Figure 5. Effects of foundation density on frequency

\[ m_0 = \frac{m_0}{D} = \text{kg/sq.M} \times \frac{1}{\text{N.M}} \times 10^{-6} \]

- \( D = 2.173 \times 10^8 \text{ N-M} \)
- \( \mu = 0.2 \)
- \( H = 40.0 \text{ meter} \)
- \( E_s = 1.8 \times 10^7 \text{ pa.} \)
- \( R = 10.0 \text{ meter} \)
- \( h = 0.5 \text{ meter} \)
- \( E = 2.5 \times 10^{10} \text{ pa.} \)
Figure 6. Effects of plate size on frequency

h=0.5 meter    H=40.0 meter
E=2.5x10^{10}  \text{pa.} \quad \mu=0.2
D=2.173x8 \text{N-M}
E_s=1.8x7 \text{pa.}
Figure 7. Effects of foundation modulus on frequency.
**NOTATIONS**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>radius of circular plate</td>
</tr>
<tr>
<td>$h$</td>
<td>thickness of plate</td>
</tr>
<tr>
<td>$D$</td>
<td>flexural rigidity of plate</td>
</tr>
<tr>
<td>$E$</td>
<td>modulus of elasticity of plate</td>
</tr>
<tr>
<td>$E_s$</td>
<td>modulus of elasticity of foundation</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Poisson's ratio of plate</td>
</tr>
<tr>
<td>$\mu_s$</td>
<td>Poisson's ratio of foundation</td>
</tr>
<tr>
<td>$H$</td>
<td>thickness of foundation</td>
</tr>
<tr>
<td>$m$</td>
<td>mass per unit plate area</td>
</tr>
<tr>
<td>$\tilde{m}_0$</td>
<td>mass per unit volume of foundation</td>
</tr>
<tr>
<td>$W(r)$</td>
<td>vertical displacement of plate</td>
</tr>
<tr>
<td>$W_f(r)$</td>
<td>vertical displacement of foundation surface beyond plate edge</td>
</tr>
<tr>
<td>$O(z)$</td>
<td>function of transverse distribution of displacement</td>
</tr>
<tr>
<td>$g$</td>
<td>decay rate of foundation</td>
</tr>
<tr>
<td>$q$</td>
<td>distributed load on surface of elastic foundation</td>
</tr>
<tr>
<td>$Q$</td>
<td>distributed load on plate</td>
</tr>
<tr>
<td>$k$</td>
<td>first elastic foundation parameter, N/Cu.M</td>
</tr>
<tr>
<td>$c$</td>
<td>second elastic foundation parameter, N/M</td>
</tr>
<tr>
<td>$K$</td>
<td>$k/D$</td>
</tr>
<tr>
<td>$C$</td>
<td>$c/2D$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>natural frequency of the plate foundation system</td>
</tr>
</tbody>
</table>
REFERENCES

1. Winkler, E., die Lehre von der Elastizitat und Festigkeite, Prague, Dominicus, 1867


