A DECISION-DIRECTED-DETECTION SCHEME
FOR PCM SYSTEMS IN A NOISY ENVIRONMENT

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1.1 Background

Improving the detection of digital transmission in the presence of noise has commanded considerable attention in recent years. A large number of schemes has been proposed and studied to achieve lower error performance in noisy environments. Prominent among these are the various pulse modulation schemes such as the pulse code modulation (PCM) family. Analysis of pulse code modulation system by Bloom, Chang, Harris, Hauptschein and Morgan [1] has indicated that a knowledge of binary transmission systems is necessary for any study of detection schemes, particularly when the level of interference becomes appreciable. Most data sources, in particular pictorial data, are highly redundant. Shannon's definition of redundancy is as follows:

"that fraction of a message which is unnecessary and hence repetitive in the sense that if it were omitted, the message would still be essentially complete or it could be completed."

Stated another way, redundancy exists whenever the sampling rate exceeds the frequency required to describe the input data in accordance with the fidelity requirements.

Pulse code modulation (PCM) schemes for detection improvement will be divided into two main categories. One is the error-correction coding scheme. This scheme consists of linear and nonlinear coding
schemes. Linear codes are weighted codes which include Hamming Codes, Hadamard Codes, Golay Codes, Reed-Muller Codes, convolutional codes and the subset of linear codes known as cyclic codes. Cyclic codes are the most important of the linear codes. They include Cyclic Hamming Codes, Cyclic (23,12) Golay Codes and Bose-Chaudhuri-Hocquenghem (BCH) Codes. The weight distributions of many linear codes were discussed by MacWilliams [2]. Nonlinear codes have not attracted much attention due to the interesting and successful results obtained in using linear codes. Nevertheless, there are a few nonlinear codes which include nongroup close-packed codes [3], optimum nonlinear codes [4], linear and nonlinear single-error-correcting q-nary perfect codes [5], and optimum nonlinear double-error-correcting codes [6].

Error-correcting coding schemes require exactly \( n \) binary digits in each code symbol, where \( m \) digits are associated with the information while the other \( k = n-m \) digits are used for the correction of errors. The \( k = n-m \) extra digits introduce redundancy. Redundancy serves as a measure of the efficiency of coding schemes as far as the transmission of information is concerned. The potential increases in system performance due to the error-correcting coding schemes are at the expense of increased bandwidth and lowered effective channel capacity for sending information. Clark and Cain [7] indicate a threshold phenomenon which exists at sufficiently low signal-to-noise ratio at which a code loses its effectiveness and actually makes the situation
worse. This thresholding phenomenon is common to all coding schemes. It is apparent that with the need for error correction, extra equipment for encoding and correcting errors will be required. Because of these considerations, applications of these coding schemes may be expected to occur in selected applications. Hamming [8] and Meggitt [9] indicate that some of these applications could be:

a. Unattended operation over long periods of time with the minimum of standby equipment.

b. Extremely large and tightly interrelated systems where a single failure incapacitates the entire installation.

c. Signaling in the presence of noise where it is either impossible or uneconomical to reduce the effect of the noise on the signal.

These situations are occurring more frequently as systems become complex. The first two are particularly true of large scale digital computing machines, while the third occurs among other places, in "jamming" situations. Due to the large amount of literature on the system performance using these coding schemes, the investigation reported in this paper will be directed towards system performance under the second category of detection improvement.

The second category is new Decision-Directed-Detection (DDD) schemes. It optimizes and improves system performance using schemes
other than the coding schemes. Optimum detection schemes consist of an optimization criteria and threshold-signal strength comparison schemes. Optimization criteria described in Couch and Ziemer and Tranter [10-11] include Bayes' Criterion, Minimax Criterion, Neyman-Pearson Criterion, and Maximum a Posteriori Criterion. They use a priori knowledge about a communication system to decide which of the many possible messages present was sent. Some criteria require more a priori knowledge than others. The criteria that use more a priori knowledge yield receivers with better performance provided that the source and channel actually have the specifications that are the design a priori conditions.

Threshold schemes compare received signal out of the processor to predetermined threshold values in order to decide which of the two possible binary signals present was sent. Most of the work documented in this area has been performed using standard pulse code modulation with a single optimum threshold value and nothing has been done to characterize system performance under double threshold schemes. That characterization is the emphasis of this work. Design procedures for the system and various trade-off studies will be an important contribution.

1.2 State of The Art

The simplified block diagram of the system to be considered in this investigation is illustrated in Figure 1-1. The analog waveform
Figure 1 - 1. Basic block diagram of the system.
X(t) representing the message is sampled and the resulting signal samples $X_i$ are quantized to one of the predetermined allowable levels in a quantizer. If binary encoding is assumed, each quantized sample $Q(X_i)$ is encoded into an $l$ bit binary code word. These bits are then encoded for transmission over the channel.

At the receiving end, optimum detection requires that precise knowledge of the bit transition time be known to the receiver before bit-by-bit detection is implemented. Since the received signal is delayed and attenuated by the transmission medium, synchronization is required in practical systems. In this study, perfect bit synchronization is assumed. Using this assumption, the received signal is processed to obtain an analog baseband signal which is sampled at the end of each bit interval. The decoding and signal reconstruction processes are then accomplished by comparing the sampled baseband signal against two predetermined symmetrical threshold values in order to make a decision as to which bit is received. When the sampled baseband signal exceeds one of the thresholds or falls below the other, the receiver can distinguish between the two possible states of a transmitted digit with high accuracy. When the sampled baseband signal falls between the two thresholds, the probability distribution of the bit position being a binary "0" or a binary "1" which is stored as a table look-up, is called upon to assist in making the distinction. The look-up table is generated by calculating the cumulative probability distribution of each bit position in the next coded
samples \( a_{i+1} \) being a binary 0 or a binary 1, using the predetermined conditional probability density function of the next sample \( X_{i+1} \) given that the present sample \( X_i \) belongs to a specified word. The statistical properties of a random noise that inevitably accompany the pulse transmission are such that there is always a finite and calculable probability that a noise peak will reach a sufficient amplitude to cause the receiver to respond incorrectly. Thus, binary errors occur at the receiver because bits are regenerated that have opposite level to those transmitted, that is, a logical "1" is regenerated as a logical "0" or vice versa. In this study, the decision making capability of the communication receiver is analyzed to determine system performance.

The analysis of the system performance is accomplished by simulation of a communication system with double symmetrical threshold. The simulation is described in Chapter 3 and the results from the simulation are given in Chapter 4.

1.3 Error

The evaluation of the overall system performance is in terms of bit error rate (BER) and mean-squared error (MSE). Smith [12] indicated that data redundancy could be used effectively for reducing statistical measure of error amplitude such as mean-squared error than for reducing the probability of error (bit error rate). This is due to the fact that data redundancy is a statistical constraint upon the
relative amplitudes of the data samples. Using mean-squared error criterion, the total error can be separated into sampling error, quantization error, and channel error. These three sources of error are additive if quantization is done in optimum manner as indicated by Totty and Clark [13].

1.4 Sampling Error

In digital communication, errors due to sampling and reconstruction are unavoidable. A considerable amount of work has been published in literature on time sampling and reconstruction. Analysis of errors in sampling bandlimited functions was performed by Papoulis [14]. Gardenhire [15] presented a method for selecting optimum sampling rates for communication systems using real data and simple realistic reconstruction filters. Liff [16] renders some interesting simple expressions for mean-squared error caused by sampling and reconstructing stationary random processes. Various classes of linear, time-invariant reconstruction filters were considered. Helms and Thomas [17] and Yao and Thomas [18] determine some upper bounds for truncation error in sampling of bandlimited functions.

1.5 Quantization and Channel Error

The reduction of quantization error has received considerable attention in the past. The minimization of mean-squared quantization error by the selection of quantizer levels results in optimum quan-
tization. Consequently, non-uniform quantizer tailored to signal statistics and/or the channel generally results. An expression for minimum mean-squared quantization error in terms of the input signal statistics was derived by Spilker [19]. Max [20] considered quantizing for minimum distortion for a signal of known probability density. By assuming the number of levels to be fixed, he derived expressions for the optimum levels and an algorithm to simplify their numerical computation. The solution of the optimum quantizing levels generally requires a considerable amount of digital computation. Using Max's results, Roe [21] derived excellent approximate expressions of practical interest. Wood [22] indicated that equi-level (uniform) quantizing yields nearly optimum results if the number of quantizing levels is large and the bit rate is fixed (buffering).

In order to transmit data in pulse code modulation mode over a noisy channel, some bit errors must be tolerated. Nuttal [23] determines the performance of communication systems in terms of bit error rate employing equi-correlated M-ary signals subjected to additive Gaussian noise for both phase-coherent and phase-incoherent reception. System performance in terms of error probability in digital data transmission systems with correlated symbols in the presence of intersymbol interference and additive noise was evaluated by Cariolaro and Pupolin [24]. Ransom and Gupta [25] describe the problems of bit detection without synchronization for signals corrupted by additive and multiplicative noise in a discrete receiver structure. Viterbi
\[26\] determines lower bounds on signal-to-noise ratio over a Gaussian channel for both coherent and non-coherent systems. Steiglitz \[27\] considers the case of transmission of a non-bandlimited analog signal over a digital channel with a fixed bit rate. Goblick \[28\] rendered some useful theoretical limitations on the transmission of data without constraining analysis to a particular communication system. Clark and Totty \[29\] have examined the contribution of channel error to the overall system performance using mean-squared error criterion. Some specific results using single error-correcting Hamming codes are also given. References \[30\] - \[38\] are additional representative, but not exhaustive investigations on various aspects of the overall problem.

As one can see from the above considerations, there exists a great amount of literature on particular aspects of the overall problem. These literatures indicate that the minimization of the overall system error and consequent efficient use of a communication system require consideration of all the integral parts of the system error under various constraints like signal-to-noise ratio, bandwidth or bit rate, power limitations and total number of bits.

1.6 Problem Statement

The research considered here is directed towards improving the detection of a transmission in the presence of noisy environment. This is accomplished by reducing the effects of channel noise on
transmission utilizing the redundancy in source data and double symmetrical variable threshold decision making (bit-by-bit detection). This necessitates the investigation of all aspects of system performance which includes sampling, quantization, and channel errors.

The investigation assumes that the statistical characteristics of the signals are known. Consequently, the input data are assumed to be from a first order Markov process with correlation function \( R(\tau) = \exp(-|\tau|) \). This process is a representative of a large number of real signals. For example, Habibi and Wintz [39] reported that video signals can be modelled by this type of source. This study provides important results which give design procedures based on easily obtainable parameters of the system such as input signal statistics, quantizer limits, and number of bits. A considerable amount of simulation work was performed to verify the theoretical results.
CHAPTER 2

SYSTEM THEORY

In this chapter, the theory involved in this work is described. It begins with modelling the input signal as Markov Gaussian process. Once the modelling is completed, the conditional probability density distribution of next sample $X_{i+1}$ given the range of the present sample $X_i$ is developed. This is useful because of the Markov process implication of the input data. The signal samples are then quantized and coded. The coding process is followed by the generation of a look-up table of probability distribution for each bit position in the next coded sample ($a_{i+1}^j$) being a binary 0 or a binary 1. This is accomplished by integrating the predetermined conditional probability density function between the limits occupied by each bit position in question. Note that the look-up table is developed for a given quantizer level and the replica of this table is stored in the receiver to help in decision making. The bits are transmitted by pulses and received with additive random channel noise. The received signal is processed by a correlation detector. The resultant analog baseband signal is sampled at bit rate interval and the decision as to which bit is received is implemented using a double symmetrical variable threshold device (a comparator). In the next chapter, the decision making capability of the receiver is analyzed and the system performance is determined by theory and simulation.
2.1 Signal Characteristics

A large number of natural phenomena which evolve in time have nondeterministic components. Such phenomena can be described in terms of an infinite collection of random variables indexed by a parameter \( t \) taking values in an infinite index set \( T \), that is, in terms of the stochastic process \( \{X(t), t \in T\} \). The classical theory of statistics dictates that repeatable experiments under identical conditions can be modelled by a sequence \( (X_n, n=1, 2, \ldots) \) of independent and identically distributed random variables (Poisson Point Process). The possibility of dependence between successive random variables introduces "chains" of random variables known as Markov chains. The sequence of independent, identically distributed variables and the Markov chains are stochastic processes. For example, the incidence of calls at a telephone exchange, the arrival of customers at a bank or other service facility, the submission of jobs to a computer terminal and random telegraph signals are frequently modelled as sequence of independent and identically distributed (Poisson) random variables. They are known as Poisson Point Process for the purpose of analyzing the performance of various modes of handling calls, customers, computer jobs or telegraph signals [40]. However, the above natural phenomena are modelled as Markov process if there exists dependency between successive random variables. Furthermore, Habibi and Wintz [39] have indicated that video signals can be represented as first order Markov process.
Consequently, many real data sources or phenomena can often be modelled as first order Markov process. With this mind, the input signal $X(t)$ in this investigation is modelled as a bandlimited first order Markov process with a known correlation function $R(\tau)$. Assuming that the random variables $X_i$ of the Markov process are Gaussian distributed, the process can be described by a two-dimensional (bivariate) Gaussian density distribution, that is,

$$p(X_{i+1}, X_i) = \frac{1}{2\pi \sigma_{i+1} \sigma_i [1-R(\tau)^2]^{1/2}} \exp \left[-\frac{1}{2[1-R(\tau)^2]} \right]$$

$$\left[ \begin{array}{c} X_{i+1} - M_{i+1} \\ \sigma_{i+1} \end{array} \right]^2 + 2R(\tau) \left[ \begin{array}{c} X_{i+1} - M_{i+1} \\ \sigma_{i+1} \end{array} \right] \left[ \begin{array}{c} X_i - M_i \\ \sigma_i \end{array} \right] + \left[ \frac{X_i - M_i}{\sigma_i} \right]^2$$

if $-\infty < X_{i+1} < \infty$, $-\infty < X_i < \infty$

(2-1)

since, the random variables $X_{i+1}$ and $X_i$ are from the same distribution, then

$$\sigma_{i+1} = \sigma_i = \sigma$$

is the standard deviation.

$$M_{i+1} = M_i = 0$$

is the mean of the Gaussian process. (2-2)

Equation (2-1) can be simplified using (2-2) to
\[ p(X_{i+1}, X_i) = \frac{1}{2\pi \sigma^2 \left[ 1 - R(T) \right]^2} \exp \left[ -\frac{1}{2\sigma^2 \left[ 1 - R(T) \right]^2} \right] \]
\[ \begin{bmatrix}
X_{i+1} - 2R(T)X_i + X_i^2
\end{bmatrix}
\]

if \(-\infty < X_{i+1} < \infty, -\infty < X_i < \infty\)  \hspace{1cm} (2-3)

where

\[ X_i, X_{i+1} \text{ = random variables} \]

\[ R(T) = \exp(-\alpha T) \text{ is the autocorrelation function} \]

\[ \alpha \text{ is a constant representing the signal bandwidth. It will be normalized for this investigation.} \]

\[ \tau \text{ is the sampling intervals.} \]

The power spectrum is given by

\[ \Phi_{XX}(\omega) = \frac{A^2 \alpha^2}{\alpha^2 + \omega^2} \quad \omega > 0 \text{ (one sided spectral distribution)} \]  \hspace{1cm} (2-4)

and is illustrated in Figure 2-1. The autocorrelation function is obtained by taking the inverse Fourier transform of the power spectrum (2-4), that is,

\[ \Phi(\tau) = \mathcal{F}^{-1} \{ \Phi_{XX}(\omega) \} = \frac{1}{2\pi} \int_{0}^{\infty} \Phi_{XX}(\omega) \exp(j\omega\tau) d\omega \]  \hspace{1cm} (2-5)
Equation (2-5) can be written using (2-4) as

\[ \phi(\tau) = \frac{1}{2\pi} \int_{0}^{\infty} \frac{A^2\sigma^2}{\alpha^2 + \omega^2} \exp(j\omega\tau) d\omega \]  

(2-6)

Equation (2-6) simplifies to

\[ \phi(\tau) = \frac{A^2\sigma}{4} \exp(-\alpha |\tau|) \]  

(2-7)

where the variance \( \sigma^2 \) is the value of the autocorrelation function (2-7) at a sampling interval \( \tau = 0 \), that is

\[ \sigma^2 = \phi(0) = \frac{A^2\sigma}{4} \]  

(2-8)

The correlation function of (2-7) is normalized by its value at \( \tau = 0 \) (2-8), to make correlation not dependent on signal power \( \sigma^2 \) but on signal bandwidth \( \alpha \) and sampling interval \( \tau \), that is,

\[ R(\tau) = \frac{\phi(\tau)}{\phi(0)} = \exp(-\alpha |\tau|) \]  

(2-9)

The signal bandwidth \( \alpha \) is normalized to 1.0 so that correlation function depends solely on the sampling interval and can be written as

\[ R(\tau) = \exp(-|\tau|) \]  

(2-10)

The correlation function of (2-10) is illustrated in Figure 2-2. Note that the sampling interval \( \tau \) should be small enough so that
Figure 2 - 1. Power spectrum of (2-4).

Figure 2 - 2. Correlation function of (2-10).
the spectrum generated by taking samples from computer has a much wider spectrum than the signal bandwidth $\alpha$, that is,

$$\int_{2\pi/\tau}^{\infty} \frac{A^2\alpha^2}{\alpha^2 + \omega^2} \, d\omega < \frac{A^2\alpha}{4}$$

This minimizes spectrum overlap which causes aliasing error or fold-over distortion that degrades the retrieved message signal. Aliasing error can also be reduced by placing a pre-sampling filter on the analog input before the sampling process. The cut-off characteristics of this filter are arranged so that the frequencies above $\alpha$ in Figure 2-1 are reduced before the sampling process is performed. An example of spectrum overlap is illustrated in Figure 2-3 for power spectrum shown in Figure 2-1.

During reconstruction, the process of determining what happened between samples is known as interpolation. Interpolation error is the amount of variance between the original data and the approximated output continuous curve. Smith [12] indicates that in order to interpolate with reasonable interpolation error of 1 percent rms error or less, it is necessary to sample sufficiently rapidly so that the correlation coefficients between adjacent samples may be 0.95 or greater. The process of sampling so often that the data does not change significantly between samples is not a satisfactory method of avoiding interpolation errors, as it leads to excessive sampling
Figure 2 - 3. Spectrum overlap that causes aliasing.
(The power spectrum was given in Figure 2 - 1 above.)
rates. The relationship between sampling rate and interpolation error is one of the fundamental tradeoffs in a sampled data system. The tradeoff is that of system accuracy versus system capacity.

The generation of signal samples is discussed in detail later in Chapter 3, Section 3.2.1.

2.2 Quantization

The basic principle in a standard PCM system is to quantize and encode the instantaneous sample values. A uniform quantizer with maximum and minimum levels $+V$ and $-V$ respectively is considered. Essman [41] indicated that for reasonably fine quantization, uniform quantization and optimum quantization yield essentially the same performance. Figure 2-4 illustrates a typical transfer characteristic of a uniform quantizer with $M = 2^L$ levels. The instantaneous input sample values are quantized into a finite number of level states. The multi-level output of a quantizer is expressed in terms of $L$-bit words $(a_{i1}, a_{i2}, \ldots, a_{iL})$. Using natural binary coding, quantizer outputs can be described by

$$Q(X_i) = V \sum_{j=1}^{L} a_{ij}2^{-j}$$

(2-11)

where $a_{ij} = \pm 1$ are the serial quantizer output bits for $i^{th}$ sample. The maximum quantizer output magnitude is
Figure 2 - 4.  a) Block diagram of a serial/parallel quantizer, and  
b) A general transfer characteristic of an 1-bit uniform quantizer.
\[ Q_{\text{max}} = \frac{V}{2} \left( 1 + \frac{1}{2} + \ldots + \frac{1}{2^{L-1}} \right) \quad (2-12) \]

Simplifying (2-12)

\[ Q_{\text{max}} = \frac{V}{2} \left[ \frac{1-(1/2^L)}{1-(1/2)} \right] = V - \Delta/2 \quad (2-13) \]

where \( \Delta \) is the quantizer step size and \( V=2^{L-1}\Delta \) as the maximum level of a quantizer. Clearly, the output signal magnitude cannot exceed this value, and any input signal sample above \( Q_{\text{max}} \) causes quantizer overload. Signal quantization using only eight quantization levels for simplicity is illustrated in Figure 2-5. A sine wave is used as the message signal \( m(t) \). At every sampling instant, a decision is made as to which of the eight discrete amplitude levels best approximates the value of the message signal at that sampling instant. The amplitude difference between the sampled value and the quantized level is called the quantization error. This error is defined as

\[ e_q \Delta X_i - Q(X_i) \quad (2-14) \]

The mean-squared quantization error is

\[
\bar{e}_q^2 = \int_{-\Delta/2}^{\Delta/2} e^2 \ p(e) \, de \quad (2-15)
\]

If the input signal is uniformly distributed over the range \( 2V \) of the quantizer, the quantizing error is also uniform, and the mean-squared quantization error is
Figure 2-5. Quantization and binary coding for PCM systems.
\[ \bar{e}_q^2 = \int_0^{\frac{\Delta}{2}} e^2 \frac{1}{\Delta} \, de = \frac{1}{3} \left( \frac{\Delta}{2} \right)^2 \]  

Equation (2-16) can be written using \( \Delta = \frac{V}{2^{L-1}} \) as

\[ \bar{e}_q^2 = \frac{V^2}{3 \cdot 2^L} \]  

Equation (2-17) holds exactly only when the input is uniformly distributed. However, McRae [42] indicated that due to the insensitivity of the uniform quantization to input probability distribution when natural coding is used, the quantization error may be calculated with negligible error as if the input distribution was uniform. This assumption is supported by simulation results obtained in [41] using first order Gaussian Markov data as the input. In practical terms, the quantization error will cause continual background crackle in sound transmission. In video transmission, the quantization error will cause the number of gray tones that exist between black and white to be limited and the picture will appear generally noisy [19]. Clearly, the approximation of the message signal can be improved by the reduction of step size and consequently increasing the number of steps that exist between the limits of the quantizer. In Figure 2-4, eight quantum steps or levels have been defined. In practical terms, such a small number of levels would provide a poor approximation of the message signal. Subjective testing by Owen [43] has established that studio quality color television can be conveyed using 512 levels,
while 64 levels still produce a reasonable color picture. Speech signals are of excellent quality if 128 levels are used, while intelligibility holds for as few as 8 to 16 levels.

For illustrative purposes and in order to minimize derivation and computation complexities, an eight level (3 bit) quantizer is used throughout this paper. The results obtained are generalized for any number of levels.

2.3 Conditional Density Function

The first-order Markov process assumption for the input signal implies that the probabilities for passing into the next state of the signal are completely determined by the present state of the signal. Therefore, assuming that the present sample $X_i$ of the input signal belongs to a finite set of input step range $[n\Delta, (n+1)\Delta]$ of a quantizer, that is, $n\Delta \leq X_i \leq (n+1)\Delta$ (see Figure 2-3), the probability of passing to the next sample $X_{i+1}$ can be described by a joint conditional probability density distribution, that is,

$$
\Pr[X_{i+1}/X_i \in [n\Delta, (n+1)\Delta]] = \frac{d}{dX_{i+1}} \left[ \frac{Pr[X_{i+1}/X_i \in [n\Delta, (n+1)\Delta]]}{Pr[n\Delta \leq X_i \leq (n+1)\Delta]} \right] (2-18)
$$

where

$X_{i+1} = \text{random variable}$

$X_{i+1} = \text{value of the random variable}$

$n = -2L-1, -2L-2, \ldots, +2L-2, 2L-1 \text{ integer values}$
\( \Delta = \frac{V}{2^{L-1}} \) is the step size

\( L \) = number of bits per sample

\( V \) = maximum quantizer limit

Equation (2-18) becomes

\[
\begin{align*}
\frac{d}{dz} \left[ \int_{-\infty}^{\infty} p(Z, X_i) dX_i dZ \right] \\
p[X_{i+1}/n\Delta \leq X_i \leq (n+1)\Delta] &= \int_{-\infty}^{\infty} p(X_{i+1}, X_i) dX_i \\
\int_{-\infty}^{\infty} dX_{i+1} \int_{n\Delta}^{(n+1)\Delta} p(X_{i+1}, X_i) dX_i
\end{align*}
\]  

\( 2-19 \)

Equation (2-19) simplifies to

\[
\begin{align*}
\int_{-\infty}^{\infty} dX_{i+1} \int_{n\Delta}^{(n+1)\Delta} p(X_{i+1}, X_i) dX_i \\
p[X_{i+1}/n\Delta \leq X_i \leq (n+1)\Delta] &= \int_{-\infty}^{\infty} \frac{(n+1)\Delta}{n\Delta} \\
\int_{-\infty}^{\infty} dX_{i+1} \int_{n\Delta}^{(n+1)\Delta} p(X_{i+1}, X_i) dX_i
\end{align*}
\]  

\( 2-20 \)

Let \( \beta = \int_{-\infty}^{\infty} dX_{i+1} \int_{n\Delta}^{(n+1)\Delta} p(X_{i+1}, X_i) dX_i \)

\( 2-21 \)

Equation (2-20) can be written using (2-21) as
\[ p\left[X_{i+1}/n\Delta \leq X_i \leq (n+1)\Delta \right] = \frac{1}{\beta} \int_{n\Delta}^{(n+1)\Delta} p(X_{i+1}, X_i) dX_i \]  
(2-22)

Since the signal is assumed to be Gaussian distributed, (2-22) can be written using (2-3) as

\[ p\left[X_{i+1}/n\Delta \leq X_i \leq (n+1)\Delta \right] = \frac{1}{2\pi \sigma^2 [1-R(\tau)^2]^{1/2}} \beta \int_{n\Delta}^{(n+1)\Delta} \exp \left[ -\frac{1}{2\sigma^2 (1-R(\tau)^2)} \right] \]

\[ \left[ X_{i+1}^2 - 2R(\tau)X_{i+1}X_i + X_i^2 \right] dX_i \]  
(2-23)

Equation (2-24) can be written in a more desirable form (see Appendix A) using (A-29), that is,

\[ p\left[X_{i+1}/n\Delta \leq X_i \leq (n+1)\Delta \right] = \frac{1}{2\pi \sigma^2 [1-R(\tau)^2]^{1/2}} \beta \exp\left[-\frac{X_{i+1}^2}{2\sigma^2 (1-R(\tau)^2)}\right] \]

\[ \int_{n\Delta}^{(n+1)\Delta} \exp \left[ -\left( X_{i+1}^2 - 2R(\tau)X_{i+1}X_i + X_i^2 \right)/2\sigma^2 (1-R(\tau)^2) \right] dX_i \]  
(2-24)

Equation (2-24) can be written in a more desirable form (see Appendix A) using (A-29), that is,

\[ p\left[X_{i+1}/n\Delta \leq X_i \leq (n+1)\Delta \right] = \frac{1}{\sqrt{8\pi} \sigma \beta} \exp\left[-\frac{X_{i+1}^2}{2\sigma^2}\right] \]

\[ \left[ \text{erf}\left(\frac{u_1}{\sqrt{2}}\right) - \text{erf}\left(\frac{u_2}{\sqrt{2}}\right) \right] \]  
(2-25)

where \( u_1 = (n\Delta - R(\tau)X_{i+1})/\sigma(1-R(\tau)^2)^{1/2} \) from (A-25)

\[ u_2 = ((n+1)\Delta - R(\tau)X_{i+1})/\sigma(1-R(\tau)^2)^{1/2} \] from (A-24)
\[ \beta = \int_{-\infty}^{\infty} p(X_{i+1}, X_1) \, dX_1 \text{ from (2-21)} \]

Equation (2-21) can be written using (2-3) as

\[
\beta = \int_{-\infty}^{\infty} dX_{i+1} \left[ \frac{1}{\sqrt{2\pi\sigma^2[1-R(\tau)^2]^2}} \exp \left[ -\frac{1}{2\sigma^2(1-R(\tau)^2)} \right] \right]
\]

\[
\left[ X_{i+1}^2 - 2R(\tau)X_{i+1}X_1^2 + X_1^2 \right] \, dX_1 \tag{2-26}
\]

The simplification of (2-26) to a more desirable form is given in Appendix A and the result (A-43) is

\[
\beta = \frac{1}{2} \text{erf}\left[\frac{(n+1)\Delta}{\sqrt{2}}\right] - \frac{1}{2} \text{erf}\left[\frac{n\Delta}{\sqrt{2}}\right] \tag{2-27}
\]

Figure 2-6 (a), (b), (c) and (d) indicate the conditional probability density functions for each given input step size in an eight level quantizer for four different correlation coefficients of 0.0, 0.6, 0.9, and 0.999 respectively. These density functions are Gaussian shaped. Note that in order to avoid repetitions, Figure 2-6 (a) shows only one density function symmetrical about the origin to represent all density functions in the eight level quantizer because the density function for each given input step size is identical when the correlation coefficient = 0.0. This figure indicates that the probability of occurrence of low signal samples is greater than that of high signal samples. The rest of the Figure 2-6 (b, c, and d)
Figure 2-6. Density functions with various correlation coefficients.

a) Correlation coefficient = 0.0, b) Correlation coefficient = 0.6, c) Correlation coefficient = 0.9, d) Correlation coefficient = 0.999.
shows that at low correlation coefficient, each density function has a large variance and almost the same peak value. But as the correlation coefficient gets closer to 1.0, the variance of each density function becomes progressively smaller and each density function peaks within their respective given input step size. This implies that the probability of occurrence of signal samples in each of these given input step sizes increases. Thus, the probability of the next sample being the same as the present sample increases and the probability of the next sample being different from the present sample decreases. Since higher correlation coefficient translates into greater probability of occurrence of signal samples in every given input step size, the probability of occurrence of low signal samples will still be greater than that of high signal samples. Consequently, the probability of the next sample being the same as the present sample increases, and the probability of the next sample being different from the present sample decreases as the given input step size approaches the origin from either side of the axis. The computed results of the probability of the next sample being the same or different from the given present sample shown in Table 2-1 for three different correlation coefficients (0.9, 0.99, 0.999) further support the deductions above.

Furthermore, two density functions superimposed on their respective given level on an eight level quantizer transfer characteristic are shown in Figure 2-7. Although only two density functions are shown, one chosen closer to the origin and the other farther away
Table 2-1. Probabilities of next sample $X_{i+1}$ being the same as or different from the given present sample $X_i$ for three different correlation coefficient.

<table>
<thead>
<tr>
<th>GIVEN STEP LEVEL</th>
<th>CORRELATION COEFFICIENT = 0.9</th>
<th>CORRELATION COEFFICIENT = 0.99</th>
<th>CORRELATION COEFFICIENT = 0.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>INPUT</td>
<td>PROBABILITY OF NEXT SAMPLE $X_{i+1}$ BEING THE SAME AS THE GIVEN PRESENT SAMPLE</td>
<td>PROBABILITY OF NEXT SAMPLE $X_{i+1}$ BEING DIFFERENT FROM THE GIVEN PRESENT SAMPLE</td>
<td>PROBABILITY OF NEXT SAMPLE $X_{i+1}$ BEING THE SAME AS THE GIVEN PRESENT SAMPLE</td>
</tr>
<tr>
<td>$-4 \leq X_i \leq -3$</td>
<td>0.4361220</td>
<td>0.54802566</td>
<td>0.80497032</td>
</tr>
<tr>
<td>$-3 \leq X_i \leq -2$</td>
<td>0.53923976</td>
<td>0.4596818</td>
<td>0.8456844</td>
</tr>
<tr>
<td>$-2 \leq X_i \leq -1$</td>
<td>0.62499541</td>
<td>0.3739927</td>
<td>0.8766528</td>
</tr>
<tr>
<td>$-1 \leq X_i \leq 0$</td>
<td>0.67442513</td>
<td>0.3245761</td>
<td>0.8936856</td>
</tr>
<tr>
<td>$0 \leq X_i \leq 1$</td>
<td>0.67459321</td>
<td>0.3244079</td>
<td>0.89389932</td>
</tr>
<tr>
<td>$1 \leq X_i \leq 2$</td>
<td>0.62546366</td>
<td>0.3735251</td>
<td>0.8772758</td>
</tr>
<tr>
<td>$2 \leq X_i \leq 3$</td>
<td>0.53991359</td>
<td>0.4590081</td>
<td>0.8466848</td>
</tr>
<tr>
<td>$3 \leq X_i \leq 4$</td>
<td>0.43688667</td>
<td>0.5471859</td>
<td>0.80630910</td>
</tr>
</tbody>
</table>
Figure 2-7. Two density functions superimposed on an 8-level quantizer characteristic.
from the origin for clarity purposes, the idea is to explain and even-tually generalize the effects of density function relative to the ori-
gin on transmission. Due to the increase in probability of occurrence of signal samples as the origin is approached from either side of the axis, signal samples in the range -1V to 0 are more likely to occur than signal samples in the range -4V to -3V. This implies that most signal samples occur near the origin than far from it. Thus, it is highly probable that most of the errors in transmission will be com-
mittted closer to the origin because the probability that the additive Gaussian noise can cause error in a word will increase due to the fact that more signal samples are available near the origin.

2.4 Coding and Bit Probability Distribution

Each sample of the input signal is coded into a binary number which is equivalent to the index of the round off (or quantization) level that is closest to the sample value. This coding process is accomplished by using a serial quantizer. The serial quantizer suc-
essively divides the ordinate into two regions. First, observation as to whether the sample is in the upper or lower half is made by dividing the axis in half. The result of this observation generates the most significant bit in the code word. The half-region in which the sample lies is then subdivided into two regions and a comparison is again performed. This generates the next bit. The process con-tinues a number of times equal to the number of bits per sample required in the encoding. Figure 2-8 shows a block diagram form of
this encoder for 3 bits of encoding and for inputs in the range -4 to 4 volts.

The coded version of sample values (3 bit/sample) is shown in Figure 2-9 on an eight level quantizer transfer characteristic. Each quantum level shows the corresponding code word which is a binary number equivalent to the index of that quantum level. The bits of each code word can be expressed by \( a_j \) where \( l \) denotes the index of each bit position in the code word. The probability of bit distribution can be evaluated by integrating the conditional probability density function of (2-25). Since there are only two possible states, binary 0 or binary 1, two probability distributions are calculated for each bit position in a code word. Furthermore, the bit distribution of the next word given a present word depends on which bit position is under consideration and will be written as

\[
P_R[a_j^{i+1}=0/n\Delta \leq X_i \leq (n+1)\Delta] \quad \text{and} \quad P_R[a_j^{i+1}=1/n\Delta \leq X_i \leq (n+1)\Delta]
\]

which are the probabilities of each bit position of the next word being a binary 0 or a binary 1 respectively given that the present sample is within an input step size of \([n\Delta,(n+1)\Delta]\).

where

\( i \) denotes the index of signal sample

\( l \) denotes the index of bit positions

Figure 2-9 shows an eight level quantizer transfer characteristic with the bits of each code word, which is used as an example
Figure 2 - 8. Serial 3-bit quantizer.

Figure 2 - 9. Eight-level quantizer transfer characteristic with bits of each code word of each level.
for the derivation of the bit probability distribution for each bit position in the next code word given that the present sample is within an input step size of \([n\Delta,(n+1)\Delta]\). It has been mentioned earlier that there are two possibilities for each bit position in the next code word. One possibility is the probability distribution of each bit position in the next code word being a binary 0 given that the present sample is within an input step size of \([n\Delta,(n+1)\Delta]\) and this process can be described as follows:

\[
P_r[a_1^{i+1}=0/n\Delta \leq X_i \leq (n+1)\Delta] = P_r[-V \leq X_{i+1} \leq 0/n\Delta \leq X_i \leq (n+1)\Delta] \tag{2-28}
\]

\[
P_r[a_2^{i+1}=0/n\Delta \leq X_i \leq (n+1)\Delta] = P_r[-V \leq X_{i+1} \leq -V/2/n\Delta \leq X_i \leq (n+1)\Delta] + P_r[0 \leq X_{i+1} \leq V/2/n\Delta \leq X_i \leq (n+1)\Delta] \tag{2-29}
\]

\[
P_r[a_3^{i+1}=0/n\Delta \leq X_i \leq (n+1)\Delta] = P_r[-V \leq X_{i+1} \leq -3V/4/n\Delta \leq X_i \leq (n+1)\Delta] + P_r[-V/2 \leq X_{i+1} \leq -V/4/n\Delta \leq X_i \leq (n+1)\Delta] + P_r[0 \leq X_{i+1} \leq V/4/n\Delta \leq X_i \leq (n+1)\Delta] + P_r[V/2 \leq X_{i+1} \leq 3V/4/n\Delta \leq X_i \leq (n+1)\Delta] \tag{2-30}
\]

Equations (2-28) - (2-30) are evaluated by integrating (2-25) over the specified ranges of \(X_{i+1}\) in (2-28) - (2-30) above. In generalizing for any number of levels (bits), let the ranges of the next sample \(X_{i+1}\) in (2-28) - (2-30) be defined as
Then, the probability distribution of each bit position being a binary 0 can be described generally as

\[
Pr[a_{ji+1}=0/nA_i^j(n+1)A_i^j] = \sum_{J=2^{J-1}}^{2^{J-1}} \int_{R_j} \frac{1}{\sqrt{8\pi} \alpha \beta} \exp\left[-\frac{X_i+1^2}{2\alpha^2}\right] \, dX_i+1
\]  

(2-32)

Equation (2-32) can be written using (2-25) as

\[
Pr[a_{ji+1}=0/nA_i^j(n+1)A_i^j] = \sum_{J=2^{J-1}}^{2^{J-1}} \int_{R_j} \frac{1}{\sqrt{8\pi} \alpha \beta} \exp\left[-\frac{X_i+1^2}{2\alpha^2}\right] \, dX_i+1
\]  

(2-33)

where

\[R_j\] are the respective ranges of the next sample \[X_i+1\] shown in (2-31).
The computed probability distribution of each bit position in the next code word being a binary 0 for each given present level in an eight level quantizer, that is, (2-28) - (2-30) are shown in Table 2-2.

The other possibility is the probability distribution of each bit position in the next code word being a binary 1 given that the present sample is within an input step size of \([n\Delta, (n+1)\Delta]\) and this process can be described as follows:

\[
P_r[a_1^{i+1}=1/n\Delta \leq X_i \leq (n+1)\Delta] = P_r[0 \leq X_{i+1} \leq V/n\Delta \leq X_i \leq (n+1)\Delta]
\] (2-34)

\[
P_r[a_2^{i+1}=1/n\Delta \leq X_i \leq (n+1)\Delta] = P_r[-V/2 \leq X_{i+1} \leq V/n\Delta \leq X_i \leq (n+1)\Delta]
+ P_r[V/2 \leq X_{i+1} \leq V/n\Delta \leq X_i \leq (n+1)\Delta]
\] (2-35)

\[
P_r[a_3^{i+1}=1/n\Delta \leq X_i \leq (n+1)\Delta] = P_r[-3/4V \leq X_{i+1} < -V/2/n\Delta \leq X_i \leq (n+1)\Delta]
+ P_r[-V/4 \leq X_{i+1} < 0/n\Delta \leq X_i \leq (n+1)\Delta]
+ P_r[V/4 \leq X_{i+1} \leq V/2/n\Delta \leq X_i \leq (n+1)\Delta]
+ P_r[3/4V \leq X_{i+1} \leq V/n\Delta \leq X_i \leq (n+1)\Delta]
\] (2-36)

Equations (2-34) - (2-36) are evaluated by integrating (2-25) over the specified ranges of \(X_{i+1}\) in (2-34) - (2-36) above. Once again, generalizing for any number of levels (or bits/sample), let the ranges of the next sample \(X_{i+1}\) in (2-34) - (2-36) be defined as

\[S_1 \overset{\Delta}{=} 0 \leq X_{i+1} < V\]

\[S_2 \overset{\Delta}{=} -V/2 \leq X_{i+1} < 0\]
Table 2-2. The probability distribution of each bit position in the next code word being a binary 0 for each given present level in an eight level quantizer. Equations (2-28) - (2-30).

<table>
<thead>
<tr>
<th>GIVEN RANGE</th>
<th>BIT PROBABILITY DISTRIBUTION FOR $a_{i+1}^1 = 0$ (1st binary 0) EQUATIONS (2-28) - (2-30)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-4 \leq X_1 &lt; -3V$</td>
<td>$P_T[a_1^{i+1} = 0 / nA \leq X_1 \leq (n+1)\Delta] = 0.99431813$</td>
</tr>
<tr>
<td>$-3 \leq X_1 &lt; -2V$</td>
<td>$0.99999803$</td>
</tr>
<tr>
<td>$-2 \leq X_1 &lt; -1V$</td>
<td>$0.99999774$</td>
</tr>
<tr>
<td>$-1 \leq X_1 &lt; 0$</td>
<td>$0.93180466$</td>
</tr>
<tr>
<td>$0 \leq X_1 &lt; 1V$</td>
<td>$0.06351697$</td>
</tr>
<tr>
<td>$1 \leq X_1 &lt; 2V$</td>
<td>$0.0$</td>
</tr>
<tr>
<td>$2 \leq X_1 &lt; 3V$</td>
<td>$0.0$</td>
</tr>
<tr>
<td>$3 \leq X_1 &lt; 4V$</td>
<td>$0.0$</td>
</tr>
</tbody>
</table>
Then, the probability distribution of each bit position being a binary 1 can be described generally as

\[ P_{r}[a_{i+1} = 1] = \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left[-\frac{X_{i+1}^2}{2\sigma^2}\right] \text{erf}(u_2/\sqrt{2}) - \text{erf}(u_1/\sqrt{2}) \]  

(2-39)

where

\[ S_j \] are the respective ranges of the next sample \( X_{i+1} \) shown in (2-37)

\[ j = 1, 2, 3, \ldots, 2^l - 1 \]
The computed probability distribution of each bit position in the next code word being a binary 1 for each given present level in an eight level quantizer, that is, \((2^{-34}) - (2^{-36})\) is shown in Table 2-3.

Due to the effect of correlation between samples discussed earlier in Section 2.3, the probability that given any present word that any bit of the next code word is the same as the corresponding bit of the present word is very high. The likelihood of the bits of the next code word being different from the corresponding bit of the present word is small but increases in small proportion as the transition is made from the most significant bit to the least significant bit in each code word as illustrated in Table 2-4. This implies that the interdependency between bits of the next code word and the corresponding bits of the present word decreases in the direction towards the least significant bit. The high correlation amongst the most significant bits results in better prediction of these bits. Consequently, the least amount of bit errors will be committed in this bit position, while most bit errors in transmission will be made in the least significant bits because the correlation amongst these bits is the lowest. This discussion indicates that the effect of using statistical dependence in data is apparently to reduce errors in the high order (most significant) bits of the code.

Furthermore, as the correlation between samples increases (i.e. tending to 1.0), the probability of the next sample being the same as
Table 2-3. The probability distribution of each bit position in the next code word being a binary 1 for each given present level in an eight level quantizer. Equations (2-34) - (2-36).

<table>
<thead>
<tr>
<th>GIVEN RANGE</th>
<th>BIT PROBABILITY DISTRIBUTION FOR $a_{i+1}^1=1$ (ie binary 1)</th>
<th>EQUATIONS (2-34) - (2-36)</th>
</tr>
</thead>
<tbody>
<tr>
<td>nΔ≤$X_i$≤(n+1)Δ</td>
<td>$P_r[a_{1}^{i+1}=1/n\Delta \leq X_i \leq (n+1)\Delta]$</td>
<td>$P_r[a_{2}^{i+1}=1/n\Delta \leq X_i \leq (n+1)\Delta]$</td>
</tr>
<tr>
<td>-4≤$X_i$&lt;-3V</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>-3≤$X_i$&lt;-2V</td>
<td>0.0</td>
<td>0.14162970</td>
</tr>
<tr>
<td>-2≤$X_i$&lt;-1V</td>
<td>0.0</td>
<td>0.97769642</td>
</tr>
<tr>
<td>-1≤$X_i$&lt;0</td>
<td>0.06582820</td>
<td>0.93299371</td>
</tr>
<tr>
<td>0≤$X_i$&lt;1V</td>
<td>0.93416840</td>
<td>0.06466621</td>
</tr>
<tr>
<td>1≤$X_i$&lt;2V</td>
<td>0.99999785</td>
<td>0.02230208</td>
</tr>
<tr>
<td>2≤$X_i$&lt;3V</td>
<td>0.99999666</td>
<td>0.85836679</td>
</tr>
<tr>
<td>3≤$X_i$&lt;4V</td>
<td>0.99419129</td>
<td>0.99419129</td>
</tr>
</tbody>
</table>
Table 2-4. Probability of the bits of the next word being the same or different from the corresponding bits of the given present word for eight level quantizer.

<table>
<thead>
<tr>
<th>THE GIVEN PRESENT WORD</th>
<th>EACH BIT POSITION</th>
<th>PROBABILITY OF THE BITS OF THE NEXT WORD BEING THE SAME AS THE CORRESPONDING BIT OF PRESENT WORD</th>
<th>PROBABILITY OF THE BITS OF THE NEXT WORD BEING DIFFERENT FROM THE CORRESPONDING BIT OF PRESENT WORD</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.99431813</td>
<td>0.0</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0.99431813</td>
<td>0.0</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>0.80497032</td>
<td>0.18794423</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.99999803</td>
<td>0.0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.85611850</td>
<td>0.14162970</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0.84566844</td>
<td>0.15308702</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0.9999774</td>
<td>0.0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.97769642</td>
<td>0.02186290</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>0.87665528</td>
<td>0.12228048</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0.93180466</td>
<td>0.06582820</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.93299371</td>
<td>0.06582820</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0.89368856</td>
<td>0.10534924</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.93416840</td>
<td>0.06351697</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.93416840</td>
<td>0.06466621</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0.89389932</td>
<td>0.10514033</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.9999785</td>
<td>0.0</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0.97725141</td>
<td>0.02230208</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0.87727588</td>
<td>0.12166339</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.9999666</td>
<td>0.0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.85836679</td>
<td>0.13939983</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>0.84666848</td>
<td>0.15209031</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.99406362</td>
<td>0.0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.99419129</td>
<td>0.0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0.80630910</td>
<td>0.18654931</td>
</tr>
</tbody>
</table>

1 in each bit position column represents the most significant bit while 3 represents the least significant bit.
the given present sample becomes higher (also tending to 1.0), and the probability of their being different becomes lower as discussed in Section 2.3. This translates into higher probability of bits in the next code word being the same as the corresponding bits of the given present word and lower probability that they are different. Therefore, errors made in the system are likely to decrease as correlation between samples gets closer to 1.0 because better prediction of the transmitted bits is guaranteed due to the higher correlation between samples.

Conclusively, the conditional probability density distribution of (2-25) allows the inaccuracy (errors) made in decoded analog samples not only to depend on which bit in the received word is erroneous but also on the given range of the present word and signal characteristics as well.

2.5 Signal Processing and Detection

Figure 2-10 illustrates a general block diagram of the Decision-Directed-Detection (DDD) communication receiver. The system performance measure for an analog system is the output-signal-to-noise ratio, and for digital systems is the probability of error of the output signal. A general simulation technique for evaluating the probability of bit error, also called the bit error rate (BER), for binary signaling is developed. This technique is later applied to a binary signaling scheme such as Phase Shift Keying (PSK) to obtain BER.
Figure 2 - 10. The block diagram of the system receiver.

THE LOOK-UP TABLE OF BIT PROBABILITIES BEING A BINARY 0 OR A BINARY 1 FOR DECISION MAKING BETWEEN THE DOUBLE THRESHOLD
In developing a general formula for the BER of a detected binary signal, let \( T \) be the length of time that it takes to transmit one bit of data. Where \( T \) can be described as

\[
T = \tau / L \tag{2-40}
\]

\( \tau = \) sampling interval

\( L = \) number of bits per sample

The transmitted signal over a bit interval of \((0, T)\) is represented by

\[
S(t) = \begin{cases} 
S_0(t), & 0 < t \leq T \text{ for a binary 0} \\
S_1(t), & 0 < t \leq T \text{ for a binary 1} 
\end{cases} \tag{2-41}
\]

where \( S_0(t) \) is the waveform that is used if a binary 0 is transmitted and \( S_1(t) \) is the waveform that is used if a binary 1 is transmitted.

The transmitted signal \( S(t) \) plus channel noise \( n(t) \) is the receiver input \( r(t) \). This produces a baseband analog waveform at the output of the processing circuits. The processing circuits in the receiver consist of low-pass filtering with appropriate amplification for baseband signaling and for bandpass signaling, such as on-off keying (OOK), Binary Phase Shift Keying (BPSK), and Frequency Shift Keying (FSK).

The processing circuits normally consist of a superheterodyne receiver containing a radio frequency (RF) amplifier, mixer, intermediate frequency (IF) amplifier, and detector. The basic block diagram of a superheterodyne receiver is illustrated in Figure 2-11. This receiver structure is used for the reception of all types of bandpass signals.
Figure 2-11. A general superheterodyne receiver.
such as television, frequency modulation (FM), amplitude modulation (AM), satellite and radar signals. The radio frequency (RF) amplifier has a bandpass characteristic that passes the desired signal and provides some rejection of adjacent channel signals and noise, although the main adjacent channel rejection is accomplished by the intermediate frequency (IF) filter. The RF amplifier characteristic also provides amplification to override additional noise that is generated in the mixer stage. An ideal mixer, illustrated in Figure 2-12, is a circuit that functions as a mathematical multiplier of two input signals. Usually, one of these signals is a sinusoidal waveform produced by a local oscillator. Mixers are used to obtain frequency translation of the input signal, that is, mixers are either down-converting or up-converting the input signal to some convenient frequency band called intermediate frequency (IF) band. The intermediate frequency filter is a bandpass filter used to select either the down-converted or the up-converted frequency of the input signal. The choice of the intermediate frequency is based on three considerations given by Couch [10].

The extraction of information from the bandpass signal plus noise, which in this case is the intermediate frequency, is accomplished by using the appropriate detector. For example, when Binary Phase Shift Keying (BPSK) signaling is used, the detector might consist of a product detector and an integrator as illustrated in Figure 2-13a. Shown in Figure 2-13b are various waveforms at the
Figure 2 - 12. A mixer followed by a filter for either up or down conversion.

\[ v_{in}(t) \rightarrow v(t) \rightarrow v(t) \rightarrow \text{FILTER} \]

\[ v(t)_{LO} \rightarrow \text{LOCAL OSCILLATOR} \]

Figure 2 - 13a. Correlation detection for BPSK.

\[ r(t) = s(t) + n(t) \]

\[ w(t) = s(t)_{1} - s(t)_{0} \]

(from carrier sync. cct)
Figure 2-13b. Various waveforms at the labelled points in Figure 2-13a.
labelled points in Figure 2-13a. A product detector is a mixer circuit that down-converts the input (bandpass signal plus noise) to baseband and the process can be described mathematically as follows:

\[ r(t) = S(t) + n(t) \tag{2-43} \]

where

- \( r(t) \) = bandpass signal plus noise (IF)
- \( s(t) \) = bandpass signal
- \( n(t) \) = channel noise

The analog baseband signal which is the output of the processing circuit denoted by \( r_0(t) \) is represented by

\[ r_0(t) = \int_0^T r(t) \cdot y(t) dt \tag{2-44} \]

where \( y(t) = S_1(t) - S_0(t) \) is the reference from the carrier synchronization circuit. Since, there are two possible transmissions (binary 0 or binary 1), the analog baseband signal of (2-44) can be written as

\[ r_0(t) = \begin{cases} 
  r_{00}(t), & 0 < t < T \text{ for a binary 0 sent} \\
  r_{01}(t), & 0 < t < T \text{ for a binary 1 sent} 
\end{cases} \tag{2-45} \]

where \( r_{00}(t) \) is the output baseband signal for a binary 0 transmission, and \( r_{01}(t) \) is the output baseband signal for a binary 1
transmission. This analog baseband waveform \( r_0(t) \) is sampled at the clocking time \( t_0 = T \), that is, at the end of the bit interval \((0, T)\) so that maximum bit energy can be obtained. The resulting baseband sample can be written as

\[
\begin{cases} 
  r_0(t_0), & t_0 = T \text{ for binary 0 sent} \\
  r_0(t_0), & t_0 = T \text{ for binary 1 sent}
\end{cases}
\]  

(2-46)

\( r_0(t_0) \) is a random variable that has a continuous distribution because the channel noise has corrupted the signal. A shortened notation for (2-46) is

\[
\begin{cases} 
  0 & \text{for a binary 0 sent} \\
  1 & \text{for a binary 1 sent}
\end{cases}
\]  

(2-47)

In order to analyze the performance of this system, it is necessary to determine the probability density function (PDF) for the two random variables \( r_0 = r_{00} \) and \( r_0 = r_{01} \). These probability density functions are actually conditional probability density functions since they depend, respectively, on a binary 0 or a binary 1 being transmitted. When \( r_0 = r_{00} \), the probability density function is \( p(r_0 / S_0 \text{ sent}) \) and when, \( r_0 = r_{01} \), the probability density function is \( p(r_0 / S_1 \text{ sent}) \). The shapes of the probability density functions depend on the characteristics of the channel noise, the specific types of filter and detector circuits used, and the type of binary signals that are transmitted. Assuming the characteristic of channel noise is
zero-mean wide-sense stationary Gaussian process and that the receiver processing circuits, except for the threshold device, are linear, then the output of the linear processor (baseband signal) will also be a Gaussian process. Thus, the conditional probability density functions \( p(r_0/S_0 \text{ sent}) \) and \( p(r_0/S_1 \text{ sent}) \) are Gaussian process as illustrated in Figure 2-14, where the mean values of \( r_0 \) are \( M_{r_00} = S_0 \) when a binary 0 is sent and \( M_{r_01} = S_1 \) when a binary 1 is sent. Assuming that the comparator (threshold) device has double symmetrical threshold (voltage) setting \( TH_1 \) and \( TH_2 \) and that the polarity of the processing circuits is such that the receiver can distinguish between the two possible transmissions (binary 0 or binary 1) outside the region encompassed by these thresholds with minimum error made and has difficulty in distinguishing between the two possible transmissions in the region encompassed by these thresholds. Therefore, within these thresholds, the bit probability distribution for each bit position in the next code word being a binary 0 or a binary 1 given the level of the present word, which is precomputed using (2-33) and (2-39) respectively and stored in the receiver, is used in making the distinction in order to minimize the possible errors that could be made here.

Consequently, if signal only is present at the receiver input, the decisions will be that when \( r_0 < TH_1 \), a binary 0 is sent, when \( r_0 > TH_2 \), a binary 1 is sent, and when \( TH_1 \leq r_0 \leq TH_2 \), it is not certain whether a binary 0 or a binary 1 is sent; therefore, the probability distribution of each bit position in the next code word being a binary
Figure 2-14. Decision regions for binary signaling with double threshold.
0 or a binary 1 given the level of the present word (2-33) and (2-39) respectively are used in the decision making. The decision will be that when \( P_r[a_t^{i+1}=0/n\Delta x_t \leq (n+1)\Delta] > P_r[a_t^{i+1}=1/n\Delta x_t \leq (n+1)\Delta] \), a binary 0 is sent and when \( P_r[a_t^{i+1}=0/n\Delta x_t \leq (n+1)\Delta] < P_r[a_t^{i+1}=1/n\Delta x_t \leq (n+1)\Delta] \), a binary 1 is sent. Note that the two threshold (TH1 and TH2) voltage settings are symmetrical in the sense that \( TH1 = -TH2 \).

2.5.1 Implementation of the System

Figure 2-15 shows the general block diagram of a Decision-Directed-Detection PCM system. It can be seen from this block diagram that the only difference between this system and the standard PCM system is the manner in which decisions are implemented during data detection process. Unlike the Decision-Directed-Detection PCM system, the standard PCM system has a single threshold at zero in the comparator (the decision making device) and decisions are simply made by comparing the strength of the sampled baseband signal \( r_0 \) to the zero threshold. If the baseband signal is less than zero, the decision will be that a binary 0 is received, and if the baseband signal is greater than zero, the decision will be that a binary 1 is received.

The block diagram for detection implementation of the Decision-Directed-Detection scheme is shown in Figure 2-16. This detection process can be generalized for any number of bits per sample, although in Figure 2-16, four bits per sample have been chosen just for illustrative purposes.
Figure 2-15. Basic block diagram of Decision-Detected-Decision system.
Figure 2-16. Implementation of decisions in the receiver.
2.5.2 Description of the Detection Process

The Decision-Directed-Detection PCM system has a double symmetrical threshold device in which decisions are made as to whether the received sampled baseband \( r_0 \) is a binary 0 or a binary 1. The decisions will be that when \( r_0 < TH_1 \), a binary 0 is received and when \( r_0 > TH_2 \), a binary 1 is received. But when \( TH_1 \leq r_0 \leq TH_2 \), there will be uncertainty as to whether the received baseband \( r_0 \) is a binary 0 or a binary 1. Therefore, the received baseband signal \( r_0 \) is used as an enable signal to two 8 bits x 1K Read Only Memory (ROM) connected in parallel, so that memory locations in both ROM can be accessed by the same address. The look-up table of the probability distribution of each bit position in the next code word being a binary 0 or a binary 1 given the present word is stored in both ROM. One of these two 8 bits x 1K ROM contain the probability distribution of each bit position in the next code word being a binary 0 given the present word, and the other 8 bits x 1K ROM contain the probability of each bit position in the next code word being a binary 1. The needed probability of each bit position in the next code word being a binary 0 or a binary 1 whenever \( TH_1 \leq r_0 \leq TH_2 \), is obtained from the ROM by the combination of the previously detected code word which is in a temporary storage and the output of the bit synchronization circuit fed into a bit counter. The temporary storage consists of a shift left register and a latch. The shift register will contain the detected previous code word at the end of each word through the shifting operation. The
shifting operation is accomplished through a timing or clock signal from the bit synchronization circuit. The latch is enabled at the end of each word so that the contents of the shift register are transferred into the latch. This can be accomplished by signal from the bit synchronization circuit fed into a bit counter and the output of the counter is fed to a logic AND gate as shown in Figure 2-16. The contents of the latch will not change until another output from the shift register is received when the latch is enabled again at the end of the next word.

The combination of the previously detected code word and the output of the bit counter will address or access the memory location in the ROM where the needed probability distribution of each bit position in the next code word being a binary 0 or a binary 1 given the present word is stored. The two required bit probability distributions of each bit position, a binary 0 or a binary 1, are compared in a comparator. If the probability of a bit position being a binary 0 is greater than the probability of the same bit position being a binary 1, the decision will be that a binary 0 is received. But if the probability of a bit position being a binary 0 is less than or equal to the probability of the same bit position being a binary 1, the decision will be that a binary 1 is received. Since some of the previously detected code words could be in error because they are obtained from the noisy signals that are present at the input of the receiver, wrong memory locations in the ROM will be accessed by any of
the erroneous previous code words. Accessing wrong bit probability value from the ROM could lead to wrong decisions in the bit probability comparator device which will result in errors. However, as the correlation between samples increases (closer to 1.0), the previous detected code word in error will decrease because high correlation between samples dictates better prediction or detection of bits of a code word as discussed earlier in Section 2.4. Consequently, the error made in using the previous detected code word is minimized.

Note that a problem with the decision as to whether a binary 0 or a binary 1 is received will arise with this kind of detection if any of the sampled baseband signal \( r_0 \) of the first received word falls within TH1 and TH2, since, at this time, there is no previously detected code word which the receiver relies on in accessing the needed bit probability values to help in decision making within this region from the memory location in the ROM. This problem is solved by requiring the receiver to use the polarity of the baseband signal \( r_0 \) instead of the bit probability distribution values in making the decision when this situation is encountered. The decision will be that if \( r_0 < 0 \) (i.e. \( r_0 \) negative), binary 0 is received and if \( r_0 \geq 0 \) (i.e. \( r_0 \) positive), binary 1 is received.

After the detection process, the decoded signals are fed into a Digital-to-Analog Converter and signal reconstruction processes are performed.
2.5.3 Synchronization

Since synchronization is an important process in the detection implementation above, a short discussion in synchronization is offered in this subsection. Synchronization is a fundamental problem in digital communications systems and the Decision-Directed-Detection PCM system is no exception. Gardner and Lindsey [44] define synchronization as the process of aligning the time scales between two or more period processes that are occurring at spatially separated points. Although there are several types of synchronization such as carrier synchronization, clock or bit synchronization, word or node synchronization, frame synchronization, and network synchronization, only three levels of synchronization are discussed in this investigation, namely:

1. Carrier synchronization
2. Bit synchronization
3. Word synchronization

Carrier synchronization concerns the generation of a reference carrier with a phase closely matching that of the data signal. This reference carrier is used at the receiver to perform a coherent demodulation operation, creating a baseband data signal. Efficient data detection requires that the receiver know when one data symbol ends and the next one begins. This is accomplished by clock or bit synchronization. Word synchronization is simply used to demarcate the serial data into digital words or groups. Perfect or noise-free
synchronization is assumed available at the receiver in this investigation. However, if the synchronization signals are obtained from noisy signals that are present at the receiver input, the reference signals are also noisy. Consequently, the bit error rate (BER) will be larger than that given when noise-free (perfect) synchronization is assumed.

2.6 System Performance

When signal plus noise is present at the receiver input, there is always a finite and calculable probability that a noise peak will reach a sufficient amplitude to cause the receiver to respond incorrectly, thus binary errors are said to occur at the receiver. In this scheme of double symmetrical threshold, there are four ways in which binary errors can occur and these can be described as follows:

I. An error occurs when \( r_0 < TH_1 \) if a binary 1 is sent, that is,

\[
P_{r_1} (\text{error}/s_1\text{sent}) = \int_{-\infty}^{TH_1} p(r_0 / s_1)dr_0
\]

(2-48)

II. An error occurs when \( r_0 > TH_2 \) if a binary 0 is sent, that is,

\[
P_{r_2} (\text{error}/s_0\text{sent}) = \int_{TH_2}^{\infty} p(r_0 / s_0)dr_0
\]

(2-49)

III. An error occurs when \( TH_1 < r_0 < TH_2 \) and

\[
P_r[a_{i+1} = 0/n\Delta \leq X_i \leq (n+1)\Delta] > P_r[a_{i+1} = 1/n\Delta \leq X_i \leq (n+1)\Delta] \]

if a
binary 1 is sent. Let \( C_{l,n}^{i+1} \triangleq P_r[a_{i+1} = 0/n \Delta \leq X_i \leq (n+1)\Delta] > 63 \)

\[ P_r[a_{i+1} = 1/n \Delta \leq X_i \leq (n+1)\Delta], \]

then

\[ P_{r3} (error/s_1\text{sent}) = C_{l,n}^{i+1} \int_{TH1}^{TH2} p(r_0/s_1\text{sent}) dr_0 \quad (2-50) \]

\( C_{l,n}^{i+1} \) for an eight level quantizer can be described as follows:

\[ C_{1,n}^{i+1} = P_r[X_{i+1} \leq v/n \Delta \leq X_i \leq (n+1)\Delta] \quad (2-51) \]

\[ n = 0, 1, 2, 3 \]

\[ C_{2,n}^{i+1} = P_r[-v/2/n \Delta \leq X_i \leq (n+1)\Delta] + P_r[0 \leq X_{i+1} \leq v/2/n \Delta \leq X_i \leq (n+1)\Delta] \quad (2-52) \]

\[ n = -2, -1, 2, 3 \]

\[ C_{3,n}^{i+1} = P_r[-v/4/n \Delta \leq X_i \leq (n+1)\Delta] + P_r[-v/2/n \Delta \leq X_{i+1} \leq -v/4/n \Delta \leq X_i \leq (n+1)\Delta] + P_r[0 \leq X_{i+1} \leq v/4/n \Delta \leq X_i \leq (n+1)\Delta] + P_r[v/2/n \Delta \leq X_{i+1} \leq 3v/4/n \Delta \leq X_i \leq (n+1)\Delta] \quad (2-53) \]

\[ n = -3, -1, 1, 3 \]

Each value of \( n \) in (2-51) - (2-53) selects the proper or valid level to be considered for each bit position \( l \) when a binary 1 is sent.

In generalizing for any number of levels and using (2-32),
\[ C_{i+1}^{n} = \sum_{j=2^{l}-1}^{2^{l}-1} \int_{R_{j}} p[\frac{X_{i+1}+1/n\Delta X_{i}}{\Delta} \leq (n+1)\Delta] dX_{i+1} \]  
(2-54)

\( n \) takes on the values shown in (2-51) - (2-53) for each bit position \((i)\).

Equation (2-54) can be written using (2-33) as

\[ C_{i+1}^{n} = \sum_{j=2^{l}-1}^{2^{l}-1} \int_{R_{j}} \frac{1}{\sqrt{8\pi} \sigma^{2}} \exp [-\frac{X_{i+1}^{2}}{2\sigma^{2}}] \text{erf}(u_{2}/\sqrt{2}) \]

\[ -\text{erf}(u_{1}/\sqrt{2})] dX_{i+1} \]  
(2-55)

IV. Finally, an error occurs when \( TH_{1} \leq r_{0} \leq TH_{2} \) and

\[ P_{r}[a_{i+1}^{n} = 0/n\Delta X_{i} \leq (n+1)] \leq P_{r}[a_{i+1}^{n} = 1/n\Delta X_{i} \leq (n+1)\Delta] \]  
if a binary 0 is sent. Let \( D_{i+1}^{n} = P_{r}[a_{i+1}^{n} = 0/n\Delta X_{i} \leq (n+1)\Delta] \)

\[ \leq P_{r}[a_{i+1}^{n} = 1/n\Delta X_{i} \leq (n+1)\Delta] \], then

\[ P_{r_{4}} \text{ (error/s0sent)} = D_{i+1}^{n} \int_{TH_{1}}^{TH_{2}} p(r_{0}/s0sent) dr_{0} \]  
(2-56)

\( D_{i+1}^{n} \) for an eight level quantizer can be described as follows:

\[ D_{1,n}^{i+1} = P_{r}[X_{i+1} \geq 0/n\Delta X_{i} \leq (n+1)\Delta] \]  
(2-57)

\[ n = -4, -3, -2, -1 \]

\[ D_{2,n}^{i+1} = P_{r}[-V/2 \leq X_{i+1} \leq 0/n\Delta X_{i} \leq (n+1)\Delta] + P_{r}[V/2 \leq X_{i+1} \leq V/n\Delta X_{i} \leq (n+1)\Delta] \]
Each value of \( n \) in (2-56) - (2-59) selects the proper or valid level to be considered for each bit position \( j \) when a binary 0 is sent.

Again generalizing for any number of levels and using (2-38),

\[
D_{n}^{i+1} = \sum_{j=2^{l-1}}^{2^{l}} \int_{S_{j}} p[X_{i+1}/n \leq X_{i} \leq (n+1)\Delta]dX_{i+1}
\]  

(2-60)

\( n \) takes on the values shown in (2-57) - (2-59) for each bit position \( l \). Equation (2-60) can be written using (2-39) as

\[
D_{n}^{i+1} = \sum_{j=2^{l-1}}^{2^{l-1}} \int_{S_{j}} \frac{1}{\sqrt{8\pi \sigma \beta}} \exp[-X_{i+1}^2/2\sigma^2][\text{erf}(u_{2}/\sqrt{2})-\text{erf}(u_{1}/\sqrt{2})]dX_{i+1}
\]

(2-61)

\( n \) values are as shown in (2-57) - (2-59). Then, the bit error rate (BER) is
Combining (2-48), (2-49), (2-50), (2-56), and (2-62), the general expression for the bit error rate is

\[
\text{BER} = \mathcal{P}_r(s_{1\text{sent}}) \int_{-\infty}^{\infty} p(r_0/s_1) dr_0 + \mathcal{P}_r(s_{0\text{sent}}) \int_{-\infty}^{\infty} p(r_0/s_0) dr_0 \\
+ \mathcal{P}_r(s_{1\text{sent}}) \mathcal{C}_r,i+1 \int_{\text{TH1}}^{\text{TH2}} p(r_0/s_1) dr_0 + \mathcal{P}_r(s_{0\text{sent}}) \mathcal{D}_r,i+1 \int_{\text{TH1}}^{\text{TH2}} p(r_0/s_0) dr_0 
\]

where \( \mathcal{P}_r(S_{1\text{sent}}) \) and \( \mathcal{P}_r(S_{0\text{sent}}) \) are known as a priori statistics which are considered to be equally likely. That is,

\[
\mathcal{P}_r(\text{binary 1 sent}) = \mathcal{P}_r(S_{1\text{sent}}) = \frac{1}{2} 
\]

\[
\mathcal{P}_r(\text{binary 0 sent}) = \mathcal{P}_r(S_{0\text{sent}}) = \frac{1}{2} 
\]

Since \( r_0 \) is a Gaussian random variable with a mean value of either \( S_0 \) or \( S_1 \), depending on whether a binary 0 or a binary 1 was sent, the \( p(r_0/S_0 \text{ sent}) \) and \( p(r_0/S_1 \text{ sent}) \) of (2-63) can be described as
\[ p(r_0/S_0) = \frac{1}{\sqrt{2\pi} \sigma_0} \exp\left[-\frac{(r_0-S_0)^2}{2\sigma_0^2}\right] \quad (2-65) \]

\[ p(r_0/S_1) = \frac{1}{\sqrt{2\pi} \sigma_0} \exp\left[-\frac{(r_0-S_1)^2}{2\sigma_0^2}\right] \quad (2-66) \]

where \( \sigma_0^2 = \bar{n}_0^2(t) \) is the average power of the output noise from the processing circuit.

Using equally likely a priori signals and substituting (2-65) and (2-66) into (2-63), the bit error rate becomes

\[
\text{BER} = \frac{1}{2} \left[ \frac{1}{\sqrt{2\pi} \sigma_0} \int_{-\infty}^{\infty} \exp\left[-\frac{(r_0-S_1)^2}{2\sigma_0^2}\right] dr_0 + \frac{1}{2} \int_{\text{TH2}}^{\infty} \exp\left[-\frac{(r_0-S_0)^2}{2\sigma_0^2}\right] dr_0 \right]
\]

\[
+ \frac{C_{i,n+1}}{2} \int_{\text{TH1}}^{\text{TH2}} \exp\left[-\frac{(r_0-S_1)^2}{2\sigma_0^2}\right] dr_0
\]

\[
+ \frac{D_{i,n+1}}{2} \int_{\text{TH1}}^{\text{TH2}} \exp\left[-\frac{(r_0-S_0)^2}{2\sigma_0^2}\right] dr_0 \quad (2-67)
\]

Equation (2-67) can be reduced to the Q(Z) functions that are defined in Section A-1 (Appendix A). Using (A-47) of Section A-2 (Appendix A), then (2-67) can be written as
Another kind of error performance is the error amplitude such as the Mean-Squared Error (MSE). The mean-squared error is evaluated for Zero-Order Hold (ZOH) reconstruction filter. Once again the sampled baseband signal $r_0$ is subjected through the four test conditions. Decisions are made as to whether the received bit is a binary 0 or a binary 1. The received binary signals are reconstructed into their analog sample values and the mean-squared error which consists of sampling error, quantization error, and channel error is derived. The expression for mean-squared error for standard PCM using uniform quantization was given by Spilker [19] as

$$MSE_{STD} = 2\left\{1 - \left[1 - \exp\left(-\alpha T\right)/\alpha T\right] + \nu^2/3.2^{2L} + 4\nu^2/3\left(1 - 2^{-2L}\right)\right\} \cdot BER_{STD} \tag{2-69}$$

where

$$BER_{STD} = 1/2 \cdot \text{erf}\left(\sqrt{S \cdot N_0 B}\right) \tag{2-70}$$

where

- $BER_{STD} =$ bit error rate for standard PCM
- $S =$ transmitted power
- $N_0 =$ one sided noise power spectral density
- $B =$ transmission rate per second
The expression in (2-69) is applicable to the Decision-Directed-Detection with bit error rate (BER) different from that given in (2-70). The bit error rate (BER) used for the Decision-Directed-Detection system is given in (2-68).

The bit error rate of (2-68) can be minimized by using appropriate double symmetrical threshold value (TH1 and TH2). Since TH1 = -TH2, the threshold (TH2) that minimizes the bit error rate can be found by solving $d(BER)/dTH2 = 0$. Integrals of (2-67) are differentiated and can be written using (A-48) of Section A-1 (Appendix A) as

$$
\frac{d(BER)}{d(TH2)} = \frac{1}{2} \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left[-\frac{(TH2-S_1)^2}{2\sigma_0^2}\right] - \frac{1}{2} \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left[-\frac{(TH2-S_0)^2}{2\sigma_0^2}\right]
$$

$$
+ \frac{C_{f,n}^{i+1}}{2} \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left[-\frac{(TH2-S_1)^2}{2\sigma_0^2}\right]
$$

$$
- \frac{C_{f,n}^{i+1}}{2} \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left[-\frac{(TH2-S_1)^2}{2\sigma_0^2}\right]
$$

$$
+ \frac{D_{f,n}^{i+1}}{2} \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left[-\frac{(TH2-S_0)^2}{2\sigma_0^2}\right]
$$

$$
- \frac{D_{f,n}^{i+1}}{2} \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left[-\frac{(TH2-S_0)^2}{2\sigma_0^2}\right] = 0
$$

(2-71)

Using (A-51) of Section A-2 (Appendix A), (2-71) can be written as
\[
\frac{\exp[-2TH2S_1/2\sigma_0^2]+C_{l,n}^i+1\exp[2TH2S_1/2\sigma_0^2]-C_{l,n}^i+1\exp[-2TH2S_1/2\sigma_0^2]}{\exp[2TH2S_0/2\sigma_0^2]+D_{l,n}^i+1\exp[-2TH2S_0/2\sigma_0^2]-D_{l,n}^i+1\exp[2TH2S_0/2\sigma_0^2]}
= \frac{\exp[-S_0^2/2\sigma_0^2]}{\exp[-S_1^2/2\sigma_0^2]}
\]

(2-72)

The threshold values in (2-72) for minimum BER in (2-68) can be found only through numerical iterations because further simplification or reduction of (2-72) in order to solve the threshold value (TH2) is not tractable. Therefore, the analysis of error performance will be done numerically by simulating a communication system with double symmetrical threshold and correlated input signal using a computer and also the relevant equations developed in this chapter. In this simulation, Phase Shift Keying (PSK) signaling is assumed.
CHAPTER 3

SIMULATION PROCESS

A thorough simulation of a communication system requires a good understanding of the operation of each device that makes up the communication system. The basic block diagram of Decision-Directed-Detection PCM system implementation was shown in Figure 2-15 and repeated in Figure 3-1 for convenience.

3.1 Summary of Important Steps in the Simulation

The simulation is achieved by systematic approach that requires some important steps. These steps, which are each represented by a block in Figure 3-1, are summarized as follows:

1. The generation of a Markov Gaussian input signal samples with a known correlation function.

2. The development of a quantizer and a binary encoder algorithm.

3. The development of a look-up table of probability distribution of each bit position in a code word being a binary 0 or a binary 1 for a chosen quantizer generated by solving (2-33) and (2-39) respectively.

4. The choice of binary waveform and the generation of varying transmission noise.
Figure 3-1. Basic block diagram of Decision-Detected-Decision system.
5. The processing of the received signal \( r(t) \), which is binary waveform \( S(t) \) plus transmission noise \( n(t) \), for binary detection.

6. Finally, the choice of the decision making device (comparator) and the calculation of the bit error rate (BER), mean-squared error and signal-to-noise ratios (SNR).

3.2 Discussion of the Simulation Steps

The simulation steps summarized in Section 3.1 are discussed in detail in this section.

3.2.1 Signal Generation

Gaussian random numbers \( Z_i \) with zero mean and standard deviation = 1.0 can be generated by a computer using the Gauss Randnr subroutine. Using these random numbers, a correlated set of samples \( X_i \) having a power spectrum given by

\[
\Phi_{XX}(\omega) = \frac{A^2 \alpha^2}{\alpha^2 + \omega^2} \quad \omega > 0 \text{ (one sided spectral distribution)}
\]  

(3-1)

and the correlation function given by

\[
R(\tau) = \exp(-|\tau|)
\]

(3-2)

is generated using the following algorithm:

\[
X_1 = Z_1 \cdot \sqrt{\sigma^2}
\]

(3-3)
where \( \{\sigma^2[1-R(\tau)^2]\}^{\frac{1}{2}} \) is the standard deviation of the error signal

\[
X_i = Z_i \cdot \{\sigma^2[1-R(\tau)^2]\}^{\frac{1}{2}} + R(\tau)X_{i-1}
\]

(3-4)

\[ i = 2, 3, 4, 5, \ldots, N \]

\( N \) is the number of signal samples

3.2.2 Quantization and Coding

Each input signal sample is coded into a set of binary numbers which are equivalent to the quantization level closest to that sample value. This coding process is accomplished using a serial quantizer. A serial quantizer successively divides the ordinate into two regions. It first divides the axis in half, and observes whether the input sample is in the upper or lower half. The results of this observation generate the most significant bit in the code word. The half-region in which the sample lives is then subdivided into two regions and a comparison is again performed. This generates the next bit. The process continues a number of times equal to the number of bits per sample required. The basic block diagram of a serial 3-bit quantizer is shown in Figure 2-9. The inputs are in the range 0 to 8 volts. Therefore, inputs in the range -4V to 4V used in this investigation are adjusted to the range 0 to 8V by adding 4V to each input sample so that the encoding process illustrated in Figure 2-9 can be used. The operation of the encoding process is illustrated for the following two input sample values: -1.27V and 2.125V. First, the two input sample
values are adjusted to 2.73V and 6.125V by adding 4V to each input sample value. For the adjusted input 2.73V, the first comparison with 8/2 would yield a No answer. Therefore, b₁ = 0. The second comparison with 8/4 would be a Yes answer. Therefore, b₂ = 1, and 8/4 is subtracted from 2.73V, leaving 0.73V. The third comparison with 8/8 would yield a No answer. Therefore, b₃ = 0, and the binary coding of -1.27V into 3 bits would be 010. For the adjusted input 6.125V, the first comparison with 8/2 would yield a Yes answer. Therefore, b₁ = 1, and 8/2 is subtracted from 6.125V, leaving 2.125V. The second comparison with 8/4 would yield a Yes answer. Therefore, b₂ = 1, and 8/4 is subtracted from 2.125V. The third comparison with 8/8 would yield a No answer. Therefore, b₃ = 0, and the binary coding of 2.125V into 3 bits would be 110.

3.2.3 Computation and Tabulation of Bit Probability Distribution

The joint probability density functions of the next sample \( X_{i+1} \) given the range of the present sample \( X_i \) and the consequent probability distribution of each bit position in a code word being a binary 0 or a binary 1 were derived in Chapter 2. Therefore, it is not necessary to repeat their derivation here. Instead, the pertinent equations to their calculation are listed below:

1. The conditional probability density function, that is, (2-25) is
2. The probability distribution of each bit position in a code word being a binary 0, that is, (2-33) is

\[ p[X_{i+1}/n\Delta \leq X_i \leq (n+1)\Delta] = \Omega \exp[-X_{i+1}^2/2\sigma^2][\text{erf}(u_2/\sqrt{2}) - \text{erf}(u_1/\sqrt{2})] \] (3-5)

3. The probability distribution of each bit position in a code word being a binary 1, that is, (2-39) is

\[ P_r[a_{i+1}^j = 0/n\Delta \leq X_i \leq (n+1)\Delta] = \sum_{j=2(\ell-1)}^{2\ell-1} \Omega \exp[-X_{i+1}^2/2\sigma^2] \int_{R_j} \text{[erf}(u_2/\sqrt{2}) - \text{erf}(u_1/\sqrt{2})]dX_{i+1} \] (3-6)

3. The probability distribution of each bit position in a code word being a binary 1, that is, (2-39) is

\[ P_r[a_{i+1}^j = 1/n\Delta \leq X_i \leq (n+1)\Delta] = \sum_{j=2(\ell-1)}^{2\ell-1} \Omega \exp[-X_{i+1}^2/2\sigma^2] \int_{S_j} \text{[erf}(u_2/\sqrt{2}) - \text{erf}(u_1/\sqrt{2})]dX_{i+1} \] (3-7)

where \( \Omega = 1/2(2\pi)^{1/2}\sigma \theta \)

Once a quantizer with a certain number of levels is chosen and coding is accomplished, the probability distribution of each bit position of each level being a binary 0 or a binary 1 is calculated by
computing (3-6) and (3-7) respectively on the computer. The results are tabulated for that quantizer. The exact replica of this tabulation is stored as a look-up table in the receiver for use whenever required. The look-up table for an eight level quantizer is shown in Table 3-1. Table 3-1 (a) shows the computed values of the probability distribution of each bit position in the next word being a binary 0 given the range of the present word while Table 3-1 (b) shows the computed values of the probability distribution of each bit position in the next word being a binary 1 given the same range of the present word.

Table 3-1. Example of the look-up table for an eight level quantizer; (a) the bit probability distribution for binary 0, and (b) the bit probability distribution for binary 1.

(a)

<table>
<thead>
<tr>
<th>GIVEN CODE WORDS</th>
<th>LOOK-UP TABLE OF BIT PROBABILITY DISTRIBUTION FOR BINARY 0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MOST SIGNIFICANT BIT POSITION</td>
</tr>
<tr>
<td>000</td>
<td>0.99431813</td>
</tr>
<tr>
<td>001</td>
<td>0.99999803</td>
</tr>
<tr>
<td>010</td>
<td>0.99999774</td>
</tr>
<tr>
<td>011</td>
<td>0.93180466</td>
</tr>
<tr>
<td>100</td>
<td>0.06351697</td>
</tr>
<tr>
<td>101</td>
<td>0.0</td>
</tr>
<tr>
<td>110</td>
<td>0.0</td>
</tr>
<tr>
<td>111</td>
<td>0.0</td>
</tr>
</tbody>
</table>
3.2.4 Binary Waveform and Transmission Noise Generation

The transmitted signal over a bit interval of \((0, T)\) is represented by

\[
S(t) = \begin{cases} 
  S_0(t) & 0 < t \leq T \text{ for binary 0} \\
  S_1(t) & 0 < t \leq T \text{ for binary 1} 
\end{cases}
\]  

(3-8)

Where \(S_0(t)\) is the waveform that is used if a binary 0 is transmitted, and \(S_1(t)\) is the waveform that is used if a binary 1 is transmitted. Antipodal rectangular pulses of constant height \(A = 1.0\) over the bit interval are chosen to represent the binary waveform \(S(t)\). The binary waveform \(S(t)\) being antipodal implies that \(S_0(t) = -S_1(t)\). Since, the binary waveform is chosen to be of constant amplitude over the bit interval, the variation of signal-to-noise ratio can be achieved by varying the transmission noise level.

The generation of transmission noise with varying noise level is accomplished using a random number generator as follows:
CALL ANORM (XN)

\[ AN = \sigma_0 \times XN \]

\[ ANP = \sigma_0^2 \] (3-9)

where ANP is the average noise power

AN is the amplitude of the noise

\( \sigma_0 \) is the standard deviation which is the parameter varied to obtain varying noise power \( \sigma_0^2 \).

3.2.5 Signal Reception and Processing

The receiver has the job of extracting source information from the received modulated signal which has been corrupted by additive noise. The received signal \( r(t) \) at the input of the receiver is binary signal \( S(t) \) pulse noise \( n(t) \), that is,

\[ r(t) = S(t) + n(t) \] (3-10)

When the received data are of the form \( r(t) = S(t) + n(t) \) and the noise is white and Gaussian, correlation detection is the optimum strategy. The block diagram of a correlation (match filter) detection was shown in Figure 2-14a. The basic operation of correlation detection is to cross-correlate the received data \( r(t) \) with a reference signal \( \omega(t) \), that is the stored replica of the binary waveform \( S_1(t) \). The cross-correlated signal is integrated to obtain analog baseband signal that is denoted by
The analog baseband signal $r_0(t)$ of (3-11) can be represented digitally by

$$BASE = \sum_{k=1}^{n} r_k \cdot \omega_k \cdot \Delta t \tag{3-12}$$

where $\Delta t = T_B/n$

$T_B = T_s/L$ is transmission time per bit
$T_s = $ sampling rate (transmission time for one sample)
$L = $ number of bits per sample
$n = $ number of discrete samples of $S(t)$ taken within a $T_B$
$\omega_k = S_1(t) - S_0(t)$ is the discrete reference signal
$r_k = $ the discrete form of binary signal plus noise

Simulation of (3-12) is accomplished by taking samples of the binary waveform $S(t)$ $n$ times within a transmission time per bit $T_B$ as illustrated in Figure 3-2. Also, taken $n$ times within $T_B$ are samples of the reference signal $\omega(t)$. This is illustrated in Figure 3-3. Furthermore, a noise sample is added to each of the $n$ samples of the binary waveform $S(t)$ to obtain received signal $r(t)$ as illustrated in Figure 3-4. Finally, each $n$ sample of the received signal $r_k$ is
Figure 3-2. Binary waveform $S(t)$.
Figure 3-3. Reference signal $\omega(t)$. 
Figure 3-4. Received signal plus noise r(t).
cross-correlated with the corresponding $k^{th}$ sample of the reference signal $\omega_k$. The resultant $n$ cross-correlated samples are summed and multiplied by $\Delta t$ to obtain the analog baseband $r_0(t)$ denoted digitally by (3-12). This is illustrated in Figure 3-5. The analog baseband is sampled at $T_B$ interval and the output of the sampler is illustrated in Figure 3-6. The sampled baseband signal is converted into a binary signal by feeding it into a decision device (a comparator).

3.2.6 Decision Level Selection and System Performance

The system performance measure is the derived error that results from the decisions made when bit-by-bit detection is performed in the decision device. The decision device chosen in this investigation has double symmetrical threshold values. The symmetry of the thresholds implies that the first threshold $TH1$ is the negative of the second threshold $TH2$. The thresholds are varied in order to determine the optimum value for a particular signal-to-noise ratio. Thus, the optimum threshold obtained when the error performance is minimum depends on the signal-to-noise ratio which in turn depends on the noise power $\sigma_0^2$. As the noise level $\sigma_0$ is varied progressively through the region of high signal-to-noise ratio (that is, region where the signal level predominates the noise level), the optimum thresholds will decreasingly approach zero. Consequently, the error performance of this system in this region will progressively approximate the error performance of a standard single (zero) threshold system. Therefore, the variation of noise level $\sigma_0$ is made to favor
Figure 3-5. Analog baseband signal $r_0(t)$. 
Figure 3-6. Sampled baseband signal $r_0$ (uT).
the low signal-to-noise ratio region (that is, the region where the noise predominates the signal).

The establishment of the double symmetrical threshold results in decision making as regards to whether the received bit is a binary 0 or a binary 1 by comparing the received signal (BASE) of (3-12) against the double threshold values. Some of these decisions will result in binary errors because bits are regenerated that have opposite level to those transmitted. There are two kinds of error performance evaluated in this study, namely: bit error rate (BER) and mean-squared error (MSE). There are four ways in which error can occur and they have been described in Section 2.6 and repeated here for convenience:

1. An error occurs when BASE < TH1 if a binary 1 is sent.
2. An error occurs when BASE > TH2 if a binary 0 is sent.
3. An error occurs when TH1 ≤ BASE ≤ TH2 and
   \[ P_T[a_{i+1} = 0 | nΔ < X_i \leq (n+1)Δ] > P_T[a_{i+1} = 1 | nΔ < X_i \leq (n+1)Δ] \] if a binary 1 is sent.
4. An error occurs when TH1 ≤ BASE ≤ TH2 and
   \[ P_T[a_{i+1} = 0 | nΔ < X_i \leq (n+1)Δ] < P_T[a_{i+1} = 1 | nΔ < X_i \leq (n+1)] \] if a binary 0 is sent.

These conditions are simulated in the computer to evaluate bit error rate performance. This error performance is computed by subjecting the baseband signal (BASE) to the above mentioned error test
conditions where decisions are made as to which received bits are in error. The number of bits in error is counted.

Let $E_i$ be the $i^{th}$ bit in error, then average bit error rate (BER) is given by

$$\text{BER} = \frac{1}{N_L} \sum_{i=1}^{K} E_i$$

(3-13)

where $N_L$ is the total number of bits

$K$ is the total number of bits in error

The other kind of error performance, the mean-squared error (MSE) is evaluated assuming a zero order hold (ZOH) reconstruction filter. This error performance is evaluated by once again subjecting the baseband signal (BASE) through four test conditions where this time, decisions are made as to whether the received bit is a binary 0 or a binary 1. The received binary signals are reconstructed into their analog sample values and the mean-squared error which comprises of sampling error, quantization error, and channel error is computed as follows:

The mean-squared sampling error ($\xi_s$) for the zero order hold reconstruction filter was given in [19] as
\[ \xi_s = 2\left\{1 - \frac{1 - \exp(-aT)}{aT}\right\} \]

The mean-squared quantization error (\(\xi_q\)) can be described as

\[ \xi_q = \frac{1}{\text{SPWR}} \sum_{i=1}^{N} (X_i - QX_i)^2 \]  

where SPWR = signal power

\(N\) = the number of signal samples

\[ QX_i = \sum_{j=1}^{L} a_{ij}2^{-j} \] is the \(i\)th quantized sample

\(a_{ij} = \pm 1\) are the serial quantizer output bits for the \(i\)th sample before the addition of the channel noise.

The mean-squared channel error (\(\xi_c\)) is described as

\[ \xi_c = \frac{1}{\text{SPWR}} \sum_{i=1}^{N} (QX_i - Y_i)^2 \]  

where \(Y_i = \sum_{j=1}^{L} \beta_{ij}2^{-j}\) is the reconstructed sample derived from the received bits \(\beta_{ij}\). \(\beta_{ij}\) may have had errors due to the addition of transmission noise. Therefore \(Y_i\) is the estimate of \(X_i\).

Totty and Clark [13] have shown that these three sources of error are additive when the quantization is done optimally.

Therefore, the mean-squared error (MSE) can be described by
3.3 Program Description and Flowcharts

Programming will be described under two headings, namely: the main programs and the subprograms.

A. Main Programs

The programming of the system described in Section 3.2 above is done in two parts. The first part of the program generates the probability density functions shown in Figure 2-8, from which a look-up table of probability distribution of each bit position of a particular quantizer level states being a binary 0 or a binary 1 is developed. As an example, the look-up table for eight level quantizer has been shown in Table 3-1. The second part of the program generates the baseband signals (BASE) on which decisions are made as to which bits are received (binary 0 or binary 1) using the double symmetrical threshold values and the pre-computed look-up table when required. The decision making leads to the evaluation of system performance in terms of two error performance criteria namely: bit error rate and mean-squared error.

Referring to the flowchart of Figure 3-7, the program starts by initializing constants. Immediately following the initialization of constants, is the reading in of the limits of integration for the computation of the probability distribution of each bit position being a

\[ \text{MSE} = \xi_s + \xi_q + \xi_c \]
Figure 3-7. The main program flowchart for the generation of the cumulative probability distribution of each bit position.
binary 0 or a binary 1. The development of the index of quantizer level states and the natural binary coding of these index level states are followed by the calculation of the given present level states of the quantizer and the computation of the value of (2-27) through error functions (EF1 and EF2). Finally, the calling of subroutine INTEG invokes the external function FUNNY which calculates and returns the values of all variables in (2-25) except the variable of integration. With the external function FUNNY invoked, the subroutine INTEG calculates and returns the values of the probability distribution of each bit position of the quantizer level states being a binary 0 or a binary 1. This is done by solving (2-33) and (2-39) respectively. These bit probability distribution values set up the look-up table which is readily available in the second part of the program and used whenever needed.

The flowchart of Figure 3-8 shows the flow diagram of the second part of the program. Once again, the program starts by initializing constants. Then it reads in the look-up table of the bit probability distribution generated in part A of the program. The subroutine SIGNAL is called at this point and the returned input signal samples from the subroutine SIGNAL are quantized and coded. The generation of varying noise levels is followed by the derivation of baseband signal (BASE) and the development of the variable double symmetrical thresholds (TH1 and TH2) against which the baseband signals (BASE) are compared in order to decide on the value of the
INITIALIZATION OF CONTENTS

READ IN THE CUMULATIVE PROBABILITY DISTRIBUTION OF EACH BIT POSITION BEING A '0' OR '1'

CALL SUBROUTINE SIGNAL. RETURN ALL SIGNAL SAMPLES.

QUANTIZE AND BINARY CODE THE SIGNAL SAMPLES

GENERATE VARYING NOISE LEVELS DERIVE BASEBAND (TESTS) SIGNAL AND CALCULATE TOTAL NOISE POWER

DEVELOP THE VARIABLE DOUBLE SYMMETRICAL THRESHOLDS (TH1 & TH2)

Figure 3-8. The main program flowchart for error performance analysis.
Figure 3-8. The main program flowchart for error performance analysis.
(page 2 of 5)
Figure 3 - 8. The main program flowchart for error performance analysis.
Figure 3-8. The main program flowchart for error performance analysis.
Figure 3-8. The main program flowchart for error performance analysis.
received binary signal (either 0 or 1). At this point, the flowchart splits into two branches. The route that each of the two branches takes depends on which kind of error performance analysis is being used. In the route taken by one of the branches, points (A) and (C) through (E) in the flow diagram, every received binary signal undergoes the four error test conditions enumerated in Section 3.2 to determine which bits are in error. Consequently, the average bit error rate (BER), average noise power (APN), and the input signal-to-noise ratio (SNRIN) are calculated. In the route taken by the other branch, points (B) and (F) through (H) in the flow diagram, every received binary signal undergoes four test conditions to determine whether the received bit is a binary 0 or a binary 1. The program then reconstructs the received binary signals into their analog sample values. Finally, the mean-squared error (MSE) described as the mean-squared quantization error plus the mean-squared channel error, average noise power (APN), input and output signal-to-noise ratios are calculated.

B. Subprograms

There are two subroutines (INTEG and SIGNAL) and an external function (FUNNY) in this simulation.

The subroutine INTEG is an integration subroutine which uses Simpson's rule approximation to calculate the probability distribution of each bit position being a binary 0 or a binary 1. A simplified
flowchart of subroutine INTEG is shown in Figure 3-9. The program starts by computing the number of integration intervals and initializing the sum of the Simpson's rule of integration to zero. After the computation of the variable of integration (which is the next sample $X_{i+1}$) and setting the appropriate Simpson's multiplication factor for integration points within a curve (1.0 for the first and the last points, 4.0 for even points and 2.0 for odd points), the probability density function to be integrated is computed through the external function FUNNY. Finally, the Simpson's rule of multiplication and summation is applied to determine the probability distribution of each bit position being a binary 0 or a binary 1.

The subroutine SIGNAL generates equi-correlated Markov Gaussian input signal samples.

The external function FUNNY calculates the argument of the joint conditional probability density function which consists of error and exponential functions. The flowchart of Figure 3-10 shows the simplified flow diagram of the external function FUNNY. The program defines and computes all necessary constants. Using these constants, the program finally computes all the unknown variables in the joint conditional probability density function with the exception of the variable of integration (that is, $X_{i+1}$) which is calculated in the subroutine INTEG.

The flowcharts of Figures 3-7, 3-8, 3-9, and 3-10 were translated into FORTRAN code and are furnished in Appendix B.
START

COMPUTE NUMBER OF INTEGRATION INTERVALS

INITIALIZE SUM TO ZERO

COMPUTE VARIABLE OF INTEGRATION

SET MULTIPLICATION FACTOR TO 1.0

IS THIS THE FIRST POINT OR THE LAST POINT OF THE CURVE?

IF AN EVEN POINT SET FACTOR TO 4.0. IF ODD SET FACTOR TO 2.0

COMPUTE FUNCTION TO BE INTEGRATED

APPLY SIMPSON'S RULE TO SUM FUNCTION

ARE ALL POINTS ON CURVE CONSIDERED?

RETURN

Figure 3 - 9. Flowchart for subroutine INTEG.
Figure 3 - 10. Flowchart for external function FUNNY.
CHAPTER 4

SIMULATION RESULTS

In this chapter, the performance results of a communication system using the Decision-Directed-Detection (DDD) PCM scheme are analyzed. The results reported here consider two measures of system performance. One measure of the system performance is the bit error rate (BER) as a function of signal-to-noise ratio measured in decibels. The other measure of the system performance is the mean-squared error (MSE) between the input signal and the output signal as a function of bit rate (L/τ), measured in bits per second. The results obtained are analyzed and compared to those of standard PCM system for both measures of the system error assuming coherent Phase Shift Keying (PSK) transmission.

4.1 Standard PCM

For the zero-order hold reconstructor, the mean-squared error for standard PCM using uniform quantization was given in (2-69) as

\[ \text{MSE}_{\text{STD}} = 2\{1-1/\alpha \tau [1-\exp(-\alpha \tau)]\} + V^2/(3.2^L) + 4V^2/3(1-2^{-2L}) \cdot \text{BER}_{\text{STD}} \] (4-1)

where as given in (2-70)

\[ \text{BER}_{\text{STD}} = 1/2\{1-\text{erf}(\sqrt{S/N_0B})\} \] (4-2)

In terms of energy per bit, (4-2) becomes

\[ \text{BER}_{\text{STD}} = 1/2\{1-\text{erf}(\sqrt{E_b/N_0})\} \] (4-3)
where

$$E_b = \frac{S}{B}$$ is the energy per bit

With these relations, useful comparisons can be made between standard PCM and Decision-Directed-Detection PCM on the basis of the signal-to-noise ratio required for a prescribed error for bit error rate (BER) measure and the transmission rate required for a prescribed error for mean-squared error measure.

4.2 Bit Error Rate for Decision-Directed-Detection

Figure 4-1 illustrates the effects of increasing the number of bits per sample on bit error rate for three different values of bits per sample. The curves are plotted for fixed transmitted power \((S)\), noise spectral density \((N_0)\) and specified bit rates \((B)\). It is evident from these error curves that there is a substantial increase in the bit error rate as the number of bits per sample is increased for a specified signal-to-noise ratio \((S/N_0B)\). This result is true due to the fact that for specified bit rates, the sampling interval \(T = L/B\) increases, which in turn, translates into lower interbit dependency as the number of bits per sample is increased. The direction of this error increase as the number of bits per sample is increased, is towards the bit error rate for standard PCM. Therefore, the bit error rate for standard PCM with any number of bits per sample sets the upper limit for the bit error rate of this system since, from (4-2), the bit error rate for standard PCM is independent of the number of
Figure 4-1. Bit error rate Vs SNR in dB for three different values of number of bits per sample L for Decision-Directed-Detection scheme. Comparison is made with standard PCM.
bits per sample for specified signal-to-noise ratios. The choice of four bits per sample in Figure 4-1 is arbitrary.

Figure 4-2 shows the effect of the correlation between samples on the bit error rate as transition is made from the most significant bit position (MSB) to the least significant bit position (LSB) for a specified signal-to-noise ratio. Evidently, the bit error rate progressively increases as transition is made towards the least significant bit position because the interbit dependency decreases in that direction. Hence, the effect of using statistical dependence in data is to reduce errors in high order (most significant) bits of the code. Since the statistical dependence is a constraint upon the relative amplitudes of the data samples, the gains afforded by Decision-Directed-Detection PCM over standard PCM using a priori data statistics will be greater for a specified error amplitude parameter such as mean-squared error than for the bit error rate. The results of system performance in terms of mean-squared error criterion are given in Section 4.4.

The error curves in Figure 4-3 illustrate the effect of correlation between samples on the system bit error rate for a specified number of bits per sample and signal-to-noise ratio. As the correlation \( \exp (-\alpha \tau) \) between samples gets closer to 1.0, the bit error rate decreases. This is possible because the higher correlation between samples results in much better prediction of the received bits since the probability of bits in the next code word being the same as the
Figure 4-2. Bit error rate Vs SNR in dB for different bit positions for a specified number of bits per sample L = 4. Comparison is made with standard PCM.
Figure 4-3. Bit error rate Vs SNR in dB for various values of correlation coefficient with $L = 4$. Comparison is made with standard PCM.
bits in the present code word increase (tending to 1.0) as the correlation between samples tends to 1.0. The results also indicate that the interbit dependence makes it possible for this system to obtain specified bit error rates with significantly less transmitted power than that required by standard PCM system.

4.3 Threshold Sensitivity

Figure 4-4 indicates the effect of signal-to-noise ratio and number of bits per sample respectively on optimum threshold for coherent PSK signaling. The results indicate that the optimum threshold approaches zero value as the signal-to-noise ratio and the number of bits per sample are increased. Consequently, in the region of high signal-to-noise ratio and large number of bits per sample respectively, the error performance of this system becomes identical to that of a standard PCM system.

4.4 Comparative Performance of Error-Correction Coding Schemes to the Decision-Directed-Detection Scheme

The results of error-correction coding techniques on channels modeled as additive, white Gaussian noise channels are analyzed. The objective is to determine the coding gain when the error-correction coding schemes are compared to uncoded system at a specified error rate. It should be noted that all coded systems have coding threshold. A coding threshold is the signal-to-noise ratio value below which a coded system provides negative coding gain (poorer performance) when compared to uncoded system.
Figure 4-4. Optimum threshold Vs SNR for different bit positions for L = 4.

DDD₁ = least significant bit position
DDD₂ = next higher order bit position
DDD₃ = most significant bit position
The BER of (23,12) triple-error-correcting Golay Code was compared to the BER without coding assuming coherent PSK by Couch II [10]. The coding gain of about 2.15 dB is achieved at BER = E-5. The coding threshold is about 3.5 dB and occurs at about BER = 2E-2. Negative coding gain is achieved for any BER above this value. The performance improvement for 1/2 convolutional codes of constraint lengths 3, 5, 6, and 7 with Viterbi decoding was given by Spilker [19] to be approximately 4 to 6 dB at BER = E-5. Specifically, Clark and Cain [7] indicated that 1/2 convolutional code of constraint length = 6 provides 5 dB of coding gain at BER = E-5. The coding threshold is about 1 dB and occurs at about BER = 6E-2. Once again, negative coding gain is achieved for any BER above this value.

Figure 4-1 indicated that for number of bits per sample 3, 4, and 5 and specified correlation coefficient, the DDD scheme achieves performance gain of approximately 1.375 to 3.25 dB over the standard PCM at BER = 6E-2. Specifically, the number of bits per sample = 4 provides 2 dB gain at BER = 6E-2. At error rates lower than 6E-2, the DDD scheme and standard PCM yield essentially the same performance. While the performance gain increases for error rates above 6E-2. For example, at BER = 1.17E-1, the gain is about 2.5 dB for number of bits per sample = 4.

Finally, it can be deduced from the above discussions that at higher error rates (BER ≥ 6E-2), the DDD scheme's performance is better than the error-correction coding schemes. Thus, very strong consideration should be given to DDD technique at very high error rates. While at very low error rates (BER ≤ E-5), the error-
correction coding scheme has considerable advantage over the DDD technique.

4.5 Mean-Squared Error

The performance of the system in terms of mean-squared error parameter is investigated in this section. The total mean-squared error is generally separated into sampling error, quantization error and channel error so that the effect of each of these component errors on the system can be investigated. This kind of analysis can provide considerable insight on the tradeoff studies involving sampling rates, quantization levels and channel noise. Furthermore, useful comparisons are made between this system and the standard PCM system on the basis of the transmission (bit) rate required for a prescribed error.

Figure 4-5 illustrates the various errors (that is, sampling error, quantization error, channel error and mean-squared error which is the addition of the other three) in Decision-Directed-Detection (DDD) PCM as a function of bit rate \((L/\tau)\). These curves are plotted for signal-to-noise density ratio \(S/N_0 = 200\) and number of bits per sample \(L = 4\). Figure 4-5 clearly indicates that there exists a threshold value in bit rate at which the overall mean-squared error is minimum. This threshold value in bit rate is referred to as the opti-
Figure 4-5. Sampling, quantization, and channel error for Decision-Directed-Detection for a specified number of bits per sample $L = 4$ and $SN_0 = 200$. 

$\begin{array}{l}
\text{DDD} = \text{Decision-Directed-Detection} \\
\text{DDD}_1 = \text{sampling error} \\
\text{DDD}_2 = \text{quantizing error} \\
\text{DDD}_3 = \text{channel error} \\
\text{DDD}_4 = \text{total mean-square error}
\end{array}$
mum bit rate. Bit rates below this optimum value are considered as low bit rates and bit rates above this optimum value are considered as high bit rates. Low bit rates translate into high signal-to-noise ratio while high bit rates translate into low signal-to-noise ratio. Consequently, the sampling error decreases and the channel error increases while the quantization error remains constant as the bit rate is increased for any specified number of bits per sample. This implies that at low bit rates, sampling error and quantization error are dominant because the effect of channel noise is very small (negligible) in this region. But as the bit rate increases above its optimum value (that is, high bit rates), assuming fixed transmitted power (S), the channel error dominates. This is the region of interest in this investigation.

Figure 4-6 compares the various errors (sampling error, quantization error, and channel error) in both Decision-Directed-Detection PCM system and standard PCM on the basis of bit rate required for a prescribed error. The comparison is made for a fixed value of signal-to-noise density ratio $S/N_0 = 200$ and a specified number of bits per sample $L = 4$. As expected, the sampling and quantization errors effect of this system are identical to those of the standard PCM system. Assuming fixed transmitted power, the results also indicate that the channel error effect of this system begins to decrease compared to that of the standard PCM as the bit rate increases above its optimum value. Below this optimum value, the channel error is the
Figure 4-6. Compares sampling, quantization, and channel error in both Decision-Directed-Detection PCM and standard PCM for a specified number of bits per sample $L = 4$ and $S/N_0 = 200$. 

- $\text{DDD} \rightarrow$ Decision-Directed-Detection PCM
- $\text{STD} \rightarrow$ Standard PCM
- $\text{DDD}_1 - \text{STD}_1$ = sampling error for both systems
- $\text{DDD}_2 - \text{STD}_2$ = quantizing error for both systems
- $\text{DDD}_3 - \text{STD}_3$ = channel error for both systems
same for both systems. Therefore, the improvement in error performance of this system over the standard PCM system is accomplished only by the reduction in channel error effect as the bit rate increases above its optimum value. This is true because with fixed value of number of bits per sample \( L \), high bit rates translate into small sampling intervals \( \tau = L/B \), which in turn, translate into high correlation between samples \( \exp(-\tau) \). Hence, better detection of the received bits. Figure 4-7 illustrates the results of plotting the rate distortion versus bit rate for both standard PCM and Decision-Directed-Detection PCM for a specified signal-to-noise density ratio \( S/N_0 = 200 \) and number of bits per sample \( L = 4 \). These curves clearly indicate that there is a substantial gain in using Decision-Directed-Detection PCM over standard PCM as the bit rate increases above its optimum value. Below the optimum bit rate, there is no significant improvement because the sampling and quantization errors dominate in this region (high signal-to-noise ratio region). Since higher bit rates translate into higher correlation between samples, Figure 4-7 in other words indicates that the mean-squared error of this system decreases considerably compared to that of standard PCM as the correlation between samples gets closer to 1.0. The system offers better detection for the received bits due to higher correlation between samples.

The results of plotting the rate distortion versus bit rate for both standard PCM and Decision-Directed-Detection PCM for three dif-
Figure 4-7. Distortion rate Vs bit rate for ZOH reconstruction for both Decision-Directed-Detection PCM and standard PCM for $L = 4$ and $S/N_0 = 200$.

DDD ➔ Decision-Directed-Detection
STD ➔ Standard PCM

$\text{DDD}_1$ = mean-square error for $S/N_0 = 200$
$\text{STD}_1$ = mean-square error for $S/N_0 = 200$
ferent values of signal-to-noise density ratio $S/N_0 = 200, 400, 600,$ and specified number of bits per sample $L = 4$ are illustrated in Figure 4-8. These curves indicate that the fixed bit rates required to produce substantial improvement in using Decision-Directed-Detection PCM over standard PCM increases as the signal-to-noise density ratio increases. The performance of this system is substantially improved in the high bit rate region which is the region of low signal-to-noise power ratio and it takes higher bit rates for this low signal-to-noise power ratio region to be reached as the signal-to-noise density ratio is increased.

Figures 4-9 and 4-10 indicate the various errors in Decision-Directed-Detection PCM and standard PCM respectively as a function of bit rate. These curves are plotted for various number of bits per sample $L = 3, 4, 5,$ fixed transmitted power and signal-to-noise density $S/N_0 = 200$ and specified bit rates. Figure 4-9 indicates that for specified bit rates, the sampling error increases as the number of bits per sample is increased because the sampling interval $\tau = L/B$ increases, which in turn, translates into lower sampling rates as the number of bits per sample is increased. The quantization error decreases because each level in the quantizer closely approximates the sample value that they represent due to the increase in the number of levels in the quantizer as the number of bits per sample is increased. Finally, the channel error is the same for the various number of bits per sample until above the optimum bit rate. Then the
Figure 4-8. Distortion rate Vs bit rate for various S/N and specified number of bits per sample L = 4 for both Decision-Directed-Detection and standard PCM.
Figure 4-9. Sampling, quantization, and channel error for various number of bits per sample for Decision-Directed-Detection for a specified signal-to-noise = 200.
Figure 4-10. Sampling, quantization, and channel error for various number of bits per sample for standard PCM for a specified signal-to-noise = 200.
channel error increases as the number of bits per sample increases for a specified bit rate above the optimum value. The fixed transmitted power and specified high bit rates will result in low energy per bit \( (i.e. E_b = S/B) \) which in turn translates into low signal-to-noise ratio. Although the correlation between samples is high for any specified high bit rate, it decreases as the number of bits per sample is increased. Hence, the effect of channel noise is high and increases as the number of bits per sample is increased in this region. While at specified low bit rates and fixed transmitted power, the energy per bit is high and so is the signal-to-noise ratio. The correlation between samples in this region is low and decreases further as the number of bits per sample is increased. For this reason, the samples tend to be uncorrelated in this region. Consequently, the effect of channel noise is small and virtually the same as the number of bits per sample is increased in this region. Figure 4-10 on the other hand shows once again that the sampling error increases and the quantization error decreases for the same reasons given above as the number of bits per sample is increased for a specified bit rate. But the channel error is the same as the number of bits per sample is increased for a specified bit rate because correlation between samples is zero (that is, samples are independent).

Finally, Figure 4-11 shows the overall mean-squared error in Decision-Directed-Detection PCM as a function of bit rate for three different values of number of bits per sample \( L = 3, 4, 5 \). These cur-
DDD → Decision-Directed-Detection PCM

DDD₁ = distortion rate for 3 bit quantizer
DDD₂ = distortion rate for 4 bit quantizer
DDD₃ = distortion rate for 5 bit quantizer
signal-to-noise = 200

Figure 4-11. Distortion rate Vs bit rate for Decision-Directed-Detection PCM for various number of bits per sample for a specified signal-to-noise = 200.
ves illustrate the effect of distortion as a function of number of bits per sample for a specified bit rate. It can be seen from these curves that there exists a threshold value above the optimum bit rate (about 80 bits per second in Figure 4-11), below which the distortion rate or the overall mean-squared error is approximately the same for various number of bits per sample. Above this threshold value, which is within the region of interest in this investigation (high bit rates), the distortion rate increases with increase in the number of bits per sample. These results indicate that it is more efficient to quantize coarsely (i.e. reducing the number of quantizer levels) and to sample more rapidly (i.e. high correlation between samples) in a noisy environment to improve performance substantially. Rapid sampling is accomplished at the expense of increased system capacity and can also lead to an excessive sampling rate. Therefore, in design, the choice of adequate sampling rate will be based on the tradeoffs between the percentage of error that can be tolerated for that kind of transmission (that is, system accuracy required) and the system capacity needed.
Closed-form expressions for the evaluation of system performance in terms of bit error rate (BER) and mean-squared error (MSE) using the Decision-Directed-Detection PCM scheme for correlated data in noisy environment are formulated. Results for the total mean-squared error of the system using the zero-order hold reconstructor were obtained assuming mutual independence of the signal, the quantization error and channel error. Finally, the performances of the Decision-Directed-Detection PCM scheme were compared to the performances of the standard PCM scheme.

Based on the results obtained from the graphs plotted, the following conclusions are offered.

1. This system takes advantage of correlation between adjacent samples of the input waveform to improve performance. As discussed in Section 4.2, interbit dependence makes it possible for this system (Decision-Directed-Detection) to obtain specified bit error rates with significantly less transmitted power than that required by standard PCM system. Consequently, the performance of this system is superior to that of the standard PCM in the region of high correlation between adjacent samples and low signal-to-noise power ratio.
2. The system error is insensitive to high signal-to-noise power ratios when compared to the standard PCM system because its system performance depends solely on the channel noise error effect, which in turn depends on the transmission channel signal-to-noise power ratio, the transmission rate and the modulation method used.

3. The advantage of this system over the error-correction coding schemes is not only in the superior performance in the region of very low signal-to-noise ratio, but also in the fact that only the detection process of the receiver end is modified. The transmitter end is unaltered when compared to that of the standard PCM as opposed to the error-correction coding schemes where extra hardwares for encoding and correcting errors are required.

4. Design formulas are obtained, in terms of well-defined system parameters (such as input signal statistics, quantizer limits, number of bits, etc.), which will allow efficient design of this system with various trade-off considerations. These trade-off considerations involve sampling rates, quantization levels and channel noise. Results from Section 4.4 conclusively show that it is more efficient to quantize coarsely and to sample more rapidly in a noisy environment to improve performance substantially. This indicates fundamental trade-offs between
system accuracy and system capacity required for a particular kind of transmission.

5. Finally, it can be concluded that simulation results support the theoretical assertions made.
REFERENCES


APPENDIX A

MATHEMATICAL IDENTITIES, AND SIMPLIFICATIONS OF
MATHEMATICAL EQUATIONS
APPENDIX A

MATHEMATICAL IDENTITIES, AND SIMPLIFICATION OF MATHEMATICAL EQUATIONS

This Appendix is intended to provide needed mathematical identities and to simplify complex mathematical equations derived in Chapter 2 to more manageable equations.

A-1

MATHEMATICAL IDENTITIES

Definite Integrals

\[
\int_{-\infty}^{\infty} \exp \left[ -(ax^2+bx+c) \right] dx = \sqrt{\pi/a} \exp(b^2-4ac)/4a \quad (A-1)
\]

\[
\int \exp (-a^2x^2) \text{erfc}(bx) dx = \frac{1}{\sqrt{\pi}a} \left\{ \exp(-a^2Z^2)\text{erfc}(aZ) - \frac{b}{(a^2+b^2)\frac{1}{2}} \text{erfc} \left( Z(a^2+b^2)^{\frac{1}{2}} \right) \right\} \quad Z > 0 \quad (A-2)
\]

Error Functions

\[
Q(Z) \triangleq \frac{1}{\sqrt{2\pi}} \int_{Z}^{\infty} \exp(-x^2/2) dx \quad (A-3)
\]

\[
Q(-Z) = 1 - Q(Z) \quad (A-4)
\]
erfc(Z) \Delta \frac{2}{\sqrt{\pi}} \int_{Z}^{\infty} \exp(-x^2)dx \quad (A-5)

erf(Z) \Delta \frac{2}{\sqrt{\pi}} \int_{0}^{Z} \exp(-x^2)dx \quad (A-6)

Q(Z) = \frac{1}{2} \text{erfc}(Z/\sqrt{2}) \quad (A-7)

Q(Z) = \frac{1}{2}[1 - \text{erf}(Z/\sqrt{2})] \quad (A-8)

Leibniz's Rule

\[ \frac{d}{dx} \left[ \int_{a(x)}^{b(x)} f(\lambda, x) d\lambda \right] = f[b(x), x] \frac{db(x)}{dx} - f[a(x), x] \frac{da(x)}{dx} \]

Bivariate Gaussian Density Function
\[ p(X_{i+1}, X_i) = \frac{1}{2\pi \sigma_{i+1} \sigma_i (1-\rho)^{1/2}} \exp \left[ -\frac{1}{2 (1-\rho^2)} \right] \]

\[
\left[ \frac{X_{i+1} - M_{i+1}}{\sigma_{i+1}} \right]^2 - 2\rho \left[ \frac{X_{i+1} - M_{i+1}}{\sigma_{i+1}} \right] \left[ \frac{X_i - M_i}{\sigma_i} \right] + \left[ \frac{X_i - M_i}{\sigma_i} \right]^2 \right] 
\]

\[
\text{if } -\infty < X_{i+1} < \infty, -\infty < X_i < \infty
\]

0 otherwise \hspace{1cm} (A-10)

where \( \sigma \) and \( M \) are standard deviation and mean of the Gaussian process respectively.

If \( \sigma_{i+1} = \sigma_i = \sigma \), and \( M_{i+1} = M_i = 0 \), then (A-10) becomes

\[ p(X_{i+1}, X_i) = \frac{1}{2\pi \sigma^2 (1-\rho^2)^{1/2}} \exp \left[ -\frac{1}{2 \sigma^2 (1-\rho^2)} \left[ \frac{X_{i+1}^2 - 2\rho X_{i+1} X_i + X_i^2}{2} \right] \right] \]

\[
\text{if } -\infty < X_{i+1} < \infty, -\infty < X_i < \infty
\]

0 otherwise \hspace{1cm} (A-11)

where \( \rho \) is correlation between variables.
SIMPLIFICATION OF MATHEMATICAL EQUATIONS

Conditional Probability Density Function

The conditional probability density function of (2-21) is simplified to a more manageable equation as follows:

\[
\frac{(n+1)\Delta}{\int_{-\infty}^{\infty} dX_{i+1} \int_{n\Delta}^{(n+1)\Delta} p(X_{i+1}, X_i) dX_i} = \frac{n\Delta}{\int_{-\infty}^{\infty} dX_{i+1} \int_{n\Delta}^{(n+1)\Delta} p(X_{i+1}, X_i) dX_i}
\]

(A-12)

Let \( \beta = \int_{-\infty}^{\infty} dX_{i+1} \int_{n\Delta}^{(n+1)\Delta} p(X_{i+1}, X_i) dX_i \) in (A-12) (A-13)

Equation (A-12) can be written using (A-13) as

\[
\frac{(n+1)\Delta}{\int_{-\infty}^{\infty} dX_{i+1} \int_{n\Delta}^{(n+1)\Delta} p(X_{i+1}, X_i) dX_i} = \frac{1}{\beta} \int_{n\Delta}^{(n+1)\Delta} p(X_{i+1}, X_i) dX_i
\]

(A-14)

Substituting (A-11) into (A-14),
\[
p[X_{i+1}/n \Delta < X_i < (n+1) \Delta] = \frac{1}{2\pi\sigma^2(a-\rho^2)^{1/2}} \beta^{(n+1)\Delta} \exp \left[ -\frac{1}{2\sigma^2(1-\rho^2)} \right]
\]
\[
\exp \left[ -\frac{X_{i+1}^2 - 2\rho X_{i+1}X_i + X_i^2}{2\sigma^2(1-\rho^2)} \right] dX_i
\]

\[
p[X_{i+1}/n \Delta < X_i < (n+1) \Delta] = \frac{1}{2\pi\sigma^2(a-\rho^2)^{1/2}} \beta^{(n+1)\Delta} \exp \left[ -\frac{X_{i+1}^2}{2\sigma^2(1-\rho^2)} \right]
\]
\[
\exp \left[ -(X_{i+1}^2 - 2\rho X_{i+1}X_i)/2\sigma^2(1-\rho^2) \right] dX_i
\]

Multiplying numerator and denominator of (A-16) by the factor \(\exp[-\rho^2X_{i+1}^2/2\sigma^2(1-\rho^2)]\), (A-16) becomes:

\[
p[X_{i+1}/n \Delta < X_i < (n+1) \Delta] = \frac{1}{2\pi\sigma^2(1-\rho^2)^{1/2}} \beta \exp \left[ -\frac{X_{i+1}^2}{2\sigma^2(1-\rho^2)} \right]
\]
\[
\exp \left[ \rho^2X_{i+1}^2/2\sigma^2(1-\rho^2) \right] dX_i
\]

\[
\exp \left[ -(X_{i+1}^2 - 2\rho X_{i+1}X_i + \rho^2X_{i+1}^2)/2\sigma^2(1-\rho^2) \right] dX_i
\]

Simplifying (A-17),
Further simplification of (A-18),

\[
p[X_{i+1}/n\Delta < X_i < (n+1)\Delta] = \frac{1}{2\pi\sigma^2(1-\rho^2)^{\frac{1}{2}}} \exp \left[ -\frac{(1-\rho^2)X_{i+1}^2}{2\sigma^2(1-\rho^2)} \right]
\]

For \( (n+1)\Delta \)
\[
\left[ \int_{n\Delta}^{(n+1)\Delta} \exp \left[ -\frac{(X_i-\rho X_{i+1})^2}{2\sigma^2(1-\rho^2)} \right] dX_i \right]
\]

\[(A-19)\]

In (A-19), let \( \lambda = (X_i-\rho X_{i+1})/\sigma(1-\rho^2)^{\frac{1}{2}} \)

\[\frac{d\lambda}{dX_i} = 1/\sigma(1-\rho^2)^{\frac{1}{2}} \]

\[dX_i = \sigma(1-\rho^2)^{\frac{1}{2}} d\lambda \]

(A-21)

Changing the limits of integration in (A-19) to reflect the new variable of integration \( \lambda \), can be done as follows:

when \( X_i = (n+1)\Delta \), \( \lambda = [(n+1)\Delta - \rho X_{i+1}]/\sigma(1-\rho^2)^{\frac{1}{2}} \) \[(A-22)\]

when \( X_i = n\Delta \), \( \lambda = [n\Delta - \rho X_{i+1}]/\sigma(1-\rho^2)^{\frac{1}{2}} \) \[(A-23)\]

In order to shorten the new integration limits of (A-22) and (A-23), let
\[ u_2 = \frac{((n+1)\Delta - \rho X_{i+1})/\sigma(1-\rho^2)^{1/2}}{\sum \cdots} \quad (A-24) \]

\[ u_1 = \frac{[n\Delta - \rho X_{i+1}]/\sigma(1-\rho^2)^{1/2}}{\sum \cdots} \quad (A-25) \]

Substituting (A-20), (A-21), (A-24) and (A-25) into (A-19) give

\[ p[X_{i+1}/n\Delta \leq X_i \leq (n+1)\Delta] = \frac{1}{2\pi\sigma\beta} \exp \left[-\frac{X_{i+1}^2}{2\sigma^2}\right] \int_{u_1}^{u_2} \exp[-\lambda^2/2]d\lambda \quad (A-26) \]

Equation (A-26) can be reduced to the Q(Z) functions that are defined in Section A-1 of this Appendix as follows:

\[ p[X_{i+1}/n\Delta \leq X_i \leq (n+1)\Delta] = \frac{1}{2\pi\sigma\beta} \exp\left[-\frac{X_{i+1}^2}{2\sigma^2}\right] \int_{u_1}^{u_2} \exp[-\lambda^2/2]d\lambda \]

Equation (A-27) can be written using (A-3) as

\[ p[X_{i+1}/n\Delta \leq X_i \leq (n+1)\Delta] = \frac{1}{\sqrt{2\pi\sigma\beta}} \exp\left[-\frac{X_{i+1}^2}{2\sigma^2}\right] [Q(u_1)-Q(u_2)] \quad (A-28) \]

Equation (A-28) can be written using (A-6) as

\[ p[X_{i+1}^2/n\Delta \leq X_i \leq (n+1)\Delta] = \frac{1}{\sqrt{8\pi\sigma\beta}} \exp\left[-\frac{X_{i+1}^2}{2\sigma^2}\right] [\text{erf}(u_2/\sqrt{2})-\text{erf}(u_1/\sqrt{2})] \quad (A-29) \]
where $\beta$ from (A-13) is

$$
\beta = \int_{-\infty}^{\infty} \left[ \int_{n\Delta}^{(n+1)\Delta} p(x_{i+1}, x_i) \, dx_i \right] \, dx_{i+1} \tag{A-30}
$$

Equation (A-30) can be written using (A-11) as

$$
\beta = \int_{-\infty}^{\infty} \left[ \int_{n\Delta}^{(n+1)\Delta} \frac{1}{2\pi\sigma^2(1-\rho^2)^{\frac{3}{2}}} \exp \left[ -\left( \frac{1}{2\sigma^2(1-\rho^2)} \right) \left( x_{i+1}^2 - 2\rho x_{i+1} x_i x_i^2 \right) \right] \, dx_i \right] \, dx_{i+1} \tag{A-31}
$$

$$(n+1)\Delta \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2(1-\rho^2)^{\frac{3}{2}}} \exp \left[ -\left( \frac{1}{2\sigma^2(1-\rho^2)} \right) \left( x_{i+1}^2 - 2\rho x_{i+1} x_i x_i^2 \right) \right] \, dx_{i+1} \, dx_i \tag{A-32}
$$

Comparing the inner integral of (A-32) with the definite integral of (A-1) gives the solution of the inner integral of (A-32), thus (A-32) becomes:

$$
\beta = \frac{\sqrt{2\pi\sigma^2(1-\rho^2)}}{2\pi\sigma^2(1-\rho^2)^{\frac{3}{2}}} \int_{n\Delta}^{(n+1)\Delta} \exp \left[ -\left( \frac{1}{2\sigma^2(1-\rho^2)} \right) \left( x_{i+1}^2 - 2\rho x_{i+1} x_i x_i^2 \right) \right] \, dx_{i+1} \tag{A-33}
$$

Simplifying (A-33) gives
\[ \beta = \frac{1}{\sqrt{2\pi} \sigma^2} \int_{n \Delta}^{(n+1) \Delta} \exp(-\frac{X_i^2}{2\sigma^2}) dX_i \]  \hspace{1cm} (A-34)

In (A-34), let \( \lambda = X_i / \sigma \) \hspace{1cm} (A-35)

\[ \frac{d\lambda}{dX_i} = \frac{1}{\sigma} \]

\[ dX_i = \sigma d\lambda \]  \hspace{1cm} (A-36)

Changing the limits of integration in (A-34) to reflect the new variable of integration \( \lambda \), can be done as follows:

when \( X_i = (n+1) \Delta \), \( \lambda = \frac{(n+1) \Delta}{\sigma} \) \hspace{1cm} (A-37)

when \( X_i = n \Delta \), \( \lambda = \frac{n \Delta}{\sigma} \) \hspace{1cm} (A-38)

Substituting (A-35), (A-36), (A-37) and (A-38) into (A-34) gives

\[ \beta = \frac{1}{\sqrt{2\pi}} \int_{n \Delta / \sigma}^{(n+1) \Delta / \sigma} \exp(-\frac{\lambda^2}{2}) d\lambda \]  \hspace{1cm} (A-39)

Equation (A-39) can be reduced to the \( Q(Z) \) functions that are defined in Section A-1 of this Appendix as

\[ \beta = \frac{1}{\sqrt{2\pi}} \left[ \int_{n \Delta / \sigma}^{\infty} \exp(-\frac{\lambda^2}{2}) d\lambda - \int_{(n+1) \Delta / \sigma}^{\infty} \exp(-\frac{\lambda^2}{2}) d\lambda \right] \]  \hspace{1cm} (A-40)
Equation (A-40) can be written using (A-30) as

$$\beta = Q[n\Delta/\sigma] - Q[(n+1)\Delta/\sigma]$$  \hspace{1cm} (A-41)

Equation (A-41) using (A-8) can be written as

$$\beta = 1/2[1-\text{erf}(n\Delta/\sqrt{2}\sigma)] - 1/2[1-\text{erf}((n+1)\Delta/\sqrt{2}\sigma)]$$  \hspace{1cm} (A-42)

$$\beta = 1/2[\text{erf}((n+1)\Delta/\sqrt{2}\sigma) - \text{erf}[n\Delta/\sqrt{2}\sigma]]$$  \hspace{1cm} (A-43)

Bit Error Rate (BER)

The bit error rate (BER) of (2-68) can be reduced to $Q(Z)$ functions that are defined in Section A-1 of this Appendix as follows: (2-68) is

$$\text{BER} = \frac{1}{2} \left[ \frac{1}{\sqrt{2\pi}\sigma_0} \int_{\infty}^{\text{TH1}} \exp\left[-(r_0-S_1)^2/2\sigma_0^2\right] dr_0 + \frac{1}{\sqrt{2\pi}\sigma_0} \int_{\text{TH1}}^{\text{TH2}} \exp\left[-(r_0-S_1)^2/2\sigma_0^2\right] dr_0 \right]$$

$$+ \frac{1}{2} \left[ \frac{1}{\sqrt{2\pi}\sigma_0} \int_{\text{TH1}}^{\text{TH2}} \exp\left[-(r_0-S_1)^2/2\sigma_0^2\right] dr_0 \right]$$

$$- \frac{1}{2} \left[ \frac{1}{\sqrt{2\pi}\sigma_0} \int_{\text{TH1}}^{\text{TH2}} \exp\left[-(r_0-S_1)^2/2\sigma_0^2\right] dr_0 \right]$$

\hspace{1cm} (A-44)
\[ \text{BER} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \sigma_0}} \exp\left[-\frac{(r_0 - S_1)^2}{2\sigma_0^2}\right] dr_0 \]

\[ \text{TH1} \]

\[ \frac{C_{i,n}^{i+1}}{2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \sigma_0}} \exp\left[-\frac{(r_0 - S_1)^2}{2\sigma_0^2}\right] dr_0 \]

\[ \text{TH2} \]

\[ \frac{D_{i,n}^{i+1}}{2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \sigma_0}} \exp\left[-\frac{(r_0 - S_0)^2}{2\sigma_0^2}\right] dr_0 \]

\[ \text{TH1} \]

\[ \frac{D_{i,n}^{i+1}}{2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \sigma_0}} \exp\left[-\frac{(r_0 - S_0)^2}{2\sigma_0^2}\right] dr_0 \]

\[ \text{TH2} \]

(A-45)

Let \( \lambda = -(r_0 - S_1)/\sigma_0 \) in the first integral \( \lambda = (r_0 - S_0)/\sigma_0 \) in the second, fifth and sixth integrals and \( \lambda = (r_0 - S_1)/\sigma_0 \) in the third and fourth integrals; then (A-45) is
BER = \frac{1}{2} \int_{-(TH1-S1)/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp[-\lambda^2/2]d\lambda + \frac{1}{2} \int_{(TH2-S0)/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp[-\lambda^2/2]d\lambda

\frac{C_{i,n}^{i+1}}{2} \int_{(TH1-S1)/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp[-\lambda^2/2]d\lambda \frac{C_{i,n}^{i+1}}{2} \int_{(TH2-S1)/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp[-\lambda^2/2]d\lambda

\frac{D_{i,n}^{i+1}}{2} \int_{(TH1-S0)/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp[-\lambda^2/2]d\lambda \frac{D_{i,n}^{i+1}}{2} \int_{(TH2-S0)/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp[-\lambda^2/2]d\lambda

Equation (A-46) can be written using (A-3) as

BER = \frac{1}{2} \left[ Q \left( \frac{TH1+S1}{\sigma_0} \right) + Q \left( \frac{TH2-S0}{\sigma_0} \right) \frac{C_{i,n}^{i+1}}{2} \left( \frac{TH1-S1}{\sigma_0} \right) - Q \left( \frac{TH2-S1}{\sigma_0} \right) \right]

\frac{D_{i,n}^{i+1}}{2} \left[ Q \left( \frac{TH1-S0}{\sigma_0} \right) - Q \left( \frac{TH2-S0}{\sigma_0} \right) \right]

Optimum Threshold

The expression for optimum threshold (2.70) is simplified until no further simplification is possible. The process can be done as follows (2.70) is
\[
\frac{1}{2} \exp\left[-\frac{(TH2-S1)^2}{2\sigma_0^2}\right] - \frac{1}{2} \exp\left[-\frac{(TH2-S0)^2}{2\sigma_0^2}\right]
\]

\[
\frac{C_{l,n}^{i+1}}{\sqrt{2\pi\sigma_0}} \frac{1}{2} \exp\left[-\frac{(TH2-S1)^2}{2\sigma_0^2}\right] - \frac{C_{l,n}^{i+1}}{\sqrt{2\pi\sigma_0}} \frac{1}{2} \exp\left[-\frac{(TH2-S1)^2}{2\sigma_0^2}\right]
\]

\[
\frac{D_{l,n}^{i+1}}{\sqrt{2\pi\sigma_0}} \frac{1}{2} \exp\left[-\frac{(TH2-S0)^2}{2\sigma_0^2}\right] = 0
\]

\[
\exp\left[-\frac{S1^2}{2\sigma_0^2}\right] + C_{l,n}^{i+1} \exp\left[-\frac{(TH2-S1)^2}{2\sigma_0^2}\right]
\]

\[
-\exp\left[-\frac{S0^2}{2\sigma_0^2}\right] + \frac{1}{2} \exp\left[-\frac{S1^2}{2\sigma_0^2}\right] + \frac{1}{2} \exp\left[-\frac{(TH2-S0)^2}{2\sigma_0^2}\right]
\]

Multiplying out the exponents in (A-49) and dividing both sides of (A-49) by \(\exp[-(TH2)^2/2\sigma_0^2]\), (A-49) becomes

\[
\exp[-\frac{S1^2}{2\sigma_0^2}] + C_{l,n}^{i+1} \exp[-2TH2S1/2\sigma_0^2] = 0
\]

\[
\exp[-\frac{S0^2}{2\sigma_0^2}] + C_{l,n}^{i+1} \exp[-2TH2S0/2\sigma_0^2] = 0
\]

\[
\exp[-\frac{S1^2}{2\sigma_0^2}] + C_{l,n}^{i+1} \exp[-2TH2S1/2\sigma_0^2] = 0
\]

\[
\exp[-\frac{S0^2}{2\sigma_0^2}] + C_{l,n}^{i+1} \exp[-2TH2S0/2\sigma_0^2] = 0
\]

(A-50)
\[
\begin{align*}
\frac{\exp[-2\theta S_1/2\sigma_0^2]+C_\alpha,n^{i+1}\exp[2\theta S_1/2\sigma_0^2]-C_\alpha,n^{i+1}\exp[-2\theta S_1/2\sigma_0^2]}{
\exp[2\theta S_0/2\sigma_0^2]+D_\alpha,n^{i+1}\exp[-2\theta S_0/2\sigma_0^2]-D_\alpha,n^{i+1}\exp[2\theta S_0/2\sigma_0^2]}&= \frac{\exp[-S_0^2/2\sigma_0^2]}{\exp[-S_1^2/2\sigma_0^2]} \quad (A-51)
\end{align*}
\]
APPENDIX B

FORTRAN PROGRAM LISTINGS
A communication system using Decision-Directed-Detection PCM scheme is simulated by translating the flowchart of Figures 3-7, 3-8, 3-9, and 3-10 into FORTRAN code. There are two main PROGRAM LISTINGS and three sub PROGRAM LISTINGS shown in this Appendix.

B.1 Main Programs

The first main program generates the look-up table of probability distribution of each bit position being a binary 0 or a binary 1 for a specified level quantizer. The bit probabilities are computed by using an external function (FUNNY) to compute the probability density function argument (2-25) and a subroutine INTEG to integrate the probability density function (2-25).

The other program simulates the operation of a communication system using Decision-Directed-Detection scheme. This program uses subroutine SIGNAL to generate Markov Gaussian input signal samples and precomputed probability distribution of each bit position being a binary 0 or a binary 1 in decision making within the double symmetrical threshold TH1 and TH2. The performance of this scheme in terms of bit error rate and mean-squared error criteria are evaluated.

The two main PROGRAM LISTINGS are given below.
MAIN PROGRAM

This program generates the look-up table of cumulative probability distribution of each bit position being a binary 0 or a binary 1 which helps in decision making within the double threshold. The cumulative probabilities are generated using an external function (FUNNY) to calculate the joint conditional probability density function argument and a subroutine (INTEGR) to integrate the joint conditional probability density function. The look-up table is generated for specified level quantizer.

Afiomah Stephen  August 9 1986

Variables

INTEGER I,IC,J,JC,JSUM,K,KC,KK,LBIT(64,6),LMID,LQUAT,LQX
INTEGER LVALUE,LZ,M,H,MA,NB,NC,NPT,NUM
REAL A,CAP,DIFF,EF1,EF2,PROB(65,6,2),QUAT,RAE(32,6,2)
REAL RANGE1,RANGE2,RAS(32,6,2),SIGMAX,SIGMIN,TAU

I,IC,J,JC,KC,M,N:LOOP INDEX.
K:NUMBER OF BITS PER A QUANTIZER LEVEL.
LBIT: BINARY 0 OR BINARY 1
LMID: HALF OF A QUANTIZER RANGE.
LQX,LVALUE: CODE NUMBER OF EACH QUANTIZER LEVEL.
LZ: LIMIT OF EACH QUANTIZER STEP SIZE.
NPT: NUMBER OF POINTS BETWEEN LIMITS.
NUM: TOTAL NUMBER OF QUANTIZER LEVEL.
A: CONSTANT.
EF1,EF2: ERROR FUNCTIONS.
PROB: THE CUMULATIVE PROBABILITY DISTRIBUTION OF
      EACH BIT POSITION (BEING 0 OR 1).
RAE,RANGE2: THE INTEGRATION END LIMITS.
RAE,RANGE1: THE INTEGRATION STARTING LIMITS.
SIGMAX: QUANTIZER MAXIMUM LIMIT
SIGMIN: QUANTIZER MINIMUM LIMIT
TAU: THE SIGNAL SAMPLING INTERVAL.

Global Variables

REAL CONST,DELTA1,DELTA2,P1,RHO,SIGMA
COMMON SIGMA,RHO,P1,DELTA1,DELTA2,CONST

CONST: THE VALUE OF EQUATION (2-27) WHICH IS COMPUTED
      THROUGH ERROR FUNCTIONS (EF1 AND EF2).
DELTA1: THE STARTING POINT OF THE GIVEN RANGE OF
      THE PRESENT LEVEL OF THE QUANTIZER.
DELTA2: THE END POINT OF THE GIVEN RANGE OF THE
      PRESENT LEVEL OF THE QUANTIZER.
RHO: THE CORRELATION BETWEEN SIGNAL SAMPLES.
SIGMA: THE STANDARD DEVIATION OF THE INPUT SIGNAL.
SUBPROGRAMS
EXTERNAL FUNNY
REAL FUNNY
EXTERNAL INTEG
INTEG: INTEGRATES THE PROBABILITY DENSITY FUNCTION (2-25).

BEGIN

INITIALIZATION OF CONSTANTS.

PI=ARCSIN(-1.0)
K=4
MPT=1025
NUM=2**K
NA=2
NB=6
NC=32
SIGHMIN=-4.0
SIGHMAX=4.0
A=1.0
TAU=0.01
SIGHMA=1.0
RHO=EXP(-TAU*A)
LM1D=2**K/2

READ IN THE LIMITS OF INTEGRATION FOR THE CALCULATION OF THE
CUMULATIVE PROBABILITY DISTRIBUTION OF EACH BIT POSITION.

DO 1 IC=1,NA
    DO 2 JC=1,NB
        DO 3 KC=1,HC
            READ(5,4)RAS(KC,JC,IC),RAE(KC,JC,IC)
    4     CONTINUE
    3     CONTINUE
    2     CONTINUE
    1     CONTINUE

THE DEVELOPMENT OF INDEX FOR QUANTIZER LEVELS AND BINARY
CODING OF THESE INDEX LEVELS.

DO 45 H=1,NUM
    LGX=N-1
    LQUAT=LGX
    DO 20 ID=1,K
        I=K-(ID-1)
        QUAT=FLOAT(LQUAT)/2.0
        LQUAT=LQUAT/2
        DIFF=AQUAT-FLOAT(LQUAT)
        IF (DIFF.EQ.0) LBIT(N,1)=0
        IF (DIFF.NE.0) LBIT(N,1)=1
    20     CONTINUE
    45     CONTINUE

THE CALCULATION OF THE GIVEN PRESENT LEVEL OF THE QUANTIZER
AND THE VALUE OF EQUATION (2-28) THROUGH ERROR FUNCTIONS

\( (E_{F1} \text{ AND } E_{F2}) \).

\[
\text{DO } 40 \text{ N=1,NUM} \\
\text{LVALUE}=0 \\
\text{DO } 50 \text{ I=1,K} \\
\quad \text{KK}=K-(I-1) \\
\text{50 LVALUE=LBIT(H,I)*2**(KK-1)+LVALUE} \\
\text{LZ}=LVALUE-LMID \\
\text{DELTA1}=\text{FLOAT}(LZ)*(\text{SIGMAX}/\text{FLOAT}(2**K)) \\
\text{EF1}=0.5*\text{ERF}(\text{DELTA1}/\text{SQRT}(2.0)) \\
\text{DELTA2}=\text{FLOAT}(LZ+1)*(\text{SIGMAX}/\text{FLOAT}(2**K)) \\
\text{EF2}=0.5*\text{ERF}(\text{DELTA2}/\text{SQRT}(2.0)) \\
\text{CONST}=E_{F2}-E_{F1} \\
\]

CALLING SUBROUTINE INTEG WHICH COMPUTES AND RETURNS
THE VALUES OF THE PROBABILITY DISTRIBUTION OF EACH
BIT POSITION BEING BINARY \( 0 \) OR BINARY \( 1 \).

\[
\text{DO } 60 \text{ I=1,K} \\
\quad \text{JSUM}=2**(I-1) \\
\text{DO } 65 \text{ J=1,2} \\
\quad \text{PROB}(N+1,I,J)=0.0 \\
\text{DO } 70 \text{ M=1,JSUM} \\
\quad \text{RANGE1=RAS(M,I,J)} \\
\quad \text{RANGE2=RRA(M,I,J)} \\
\quad \text{CALL INTEG(FUNNY,SINTZ,RANGE1,RANGE2,HPT)} \\
\quad \text{PROB}(N+1,I,J)=\text{PROB}(N+1,I,J)+\text{SINTZ} \\
\text{70 } \text{CONTINUE} \\
\text{65 } \text{CONTINUE} \\
\text{60 } \text{CONTINUE} \\
\text{WRITE(6,80)((PROB(N+1,I,J),I=1,K),J=1,2)} \\
\text{80 FORMAT(1X,8F15.8)} \\
\text{40 CONTINUE} \\
\text{STOP} \\
\text{END} \\
\]
MAIN PROGRAM

THIS PROGRAM SIMULATES THE OPERATION OF A COMMUNICATION SYSTEM USING DECISION-DIRECTED-DETECTION SCHEME SO THAT THE PERFORMANCE OF THIS SCHEME IN TERMS OF BIT ERROR RATE CRITERION AND ERROR AMPLITUDE CRITERION SUCH AS MEAN-SQUARED ERROR ARE EVALUATED. THIS SCHEME USES SUBROUTINE (SIGNAL) TO GENERATE MARKOV GAUSSIAN INPUT SIGNAL SAMPLES AND PRECOMPUTED PROBABILITY DISTRIBUTION OF EACH BIT POSITION BEING A BINARY 0 OR A BINARY 1 IN DECISION MAKING WITHIN THE DOUBLE SYMMETRICAL THRESHOLD TH1 AND TH2.

AFIONAH STEPHEN    AUGUST 9 1986

VARIABLES

INTEGER 1, IN, INDEX, J, K, KK, LBIT, LQX(20001), M, MOM, N, NET, MOT, NUM
REAL A, ACE, AMSE, AN, ASE, BASE, BER, BRATE, CAP, CER, CSUM, DEV
REAL ERROR, PAN, PROB(33, 5, 2), QER, QREN, QSUM, QX, QY, QAP, RATE
REAL FRI, REI, REP, REF, FHO, S0, S1, SIGMA, SIGMAX, SIGMIN, SHR, SPD
REAL TAP, TAUE, TEIT, TH1, TH2, Y(20001), XADJ

1, IN, J, M, H: LOOP INDEX
INDEX: NUMBER OF LEVELS IN A QUANTIZER
K: NUMBER OF BITS PER SAMPLE
LBIT: BINARY 0 OR BINARY 1
LQX: CODE NUMBER OF PREVIOUSLY DETECTED CODEWORD
MOM: NUMBER OF SIGNAL SAMPLES GENERATED
NUM: NUMBER OF SIGNAL SAMPLES USED
A: CONSTANT
ACE: AVERAGE CHANNEL ERROR.
AMSE: MEAN-SQUARED ERROR.
AN: NOISE LEVEL.
AGE: AVERAGE QUANTIZATION ERROR.
ASE: AVERAGE SAMPLING ERROR.
BASE: ANALOG BASEBAND SIGNAL.
BER: BIT ERROR RATE.
BRATE: TIME PER BIT IN A SAMPLE.
CER: CHANNEL ERROR.
CSUM: TOTAL CHANNEL ERROR.
DEV: STANDARD DEVIATION OF THE NOISE.
ERROR: BIT ERROR COUNT.
PROB: PROBABILITY DISTRIBUTION OF EACH BIT POSITION BEING A BINARY 0 OR A BINARY 1.
QER: QUANTIZATION ERROR.
QREN: QUANTIZER RANGE COVERAGE.
QSUM: TOTAL QUANTIZATION ERROR.
QX: TRANSMITTED SIGNAL SAMPLE.
QY: RECEIVED SIGNAL SAMPLE.
RATE: BIT RATE
RB1: RECEIVED LEVEL FOR BINARY 0 OR BINARY 1
RECD: SAMPLED BASEBAND SIGNAL.
REF: REFERENCE SIGNAL.
RH0: CORRELATION BETWEEN SIGNAL SAMPLES.
S0: AMPLITUDE OF TRANSMITTED BINARY 0.
SI: AMPLITUDE OF TRANSMITTED BINARY 1.
SIGMA: STANDARD DEVIATION OF THE INPUT SIGNAL.
SIGMAX: QUANTIZER MAXIMUM LIMIT.
SIGMIN: QUANTIZER MINIMUM LIMIT.
SNR: SIGNAL-TO-NOISE RATIO.
SPD: SPECTRAL NOISE DENSITY.
TAU: SAMPLING INTERVAL.
TEIT: TRANSMITTED LEVEL FOR BINARY 0 OR BINARY 1.
TH1 AND TH2: THE DOUBLE SYMMETRICAL THRESHOLD.
X: SIGNAL SAMPLES.
XADJ: ADJUSTED SIGNAL SAMPLE VALUE FOR QUANTIZATION AND CODING.
Y: TRANSMITTED SIGNAL PLUS NOISE.

SUBPROGRAMS
SUBROUTINE SIGNAL
SIGNAL: SIGNAL SAMPLE GENERATOR
BEGIN

INITIALIZATION OF CONSTANTS.

PI = ARCOS(-1.0)
K = 4
INDEX = 2**K
MOM = 20001
M = 2
MO1 = 20
NUM = MOM - 1
SIGMIN = -4.0
SIGMAX = 4.0
GRAN = SIGMAX - SIGMIN
A = 1.0
RATE = 120.00
SPD = 0.005
TAU = FLOAT(K)/RATE
SIGMA = 1.0
S0 = -A
S1 = A
RHO = EXP(-TAU*A)
BRATE = TAU/FLOAT(K)
DEV = SQRT(SPD*RATE)
REF = S1 - S0

READ IN THE PRECOMPUTED VALUES OF THE PROBABILITY DISTRIBUTION
OF EACH BIT POSITION BEING A BINARY 0 OR A BINARY 1 FOR THE
SPECIFIED QUANTIZER.

DO 10 M = 1, INDEX
   READ(5, 15)((PROB(H,1,J), I=1,K), J=1,2)
15   FORMAT(1X,8F15.8)
10  CONTINUE

WRITE(25)
25  FORMAT(1X,2X,'SIGMA',8X,'SNR',15X,'BER',9X,'THRESHOLD')
CALLING SUBROUTINE (SIGNAL) WHICH RETURNS SIGNAL SAMPLE VALUES
GENERATED IN THE SUBROUTINE (SIGNAL).

CALL SIGNAL(RHO,A,MOM,SIGMA,X)
DO 50 M=1,10
CSUM=0.0
ERROR=0.0
QSUM=0.0
TH2=FLOAT(M)/FLOAT(2000)
TH1=-TH2

SERIAL QUANTIZATION AND BINARY CODING OF THE INPUT SIGNAL SAMPLES.

DO 55 N=1,NUM
LUX(N+1)=0
RAP=0.0
TAP=0.0
IN=LUX(N)+1
IF(X(N+1).GT.SIGMAX)X(N+1)=SIGMAX
XADJ=X(N+1)+SIGMAX
DO 60 I=1,K
BASE=0.0
KK=K-(I-1)
CAP=2.0**I
PAN=QRAN/CAP
IF(XADJ.GT.PAN)GO TO 65
TBIT=0
GO TO 70
65
TBIT=1
XADJ=XADJ-PAN
70
TAP=TAP+TBIT/CAP

GENERATION OF GAUSSIAN NOISE.SIGNAL PLUS NOISE AND PROCESSING THE RECEIVED SIGNAL PLUS NOISE TO OBTAIN SAMPLED BASEBAND SIGNAL (RECD).

DO 75 J=1,NET
CALL ANORM(XH)
AN=DEV*XH
Y=TBIT+AN
BASE=BASE+Y*REF
CONTINUE

IMPLEMENTATION OF THE DECISION MAKING OF THE RECEIVER USING DOUBLE SYMMETRICAL THRESHOLD TH1 AND TH2 IN THE COMPARATOR DEVICE.

RECD=BASE+BASE
IF(RECD.LE.TH1)GO TO 30
IF(RECD.GT.TH2)GO TO 35
IF(RECD.GE.TH1.AND.RECD.LE.TH2)GO TO 40

LOOP UNTIL TEIT EQUALS 81
CONTINUE
C

30         LBIT=0
           RBIT=S0
           LQX(N+1)=LBIT*2**(KK-1)+LQX(N+1)
           RAP=RAP+RBIT/CAP
           IF(TBIT.EQ.S1)GO TO 45
           GO TO 60

C

35         LBIT=1
           RBIT=S1
           LQX(N+1)=LBIT*2**(KK-1)+LQX(N+1)
           RAP=RAP+RBIT/CAP
           IF(TBIT.EQ.S0)GO TO 45
           GO TO 60

40         IF(N.EQ.1)GO TO 80
           IF(PROB(IN,1,1).GT.PROB(IN,1,2))GO TO 30
           IF(PROB(IN,1,1).LE.PROB(IN,1,2))GO TO 35

80         IF(RECD.LT.0.0)GO TO 30
           IF(RECD.GE.0.0)GO TO 35
           ERROR=ERROR+A
           CONTINUE

C

COMPUTATION OF SYSTEM ERROR PERFORMANCE (BIT ERROR
C RATE, SAMPLING ERROR, QUANTIZATION ERROR, CHANNEL
C ERROR, AND OVERALL MEAN-SQUARED ERROR).

C

QX=SIGMAX*TAP
GY=SIGMAX*RAP
QER=(X(N+1)-QX)**2
CER=(QX-GY)**2
QSUM=QSUM+QER
CSUM=CSUM+CER

CONTINUE

BER=ERROR/FLOAT(NUM*K)
SNR=10.0*ALOG10(A/DEV**2)
WRITE(6,90)DEV,SNR,BER,TH2

90 FORMAT(1X,4F15.8)
ASE=2.0*(A-(A-RHO)/TAU)
AGE=QSUM/FLOAT(NUM)
ACE=CSUM/FLOAT(NUM)
AMSE=ASE+AGE+ACE
WRITE(8,95)RATE,ASE,AGE,ACE,AMSE,TH2

95 FORMAT(1X,6F15.8)

CONTINUE
STOP
END

EOF
B.2 Subprograms

The three subprograms consist of an external function (FUNNY) and two subroutines (INTEG and SIGNAL).

The external function FUNNY computes the argument of the joint conditional probability density function of (2-25).

INTEG is an integration subroutine which uses Simpson's rule approximation to compute the probability distribution of each bit position being a binary 0 or a binary 1 by integrating (2-25).

The subroutine SIGNAL generates Markov Gaussian input samples.

The three SUBPROGRAM LISTINGS are given below.
SUBROUTINE INTEG(FUN,SUMT,START,END,NPT)

INTEGRATION USING SIMPSON'S RULE APPROXIMATION TO COMPUTE
THE PROBABILITY DISTRIBUTION OF EACH BIT POSITION BEING
A BINARY 0 OR A BINARY 1.

AFIOMAH STEPHEN AUGUST 9 1986

PARAMETERS
INTEGER NPT
REAL SUMT,START,END

NPT: NUMBER OF SUMMATION POINTS BETWEEN LIMITS (IN ONLY).
SUMT: RESULTS OF THE INTEGRATION (OUT ONLY).
START: LOWER LIMIT OF INTEGRATION (IN ONLY).
END: UPPER LIMIT OF INTEGRATION (IN ONLY).

VARIABLES
INTEGER LP
REAL H,SUM1,SUM2,SUM3,SUM4

LP: LOOP INDEX
H: SIMPSON'S RULE INTERVALS BETWEEN INTEGRATION LIMITS.
SUM1: COMPUTATION OF FIRST SUMMATION POINTS.
SUM2: COMPUTATION AND SUMMATION OF EVEN POINTS.
SUM3: COMPUTATION AND SUMMATION OF ODD POINTS.
SUM4: COMPUTATION OF LAST SUMMATION POINTS.

SUBPROGRAMS
EXTERNAL FUN
REAL FUN

FUN: PROBABILITY DENSITY ARGUMENT (IN ONLY)

BEGIN

H=(END-START)/FLOAT(NPT-1)
SUM1=FUN(START)
SUM2=0.0
SUM3=0.0
DO 2 LP=2,NPT,2
   SUM2=SUM2+FUN(START+FLOAT(LP)*H)
   IF ((LP+1).EQ.NPT) GO TO 3
   SUM3=SUM3+FUN(START+FLOAT(LP+1)*H)
3   SUM4=FUN(END)
2 CONTINUE
SUMT=(SUM1+2.0*SUM2+4.0*SUM3+SUM4)*H/3.0
RETURN
END
REAL FUNCTION FUNNY(Z)

THIS SUBPROGRAM COMPUTES THE ARGUMENT OF JOINT
CONDITIONAL PROBABILITY DENSITY OF (2-25).

AFIOMAH STEPHEN AUGUST 9 1986

PARAMETERS
REAL Z

Z:POINTS IN ARGUMENT FUNNY

VARIABLES
REAL ARG1, ARG2, C1, C2, T1, T2, VAR

GLOBAL VARIABLES
REAL CONST, DELTA1, DELTA2, PI, RHO, SIGMA
COMMON SIGMA, RHO, PI, DELTA1, DELTA2, CONST

CONST: THE VALUE OF EQUATION (2-27) WHICH IS COMPUTED
USING ERROR FUNCTIONS (EF1 AND EF2).
DELTA1: THE STARTING POINT OF THE GIVEN RANGE OF THE
PRESENT LEVEL OF THE QUANTIZER.
DELTA2: THE END POINT OF THE GIVEN RANGE OF THE PRESENT
LEVEL OF THE QUANTIZER.
RHO: THE CORRELATION BETWEEN SIGNAL SAMPLES.
SIGMA: THE STANDARD DEVIATION OF THE INPUT SIGNAL.

BEGIN

C1 = SIGMA * SQRT(1.0 - RHO**2)
C2 = 2.0 * SQRT(2.0 * PI) * CONST
T1 = (DELTA1 - RHO * Z) / C1
T2 = (DELTA2 - RHO * Z) / C1
ARG1 = T1 / SQRT(2.0)
ARG2 = T2 / SQRT(2.0)
VAR = EXP(-Z**2 / 2.0)
FUNNY = VAR * (ERF(ARG2) - ERF(ARG1)) / C2
RETURN
END

EOF:
SUBROUTINE SIGNAL(R,A,MOM,SIGMA,X)
FIRST ORDER MARKOV SIGNAL GENERATION.

AFIOMAH STEPHEN       AUGUST 9 1986

PARAMETERS
INTEGER MOM
REAL R,A,SIGMA,X(20001)

MOM: NUMBER OF SIGNAL SAMPLES GENERATED (IN ONLY)
R: CORRELATION PARAMETER (IN ONLY)
A: DUMMY
SIGMA: SIGNAL STANDARD DEVIATION (IN ONLY)
X: SIGNAL SAMPLES (OUT ONLY)

VARIABLES
INTEGER I
REAL C,C1,Z

I: LOOP INDEX
Z: RANDOM NUMBER

SUBPROGRAMS
EXTERNAL ANORM
ANORM: RANDOM NUMBER GENERATOR

BEGIN

C=1.0-R*R
C1=SQRT(SIGMA*C)
CALL ANORM(Z)
X(1)=Z*SQRT(SIGMA)
DO 300 I=2,MOM
   CALL ANORM(Z)
   X(I)=C1*Z+R*X(I-1)
300 CONTINUE
RETURN
END