A COMPARATIVE STUDY OF ART AND THE CONVOLUTION METHOD AS APPLIED TO CROSS BOREHOLE GEOPHYSICAL TOMOGRAPHY

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I. INTRODUCTION

In many fields of science and engineering, it is often desirable to be able to determine the internal structure of an object or environment from data collected by external measurements. The techniques used to solve internal imaging problems are based upon a concept called image reconstruction from projections. This method, also known as tomography, produces a two-dimensional image of a thin cross-section of the region under investigation. These thin cross-sectional images are then sometimes stacked together to give the complete internal image of the object or environment.

Tomography has found applications in many scientific fields. For example, in the world of microbiology, researchers use image reconstruction methods to determine the molecular structure of bacteriophages [1]. In contrast, astrophysicists use the same method to measure the brightness distribution of radio sources in space [2]. Other fields in which tomography has made useful contributions include nuclear science, environmental science, and solar physics [3]-[5].

By far the most familiar application of tomography is Computerized Axial Tomography. Commonly known as the CAT scan, this process has been routinely used in medical diagnostics since the early 1970's when the technique was developed by G. N. Hounsfield and J. Ambrose. In 1979, Hounsfield and Cormack were awarded the Nobel Prize in medicine for their pioneering efforts toward the development of the CAT scanner [6]-[8].
Shortly after the development of Computerized Axial Tomography, researchers at the Lawrence Livermore National Laboratory adapted the CAT concept to investigate underground regions of the earth. This adaptation known as electromagnetic geophysical tomography (often referred to as geotomography in the literature) has many applications in geophysical exploration. Perhaps the most useful application of electromagnetic geophysical tomography (EGT) is rock fracture detection. Fractures in rock often are the major pathways for underground fluid flow. Knowledge of the locations of these fluid flow paths is valuable information in many instances, such as choosing a site for nuclear or chemical waste disposal. Unknown rock fractures often give rise to potential leakage problems. Also, rock fractures conduct valuable natural resources such as petroleum or natural gas. Remote detection of these fossil fuels can offer great savings in both time and money to governments and oil exploration companies.

Another useful application of EGT is the monitoring and mapping of the burn front in a coal gasification process. If the burn front can be monitored, then large amounts of ungasified coal cannot go undetected [9].

Although EGT is similar to Computerized Axial Tomography in concept, there are important differences in the way the techniques are implemented. Since geophysical tomography is used to reconstruct images on a larger physical scale then CAT (1-100 meters scale length compared to a few centimeters for medical imaging), lower frequencies
must be used. EGT operates at frequencies in the megahertz range compared to frequencies in the X-ray range for CAT. Also, EGT requires a different scanning geometry than CAT. When using CAT, the object under investigation is a cross-section of the human body. This cross-section of tissue is scanned using a machine which can be rotated through a full circle around the body. Therefore, the region under investigation can be scanned from a full field of view. In comparison, EGT is used to scan an underground region between two parallel boreholes. In this geometry a full field of view can never be achieved [10].

This difference in basic scanning geometry characteristics gives rise to the most fundamental difference between the EGT and CAT methods, the mathematical reconstruction algorithms used to reconstruct the images. The full field of view of a CAT scan is very well suited to a mathematical transform algorithm called the convolution method. The limited view environment of EGT is better suited to a series expansion algorithm known as the algebraic reconstruction technique (ART). However, the convolution method does have advantages over ART, such as ease of implementation and drastically reduced computational time. Therefore, it is the purpose of this study to examine the possible applications of the convolution method to the EGT environment, and to make detailed comparisons between ART and the convolution method under geophysical scanning geometries.
II. FUNDAMENTALS OF COMPUTERIZED GEOPHYSICAL TOMOGRAPHY

A. Basic Concepts

The objective of geophysical tomography is to produce a two-dimensional spatial distribution of some unknown material parameter that is inherent to the underground region of interest. Usually these material parameters are either velocity (signal delay) or attenuation rate (signal loss). The measured data needed to determine the material parameters are generated by transmitting either electromagnetic or seismic energy through the region. Seismic tomography analyses usually employ signal delay parameters to image the region of interest. In contrast, electromagnetic tomography analyses mainly rely upon attenuation rate parameters to construct images. In this investigation, only electromagnetic scanning will be considered and the only unknown parameter to be considered will be the attenuation rate. Since the attenuation constants for many materials and substances found within the earth are well known, the attenuation rate maps can be used to produce a "picture" of the material structure of many underground environments.

B. Propagation Characteristics of the Scanning Signal

The scanning signal propagated through the region of interest is an electromagnetic wave in the frequency range of 10-100 MHz. The signal originates at a transmitting antenna and is received by a similar antenna positioned at the opposite side of the region being
scanned. The electromagnetic wave propagation between the transmitting antenna and a receiving antenna in the far field can be expressed by Friis Transmission Equation for lossy media [11],

\[
\frac{P_r}{P_t} = e_{tt} e_{tr} \left[ \frac{\lambda}{4\pi R} \right]^2 D_{gt} D_{gr} |\hat{\rho}_t \cdot \hat{\rho}_r|^2 e^{-2\alpha R}
\]  

(2-1)

where \( P_r, P_t \) = power received, power transmitted (watts)

\( e_{tt}, e_{tr} \) = efficiency of transmitting antenna, receiving antenna

\( \lambda \) = wavelength (m)

\( R \) = distance between antennas (m)

\( \alpha \) = attenuation constant (nepers/m)

\( D_{gt}, D_{gr} \) = directive gain of transmitting antenna, receiving antenna

\( |\hat{\rho}_t \cdot \hat{\rho}_r|^2 \) = polarization loss factor

The Friis Transmission Equation relates power received to power transmitted by taking into account wave attenuation losses (\( e^{-2\alpha R} \) term), losses due to the spherical spreading of the energy by the antenna ((\( \lambda/4\pi R \))² term), and all antenna parameters (efficiency, directive gain, and polarization loss). The far field is defined by \( R > 2D^2/\lambda \), where \( D \) is the largest dimension of either antenna.

If source and receiver antennas are aligned vertically (to prevent polarization loss) and the antennas are assumed to be isotropic, equation (2-1) becomes,

\[
P_r = k \frac{e^{-2\alpha R}}{R^2}
\]  

(2-2)
where $k = \left( \frac{\lambda}{4\pi} \right)^2 e_{tt} e_{tr} P_t$

Thus,

$$e^{2\alpha_R} = \frac{k}{R^2 P_r}$$  \hspace{1cm} (2-3)

and as a result,

$$\alpha_R = \frac{1}{2} \ln \left( \frac{k}{R^2 P_r} \right)$$  \hspace{1cm} (2-4)

It should be noted that the term $\alpha_R$ can be considered as the integrated attenuation along the entire ray path from transmitter to receiver.

$$\alpha_R = \int_{R} \alpha(x,y) dR$$  \hspace{1cm} (2-5)

During an electromagnetic scanning process the integrated attenuation is calculated from measurements for each ray path passing through the region of reconstruction. These data are then stored in a computer and retrieved later to be used in the reconstruction process [10], [12].

In this study, a straight-ray optical model is assumed for the rays passing through the underground region of interest. A straight-ray optical model assumes that the ray experiences no refraction, diffraction, or reflection as it passes from transmitter to receiver. If any combination of refraction, diffraction, or reflection is significant and the straight-ray model is assumed, the reconstructed
pictures will contain distortions. Three criteria are required for the straight-ray model to be a valid approximation of the physical situation [10]:

1. The separation distance between transmitter and receiver should be much larger than $\lambda/2\pi$, where $\lambda$ is the wavelength in the medium.

2. The refractive index should vary slowly enough with distance that the field behavior is adequately described by straight-line geometrical optics.

3. $\lambda \ll \pi \delta$, where $\delta$ is the skin depth in the medium.

Fortunately, many naturally occurring underground geological structures adhere to these criteria, and often experimental data closely resemble computer simulated models [10].

C. Data Collection and Interpretation

To collect the necessary data for image reconstruction, the transmitting and receiving antennas must be lowered into boreholes on each side of the region of interest. Usually only one antenna pair is used to scan the region. The region is scanned by systematically repositioning the pair, before each new measurement, so as to produce a parallel scanning geometry. Each new position of the antenna pair constitutes a coupled ray path and each different angular orientation of the ray paths is known as a projection (see Figure 2-1). The area
Figure 2-1 Parallel beam scanning geometries.
under investigation is usually scanned at many projection angles. A computer is used to monitor and record the power transmitted, power received, and antenna locations for each projection. A typical raw data collection system is shown in Figure 2-2 [9].

After all the raw data have been collected, the image reconstruction process begins. In order to form the image, the area under investigation is represented by a two-dimensional array of elements called pixels (see Figure 2-3). Each pixel is assumed to contain a constant attenuation rate, which can be expressed as

$$\alpha = \omega \sqrt{\mu \varepsilon} \left[ \frac{1}{2} \left[ \sqrt{1+\left(\frac{\sigma}{\omega \varepsilon}\right)^2} - 1 \right] \right]^{\frac{1}{2}} \text{ nepers/m} \quad (2-6)$$

where $\omega$ = angular frequency (rad/sec)

$\mu$ = permeability (H/m)

$\varepsilon$ = permittivity (F/m)

$\sigma$ = conductivity (S/m)

In this cellular model, the permeability and permittivity are assumed constant over the entire region of investigation and the conductivity is assumed constant over each individual cell. Therefore, the complete imaging model for this investigation is based upon the following criteria:

1. cellular division of reconstruction space
2. uniform constitutive parameters in each cell
3. straight-ray electromagnetic propagation
Figure 2-2 Raw data collection system.
Figure 2-3 Cellular discretization of the reconstruction area.
4. no polarization loss
5. isotropic sources

When using series expansion techniques such as ART to estimate the unknown cellular attenuation constants, the Friis Transmission Equation can be "digitized" in the following manner [12]:

\[ \alpha R = \sum_{\text{one path}} D_{ij} X_j \]  

(2-7)

where \( D_{ij} \) is defined to be the length of the \( i \)th path through the \( j \)th cell and \( X_j \) is the unknown attenuation constant \( \alpha \) (nepers/m) for the \( j \)th cell. From equation (2-4) it can also be said that,

\[ \sum_{\text{one path}} D_{ij} X_j = Y_i = \frac{1}{2} \ln \left[ \frac{k}{\text{Prec}_i L_i^2} \right] \]

(2-8)

where \( Y_i \) = integrated attenuation along the \( i \)th ray path

\( L_i \) = the \( i \)th path length through the area of reconstruction

\( \text{Prec}_i \) = the power received in the \( i \)th measurement

This "digitization" allows the integrated attenuation to be expressed as a linear system of equations of the form (see Figure 2-4).

\[ [D_{ij}] [X_j] = [Y_i] \]

(2-9)

Since the \( Y \) vector can be calculated from measurements and the \( D \) matrix is determined by cell division and antenna placement, it
Figure 2-4  Geometrical illustration of the variables contained in equation (2-9).
would appear that (2–9) could be solved by direct matrix inversion. Unfortunately, in most reconstruction applications the number of equations is extremely large and the D matrix very sparse. As a result, the D matrix is often found to be singular or nearly singular, which precludes the application of direct matrix inversion techniques. However, series expansion techniques can be used to estimate the X vector indirectly, based upon the information contained in the Y vector and the D matrix. In contrast, the convolution transform method can estimate the X vector without knowledge of the D matrix. The convolution estimate is derived from Y vector measurement data and the interpolating distances between ray paths and cell center points. In the next chapter several indirect reconstruction algorithms (both series expansion and transform methods) will be defined and discussed in detail.
III. MATHEMATICAL FOUNDATIONS OF THE RECONSTRUCTION TECHNIQUES

A. Basic Concepts of Reconstruction Algorithms

A reconstruction algorithm can be defined as a mathematical process that can estimate a function \( f(x,y) \) on some discrete set of points \((x_i, y_i)\) in a plane, when given as its input the values of the line integrals of the function \( f(x,y) \) for a collection of parallel lines in the same plane. In this investigation, \( f(x,y) \) is the attenuation rate at \((x_i, y_i)\) and the line integrals for the collection of parallel lines are defined by equation (2-5) as the integrated attenuation along the ray paths. If the number of projections is defined as \( N \), and the number of ray paths per projection is defined as \( M \), then the total number of ray paths in any given collection equals \( N \cdot M \). Of course, any practical reconstruction algorithm must be coded into a computer program. This means that in final form the reconstruction algorithm must consist of a series of step-by-step unambiguous instructions.

Several basic reconstruction algorithms have gained popularity in the field of image reconstruction. Although there are many variations of these basic algorithms, all of the reconstruction techniques may be divided into two distinct classes:

(1) transform methods
(2) series expansion methods

These two methods take vastly different mathematical approaches to the problem of image reconstruction from projections. However, both methods have found applications in various areas of tomography.
B. Transform Method Theory

Transform method algorithms can be defined as a numerical implementation of a closed-form inversion formula. In tomography, the inversion formula is usually some form of the inverse Radon formula. This inversion formula can be manipulated into several different forms, each of which may be discretized to obtain different algorithms for reconstruction from measured data. In the following sections, the basic mathematical concepts of transform methods will be discussed. In addition, two of the more sophisticated transform methods will be presented.

1. The Radon Transform

The Radon transform is a mathematical operator that associates a given function \( f \) of two polar variables with another function \([Rf]\) of two variables. This transform was first derived by German mathematician Johan Radon in 1917 [13], and now forms the basis for all transform reconstruction techniques. In this section, a detailed mathematical discussion of the Radon and inverse Radon transforms will be presented [14].

The Radon transform can be defined as follows:

\[
[Rf](\ell, \theta) = \int_{-\infty}^{\infty} f(\sqrt{\ell^2 + z^2}, \theta + \tan^{-1}(z/\ell))dz ; \text{ if } \ell = 0
\]
\[ [Rf](0, \theta) = \int_{-\infty}^{\infty} f(z, \theta + \frac{\pi}{2}) dz \] ; if \( l = 0 \) \tag{3-1} \]

Where \([Rf]\) is the Radon transform of some function \(f\) and the value of \([Rf]\) at the point \((l, \theta)\) in its domain is denoted by \([Rf](l, \theta)\). From Figure (3-1) it can be seen that the Radon transform is nothing more than the line integral of \(f\) along the line \(L\). Also, it should be noted that the domains of the function \(f\) and \(Rf\) have one important difference. The function \(f\), also known as the picture function, is defined for pairs of real numbers \((r, \phi)\) which are the polar coordinates of points in the plane. The value of \(f\) at the point \( (0, \phi) \) is always the same regardless of the value of \(\phi\). The coordinates \((0, \phi)\) will always represent the origin. This is not true for \(Rf\). The value for \(Rf\) for the pair \((0, \theta)\) is the line integral of \(f\) along a line passing through the origin at an angle of \(\theta\) with the positive y axis. In general, the value of the line integral will depend on the angle \(\theta\). Therefore, the pair of real numbers \((l, \theta)\) are not to be interpreted as polar coordinates of points in a plane. The relationship between the \((r, \phi)\) space and the \((l, \theta)\) space is shown in Figure 3-2.

Keeping the preceding discussion in mind, the Radon transform can be summarized as follows:

(1) The operator \(R\) associates with a function \(f\) over the \((r, \phi)\) space another function \(Rf\) over the \((l, \theta)\) space.

(2) A single point in the \((l, \theta)\) space corresponds with a line
Figure 3-1  A physical representation of the variables contained in equation (3-1).
Figure 3-2  A pictorial representation of the relationship between the (r,\(\phi\)) space and the (\(l,\theta\)) space.
L at a perpendicular distance \( l \) from the origin and making an angle \( \theta \) with the positive y axis in the \((r,\phi)\) space.

(3) \([R_f](l,\theta)\) is the integral of \( f \) along \( L \).

When applying the Radon transform to geophysical tomography, the values of \([R_f](l,\theta)\) are estimated integrated attenuation values (based on instrument measurements made in the field) for a finite number of pairs \((l,\theta)\). These finite number of pairs are expressed as \((l_i,\theta_i)\) for \( 1 \leq i \leq I \), where \( I \) equals the total number of ray paths in the collection. Therefore, the Radon transform at the coordinates \((l_i,\theta_i)\) is defined as

\[
R_{l}f = [R_f](l_i,\theta_i) \tag{3-2}
\]

Often, in the literature, the available estimate of \( R_{l}f \) is referred to as \( Y_i \) (refer to Chapter II), and the vector \( Y \) is denoted as the \( I \)-dimensional column vector whose \( i \)th element is \( Y_i \). This column vector is known as the measurement vector. Thus, the task of any transform reconstruction method is given the measurement vector \( Y \), estimate the picture \( f \). The estimate of \( f \) is denoted as \( f^* \).

All transform methods attempt to give an estimate \( f^*(r,\phi) \) of \( f(r,\phi) \) in terms of \( r,\phi \), and the \( Y \) vector. As a result, all transform reconstruction methods are basically "discretized" versions of a Radon inversion formula. Such inverse transforms are notated as \( R^{-1} \) and are defined to operate on \( Rf \) such that \( R^{-1}Rf = f \). One such inversion formula can be expressed as
\[
[R^{-1}p](r,\phi) = \frac{1}{2\pi^2} \int_0^\pi \int_{-\infty}^{\infty} \frac{1}{r\cos(\theta-\phi)-\ell} p_1(\ell,\theta) \, d\ell \, d\theta \quad (3-3)
\]

This formula defines \( R^{-1}p \) at points \((r,\phi)\) for functions \(p\) of two real variables. The term \( p_1(\ell,\theta) \) denotes the partial derivative of \( p(\ell,\theta) \) with respect to \( \ell \), where for purposes of notational simplicity \( p(\ell,\theta) \) is defined to be the Radon transform \([Rf](\ell,\theta)\) of a picture.

The operator \( R^{-1} \) expressed in equation (3-3) can be broken up into a sequence of simpler operators.

An operator denoted as \( D_\ell \) can be used to represent the partial differentiation with respect to \( \ell \) of the function \( p(\ell,\theta) \). Thus for any \( p(\ell,\theta) \) where \( \ell \) and \( \theta \) are real variables,

\[
[D_\ell p](\ell,\theta) = \lim_{\Delta \ell \to 0} \frac{p(\ell+\Delta \ell,\theta)-p(\ell,\theta)}{\Delta \ell} \quad (3-4)
\]

Equation (3-4) must assume that the limit on the right-hand side exists and that \( D_\ell Rf \) is defined for all \((\ell,\theta)\).

The next operator used is the Hilbert transform \( H_q \) with respect to the first variable of a function \( q \) of two variables. This operator can be defined for any real number pair \((\ell,\theta)\) as

\[
[H_q](\ell^-,\theta) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{q(\ell,\theta)}{\ell^--\ell} \, d\ell \quad (3-5)
\]

Since the integrand diverges at \( \ell=\ell^- \), equation (3-5) should be evaluated in the Cauchy principle value sense as follows:
In tomography applications, \( q \) is \( D_y R f \) for some picture \( f \). Again, equation (3-6) is based on the assumption that the limit on the right-hand side exists.

Lastly, an important operator known as backprojection is used. The operator \( B_t \) is used to operate on any given function \( t \) of two variables. Once given a function \( t \) of any two variables, \( B_t \) is another function of two polar variables whose value at any point \((r, \phi)\) is defined by

\[
[B_t](r, \phi) = \int_0^\pi t(r \cos(\theta - \phi), \theta) d\theta
\]  

(3-7)

A more detailed discussion of the important concept of backprojection will be given in the next section.

Combining (3-4), (3-5), and (3-7) the following equality is obtained:

\[
[B_H q](r, \theta) = - \frac{1}{\pi} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{q(l, \theta) \text{d}l}{l - l^*} \text{d}l^* \right) \text{d}l \right]
\]  

(3-6)

If the right-hand side of equation (3-8) is multiplied by \(-1/2\pi\), the following operator equation is valid:
In summary, to obtain the inverse transform of a function \( p \) of two variables, the following sequence of operations should be used:

1. take the partial derivative of the Radon transform \( Rf \) with respect to \( \theta \) to obtain a function \( q \),
2. take the Hilbert transform of \( q \) with respect to its first variable to obtain a function \( t \),
3. backproject \( t \), and
4. multiply by the multiplicative constant \(-1/(2\pi)\).

An alternative expression for the inverse Radon transform given in equations (3-3) and (3-9) is,

\[
\mu(x,y) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\epsilon}^{\infty} \int_{0}^{2\pi} m_1(x\cos\theta + y\sin\theta + q, \theta) \, d\theta \, dq \tag{3-10}
\]

In this equation \( \mu(x,y) \) can be defined as the relative attenuation at the Cartesian coordinates \((x,y)\). Again, \( m_1(x\cos\theta + y\sin\theta + q, \theta) \) denotes the partial derivative of \( m(x\cos\theta + y\sin\theta + q, \theta) \) where \( m \) is the Radon transform defined as,

\[
m = \int_{0}^{D} \mu(x,y) \, dz \tag{3-11}
\]

where \( D \) is the length of the path through the reconstruction space.

For the more mathematically adventurous, a detailed proof of the
validity of equations (3-3) and (3-10) can be found in Chapter 16 of [14].

While the exact details of the preceding discussion are likely to be vague to the non-mathematician, the implication should be clear that the distribution of relative linear attenuation rates in an infinitely thin slice is uniquely determined by the set of all its line integrals. This is the idealized solution to the problem of image reconstruction from projections. Real world reconstruction algorithms must contend with the following inaccuracies:

(1) In tomography, only a finite number of line integrals are known.

(2) The measurements in tomography are only estimates. In geotomography, specifically, measurement errors may result when the rays do not propagate along a straight path or from antenna positioning inaccuracies. These are only two sources of error among several others.

As a result of these inaccuracies, the task of finding an efficient transform algorithm for all applications is most difficult. However, several researchers in the field have invested a great amount of effort in the development of algorithms that are fast when implemented on a computer and produce acceptable reconstructions in spite of the inaccuracies inherent to tomography.
2. Continuous Backprojection

Backprojection is the most fundamental of all reconstruction techniques. In general, the backprojection technique will not produce finely detailed reconstructions. However, the nature of the backprojection concept needs to be understood in order to appreciate the need for more sophisticated reconstruction techniques.

The backprojection operator as defined by equation (3-7) is simply the integrated sum, over all projection angles, of the function \( t(r \cos(\theta - \phi), \theta) \). In the previous discussion, the function \( t \) was defined as the Hilbert transform of the partial derivative (with respect to \( f \)) of the Radon transform. When using the backprojection operator alone as a reconstruction technique, the function \( t \) will be defined as the integrated attenuation value \( p \). Also, if the reconstruction area of interest is defined in Cartesian coordinates, it is often desirable to express equation (3-7) as a function of the Cartesian coordinates \((x, y)\). This may be accomplished using a common trigonometric identity, which can be used to define the following equality:

\[
l = r \cos(\theta - \phi) = x \cos \theta + y \sin \theta \quad (3-12)
\]

Thus, equation (3-7) can be expressed in Cartesian coordinates as

\[
[Bp](x, y) = \int_{0}^{\pi} p(x \cos \theta + y \sin \theta, \theta) d\theta \quad (3-13)
\]

If the function \( p \) is redefined as the integrated attenuation at a fixed projection angle \( \theta_i \), denoted as \( p_{\theta_i} \), then equation (3-13) may be
numerically evaluated by the sum

\[
\Delta \sum_{i=1}^{N} p_{\theta_i} (x \cos \theta_i + y \sin \theta_i) \tag{3-14}
\]

where \( \Delta \) is the value of the angular increment between successive viewing angles. It is the use of this Riemann sum that allows the continuous backprojection operator to be implemented as a reconstruction algorithm [14]. Again, the integrated attenuation values are estimated from measured data.

It should be noted that the value of \( p_{\theta_i}(\ell) \) for any ray path \( \ell \) will be the same for all points \((x,y)\) along \( \ell \). If the value of \( p_{\theta_i}(\ell) \) is assigned to all points of interest along \( \ell \), then the integrated attenuation value \( p_{\theta_i}(\ell) \) is said to be "backprojected" or "smeared" over the picture plane (see Figure 3-3). Thus, equation (3-14) states: that for any point \((x,y)\), the attenuation rate can be estimated by summing the values for \( p_{\theta_i}(\ell) \) for all ray paths passing through \((x,y)\) and then multiplying that sum by the value of angular increment between viewing angles.

Some backprojection techniques average the integrated attenuation summation by dividing the summation by the number of viewing angles rather than multiplying the summation by \( \Delta \) [15]. This variation of the backprojection technique can be expressed as
Figure 3-3 Illustration of the backprojection concept.
\[
\frac{1}{N} \sum_{i=1}^{N} P_{\theta_i}(x\cos\theta_i + y\sin\theta_i)
\]  
(3-15)

where \( N \) is the number of viewing angles. However, when backprojection is implemented as one step in a more sophisticated algorithm it will be defined as in equation (3-14).

In all practical image reconstruction schemes, the points \((x,y)\) at which the estimate \( f^*(x,y) \) is desired is the center point of a cell or pixel. Usually, the reconstruction area has to be divided into cells of constant attenuation rate in order to "digitize" the problem. When using a finite number of ray paths, it is often the case that no ray paths will pass through the desired cell center points. Therefore, it is necessary to use some type of interpolation method to find the estimate at the cell center points. The two basic forms of interpolation used in tomography are the "nearest neighbor" and "linear" interpolations. The nearest neighbor method is applied as follows: add together the ray sums (one form each viewing angle) that pass nearest to the cell center point, then simply average the total ray sum to estimate \( f^*(x,y) \). Linear interpolation is slightly more involved. It involves adding linear interpolates of the ray sums of the two rays that lie on either side of the cell center point. Thus, to find \( f^*(x,y) \), the linear interpolates (one for each viewing angle) are added and then averaged.

Although backprojection is an essential part of all the more sophisticated transform reconstruction techniques, it will not produce
acceptable reconstructions when used alone. To demonstrate this, consider a reconstruction area consisting of nine cell center points (see Figure 3-4). Let cell center point 5 have an attenuation rate of 1 N/m, and all other points within the area of reconstruction have attenuation rates of 0 N/m. Also, for simplicity, let all the area outside of the reconstruction phantom have an attenuation rate of 0 N/m. In order to digitize the reconstruction space, the attenuation rate of the cell center point is assumed constant over the entire cell. If the backprojection algorithm of equation (3-14) is applied using 3 ray paths at projection angles -90°, -45°, 0°, 45° and 90° (see Figure 3-5), the reconstructed image will be produced as shown in Figure 3-6. As can be seen from Figure 3-6, the backprojection method tends to "smear" or "streak" the original image. Also, it should be noted that if the backprojection algorithm of equation (3-15) is applied to the phantom of Figure 3-4, the reconstruction is somewhat better (see Figure 3-7). In most cases, this algorithm tends to work better with small amounts of measured data, such as in this example. Finally, it should be mentioned that if this reconstruction was to be carried out in the geophysical environment, the views at -90° and 90° would not be available. This is due to the cross borehole scanning geometry used in geotomography.

To produce a more detailed reconstruction, mathematical techniques that perform the operations Dy and Hy must be applied. These methods, in effect, filter the backprojection data before producing the reconstructed image. Techniques that modify the backprojection procedure will be discussed in the next two sections.
Figure 3-4 Phantom reconstruction area for backprojection example.
a) Projection angle of $+90^\circ$ and $-90^\circ$

b) Projection angles of $+45^\circ$ and $-45^\circ$

c) Projection angle of $0^\circ$

Figure 3-5  Scanning geometry for backprojection example.
<table>
<thead>
<tr>
<th>0.79 N/m</th>
<th>1.57 N/m</th>
<th>0.79 N/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.79 N/m</td>
<td>3.93 N/m</td>
<td>0.79 N/m</td>
</tr>
<tr>
<td>0.79 N/m</td>
<td>1.57 N/m</td>
<td>0.79 N/m</td>
</tr>
</tbody>
</table>

Figure 3-6  Image reconstruction using an algorithm based on equation (3-14).
Figure 3-7  Image reconstruction using an algorithm based on equation (3-15).
3. The Fourier Method

The Fourier method of reconstruction is based upon a concept known as the "projection slice theorem." It proposes a basic relationship between the Radon transform, the two-dimensional Fourier transform, and an operator which will be called $F_y$. This theorem can be stated as follows [15]:

If $p(l, \theta)$ represents the integrated attenuation at $(l, \theta)$ for a picture $f(x,y)$, then the one-dimensional Fourier transform of $p(l, \theta)$ (with respect to the variable $l$), denoted as $F_y r f$, is equal to the two-dimensional Fourier transform, denoted as $F_2$.

In operator notation the above statement can be represented as [14],

$$F_2 = F_y r f$$  \hspace{1cm} (3-16)

In this operator equation, the two-dimensional Fourier transform is defined as

$$[F_2 f](R, \Phi) = \int_{0}^{\pi} \int_{-\infty}^{\infty} r \cdot f(r, \phi) \exp[-2\pi irR\cos(\Phi - \phi)] dr d\phi$$  \hspace{1cm} (3-17)

and the operator $F_y r f$ is defined as

$$[F_y r f](l) = \int_{-\infty}^{\infty} p(l, \theta) \exp[-2\pi i l R] dl$$  \hspace{1cm} (3-18)

Thus, from equation (3-16) it can be seen that for a function $f$ of two polar variables,
\[ f = F_2^{-1} F_y Rf \]  \hspace{1cm} (3-19)

where \( F_2^{-1} \) is defined as

\[
[F_2^{-1} F](r, \phi) = \int_{0}^{\pi} \int_{-\infty}^{\infty} R R \exp[2n\pi R \cos(\phi - \phi')] dR d\phi \quad (3-20)
\]

The Fourier method can be thought of as a backprojection filtering technique using Fourier transforms, and equation (3-19) can be viewed as the mathematical idealization of this method.

The Fourier method, while simple in concept, is often difficult to implement using real data. If a parallel mode of data collection is used, then \( p(\ell, \theta) \) is known at points \((m_\Delta, n\Delta)\), where \( \Delta \) is the distance between ray paths and \( \Delta \) is the angular increment between successive projection views for \(-M \leq m \leq M\) and \(0 \leq n \leq N-1\) (see Figure 3-8). For any value of \( R \), \( F_y Rf \) may be calculated as

\[
[F_y p](R, n\Delta) = \int_{-E}^{E} p(\ell, n\Delta) \exp[-2n\pi R \ell] d\ell \quad (3-21)
\]

where \( p(\ell, \theta) = 0 \) if \( |\ell| \geq E \). The right-hand side of (3-21) has to be numerically evaluated for selected values of \( R \). A Riemann sum approximation using the sample points gives the expression:

\[
[F_y p](R, n\Delta) \approx \sum_{m=-M}^{M} p(m\Delta, n\Delta) \exp[-2n\pi R m\Delta] \quad (3-22)
\]

However, this approximation is accurate for only a limited range of \( R \) values. It has been found that equation (3-22) can only be an
Figure 3-8 Figure representing the variables used by the Fourier method in the parallel mode of data collection.
accurate approximation of (3-21) if \( R < 1/2d \), and it may even be inaccurate near the edges of that interval. For \( |R| > 1/2d \), it is usually better to approximate by zero than by the use of (3-22).

The next task, when using the Fourier method, is to choose which points \( R \) in the interval \(-1/2d < R < 1/2d\), at which \([Fyp]\,(R,n\Delta)\) is to be evaluated. It has been found that it is computationally desirable to choose values of \( R \) which are multiples of \( 1/(2M+1)d \). It should be noted that this method of selecting \( R \) places no restriction on how close the sample points are spaced, since the evaluation of (3-22) would not change for a larger value of \( M \).

If the above guidelines for selecting \( R \) are followed, then the first step of the Fourier method is to estimate, for \( 0 \leq n < N-1, -M \leq m < M \),

\[
[Fyp](m'/2M+1)d,n\Delta) = d \sum_{m=-M}^{M} p(md,n\Delta)\exp[-2\pi imm'/(2M+1)]
\]  

(3-23)

The right-hand side of (3-23) is \( d \) multiplied by the discrete Fourier transform in the first variable of the sampled version of \( p(l,\theta) \). The evaluation of equation (3-23) is required at the \( N(2M+1) \) points \( (0 \leq n < N-1, -M \leq m < M) \). To evaluate these points each view is dealt with separately. Using conventional calculations, the \( 2M+1 \) different values of (3-23) for a fixed \( n \) and varying \( m \) are evaluated from the \( 2M+1 \) values of \( p(md,n\Delta) \) for the same fixed \( n \) and varying \( m \). This method evaluates the expression for each value of \( m \) separately.
However, for most Fourier algorithms a fast Fourier transform (FFT) is used to evaluate (3-23). Using the FFT, all of the $2M+1$ values of $m'$ between $-M$ and $M$ are calculated for a fixed $n$. Employing the FFT will usually reduce the number of multiplications needed within a program by at least two orders of magnitude.

The next problem faced with the Fourier method is that of numerically evaluating $F_{2^{-1}}F_{yp}$ from the estimated values of $F_{yp}$ at the points $(m/(2M+1)d,n\Delta)$. One technique for doing this is with a Riemann sum approximation of (3-20) which gives the estimated picture

$$f^*(r,\phi) = \frac{\Delta}{2M+1} \sum_{n=0}^{N-1} \sum_{m=M}^{M} \left[ F_{yp}(m/2M+1)d,n\Delta \right]$$

$$\cdot \exp \left[ 2\pi i \frac{m}{(2M+1)d} r\cos(\phi-n\Delta) \right]$$

(3-24)

However, due to the large number of multiplication calculations needed to evaluate (3-24), most Fourier reconstruction algorithms do not use this method. Most, in fact, make use of the two-dimensional fast Fourier transform. This is a version of the FFT which can be used to estimate the values of the two-dimensional inverse Fourier transform $F_{2^{-1}}F_{yp}$ at $I$ points from the values of $F_{yp}$ at $I$ points. There is one major difficulty with this process. The points at which $F_{2^{-1}}F_{yp}$ are to be evaluated and the points at which the values of $F_{yp}$ have to be given must lie in rectangular arrangement. In the output this is desirable since the object of the reconstruction is to estimate
f' = F^2 \cdot F_{yp} at points which are centers of pixels positioned in a rectangular geometry. However, the input is not in a rectangular arrangement. The first step of the Fourier method provides the input values at points with polar coordinates \((m/2M+1)d, n\Delta\). These points lie in a radial arrangement (see Figure 3-9). Thus, the Fourier reconstruction algorithm has three major subprocesses [16]:

1. Evaluation of the measurement data using an FFT.
2. Interpolation from a polar to a Cartesian grid in Fourier space.
3. Two-dimensional inverse transform to obtain the reconstructed image.

The major disadvantage of the Fourier method is that it often requires excessive amounts of computer memory. The difficulty arises from the fact that the FFT algorithm only provides an estimate of the two-dimensional Fourier transform. To compensate for the inaccuracies in the FFT, the FFT must be calculated for a number of points much larger than the number of pixels in the final display. The additional pixels are introduced to make the picture region \(1/\delta \times 1/\delta\) instead of \(E \times E\), where \(\delta\) is the spacing between the points at which values of \(F_{yp}\) are interpolated and \(E\) is the height of the reconstruction area. The increase may be as large as a factor of 9 (if \(1/\delta = 3E\)). This increase often requires large amounts of computer memory which negates the memory saved by using the FFT.

Also, in order to have accurate estimates of \(F_{yp}\) at the rectangular grid with spacing \(\delta\), the first step of the Fourier method may
a) The arrangement of points at which estimates of the values of $F_y p$ are known.

b) The arrangement of the interpolated points needed before the inverse two-dimensional FFT can be applied.

Figure 3-9 Illustration of the points at which estimates of $F_y p$ are known before and after interpolation.
need to be evaluated for a larger number of rays per view. This will result in a further increase in computational cost.

4. The Convolution Method

The most commonly used reconstruction transform method is the convolution method. This method is easy to implement and often provides good reconstructions while using minimal amounts of computer time. The convolution method derives its name from the fact that when it is used to solve the inverse Radon equation, the taking of the derivative and the Hilbert transform are approximated by the use of a single convolution. Once the convoluted data are determined, they are backprojected.

The mathematical foundation of this method was first proposed by Bracewell and Riddle [17], as a method to determine the brightness distribution over a stellar radio source. This technique was later rediscovered by Ramachandran and Lakshminarayanan [18], who used the method to determine the optical density of an object from electron micrographs. They also gave the method a form that could be easily implemented on a computer. While the convolution method has numerous variations, the derivation of Ramachandran and Lakshminarayanan will be followed to describe the underlying mathematical theory of the method.

The starting point for any convolution method lies in the "projection slice theorem" for Fourier transforms. As a result of
this, the convolution method is often viewed as an "offshoot" of the Fourier method. The "projection slice theorem" can be expressed by the following Fourier transform relationships:

\[
F(R, \theta) = \int_{-\infty}^{\infty} p(l, \theta) \exp(2\pi i R l) dl \tag{3-25}
\]

\[
f(r, \phi) = \int_{0}^{2\pi} \int_{-\infty}^{\infty} F(R, \theta) \exp[-2\pi i R r \cos(\theta - \phi)] Rd\theta \tag{3-26}
\]

where \(p(l, \theta)\) is the integrated attenuation at a point \((l, \theta)\) in the \(Rf\) domain. The object of this derivation is to rewrite equations (3-25) and (3-26) in a form where Fourier transforms do not appear explicitly, but contain only integrals of functions defined in the real space of observation. Equation (3-26) can be rewritten into the form,

\[
f(r, \phi) = \int_{0}^{\pi} \int_{-\infty}^{\infty} |R| F(R, \theta) \exp[-2\pi i R r \cos(\theta - \phi)] Rd\theta \tag{3-27}
\]

If \(p'(l, \theta)\) is defined as

\[
p'(l, \theta) = \int_{-\infty}^{\infty} |R| F(R, \theta) \exp(-2\pi i R l) dR \tag{3-28}
\]

then equation (3-27) becomes

\[
f(r, \phi) = \int_{0}^{\pi} p'[\cos(\theta - \phi), \theta] d\theta \tag{3-29}
\]
Now, the task is to express \( p'(l, \theta) \) in terms of the measurement data \( p(l, \theta) \). This is done in the following manner: Fourier-inverting equation (3-25) gives

\[
p(l, \theta) = \int_{-\infty}^{\infty} F(R, \theta) \exp(-2\pi i Rl) dR
\]  

(3-30)

From equations (3-28) and (3-30) it can be seen that the Fourier transform of \( p(l, \theta) \) is \( F(R, \theta) \), while the Fourier transform of \( p'(l, \theta) \) is \( R | F(R, \theta) \). Thus the Fourier transform (F.T.) of \( p'(l, \theta) \) can be expressed as follows:

\[
\text{F.T. of } p'(l, \theta) = [\text{F.T. of } p(l, \theta)] \cdot [\text{F.T. of } q(l)]
\]  

(3-31)

where \( |R| \) is the Fourier transform of the function \( q(l) \) expressed as

\[
|R| = \int_{-\infty}^{\infty} q(l) \exp(2\pi i Rl) dl
\]  

(3-32)

Next, the convolution theorem for the inverse Fourier transform of the product of Fourier transforms is employed. This theorem states that the inverse transform of the product of the Fourier transforms of two functions is equal to the convolution of those two functions [19]. In this case the functions are \( p(l, \theta) \) and \( q(l) \); thus \( p'(l, \theta) \) can be expressed as
To evaluate the function $p'(l, \theta)$ as a convolution integral, the function $q(l)$ is required explicitly. If equation (3-32) is inverted, $q(l)$ can be expressed as

$$q(l) = \int_{-\infty}^{\infty} |R| \exp(-2\pi i R l) dR$$  \hspace{1cm} (3-34)$$

However, this integral is impossible to evaluate as the integrand diverges. This difficulty can be overcome by noting that the high frequency components are of little interest in a reconstruction problem [20]. Hence, the limits $-\infty$ and $+\infty$ in equation (3-34) may be replaced by $-A(\theta)/2$ and $A(\theta)/2$, where $A(\theta)$ is some finite number and is a function of the projection angle. Thus equation (3-31) becomes

$$q(l) = \int_{-A(\theta)/2}^{+A(\theta)/2} |R| \exp(-2\pi i R l) dR$$  \hspace{1cm} (3-35)$$

To evaluate equation (3-35), $q(l)$ is defined as $q_{0j}(kaj)$, where $aj$ is the distance between ray paths for the $j$th projection ($1 \leq j \leq N$, $1 \leq k \leq M$). $N$ is defined as the total number of viewing angles and $M$ is the total number of ray paths per projection (see Figure 3-10). If the evaluation is carried out choosing $A(\theta)=1/aj$, the solution is
Figure 3-10  Pictorial illustration of the variables used by the convolution method.
If the measured data \( p(\ell, \theta) \) are known at a set of equally spaced points \( \ell = \text{maj} \) (where \( m \) is a positive or negative integer), then equation (3-33) can be evaluated numerically and expressed in the form of an infinite sum as

\[
p^{\prime}(ka_j, \theta) = a_j \sum_{m=-\infty}^{\infty} p(\text{maj}, \theta) q[(m-k)a_j]
\]

When using equation (3-36) as the values of \( q_{\theta_j}(ka_j) \), equation (3-37) becomes

\[
p^{\prime}(ka_j, \theta) = p(ka_j, \theta)/4a_j - (1/\pi^2 a_j) \sum_{t=0}^{\infty} p[(k+t)a_j, \theta]/t^2
\]

Equation (3-38) is referred to as the convoluted projection data, and is calculated for each ray path contained in each fixed projection. This is illustrated more easily if equation (3-38) is expressed in the following form:

\[
P^{\prime}j,k = \frac{1}{a_j} \left[ \frac{P_{j,k}}{4} - \frac{1}{\pi^2} \sum_{t=0}^{\infty} \frac{P_{j,k+t}}{t^2} \right] \]

where \( P^{\prime}j,k \) is the convoluted projection data for the kth ray of the jth projection.

\( P_{j,k} \) is the measured data for the kth ray of the jth projection.
It should be noted that equation (3-39) is an approximation for two reasons:

(1) The measurement data provided is only an approximation.
(2) The results of the numerical evaluation of the integral is not exact.

Once the convoluted projection data has been determined, it must be backprojected. This is accomplished by numerically evaluating (3-29) using the trapezoidal rule, which gives

\[ f^* (r, \phi) = \sum_{j=1}^{N} p^- [r \cos(\theta_j - \phi), \theta_j] (\theta_{j+1} - \theta_{j-1})/2 \]  

(3-40)

Notice that the weighting factor \((\theta_{j+1} - \theta_{j-1})/2\) is used to compensate for the possibility of unequally spaced projections. If the projections are equally spaced then the weighting factor becomes the value of the angle increment between successive projection angles.

In most cases the value of \(p^- [r \cos(\theta_j - \phi), \theta_j]\) needed to give the values of \(f^*\) at the pixel center points is not available. Thus, the convoluted data needs to be interpolated to give the desired values. To perform the interpolation, suppose the center of the \(i\)th pixel, whose density attenuation rate \(f\) which is being estimated with \(f^*\), lies between the \(k\)th and \((k+1)\)th ray of the \(j\)th projection. If the distance from the center point to the \(k\)th ray is said to be \(b_{ij}\) and the distance from the center point to the \((k+1)\)th ray is said to be
\[ c_{ij}, \text{ where } b_{ij} + c_{ij} = a_j, \text{ then the interpolated convoluted projection data is expressed as} \]

\[ x_{ij} = \frac{1}{a_j} \left( c_{ij}p^j_{j,k} + b_{ij}p^j_{j,k+1} \right) \tag{3-41} \]

Thus, the final assignment of an attenuation rate value to each pixel can be expressed as

\[ f^*_i = \sum_{j=1}^{N} x_{ij} \frac{\theta_{j+1} - \theta_{j-1}}{2} \tag{3-42} \]

To obtain this final form two additional approximations were made: the numerical integration and the linear interpolation which estimates \( p'[rcos(\theta j-\phi), \theta j] \). This algorithm is coded in Appendix B, and is the transform method used in the comparisons made in the next chapter.

C. Series Expansion Method Theory

The series expansion approach to image reconstruction differs fundamentally from the transform method approach. When using series expansion methods the reconstruction problem is discretized at the very beginning. In contrast, when using a transform method the continuous problem is solved until the end, then the final formulas are "discretized" for implementation on the computer.

The discretized model required for use with the series expansion techniques was introduced in Chapter II. The model can be sum-
marized in the following way [21]: a Cartesian grid of picture elements, called pixels, is used to represent the region of reconstruction. The pixels are numbered from 1 (top left corner pixel) to n (bottom right corner pixel). The electromagnetic attenuation rate is assumed to be a constant value $X_j$ throughout the $j$th pixel for $j=1, 2, 3, \ldots, n$. The transmitter and receiver are assumed to be points and the ray paths between them are assumed to be lines. In addition, the length of intersection of the $i$th ray with the $j$th pixel is denoted by $D_{ij}$ for all $i=1, 2, \ldots, m$, $j=1, 2, \ldots, n$. This represents the weight of the contribution of the $j$th pixel to the total attenuation along the $i$th ray. The measured total attenuation of the $i$th ray is denoted by $V_i$. This represents the integrated attenuation along the path of the $i$th ray (see Figure 3-11).

This model can be described by a system of linear equations. In the following sections, four series expansion methods are discussed and evaluated for their effectiveness in solving the sets of linear equations that are generated in geotomography.

1. The Direct Matrix Inversion Technique

In Chapter II it was mentioned that direct matrix inversion techniques are generally ineffective when applied to image reconstruction problems. In this section the problems associated with the direct matrix inversion technique will be discussed in more detail.

If a system of linear equations is to be solved using direct matrix inversion, then a solution must exist in the ordinary sense
Figure 3-11 The discretized model required for series expansion techniques.
X = D⁻¹Y. Here, D⁻¹ is defined to be the inverse of D. It is often the case that D⁻¹ cannot be evaluated. The reason for this is that the system is rarely even determined. Often, in order to produce a good image resolution, the system is greatly overdetermined. This will always exclude a solution in the ordinary sense.

If, however, the reconstruction problem could be modeled in such a way to give an even determined system, it would still be highly unlikely that the inverse of the D matrix could be solved. The D matrix of the system is usually very sparse. This is to say that only a few pixels have a non-empty intersection with each individual ray. In most cases the non-zero elements of the D matrix is less than 10 percent. The sparseness of the matrix will increase the probability that the matrix will be found to be singular or nearly singular.

Finally, even if the system could be solved in the ordinary sense, an enormous amount of computing time would be required to solve for the extremely large number of equations that are typical of a reconstruction problem. For example, it is not unusual for at least 1700 measurements to be performed when scanning an underground region of interest [10]. Then, to make the system even determined, the reconstruction region needs to be divided into 1700 pixels. This means that the D matrix will consist of 1700×1700=2.89×10⁴ elements. The amount of computer time needed to invert a matrix of this size is excessive. In addition, once the matrix has been inverted, more computer time is needed to perform the final matrix multiplication needed
to give the attenuation rates. Lastly, the need to store the $2.89 \times 10^6$ values of the inverted D matrix usually exceeds the available memory of most computers.

As a result of the inapplicability of direct matrix inversion techniques to image reconstruction problems, researchers have developed several indirect techniques that have found acceptance in the field of tomography. Three important indirect series expansion techniques will be presented in the following discussion.

2. The Discrete Backprojection Technique

The discrete backprojection technique can be thought of as the series expansion method equivalent of the continuous backprojection transform method. Instead of producing a digitized picture by numerically evaluating the integral in equation (3-13); the backprojection is computed using a weighted averaging process based on the series expansion discretized model.

Discrete backprojection, like continuous backprojection, usually does not produce images that have good resolution. However, of all the indirect series expansion techniques, discrete backprojection is the simplest to implement and the least time consuming due to the noniterative nature of the technique. The technique has been researched extensively in nuclear medicine applications since about 1956 and is usually referred to in the literature as "transverse section scanning" [22].
The discrete backprojection algorithm can be implemented in the following manner [12]. First, a preliminary $X$ value is calculated for the $j$th pixel, for each of the $i$th ray paths. This preliminary value is expressed as

$$X_{ij} = \frac{Y_i}{L_i}$$  \hspace{1cm} (3-43)

where $Y_i$ is related to the measured power (see Chapter II), and $L_i$ is the length of the $i$th path through the region of reconstruction. The preliminary values $X_{ij}$ are then weighted by the length of the $i$th path through the $j$th pixel to give an intermediate value

$$X^*j = \sum_{i} X_{ij} D_{ij}$$  \hspace{1cm} (3-44)

This equation shows that the contribution of the $i$th ray to a pixel is proportional to $Y_i$, and depends on the ratio of the length of the $i$th path through the $j$th cell compared to the total length of the $i$th path through the area of reconstruction. The intermediate value $X^*j$ can then be thought of as the sum of all the contributions made by each ray that passes through the $j$th pixel. To obtain the final estimates of $X$, the intermediate value $X^*j$ is divided by the total path lengths through the $j$th pixel by all the paths. This, in effect, averages the sum and can be expressed as

$$X_j = \frac{X^*j}{\sum_{i} D_{ij}}$$  \hspace{1cm} (3-45)
3. The Algebraic Reconstruction Technique

The poor image resolution that is inherent to the discrete backprojection technique prompted researchers to develop more sophisticated iterative schemes to use with image reconstruction problems. The Algebraic Reconstruction Technique (ART) was first introduced by Gordon, Bender, and Herman for solving image reconstruction problems found in electron microscopy and radiology [23]. Since that time, the ART and variations of the ART have been applied in many areas of the tomography field, and is used almost exclusively in the field of geotomography.

The mathematical basis of the ART is well documented and can be implemented easily into a computer code [21],[24]. The ART algorithm can be developed by first arbitrarily assigning values of $X$ to each pixel. These values are usually zero unless there is already some prior knowledge that would suggest that a more accurate value for $X$ could be chosen. Then an estimate of $Y_i$ is calculated for the $i$th path in the $q$th iteration using the formula

$$Y_i^q = \sum_j D_{ij} X_j^q$$ (3-46)

where $Y_i^q$ is the estimated value of $Y_i$ in the $q$th iteration and $X_j^q$ are the estimated values of $X_j$. Usually these $X_j$ are not correct and the resulting $Y_i^q$ will not equal the $Y_i$ obtained by measurement. Thus, to improve the estimate, a correction factor for each cell is calculated in the following manner [12]:
where $C^q_j$ is defined as the correction factor for the $j$th cell in the $q$th iteration.

This correction factor is calculated for one ray path at a time. After the corrections for all the cells intersected by the $i$th ray path are found, the correction values are then added to the existing $X$ estimates and the algorithm operates on the next path. One iteration is completed when this process has been performed for all ray paths. The algorithm then starts another iteration by returning to the first ray path and repeating the procedure again. The iterative process continues until convergence is obtained or the until the process is halted in accordance to a predetermined criterion.

It has been found that as the number of iterations increase that the ART will occasionally converge to a best case solution and then diverge from this best case if noise is appreciable. As a result, several criteria for convergence have been proposed [24]. However, it has been determined in further studies that if the noise level is low, the ART usually did not diverge. In this case the ART algorithm should be allowed to iterate until there is little change in the $X$ estimates for each further iteration.

To illustrate the use of the ART algorithm, a simple example can be constructed. Consider the reconstruction profile of four
pixels and four ray paths shown in Figure 3-12. If the initial \( X \)
estimates are set equal to zero and the correction factors are imple­
mented for the first ray path, the reconstructed image will appear as
in Figure 3-13a. Now the algorithm is implemented on the second ray
path and the half finished reconstructed image is shown in Figure
3-13b. Next, after the third ray is considered, the \( X \) values for
cells 1 and 3 have converged. Finally, after the fourth ray has been
considered, the \( X \) values for cells 2 and 4 are obtained (see Figure
3-13d).

In this simple example convergence was achieved after one
iteration. Of course, in actual image reconstruction problems many
iterations are needed to produce an accurate estimate of a given con­
ductivity profile and total convergence is rarely obtained. In the
computer simulations discussed in the next chapter, the ART was imple­
mented using fifty iterations. A FORTRAN program for this algorithm
can be found in Appendix C.

4. The Simultaneous Iterative Reconstruction Technique

One final iterative reconstruction technique that deserves to be
mentioned is the Simultaneous Iterative Reconstruction Technique
(SIRT). The additive SIRT algorithm was first introduced by Gilbert
in 1972 as an alternate method to the ART [25]. The SIRT differs from
the ART in that at each iteration the \( X \) values are altered by using
data from all rays that pass through the cell rather than changing the
\( X \) values by considering one ray at a time.
Figure 3-12 Profile for ART example.
<table>
<thead>
<tr>
<th>1.5N/m</th>
<th>1.5N/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0N/m</td>
<td>0.0N/m</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
S &= (6-0)/2 = 1.5 \\
S_1 &= (14-0)/2 = 7.0 \\
S_2 &= (8-10)/2 = -1.0 \\
S_3 &= (12-10)/2 = 1.0
\end{align*}
\]

Figure 3-13 Reconstruction example using the ART.
To implement the SIRT, the initial values for $X$ are set and an estimate $y_i^q$ is calculated for each path in the $q$th iteration using equation (3-46). The preliminary correction factors for the $i$th path are then calculated in the following manner [12]:

$$c_{ik}^q = (y_i - y_i^q) \frac{d_{ik}}{\sum_{j}(d_{ij})^2}$$

(3-48)

The variable $k$ is used to signify that only the paths passing through cell $k$ are being considered.

The final correction factor is determined by averaging all the preliminary values and is expressed as

$$c_k^q = \frac{1}{N_k} \sum_{i} c_{ik}^q$$

(3-49)

where $N_k$ is the number of ray paths passing through cell $k$. Once all the $c_k^q$'s have been calculated, they are added to the existing estimates $X_j^q$ and another iteration begins.
IV. COMPUTER SIMULATIONS

A. Computer Modeling and Simulation Concepts

The objective of this investigation was to determine the applicability, if any, of the convolution algorithm to the cross borehole scanning geometries of geotomography. Due to a lack of funds and facilities, an experimental investigation was impossible. Therefore, computer simulations provided the only means available by which to test the convolution method. Using a model that simulated geophysical scanning geometries, the convolution algorithm was tested and then compared to the widely accepted ART algorithm. This chapter will contain a detailed description of the simulation process, all of the relevant reconstruction data obtained by the simulations, and a discussion of the methods used to evaluate the data.

1. Determining the Test Profiles and Scanning Geometries

A test profile (sometimes called a test phantom) is a rectangular grid of pixels that simulates the underground region to be investigated. In this investigation the test profiles were specified to be 18 meters high and 30 meters wide. The cell size was specified to be 3 meters in the horizontal direction, and 2 meters in the vertical direction. This choice of dimensions was based on the findings of a previous investigation which seemed to indicate that the 3 to 2 ratio produced better reconstructions than other ratios tested [12]. Thus, the test profiles can be described as grids of 90 cells; 10 in
the horizontal direction and 9 in the vertical direction. All test profiles used in this investigation adhere to the above dimensions.

Once the test profile dimensions have been determined, different profiles can be constructed by assigning constant attenuation values (in nepers/m) to each pixel. In this study two profiles were constructed in an attempt to simulate different underground structures. Profile 1 is a simulation of a horizontal rock fracture (see Figure 4-3). Due to fluid flow in the fracture (such as ground water or nuclear waste fluid), the fractured region usually has a much higher attenuation constant. In contrast, Profile 2 is a simulation of an underground region containing two anomalies of different attenuation constants (see Figure 4-25). The anomalies in this situation can represent mineral, ore deposits, or solid toxic waste.

When scanning an underground region of interest, there are several controllable variables that can affect the measurement data obtained and thus affect the quality of the subsequent reconstruction. These variables are known as the scanning conditions and are listed below:

1) The scanning range (also called viewing range).
2) The angular increment between successive viewing angles.
3) The number of measurements per view.
4) The spacing, in the borehole, between successive antenna positions.
The scanning range can be defined as the difference between the maximum viewing angle and the minimum viewing angle. In this study all viewing angles were defined relative to the horizontal boundaries of the reconstruction area. A positive viewing angle is defined as a view in which the ray paths have a positive slope when going from source to receiver and a negative viewing angle is defined as a view that has a negative slope (see Figure 2-1). A full viewing range is defined as 180.0°, in which the reconstruction area is scanned between both vertical extremes (+90.0° to -90.0°). The convolution algorithm works ideally under full viewing range conditions. The ART algorithm, on the other hand, can provide quality reconstructions when used under very limited viewing range conditions. In addition, if the underground region of interest is at a shallow depth beneath the surface, a large viewing range may be impossible to obtain. Therefore, when determining what scanning range is best for a particular application, the type of algorithm used, as well as the constraints of the physical environment must be taken into consideration. In this investigation the convolution algorithm was tested and compared with the ART over the viewing ranges of both 150.0° (+75.0° to -75.0°) and 100.0° (+50.0° to -50.0°).

The angular increment between successive viewing angles is determined by dividing the viewing range by the number of views desired. Thus, a small angular increment between views implies a more complete coverage of the reconstruction region by the scanning proce-
dure. When a large number of views (also called projections) are used in a reconstruction, a large amount of measurement data will be obtained as a result. In general, large amounts of measurement data will increase the quality of the reconstruction. However, when choosing an angular increment (and thus the number of projections) for any reconstruction, the extra time and cost required to take a large amount of measurements must also be considered. Thus, the choice is a trade-off between cost and quality. In this study, 21 projections were implemented for each reconstruction case. As a result, the angle increment for the reconstructions produced from a 150.0° viewing range was 7.5°, and the angle increment for the reconstructions produced from a 100.0° viewing range was 5.0°.

When deciding upon the number of measurements (ray paths) per view, again the general rule is that the more measurements taken, the better the quality of the reconstruction. However, there are other considerations depending on the algorithm that is implemented. When using ART it is good practice to provide enough ray paths to ensure that each cell is intersected by at least one ray path. To ensure a quality reconstruction, all cells should be intersected many times. This is also a good rule of thumb when using the convolution algorithm. However, when using the convolution method there is one other stipulation to which one must be adhere. Each viewing angle must contain enough ray paths to ensure that each cell center point has a ray path positioned on either side of it. This stipulation
results from the method used by the convolution algorithm to determine
the convoluted data at a cell center point. It uses an interpolation
scheme that requires the convoluted data for the ray paths that lie on
both sides of the cell center points. Thus, a special consideration
must be made concerning the number of measurements per view when
implementing the convolution method. In this investigation it was
necessary to use 93 ray paths per projection to ensure a complete
coverage of the cell center points at the extreme angles of the 150.0°
viewing range. When scanning over the 100.0° viewing range, only 57
ray paths per projection were needed to ensure complete coverage.

The final controllable scanning variable is the distance be-
tween successive antenna positions (the distance between successive ray
paths at the borehole). The ideal choice of this variable, again,
depends on the type of algorithm being used. When using ART the
spacing has little bearing on the quality of the reconstruction, as
long as at least one ray path passes through each cell. As was men-
tioned before, the more rays that pass through each cell, the better
the quality of the reconstruction. However, when using the con-
volution algorithm there is always an optimum ray spacing that will
give a best case reconstruction as defined by Euclidean distance. The
Euclidean distance is a measure of error used to evaluate the quality
of a reconstruction; it will be mathematically defined later.

During the course of this study it was found that when a
sequence of ray spacings were investigated, the reconstruction would
converge to a best case Euclidean distance at an optimum ray spacing and then diverge away from the best case. The sequence used would start with large values of ray spacings (2-3 meters) and then be decremented through successive ray spacing values at regular intervals until smaller values were reached (0.6-0.7 meter). The values defined as large and small ray spacings are relative to the size of the reconstruction area (18 by 30 meters). After many trial and error investigations, it was found that a good criterion for choosing an optimum value of ray spacing, when using the convolution method, can be expressed as follows:

\[
SP = \frac{(INC)(PROJ)}{1.94}
\]  

(4-1)

where \( SP \) = ray spacing (m)  
INC = angle increment between successive viewing angles in radians  
PROJ = number of projections

This criterion was applied to three separate profiles at four different viewing ranges (150.0°, 130.0°, 100.0°, and 70.0°) for various numbers of projections and angle increment values. It was found that the Euclidean distances obtained for the reconstructions generated using the ray spacing of equation (4-1) had an average deviation from the best case Euclidean distance of only 12.3%. In other words, the criterion proposed in equation (4-1) always produced a reconstruction within a reasonable range of the best possible
reconstruction. Also, in preliminary investigations, it was found
that this criterion not only applies to test profiles made up of
pixels with the dimensions of 3 by 2 meters, but also applies to any
test profile that has a cell size with the ratio of 3 by 2 (such as 6
by 4 meters or 1.5 by 1 meters).

Using the criterion proposed in equation (4-1), a ray spacing
of 1.42 meters was implemented when the viewing range of 150.0° was
used and a ray spacing of 0.95 meter was implemented in conjunction
with the 100.0° viewing range.

2. Generation of the Simulated Measurement Data

In geotomography the measurement data are derived from measure­
ments made in the field. In the computer simulations performed during
the course of this investigation, the measurement data are calculated
using equation (2-9). It should be mentioned that in order to make
the simulation less complicated, the attenuation constants outside the
region of reconstruction are assumed to be zero. Thus, for those ray
paths that only partially intersect the region of reconstruction, the
measurement data are calculated as if the total length of the ray path
is that portion that lies within the reconstruction region. If
reconstructions were being produced based upon field data, a larger
reconstruction area would have to be considered in order to produce
accurate measurement data for the region of interest (see Figure 4-1).
However, the methods of reconstruction should perform the same,
regardless of the size or shape of the reconstruction region.
Figure 4-1 Total reconstruction area needed to obtain accurate measurement data for the region of interest.
3. Injection Of Random Noise

In this study a straight-ray model was implemented. This model assumes that the ray experiences no refraction, diffraction, or reflection as it passes from transmitter to receiver (refer to Chapter II). However, in some physical applications the straight-ray model may not be valid. Therefore, it is often useful to examine the performance of a reconstruction algorithm when the straight-ray model has been altered in some way.

In this investigation it was decided that the effects of refraction, diffraction, and reflection could be approximated by injecting random noise into the measurement data. To do this, first the values of power received obtained under noiseless conditions (using the straight-ray model) are calculated for each ray path using equation (2-8). The constant k in equation (2-8) is calculated assuming that the transmitters are perfect isotropic sources transmitting a 50 Megahertz signal at 1000 watts of power. Next, the original value of power received, for each ray path, is multiplied by some random number to give a new value of power received. These new values of power received are used to recalculate the measurement data under noisy conditions.

To determine the random number multiplier, first a random number generator is used to obtain a uniformly distributed random number X between 0 and 1. This random number is then inserted in the following equation.
\[ Pr'_i = Pr_i (1+XNF \cdot X)^{-N} \]  

(4-2)

where \( Pr_i \) = the power received for the ith ray path

\( Pr'_i \) = the new "noisy" value of power received

\( XNF \) = a user defined constant

\( X \) = the random number between 0 and 1

\( N \) = an integer representing the truncated value of \( X \cdot 10 \)

From this equation it can be seen that the random number multiplier injected into the simulated measurements is between 1/(XNF+1) and (XNF+1). The value of (XNF+1) is defined as the noise factor.

In this investigation all reconstructions were carried out under noiseless conditions as well as noisy conditions in which the noise factor was set at 5 and 10 respectively. A FORTRAN program that injects random noise into the measurement data can be found in Appendix D.

4. Selective Smoothing

Selective smoothing is a type of weighted averaging process that can be used to sharpen the boundary definition between the anomalies and the ambient medium. In this study it was used as a post-
reconstruction scheme in an attempt to improve the quality of the reconstructions generated by the convolution algorithm. It was expected that the effectiveness of the algorithm would be decreased by the lack of a full viewing range in the geophysical model.

In order to understand the smoothing process, consider the block of variables shown below:

\[
\begin{align*}
X_6 & \quad X_2 & \quad X_7 \\
X_3 & \quad X_1 & \quad X_4 \\
X_8 & \quad X_5 & \quad X_9
\end{align*}
\]

where \(X_1, X_2, X_3, \ldots, X_9\) represent the values of the attenuation constants in cell 1 and its surrounding cells. The smoothed value of \(X\) in cell 1 is defined as

\[
X_{1\text{ smoothed}} = \frac{w_1 X_1 + w_2 \sum_{i=2}^{5} f_i X_i + w_3 \sum_{i=6}^{9} f_i X_i}{w_1 + w_2 \sum_{i=2}^{5} f_i + w_3 \sum_{i=6}^{9} f_i}
\]

(4-3)

where

\[
f_i = \begin{cases} 1 & \text{if } |x_i - x_1| \leq t \\ 0 & \text{otherwise} \end{cases}
\]

(4-4)

and

\[t = \text{threshold}\]

\[w_1, w_2, w_3 = \text{smoothing weights}\]

This procedure is executed a number of times (50 in this study) for all the cells. If the cell being smoothed is on the border of the
reconstruction region, then \( f_1 = 0 \) for the missing cells.

The objective of the smoothing process is to average only selected neighboring cells in such a way as to not smooth across the boundaries between anomalies and the ambient medium. A neighboring cell is included in the average only if the difference between the center cell and the neighbor cell is less than or equal to the predetermined threshold level. Thus, the effectiveness of the smoothing procedure is greatly influenced by the choice of \( w_1, w_2, w_3, \) and \( t \). The weights chosen for this study were \( w_1 = 9, w_2 = 4, \) and \( w_3 = 1 \). These choices were based upon previous studies in which these weighting values were used to improve the output of medical tomograms produced by transform methods [14]. This choice of weights tends not to make major changes in the reconstruction other than to sharpen the boundaries. The threshold level chosen was one-sixth of the difference between the highest and lowest \( X \) values in the reconstruction. It has been found that this is a good choice for test profiles in which the anomaly has a higher attenuation constant than the surrounding ambient medium. Again, this choice was based upon a previous investigation [24]. A FORTRAN program used to implement the smoothing procedure can be found in Appendix E.

B. Methods Of Evaluation

In order to evaluate the quality of any particular reconstruction some basis of comparison had to be defined. The methods of com-
parison are used to compare the performance of the convolution algorithm with ART. They are also used to compare the performance of the convolution algorithm with itself when applied to different profiles. In the following discussion the three methods of evaluation chosen for this study will be examined in detail.

1. Euclidean Distance

The Euclidean distance was proposed by Gordon, Bender, and Herman [23] and is defined as follows:

$$\delta = \sqrt{\frac{1}{m} \sum_{j=1}^{m} (X_j - X_j^t)^2}$$  \hspace{1cm} (4-4)

where

- \(m\) = the number of cells
- \(X_j\) = the reconstruction estimate for the jth cell
- \(X_j^t\) = the original test profile value for the jth cell

The Euclidean distance is a common measure of error that has been used in many image reconstruction investigations. It is popular because it is more sensitive to large changes in fine detail than to small changes over a large area. This property is desirable since the object of geotomography is to locate distinct boundaries of the subsurface anomalies rather than to obtain precise values of attenuation constants.
2. Minimum Variance Partitioning (MVP)

Minimum variance partitioning is a technique for picking out anomalies from their surrounding medium [27]. MVP involves separating all the attenuation constants for any given reconstruction into groups that give the minimum sum of squared errors between each group member and the group mean. By using this technique, the MVP attempts to define the anomalies and the ambient medium. In this study there will be two groups for Profile 1 and three groups for Profile 2.

To help explain the MVP concept, consider a two group reconstruction in which the attenuation constants can be thought of as a set of arbitrary numbers. The first step of implementation is to sort the numbers in ascending order. In order to find the MVP, all possible divisions of this list into two groups must be considered. For N entries in the list of numbers there are N-1 possible partitions. For each partition, the means ($\mu_1, \mu_2$) of both groups are found. Finally, the sum of the squared errors between each group member and its group mean is calculated as follows:

$$S_e = \sum_{i=1}^{2} \left\{ \sum_{X_j \in \text{group}_i} |X_j - \mu_i|^2 \right\}$$

(4-6)

The grouping that gives the minimum $S_e$ is defined as the minimum variance partition that separates the anomaly from the medium.

In the case of a three group reconstruction, the anomalies are defined by the two lowest values of $S_e$. FORTRAN programs for two and three group MVP can be found in Appendixes F and G respectively.
3. Computational Time

All reconstructions in this study were evaluated based on the virtual CPU time needed by the algorithm to complete the reconstruction process. It is obvious that reconstructions requiring smaller amounts of CPU time will result in lower computational cost. At Ohio University, private business and industry are charged $0.18 for each second of CPU time used on the IBM 370.

Also, computational speed may be important if a rapidly changing environment is being monitored. Waste leakage and coal gasification are two examples of such an environment. In these cases a small computer may be used in the field. Therefore, it is important to have a fast algorithm in order to monitor the environment constantly.

C. The Simulation Procedure

The simulation procedure was implemented by interconnecting seven different computer programs. These programs generate the reconstructions and then evaluate the reconstructions based on the methods discussed previously.

The simulation process begins with the DMATRIX program. The input variables needed for the DMATRIX program are as follows:

1. W - The width of each cell in meters.
2. T - The Height of each cell in meters.

3. IAX - The number of cells in the X-direction.

4. IAY - The number of cells in the Y-direction.

5. IPRS - The number of ray paths per projection.

6. SP - The spacing between ray paths at the borehole in meters

7. TMAX - The maximum projection angle in degrees.

8. TMIN - The minimum projection angle in degrees.

9. TINC - The angle increment between successive viewing angles in degrees.

10. N4 - Noise indicator. N4=0 indicates that no noise is to be added. N4=1 indicates that noise will be added.

11. XE - The attenuation values in Nepers/meter for each cell of the test profile.

Using these input variables the DMATRIX program calculates the D matrix and the simulated measurement data for the given test profile. The necessary input information for the CONV, ART, and NOISE programs are produced by the DMATRIX program.

The CONV program generates a reconstruction of the test profile using the convolution method. It then calculates the Euclidean
distance of the reconstruction. The ART program generates a reconstruction of the test profile using the ART algorithm. It also calculates the Euclidean distance of the reconstruction. For both programs, the computational time is monitored and displayed by the computer. The NOISE program is used to calculate noisy measurement data for any noise factor the user implements. The noise contaminated measurement data can then be inputed into either the ART or CONV programs if the user wishes.

The SMOOTH program implements the selective smoothing algorithm. This program can be used as a post-reconstruction process for the convolution method if the user so desires. Finally, the MVP2 and MVP3 programs find the minimum variance partition for any given set of reconstructed attenuation values. ART, CONV, and SMOOTH provide inputs for the minimum variance partition programs.

A diagram showing the flow of information between all seven programs is presented in figure (4-2). The reconstructions generated by this simulation process are shown in figures (4-3) through (4-46). The reconstructions are grouped based on test profile and scanning conditions. Each figure is labeled with information concerning the type of reconstruction method used (including an indication if smoothing was used after the convolution method), the noise factor used, and the value of Euclidean distance obtained. The average value of computational time for each method will be presented in the next chapter as part of the results and conclusions. Also, the MVP evaluation will be discussed in the next chapter.
Figure 4-2 Information flow between programs.
Figure 4-3 Test Profile 1.

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Figures 4-4 through 4-12 are reconstructions of Profile 1 made under the following scanning conditions:

- Scanning range - 150.0° (75.0° to -75.0°)
- Angular increment - 7.5° (21 projections)
- Measurements per view - 93 (1953 total measurements)
- Ray path spacing - 1.42 m
## Figure 4-4
Reconstruction of Profile 1 using the convolution method, no noise.  \( \delta = 0.1177 \)

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Figure 4-5 Reconstruction of Profile 1 using the convolution method with smoothing, no noise. δ = 0.0700
Table 4-6 Reconstruction of Profile 1 using the ART, no noise.

\[ \delta = 0.0017 \]
Figure 4-7  Reconstruction of Profile 1 using the convolution method, noise factor = 5.  $\delta = 0.1212$
Figure 4-8 Reconstruction of Profile 1 using the convolution method with smoothing, noise factor = 5. \( \delta = 0.0640. \)
Figure 4-9 Reconstruction of Profile 1 using the ART, noise factor = 5. \( \delta = 0.0479 \).
Figure 4-10 Reconstruction of Profile 1 using the convolution method, noise factor = 10. \( \delta = 0.1307 \).
Figure 4-11 Reconstruction of Profile 1 using the convolution method with smoothing, noise factor = 10. $\delta = 0.0615$. 

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Figure 4-12 Reconstruction of Profile 1 using the ART, noise factor = 10. $\delta = 0.0627$. 

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Figures 4-13 through 4-21 are reconstructions of Profile 1 made under the following scanning conditions:

Scanning range - 100.0° (50.0° to -50.0°)
Angular increment - 5.0° (21 projections)
Measurements per view - 57 (1197 total measurements)
Ray path spacing - 0.95 m
Figure 4-13 Reconstruction of profile 1 using the convolution method, no noise.  \( \delta = 0.1426 \)
Figure 4-14 Reconstruction of Profile 1 using the convolution method with smoothing, no noise. $\delta = 0.0942$
Figure 4-15 Reconstruction of Profile 1 using the ART, no noise. $\delta = 0.0038$
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Figure 4-16 Reconstruction of Profile 1 using the convolution method, noise factor = 5. $\delta = 0.1476$
Figure 4-17 Reconstruction of Profile 1 using the convolution method with smoothing, noise factor = 5. $\delta = 0.0872$
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Figure 4-18 Reconstruction of Profile 1 using the ART, noise factor = 5. $\delta = 0.0607$
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Figure 4-19 Reconstruction of Profile 1 using the convolution method, noise factor = 10. $\delta = 0.1521$. 
Figure 4-20 Reconstruction of Profile 1 using the convolution method with smoothing, noise factor = 10. $\delta = 0.0823$. 

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Figure 4-21 Reconstruction of Profile 1 using the ART, noise factor = 10. \( \delta = 0.0720 \).
Figures 4-22 through 4-24 are reconstructions of Profile 1 using the following scanning conditions:

- Scanning range - 100.0° (50.0° to -50.0°)
- Angular increment - 10.0° (11 projections)
- Measurements per view - 9 (99 total measurements)
- Ray path spacing - 1.42 m
Figure 4-22 Reconstruction of Profile 1 using the ART, no noise. $\delta = 0.0504$. 
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Figure 4-23 Reconstruction of Profile 1 using the ART, noise factor = 5. \( \delta = 0.1019 \).
Figure 4-24 Reconstruction of Profile 1 using the ART, noise factor = 10. $\delta = 0.1294$. 

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Figure 4.25 Test Profile 2
Figures 4-26 through 4-34 are reconstructions of Profile 2 made under the following scanning conditions:

Scanning range - 150.0° (75.0° to -75.0°)
Angular increment - 7.5° (21 projections)
Measurements per view - 93 (1953 total measurements)
Ray path spacing - 1.42 m
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Figure 4-26 Reconstruction of Profile 2 using the convolution method, no noise. $\delta = 0.3181$. 
Figure 4-27 Reconstruction of Profile 2 using the convolution method with smoothing, no noise. \( \delta = 0.2365 \).
Figure 4-28 Reconstruction of Profile 2 using the ART, no noise. $\delta = 0.0019$
Table 4-29 Reconstruction of Profile 2 using the convolution method, noise factor = 5. $\delta = 0.3239$.
Figure 4-30  Reconstruction of Profile 2 using the convolution method with smoothing, noise factor = 5. \( \delta = 0.2388 \).
Figure 4-31 Reconstruction of Profile 2 using the ART, noise factor = 5. \( \delta = 0.0489 \).
Figure 4-32 Reconstruction of Profile 2 using the convolution method, noise factor = 10. \( \delta = 0.3315 \).
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Figure 4-33 Reconstruction of Profile 2 using the convolution method with smoothing, noise factor = 10. $\delta = 0.2414$. 
Figure 4-34 Reconstruction of Profile 2 using the ART, noise factor = 10. \( \delta = 0.0675 \).
Figures 4-35 through 4-43 are reconstructions of Profile 2 made under the following scanning conditions:

Scanning range - 100.0° (50.0° to -50.0°)
Angular increment - 5.0° (21 projections)
Measurements per view - 93 (1197 total measurements)
Ray path spacing - 0.95 m
Figure 4-35  Reconstruction of Profile 2 using the convolution method, no noise. \( \delta = 0.6643 \).
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**Figure 4-36**  Reconstruction of Profile 2 using convolution method with smoothing, no noise. $\delta = 0.6293$. 
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Figure 4-37  Reconstruction Profile 2 using the ART, no noise.  δ = 0.0189.
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Figure 4-38  Reconstruction of Profile 2 using the convolution method, noise factor = 5.  \( \delta = 0.6649 \).
Figure 4-39  Reconstruction of Profile 2 using the convolution method with smoothing, noise factor = 5.  \( \delta = 0.6265. \)
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Figure 4-40 Reconstruction of Profile 2 using the ART, noise factor = 5. $\delta = 0.0558$. 
Figure 4.41 Reconstruction of Profile 2 using the convolution method, noise factor = 10. $S = 0.6657$. 
Figure 4-42. Reconstruction of Profile 2 using the convolution method with smoothing, noise factor = 10. $\delta = 0.6069.$
Figure 4-43 Reconstruction of Profile 2 using the ART, noise factor = 10. \( \delta = 0.0662. \)
Figures 4-44 through 4-46 are reconstructions of Profile 2 made under the following scanning conditions:

- Scanning range - 100.0° (50.0° to -50.0°)
- Angular increment - 10.0° (11 projections)
- Measurements per view - 9 (99 total measurements)
- Ray path spacing - 1.42 m
Figure 4-44 Reconstruction of Profile 2 using the ART, no noise. $\delta = 0.2117$. 

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Figure 4-45  Reconstruction of Profile 2 using the ART, noise factor = 5. \( \delta = 0.2532 \).
Figure 4-46  Reconstruction of Profile 2 using the ART, noise factor = 10.  δ = 0.2913.
V. RESULTS AND CONCLUSIONS

In this investigation the convolution algorithm was tested using two different test profiles, with two viewing ranges for each profile. Due to the mathematical foundations of the convolution method, it was expected that the quality of the reconstructions it produced would suffer under the limited viewing range constraints inherent to geotomography. To compensate for this, a post-reconstruction smoothing algorithm was implemented to help define the boundaries between the anomaly and the ambient medium. The reconstructions generated using the convolution method (both with and without smoothing) were then compared to reconstructions produced using the ART under the same scanning conditions. Also, the convolution reconstructions were compared to ART reconstructions that were generated using significantly less simulated measurement data than that needed by the convolution method.

A summary of the results obtained from the reconstructions of Profile 1 is presented in Table 5-1. When operating over a viewing range of 150.0°, the convolution algorithm (without smoothing) produced a good reconstruction having a Euclidean distance (E.D.) of 0.1177 when no noise was added. The E.D. deteriorated 11.05% to a value of 0.1307 when the noise factor was increased to 10. After the smoothing algorithm was implemented, the Euclidean distance improved to 0.700 for the no noise case and continued to improve even when noise was added. The ART, on the other hand, produced a perfect
Table 5-1 Tabulated results for Profile 1.

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<td>100.0°</td>
<td>5</td>
<td>0.0607</td>
<td>P</td>
<td>0.1476</td>
<td>2.87</td>
<td>0.0872</td>
<td>P</td>
</tr>
<tr>
<td>100.0°</td>
<td>10</td>
<td>0.0720</td>
<td>P</td>
<td>0.1521</td>
<td>P</td>
<td>0.0823</td>
<td>P</td>
</tr>
</tbody>
</table>
reconstruction (to two decimal places) having an E.D. of 0.0017 for the no noise case. However, the Euclidean distance deteriorated by approximately a factor of 36 to a value of 0.0627 when the noise factor was increased to 10. This Euclidean distance is comparable to the E.D. value obtained using the convolution method with smoothing, when the noise factor was set to 10.

When operating over a viewing range of 100.0°, the reconstructions generated using the convolution method became slightly worse but were still able to define the boundaries of the anomaly quite adequately. The convolution method (without smoothing) produced a reconstruction having an E.D. of 0.1426 for the no noise case. The Euclidean distance deteriorated only 6.66% to a value of 0.1521 when the noise factor was set at 10. When the smoothing algorithm was implemented the E.D. improved to a best case value of 0.0823 (for noise factor = 10). In contrast, the ART produced a near perfect reconstruction for the noiseless case. However, the E.D. deteriorated by approximately a factor of 18 from 0.0038 to 0.0720, when the noise factor was increased to 10.

For all reconstructions of Profile 1, the boundaries of the anomaly were defined perfectly using minimum variance partitioning. This is signified by a "P" in the table.

A summary of the results obtained from the reconstructions of Profile 2 is presented in Table 5-2. The convolution method, when operating over a viewing range of 150.0°, produced a adequate
<table>
<thead>
<tr>
<th>Scanning Range</th>
<th>Noise Factor</th>
<th>E.D.</th>
<th>MVF</th>
<th>Ave.CPU Time (s)</th>
<th>E.D.</th>
<th>MVF</th>
<th>Ave.CPU Time (s)</th>
<th>E.D.</th>
<th>MVF</th>
<th>Ave.CPU Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>150.0°</td>
<td>No Noise</td>
<td>0.0019 P</td>
<td>0.3181 P</td>
<td>0.2365 P</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>150.0°</td>
<td>5</td>
<td>0.0489 P</td>
<td>0.3239 P</td>
<td>5.13  P</td>
<td>0.2388 P</td>
<td>6.23</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>150.0°</td>
<td>10</td>
<td>0.0675 P</td>
<td>0.3315 P</td>
<td>0.2414 P</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100.0°</td>
<td>No Noise</td>
<td>0.0189 P</td>
<td>0.6643 F</td>
<td>0.6293 F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100.0°</td>
<td>5</td>
<td>0.0558 P</td>
<td>0.6649 F</td>
<td>2.87  F</td>
<td>0.6266 F</td>
<td>3.94</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100.0°</td>
<td>10</td>
<td>0.0662 P</td>
<td>0.6657 F</td>
<td>0.6069 F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5-2  Tabulated results for Profile 2.
reconstruction having a Euclidean distance of 0.3181 (for the no noise case). The E.D. deteriorated only 4.21% to a value of 0.3315 when the noise factor was set at 10. When the smoothing algorithm was added, the Euclidean distance improved to 0.2365 for the no noise case. Again, the ART produced a perfect reconstruction (to two decimal places) for the no noise case. Once again, however, the E.D. deteriorated drastically when noise was added; decreasing by approximately a factor of 35 from 0.0019 to 0.0675. For all reconstructions of Profile 2 made using the 150.0° viewing range, MVP was able to define the boundaries of the anomalies perfectly.

When the viewing range was decreased to 100.0°, the convolution method was unable to produce acceptable reconstructions. The Euclidean distance obtained using the convolution method (without smoothing) for the no noise case is 0.6643 and changes very little to a value of 0.6657 when the noise factor is increased to 10. The large values of E.D. obtained can be attributed to the algorithms inability to adequately reconstruct the anomaly which is longer in the vertical direction than in the horizontal direction. The smoothing algorithm improved the Euclidean distance only slightly, although it did tend to sharpen the boundaries of the anomalies. However, in each case the smoothing algorithm was unable to adequately smooth two or four cells, of the ambient medium, lying on the border of the vertically oriented anomaly. Thus, the exact boundaries of the anomaly were still unclear. In addition, MVP failed to define the boundaries of the anomalies in each case in which the convolution method was used at a
viewing range of 100.0°. This is signified with a "F" in the table. The MVP tended to group both anomalies together as one cluster and then define a group of cells lying in the ambient medium as the second cluster. In contrast the ART still produced a near perfect reconstruction for the no noise case, having a E.D. of 0.0189. This value deteriorated 250.26% when the noise factor was increased to 10. Finally, MVP defined all boundaries of the anomalies perfectly for the ART reconstructions.

The average amount of virtual CPU time required to complete a reconstruction, when 1953 simulated measurements were used was found to be 233.15 seconds for the ART, 5.13 seconds for the convolution method alone, and 6.23 seconds for the convolution method with smoothing. For the reconstructions generated using 1197 measurements these values decreased to 152.93 seconds, 2.87 seconds, and 3.94 seconds respectively.

The discussion up to this point would seem to imply that the ART produces extremely accurate reconstructions, but is laboriously slow in doing so (approximately 37 times slower than the convolution method with smoothing). In contrast, the reconstruction data seems to suggest that the convolution method produces acceptable reconstructions (at the 150.0° viewing range) in a very fast manner. However, it should be remembered that the large amount of simulated measurement data used in these reconstructions are necessary only to ensure complete coverage of the cell center points for the convolution method.
(refer to Chapter IV). The ART does not need large amounts of measurement data in order to produce an adequate reconstruction. Also, it should be remembered that the ART can produce acceptable reconstructions using almost any set of arbitrary scanning conditions. Thus, in order to evaluate the ART algorithm fairly, it was decided to decrease the amount of simulated measurement data available to the algorithm until a Euclidean distance comparable to that obtained using the convolution method with smoothing (using a 150.0° viewing range) was reached. This was done using the set of scanning conditions given for Figures 4-22 through 4-24 and Figures 4-44 through 4-46. Once a comparable value of Euclidean distance had been found, then the computational time required for each method would be compared again.

The ART, operating on only 99 total measurements, produced a reconstruction of Profile 1 having a E.D. of 0.0504 under no noise conditions. This value of Euclidean distance is comparable to the 0.700 value obtained using the convolution method with smoothing. However, when the noise factor was increased to 10 the E.D. deteriorated 156.75% to a value of 0.1294 (see figures 4-22 through 4-24). In constrast, the Euclidean distance does not deteriorate with noise when the convolution method with smoothing is used to reconstruct Profile 1.

The ART (again operating on only 99 total measurements) produced a reconstruction of Profile 2 having a E.D. of 0.2117 under no noise conditions. Again, this value is comparable to the E.D. of
0.2365 obtained when using the convolution method with smoothing under similar noise conditions. When the noise factor was increased to 10, the ART produced a reconstruction having a Euclidean distance of 0.2913 (see figures 4-44 through 4-46). A deterioration of 37.60%. The E.D. only deteriorates 2.07% when generating the reconstruction using the convolution method with smoothing.

The average amount of virtual CPU time required to produce these ART reconstructions using 99 measurements was found to be 15.09 seconds. This is still 2.4 times slower than the convolution method with smoothing.

The object of this investigation was to determine if the convolution method of reconstruction had any applicability to the cross-borehole scanning geometries of geotomography. After reviewing the results of this investigation it has been determined that the convolution method can be applied to geotomography if any of the following conditions exist:

1. If a large viewing range of 150.0° can be obtained the convolution method will produce acceptable reconstructions. In this investigation the profiles would have to be approximately 115 meters underground in order to obtain this large of a viewing range. If the region of interest is any shallower, the antennas can not be positioned within the borehole for the measurements taken at the extreme angles. In addition, the cost of extra com-
putational time required for the ART algorithm must exceed the cost of the extra measurements needed for the convolution method in order for the convolution algorithm to be feasible.

(2) If it is known that the anomalies to be investigated are horizontally oriented (long in the horizontal direction) the convolution method is applicable. For example, if a region was being investigated to detect only horizontal rock fractures (such as Profile 1) then the convolution method could be used without the constraint of having to have a large viewing range. In fact, the convolution method produced good results for Profile 1 even when the viewing range was as small as 60.0°.

(3) If the underground region of interest does not adhere to the straight-ray model the convolution method can be implemented. In all cases reconstructions produced using the convolution method deteriorated significantly less than those made using the ART, under noisy conditions. Again, a large viewing range is required to make the necessary measurements.
VI.  RECOMMENDATIONS

In this investigation it was shown that the convolution method can be applied to geotomography under certain conditions. However, there are several facets of this investigation which deserve further study. The areas in which further research is recommended are listed below:

1. The possible implementation of special purpose convolution algorithms which are designed to operate over a limited field of view should be investigated [28] [29]. These algorithms attempt to estimate the missing measurement data based upon the existing set of measurement data obtained from the limited field of view. This estimation often depends strongly on the availability of a priori information concerning the region under investigation.

2. In practical applications of geotomography, the effects of reflection and refraction may be significant. However, there are methods that can be implemented to correct for these effects [26]. The possibility of incorporating these methods into the convolution process should be investigated.

3. An extension of this investigation to include three-dimensional image reconstruction is needed.

4. A modification of the DMATRIX program in which the geometry of the region of interest is not restricted to
rectangular shapes could be developed. This would result in a more accurate model of the reconstruction area that would occur in practical applications (see Figure 4-1).

(5) An investigation into the effectiveness of the convolution and ART algorithms when implemented on a portable personal computer could prove useful. Portable computers would be required if constant monitoring in the field is necessary.
VII. BIBLIOGRAPHY


APPENDIX A

FORTRAN LISTING OF DMATRIX
THE INPUT VARIABLES NEEDED ARE AS FOLLOWS:

1.) W-THE WIDTH OF EACH CELL.
2.) T-THE HEIGHT OF EACH CELL.
3.) IAX-THE NUMBER OF CELLS IN THE X-DIRECTION.
4.) IAY-THE NUMBER OF CELLS IN THE Y-DIRECTION.
5.) IPRS-THE NUMBER OF TRANSMISSION PATHS AT ANY GIVEN ANGLE.
6.) SP-THE SPACING BETWEEN TRANSMISSION PATHS.
7.) TMAX-THE MAXIMUM ANGLE OF INCIDENCE OF THE TRANSMISSION PATHS UPON THE AREA OF RECONSTRUCTION.
8.) TMIN-THE MINIMUM ANGLE OF INCIDENCE OF THE TRANSMISSION PATHS UPON THE AREA OF RECONSTRUCTION.
9.) TINC-THE AMOUNT OF ANGLE INCREMENT WHEN SCANNING FROM TMIN TO TMAX.

WHEN USING THIS PROGRAM THERE ARE TWO STIPULATIONS:

1.) THE NUMBER OF CELLS IN THE Y-DIRECTION (IAY) MUST BE ODD.
2.) THE NUMBER OF TRANSMISSION PATHS AT ANY GIVEN ANGLE MUST BE ODD.

AT THETA=0 THE PROGRAM ORIENTS THE CENTER TRANSMISSION PATH AT (IAY/2). THE OTHER PATHS ARE SYMMETRICALLY SPACED ABOUT THE CENTER PATH--AT A SPACING OF (SP) BETWEEN EACH PATH.

*****************************************************************************

DIMENSION ANSW(50,50),P(82,2),D(1953,99),Y1(1103),YM(1953),
XE(90)
COMM N 3,YM,XE
DOUBLE PRECISION THETA,TINCA
READ INPUT VARIABLES
READ(S,16)W,T,SP,TMAX,TMIN,TINC,IPRS,IAX,IAY,N4
16 FORMAT(3(F7.4,1X),3(F6.2,1X),4(I3,1X))
PI=3.14159265358979
IV=IAX*IAY
IZ=IAX*IAY*2
IV=IPRS/2
READ(S,17)(YE(J),J=1,IV)
17 FORMAT(F4.1)
CALCULATE POSITION OF TRANS. PATHS AT THETA=0
DO 50 JI=1,IPRS
Y1(JI)=PI*IAY/2+J*EP
IX=IX-1
CONTINUE
JAY=IAY+1
YK=0
DO 72 IR=1,JAY
DO 73 JR=1,IPRS
1 IF(YK.EQ.Y1(JR)) GOTO 73
72 CONTINUE
73 CONTINUE
  \$x += \$t

72 CONTINUE
X1 = \$max - \$min
\$a = \$max - \$a
\$theta = \$min - \$180.0
\$t =
DO \$b1 = \$I \$t
DO \$j = \$I \$\$t
  \$\$w[1,1] = 0
  CONTINUE
DO \$2 \$i = \$I \$\$t
  P[\$j,1] = 0
  CONTINUE
DO \$3 \$k = \$I \$\$t
  D[\$l,1] = 0
  CONTINUE
3 CONTINUE

2 INITIALIZE ARRAYS TO ZERO
DO \$1 \$i = \$I \$\$t
  C[\$j,1] = 0

1 CONTINUE
DO \$2 \$i = \$I \$\$t
  \$\$w[1,1] = 0
  \$\$w[1,1] = \$0
  CONTINUE
DO \$3 \$k = \$I \$\$t
  \$\$w = \$0
  CONTINUE
3 CONTINUE

CALCULATE THE POINT WHERE THE TRANS. PATH ENTERS THE AREA
OF RECONSTRUCTION(\$x1,\$y1) AND ALSO THE POINT WHERE THE TRANS.
PATH EXITS THE AREA OF RECONSTRUCTION(\$x2,\$y2).
\$a = \$\$a * \$\$a + \$\$a
\$y1 = \$y1 + \$\$a
\$y2 = \$y2 + \$\$a
IF (\$y1 < 0.0) AND (\$y1 > 0.0)\n  GOTO 92
IF (\$y2 > 0.0) AND (\$y2 < 0.0)\n  GOTO 93
IF (\$y2 < 0.0) AND (\$y2 <= 0.0)\n  GOTO 94
IF (\$y2 > 0.0) AND (\$y2 <= 0.0)\n  GOTO 95
IF (\$y2 > 0.0) AND (\$y2 < 0.0)\n  GOTO 96
IF (\$y2 > 0.0) AND (\$y2 <= 0.0)\n  GOTO 97
IF (\$y2 > 0.0) AND (\$y2 <= 0.0)\n  GOTO 98

92 \$y1 = ABS(\$y1)
\$x1 = \$y1 / \$\$a
\$y1 = 0.0
\$y2 = \$y2 - \$\$a
\$x2 = \$y2 / \$\$a
\$x2 = \$\$a - \$\$a
\$y2 = \$\$a
GOTO 71

93 \$y1 = \$y1 - \$\$a
\$x1 = \$y1 / \$\$a
\$y1 = \$\$a
\$y2 = \$y2 - \$\$a
\$x2 = \$y2 / \$\$a
\$x2 = \$\$a - \$\$a
\$y2 = \$\$a
GOTO 71

94 IF (\$y1 < 0.0) GOTO 100
IF (\$y1 > 0.0) GOTO 101
\$x1 = 0.0
Y1=VF
Y2=W*IAX
Y2=YL
GOTO 71
100 VF1=ABS(VF)
X1=VF1/DTAN(THETA)
Y1=0.0
X2=W*IAX
Y2=YL
GOTO 71
101 VF1=VF-(T*IAY)
XX=VF1/DTAN(THETA)
X1=ABS(XX)
Y1=T*IAY
X2=W*IAX
Y2=YL
GOTO 71
95 IF(YL.LT.0) GOTO 102
IF(YL.GT.(T*IAY)) GOTO 103
X1=0.0
Y1=VF
X2=W*IAX
Y2=YL
GOTO 71
102 VF2=ABS(VL)
XX1=VF2/DTAN(THETA)
XX=ABS(XX1)
Y2=0.0
X1=0.0
Y1=VF
GOTO 71
103 VF2=YL-(T*IAY)
XX=VF2/DTAN(THETA)
X2=W*IAX-XX
Y2=T*IAY
X1=0.0
Y1=VF
C FIND INTERSECTION POINTS
71 SLP=(Y2-Y1)/(X2-X1)
IF(SLP.EQ.0) GOTO 22
IF(SLP.GT.0) GOTO 191
N1=IAX+1
R1=0
A=1
DO 702 I=1,N1
IF(X1.GT.R1) GOTO 170
IF(Y2.LT.R1) GOTO 502
X0=R1+1
Y0=Y0*DTAN(THETA)
YC1=ABS(YC)
Y=Y1-YC1
P(A,1)=R1
P(A,2)=Y
702 CONTINUE
170 DO 170 R1=1,X0
CONTINUE
191 DO 191 R1=1,Y0
CONTINUE
502 DO 502 R1=1,X0
CONTINUE
A=1
170 R1=R1+1
702 CONTINUE
M1=IA1Y+1
U1=0
23 DO I=1,M1
IF(Y1.GT.U1) GOTO 171
IF(Y1.LT.U1) GOTO 302
Y=U1-Y2
XC=Y1/TAN(THETA)
XC1=ABS(XC)
X=XC1-Y1
P(A,1)=X
P(A,2)=U1
A=A+1
171 U1=U1+T
703 CONTINUE
S=A-1
GOTO 0
191 N1=IA1X+1
R1=0
A=1
DO 310 I=1,N1
IF(X1.GT.R1) GOTO 175
IF(X2.LT.R1) GOTO 310
X=U1-X1
Y1=K1/DY1(THETA)
Y=Y1+Y1
P(A,1)=R1
P(A,2)=Y
A=A+1
175 R1=R1+W
310 CONTINUE
M1=IA1Y+1
U1=0
DO 311 I=1,M1
IF(Y1.GT.U1) GOTO 176
IF(Y1.LT.U1) GOTO 311
Y=U1-Y1
XC=Y1/TAN(THETA)
X=X1+XC
P(A,1)=X
P(A,2)=U1
A=A+1
176 U1=U1+T
311 CONTINUE
S=A-1
GOTO 0
22 Y=Y1
X=0
S=0
A=1
NA=IA1X+1
SORT IN ASCENDING X VALUES

IR=S-1
DO 10 L=1,IR
N=S-L
DO 10 M=1,N
IF(P(M,1).LE.P(M+1,1)) GOTO 10
F=P(M+1,1)
S=P(M+1,2)
P(M+1,1)=P(M,1)
P(M+1,2)=P(M,2)
P(M,1)=F
P(M,2)=S
10 CONTINUE
NS=S
DO 330 I=1,NS
DO 330 J=1,2
D3=INT(P(I,J))
CS=P(I,J)-D3
IF(CS.95.0.9999)) GOTO 331
IF(CS.LT.0.0001)) GOTO 332
GOTO 330
331 P(I,J)=D3+1
GOTO 330
332 P(I,J)=D3
330 CONTINUE
N=N+1
M=N+1
E=SQRT((P(M,2)-P(N,2))**2+(P(M,1)-P(N,1))**2)
IF(E.EQ.0) GOTO 13
IDX=(P(N,1)/M)+1
IF(SLP.GT.0) GOTO 11
IDY=(P(M,2)/T)+1
GOTO 12
11 IDY=(P(N,1)/T)+1
12 ANSW(IDX,IDY)=E
13 CONTINUE
STORE IN D ARRAY
M=1
DO 14 N=1,IAY
L=IAY+1-K
DO 14 I=1,IAK
DIIT,M1=ANSW(I,L)
M=M+1
14 CONTINUE
GOTO 26
19 DO 32 NN=1,IV
DIIT,NN)=0
DO 120 I=1,15
C $WF.~ I TE ( 12! 121
j
(D { I ,J) 
J = 1 ,
V)
C120 CONTINUE
DO 125 1=1,:3
WRITE(13,126) {DfI,J) ,J=1,IV)
126 FORMAT(10(F5.2,lX)}
125 CONTINUE
WRITE(11,215) (YM(Jj ,J=i,ISi
401 CONTINUE
WRITE(11,402) (YM(J),J=1,IS)
400 CONTINUE
DO 401 I=1,15
DO 401 J=1,IV
Y=0{I,J}f.XE{J)
YM(I)=YM(I)+Y
401 CONTINUE
WRITE(11,402) (YM(J),J=1,IS)
402 FORMAT(FB.3)
WRITE(13,405) (YM(J),J=1,IS)
405 FORMAT(F8.3)
D WRITE(12,409) (YM(J),J=1,IS)
C107 FORMAT(F8.3)
407 WRITE(12,409) (YM(J),J=1,IS)
C107 FORMAT(F8.3)
426 FORMAT(F8.3)
GOTO 64
55 WRITE(6,66)
60 FORMAT('0', 'AT THETA=0, ONE OF THE TRANSMISSION PATHS COINCIDES WIT
# A GRID LINE. PLEASE ADJUST (SP) VARIABLE. ')
64 STOP
END
APPENDIX B

FORTRAN LISTING OF CONV
DIMENSION YM(100),SP(100,200),VIN(100),OUT(100),PII(100),R(100)
I,II,III,IV,IPRS,AA,II,XE(100),XY(100),THET(100),YI(100)
READ(1,100)N4,XY,IV,IPRS,IAV,SP,TMIN,TINCA
100 FORMAT(11,104) WRITE(7,107) IV,IAV
107 FORMAT(2(I4)) READ(11,105) XE(5),J=1,IV
105 FORMAT(F3.3) WRITE(8,108) XE(1),J=1,IV
108 FORMAT(10(F7.2)) READ(11,104)(YM(I),J=1,IV)
104 FORMAT(FB3.3) DOUBLE PRECISION TH,THET
IX=IPRS/2 DO J=1,IPRS
YI(J)=T*IAY/2+IX*SP
IX=IX-1
305 CONTINUE
THET=TMIN*PI/180.0 DO J=1,JX
AA(I)=SP*DSIN(P1/2-THET)
THET=THET+TINCA
301 CONTINUE
THET=TMIN*PI/180.0 IT=1 DO J=1,JX
DO 303 IN=1,IPRS
AA=(W*IAX)*DTAN(THET)
VIN(IT)=VIN(IN)+AA
YOUT(IT)=YI(IN)+AA/2
IT=IT+1
303 CONTINUE
THET=THET+TINCA
302 CONTINUE
IF(N4.EQ.0) GOTO 2 READ(14,3)(YNO(I),I=1,IV)
3 FORMAT(FS3.3) DO 4 I=1,IV
YM(I)=YNO(I)
4 CONTINUE
2 I=1 DO 5 J=1,JX
DO 5 K=1,IPRS
R(I,X)=YM(I)
I=I+1
5 CONTINUE
NN=1 DO 10 N=1,IPRS,2
QDO(NN)=N
NN=NN+1
10 CONTINUE
N1=NN-1 DO 20 J=1,JX
DO 20 K=1,IPRS
IF K.EQ.ODD(N)
GOTO 40
CNTLUE
GOTO 5

F(T~)
GOTO 45

KK=K+NT
RS=RS+R(J,KK)/NT**2
NT=NT+2
GOTO 35

IF(N.T.BT.(IPRS-K))GOTO 45

DO 48 I=1,IV
        PI(1)=0.0

THETA=TMIN+PI/180.0
KI=1
IP=IPRS
DO 50 J=1,JX
        II=1
        VC=(IAY+T)-T/2
        DO 60 L=1,IAY
            XC=W/2
            DO 65 M=1,IAX
                TH=ABS(THETA)
                D=DTAN(TH)*XC
                DO 70 I=K,IP
                    IF(THETA.GT.0.0) GOTO 78
                    3=D*TAN(TH)*XC
                    DO 75 I=1,IP
                        IF(YIN(I)+D) GOTO 75
                        IF(YIN(I)+D) GOTO 75
                        CONTINUE
                        IF(YIN(I)+D) GOTO 75
              250

200 RR=YIN(I)+D-YC
205 B10=DDOS(TH)*RR
        K=IPRS
        XIJ=(RF(J,K)/B10)/AI(J)
        GOTO 250
IF (I.EQ.XI) GOTO 70
  I=1-1
  IF (THETA .GT. (O.O)) GOTO 71
  RR=YE(I)+C+D
  GOTO 72
71 RR=YE(I)+D-YC
72 XI=DCOS(THA)*RR
   CII=AI(J)-XI
   X=1-K+1
   XI=1/(AI(J))*((CII*RP(J,K)+XI)*RP(J,K+1))
   GOTO 250
63 IF (THETA .GT. (O.O)) GOTO 210
   RR=VC-YE(1+1+1)
   GOTO 220
210 XI=VC-YE(1+1-1)
220 CII=DCOS(THA)+RR
   X=1
   X=1/(RP(J,K)/CII)/AI(J)
   XI=1/(XI)*((THETA-TINC)+(THETA+TINC))/2
   C=C+C#
   I=I+1
   CONTINUE
   YC=VC-T
   CONTINUE
   THETA=THETA+TINC
   KI=KI+IPRS
   IP=IP+IPRS
50 CONTINUE
   THETA=THETA-TINC
   THETA1= THETA1+180.0/PI
   TINC=TINC+180.0/PI
   DO 55 =1,IV
     IF (PII(I).GE.(O.O)) GOTO 55
     PII(I)=O.O
   55 CONTINUE
   WRITE(6,110) THETA1, TMIN, TINC, IPRS, JX
110 FORMAT(1X, 'THE MAX. ANGLE OF INCIDENCE=',F7.2,'O',
&  'THE MIN. ANGLE OF INCIDENCE=', F7.2,'O',
&  'THE AMOUNT OF ANGLE INCREMENT=', F7.2,'O',
&  'THE NUMBER OF RAYS=', I3,'O',
&  'THE NUMBER OF PROJECTIONS=', I3,'O')
   WRITE(6,30) (PII(J), J=1,IV)
30 FORMAT(0',10(F7.2))
   ERR=O.O
   WRITE(6,94) (PII(J), J=1,IV)
94 FORMAT(10(F7.2))
   WRITE(6,200) (X(J), J=1,IV)
200 FORMAT(10(F7.2))
   DO 31 J=1,IV
     ERR=ERR+(PII(J)-X(J))**2
   31 CONTINUE
   ED=SQRT(ERR/IV)
   WRITE(6,82) ED
82 FORMAT(0', 'THE EUCLIDEAN DISTANCE IS=',F8.4)
GOTO 15
90 WRITE (6,75)
95 FORMAT ('RETURN RE-ADJUST LINE NUMBER OR SPACING.'), RETURN
END
APPENDIX C

FORTRAN LISTING OF ART
RE~IND

READ(13,14)X,Y,IZ,IY,IAX,T,W,TMIN,TMAX,TINC,TINC
1 FORMAT(7(I4),6(F6.12),2(F10.6))
WRITE(6,150)T,IAX,IAY,IPRS,TMAX,TMIN,TINC
150 FORMAT(1X,'THE WIDTH OF EACH CELL IN THE X DIRECTION=',F7.4/'0',
&'THE HEIGHT OF EACH CELL IN THE Y DIRECTION=',F7.4/'0',
&'THE NUMBER OF CELLS IN THE X DIRECTION=',I3/'0',
&'THE NUMBER OF CELLS IN THE Y DIRECTION=',I3/'0',
&'THE NUMBER OF TRANSMISSION PATHS=',I3/'0',
&'THE SPACING BETWEEN THE TRANSMISSION PATHS=',F7.4/'0',
&'THE MAX. ANGLE OF INCIDENCE=',F7.2/'0',
&'THE MIN. ANGLE OF INCIDENCE=',F7.2/'0',
&'THE AMOUNT OF ANGLE INCREMENT=',F7.2)
DO 200 J=1,IS
READ(13,20)D(I,J),J=1,IV
200 CONTINUE
READ(13,20)1YM(J),J=1-IS
20 FORMAT(10(F5.2,1X))
DO 5 J=1,IV
READ (13,10)XE(J)
10 FORMAT(F4.1)-
WRITE(6,15)XE(J),J=1,IV
15 FORMAT('0',10(F7.1))
WRITE(6,25)
25 FORMAT('0')
IF(N4.EQ.0) GOTO 32
READ(14,33)YNO(I),I=1-IS
33 FORMAT(F8.3)
DO 34 I=1,IS
YM(I)=YNO(I)
34 CONTINUE
WRITE(6,31)
31 FORMAT('0','BUT THIS FAR')
DO 40 J=1,IV
%REC(J)=0.0
40 CONTINUE
DO 42 I=1,IS
DSO(I)=0.0
DO 42 J=1,IV
DSO(I)=DSO(I)-D(I,J)**2
42 CONTINUE
WRITE(6,44)(DSO(I),I=1,IS)
C 44 FORMAT(1X,9(F7.2))
DO 48 K=1,50
DO 50 I=1,IS
YEST(I)=0.0
DO 55 J=1,IV
YEST(I)=YEST(I)+D(I,J)*%REC(J)
55 CONTINUE
DO 60 J=1,IV
60 CONTINUE
IF(DSQ(I),EQ.,0.)) GOTO 58
CORR(J)=(Y(I)-XST/I)*D(I,J)/DSQ(I)
GOTO 56
56 CORR(J)=0.0
56 XREC(J)=XREC(J)-CORR(J)
IF(XREC(J),L.T.,0.001) XREC(J)=0.001
CONTINUE
CONTINUE
WRITE(6,70) (XREC(J),J=1,IV)
70 FORMAT(18*,10(F7.2))
WRITE(8,100) (XREC(J),J=1,IV)
100 FORMAT(10(F7.2))
WRITE(8,110) (XJ(E,J),J=1,IV)
110 FORMAT(10(F7.2))
ERR=0.0
DO 80 J=1,IV
ERR=ERR+XREC(J)-XE(J)="#2
80 CONTINUE
ED=SQRT(ERR/IV)
WRITE(6,82)ED
82 FORMAT(18, "THE EUCLIDEAN DISTANCE IS",F8.4)
RETURN
END
APPENDIX D

FORTRAN LISTING OF NOISE
DIMENSION D(1953,90),YM(1953),YNO(1953)

C

THIS ROUTINE READS IN THE D MATRIX AND THE SIMULATED Y VECTOR
IT THEN PERTURBS THE Y VECTOR WITH RANDOM NOISE BASED ON A USER
SET NOISE FACTOR,NF....THE POWER RECEIVED IN EACH SIMULATED
MEASUREMENT IS MULTIPLIED BY A RANDOM NUMBER BETWEEN 1/NF & NF

SET NOISE FACTOR

NF=2
XNF=FLOAT(NF)- 1.0

C

REWIND 13
READ(12,10)N4,JX,IS,IV,IPRS,IAY,IAX,T,M,TMIN,SP,TMAX,TINC,TINCA,PI
10 FORMAT(7(I4),6(F6.2),2(F10.6))
DO 20 I=1,1S
READ(12,30)D(I,J),J=1,IV
30 FORMAT(10(F5.2,1X))
20 CONTINUE
READ(12,40)YM(I),I=1,IS
40 FORMAT(F6,3)
FREQ=.00
BETA=2.*PI*FREQ*1.0E6*SQRT(4.0E-7*PI*9.0*8.35E-12)*SQRT((1.0+SQRT
&(-1.0-P1*PI*FREQ*1.0E6*9.0*8.35E-12**2)))/2.0)
WL=-2.0*F1/BETA
CORN=(WL/(4.0*PII)**2)*1000.0
DO 50 I=1,1S
AL=0.0
50 CONTINUE
IF(AL.EQ.0.0) GOTO 80
IF(AL.LT.(W*IAX/2.0)) GOTO 77
PREC=COND/((AL**2)*EXP(2*YM(I)))))
CALL ARAND(X)
N=X**10
PREC=PREC/((1.0+XNF*X)**((-1)**N))
YNO(I)=0.5*ALOG(PREC/(AL**2))
GOTO 50
80 YNO(I)=0.0
GOTO 30
77 YNO(I)=YM(I)
50 CONTINUE
WRITE(14,70)(YNO(I),I=1,IS)
70 FORMAT(F6,3)
STOP
END
APPENDIX E

FORTRAN LISTING OF SMOOTH
DIMENSION X(9), I(90), XEST(90), XREC(90), XE(90)
READ(7,12) IV, IAX, W1, W2, W3, I
12 FORMAT(2(I4), 4(F6.2))
READ(8,14) (XEST(J), J=1, IV)
14 FORMAT(10(F7.2))
READ(9,15) (XE(J), J=1, IV)
15 FORMAT(10(F7.2))
DO 5 IT=1, 50
DO 10 I=i, IV
IF(I.EQ.IAV) GOT0 20
IF(I.EQ.IAX) GOT0 30
IF(I.EQ. IV) GOT0 40
IF(I.EQ.(IV-(IAX-1))) GOT0 50
IF(I.LT.IAX) GOT0 60
IF(I.GT.(IV-(IAX-1))) GOT0 70
M=2*IAX
DO 90 J=M, IV, IAX
IF(I.EQ.J) GOT0 80
90 CONTINUE
M=IAX+1
N=IV-(IAX-1)
DO 100 J=M, N, IAX
IF(I.EQ.J) GOT0 110
100 CONTINUE
X(1)=XEST(I)
X(2)=XEST(I-IAX)
X(3)=XEST(I-1)
X(4)=XEST(I+1)
X(5)=XEST(I+IAX)
X(6)=XEST(I-(IAX+1))
X(7)=XEST(I-(IAX-1))
X(8)=XEST(I+(IAX-1))
X(9)=XEST(I+(IAX+1))
GOT0 120
20 X(1)=XEST(I)
X(4)=XEST(I+1)
X(5)=XEST(I+IAX)
X(9)=XEST(I+(IAX+1))
X(2)=1000.0
X(3)=1000.0
X(6)=1000.0
X(7)=1000.0
GOT0 120
30 X(1)=XEST(I)
X(3)=XEST(I-1)
X(8)=XEST(I+(IAX-1))
X(5)=XEST(I+IAX)
X(6)=1000.0
X(2)=1000.0
X(7)=1000.0
X(4)=1000.0
X(9)=1000.0
GOT0 120
40 X(1)=XEST(I)
   X(2)=XEST(I-IAAX)
   X(6)=XEST(I-(IAAX+1))
   X(9)=XEST(I-IAAX)
   X(10)=1000.0
   X(11)=1000.0
   X(12)=1000.0
   X(13)=1000.0
   GOTO 120
50 X(1)=XEST(I)
   X(4)=XEST(I+1)
   X(2)=XEST(I-IAAX)
   X(7)=XEST(I-(IAAX-1))
   X(6)=1000.0
   X(13)=1000.0
   X(12)=1000.0
   X(11)=1000.0
   X(1)=1000.0
   GOTO 120
60 X(6)=1000.0
   X(2)=1000.0
   X(7)=1000.0
   X(3)=XEST(I-1)
   X(1)=XEST(I)
   X(4)=XEST(I+1)
   X(8)=XEST(I+IAAX-1)
   X(5)=XEST(I+IAAX)
   X(9)=XEST(I+IAAX+1)
   GOTO 120
70 X(8)=1000.0
   X(5)=1000.0
   X(9)=1000.0
   X(6)=XEST(I-(IAAX+1))
   X(2)=XEST(I-IAAX)
   X(7)=XEST(I-(IAAX-1))
   X(3)=XEST(I-1)
   X(1)=XEST(I)
   X(4)=XEST(I+1)
   GOTO 120
80 X(7)=1000.0
   X(4)=1000.0
   X(3)=XEST(I-1)
   X(1)=XEST(I)
   X(5)=XEST(I+IAAX)
   X(9)=XEST(I+IAAX)
   GOTO 120
90 X(7)=1000.0
   X(4)=1000.0
   X(3)=XEST(I-1)
   X(1)=XEST(I)
   X(5)=XEST(I+IAAX)
   X(9)=XEST(I+IAAX)
   GOTO 120
100 X(6)=1000.0
   X(7)=1000.0
   X(8)=1000.0
   X(2)=XEST(I-IAAX)
X(7) = XEST(I-(IAX-1))
X(1) = XEST(I)
X(4) = XEST(I+1)
X(5) = XEST(I+IAX)
X(9) = XEST(I+(IAX+1))
120 FISUM = 0.0
SUM = 0.0
DO 130 J = 2, 5
DIFF = ABS(X(J) - X(I))
IF (DIFF .LE. T) GOTO 140
FI = 0.0
GOTO 150
140 FI = 1.0
150 SUM = SUM + (FI * X(J))
FISUM = FISUM + FI
130 CONTINUE
FISUM1 = 0.0
SUM1 = 0.0
DO 160 J = 6, 9
DIFF = ABS(X(J) - X(I))
FI = 0.0
GOTO 180
170 FI = 1.0
180 SUM1 = SUM1 + (FI * X(J))
FISUM1 = FISUM1 + FI
160 CONTINUE
XREC(I) = (W1 * X(I) + W2 * SUM + W3 * SUM1) / (W1 + W2 + FISUM + W3 * FISUM1)
10 CONTINUE
DO 190 J = 1, IV
XEST(J) = XREC(J)
190 CONTINUE
5 CONTINUE
WRITE (6, 200) (XEST(J), J = 1, IV)
200 FORMAT ('X', 10(F7.2))
ERR = 0.0
DO 81 J = 1, IV
ERR = ERR + (XEST(J) -XE(J))**2
81 CONTINUE
ED = SQRT (ERR / IV)
WRITE (6, 32) ED
32 FORMAT ('THE EUCLIDEAN DISTANCE IS', F8.4)
STOP
END
APPENDIX F

FORTRAN LISTING OF MVP2
DIMENSION PII(100)
DOUBLE PRECISION JEl(100), AVG1, AVG2
IV=90
READ(8,100)(PII(J), J=1, IV)
100 FORMAT(10(F7.2))
CALL SORT(PII, IV)
N=IV
K=N-1
DO 1 I=1,K
II=I+1
JE(I)=0.0
AVG1=0.0
DC 2 J=1, I
2 AVG1=AVG1+PII(J)
AVG1=AVG1/FLOAT(I)
AVG2=0.0
DO 3 J=II,N
3 AVG2=AVG2+PII(J)
AVG2=AVG2/FLOAT(N-I)
DO 4 J=1, I
4 JE(I)=JE(I)+CASE(AVG1-PII(J))**2
DO 5 J=II,N
5 JE(I)=JE(I)+CASE(AVG2-PII(J))**2
CONTINUE
M=1
DO B I=2, K
8 IF(JE(I).LT.JE(M)) M=I
WRITE(6,15)
15 FORMAT(10X,/," GROUP 1: ",/)
WRITE(6,16) (PII(I), I=1, M)
MM=MM+1
WRITE(6,9)
7 FORMAT(10X,/," GROUP 2: ",/)
WRITE(6,25) (PII(I), I=MM, N)
16 FORMAT(10F2.3)
STOP
END
DO 31 L=1,IV
31 PII(L)=XX(L)
RETURN
END
DIMENSION PII(100)
DOUBLE PRECISION VPM(100),AVG1,AVG2,AVG3
IV=90
READ(8,100)(PII(J),J=1,IV)
100 FORMAT(10IF7.2))
CALL SORT(PII,IV)
K=IV-2
J1=1
DO 1 I=1,K
II=I+1
N=I
20 VP=0.0
AVG1=0.0
DO 2 I=1,II
2 AVG1=AVG1+PII(I)
AVG1=AVG1/FLOAT(I)
AVG2=0.0
DO 3 I=II,N
3 AVG2=AVG2+PII(I)
AVG2=AVG2/FLOAT(N-I)
I2=N+1
AVG3=0.0
DO 4 I=I2,IV
4 AVG3=AVG3+PII(I)
AVG3=AVG3/FLOAT(N-I2)
DO 5 I=1,K
5 VP=VP+DABS(AVG1-PII(I))**2
DO 6 I=II,N
6 VP=VP+DABS(AVG2-PII(I))**2
DO 7 I=I2,IV
7 VP=VP+DABS(AVG3-PII(I))**2
IF(J1.EQ.1) GOTO 12
IF(VP.EQ.1) GOTO 10
12 VP(I)=VP
M1=I
M2=N
10 N=N+1
J1=J1+1
IF(N.EQ.IV) GOTO 1
GOTO 20
1 CONTINUE
WRITE(1,30)
30 FORMAT(10X,'\\nGROUP 1 :''\\nWRITE(6,18)(PII(I1),I1=1,M1)
HH=M1+1
WRITE(6,40)
40 FORMAT(10X,'\\nWRITE(6,15)(PII(I1),I1=HH,M2)
MN=M2+1
WRITE(6,50)
50 FORMAT(10X,'\\nWRITE(6,15)(PII(I1),I1=MN,IV)
16 FORMAT(10F5.3)
STOP
SUBROUTINE SORT (PII, IV)
DIMENSION PII(IV), XX(100)
LAST = IV - 1
DO 20 I = 1, LAST
   XMIN = PII(I)
   JMIN = I
   JFIRST = I + 1
   DO 10 J = JFIRST, IV
      IF (XMIN .LE. PII(J)) GO TO 10
      XMIN = PII(J)
      JMIN = J
10 CONTINUE
   PII(JMIN) = PII(I)
   PII(I) = XMIN
20 CONTINUE
   DO 30 L = 1, IV
      XX(L) = PII(IV + 1 - L)
   DO 31 L = 1, IV
51 PII(L) = XX(L)
   RETURN
END