WAVE REFLECTION IN UNIAXIALLY ANISOTROPIC MEDIA

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Master of Science

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ABSTRACT

The science of optics is very old, but now there have been a number of new and far reaching developments. The most notable among these is the laser and its applications. The field of fiber optic communications has also been a rapidly changing one. These, and other innovations have resulted in a remarkable upsurge in the importance of optics in both pure science and in technology. The technological improvement in the fabrication of optical fibers, interconnection devices, cables, source and detectors has moved from the early stage into a major industry in the 1980s[7].

The theory of propagation and excitation of electromagnetic waves have been studied for a long period of time. Most of the previous studies on these subjects considered in solving the problem of wave reflection from the surface of an anisotropic medium and propagation in an anisotropic medium by the incident wave in isotropic medium. [3,4,5]

The object of this thesis is to present a study of the reflection in a uniaxially anisotropic medium from the surface of an isotropic medium. These media are considered source-free and homogeneous. We adopt the coordinate-free approach, introduced by Chen [6], as a mathematical means that greatly facilitates the solutions of our problems. This method is based on the direct manipulation of vectors, dyadics, and their invariants, eliminates the use of coordinate systems since the coordinate method renders not only the computation extremely tedious and complicated, but also the final results are difficult to interpret. [8]
Chapter 1

Introduction

Due to the rapid advance in technology, more and more media used in applications are anisotropic. Wave propagation in anisotropic media such as plasmas, ferrites, etc., has become a subject of intense research [2,10,11,12,13,17,19]. In this thesis, we will present a coordinate-free method and graphical concepts to solutions of wave reflection in a uniaxially anisotropic medium from an isotropic medium when the given incident wave can be either ordinary or extraordinary wave. We will review wave propagation in a uniaxially anisotropic medium in chapter 2 [6]. We consider only the case when $\mathbf{e}$ is a tensor while $\mu$ is a scalar. The method applies equally well to the dual case of ferrites. Chapter 3 will begin with the laws of reflection and refraction and determination of the wave vectors at the interface. We will introduce the geometrical consideration in the middle of this chapter to make the material easy to understand. Since the electric and magnetic fields are vector quantities, and they are related by the vector Maxwell equations and constitute relations, we can find vector solutions directly from these vector equations. Based on the coordinate-free approach, introduced by Chen [6], the method eliminates the use of coordinate systems. It facilitates solutions, condenses exposition, and provides results in a greater generality. In the last chapter, the reflectivities and transmittivity are plotted versus the angle of incidence to study graphically the existence of the reflected waves. With some particular orientation of the optic axis, we can obtained only one reflected wave for the given incident wave.
Chapter 2

Wave Propagation in Uniaxial Media.

Introduction

In this chapter we will review wave propagation in a uniaxial medium. Starting with the Maxwell's equations and the constitutive relations for a uniaxial medium. We will find that the electric property of a uniaxial medium is characterized by a real positive definite, symmetric dielectric tensor $\varepsilon$. This tensor can be transformed to diagonal form in an orthogonal coordinate system, where two of the three diagonal elements are equal. In Sec. 3, we will find that the dispersion equation can be factored into a product of two terms which indicates the existence of two waves: ordinary and extraordinary, these two waves will equal only when the wave normal of the extraordinary wave coincide with the optic axis $c$. Therefore the two values of the phase velocity are equal. By those two wave equations, we can find the direction of the field vectors that they can exist. At the end of this chapter the polarization of waves will be reviewed and the time-averaged poynting vectors of both ordinary and extraordinary waves will be found. We will leave the boundary value problem for the next chapter. The detail of the material presented in this chapter is given in the book by Chen [6].

2.1 Maxwell’s Equations.

In the International System (SI) of units, the Maxwell equations in differential forms are [1,6,15]:

Faraday’s law of induction:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

(2.1)

generalized Ampere’s law:
\[ \nabla \times \mathbf{H} = \left( \frac{\partial \mathbf{D}}{\partial t} \right) + \mathbf{J} \]  
(2.2)

Gauss' law for the magnetic field:

\[ \nabla \cdot \mathbf{B} = 0 \]  
(2.3)

Gauss's law for the electric field:

\[ \nabla \cdot \mathbf{E} = \rho \]  
(2.4)

where \( \mathbf{E} \) = electric field intensity (vector), volts/meter (V/m)

\( \mathbf{H} \) = magnetic field intensity (vector), amperes/meter (A/m)

\( \mathbf{D} \) = electric flux density (vector), coulombs/meter (C/m)

\( \mathbf{B} \) = magnetic flux density (vector), webers/meter (Wb/m)

\( \mathbf{J} \) = electric current density (vector), amperes/meter (A/m)

\( \rho \) = electric charge density (scalar), coulombs/meter (C/m)

Taking the divergence of Eq.(2.2), noting that \( \nabla \cdot (\nabla \times \mathbf{H}) = 0 \) and using Eq.(2.4), we obtain the equation of continuity

\[ \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \]  
(2.5)

To transform Maxwell's equations of a monochromatic field from time to frequency domain, we choose the time-dependent factor \( e^{-i\omega t} \) and adopt the convention

\[ \mathbf{E} = \text{Re}(\mathbf{E} e^{-i\omega t}) \]

where the time domain \( \mathbf{E} \) is a real vector function of the position vector \( \mathbf{r} \) and time \( t \), and the frequency domain \( \mathbf{E} \) is a complex vector whose cartesian components are complex functions of \( \mathbf{r} \). Re is a shorthand for the "real part of," \( \omega = 2\pi f \) is the angular frequency and \( f \) is the frequency.

By the following rule:

\[ \mathcal{L}(\text{Re } g) = \text{Re } (\mathcal{L} g) \]

where \( \mathcal{L} \) is a real linear operator and \( g \) is a complex function

we can obtain the complex Maxwell equations in frequency domain.
\[ \nabla \times E = \omega \mu B \]  
\[ \nabla \times H = -\omega \mu D + J \]  
\[ \nabla \cdot B = 0 \]  
\[ \nabla \cdot D = \rho \]  

and from Eq. (2.5), the equation of continuity

\[ \nabla \cdot J = \omega \mu \rho \]

For an anisotropic, nondispersive medium, the constitutive relations are

\[ D = \varepsilon_0 \varepsilon E \]
\[ B = \mu_0 \mu H \]

where the constant

\[ \varepsilon_0 \approx \frac{1}{36\pi} \times 10^{-9} \quad \text{F/m} \]

is the dielectric constant of free space, and

\[ \mu_0 = 4\pi \times 10^{-7} \quad \text{H/m} \]

is the magnetic permeability of free space.

\( \varepsilon \) and \( \mu \) are the dielectric tensor and relative permeability constant respectively which will be discussed in the following section.

2.2 Property of the dielectric tensor of a uniaxial medium.[6,14]

Uniaxial medium is the medium whose electrical property depends upon the directions of field vectors. The constitutive relations for a homogeneous, lossless anisotropic medium are

\[ D = \varepsilon_0 \varepsilon E \]  

and
where $\varepsilon$ is the real constant dyadic or tensor of rank 2 (3x3 matrix) and is called the dielectric tensor. Equation (2.11) clearly indicates that the electric flux density $D$ is not in the direction of electric field intensity $E$; instead, each component of $D$ is linearly related to the components of $E$.

The law of conservation of energy, that the constitutive relation (2.11) should always obey, implies that the dielectric tensor for a lossless uniaxial medium must be symmetric, i.e., [6]

$$\varepsilon^\top = \varepsilon \quad (2.13)$$

Thus, the time-averaged electric and magnetic energy densities for a lossless, nondispersive uniaxial medium are [6]

$$\langle W_e \rangle = \varepsilon_0 E^\ast \cdot \varepsilon E / 4 \quad (2.14)$$

and

$$\langle W_m \rangle = \mu_0 \mu H^\ast \cdot H / 4 \quad (2.15)$$

respectively. Since the electric and magnetic energy densities must always be positive and are zero only when $E$ and $H$ are zero, we conclude that $\varepsilon$ must be positive, definite, symmetric tensor.

According to eigenvalue problems, every real symmetric matrix can always be reduced to a diagonal form in an orthogonal coordinate system formed by the eigenvectors. Thus, the dielectric tensor of a uniaxial medium takes the matrix form

$$\varepsilon = \begin{bmatrix} \varepsilon_\perp & 0 & 0 \\ 0 & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_\parallel \end{bmatrix} \quad (2.16)$$
where two of the three diagonal elements are equal and the real constants \( \varepsilon_\perp \), \( \varepsilon_{//} \) are the eigenvalues of \( \overline{\mathbf{e}} \) and are called the principal dielectric constant. The orthogonal form are referred to as the principal dielectric axis. As a result of the positive definiteness of \( \overline{\mathbf{e}} \), all the eigenvalues must be positive and nonzero. Hence the determinant of \( \mathbf{e} \) must always be positive and different from zero:

\[
|\overline{\mathbf{e}}| = \varepsilon_\perp^2 \varepsilon_{//} > 0
\]

(2.17)

In other words, \( \mathbf{e} \) is nonsingular and thus the inverse of \( \mathbf{e} \) always exist. Because of this characteristic, the constitutive relation (2.11) may be written as

\[
\varepsilon_0 \mathbf{E} = \overline{\mathbf{e}}^{-1} \mathbf{D}
\]

(2.18)

We may express Eq. (1.16) in dyadic form, [2]

\[
\overline{\mathbf{e}} = \varepsilon_\perp \overline{\mathbf{T}} + (\varepsilon_{//} - \varepsilon_\perp) \mathbf{c} \mathbf{c}
\]

(2.19)

where \( \mathbf{c} \) is a unit eigenvector of \( \overline{\mathbf{e}} \) corresponding to the nonrepeated eigenvalue \( \varepsilon_{//} \), and \( \varepsilon_\perp \) (repeated) is the eigenvalue of \( \overline{\mathbf{e}} \). \( \overline{\mathbf{T}} \) is the unit matrix. From Eqs. (2.19) and some identities of Linear Analysis (detail in [6]), we easily find

\[
|\overline{\mathbf{e}}| = \varepsilon_\perp^2 \varepsilon_{//}
\]

\[
\text{adj} \mathbf{e} = \varepsilon_\perp (\varepsilon_{//} \overline{\mathbf{T}} + (\varepsilon_\perp - \varepsilon_{//}) \mathbf{c} \mathbf{c})
\]

(2.20)

\[
(\text{adj} \mathbf{e})_\perp \overline{\mathbf{T}} - \text{adj} \mathbf{e} = \varepsilon_\perp (\varepsilon_{//} \overline{\mathbf{T}} + \overline{\mathbf{e}})
\]

2.3 **Dispersion Equation** [6]

Let us consider a monochromatic field whose variation in space is also sinusoidal. In this case, the instantaneous vector \( \mathbf{E} \) takes the form
\[ \mathbf{E} = \text{Re}(\mathbf{E} e^{-i\omega t}) \]  
(2.21)

where the complex vector \( \mathbf{E} \) depends on the position vector \( \mathbf{r} \) and is given by

\[ \mathbf{E} = E_0 e^{-i\mathbf{k} \cdot \mathbf{r}} \]  
(2.22)

In Eq.(2.22), \( E_0 \) is a complex-constant amplitude vector, independent of \( \mathbf{r} \), and the constant vector \( \mathbf{k} \) is called the wave vector. Substituting \( \mathbf{E} \) in eq.(2.22) back into eq.(2.21) we can obtain another form of the instantaneous vector \( \mathbf{E} \), i.e.,

\[ \mathbf{E} = \text{Re}(E e^{i\phi}) \]  
(2.23)

where \( \phi = \mathbf{k} \cdot \mathbf{r} - \omega t \) is the phase of the wave. In a source-free region, the Maxwell equations for uniaxial medium with the above space-time variations can be written as

\[ \omega \mathbf{D}_0 = \omega \varepsilon_0 \{ \varepsilon_{\perp} \mathbf{1} + (\varepsilon_{\parallel} - \varepsilon_{\perp})\mathbf{cc} \} \mathbf{E}_0 = -\mathbf{kxH}_0 \]  
(2.24)

\[ \omega \mathbf{B}_0 = \omega \mu_0 \mathbf{H}_0 = \mathbf{kxE}_0 \]  
(2.25)

With the aid of the antisymmetric matrix \( \mathbf{kx1} \), we may rewrite Maxwell’s equations as

\[ \omega \varepsilon_0 \mathbf{E} \cdot \mathbf{E}_0 = -\left( \mathbf{kx1} \right) \cdot \mathbf{H}_0 \]  
(2.26)

\[ \omega \mu_0 \mathbf{H}_0 = \left( \mathbf{kx1} \right) \cdot \mathbf{E}_0 \]  
(2.27)

After eliminating either \( \mathbf{H}_0 \) from equations (2.26), we obtain

\[ [k_0^2 \mu \{ \varepsilon_{\perp} \mathbf{1} + (\varepsilon_{\parallel} - \varepsilon_{\perp})\mathbf{cc} \} + (\mathbf{kx1})^2] \mathbf{E}_0 = 0 \]  
(2.28)

or

\[ [k_0^2 \mu \varepsilon_{\perp} \mathbf{1} + \{ \varepsilon_{\parallel} \mathbf{1} + (\varepsilon_{\perp} - \varepsilon_{\parallel})\mathbf{cc} \} + (\mathbf{kx1}) \mathbf{H}_0 = 0 \]  
(2.29)

where \( k_0^2 = \omega^2 \mu_0 \varepsilon_0 \), or, after dot-premultiplying both sides of eq.(2.28) by \( \mathbf{E} \), we have
Also since \( \varepsilon_0 = \varepsilon^{-1} D_0 / \varepsilon_0 \), Equation (2.28) may be expressed as

\[
[k_0^2 \mu \varepsilon_+ \varepsilon_+ \varepsilon_+ ( \varepsilon_+ - \varepsilon_+ ) \mathbf{c} c \cdot ( \mathbf{k} \times \mathbf{T} )^2 ] E_0 = 0 \quad (2.30)
\]

For a uniform plane wave with \( \mathbf{k} = \mathbf{k} \mathbf{u} \), we may write Eq. (2.30) in the form of an eigenvalue problem, namely,

\[
[(\varepsilon_+/+ (\varepsilon_+ - \varepsilon_+) \mathbf{c} c \cdot ( \mathbf{k} \times \mathbf{T} )^2 ] E_0 = \lambda E_0 \quad (2.32)
\]

where \( \lambda = -k_0^2 \mu \varepsilon_+ / k^2 \) is an eigenvalue of the matrix \([\varepsilon_+/+ (\varepsilon_+ - \varepsilon_+) \mathbf{c} c \cdot ( \mathbf{k} \times \mathbf{T} )^2 \] and \( E_0 \) is the corresponding eigenvector. Similarly, Eqs. (2.29) and (2.31) may be written as

\[
[(\mathbf{k} \times \mathbf{T} ) \cdot \{ \varepsilon_+/+ (\varepsilon_+ - \varepsilon_+) \mathbf{c} c \cdot \varepsilon_+/+ ] H_0 = \lambda H_0 \quad (2.33)
\]

\[
[(\mathbf{k} \times \mathbf{T} )^2 \cdot \{ \varepsilon_+/+ (\varepsilon_+ - \varepsilon_+) \mathbf{c} c \} ] D_0 = \lambda D_0 \quad (2.34)
\]

respectively.

We note that the vector equation (2.29) [or (2.30) or (2.31)] represents a set of three linear homogeneous equations with three unknown components of the field vector \( E_0 \) (or \( H_0 \) or \( D_0 \)). For the homogeneous equation (2.30) to have a nonzero vector solution \( E_0 \), it is necessary that the determinant of the coefficient matrix vanishes, i.e.,

\[
|k_0^2 \mu \varepsilon_+ \varepsilon_+ \varepsilon_+ ( \varepsilon_+ - \varepsilon_+ ) \mathbf{c} c \cdot ( \mathbf{k} \times \mathbf{T} )^2 | = 0 \quad (2.35)
\]

This is the dispersion equation. Similarly, for nonzero \( H_0 \) and \( D_0 \) to exist, we must have [see Eqs. (2.29) and (2.31)]

\[
|k_0^2 \mu \varepsilon_+ \varepsilon_+ \varepsilon_+ ( \mathbf{k} \times \mathbf{T} ) \cdot \{ \varepsilon_+/+ (\varepsilon_+ - \varepsilon_+) \mathbf{c} c \} \cdot ( \mathbf{k} \times \mathbf{T} ) | = 0 \quad (2.36)
\]
Based on the Maxwell equations (2.24) and (2.25) showing that the existence of any one of the nonzero vectors $E_0$, $H_0$ and $D_0$ implies the existence of all other nonzero vectors. That is, the dispersion equations (2.35), (2.36) and (2.37) are all equivalent, yield

$$\begin{align*}
|k_0^2 \mu \varepsilon_1 \varepsilon_2 + \{\varepsilon_0 \varepsilon_2 + (\varepsilon_1 - \varepsilon_2) \varepsilon_0\} & = 0 \quad (2.37)
\end{align*}$$

To prove Eq. (2.38) we may use these facts [6]:

a) The identity

$$|A + \lambda T| = \pm \lambda^3 + A \lambda^2 + (\text{adj} A) \lambda + |A| \quad (2.29)$$

b) Tensors $\{\varepsilon_0 \varepsilon_2 + (\varepsilon_1 - \varepsilon_2) \varepsilon_0\} \{\varepsilon_0 \varepsilon_2 + (\varepsilon_1 - \varepsilon_2) \varepsilon_0\}$, $(kx \vec{T})^2$, $(kx \vec{T})$, $(kx \vec{T})$, $(kx \vec{T})$, $(kx \vec{T})$ and $(kx \vec{T})$ are the products of tensors $\{\varepsilon_0 \varepsilon_2 + (\varepsilon_1 - \varepsilon_2) \varepsilon_0\}$, $(kx \vec{T})$ and $(kx \vec{T})$ in cyclic order.

c) The trace and the determinant of a product of three matrices in cyclic order are invariant.

Noting that we can get the dispersion equation expressed in various equivalent forms. First, let us obtain the dispersion equation in explicit form. Expanding the determinant in Eq.(2.35) according to Eq.(2.39), we have

$$\begin{align*}
|k_0^2 \mu \varepsilon_1 \varepsilon_2 + \{\varepsilon_0 \varepsilon_2 + (\varepsilon_1 - \varepsilon_2) \varepsilon_0\} & = 0 \quad (2.37)
\end{align*}$$
By using some identities of Linear Analysis (detail in [6]), we finally obtain the explicit form of the dispersion equation in the form

\[
\begin{align*}
k_0^4 \mu^2 \varepsilon_{\perp}^2 \varepsilon_{\parallel} & - k_0^2 \mu \varepsilon_{\perp} \{ (\varepsilon_{\parallel} + \varepsilon_{\perp}) + (\varepsilon_{\parallel} - \varepsilon_{\perp})(k \cdot c)^2 \} k^2 \\
+ \{ \varepsilon_{\perp} + (\varepsilon_{\parallel} - \varepsilon_{\perp})(k \cdot c)^2 \} k^4 & = 0 \quad (2.40)
\end{align*}
\]

or

\[
(k_0^2 \varepsilon_{\perp} \mu - k^2)[k_0^2 \varepsilon_{\perp} \varepsilon_{\parallel} \mu \{ (\varepsilon_{\parallel} + \varepsilon_{\perp}) + (\varepsilon_{\parallel} - \varepsilon_{\perp})(k \cdot c)^2 \} k^2] = 0
\]

and, in terms of the refractive index vector \( n = n \hat{k} = k/k_0 \)

\[
\begin{align*}
\varepsilon_{\perp} + (\varepsilon_{\parallel} - \varepsilon_{\perp})(k \cdot c)^2 \right) n^4 - \mu \varepsilon_{\perp} \{ (\varepsilon_{\parallel} + \varepsilon_{\perp}) + (\varepsilon_{\parallel} - \varepsilon_{\perp})(k \cdot c)^2 \} n^2 \\
+ \mu^2 \varepsilon_{\perp}^2 \varepsilon_{\parallel} & = 0 \quad (2.41)
\end{align*}
\]

or

\[
(k_0^2 \varepsilon_{\perp} \mu - k^2)[\varepsilon_{\perp} \varepsilon_{\parallel} \mu \{ (\varepsilon_{\parallel} + \varepsilon_{\perp}) + (\varepsilon_{\parallel} - \varepsilon_{\perp})(k \cdot c)^2 \} n^2] = 0
\]

where \( n = k/k_0 \) is the index of refraction.

The same dispersion equation can be obtained by expanding the determinants in Eq.(2.36) or Eq.(2.37). Equation (2.42) is a quadratic equation in \( n^2 \) with one of the coefficients depending on the direction of wave normal \( k \), i.e. the dispersion equation alone does not uniquely determine the vector \( n \).

On the other hand, if \( [k_0^2 \mu \{ \varepsilon_{\perp} \varepsilon_{\parallel} \} + (\varepsilon_{\parallel} - \varepsilon_{\perp})\varepsilon c c \} + (k \varepsilon_{\perp} T) ]^2 \) is nonsingular (that is, \( |k_0^2 \mu \{ \varepsilon_{\perp} \varepsilon_{\parallel} \} + (\varepsilon_{\parallel} - \varepsilon_{\perp})\varepsilon c c \} + (k \varepsilon_{\perp} T) | \neq 0 \)) we may divide Eq.(2.28) by \( |k_0^2 \mu \{ \varepsilon_{\perp} \varepsilon_{\parallel} \} + (\varepsilon_{\parallel} - \varepsilon_{\perp})\varepsilon c c \} - k^2 \varepsilon_{\perp} T | \) and obtain the dispersion equation in the form of

\[
1 + k \{ k_0^2 \mu \{ \varepsilon_{\perp} \varepsilon_{\parallel} \} + (\varepsilon_{\parallel} - \varepsilon_{\perp})\varepsilon c c \} - k^2 \varepsilon_{\perp} T \}^{-1} k = 0 \quad (2.43)
\]
2.4 **Determination of the directions of the field vectors.**

From the homogeneous equation (2.28), we may rewrite it in another form as

\[ \overline{W}_u(k).E_0 = 0 \]  
(2.44)

where \( \overline{W}_u(k) \), the wave matrix of a uniaxial medium, is a function of \( k \) and is given by

\[ \overline{W}_u(k) = (k_0^2\varepsilon_\perp \mu - k^2)\overline{T} + \overline{k}k + k_0^2\mu(\varepsilon_\parallel - \varepsilon_\perp)cc \]  
(2.45)

From Eq. (2.41), for those values of \( k \) we can obtain nonzero solutions which are proportional to the columns of \( \text{adj.} \overline{W}_u(k) \), i.e.,

\[ E_0 = \{\text{adj.} \overline{W}_u(k)\}.u \]  
(2.46)

where \( u \) is an arbitrary vector and by using some identities of Linear Analysis (detail in [6]), we can find the adjoint of the wave matrix \( \overline{W}_u(k) \) as

\[ \text{adj.} \overline{W}_u(k) = (k_0^2\varepsilon_\perp \mu - k^2)[k_0^2\varepsilon_\parallel \mu - k_0^2\varepsilon_\perp]k_0^2\mu(\varepsilon_\parallel - \varepsilon_\perp)cc - kk \]

\[ + k_0^2\mu(\varepsilon_\parallel - \varepsilon_\perp)(kxc)(kxc) \]  
(2.47)

Once the direction of \( E_0 \) is determined, the remaining field vectors can be found from Maxwell's equations and the constitutive relations:

\[ H_0 = \frac{(kxE_0)}{\omega\mu_0\mu} \]
\[ B_0 = \mu_0\mu H_0 \]  
(2.48)

\[ D_0 = \varepsilon_0(\varepsilon_\parallel - (\varepsilon_\parallel - \varepsilon)cc).E_0 = -kxH_0/\omega \]

Now considering the dispersion equation (2.41), it consists of a product of two terms so we can rewrite it in another form as
For a given direction of wave normal $\mathbf{k}$, these equations determine two values of wave numbers $k$ and also the two phase velocities. Since the wave number defined by Eq. (2.49) does not depend on the direction of wave normal $\mathbf{k}$ and the wave number defined by Eq. (2.50) do depend on the direction of wave normal $\mathbf{k}$, we will call the wave that defined by Eq. (2.49) as the ordinary wave and call the wave that defined by Eq. (2.50) as the extraordinary wave, respectively. According to Eq. (2.49), the wave number of the ordinary wave denoted by $k_1$ is

$$k_1 = k_0 \sqrt{\epsilon_{\perp} \mu} \tag{2.51}$$

and according to Eq. (2.50), the wave number of the extraordinary wave denoted by $k_2$ is

$$k_2 = k_0 \sqrt{\epsilon_{\perp} \epsilon_{\parallel} \mu / \{\epsilon_{\parallel} + (\epsilon_{\perp} - \epsilon_{\parallel})(\mathbf{k} \cdot \mathbf{c})^2\}} \tag{2.52}$$

If the wave normal $\mathbf{k}_2$ coincide with the vector $\mathbf{c}$, then $(\mathbf{k}_2 \cdot \mathbf{c})^2 = 0$, and

$$k_1 = k_0 \sqrt{\epsilon_{\perp} \mu} = k_2 \tag{2.53}$$

We also can obtain Eq. (2.53) by letting $k_1 = k_2$, a comparision of Eq. (2.51) and Eq. (2.52) shows that $(\mathbf{k}_2 \cdot \mathbf{c})^2 = 0$, i.e. $\mathbf{k}_2 = \mathbf{c}$. It means that the wave numbers and the phase velocities of the two isonormal waves coincide only when the wave normal $\mathbf{k}_2$ is along $\mathbf{c}$, an eigenvector of the tensor. Those direction of wave normal that yield two equal wave numbers are called the optic axes of the uniaxial medium. Therefore $\mathbf{c}$ is the only optic axis of the
medium.

For the ordinary wave $k_1 = k_1 k$, the adjoint of the wave matrix (2.47) is

$$\text{adj.} \overline{W}(k_1) = k_0^2 \mu (\epsilon_{//} - \epsilon_{\perp})(k_1 x c)(k_1 x c)$$

(2.54)

According to Eq.(2.46), we may choose the direction of $E_0$ as

$$e_1 = k_1 x c$$

(2.55)

Substituting Eq.(2.55) into Eq.(2.48), we obtain the directions of the remaining field vectors:

$$h_1 = \{k_1 x (k_1 x c)\} / \omega \mu_0 \mu$$

$$b_1 = \{k_1 x (k_1 x c)\} / \omega$$

$$d_1 = \epsilon_0 \epsilon_{\perp} (k_1 x c)$$

Figure 2.1 Orientations of field vectors of the ordinary wave in a uniaxial medium.
From the above equations (2.55) and (2.56), we found that, for the case of the ordinary wave propagating in uniaxial media, vectors $e_1$ and $d_1$ are perpendicular to the plane formed by vectors $k_1$ and $c$ while $h_1$ and $b_1$ lie on the plane as shown in Fig. 2.1.

For the extraordinary wave $k_2 = k_2 \mathbf{c}$ the adjoint of the wave matrix (2.47) is

$$\text{adj} \mathbf{W}_0(k_2) = (k_0^2 \epsilon_\perp \mu - k_2^2)\{(k_0^2 \epsilon_\parallel \mu - k_0^2 \epsilon_\perp \epsilon_\perp)cc - k_2k_2\}$$

$$+ k_0^2 \mu (\epsilon_\parallel - \epsilon_\perp) (k_2xc)(k_2xc)$$

(2.57)

For simplicity we choose the arbitrary vector $u = c$ in Eq. (2.47). We obtain the direction of $E_0$ for the extraordinary wave as

$$e_2 = k_0^2 \epsilon_\perp \mu c - (k_2 c) k_2$$

(2.58)

Substituting Eq. (2.58) into Eq. (2.48), we obtain the direction of the remaining field vectors

$$h_2 = \omega_0 \epsilon_\perp (k_2 xc)$$

$$b_2 = \omega_0 \epsilon_\perp \mu (k_2 xc)$$

(2.59)

$$d_2 = \epsilon_\perp (k_2 xc(k_2 xc))$$

We found that the orientation of the field vectors of the extraordinary wave in uniaxial medium, vectors $e_2$ and $d_2$ lie on the plane formed by the wave vector $k_2$ and the optic axis $c$, but $h_2$ and $b_2$ are perpendicular to it as shown in Fig. 2.2.

Figs. 2.1 & 2.2 show that the vectors $E_0$, $D_0$ and the optic axis $c$ of either the extraordinary or ordinary waves are coplanar.
2.5 **Polarization of Waves in Uniaxial medium, Optic Axes** [6,15]

Let us examine the polarizations of field vectors in lossless crystals. For a monochromatic uniform plane wave, the Maxwell equations in lossless crystals are

\[
\begin{align*}
\omega D_0 &= \omega \varepsilon_0 (\varepsilon_\perp \mathbf{T} + (\varepsilon_{//} - \varepsilon_\perp) \mathbf{c c}) \mathbf{E}_0 = - \mathbf{kH}_0 \\
\omega B_0 &= \omega \mu_0 \mu_0 \mathbf{H}_0 = \mathbf{kE}_0
\end{align*}
\]  

(2.60)

(2.61)

where \( k \) and \( \omega \) are real. The complex conjugates of Eqs (2.60) and (2.61) give

\[
\begin{align*}
\omega D_0^* &= \omega \varepsilon_0 (\varepsilon_\perp \mathbf{T} + (\varepsilon_{//} - \varepsilon_\perp) \mathbf{c c}) \mathbf{E}_0^* = - \mathbf{kH}_0^* \\
\omega B_0^* &= \omega \mu_0 \mu_0 \mathbf{H}_0^* = \mathbf{kE}_0^*
\end{align*}
\]  

(2.62)

(2.63)

Cross multiplying Eqs (2.60) and (2.62) and using some identities of Linear Analysis (detail in [6]), we get
\[
D_0 \times D_0^* = \epsilon_0^2[(\epsilon_\perp \bar{T} + (\epsilon_\parallel - \epsilon_\perp)c)(E_0) \times [(\epsilon_\perp \bar{T} + (\epsilon_\parallel - \epsilon_\perp)c)E_0]
\]

\[
= \epsilon_0^2\epsilon_\perp \{\epsilon_\parallel \bar{T} + (\epsilon_\parallel - \epsilon_\perp)c\}(E_0 \times E_0^*)
\]

\[
= (kxH_0) \times (kxH_0^*) / w^2 = kk.(H_0 x H_0^*) / w^2
\]  

(2.64)

Hence

\[
E_0 \times E_0^* = \{\epsilon_\perp \bar{T} + (\epsilon_\parallel - \epsilon_\perp)c\}. kk.(H_0 x H_0^*) / w^2 \epsilon_\parallel^2 \epsilon_\perp
\]  

(2.65)

Similarly, cross-multiplying Eqs (2.61) and (2.63) we find

\[
H_0 x H_0^* = kk.(E_0 x E_0^*) / w^2 \mu_0^2 \mu
\]  

(2.66)

Eliminating either \(H_0 x H_0^*\) or \(E_0 x E_0^*\) from Eqs. (2.65) and (2.66) we obtain

\[
[k_0^4 \mu^2 \epsilon_\perp^2 \epsilon_\parallel \bar{T} - k^2 \{\epsilon_\perp \bar{T} + (\epsilon_\parallel - \epsilon_\perp)c\}]. kk.(E_0 x E_0^*) = 0
\]  

(2.67)

\[
[k_0^4 \mu^2 \epsilon_\perp^2 \epsilon_\parallel \bar{T} - (k.\{\epsilon_\perp \bar{T} + (\epsilon_\parallel - \epsilon_\perp)c\}. kk).(H_0 x H_0^*) = 0
\]  

(2.68)

Equations (2.67) and (2.68) are two homogeneous equations. Thus for nonzero vector \(E_0 x E_0^*\) or \(H_0 x H_0^*\) to exist, the determinant of the coefficient matrix of Eq. (2.67) or (2.68) must vanish; i.e.,

\[
|k_0^4 \mu^2 \epsilon_\perp^2 \epsilon_\parallel \bar{T} - k^2 \{\epsilon_\perp \bar{T} + (\epsilon_\parallel - \epsilon_\perp)c\}]. kk | = 0
\]  

(2.69)

or

\[
|k_0^4 \mu^2 \epsilon_\perp^2 \epsilon_\parallel \bar{T} - (k.\{\epsilon_\perp \bar{T} + (\epsilon_\parallel - \epsilon_\perp)c\}. kk)| = 0
\]  

(2.70)

Which the determinants in Eqs. (2.69) and (2.70) yield the same result

\[
k_0^4 \mu^2 \epsilon_\perp^2 \epsilon_\parallel - \{\epsilon_\perp k^2 + (\epsilon_\parallel - \epsilon_\perp)c^2\} k^2 = 0
\]  

(2.71)

or

\[
\mu^2 \epsilon_\perp^2 \epsilon_\parallel - \{\epsilon_\perp + (\epsilon_\parallel - \epsilon_\perp)c^2\} n^4 = 0
\]  

(2.72)

It means that \((E_0 x E_0^*)\) and \((H_0 x H_0^*)\) are either all zero or nonzero depending
on whether or not the wave vector $\mathbf{k}$ satisfies Eq.(2.71). From the condition for linear polarization, we can conclude that for every direction of propagation, the uniform plane waves in a lossless crystal are always linearly polarized except when the wave vector $\mathbf{k}$ satisfies the condition (2.71). In the latter case, the wave can have any polarization. But the wave vector $\mathbf{k} = k_0 \mathbf{n}$ must also satisfy the dispersion equation (2.42), the solutions of which are

$$n^2 = \left[ \mu \varepsilon_\perp \{(\varepsilon_{\|} + \varepsilon_\perp) + (\varepsilon_{\|} - \varepsilon_\perp)(\mathbf{k} \cdot \mathbf{c})^2 \} \pm \sqrt{\Delta} \right] / 2 \{\varepsilon_\perp + (\varepsilon_{\|} - \varepsilon_\perp)(\mathbf{k} \cdot \mathbf{c})^2\}$$

(2.73)

where the discriminant $\Delta$ is given by

$$\Delta = [\mu \varepsilon_\perp \{(\varepsilon_{\|} + \varepsilon_\perp) + (\varepsilon_{\|} - \varepsilon_\perp)(\mathbf{k} \cdot \mathbf{c})^2\}]^2 - 4 \mu^2 \varepsilon_\perp^2 \varepsilon_{\|}\{\varepsilon_\perp + (\varepsilon_{\|} - \varepsilon_\perp)(\mathbf{k} \cdot \mathbf{c})^2\}$$

(2.74)

Condition (2.72) and the dispersion equation (2.42) imply that

$$\mu \varepsilon_\perp \{(\varepsilon_{\|} + \varepsilon_\perp) + (\varepsilon_{\|} - \varepsilon_\perp)(\mathbf{k} \cdot \mathbf{c})^2\}n^2 = -2 \mu^2 \varepsilon_\perp^2 \varepsilon_{\|}$$

(2.75)

and substitution of Eqs. (2.72) and (2.75) into Eq.(2.74) yields

$$\Delta = 0$$

(2.76)

That is, when the refractive index vector $\mathbf{n}$ satisfies the condition (2.72). The two solutions in $n^2$ of the dispersion equation become equal. Those directions of the wave normal that cause the discriminant (2.74) to vanish and yield two equal roots for $n^2$ are called the optic axes of the uniaxial medium. In other words, optic axes are the directions in the crystals for which the two values of the phase velocity are equal.

For each given direction of wave normal $\mathbf{k}$ there are two linearly polarized waves propagating at different phase velocities in the crystals. Each wave is characterized by a set of fixed directions of amplitude vectors.
\(E_0, D_0, H_0, B_0\). Since the wave are linearly polarized, without loss of generality we may assume that all the amplitude vectors are real. Hence, according to Maxwell's equations (2.60) and (2.61), the vector \(H_0\) is perpendicular to \(E_0, D_0,\) and \(k\), which must therefore be coplanar and also the same \(D_0\) is perpendicular to \(H_0, B_0\) and \(k\), which must be coplanar. From the constitute relation (2.11), (2.12) and Eqs (2.60), (2.61), we have

\[
\mathbf{k} \cdot D_0 = \varepsilon_0 \cdot \mathbf{k} \cdot (\varepsilon_{\perp} I + (\varepsilon_{\parallel} - \varepsilon_{\perp}) \mathbf{cc}) \cdot E_0 = 0 \tag{2.77}
\]

and

\[
\mathbf{k} \cdot B_0 = \mu_0 \mathbf{k} \cdot H_0 = 0 \tag{2.78}
\]

Figure 2.3 Orientation of field vectors, wave vector, and the time averaged Poynting vector. that is, unlike isotropic media, electric field intensity is not perpendicular to \(k\), but to the vector \((\overline{E} \cdot k)\). The time-averaged Poynting vector is given by
\[ \langle p \rangle = \frac{(E \times H_0)}{2} \]  

which is perpendicular to both \( E \) and \( H_0 \). Figure 2.3 shows the relative orientation of these vectors.

Substitute Eqs. (2.55) and (2.56) into Eq. (2.79) we obtain the time-averaged poynting vector for the ordinary wave as:

\[ \langle p_1 \rangle = (k_1 x e)^2 k_1 / 2 \omega \mu_0 \mu \]  

Substitute Eqs. (2.58) and (2.59) into Eq. (2.79) we obtain the time-averaged poynting vector for the extraordinary wave as:

\[ \langle p_2 \rangle = \omega \varepsilon_0 \varepsilon_{\perp} (k_2 x e)^2 (k_2 e) / 2 \varepsilon_{\parallel} \]
Chapter 3

Reflection in uniaxial media.

Introduction

In chapter 2 we have reviewed plane wave propagation in an unbounded medium. In this chapter we will review the laws of reflection and refraction in the first section. In the second section we will determine the incidence, reflected and transmitted wave vectors at the uniaxial-isotropic interface. We will assume that we can have either ordinary or extraordinary wave for the incidence wave, one reflected ordinary wave and one reflected extraordinary wave in the uniaxial medium and in the isotropic medium we have one transmitted wave. Section 3.3 we will consider the existence of the reflected and transmitted waves by means of the geometry of the boundary surfaces. For simplicity in the first part of section 3.4 we will apply the boundary condition at the interface of general anisotropic-isotropic interface, we will obtain amplitude of the reflected and transmitted waves in term of the given amplitude of the incidence wave. In the second part we will replace the general field vectors by the field vectors that we already found in chapter 2 for uniaxial medium. We will also introduce some constants for the incident wave since we can have either ordinary or extraordinary wave that can propagate at a time. Before we leave this section we will find out the unknown amplitudes in the case of normal incidence. Section 3.5 we will give the normal component of the time-averaged poynting vector of the incident, reflected and transmitted wave in general form. In the last section we will talk about some special cases that we can obtain only one reflected wave for a particular orientation of the optic axis \( \mathbf{c} \) in two separate cases of the given incident wave by using
the resulting solution of the unknown amplitude in section 2.4.

3.1 Laws of reflection and refraction.

In this section we will review the general laws of reflection and refraction. In practice, the medium has finite dimension, we shall examine the effects of boundary surface (surface due to an abrupt change in physical properties) on the wave propagation. Suppose that a monochromatic plane wave, propagating in the first medium is incident upon a plane interface of discontinuity to the second medium of different electrical and magnetic properties. The incident wave splits into transmitted waves proceeding into the second medium and reflected waves propagating back into the first medium. The existence of these waves is necessary in order to satisfy the boundary conditions imposed on the field vectors at the interface.

According to the boundary value problem [6,7,15], the boundary conditions are linear relations. We may write them in frequency domain as

\[
\begin{align*}
E_{1\mathbf{q}} &= E_{2\mathbf{q}} \\
H_{1\mathbf{q}} &= H_{2\mathbf{q}} \\
B_{1\mathbf{q}} &= B_{2\mathbf{q}} \\
D_{1\mathbf{q}} &= D_{2\mathbf{q}}
\end{align*}
\]

where \( \mathbf{q} \) is a unit vector normal to the interface pointing from the first to the second medium. Assuming that there are no surface charge and current densities on the interface. The subscripts 1 and 2 denote the total complex field vectors in the first and second medium respectively. These boundary conditions hold only for points on the interface. Finally the general laws of reflection and refraction can be expressed in vector forms as
\[ k_1 q = k_{r1} q = k_{r2} q = \ldots = k_{l1} q = k_{l2} q = \ldots \] (3.5)

where \( k_{r1}, k_{r2}, \ldots \) are the wave vectors of the first, second, \ldots reflected waves respectively, and \( k_{l1}, k_{l2}, \ldots \) are the wave vectors of the first, second, \ldots transmitted waves respectively.

Eq.(3.5) states that across the interface of any two linear, homogeneous media, the tangential components of the wave vectors of the incident, reflected, and transmitted waves, must all be equal. Now denoting the constant vector

\[ k_1 q = a \] (3.6)

By taking the magnitude of the cross products in Eq.(3.5), we obtain the form of the laws of reflection and refraction in terms of angles as

\[ a = k_1 \sin \theta_i = k_{r1} \sin \theta_{r1} = k_{r2} \sin \theta_{r2} = \ldots = k_{l1} \sin \theta_{l1} = k_{l2} \sin \theta_{l2} = \ldots \] (3.7)

where \( a \) is the magnitude of the constant vector \( a \).

\( k_1 \) is the wave number of the incident wave, and \( \theta_i \) is the angle between the vector \( k_1 \) and \( q \) and is called the angle of incident.

\( k_{r1}, k_{r2}, \ldots \) are the wave numbers of the reflected waves, and \( \theta_{r1}, \theta_{r2}, \ldots \) are the corresponding angles of reflection formed by the vectors \( k_{r1}, k_{r2}, \ldots \) with \( q \), respectively.

A simple geometrical construction of the wave vectors can be obtained by taking the cross product of \( q \) with Eq.(3.5) and expanding the vector triple product, yielding
\[ \begin{align*}
  k_i &= b + q_i q \\
  k_{r1} &= b + q_{r1} q \\
  k_{r2} &= b + q_{r2} q \\
  k_{t1} &= b + q_{t1} q \\
  k_{t2} &= b + q_{t2} q
\end{align*} \] (3.8)

where \( b = q \times a \) \hspace{1cm} (3.9)

In summary, Eq.(3.8) states that

1) All the wave vectors of the incident, reflected and transmitted waves lie on the plane of incident which is defined by vectors \( b \) and \( q \) (or \( k_i \) and \( q \) as shown in Fig.3.1.)
2) The projections of the wave vectors $k_1, k_{r1}, k_{r2}, -$ and $k_{t1}, k_{t2}$ on the interface are equal to the same constant vector $b (|b| = |a|)$. Based on these, letting $o$ be a fixed origin located on the interface, the tips of all the wave vectors of the incident, reflected and transmitted waves drawn from $o$ must lie on a straight line which passes through the end point of vector $b$ and is parallel to $q$ as shown in Fig.3.2.

![Figure 3.2 Geometrical construction of wave vectors.](image)

3.2 Determination of wave vectors at uniaxial - isotropic interface.

In this section we will find the wave vectors of the incident, reflected and transmitted waves. Let us assume that a uniaxial medium characterized by the dielectric tensor (2.19) and relative magnetic permeability $\mu_1$ is in medium 1 and that medium 2 is an isotropic medium. From Section 2.4 Eqs. (2.49) and (2.52) we already found the two dispersion
equations that gave us two wave numbers. According to that two dispersion equations, in medium 1 we can have two kind of waves that can propagate, one is ordinary wave and another is extraordinary wave. So we can have either ordinary wave or extraordinary wave for the incident wave and we have both ordinary wave and extraordinary wave for the reflected wave.

Firstly we determine the wave vector

\[ \mathbf{k}_{i1} = \mathbf{b} + q_{i1} \mathbf{q} \]  \hspace{1cm} (3.10)

of the incident ordinary wave, substituting Eq.(3.10) into the dispersion equation (2.49) we get

\[ q_{i1} = \sqrt{k_0^{-2} \epsilon_1 \mu_1 - a^2} \]  \hspace{1cm} (3.11)

Secondly we determine the wave vector

\[ \mathbf{k}_{i2} = \mathbf{b} + q_{i2} \mathbf{q} \]  \hspace{1cm} (3.12)

of the incident extraordinary wave, we substitute Eq.(3.12) into the dispersion equation (2.50) and obtain

\[ \{\epsilon_\perp + (\epsilon_// - \epsilon_\perp)(\mathbf{q}, \mathbf{c})^2\}q_{i2}^2 + 2(\epsilon_// - \epsilon_\perp)(\mathbf{b}, \mathbf{c})(\mathbf{q}, \mathbf{c})q_{i2} \]

\[ + b^2(\epsilon_\perp + (\epsilon_// - \epsilon_\perp)(\mathbf{b}, \mathbf{c})^2) - k_0^2 \epsilon_\perp \epsilon_// \mu_1 = 0 \]  \hspace{1cm} (3.13)

which is a quadratic equation in \( q_{i2} \). From the fact that the energy carried by the incident wave flows toward medium 1 to the interface, we have to choose solution \( q_{i2} \) so that \( \langle \mathbf{p}_{i2}, \mathbf{q} \rangle > 0 \). This condition corresponds to

\[ \mathbf{k}_{i2} \cdot \mathbf{c} \cdot \mathbf{q} > 0 \], or

\[ q_{i1} > - (\epsilon_// - \epsilon_\perp)(\mathbf{b}, \mathbf{c})(\mathbf{q}, \mathbf{c}) / \{\epsilon_\perp + (\epsilon_// - \epsilon_\perp)(\mathbf{q}, \mathbf{c})^2\} \]  \hspace{1cm} (3.14)

From Eqs. (3.13) and (3.14) we obtain

\[ q_{i2} = \{ - (\epsilon_// - \epsilon_\perp)(\mathbf{b}, \mathbf{c})(\mathbf{q}, \mathbf{c}) + \mathcal{L} \} / \{\epsilon_\perp + (\epsilon_// - \epsilon_\perp)(\mathbf{q}, \mathbf{c})^2\} \]  \hspace{1cm} (3.15)
where
\[
\mathcal{E} = \left[ \left( \varepsilon_{//} - \varepsilon_{\perp} \right) (b \cdot c) (q \cdot c) \right]^2 - \left( \varepsilon_{\perp} + (\varepsilon_{//} - \varepsilon_{\perp}) (q \cdot c)^2 \right) \\
\left[ b^2 \left( \varepsilon_{\perp} + (\varepsilon_{//} - \varepsilon_{\perp}) (b \cdot c)^2 \right) - k_0^2 \varepsilon_{\perp} \varepsilon_{//} \mu_{\perp} \right]^{1/2}
\]
(3.15a)
Substituting Eq.(3.15) into Eq.(3.12), finally we obtain
\[
k_{r2} = \left[ \left( \varepsilon_{\perp} q + (\varepsilon_{//} - \varepsilon_{\perp}) (q \cdot c) x a + \mathcal{E} q \right) / \left( \varepsilon_{\perp} + (\varepsilon_{//} - \varepsilon_{\perp}) (q \cdot c)^2 \right) \right] (3.16)

Thirdly we determine the wave vector
\[
k_{r1} = b - q_{r1} q
\]
(3.17)
of the reflected ordinary wave. Substituting Eq.(3.17) into the dispersion equation (2.49) we obtain
\[
q_{r1} = \sqrt{k_0^2 \varepsilon_{\perp} \mu_{\perp} - \alpha^2}
\]
(3.18)
Fourthly we determine the wave vector
\[
k_{r2} = b - q_{r2} q
\]
(3.19)
of the reflected extraordinary wave. Nearly the same as in the case of incident extraordinary wave, we obtain a quadratic equation in \(q_{r2}\) as
\[
\left( \varepsilon_{\perp} + (\varepsilon_{//} - \varepsilon_{\perp}) (q \cdot c)^2 \right) q_{r2}^2 - 2 (\varepsilon_{//} - \varepsilon_{\perp}) (b \cdot c) (q \cdot c) q_{r2} \\
+ b^2 \left( \varepsilon_{\perp} + (\varepsilon_{//} - \varepsilon_{\perp}) (b \cdot c)^2 \right) - k_0^2 \varepsilon_{\perp} \varepsilon_{//} \mu_{\perp} = 0
\]
(3.20)
Since the energy carried by the reflected wave bounces back from the interface to the uniaxial medium, we have to choose solution \(q_{r2}\) so that
\[\langle p_{r2} \rangle \cdot q < 0\] which the same corresponds to \(k_{r2} \cdot \varepsilon \cdot q < 0\) or
\[q_{r2} > (\varepsilon_{//} - \varepsilon_{\perp}) (b \cdot c) (q \cdot c) / \left( \varepsilon_{\perp} + (\varepsilon_{//} - \varepsilon_{\perp}) (q \cdot c)^2 \right)
\]
(3.21)
From Eqs. (3.20) and (3.21) we obtain
\[
q_{r2} = \left( (\varepsilon_{//} - \varepsilon_{\perp}) (b \cdot c) (q \cdot c) + \mathcal{E} \right) / \left( \varepsilon_{\perp} + (\varepsilon_{//} - \varepsilon_{\perp}) (q \cdot c)^2 \right)
\]
(3.22)
where \( \mathcal{Q} \) is given by Eq.(3.15a). We substitute Eq.(3.22) into Eq.(3.19) and obtain

\[
\mathbf{k}_{r_2} = \left[ \left\{ \varepsilon_{\parallel} \mathbf{q} + (\varepsilon_{\parallel} - \varepsilon_{\perp}) (\mathbf{q} \cdot \mathbf{c}) \mathbf{c} \right\} \mathbf{x}_a + \mathcal{Q} \mathbf{q} \right] / \left\{ \varepsilon_{\perp} + (\varepsilon_{\parallel} - \varepsilon_{\perp}) (\mathbf{q} \cdot \mathbf{c})^2 \right\} \quad (3.23)
\]

And the last the wave vector

\[
\mathbf{k}_t = \mathbf{b} + q_t \mathbf{q} \quad (3.24)
\]

of the transmitted wave in an isotropic medium 2. Since medium 2 is isotropic, characterized by the dielectric constant \( \varepsilon_2 \) and relative permeability \( \mu_2 \), the dispersion equation is in the form of

\[
\mathbf{k}_t^2 = k_0^2 \varepsilon_2 \mu_2 \quad (3.25)
\]

Substitute Eq.(3.24) into the dispersion equation (3.25), we obtain

\[
q_t = \sqrt{k_0^2 \varepsilon_2 \mu_2 - a^2} \quad (3.26)
\]

Figure 3.3 Geometrical determination of wave vectors at the interface of a uniaxial and an isotropic medium.

where \( \varepsilon_2 \) and \( \mu_2 \) are the dielectric constant and relative permeability
constant of the isotropic medium 2, respectively. The geometrical determination of the wave vectors at the interface of a uniaxial crystal and isotropic medium is shown in Fig. 3.3.

3.3 Geometrical consideration of wave vectors at uniaxial-isotropic interface.

Now we shall consider the existence of the reflected and transmitted waves at the uniaxial-isotropic interface for the given incident wave by considering the projection of the given incident wave vector in the plane of incidence onto the plane of interface, i.e. the wave vector \( b \) which its magnitude is given by

\[ b = k_i \sin \theta_i = a \]  

(3.27)

As we know from figure 3.3 the geometrical determination point of view of wave vectors at the interface of a uniaxial-isotropic medium, the existence of reflected and transmitted wave vectors do depend on the wave vector \( b \). If the wave vector \( b \) is longer than any of the boundary surface of the ordinary or extraordinary reflected waves on transmitted wave, this will give rise to the problem of the over-critical angle at the interface (surface wave). As we shall see by Eqs. (3.7) and (3.27), that is, \( k_i \sin \theta_i \) is greater than \( k_r, k_r, k_r \) or \( k_t \), then the corresponding angles \( \theta_{r1}, \theta_{r2} \) or \( \theta_t \) will be \( \sin^{-1}(k_i \sin \theta_i/k_{r1}) \), \( \sin^{-1}(k_i \sin \theta_i/k_{r2}) \) or \( \sin^{-1}(k_i \sin \theta_i/k_{r1}) \) which these angles don't exist (because the value of \( \arcsin \) is greater than one).

To reduce the enormous number of cases that we will examine, we consider the boundary surface of the uniaxial medium 1. According to Eqs. (2.53) and (2.54) the wave number of the ordinary and extraordinary waves
respectively, the boundary surface of the ordinary wave is the circle whose radius is equal to \( k_0 \sqrt{\varepsilon_\perp \mu_1} \) and the boundary surface of the extraordinary wave is the ellipse whose radius of the minor and major axes are either \( k_0 \sqrt{\varepsilon_\parallel \mu_1} \) or \( k_0 \sqrt{\varepsilon_\perp \mu_1} \), or in other words whether \( \varepsilon_\perp \) is greater than \( \varepsilon_\parallel \) or not [14]. By this reason we conclude that

a) if \( \varepsilon_\parallel \) is greater than \( \varepsilon_\perp \) (negative uniaxial crystal [14]), the radius of the minor axis of the ellipse is equal to the radius of the circle

b) if \( \varepsilon_\perp \) is greater than \( \varepsilon_\parallel \) (positive uniaxial crystal [14]), the radius of the major axis of the ellipse is equal to the radius of the circle.

Fig 3.4 The boundary surface of the ordinary and extraordinary waves.

Fig. 3.4 a and b show the boundary surface as we discussed above.

From now on we will consider only the case a. \((\varepsilon_\parallel > \varepsilon_\perp)\) since what we will examine will hold for the case b. \((\varepsilon_\parallel < \varepsilon_\perp)\) also, just interchanging the
boundary surface as shown in Fig. 3.4.

Now we shall examine the geometrical consideration in three cases since the boundary surface for the ordinary and extraordinary waves of the uniaxial medium 1 is given by the characteristic of the medium and the optic axis, the dispersion equation (2.51) and (2.52), and for the transmitted wave of the isotropic medium 2 is given by the dispersion equation (3.25). For simplicity, we assume that the given incidence wave has the same sign on both $b$ and $q$ - directions.

A. When the optic axis $c$ has the same sign on both $b$ and $q$ - directions.

1) $k_t < k_{r1}, k_{r2}$

If the projection of the given incident wave on the unit vector $b$ direction is within (See figure 3.5 a)

a) region 1, we can have all, two reflected ordinary and extraordinary waves and one transmitted wave.

b) region 2, we can have only two reflected ordinary and extraordinary waves.

c) region 3, we can have only one reflected extraordinary wave. Noting that in this case for the given incident wave, it cannot be ordinary wave, it have to be extraordinary wave (because of the limitation of the boundary surface).

d) region 4, this region is unnecessary since the projection of the given incident wave on the direction of the unit vector $b$ cannot exist.

2) $k_{r1} < k_t < k_{r2}$

If the projection of the given incident wave on the unit vector $b$ direction is within (See Fig. 3.5 b)

a) region 1 , we can have all, two reflected ordinary and extraordinary
waves and one transmitted wave.

b) region 2, we can have only one reflected extraordinary wave and one transmitted wave.

c) region 3, we can have only one reflected extraordinary wave.

d) region 4, this region is unnecessary since the projection of the given incident wave on the direction of the unit vector \( \mathbf{b} \) cannot exist.

3) \( k_t > k_{r1}, k_{r2} \)

If the projection of the given incident wave on the unit vector direction is within (See fig. 3.5 c)

a) region 1, we can have all, two reflected ordinary and extraordinary waves and one transmitted wave.

b) region 2, we can have only one reflected extraordinary wave and one transmitted wave.

c) region 3, this region is unnecessary since the projection of the given incident wave on the direction of the unit vector \( \mathbf{b} \) cannot exist.

Summary: From the three cases above we can conclude that the projection of the given incident wave on the unit vector \( \mathbf{b} \) direction and the boundary surfaces give the limitation of the existence of the reflected ordinary and extraordinary waves and the transmitted wave. Noting that the given incident wave also has the limitation by itself, i.e. it depends on whether it is ordinary or extraordinary wave.

B. When the optic axis \( \mathbf{c} \) has the difference sign on \( \mathbf{b} \) and \( \mathbf{q} \) – direction.

Now our area of concentration is only on area 2 (See Fig. 3.5d) for the given incidence extraordinary wave. For area 1 we can consider as the same as when the optic axis \( \mathbf{c} \) has the same sign on both \( \mathbf{b} \) and \( \mathbf{q} \) – direction, except the region that the projection of the given incidence wave on the
direction of unit vector $b$ does not exist.

1) $k_t < k_{r_2}$ (See Fig. 3.5e)

We can not have either transmitted or reflected waves since the angle of refraction and reflection do not exist. We will have only the surface wave.

2) $k_t > k_{r_2}$ but within area 2. (See Fig. 3.5f)

We will have only total transmission in area 3 and only surface wave (no transmission and reflection) in area 4.

3) $k_t > k_{r_2}$ and the projection of the boundary surface of extraordinary wave. (See Fig. 3.5g)

We will always have total transmission.

Summary: from the three cases above we can conclude that if the given incidence extraordinary wave is within the area 2, we have no reflection since the angle of reflection does not exist.
\[ \mu_2, \varepsilon_2 \]
Medium 2

\[ \mu_1, \varepsilon \]
Medium 1

\[ k_{r1} < k_l < k_{r2} \]

\[ k_l > k_{r1}, k_{r2} \]
$\mu_2, \varepsilon_2$
Medium 2

$\mu_1, \varepsilon$
Medium 1

d) area of concentration is only on area 2

e) $k_t < k_{r2}$
Fig. 3.5 Geometrical consideration of wave vectors at the interface of a uniaxial and an isotropic medium.
3.4 Amplitude of the reflected and transmitted waves.

From the previous section we know the wave vectors of all the waves, we now will proceed to determine the amplitude of the reflected and transmitted waves at the interface of an uniaxial - isotropic medium in term of the given amplitude of the incident wave. We assume that a uniform plane wave of arbitrary polarization is incident from a uniaxial crystal (medium 1) into an isotropic medium 2. The incident wave gives rise to two reflected waves in the medium 1, characterized by the dielectric tensor (2.19) and relative permeability constant $\mu_1$, and one transmitted wave in the isotropic medium 2, characterized by the dielectric constant $\varepsilon_2$ and relative permeability constant $\mu_2$. Fig.3.6 shows the orientation of the wave vectors with respect to the normal to the interface of the media.

![Figure 3.6 Reflection and transmission of waves at the interface of a uniaxial - isotropic medium.](image-url)
In the uniaxial crystal medium 1, we can have two kinds of wave that can propagate, let subscripts 1 and 2 denote these two waves. The two solutions $q_1$ and $q_2$ are selected such that they satisfy either the condition $\langle p \cdot q \rangle > 0$ (for incident wave) or the condition $\langle p \cdot q \rangle < 0$ (for reflected wave). The corresponding wave vectors $k_1$ and $k_2$ are then found from Eq.(3.8). Substituting these wave vectors into Eqs. (2.57) and (2.60) respectively, we can determine the directions of the electric field intensities $e_1$ and $e_2$. Since we will discuss in the case of having one incident wave that can propagate at a time so it can be either kind of these two waves. We will discuss in each case separately later, from now on we assume that the subscript $i$ can be either 1 or 2 for the incident wave in which it has to satisfy the condition $\langle p \cdot q \rangle > 0$ since the energy flow to the interface.

3.4.1 At the anisotropic - isotropic interface.

For simplicity we will give all of the field vectors in general form first and after we obtain the solution of the unknown coefficients, we will replace every field vectors in uniaxial case. The incident wave can be written as

$$E_{oi} = A e_i$$

$$H_{oi} = \frac{(k \times E_{oi})}{\omega \mu_0 \mu_1} = A h_i$$

where

$$h_i = \frac{(k \times e_i)}{\omega \mu_0 \mu_1}$$

The two reflected waves, which have to satisfy the condition $\langle p \cdot q \rangle < 0$, due to the energy bouncing back from the interface, can be written as

$$E_{01} = B_1 e_i$$
\[ H_{o1} = (k_1 x E_{o1}) / \omega \mu_0 \mu_1 = B_1 h_1 \] (3.29)

where
\[ h_1 = (k_1 x e_1) / \omega \mu_0 \mu_1 \]

and
\[ E_{o2} = B_2 e_2 \]

\[ H_{o2} = (k_2 x E_{o2}) / \omega \mu_0 \mu_1 = B_2 h_2 \] (3.30)

where
\[ h_2 = (k_2 x e_2) / \omega \mu_0 \mu_1 \]

B_1 and B_2 are two arbitrary constants.

We decompose the amplitude vectors of the transmitted wave in an isotropic medium 2 into components perpendicular and parallel to the plane of incidence, namely, the transmitted wave

\[ E_{ot} = C \perp a + C \parallel (k_1 x a) \]

\[ H_{ot} = k_i (C \perp (k_1 x a) - C \parallel a) / \omega \mu_0 \mu_2 \] (3.31)

At the interface \( \mathbf{r} \cdot \mathbf{q} = 0 \) the field vectors satisfy the boundary conditions. We may write them in frequency domain as

\[ (E_{o1} + E_{o1} + E_{o2} - E_{ot}) \mathbf{q} = 0 \]

or
\[ E_{o1} + E_{o1} + E_{o2} - E_{ot} = \alpha q \] (3.32)

and

\[ (H_{o1} + H_{o1} + H_{o2} - H_{ot}) \mathbf{q} = 0 \]

or
\[ H_{o1} + H_{o1} + H_{o2} - H_{ot} = \beta q \] (3.33)

where \( \alpha \) and \( \beta \) are two constants. Substituting Eqs. (3.28) - (3.31) into Eqs. (3.32) and (3.33), we obtain

\[ A e_1 + B_1 e_1 + B_2 e_2 - C \perp a - C \parallel (k_1 x a) = \alpha q \] (3.34)

and
\[ A h_1 + B_1 h_1 + B_2 h_2 - k_i (C \perp (k_1 x a) - C \parallel a) / \omega \mu_0 \mu_2 = \beta q \] (3.35)
We will determine the amplitudes $B_1$, $B_2$, $C_\perp$ and $C_\parallel$ of the reflected and transmitted waves in terms of the known amplitude $A$ of the incident wave. Taking the dot product of Eqs. (3.34) and (3.35) with vectors $\mathbf{a}$ and $\mathbf{b} = q_1 \mathbf{a}$ and then eliminate $C_\perp$ and $C_\parallel$ from the equation that we obtain, reorganize in the form that can be solved by Cramer's Rule \cite{16}. Finally we obtain the unknown amplitudes $B_1$ and $B_2$ as

$$B_1 = \frac{A}{\Delta} [\mathbf{a} \cdot \mathbf{e}_i (\mu_1 q_t - \mu_2 q_i) (\omega \mu_0 \mu_2 q_t \mathbf{a} \cdot \mathbf{h}_1 + k_t^2 \mathbf{b} \cdot \mathbf{e}_2)$$

$$- \mathbf{a} \cdot \mathbf{e}_2 (\mu_2 q_2 + \mu_1 q_1) (\omega \mu_0 \mu_2 q_2 \mathbf{a} \cdot \mathbf{h}_1 + k_t^2 \mathbf{b} \cdot \mathbf{e}_1)] \tag{3.36}$$

$$B_2 = \frac{A}{\Delta} [\mathbf{a} \cdot \mathbf{e}_i (\mu_1 q_t + \mu_2 q_1) (\omega \mu_0 \mu_2 q_t \mathbf{a} \cdot \mathbf{h}_1 + k_t^2 \mathbf{b} \cdot \mathbf{e}_1)$$

$$+ \mathbf{a} \cdot \mathbf{e}_1 (\mu_2 q_2 - \mu_1 q_1) (\omega \mu_0 \mu_2 q_2 \mathbf{a} \cdot \mathbf{h}_1 + k_t^2 \mathbf{b} \cdot \mathbf{e}_1)] \tag{3.37}$$

where $\Delta = -\mathbf{a} \cdot \mathbf{e}_1 (\mu_2 q_1 + \mu_1 q_t) (\omega \mu_0 \mu_2 q_t \mathbf{a} \cdot \mathbf{h}_2 + k_t^2 \mathbf{b} \cdot \mathbf{e}_2)$

$$+ \mathbf{a} \cdot \mathbf{e}_2 (\mu_1 q_t + \mu_2 q_2) (\omega \mu_0 \mu_2 q_1 \mathbf{a} \cdot \mathbf{h}_1 + k_t^2 \mathbf{b} \cdot \mathbf{e}_1) \tag{2.38}$$

Substitute Eqs. (3.36) and (3.37) into Eqs. (3.34) and (3.35) after dot-multiplication with vectors $\mathbf{a}$ and $\mathbf{b}$. We obtain the coefficients $C_\perp$ and $C_\parallel$ as

$$C_\perp = (\mu_2 A/\Delta) [(\mathbf{a} \cdot \mathbf{e}_1) (\mathbf{a} \cdot \mathbf{e}_2) (q_1 - q_2) (\omega \mu_0 \mu_2 q_t \mathbf{a} \cdot \mathbf{h}_1 + k_t^2 \mathbf{b} \cdot \mathbf{e}_1)$$

$$+ (\mathbf{a} \cdot \mathbf{e}_1) (\mathbf{a} \cdot \mathbf{e}_2) (q_2 + q_1) (\omega \mu_0 \mu_2 q_1 \mathbf{a} \cdot \mathbf{h}_1 + k_t^2 \mathbf{b} \cdot \mathbf{e}_1)$$

$$- (\mathbf{a} \cdot \mathbf{e}_1) (\mathbf{a} \cdot \mathbf{e}_2) (q_1 + q_2) (\omega \mu_0 \mu_2 q_2 \mathbf{a} \cdot \mathbf{h}_1 + k_t^2 \mathbf{b} \cdot \mathbf{e}_1)] \tag{3.39}$$

and $C_\parallel = (k_t A/\omega_0 \Delta) [(\omega \mu_0 \mu_2 q_t \mathbf{a} \cdot \mathbf{h}_1 + k_t^2 \mathbf{b} \cdot \mathbf{e}_1)$

$\{ (\mathbf{b} \cdot \mathbf{e}_1) (\mathbf{a} \cdot \mathbf{e}_2) (\mu_1 q_t - \mu_2 q_1) + (\mathbf{b} \cdot \mathbf{e}_2) (\mathbf{a} \cdot \mathbf{e}_1) (\mu_1 q_t + \mu_2 q_1) \}$

$$+ (\omega \mu_0 \mu_2 q_t \mathbf{a} \cdot \mathbf{h}_1 + k_t^2 \mathbf{b} \cdot \mathbf{e}_1)$

$\{ (\mathbf{b} \cdot \mathbf{e}_1) (\mathbf{a} \cdot \mathbf{e}_2) (\mu_1 q_t + \mu_2 q_2) + (\mathbf{b} \cdot \mathbf{e}_2) (\mathbf{a} \cdot \mathbf{e}_1) (\mu_1 q_t - \mu_2 q_1) \}$
3.4.2 At uniaxial - isotropic interface.

Now we will replace the field vectors \( e_i, h_i, e_1, h_1 \) and \( e_2, h_2 \) in Eqs. (2.36) - (2.41) by the field vectors that we obtained in Section 2.4. For the reflected ordinary and extraordinary waves, the field vectors \( e_1, h_1 \) and \( e_2, h_2 \) were given by Eqs. (2.57), (2.58) and (2.60), (2.61), respectively. Since we consider only one incident wave propagate at a time, as we already mentioned, we can have either ordinary or extraordinary waves on that. We will introduce \( y_o \) and \( y_e \) into the field vectors \( e_i \) and \( h_i \); when we have an ordinary incident wave propagate, we will have \( y_o = 1 \) and \( y_e = 0 \); on the other hand, if we have an extraordinary wave propagate, we will have \( y_e = 1 \) and \( y_o = 0 \). The field vectors \( e_i \) and \( h_i \) of the incident wave now become

\[
e_i = y_o (k_i x c) + y_o \{ k_0^2 \varepsilon_1 \mu_1 c - (k_i c) k_i \} \tag{3.42}
\]

and

\[
h_i = y_o (k_i x (k_i x c)) / \omega \mu_0 \mu_1 + y_e \{ \omega \varepsilon_0 \varepsilon_1 (k_i x c) \} \tag{3.43}
\]
We will introduce the optic axis $c$ by using coordinate free approach (See Fig. 3.7), we can decompose it into the components of the unit vectors $b$, $q$ and $a$ where the plane that formed by the unit vectors $b$ and $q$ is the plane of incident and the unit vector $a$ is the unit vector that perpendicular to that plane. The relationship between $b$, $q$ and $a$ is given by $b = q \times a$, so the optic axis can be given in the form of

$$c = \cos \theta \sin \phi \ a + \sin \theta \sin \phi \ b + \cos \phi \ q$$

(3.44)

where $\theta$ is the counter-clockwise angle between the unit vector $a$ to the projection of the optic axis $c$ on the plane that formed by the unit vector $a$ and $b$.

and $\phi$ is the angle between the unit vector $q$ to the optic axis $c$.

Figure 3.7 The orientation of the optic axis $c$.

After we introduce the optic axis $c$, given by Eq.(3.44), into the equations of the field vectors $e_1$, $h_1$, $e_1$, $h_1$ and $e_2$, $h_2$ (as the above) and
decompose the wave vectors $k_1, k_2$ and $k_3$ into the plane of incident (formed by vectors $\mathbf{q}$ and $\mathbf{r}$), finally we obtain:

The incident wave:

$$
e_i = y_o [k_i (\sin \theta_1 \cos \phi - \cos \theta_1 \sin \theta \sin \phi) \mathbf{a}$$

$$+ \cos \theta_1 \cos \theta \sin \phi \mathbf{b} - \sin \theta \cos \theta \sin \phi \mathbf{q}]$$

$$+ y_o [k_0^2 \mathbf{e} - \mu_1 \cos \theta \sin \phi \mathbf{a}$$

$$+ \{k_0^2 \mathbf{e} - \mu_1 \sin \theta \sin \phi - k_1^2 \sin \theta_1 \sin \theta \sin \phi + \cos \theta_1 \cos \phi \} \mathbf{b}$$

$$+ \{k_0^2 \mathbf{e} - \mu_1 \cos \phi - k_2^2 \cos \theta \sin \theta \sin \phi + \cos \theta \cos \phi \} \mathbf{q}]$$

$$h_i = y_o [w e_0 e - \mu_1 \cos \theta \sin \phi \mathbf{a} + \cos \theta \sin \theta \sin \phi \mathbf{b}$$

$$- \sin \theta_1 \sin \theta \sin \phi - \cos \theta \sin \theta \sin \phi]$$

$$+ y_o [w e_0 e - \mu_1 \{\sin \theta_1 \sin \phi - \cos \theta_1 \sin \theta \sin \phi \} \mathbf{a}$$

$$+ \cos \theta_1 \cos \theta \sin \phi \mathbf{b} - \sin \theta_2 \cos \theta \sin \phi \mathbf{q}]$$

The reflected ordinary wave:

$$e_1 = k_1 (\sin \theta_2 \cos \phi + \cos \theta_1 \sin \theta \sin \phi) \mathbf{a}$$

$$- \cos \theta_1 \cos \theta \sin \phi \mathbf{b} - \sin \theta_1 \cos \theta \sin \phi \mathbf{q}$$

$$h_1 = w e_0 e - \mu_1 \{\cos \theta \sin \phi - \cos \theta_1 \sin \theta \sin \phi \} \mathbf{b}$$

$$- \sin \theta_1 \sin \theta \sin \phi - \cos \theta \sin \theta \sin \phi \mathbf{q}]$$

The reflected extraordinary wave:

$$e_2 = k_0^2 \mathbf{e} - \mu_1 \cos \theta \sin \phi \mathbf{a}$$

$$+ \{k_0^2 \mathbf{e} - \mu_1 \sin \theta \sin \phi - k_2^2 \sin \theta_2 \sin \theta \sin \phi - \cos \theta_2 \cos \phi \} \mathbf{b}$$

$$+ \{k_0^2 \mathbf{e} - \mu_1 \cos \phi + k_2^2 \cos \theta_2 \sin \theta \sin \phi - \cos \theta_2 \cos \phi \} \mathbf{q}$$
\[ h_2 = w e_0 e_\perp k_2 \{ (\sin \theta_2 \cos \phi + \cos \theta_2 \sin \theta_2 \sin \phi) a - \cos \theta_2 \cos \phi \sin \theta_2 b - \sin \theta_2 \cos \phi \sin \theta_2 q \} \] (3.50)

Substitute Eqs. (3.45) - (3.50) into Eqs. (3.36) - (3.41), noting that \( k_x \sin \theta_x = a \) and \( k_x \cos \theta_x = q_x \) where \( x \) can be any of subscripts \( i, j, 1, 2 \) or \( t \); the final result of the amplitude of the reflected and transmitted waves can be obtained as:

\[ B_1 = (A/\Delta) \left[ y_0 [-k_0^2 e_\perp \mu_1 (a.c)^2 (\mu_2 q_2 + \mu_1 q_t) (k_t^2 q_1 - k_0^2 e_\perp \mu_2 q_t)] + \{ b.c (k_t^2 q_1^2 + k_0^2 e_\perp \mu_2 q_t q_2) + a^2 q.c (k_t^2 q_2 + k_0^2 e_\perp \mu_2 q_t) \} (q_1 b.c - a^2 q.c) (\mu_2 q_1 - \mu_1 q_t) \] + \( y_0 k_0^2 e_\perp \mu_1 a.c q_2 \mu_2 b.c (k_t^2 q_1^2 + k_0^2 e_\perp \mu_1 q_t^2) + q_t a^2 q.c (k_t^2 \mu_1 - k_0^2 e_\perp \mu_2^2) \] (3.51)

\[ B_2 = (A/\Delta) \left[ y_0 q a.c (\mu_2 b.c (k_t^2 q_1^2 - k_0^2 e_\perp \mu_1 q_t^2) + q_t a^2 q.c (k_t^2 \mu_1 - k_0^2 e_\perp \mu_2^2)) + y_0 k_0^2 e_\perp \mu_1 (a.c)^2 (\mu_1 q_t - \mu_2 q_t) (k_t^2 q_1 + k_0^2 e_\perp \mu_2 q_t) + \{ b.c (k_t^2 q_1^2 - k_0^2 e_\perp \mu_2 q_t q_1) + a^2 q.c (k_t^2 q_1 - k_0^2 e_\perp \mu_2 q_t) \} (q_1 b.c + a^2 q.c) (\mu_2 q_1 + \mu_1 q_t) \] (3.52)

\[ C_\perp = (A/a^2 \Delta) \left[ y_0 q \mu_2 a.c [k_0^2 e_\perp \mu_1 (a.c)^2 (b.c (k_t^2 q_1^2 + k_0^2 e_\perp \mu_2 q_t q_2) - a^2 q.c (k_t^2 q_2 + k_0^2 e_\perp \mu_2 q_t))] + \{ q^{-2} b.c - a^4 (q.c)^2 \} (b.c (k_t^2 q_1^2 + k_0^2 e_\perp \mu_2 q_t q_2) + a^2 q.c (k_t^2 q_2 + k_0^2 e_\perp \mu_2 q_t)) \right] - y_0 k_0^2 e_\perp \mu_1 \mu_2 a.c (q_2 + q_1) (k_t^2 q_1 + k_0^2 e_\perp \mu_2 q_t) \] (3.53)
\[-k_0^2\epsilon_\perp\mu_1(a.c)^2 + (q.b.c + a^2q.c)^2\] 

\[C_{II} = k_0^2\epsilon_\perp\mu_2 q_1[-y_0 q_1 a.c \[2a^2(b.c)(q.c)(\mu_2 q_1^2 + \mu_1 q_t q_2) \]
+ (\mu_2 q_2 + \mu_1 q_t)(k_0^2\epsilon_\perp\mu_1(a.c)^2 + q_1^2(b.c)^2 + a^4(q.c)^2)] \quad (3.54)\]

\[+ y_q (\mu_1 q_t + \mu_2 q_1)(q_2 + q_t) (-q_1 b.c + a^2 q.c) \]
\[\{ -k_0^2\epsilon_\perp\mu_1(a.c)^2 + (q_1 b.c + a^2 q.c)^2\} (k_t A / q_t a^4 A) \]

where
\[
\Delta = -k_0^2\epsilon_\perp\mu_1(a.c)^2(\mu_2 q_2 + \mu_1 q_t)(k_t^2 q_1 + k_0^2\epsilon_\perp\mu_2 q_t) \]
\[-(q_1 b.c + a^2 q.c)(\mu_1 q_t + \mu_2 q_1) \quad (3.55)\]

\[b.c = \cos\theta \sin\phi \]
\[q.c = \cos\phi \quad (3.56)\]

and \[a.c = \cos\theta \sin\phi\]

In the case of normal incidence, the resulting formulas (3.51) - (3.55) are no longer valid because the concept of the plane of incidence loses its meaning. In this case, the wave vectors take the form

\[k_i = k_1 = k_t q \quad , \quad k_1 = -k_1 q \quad , \quad k_2 = -k_2 q \quad , \quad k_l = k_l q \quad (3.57)\]

where
\[k_i = k_1 = k_0\sqrt{\epsilon_\perp\mu_1} \quad \text{for incident ordinary wave} , \]
\[k_i = k_2 = k_0\sqrt{\epsilon_\perp\epsilon_\parallel\mu_1}/\{\epsilon_\parallel + (\epsilon_\perp - \epsilon_\parallel)(q_x c)^2 \} \quad \text{for incident extraordinary wave} \quad \text{and} \quad k_l = k_0\sqrt{\epsilon_\parallel\mu_1} \quad (3.58)\]

As an alternative approach, we will treat the plane formed by vectors \(c\) and \(q\) as it were the plane of incidence. Therefore the subscripts \(\perp\) and \(//\) will be
used in this sense. According to Eqs. (3.28), (3.42) and (3.43), the incident wave becomes

\[
E_{0i} = A \left( y_0 k_i (q x c) + y_0 k_0^2 \epsilon_\perp \mu_1 c - k_2^2 (q . c) q \right)
\]

\[
H_{0i} = A \left( k_i x E_{0i} \right) j \omega \mu_0 \mu_1
\]

\[
= A \left( y_0 w \epsilon_0 \epsilon_\perp (q x (q x c)) + y_0 w \epsilon_0 \epsilon_\perp k_i (q x c) \right)
\]  

(3.59)

According to Eqs (3.29), (3.30) and (2.57)-(2.61), the reflected waves are

Ordinary wave:

\[
E_{01} = - B_1 k_i (q x c)
\]

\[
H_{01} = B_1 w \epsilon_0 \epsilon_\perp (q x (q x c))
\]  

(3.60)

Extraordinary wave:

\[
E_{02} = B_2 \left( k_0^2 \epsilon_\perp \mu_1 c - k_2^2 (q . c) q \right)
\]

\[
H_{02} = - B_2 w \epsilon_0 \epsilon_\perp k_i (q x c)
\]  

(3.61)

We decompose the field vectors of the transmitted wave into components perpendicular and parallel to the plane formed by vector \( c \) and \( q \) as shown in Figs. 3.8 and 3.9. Thus, the transmitted wave becomes

\[
E_{0t} = C_\perp (q x c) + C_{//} (q x (q x c))
\]

\[
H_{0t} = (k_i x E_{0t}) / j \omega \mu_0 \mu_2 = k_i [C_\perp (q x (q x c)) - C_{//} (q x c)] / j \omega \mu_0 \mu_2
\]  

(3.62)

Substituting Eqs. (3.59) - (3.62) into the boundary conditions (3.32) and (3.33), we obtain

\[
(A y_0 k_i - B_1 k_1 - C_\perp) (q x (q x c)) + (k_0^2 \epsilon_\perp \mu_1 (A y_0 + B_2) + C_{//}) (q x c) = 0
\]  

(3.63)

\[
(k_0^2 \epsilon_\perp \mu_2 (A y_0 k_i - B_2 k_2) - C_{//} k_i) (q x (q x c)) - (k_0^2 \epsilon_\perp \mu_2 (A y_0 + B_1) + C_\perp k_i) (q x c) = 0
\]  

(3.64)
Figure 3.8 Orientations of the interface and the plane formed by vectors $q$ and $c$

In the case of normal incidence.

Figure 3.9 The wave vectors of the incident, the reflected and the transmitted waves

in the case of normal incidence.

respectively. Since $qc$ and $q(xqc)$ are two linearly independent vectors,

it follows that
From Eqs. (3.65) - (3.68), finally we obtain

\[ B_1 = \frac{A_{y_0} (k_1 k_t - k_0^2 e_{\perp} \mu_2) / (k_1 k_t + k_0^2 e_{\perp} \mu_2)}{k_0^2 e_{\perp} \mu_2 (A_{y_0} k_1 - B_2 k_2) + C_{\|} k_t} \] (3.69)

\[ B_2 = \frac{A_{y_0} (\mu_2 k_2 - \mu_1 k_t) / (\mu_2 k_2 + \mu_1 k_t)}{k_0^2 e_{\perp} \mu_2 (A_{y_0} k_1 - B_2 k_2) + C_{\|} k_t} \] (3.70)

\[ C_{\perp} = \frac{2A_{y_0} k_1 k_0^2 e_{\perp} \mu_2 / (k_1 k_t + k_0^2 e_{\perp} \mu_2)}{k_0^2 e_{\perp} \mu_2 (A_{y_0} k_1 - B_2 k_2) + C_{\|} k_t} \] (3.71)

\[ C_{\|} = -\frac{2A_{y_0} k_0^2 e_{\perp} \mu_1 \mu_2 k_2 / (\mu_2 k_2 + \mu_1 k_t)}{k_0^2 e_{\perp} \mu_2 (A_{y_0} k_1 - B_2 k_2) + C_{\|} k_t} \] (3.72)

As we can see from the Eqs. (3.69) and (3.70) for the reflected waves, we can obtain only one of the reflected waves. This reflected wave depends on the given incident wave, i.e.

1) if the given incident wave is ordinary wave, the reflected wave is ordinary wave, and

2) if the given incident wave is extraordinary wave, the reflected wave is extraordinary wave.

### 3.5 Energy Relation at the Interface

We will examine the energy balance when waves are reflected and transmitted at the uniaxial - isotropic interface. By the law of conservation of energy, i.e. the power flux incident on any part of the interface must be equal to the sum of power fluxes leaving that interface:

\[ (p_i + p_{r1} + p_{r2} - p_t) \cdot q = 0 \] (3.73)
By taking the time average on both sides of Eq.(3.73), we obtain

\[ \langle p_i \rangle \cdot q + \langle p_{r1} \rangle \cdot q + \langle p_{r2} \rangle \cdot q = \langle p_t \rangle \cdot q \]  \hspace{1cm} (3.74)

which indicates that the normal component of the time-average energy flow across the interface must also be continuous.

The ratios

\[ r = -\{ \langle p_{r1} \rangle \cdot q + \langle p_{r2} \rangle \cdot q \} / \langle p_i \rangle \cdot q \]  \hspace{1cm} (3.75)

and

\[ t = \langle p_t \rangle \cdot q / \langle p_i \rangle \cdot q \]  \hspace{1cm} (3.76)

are called the reflectivity (energy reflection coefficient) and transmissivity (energy transmission coefficient) respectively. It follows from Eq.(3.74) that

\[ r + t = 1 \]  \hspace{1cm} (3.77)

Using Eqs. (3.28) - (3.31) and Eqs. (3.45) - (3.50) for the field vectors and directions of the incident, reflected and transmitted waves and the definition of the time-averaged Poynting vector in Section 2.5. Finally we obtain the normal component of the time-averaged Poynting vectors as:

For the incident wave:

\[ \langle p_i \rangle \cdot q = (A^2/2)[y_0 \omega \varepsilon_0 \varepsilon_\perp k_1 \cos \theta_1 \{ (\sin \theta_1 \cos \phi - \cos \theta_1 \sin \theta_1 \sin \phi)^2 + (\cos \theta_1 \sin \phi)^2 \} \]

\[ + y_0 \omega \varepsilon_0 \varepsilon_\perp k_1 \{ k_0^2 \varepsilon_\perp \mu_1 \sin \phi (\cos \theta_1 \sin \phi - \sin \theta_1 \sin \theta_1 \cos \phi) \}

\[ + k_1^2 \sin \theta_1 \{ \sin \theta_1 \sin \phi \cos \phi (1 - 2 \cos^2 \theta_1) + \sin \theta_1 \cos \theta_1 \{ \cos^2 \phi - (\sin \theta_1 \sin \phi)^2 \} \} \} \]  \hspace{1cm} (3.78)

For the ordinary reflected wave:

\[ \langle p_{r1} \rangle \cdot q = (-B_1^2/2)[\omega \varepsilon_0 \varepsilon_\perp k_1 \cos \theta_1 \{ (\sin \theta_1 \cos \phi + \cos \theta_1 \sin \theta_1 \sin \phi)^2 \]

\[ + (\cos \theta_1 \sin \phi)^2 \} \]  \hspace{1cm} (3.79)

For the extraordinary reflected:
\[ \begin{align*}
\langle p_{r_2} \rangle \cdot q &= \left( B_2^2 / 2 \right) \omega \epsilon_0 \epsilon_1 \left\{ k_0^2 \epsilon_1 \mu \sin \phi \left( \cos \theta \sin \phi - \sin \theta \cos \phi \right) 
+ k_2^2 \sin \theta \left[ \sin \theta \sin \phi \cos \phi (1 - 2 \cos^2 \theta) - \sin \theta \cos \theta \{ \cos^2 \phi - (\sin \theta \sin \phi)^2 \} \right] \right\} \\
& \quad + k_2^2 \sin \theta \left[ \sin \theta \sin \phi \cos \phi (1 - 2 \cos^2 \theta) - \sin \theta \cos \theta \{ \cos^2 \phi - (\sin \theta \sin \phi)^2 \} \right] \\
& \quad \left( \sin \phi \right) \left( \cos \phi \right) \\
& \left( \sin \theta \right) \left( \cos \theta \right)
\end{align*} \]

For the transmitted wave:
\[ \langle p_t \rangle \cdot q = \left\{ (C_\perp^2 + C_{\parallel}^2) / 2 \right\} \left( k_t^3 \sin^2 \theta_t \cos \theta_t \right) \quad (3.81) \]

### 3.5 Some Special Cases.

In this section by using the resulting Eqs. (3.51) - (3.55) and (3.78) - (3.81) we will find that at a particular orientation of the optic axis \( c \), we obtain only one reflected wave and also we will find the transmitted coefficient, the reflectivity and transmittivity in each case.

**For the incident ordinary wave:**

**Optic Axis \( c \) parallel to the interface.**

1) When optic Axis \( c \) is parallel to the interface but perpendicular to the plane of incident, i.e. \( c = a \); \( \theta = 0^\circ \); \( \phi = 90^\circ \). (See Fig. 3.10)

By Eqs. (3.51) - (3.55) we obtain
\[ \Delta = -k_0^2 \epsilon_1 \mu_1 a^2 \left( \mu_2 q_2 + \mu_1 q_t \right) \left( k_t^2 q_1 + k_0^2 \epsilon_1 \mu_2 q_t \right) \]
\[ B_1 = A \left( k_t^2 q_1 - k_0^2 \epsilon_1 \mu_2 q_t \right) / \left( k_t^2 q_1 + k_0^2 \epsilon_1 \mu_2 q_t \right) \]
\[ B_2 = 0 \]
\[ C_\perp = 0 \]
\[ C_{\parallel} = 2Ak_0^2 \epsilon_1 \mu_2 q_1 k_t / a \left( k_t^2 q_1 + k_0^2 \epsilon_1 \mu_2 q_t \right) \quad (3.82) \]

2) When optic Axis \( c \) is parallel to the interface and the plane of incident, i.e. \( c = b \); \( \theta = 90^\circ \); \( \phi = 90^\circ \). (See Fig. 3.11)
Figure 3.10 Optic axis $c$ is parallel to the interface but perpendicular to the plane of incidence.

Figure 3.11 Optic axis $c$ is parallel to the interface and the plane of incidence.
Figure 3.12 Optic axis $c$ is parallel to the plane of incidence but perpendicular to the plane of interface.

Figure 3.13 Optic axis $c$ is parallel to the plane of incidence but is arbitrary oriented with respect to the interface.
By Eqs. (3.51) - (3.55) we obtain

\[ \Delta = -q_1 a^2 (\mu_1 q_l + \mu_2 q_1)(k_t^2 q_2^2 + k_0^2 e_{\perp} \mu_2 q_2) \]

\[ B_1 = A (\mu_1 q_l - \mu_2 q_1)/(\mu_1 q_l + \mu_2 q_1) \]

\[ B_2 = 0 \]  

\[ C_{\perp} = 2A \mu_2 q_1^2/a(\mu_1 q_l + \mu_2 q_1) \]

\[ C_{\parallel} = 0 \]

**Optic Axis** \( c \)** parallel to the plane of incidence**

1) When optic Axis \( c \) is parallel to the plane of incidence but perpendicular to the plane of interface (in the other word, the optic axis \( c \) is normal to the interface), \( i.e. c = q \); \( \phi = 0^\circ \). (See Fig. 3.12). By Eqs. (3.51) - (3.55) we obtain

\[ \Delta = -a^2 (\mu_1 q_l + \mu_2 q_1)(k_t^2 q_2^2 + k_0^2 e_{\perp} \mu_2 q_2) \]

\[ B_1 = A (\mu_1 q_l - \mu_2 q_1)/(\mu_1 q_l + \mu_2 q_1) \]

\[ B_2 = 0 \]  

\[ C_{\perp} = 2A \mu_2 q_1^2/(\mu_1 q_l + \mu_2 q_1) \]

\[ C_{\parallel} = 0 \]

2) When optic Axis \( c \) is parallel to the plane of incidence but is arbitrary oriented with respect to the interface, \( i.e. \; c = sin\phi \; b + cos\phi \; q \); \( \Theta = 90^\circ \). (See Fig. 3.13)

By Eqs. (3.51) - (3.55) we obtain

\[ \Delta = -a^2 (q_1 sin\phi + a cos\phi)(\mu_1 q_l + \mu_2 q_1) \]

\[ \{sin\phi(k_t^2 q_2^2 + k_0^2 e_{\perp} \mu_2 q_2) + a cos\phi(k_1^2 q_2^2 + k_0^2 e_{\perp} \mu_2 q_2)\} \]
\[
B_1 = Ay_0(q, \sin \phi - \cos \phi)(\mu_1 q_l - \mu_2 q)/(q, \sin \phi + \cos \phi)(\mu_1 q + \mu_2 q_l)
\]
\[
B_2 = 0
\] (3.85)

\[
C_{\perp} = -2Ay_0 \mu_2 q_1 (q, \sin \phi - \cos \phi)/a(\mu_1 q + \mu_2 q_1)
\]
\[
C_{//} = 0
\]

As we can see from Eqs. (3.82) - (3.85), the amplitude of the reflected extraordinary wave becomes zero, i.e. the reflected extraordinary wave doesn't exist.

**For the incident extraordinary wave:**

**Optic Axis \textbf{c} parallel to the interface.**

1) When optic Axis \textbf{c} is parallel to the interface but perpendicular to the plane of incident, i.e. \(c = a\); \(\theta = 0^\circ\); \(\phi = 90^\circ\). (See Fig. 3.10)

By Eqs. (3.51) - (3.55) we obtain

\[
\Delta = -k_0^2 \varepsilon_\perp \mu_1 a^2(\mu_2 q_2 + \mu_1 q_l)(k_l q_1^2 + k_0^2 \varepsilon_\perp \mu_2 q_l)
\]

\[
B_1 = 0
\]

\[
B_2 = A(\mu_2 q_2 - \mu_1 q_l)/(\mu_2 q_2 + \mu_1 q_l)
\] (3.86)

\[
C_{\perp} = 2Ak_0^2 \varepsilon_\perp \mu_1 \mu_2 q_2/a(\mu_2 q_2 + \mu_1 q_l)
\]

\[
C_{//} = 0
\]

2) When optic Axis \textbf{c} is parallel to the interface and the plane of incident, i.e. \(c = b\); \(\theta = 90^\circ\); \(\phi = 90^\circ\). (See Fig. 3.11)

By Eqs. (3.51) - (3.55) we obtain

\[
\Delta = -q_1 a^2(\mu_1 q_l + \mu_2 q_1)(k_l q_1^2 + k_0^2 \varepsilon_\perp \mu_2 q_l q_2)
\]

\[
B_1 = 0
\]
\[ B_2 = -A(k_t^2q_1^2 - k_0^2\varepsilon\mu_2q_2)/(k_t^2q_1^2 + k_0^2\varepsilon\mu_2q_2) \]  \hspace{1cm} (3.87)
\[ C_\perp = 0 \]
\[ C_\parallel = 2Ak_0^2\varepsilon\mu_2k_tq_1q_2/a(k_t^2q_1^2 + k_0^2\varepsilon\mu_2q_2) \]

**Optic Axis \( \mathbf{c} \) parallel to the plane of incidence.**

1) When optic Axis \( \mathbf{c} \) is parallel to the plane of incidence but perpendicular to the plane of interface (in the other word, the optic axis \( \mathbf{c} \) is normal to the interface), i.e. \( \mathbf{c} = \mathbf{q} \); \( \phi = 0^\circ \). (See Fig. 3.12).

By Eqs. (3.51) - (3.55) we obtain
\[ \Delta = -a^4(\mu_1q_t + \mu_2q_1)(k_t^2q_2 + k_0^2\varepsilon\mu_2q_t) \]
\[ B_1 = 0 \]
\[ B_2 = A(k_t^2q_2 - k_0^2\varepsilon\mu_2q_t)/(k_t^2q_2 + k_0^2\varepsilon\mu_2q_t) \]  \hspace{1cm} (3.88)
\[ C_\perp = 0 \]
\[ C_\parallel = 2Ak_0^2\varepsilon\mu_2q_1k_t/(k_t^2q_2 + k_0^2\varepsilon\mu_2q_t) \]

2) When optic Axis \( \mathbf{c} \) is parallel to the plane of incidence but is arbitrary oriented with respect to the interface, i.e.; \( \mathbf{c} = \sin\phi \mathbf{b} + \cos\phi \mathbf{q} \)
\( \theta = 90^\circ \). (See Fig. 3.13)

By Eqs. (3.51) - (3.55) we obtain
\[ \Delta = -a^2(q_1\sin\phi + \cos\phi)(\mu_1q_t + \mu_2q_1) \]
\[ \{\sin\phi(k_t^2q_1^2 + k_0^2\varepsilon\mu_2q_2) + \cos\phi(k_t^2q_2 + k_0^2\varepsilon\mu_2q_t)\} \]
\[ B_1 = 0 \]
\[ B_2 = -A\{\sin\phi(k_t^2q_1^2 - k_0^2\varepsilon\mu_2q_t) - \cos\phi(k_t^2q_1 - k_0^2\varepsilon\mu_2q_t)\}/ \]
\[ \{ \sin \phi (k_1^2 q_1^2 + k_0^2 \varepsilon_{2} \mu q_2 q_3) + \cos \phi (k_1^2 q_2 + k_0^2 \varepsilon_{1} \mu q_1) \} \]

\begin{align*}
C_{\perp} &= 0 \\
C_{//} &= A k_0^2 \varepsilon_{1} \mu_2 k_1 (q_2 + q_3) (q_1^2 \sin^2 \phi - a^2 \cos^2 \phi) \\
&+ a \{ \sin \phi (k_1^2 q_1^2 + k_0^2 \varepsilon_{2} \mu q_2 q_3) + \cos \phi (k_1^2 q_2 + k_0^2 \varepsilon_{1} \mu q_3) \} 
\end{align*}

As we can see from Eqs. (3.86) - (3.89), the amplitude of the reflected ordinary wave become zero, i.e. the reflected ordinary wave doesn't exist.

**Summary**: We can conclude that

- a) for a given ordinary incident wave in the particular orientation of the optic axis \( \mathbf{c} \) (as we already discussed) we can obtain only ordinary wave for the reflected wave, and

- b) for a given extraordinary incident wave in the particular orientation of the optic axis \( \mathbf{c} \) (as we already discussed) we can obtain only extraordinary wave for the reflected wave.
Chapter 4

Numerical Application.

Introduction

In the previous chapter, we obtain the mathematical solution of the amplitude of the reflected and transmitted waves at the interface of uniaxial-isotropic media and by using these solutions we can find the energy relation at the interface. It is still hard to visualize the characteristic of these energy coefficients. Even we can solve these problems by hand, it will take long time to solve just only one particular problem with particular orientation of the optic axis and also angle of incidence. With the aid of the computer we can solve these problems a lot more faster also with the changing of the angle of incidence and plot the result to make it able to visualize. We will examine graphically the reflected and transmitted power coefficients at the interface of uniaxial-isotropic media (referred to the characteristic of the media as given in appendix A, the computer program). We will plot these coefficients versus the angles of incidence and compare each pair of them since we can have two kind of incidence waves propagate in the uniaxially anisotropic medium.

4.1 General Equations.

Since we have already expressed all of the equations in Chapter 3 in arithmetical forms which the angles between the coordinate axises and the optic axis have been given by Eq.(3.44), as shown in Fig. 3.7, now we are able to feed these expressions to the computer program directly. But there is one important thing that we need to generalize first, i.e. the angle of reflected extraordinary wave. From Eq.(3.7), the Snell's law of reflection
and refrection, we consider only the reflection of the reflected extraordinary wave so it can be rewritten as

\[ k_i \sin \theta_i = k_2 \sin \theta_2 \]  \hspace{1cm} (4.1)

where \( k_i, \theta_i \) and \( k_2, \theta_2 \) are the incident wave number, angle of incidence and the reflected extraordinary wave number, angle of reflected extraordinary wave, respectively.

Now we will study about the angle of reflected extraordinary wave. From Eq.(4.1), for a given incidence wave we have already known \( k_i, \theta_i, k_2, \) as given by Eq.(2.54), does depend on the angle of reflected and the orientation of the optic axis. Since the optic axis has to be given fixed for each case so the angle of reflection is the only unknown variable that we need to find.

After we solved Eq.(4.1) looking for the angle of reflected extraordinary wave, we obtained a quadratic equation of fourth degree as

\[ A \sin^4 \theta_2 - B \sin^2 \theta_2 + C = 0 \]  \hspace{1cm} (4.2)

where

\[ A = \left( k_o^2 \varepsilon_1 e_{//} / k_i^2 \sin^2 \theta_i (e_{//} - e_{\perp}) + \cos^2 \phi - \sin^2 \theta \sin^2 \phi \right) \]

\[ + 4 \sin^2 \theta \sin^2 \phi \cos^2 \phi \]

\[ B = 2 \left( k_o^2 \varepsilon_1 e_{//} / k_i^2 \sin^2 \theta_i (e_{//} - e_{\perp}) + \cos^2 \phi - \sin^2 \theta \sin^2 \phi \right) \]

\[ \left( e_{\perp} / (e_{//} - e_{\perp}) + \cos^2 \phi \right) + 4 \sin^2 \theta \sin^2 \phi \cos^2 \phi \]

\[ C = \left( e_{\perp} / (e_{//} - e_{\perp}) + \cos^2 \phi \right)^2 \]

so the final solution for \( \theta_2 \) will be

\[ \theta_2 = \arcsin(B \pm \sqrt{B^2 - 4AC}) / 2A \]  \hspace{1cm} (4.3)

Some important thing arises with the choice of \( \pm \) sign in Eq.(4.3). For simplicity, we assume that the given incident wave propagates in the
direction of $\mathbf{b}$ and $\mathbf{q}$ which have the same sign, i.e. within quadrant 3. (See Figure 4.1)

We will have the following two cases:

1) When $\mathbf{b}$ and $\mathbf{q}$ (in Eq. 3.44) have the same sign, the optic axis $\mathbf{c}$ will be in the first or third quadrants (See fig. 4.2a). In this case, we choose negative sign in Eq. 4.3.

2) When $\mathbf{b}$ and $\mathbf{q}$ (in Eq. 3.44) have opposite sign, the optic axis $\mathbf{c}$ will be in the second or fourth quadrants (See fig. 4.2b). In this case, we choose positive sign in Eq. 4.3.

The reason in both cases is shown in figure 4.2a and b as a geographical point of view.

Now, using the computer program given in appendix A, we will plot the power ratios versus the angles of incidence at the interface of uniaxial
Figure 4.2 The particular orientation of the optic axis.
- isotropic media for both incident ordinary and extraordinary waves, starting with some special cases (particular orientation of the optic axis) that give only one reflected wave and ending with general cases that give both reflected ordinary and extraordinary waves.

4.2 **Some special cases.**

Some particular orientation of the optic axis that will give only one reflected wave.

4.2.1 **Optic axis is in the directions of the coordinate.**

In these cases the optic axis will be in $a$, $b$ or $q$ - direction (See figures G1 - G6). We can obtain only one reflected wave which depends on the given incident wave as discussed in section 3.5. The numerical results showed the following:

1) For the given incident ordinary wave we can obtain total transmission (at Brewster's angle) only when the optic axis is in $a$ - direction (See figure G1), i.e. this incident wave acts as it has parallel polarization in case of isotropic - isotropic interface [9,15,18], otherwise this incident wave will act as it has perpendicular polarization in case of isotropic - isotropic interface [9,15,18]. (See figure G1-3)

2) For the given incident extraordinary wave we can obtain total transmission (at Brewster's angle) when the optic axis is either in $b$ or $q$ - direction (See figures G5 and G6), in $a$ - direction we have no total transmission at all. (See figure G4)

3) By comparing figures G1 - G6 we can conclude that in the particular orientation of the optic axis as discuss above, we can obtain total transmission only in one particular case of the given incident wave. In
other words, we cannot obtain total transmission for both incident ordinary and extraordinary waves when the optic axis is in \( a \), \( b \) or \( q \) - direction. To get the idea we can compare as pairs of figures G1 & G4, figures G2 & G5 and figure G3 & G6.

4.2.2 Optic axis is on the plane of incidence. (See figures G7 - G12)
In this case we also obtain only one reflected wave as expected. Each given incident wave will act nearly the same as the optic axis is in the \( b \) or \( q \) - direction. It is hard to see the difference of the Brewster's angle or in other words we have no variation of the Brewster's angle for the given incident extraordinary wave.

4.3 Optic axis is on the plane of interface.
In this case we obtain both reflected ordinary and extraordinary waves. The numerical results showed the following:

4.3.1) For a given incident ordinary wave, starting with the orientation of the optic axis is on \( a \) - direction and rotates in counter-clockwise direction on the plane of interface to the \( b \) - direction. As \( \Theta \) increase, the zero ordinary reflectivity angle will shift toward to the right until it has no zero ordinary reflectivity angle (See figures G1, G2 and G13-G15). While the extraordinary reflectivity does not exist on both starting and ending points (as we discussed in case of the optic axis is on \( a \) and \( b \) - directions). As \( \Theta \) increases the magnitude of the extraordinary reflectivity varies up to the peak point and then down, starts from zero and goes back to zero. Nothing is interesting for the extraordinary reflectivity in this case.

4.3.2) For a given incident extraordinary wave, again the same we
considered at the optic axis is in \( a \) - direction and rotates counter-clockwise to \( b \) - direction. Starting with no zero extraordinary reflectivity angle, as the optic axis is rotating at some particular point we will obtain the zero extraordinary reflectivity angle and this zero extraordinary reflectivity angle will shift to the left (See figures G4, G5 and G16 - G18). Nothing is interesting for the ordinary reflectivity in this case.

4.3.3) Compare the pattern of both incident ordinary and extraordinary waves, we can see as its interchanging its pattern by the time \( \Theta \) increases for the ordinary reflectivity and extraordinary reflectivity of the incident ordinary and extraordinary waves, respectively.

4.4 Optic axis is on the plane given by vectors \( q \) and \( q \).

In this case, we obtain both reflected ordinary and extraordinary waves (See figures G1, G3, G4, G6 and G19 - G24). The numerical results showed the following:

4.4.1) For a given incident ordinary wave, starting to consider with the optic axis is in \( q \) - direction and then varies to \( a \) - direction. The reflected ordinary wave will play an important role on this. With the optic axis in \( q \) - direction we varies the optic axis against \( a \) - direction, the zero ordinary reflectivity angle will shift to the right until it has no zero ordinary reflectivity angle as we can see from figures G1, G3 and G19 - G21. Nothing is interesting for the extraordinary reflectivity in this case.

4.4.2) For a given incident extraordinary wave, again starting to consider with the optic axis is in \( q \) - direction and then varies to \( a \) - direction. The reflected extraordinary wave will play an important role on this. With the optic axis in \( q \) - direction we have no zero extraordinary
reflectivity angle. As the optic axis is rotating at some particular point we will obtain the zero ordinary reflectivity angle and this zero extraordinary reflectivity angle will shift to the left as we can see from figures G4, G6 and G22 - G24. Nothing is interesting for the ordinary reflectivity in this case.

3.4.3 Compare the pattern of both incident ordinary and extraordinary waves, we can conclude the results in the same manner as in section 4.3.3.

4.5 **General case.** (See figure G25 - G36)

In this case we considered the optic axis is in arbitrary direction, i.e. the direction of the optic axis is somewhere other than the particular direction that we have discussed in the previous three sections. The numerical results showed the following:

4.5.1) For a given incident ordinary wave, in general we can obtain both reflected ordinary and extraordinary waves but at some particular angle of incidence of the given incident wave we obtained only one reflected wave (See figure G25 - G28). In the other words, at a particular angle of incidence we obtained no reflected ordinary wave and at another particular angle of incidence we obtained no reflected extraordinary wave. As the changing of optic axis we can also find total transmission (at Brewster's angle) since the zero reflectivity angle of the reflected waves move in opposite direction as the optic axis moves.

4.5.2) For a given incident extraordinary wave, the reflected ordinary wave always exists and at some particular orientation of the optic axis and angle of incidence we can have no reflected extraordinary wave.

4.5.3) We can conclude that for a particular direction of the arbitrary
orientation of the optic axis we can obtain total transmission (at Brewster's angle) if the given incident wave is the ordinary wave but we cannot obtain total transmission if the given incident wave is the extraordinary wave.

4.6 Summary.

The major variation of the magnitude of the reflected waves can be summarized into 3 parts, depending on which plane the optic axis lies as the following:

1) If the optic axis is on the plane of incident, there is no variation at all.

2) If the optic axis is on the plane of interface, the variation of the zero reflectivity angle (either reflected ordinary or extraordinary wave) is only on the right hand side of the reference point (considered the reference point as in the case of the optic axis is in the directions of the coordinate, i.e. $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{q}$ - directions).

3) If the optic axis is on the plane given by vectors $\mathbf{a}$ and $\mathbf{q}$, the variation of the zero reflectivity angle (either reflected ordinary or extraordinary wave) is only on the left hand side of the reference point (considered the reference point as in the case of the optic axis is in the directions of the coordinate, i.e. $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{q}$ - directions).

So we can conclude that in general (orientation of the optic axis) we can expect the pattern of the reflected waves as the influence of each plane as we discussed above.
G1. ORDINARY INC., OPTIC AXIS IN $a$-DIRECTION.

G2. OR-INC., OPTIC AXIS IN $b$-DIRECTION.
**Legend**
- • OR-REFLECTED
- □ EX-REFLECTED
- ◇ TRANSMITTED

**G3.** OR-INC., OPTIC AXIS IN 9-DIRECTION.

**G4.** EXTRAORDINARY INC., OPTIC AXIS IN 9-DIRECTION.
**G5.** EX-INC., OPTIC AXIS IN b-DIRECTION.

**G6.** EX-INC., OPTIC AXIS IN q-DIRECTION.
G7. OR-INC., OPTIC AXIS (90, 30).

G8. OR-INC., OPTIC AXIS (90, 45).
69

69. DR-INC., OPTIC AXIS (90, 60).

610. EX-INC., OPTIC AXIS (90, 70).

Legend

III DR-REFLECTED
□ EX-REFLECTED
○ TRANSMITTED
G11. EX-INC., OPTIC AXIS (90, 45).

G12. EX-INC., OPTIC AXIS (90, 60).
G13. OR-INC., OPTIC AXIS (30, 90).

Legend

- OR-REFLEC.
- EX-REFLEC.
- TRANS.

G14. OR-INC., OPTIC AXIS (45, 90).

Legend

- OR-REF.
- EX-REF.
- TRANS.
**G17.** EX-INC., OPTIC AXIS (45,90).

**G19.** EX-INC., OPTIC AXIS (60,90).
G19. OR-INC., OPTIC AXIS (00, 30).

G20. OR-INC., OPTIC AXIS (00, 45).
G21. OR-INC., OPTIC AXIS (00,60).

G22. EX-INC., OPTIC AXIS (00,30).
G23. EX-INC., OPTIC AXIS (00, 45).

G24. EX-INC., OPTIC AXIS (00, 60).

Legend
- DOR-REF.
- EX-REF.
- TRANS.
G25. OR-INC., OPTIC AXIS (30, 30).

G26. OR-INC., OPTIC AXIS (45, 30).
POWER RATIO

ANGLE OF INCIDENCE

Legend

OR-REF.

EX-REF.

TRANS.

G27. OR-INC., OPTIC AXIS (45, 45).

POWER RATIO

ANGLE OF INCIDENCE

Legend

OR-REF.

EX-REF.

TRANS.

G28. OR-INC., OPTIC AXIS (60, 30).
**G24. OR-INC., OPTIC AXIS (60, 45).**

Legend:
- □ OR-REF.
- □ EX-REF.
- □ TRANS.

**G30. OR-INC., OPTIC AXIS (60, 60).**

Legend:
- □ OR-REF.
- □ EX-REF.
- □ TRANS.
G31. EX-INC., OPTIC AXIS (30, 30).

G32. EX-INC., OPTIC AXIS (45, 30).
**G33. EX-INC., OPTIC AXIS (45, 45).**

**G34. EX-INC., OPTIC AXIS (60, 30).**
G35. EX-INC., OPTIC AXIS (60, 45).

G36. EX-INC., OPTIC AXIS (60, 60).
CHAPTER 5
OVERALL SUMMARY AND
RECOMMENDATION FOR THE FUTURE RESEARCH.

By using Coordinate-Free Approach, introduced by Chen [6], we found that the general solution of the amplitude of the reflected and transmitted waves in the form of some constants and dot product between the optic axis and some other vectors. This will enable us to analyze and draw the picture of what's going on when we have the given incident wave and know the direction of the optic axis of the system.

First, we assumed that we can have two reflected waves, one reflected ordinary wave and the other reflected extraordinary wave, at the interface. Later on we found out that at some particular orientation of the optic axis we can have only one reflected wave, and this reflected wave is the same kind as the given incident wave, i.e., if the given incident wave is ordinary wave we can obtain only reflected ordinary wave but, if the given incident wave is extraordinary wave we can obtain only reflected extraordinary wave.

$$\mu_2, \varepsilon_2$$

![Diagram](image.png)

Figure 5.1 Spliting the guided wave by changing the guide directional.
Also this phenomenon is valid for the normal incident with any arbitrary orientation of the optic axis. It allows the idea for the application that we can split the given incident wave into two waves, one is ordinary wave and the other is extraordinary wave by changing the orientation of interface (See Figure 5.1), i.e., first we control the given incident wave to propagate at the single reflected wave (by the particular orientation of the optic axis) then we change the direction of the guiding interface back to let it propagate at the single reflected wave. In this case we will have two kinds of waves that can propagate with different phase velocities.

Figure 5.2 The future possibility for the research in the parallel plane.

There also would be another future possibility for the research to study on condition to guide wave, for instant in the wave guide as shown in Figure 5.1 or in the parallel plane as shown in Figure 5.2, etc.
REFERENCES:


APPENDIX A
COMPUTER PROGRAM

This program computes the amplitude of the reflected and transmitted wave, the ordinary and extraordinary reflectivity, and the transmissivity at the uniaxial-isotropic interface with the variation of the angle of incidence. The following parameters used in this program are defined as:

EPER, EPAR and U1 are the repeated and nonrepeated diagonal elements of the permittivity, and permeability constant of the first medium, respectively.

E2 and U2 are the permittivity and the permeability of the second medium.

ZIN, ZTA and APHI are, respectively, the angle of incidence, the angle $\theta$ and $\phi$ of the optic axis as given by Eq.(3.44). (Also See Fig. 3.7)

AK1, AK1, AK2 and AKT are the wave number of incident, ordinary reflected, extraordinary reflected and transmitted wave, respectively.

ZTA1, ZTA2, ZTAT and ZINC are the angle of ordinary reflected, extraordinary reflected, transmitted waves and critical angle, respectively.

B1, B2, CPER and CPAR are the amplitude of the reflected and transmitted waves as defined by Eqs (3.51)-(3.54).

API, AP1, AP2 and APT are the normal component of the time-averaged Poynting vectors of the incident, ordinary reflected, extraordinary reflected and transmitted waves as defined by Eqs.(3.87)-(3.81).

AP11, AP21and APT1 are the ordinary reflectivity, extraordinary reflectivity and transmissivity, respectively.
THE GIVEN VARIABLES
DATA AX=0.062831857,EPER=4.57,EPAR=4.00,Y0=0.0,YE=1.0,
ZIN=5.02,ETA=45.0,APHI=45.0,ASI=1.0,E=3.067,
JL=1.0,Y0=0.0,ET2=1.0,ET2=1.0
WRITE(6,3)
3 FORMAT (5x,'NEGATIVE SQRT ',3x,'EX-OR1')
WRITE(6,5) ZTA,APHI
5 FORMAT (5x,2(F5.2,3X))
PI = 2.0*ARCCOS(-1.0)
W = 2.0*PI*
Q = 6.0*P-I(15,1,5,-7)
EPSF=(1.0/30.0)*P11P11*P1.E-9
AK1 = AK*SQRT(EPSF2)
ZTAK = 71.0
APHI=APHI

ZIN = ZIN
PRINT ZIN
STOP
CONE TO DEGREES TO RADIANS

ZIN = (P1/180.)*3.141
ZTA = (P1/180.)*ZTAK
APHI=(P1/180.)*APHI

COMPUTE AK1,AK2,ZTA1,ZTA2,ZTAT,ZINC

AK00 = EPP*EPAR*AP21
AK01 = PI*AC - PI*W
AK02 = SIN(PI)*SIN(ZTA)*SIN(APHI)
IF(AK02.LE.0.000013) AK02 = 0.0
AK03 = COS(PI)*COS(APHI)
IF(AK03.LE.0.000013) AK03 = 0.0
AK04 = PI*AK1 + AK02*(AK02+AK03)*(AK02+AK03)

AK1 = AK01*Y*Y*PI*(EP20*AK1) + YE*SQRT(AK00/AK03)

AK05 = (AK05AK)*SFP*EPAR*AP201/(AK1*AK1)/(AK01*AK03)/(AK01*AK03)/(PI)
AK06 = COS(PI1)*COS(APHI)
AK07 = SIN(ZTA)*SIN(PI)*SIN(APHI)*SIN(APHI)
IF(AK07.LE.0.000013) AK07 = 0.0
IF(AK07.LE.0.000013) AK07 = 0.0
AK08 = PI - PI
AK09 = 2.0*PI*(ZTA)*SIN(APHI)*SIN(APHI)
IF(AK09.LE.0.000013) AK09 = 0.0
AK10 = (EPSZ/AK01) + PI
AK11 = AK01 + AK06
AK12 = 2.0*AK02*AK08 + AK07*AK07
AK13 = AK08 + AK07 + AK06
AK14 = AK08 + AK07
AK15 = AK05AK + AK02009 + AK07*AK07
AK16 = AK08 - AK07 + AK08*(AK05AK) - 4.0*AK01*(AK02009)
AK17 = AK05AK
ZINC = AK05AK/(AK1*AK1)

ZTAK = ZIN
PRINT ZIN
STOP
CHANGE RADIAN TO DEGREE TO PLOT GRAPH

\[
\begin{align*}
\theta_1 &= 180.0 \cdot \sin(\theta_1) \\
\theta_2 &= 180.0 \cdot \sin(\theta_2) \\
\theta_3 &= 180.0 \cdot \sin(\theta_3) \\
\theta_4 &= 180.0 \cdot \sin(\theta_4)
\end{align*}
\]