DESIGN AND EVALUATION OF A CORRECTIVE MEASURE
FOR STUDENTS' DEFICIENCIES IN
BASIC ENGINEERING CALCULUS

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# TABLE OF CONTENTS

**Chapter**

1. **INTRODUCTION** ................................................. 1
2. **STATEMENT OF PROBLEM** ................................. 5
3. **LITERATURE REVIEW** ..................................... 10
4. **METHODOLOGY** ............................................... 36
5. **RESULTS AND DISCUSSION** ............................. 57
6. **RECOMMENDATIONS** ........................................ 99

**BIBLIOGRAPHY** .................................................. 101

**APPENDIX** ...................................................... 106

A. **USER'S MANUAL**  
B. **HYPOTHESIS TESTS**  
C. **PROGRAMMED INSTRUCTION**  
D. **TESTS**  
E. **REVISED PROGRAMMED INSTRUCTION**  
F. **QUESTION BANK**
LIST OF Tables

Table

1. Test 1 - statistics .................................................. 63
2. Test 2 - statistics .................................................. 64
3. Test 1 - levels of learning vs percent correct responses ..................................... 73
4. Test 2 - levels of learning vs percent correct responses ..................................... 74
5. Final test -- pre-test statistics ............... 82
6. Final test -- post-test statistics ............... 83
7. Final test -- pre-test vs post-test statistics .......... 84
8. Final test -- pre-test - levels of learning vs percent correct responses 85
9. Final test -- post-test - levels of learning vs percent correct responses 86
10. Final test - levels of learning vs percent correct responses .................................. 87
LIST OF FIGURES

Figure

1. Test 1 -- Levels of learning vs percent correct responses ....................... 75
2. Test 2 -- Levels of learning vs percent correct responses ....................... 76
3. Final test - Levels of learning vs percent correct responses ..................... 89
4. Final test - pre-test vs post-test - Knowledge ... 90
5. Final test - pre-test vs post-test - Comprehension 91
6. Final test - pre-test vs post-test - Application . 92
7. Final test - pre-test vs post-test - Analysis .... 93
CHAPTER 1
INTRODUCTION

Virtually all engineering disciplines, demand a strong background in basic calculus. Basic calculus includes limits and functions, infinite series, the differentials, and the indefinite, definite and multiple integrals. The student is expected to possess not only manipulative or computational skills but also a sound understanding of the principles, constraints and contextual issues inherent in the content of the course (Trollip S.R. & Lippert R.C. 1987). In other words, a clear understanding of the concept and the theory behind a principle or generalization is essential; not just memorization of the definitions and formulas. More specifically, the student should be able to (1) recall or recognize the principles and rules studied, (2) use the appropriate information from memory to tackle a specific situation, and (3) discern a relationship between a new situation and a familiar problem. The afore-mentioned objectives are known as knowledge and comprehension, application, and analysis respectively. These learning outcomes are part of a comprehensive set of educational objectives categorized by Bloom (A Committee of College and University Examinees, 1956).

Bloom classified the educational objectives into three domains - cognitive, affective and psychomotor. He further
classified these domains into sub-areas. The cognitive domain is comprised of knowledge, comprehension, application, analysis, synthesis and evaluation. The affective domain deals with the interests and attitudes of the learner. These are intangible and difficult to measure, hence this domain is not usually considered for testing. The psychomotor domain deals with the small and large muscle skills, and coordination. The testing of these skills is not appropriate for a mathematics course.

The definitions of the levels of learning under the cognitive domain are slightly modified to suit the objectives of this investigation. Knowledge is the remembering, either by recognition or recall, of ideas, material or principles learned earlier. Comprehension is the understanding of a message or an idea. The learner reorganizes the idea and stores it in memory in a form meaningful to him/her. He/she should be capable of demonstrating its use. Application is a step above comprehension. It is the recall of the right idea or method which applies to the given problem. Analysis relates to the nonroutine application of concepts. It involves decomposing the material into elements, discovering relationships among them, and recognizing the way in which they are organized. Synthesis and evaluation, the highest two levels of learning are not tested in this study since they are usually too
complex to be achieved at the undergraduate level of education.

If a student does not exhibit sufficient skill in at least the lower three levels of learning with regard to the different subject areas in basic calculus, it will be very hard for him/her to fully comprehend the concepts involved in the various engineering classes. Hence, a need exists to determine in which areas and at what levels of learning deficiencies exist so that corrective measures can be developed.

The typical evaluation methods, which report the total scores on an achievement test, determine only the overall capability of a student. Such evaluation techniques do not help the student or the teacher in determining the specific problem area i.e., in what particular subject area or at what particular learning skill, the student is weak. Hence, an effective evaluation method needs to be developed to detect specific deficiencies. This would help in designing a corrective method based on the deficiency.

Some of the available corrective measures include programmed instruction, computer programmed instruction, audio-tutorial instruction and video instruction. In programmed instruction the entire subject matter is broken into small units and presented in logical sequence. The student learns progressively by reading through the program. Computer programmed instruction is basically programmed
instruction where the presentation of the material is done on a computer screen instead of in print. Audio-tutorial instruction is quite similar to programmed instruction, but for the method of presentation. In this method a tape recorder is used in place of print. Video instruction uses overheads, slides, films or a video tape to present the programmed material, thus providing stimuli through both visual and auditory senses.

To summarize, there is a need to (1) design a test which identifies the specific areas in basic calculus where a student may have a deficiency, (2) develop corrective measures to rectify the above deficiencies, and (3) verify the adequacy of the corrective measures. This is the focus of this research.
CHAPTER 2

STATEMENT OF PROBLEM

The three main objectives of this investigation are:

1. the design of a test which identifies the specific deficiencies existing at the different levels of learning in each subject area.

2. the design of a corrective measure(s) based on the type of deficiency.

3. the design of a test which checks the effectiveness of the above corrective measure(s).

The first objective of this investigation involves identifying the subject areas and the levels of learning to be tested, the type of test to be used, and the format of the test.

The major subject area and sub-topics, in which the student is to be tested, are chosen based on the requirements of the introductory courses in the engineering program. They are:

1. Limits and Functions
   - L'Hôpital's rule
   - Indeterminate forms such as \(0^0, \infty^\infty, 1^-
   - Continuity and discontinuity of functions
   - Increasing/Decreasing functions
(2) Differentiation

-- Relationship between continuity and differentiability of a function.
-- Product rule and quotient rule
-- Chain rule
-- Power rule
-- Trigonometric functions
-- Logarithmic and exponential functions
-- Inverse functions
-- Partial derivatives
-- Higher order derivatives

(3) Infinite series

-- Power series representation of a function
-- Taylor series
-- Maclaurin series

(4) Integration

-- Indefinite integral
-- Method of substitution
-- Definite integral
-- Fundamental theorem of integral calculus
-- Mean value theorem
-- Trigonometric functions
-- Power rule
-- Logarithmic and exponential functions
-- Integration by parts
-- Trigonometric substitution
-- Partial fractions
-- Discontinuous integrands
-- Area bounded by curves
-- Double integral
-- Iterated double integral

The different levels of learning that will be tested in each of the major subject areas listed above are knowledge, comprehension, application and analysis. These levels are sub-divided as follows:

(1) Knowledge

- knowledge of specific facts
- knowledge of terminology
- ability to carry out simple algorithms
(2) Comprehension

- knowledge of concepts
- knowledge of principles, rules and generalizations
- knowledge of mathematical structure
- ability to transform problem elements from one mode to another
- ability to follow a line of reasoning
- ability to read and interpret a mathematics problem

(3) Application

- ability to solve routine problems
- ability to make comparisons
- ability to analyze data
- ability to recognize patterns, isomorphisms, and symmetries

(4) Analysis

- ability to solve nonroutine problems
- ability to discover relationships
- ability to construct proofs
- ability to criticize proofs
- ability to formulate and validate generalizations

Synthesis and evaluation are the highest and most complex forms of learning outcomes. It is difficult to measure these learning outcomes in achievement tests (Brightman, 1984). In addition, the focus of this investigation is not to test the complex learning levels. Hence these two levels of learning are not considered.

The types of tests most commonly used are essay, short answer, and objective, with various advantages and limitations. The formats of these tests depend on the objectives of the subject area tested.

The second objective of this investigation, is the design of a corrective measure. This corrective measure is based on the nature of the deficiency, i.e. the learning
level and the subject area in which the student lacks skill, as determined by the pre-test. Programmed instruction, either in print, or on a microcomputer, audio-tutorial instruction, and video instruction, are some of the corrective measures available.

In programmed instruction, information is presented to the student in units, or frames, as they are usually called. The student works through the programmed instruction, answering the test questions given after a few frames. He/she receives immediate feedback as to the correctness of his/her responses and thus learns the subject matter progressively. The material can be presented either in print or on a microcomputer screen.

Audio-tutorial instruction involves no visuals. The same information which is presented in print in the programmed instruction method is presented by using audio devices such as tape recorders.

Video instruction includes slides, overheads, films and video tapes. The subject matter is presented in small units, as in the programmed instruction. These units are presented on overheads or slides and are assisted by commentary, thus providing both auditory and visual stimuli to the viewer. The entire lecture is recorded on video tape for further viewing.

The final objective is the design of a retest or a post-test to determine how effective the corrective
measure(s) is(are) in rectifying the deficiencies at the different learning levels in each subject area. All the subject areas and the learning levels covered in the earlier test (pre-test) must be included in the post-test. The post-test must be similar in difficulty to the pre-test so as to enable comparison of performances of students between the two tests.

In brief, the objectives of this investigation are the design of a diagnostic test for testing the required topics in basic calculus; the determination of specific deficiencies for each student in each learning level and subject area, through the results of the pre-test; the development of corrective measures for each student to help correct these weaknesses; and finally, the design of a post-test to test how well the corrective measure performed.
CHAPTER 3

LITERATURE SURVEY

This investigation requires the selection of a test format, an evaluation procedure, and a corrective method for instruction in basic calculus. Empirical results provide some guidelines on how to make this selection. The literature regarding test formats, evaluation procedures and corrective measures is discussed in the following sections.

TEST FORMATS

There is no one best form of test which is ideal for all purposes. Based on the complexity of learning objectives and the subject content to be tested, different forms of tests are used. The two major types of tests are essay and objective.

Essay test

The essay test requires the examinee to write the answers in own words or to work out a problem step by step, as in the case of a mathematics course. The different forms of essay tests as listed by Monroe and Carter are discussion, criticism, comparison, summary, outline, problem solving, construction of proofs, etc. (Bennion, 1973). The chief characteristic of all these forms is - they emphasize
the higher levels of learning, such as synthesis, analysis, and evaluation rather than the lower learning levels, such as knowledge, comprehension and application.

Objective test

In objective tests, the student is not required to construct own responses, but must choose a response from among the several options provided for each item (Furst, 1958). Objective tests are of various types, such as true-false, matching, multiple-choice, and fill in the blanks, each with its own advantages and limitations. Whatever be the form of the objective test constructed, much skill is required in preparing a high quality test (Ebel, 1965). Objective tests can be designed to test any level of cognitive thinking (Bennion, 1971).

Even though both essay and objective tests are used extensively in evaluation, there are many differences between them with respect to the objectives achieved, scoring, versatility, and student attitudes.

Comparison of Essay and Objective tests

1. Essay tests are appropriate for measuring the writing abilities and the ability to select and organize ideas, while objective tests are efficient for measuring knowledge, understanding, thinking skills and other complex outcomes (Gronlund, 1971).
2. Essay tests encourage pupils to organize their own ideas and express them effectively, whereas the objective tests encourage pupils to build a broad background of knowledge and abilities (ETS, 1973).

3. Essay type tests do not measure factual information efficiently. The objective type, in addition to measuring knowledge of facts, can also measure high levels of learning (ETS, 1973).

4. Essay and short-answer questions give a better estimate of level of competency while choice type questions permit a wide sampling in a relatively short period of time (Furst, 1958).

5. Essay and short answer tests are not as versatile as objective tests (Aiken 1987).


7. Students attitudes are more favorable towards objective tests as compared to essay type formats, particularly with respect to perceived difficulty, anxiety, success expectancy, complexity and feeling at ease with the format (Moshe Zeidner, 1987).

TEST FORMAT SELECTION

There are no set rules as to the type of test to use. However a knowledge of the characteristics of the test formats helps in choosing the most appropriate one for the
purpose at hand (ETS, 1973). The following elements may be considered in choosing the most applicable test format for a given purpose - the level of learning measured, the scope of the test, the purpose of the test, the ease of testing, the reliability of the test, and the reusability of the test (ETS, 1973).

1. Level of learning measured: Essay tests measure complex levels of learning more effectively than the lower levels of learning as opposed to objective tests, which measure both levels of learning with equal efficacy.

2. Scope of the test: Essay tests cover only a limited number of topics in any one test, whereas objective tests cover a multitude of topics in one test.

3. Purpose of the test: Essay tests are useful in detecting the overall competency of the student, whereas objective tests are helpful in detecting the specific deficiencies in a student.

4. Ease of testing: Essay tests are easy to construct but difficult to score. In contrast, objective tests are difficult to develop but easy to score, even by a clerk using a score key.

5. Reliability of test scores: Objective tests produce very reliable scores as compared with essay tests, where an element of subjectivity exists in scoring.
6. Reusability of the test: Essay tests are used in places when the test may not be reused as opposed to objective tests, where the same test can be used several times (Ebel, 1965).

Some other factors which determine the type of test to be used are: the nature of the pupil population, the readiness, willingness, and ability of the school staff to administer tests and interpret scores, and the amount of time and money available for testing (Katz, 1972).

The above discussion indicates that for this investigation, where a wide variety of topics in basic calculus is to be tested, objective tests seem to be more appropriate than essay tests. Hence, objective tests are discussed in greater detail in the following paragraphs.

There are several formats for objective tests. These include true-false, matching, multiple-choice, and fill in the blanks.

True-false items present a simple statement and the examinee is required to indicate if it is true or false. True-false tests are the least favored of the objective tests because they are uncritical, susceptible to chance error due to guessing, encourage rote memory (Ebel, 1965) and test only the lower levels of cognitive behavior (Bennion, 1973).

Matching tests present several problems in one column which are to be matched with answers in another column. All
the items should be part of a major idea and closely related. The difficulty lies in collecting homogenous material with related ideas (Gronlund, 1971).

Multiple-choice tests present a question with two or more answers from which to choose the correct response(s). The incorrect answers must be carefully chosen so as to be seemingly correct, thus misleading the examinee with superficial knowledge.

Fill-in-the-blank or completion items present part of an idea and require the examinee to supply the rest of it.

Among these objective test formats, the multiple-choice type is favored by both teachers and students (Aiken, 1987). The several arguments in favor of multiple-choice tests are:

1. Multiple-choice tests examine more subject areas in one test and are more efficient to administer and score than other formats (Aiken, 1987).

2. A multiple-choice test is more reliable than a true-false test since the chances of guessing the correct answer are reduced in the case of the former (Gronlund, 1971).

3. Multiple-choice tests provide more analytic data than the true-false test and provide the basis upon which judgments were made (Furst, 1958).

4. Multiple-choice questions are less ambiguous than completion or true-false items (Ebel, 1965).
5. The difficulty of collecting homogenous material which includes a series of related ideas, as in the case of constructing matching items does not exist in the case of multiple-choice items (Gronlund, 1971).

Since all these features of multiple-choice tests met the design criteria of the test format to be used in this investigation, the multiple-choice test was chosen.

MULTIPLE-CHOICE TESTS

A multiple-choice test is not just a collection of questions with each question having more than one response from which to choose. A great deal of planning has to be done to prepare a competent test.

The "art" of test-item preparation depends primarily on knowledge of the subject matter, an understanding of what students should know and are not likely to know about the subject, and skill in asking and writing items (Lewis R. Aiken, 1987). Besides a knowledge of the subject, a knowledge of how the subject is taught at the educational level for which the question is intended, is required for writing a multiple-choice question (ETS, 1973).

The quality and effectiveness of the test may be considerably influenced by the following factors:

1. selection of distracters
2. the number of options provided for each question
3. use of "none of the above" and "all of the above" options
4. the level of
difficulty of the test. Decisions regarding these aspects are based on empirical results of tests conducted by researchers.

(1) Selection of distracters: The primary factor in determining the effectiveness of multiple-choice items as measures of ability is the selection of distracters (Weitzman & McNamara, 1946). Distracters are choices seemingly correct, provided to mislead the pupil who does not know the answer. The answer choices should be provided in such a way that students who have misinformation or misconceptions will be differentiated from those who are knowledgeable (ETS, 1973). All the distracters should be relevant to the test-item and unnecessary cues, either due to absurdity of the distracters or the length of the distracters, should not be present. There are principally two methods used in selecting distracters for an item: (1) responses from earlier tests which are not chosen by any student are thrown out while the remaining options are possible distracters; and (2) individual judgment can be used. Individual judgment is a better method of selecting distracters than the former method (Hanna and Johnson, 1978).

(2) Number of options provided: There is no hard and fast rule as to the ideal number of choices/options to be provided for each item. There is no difference in mean item difficulty, mean discrimination, or total test score between
a 3-option item test and a 5-option item test (Steven V. Owen & Robin D. Froman, 1987). There is no decline in the effectiveness of the test from a statistical viewpoint, when the number of distracter are changed from four to three (Costin, 1972). But as the number of choices decreases, the probability of correctly guessing the answer increases. In addition, too many options are also not advisable since they make the test too long. These arguments indicate that four is a reasonable number of alternatives on a multiple-choice test. It is a compromise between a 3-item test and a 5-item test.

(3) Use of none-of-the-above and all-of-the-above options: The next feature to be considered regarding multiple choice tests is the suitability of "none of the above", and/or "all of the above" options. In many cases, options like "none of the above" and/or "all of the above" do not serve any useful purpose on a test. The test format which uses "none of the above" are much more difficult than the one correct answer format (Nona Tollefson, 1987). Lewis R. Aiken (1987) also shares the view that options such as "all of the above," "none of the above," "two of the above," "all but one of the above," etc. can make the multiple-choice test more complex. In a mathematics test, an occasional use of these options may make the examinee solve the problem completely.

(4) Complexity of the test: The complexity of the test can be increased by using all/none of the above options, by
giving more than one correct answer, or by asking the examinee to choose the best answer from several correct answers. There are many other ways to make the test more difficult. Items with highly similar alternatives are difficult even when the individual has sufficient mental capacity and the essential knowledge (Maier, N.R.F., 1970). Item difficulty can be manipulated through homogeneity of options (Kathy Green, 1984). Therefore, depending on the target group, and the objectives of the course, the above techniques can be used to alter the complexity of the test.

The afore-mentioned literature gives some background on the type and structure of the test that best achieves the educational objectives of a mathematics course.

LIMITATIONS OF MULTIPLE-CHOICE TESTS

Multiple choice tests are not flawless. Some of the criticisms made by researchers are:

1. Overuse of these types of tests retards the reading and writing skills of the students (David, 1981), hence a mixture of objective and essay tests are good when measuring the attainment of educational objectives (Aiken, 1987).

2. Banesh and Hoffman (1962) claim that multiple choice tests penalize the more subtle, creative, profound person, while favoring the shrewd, nimble-witted, rapid reader. However, this is refuted by Alker, Carlson and
Herman (1961) who found that test-wiseness was positively related to scores on multiple-choice tests.

3. The act of responding to multiple choice items can condition examinees to the ensuing response, even when the response is incorrect, and the more intelligent examinees tend to be more susceptible to such conditioning (Preston, 1965). David (1985) states that multiple-choices offer a form of help to students that guide their thinking to certain lines, but can be misleading and often result in ambiguous choices as a result of poor menu construction.

4. Multiple choice test, or any other test results do not always reflect the true skills of the examinee, because they depend on so many extraneous factors. Test performance may be influenced by many factors such as amount and quality of certain kinds of training, distractions during testing, sensory defects, inability to hear instructions because of poor administration, inappropriate language in instructions or in the test, inability to read, brain damage, motivation level, illumination level, cultural background of the examinee, or test-taking strategies (APA, 1974).

Many of the limitations of the multiple-choice tests can be overcome by constructing the test with care, adhering to the design principles, and creating good testing conditions. Multiple-choice tests are increasingly used in
testing, even as the search for a perfect test format is going on simultaneously (Aiken, 1987).

EVALUATION OF THE TEST

There are several methods for scoring a test. Some of the most common methods are correction-for-guessing formulas, number right formula, response weighted scoring, and elimination scoring.

Correction-for-guessing formulas developed by Reid and Little yield over corrected scores when examinees are less familiar with the test material and under corrected scores when they are more familiar with it (Aiken, 1987). Further, when guessing is penalized, there is an increase in the number of omitted items (Waters and Waters, 1971) because there is a danger that the examinee might not attempt to answer a question about which he is even slightly unsure. This could result in a high risk taker getting affected significantly as compared to a low risk taker.

In the number right formula, as the name indicates, the final score is the number of items answered correctly. Here guessing is not penalized.

In response weighted scoring, each item and each distracter for every item is given a different weighting and so based on the correctness of a response, either full or partial credit is given. This encourages the examinees to attempt every item. However, scoring differentially
weighted items is expensive and time consuming and not recommended (Aiken, 1987).

In elimination scoring, examinees respond by indicating which options are wrong. Correct and incorrect options are scored separately and the resulting scores are given equal importance in the calculation of the final score. Both the weighted scoring and elimination scoring are not very reliable or valid when compared with the conventional scoring method (Aiken, 1987). All things considered, the conventional number right formula is the best overall approach to scoring multiple-choice tests (Aiken, 1987).

Reliability of the test results is another characteristic which must be considered. Reliability is defined as the ability of a test to be consistent in measuring student skills or knowledge. If a test is reliable a student's score, compared to that of the other examinees, should be similar to his relative score on another test measuring the same information (Bennion, 1973). High test reliability can be achieved by (1) narrowing the focus of the test and attaining high homogeneity (Humphreys, 1967), (2) allowing several or all options of a given item to be correct (Willson V. L. 1982) and (3) decreasing the number of options (Kathy Green, Gilbert Sax et al, 1982).

Test reliability may be estimated by the following methods - (1) Test-retest method (2) equivalent-forms method (3) split-half method and (4) Kuder-Richardson method. The
first two methods measure the stability and equivalence respectively of two test forms, whereas the last two measure the internal consistency of the test.

In the **Test-retest** method, the same test is administered to the same individual after a lapse of time and the extent to which the score fluctuates is determined. There are several drawbacks with this method. Firstly, the scores are influenced by extraneous elements if the interval is too long and by practice effects if it is too short. Secondly, the test conditions may vary between the two tests and may result in a large variance in the scores (Fogiel, 1984).

The **equivalent forms** method uses two equivalent forms of the test which are given to the same group in close succession and the resulting scores are correlated (Gronlund, 1971). The problems faced in the test-retest method are reduced in this method, though not eliminated completely. The difficulty involved in preparing two equivalent tests in terms of subject content, difficulty level, maintaining the same test conditions for both the tests; and the fatigue experienced by examinees in taking two tests in quick succession are some of the problems encountered by the equivalent forms method.

The **split-half** method divides the test into halves. The two scores, when correlated, give a measure of internal consistency (Kuder & Richardson, 1967). The reliability
coefficient is calculated using the Spearman-Brown formula

\[
\text{Reliability on full test} = \frac{2 \times \text{Reliability on 1/2 test}}{1 + \text{Reliability on 1/2 test}}
\]

(Gronlund, 1971).

This method gives high estimates of the reliability coefficient and a coefficient value which is not unique. The latter occurs because there are \( \frac{n!}{2 \left(\frac{n}{2}\right)!^2} \) ways of splitting a test of \( n \) items into halves and each of these give a different estimate of the reliability coefficient. Thus there are large fluctuations in the reliability coefficient based on the manner in which the test is split (Kuder & Richardson, 1967).

The Kuder-Richardson method also employs a single form of test but it does not require splitting the test. The reliability coefficient is calculated using the Kuder-Richardson formula:

\[
\text{Reliability Coefficient} = \frac{k}{k-1} \left[1 - \frac{\Sigma pq}{\sigma^2}\right]
\]

where
- \( k = \text{number of items in the test} \)
- \( p = \text{proportion of the responses to one item which is correct} \)
- \( q = \text{proportion of the responses which is not correct} \)
- \( \sigma^2 = \text{variance of the scores on the test} \)
A simplified version of this formula is:

\[
\text{Reliability Coefficient} = \frac{k}{k-1} \left[ 1 - \frac{M (K-M)}{Ks^2} \right]
\]

where 
- \( K \) = the number of items in the test
- \( M \) = the mean of the test scores
- \( s \) = the standard deviation of the test scores

(Gronlund, 1971).

When there is moderate or no variation between the items of the test, the above formulas give the same value for the reliability coefficient. For a test where items vary widely the latter will underestimate the reliability (Ebel, 1965).

Anastasi (1976) claims that this method is applicable to tests whose items are scored as right or wrong, while Gronlund (1971) states that this method is not applicable for speeded tests where all the examinees cannot finish the test in the stipulated time.

The reliability of a test may also be determined by using a variance technique as described by Hoyt (1967). This technique uses analysis of variance and gives the same results as the Kuder-Richardson method. The scores of the examinees are tabulated item-wise and the following formulas are used to determine the reliability coefficient.
The sum of squares "among students" is
\[ \sum_{i=1}^{k} \left( \frac{\sum_{i=1}^{n} t_i^2 - \sum_{i=1}^{n} p_i^2}{nk} \right) \]

The sum of squares "among items" is
\[ \sum_{i=1}^{k} \left( \frac{\sum_{i=1}^{n} p_i^2 - \sum_{i=1}^{n} t_i^2}{nk} \right) \]

The total sum of squares is
\[ \sum_{i=1}^{k} \left( \frac{(\sum_{i=1}^{n} t_i)^2 - (\sum_{i=1}^{n} t_i)^2 - (\sum_{i=1}^{n} p_i)^2}{nk} \right) \]

where \( t_i = \) total score of an individual
\( p_i = \) total score for an item
\( n = \) number of examinees
\( k = \) number of items on the test

By subtracting the "among students" and the "among items" sums of squares from the total sum of squares, the residual sum of squares is obtained which is used as the basis for estimating the discrepancy between the obtained variance and the true variance. The coefficient of reliability of the test may be expressed as the ratio of the difference of "among students" and "remainder" variances to the "among students" variances.
This method of estimating the reliability of a test gives a better estimate than the split-half method (Hoyt, 1967).

Reliability is a necessary, but not a sufficient, condition for a test to have validity (Gronlund, 1971). Even though the reliability indicates the accuracy of the test, it has two shortcomings; (1) it depends on the spread of the scores to a great extent and (2) it does not help in evaluating the individual scores (Doppelt, 1972). The standard error of measurement does not have these shortcomings. The standard error of measurement is the estimate of the amount of variation to be expected in test scores due to chance. Reasonable limits can be established for an individual's true score using this statistic. The standard error of measurement is found by multiplying the standard deviation of the scores and the square root of the difference between one and the reliability coefficient (Ebel, 1965). The standard error of measurement has two special advantages. Firstly, the estimates are in the same units as the test scores, which make it easy to interpret individual scores and secondly, the standard error is likely to remain fairly constant even as the group changes (Gronlund, 1971).

The reliability coefficient is a better measure, when comparing the reliability of different tests while the
standard error of measurement is more appropriate when interpreting individual scores (Anastasi, 1976).

CORRECTIVE MEASURES

Corrective instruction is given for the purpose of remedying deficiencies in skill mastery that are interfering with adequate achievement (Otto & McMenemy, 1966). Programmed instruction in print, computer programmed instruction, audio-tutorial instruction, and video instruction are some of the several corrective measures available.

Programmed instruction, either in print or computer-based, is the presentation of a large text of material in small units, so that the student learns progressively. Wilbur Schramm (1962) summarized the elements of programmed instruction as:

1. an ordered sequence of stimulus items
2. to each of which a student responds in some specified way
3. his response being reinforced by immediate knowledge of results
4. so that he moves by small steps
5. therefore making few errors and practicing mostly correct responses
from what he knows, by a process of successively closer approximations, toward what he is supposed to learn from the program.

In other words, programmed instruction is the process of arranging materials to be learned in a series of small steps designed to lead a student through self-instruction from what he knows to the unknown of new and more complex knowledge and principles (Lysaught and Williams, 1963).

In audio-tutorial instruction, the print media is substituted by an audio media, for example, a tape recorder. S. N. Postlethwait at Purdue University developed the tape controlled independent study in the early 1960s. The students listen to a taped presentation which sounds more conversational than a lecture. Learners proceed at their own pace. Individualization and personalization are critical elements of this sort of system (Heinich et al., 1982). Some of the shortcomings of an audio instruction are (1) the lesson content is fixed and every time the material is revised, new copies have to be made (2) long sessions could be boring (3) development of quality scripts can be time consuming and require specialized skills and (4) it may not be possible to describe the computations involved in a mathematics course (Anderson, 1983).

Video instruction is an audio-visual presentation of the programmed instruction material. This form of presentation helps attract the attention of the learner
better than any other media. In other words, it has a captive audience. It presents the same information to various sized audiences simultaneously and also produces self-paced instruction for individualized study (Anderson, 1983). Even though video instruction is very effective in providing the necessary instruction, there are several drawbacks associated with it. According to Anderson (1983) these drawbacks are:

1. video script writing is difficult and time consuming.
2. cost of producing a video is high and much expertise is required.
3. the amount of lettering for video is limited to about one-half that of film or still visuals.
4. rapid changes in technology make obsolescence of video systems a continuing problem.

In each of these methods of instruction, the basic technology of instruction is programmed instruction. Only the presentation media differ, each with its own advantages and limitations.

Programmed Instruction

Individual differences in performance during learning remain large. Programmed instruction was empirically found to be more efficient than conventional instruction even when individual differences in final ability are substantial.
Programmed instruction has received a tremendously enthusiastic reception from educators (Calvin, 1969).

The most inexpensive media for presenting the programmed text are print and computer. Between these two media, there are several arguments put forth by researchers that information presented in print is as effective as, or slightly better than, the same information presented on a microcomputer screen. Some of them are:

1. For following directions, no differences in comprehension were found between print and microtext (text presented on microcomputer screen). Near-significant differences favoring print were found for informational material, and are not attributable to reading speed (Fish & Feldman, 1987).

2. There are no significant differences in mastery between the computer and programmed-text forms of presentation. There is a time-saving feature connected with the use of programmed textbooks (Goldstein & Gotkin, 1962).

3. Reading from CRT displays can be made as fast as from print by increasing the image quality of the characters and by increasing the resolution of the display; but the trade-offs have not been determined and there are some inherent defects in CRT technology in order to achieve this (Gould et al., 1987).
Besides the mode of presentation, there are many other variables which affect the effectiveness of programmed instruction. Among them are the type of response from the student, the sequence of presentation, and the construction of individual frames, to name just a few.

Some of the findings of several experiments conducted by researchers on the above elements are:

**Type of response:** Experimental findings could not show that a programmed format requiring student responses was superior to a textual format which required only reading (Alter & Silverman, 1962). The further the material is from the repertoire of the student, that is, the higher the information level is, the more important is the requirement of an overt response. Experimental findings suggest that potentially greater information can be received when overt responses are made than when covert responses are made. One obvious explanation for the greater information-reception capacity produced by overt responses is that these responses require more time than covert responses (Eigen & Margulies, 1963). A 305-frame semi-technical linear program given to AFROTC cadets showed that (1) those who wrote the program responses did far better on an immediate recall test than those who only read the program and (2) after ten weeks, retention dropped severely, and the difference between those who wrote responses and those who did not is greatly reduced (Cartier, 1963).
Model used: There are two major models of programming, the linear programming model and the flip programming model. There are a number of other techniques which are modifications of these two. Linear programming guides the students through one and only one path. The frames are sequentially arranged so that a student cannot go to the following frame until he finishes the preceding frame. In the flip programming approach, students branch to different segments or branches of the program, depending upon their responses to the questions at the end of each frame. Calvin (1969) lists the several advantages of linear programming when compared with flip programming as follows:

1. The flip or branching program is less enjoyable.
2. A good linear program is challenging and the challenge comes through trying to get the correct answers fast.
3. For the branches to be of any use, some students must get to them; and the only way to get to the branches is to make mistakes. A serious drawback of this is that making an erroneous response imprints the response on the student's mind and there is a high probability that the same erroneous response will be produced when the question is asked again. In addition to this, a number of incorrect responses shatters the learner's confidence. A linear program does not have this disadvantage.
4. A linear program helps by providing positive reinforcement.

Gray's (1987) findings that students in the linear sequence condition performed better on a retention measure than those in the flip sequence condition strengthens this argument in favor of linear programming.

Order of frames: Another feature of the programmed instruction is the order or the sequence of frames. The order or arrangement of the frames can be made to follow a RULEG system, or a mathetics system, or any modification of these. In the RULEG system, the entire subject material is written in the form of rules and examples and then arranged in a logical sequence. After all the rules and examples are collected, several variations can be tried in the arrangement. The student solves the examples by following a few rules and then composes own formulations of the appropriate rules for each example, resulting in inductive learning (Lysaught & Williams, 1963).

Mathetics is a "training system which provides the programmer with a set of procedures with which to diagnose training problems." (Callender, 1969). Based on the diagnosis, a decision is made on what, whether and how to program.

Between these two sequencing systems, the RULEG system is preferred over the mathetics system for several reasons.
(1) Mathetics is too involved. The method is more exhaustive than is normally needed in classroom instruction (Lysaught & Williams, 1963). (2) The writing of a mathetics program is a very complex task, confounded by remarkably difficult terminology and associated mystique (Hartley, 1972). (3) The urgencies of time and money are strong arguments in favor of the RULEG system, since it is very effective as well as efficient (Lysaught & Williams, 1963). (4) In the RULEG system, there is a learning advantage in teaching the rule followed by examples (inductive approach). Students generally prefer the inductive approach over the deductive approach, since they are more accustomed to this method of learning (Hartley, 1972).

It can be inferred from the previous discussion that a linear program model with the frames arranged according to RULEG system is the most applicable style of programmed instruction for a course which is intended to improve the quantitative abilities.

With the present state of the art of programmed instruction, short programs to fit a specific curriculum and/or pupil need are more successful and create fewer problems than do full-year programs (Welch, 1964). Hence, a programmed instructional material written to address specific pupil needs is designed, rather than a comprehensive program spread over a period of time.
CHAPTER 4

METHODOLOGY

Design challenges

The objective of this project is the design of a system which identifies and corrects problem areas for individual students in basic calculus. A systems design approach is used. The various steps are:

1. Identification of the problem
2. Development of goals for the corrective action
3. Design of a corrective action
4. Evaluation of the corrective action.

Identification of the problem involves the determination of specific problem areas for students in the fundamentals of basic calculus. Based on the nature of the problems identified and the goals of the system, a suitable corrective action is designed. Finally, an evaluation of the corrective action is done to assess its effectiveness in rectifying the deficiencies of the students. This process is repeated until the student demonstrates an understanding of all areas in basic calculus.

Step 1: Identification of the problem

The identification of the problem is done by means of a test in calculus. To construct a test, it is first
necessary to determine both the subject areas to be tested and the appropriate levels of learning.

The determination of the appropriate subject areas and levels of learning are based on discussions with several members of both the engineering and the mathematics faculty. These discussions reveal the nature and depth of knowledge required for an engineering student in each subject area of calculus. Depending upon the extent to which they are used in the introductory and higher level engineering courses, different sub-areas are given different priorities. For example, concepts and rules such as L'Hôpital's rule, discontinuity of a function, integration by parts, evaluation of double integrals, and differentiation and integration of logarithmic and exponential functions are given more emphasis than other sub-areas. For a detailed list of the major topics and sub topics, please refer to the "Statement of the Problem." Additionally, it is determined that a greater emphasis has to be placed on computational skills rather than on the ability to memorize formulas and theorems. In other words, of the six levels of learning discussed in the literature review, comprehension and application are to be given a higher priority than the other learning levels.

After the subject areas and the learning levels are identified, a table of specifications is designed (See appendix A). A table of specifications contains the
subject matter and the levels of learning that an instructor considers important for a unit of the course. On one axis of the matrix-like table of specifications are the important sub topics and on the other axis are the different levels of learning. Each cell in the table represents one sub area and one level of learning. Having formed a comprehensive table of specifications containing a list of what is to be tested and at what level of difficulty, the next step is to determine the appropriate test format.

Design criteria for the test format

The major design criteria for determining the test format are:

1. The test should encompass a wide variety of subject areas and different levels of learning.
2. The test should assess both recall and recognition of the correct answer.
3. The test should provide for easy, rapid, and objective scoring.
4. The test should lend itself easily to statistical analyses of the scores.

Multiple-choice tests

A multiple-choice test meets the above criteria better than any other single test type. A multiple-choice item (question) consists of a problem statement, in the form of a
direct question or an incomplete statement, and a list of possible solutions. The problem statement is called the stem of the item and the solutions are called alternatives. The correct alternative is the answer and the other alternatives are distracters. Distracters are intended to "distract" those pupils who are in doubt about the correct answer.

Some of the reasons for selecting a multiple-choice format are listed below:

1. The multiple-choice item is less vague and more certain than other types of test items.

2. The pupils' knowledge and comprehension of several different subject areas can be tested in a short time, without worrying about the homogeneity of the test material.

3. A major advantage of multiple-choice items is its wide applicability in measuring various phases of achievement.

4. They are free from response sets. That is, pupils generally do not have a tendency to favor a particular alternative when they don't know the answer.

5. They set up a forced-choice situation which requires that the pupil demonstrate the specific ability called for by each item. The pupil cannot deliberately avoid coming to grips with the question, preferring to conceal his/her ignorance or ineptitude.
6. They permit rapid, easy, and highly objective scoring.
7. A careful analysis of the results gives the evaluator an idea as to the type of mistakes commonly committed. This helps in the development of remedial or corrective actions.
8. They lend themselves more readily to statistical analyses.

Design principles

The advantages of a multiple-choice test are realized only when the test is designed with a great deal of care (Gronlund, 1971). The following design principles, as formulated by Trump et al. (Trump, 1952), serve as a useful guide for this purpose. Most of these rules are governed by logic or common sense.

1. The stem of the item should be meaningful by itself and should present a definite problem.
2. The stem should include as much of the item as possible and should be free of irrelevant material.
3. A negatively stated item should be used only when significant learning outcomes require it.
4. All alternatives should be grammatically consistent with the stem of the item.
5. The alternatives should be reasonably similar.
6. The alternatives should be as brief as possible.
7. All distracters should be plausible.
8. An item should contain only one correct or clearly best answer.

9. Verbal association between the stem and the correct answer should be avoided.

10. The relative length of the alternatives should not provide a clue to the answer.

11. The correct answer should appear in each of the alternative positions approximately an equal number of times, but in random order.

12. Special alternatives such as "none of the above" or "all of the above" should be used sparingly.

Actual design of the test

The design of the multiple-choice test used in this study is based on the design principles discussed in the previous section. The table of specifications (see appendix A) already described is used for determining the content and complexity of the test.

At least one question is designed for most of the cells in the table of specifications. No questions are designed for some cells because of two reasons - (1) a very low importance is attached to that cell owing to its complexity or irrelevance or (2) it is difficult to write multiple-choice questions for that cell.

The majority of the problem statements (stem) of the multiple-choice test items are of the incomplete statement
form. Though it is difficult to develop this form, it is more concise and easy to understand than the direct question form. The direct question form is used occasionally wherever the incomplete statement form cannot be written.

The stem of an item is written in such a way that it makes sense without the help of the alternatives. That is to say, it is complete enough to serve as a short-answer item. Thus, a well written item makes the problem statement clear and also reduces the reading time required. To further improve clarity and increase comprehensibility, and to avoid the possibility of pupils' overlooking words such as, "no," "not," "least" and "similar," the problem statement is written in positive terms where possible. In places where such words are unavoidable, they are underlined so that the student does not overlook them. In some problems, irrelevant material is intentionally included to determine whether pupils are capable of identifying and selecting only that material which is relevant to the solution of the problem.

The efficiency and reliability of the test depends, to a great extent, on the way the distracters or alternatives are chosen. They should be similar in content, grammatical form, degree of precision, and length. To the pupil who is not well prepared for the test, the distracters should be as attractive a choice as the correct answer.
The distracters are chosen following two methods. First, the results of the Advanced Placement Examination in calculus, conducted by ETS are collected over three years. These results indicate the types and frequency of responses chosen by students. Second, examination papers of students in the calculus courses taught at the undergraduate level at Ohio University are collected. A careful examination of these results show that many errors committed by students are due to not reading the stem of the item in full, not solving the problem completely, not testing the correctness of all the distracters, assuming wrong data, and most of all, making algebraic errors. All the alternatives, plausible and relevant to the item concerned are chosen as distracters, which eliminates the possibility of using any alternatives which are completely unreasonable.

The "none of the above" option is provided as a distracter in a limited number of cases to make the pupil read and consider all the alternatives. This option appears as the correct answer in at least a few of the items in which it is used. Because this option increases the difficulty of an item, it is used only at places where it is absolutely necessary, specifically, it is used for items which are exceptions to a rule or a generalization. This encourages the student to examine the distracters more carefully before answering the question.
Depending on the complexity of the item, either four or five alternatives are given. As has already been discussed in the literature review, the fewer the number of alternatives, the greater the probability of guessing right. Hence, the simpler items are provided with five alternatives to make them more difficult, and the more difficult items, with four alternatives to make them easier. Thus, there is an effort to make the difficulty level of all the items similar.

Finally, only one correct answer is provided for each item. This makes the evaluation easier since the number of correct answers marked for an item does not vary from one pupil to another. This also removes any ambiguity in the problem statement and makes the item easier and faster to solve. The correct choices are randomized so that all of the alternatives appear approximately an equal number of times as the correct choice in the entire test. This helps reduce the probability of a student correctly guessing the answer.

Structure of the test

A general procedure for preparing a test is described in the "User's Manual." The following section is a description of the structure of the test used in this investigation.
Based on the previous design principles, a number of multiple-choice test items (question bank) is constructed. In each cell of the table of specifications - 2, the question numbers of the items in the question bank which pertain to that cell, are listed.

The test is constructed by choosing questions from the appropriate cells based on the specific requirements of the test. The test items are arranged randomly, so that the examinee is kept guessing as to the subject area and level of learning of the next item and thus the interest level is sustained.

The examinees are allowed to mark their responses on the test paper itself instead of on a separate answer sheet. This saves some time for the examinees and also prevents them from making errors while marking their choices.

Administration of the Test

The students are told in advance as to when they are going to take the test, what it would encompass and what kind of test it would be. Since the test is in print and not presented on a computer screen, the examinee may answer the test items in any order and change an answer any number of times.

Instructions are clearly provided to the examinee as to (1) the duration of the test (2) how to mark the answers (3) the penalty for guessing wrong and (4) the maximum points.
Evaluation

The conventional number right formula is used to score the test. The merits of this evaluation method have already been discussed in the literature review. Hand scoring is employed since machine scoring is not economical with fewer than 100 answer sheets (Furst, 1958), and it also requires the use of pencils and specially printed answer sheets. A previously prepared answer key makes the hand scoring easy and rapid.

The table of specifications described earlier is used to report the scores. The scores are summarized and reported in three ways. One is an overall score, another is the score in each major topic area and the last is the score in each level of learning. This helps to determine in which level of the cognitive domain or in which subject area, the examinees are strong and in which they are weak.

Step 2: Development of goals for the corrective action

Before generating a corrective action, the target population and the objectives to be achieved through the corrective action are defined.

The nature of the target population, with regard to its knowledge level, is determined through the results of the pre-test. The results of the pre-test indicate the subject areas which need to be emphasized, common misconceptions, errors of judgment, and faulty reasoning. Based on the
nature of the population, the content and structure of the corrective actions vary.

The objective of the corrective action is to help the learner recall, interpret, compare and compute any problem in basic calculus. The student should recognize the solution, even if the actual definition or generalization is not recalled verbatim. Contrary to text books, these corrective actions should emphasize only the specific problem areas.

A clear definition of the target population and the objectives to be achieved through the corrective action enable the design of a corrective action which adapts to the needs of the individual rather than the individual adapting to the style of the corrective action.

Step 3: Design of a Corrective action

The major design criteria for choosing the corrective action are as follows:

1. The corrective action should be precise with minimum redundancy.

2. It should provide instruction in all sub topics of the subject and at all levels of learning.

3. The learner should be given an opportunity to actively participate in the learning process. That is, in addition to providing stimuli to the learner, it
should also allow the learner to test and assess his/her understanding of the subject.

4. The learner should be free to flip back and forth through the instructional material or skip familiar sections or start with any major topic without being concerned about the continuity.

5. It should be cost effective.

PROGRAMMED INSTRUCTION

Programmed instruction is the most inexpensive and efficient corrective action available. In programmed instruction, the body of instructional material is divided into steps or frames. After several frames, a few multiple-choice questions are given. The student answers these questions before moving on to subsequent frames. The correctness of the response is immediately known and a correct response acts as a positive reinforcement. The student thus reads through the program, actively participating until the programmed instruction is completed.

Programmed instruction is deemed the appropriate corrective action, for the following reasons:

1. Programmed instruction provides the stimuli, response, feedback and reinforcement, whereas the audio-visual aid provides either response or stimuli (Lysaught, 1963).
2. It is very easy to write programmed instruction for a mathematics course, since the items in a mathematics course are logically related, and the student's response is specific.

3. The learner can pace through the program at a rate that behoves him/her. The differences in learning capacities among students are leveled by a program (Lysaught, 1963).

4. The learner can flip back and forth through the program with ease.

5. Motivation of the learner is possible since feedback regarding the correctness of the responses chosen is given immediately. This acts as a positive reinforcement.

6. The program can be revised and edited any number of times, based on the changing needs of the learners, with minimum expense.

Design of the programmed instruction

The first step in developing programmed instructional material, the selection of the subject matter, is done in the table of specifications in step 1. The second step, that of defining the target audience and the objectives of the corrective action, is done in step 2. The next step is the actual design of the programmed material (frames).
The salient features of programmed instruction are:

1. Presentation of a frame
2. The connectivity among individual frames
3. The sequence of the presentation of frames
4. The medium of presentation.

(1) Presentation of a frame:

The length of a frame is affected by three different variables, i.e., the time needed to work on a single frame, the number of sentences in a frame, and the number of ideas contained in a frame (Fry, 1963). A balance is struck among these variables in determining the length of a single frame. One or more ideas are discussed in a single frame depending upon their complexities. If a complex idea is discussed, more than one frame may be devoted to that idea.

For individualized instruction, only parts of the programmed instruction are given to each student. Hence, the individual frames are written in such a way that every frame makes sense when studied in isolation, as well as provides continuity when in series with the other frames. In addition to the item length, the readability of the item affects the interest of the reader. Readability depends on the style of writing, the complexity of ideas involved and the neatness of presentation (Fry, 1963). Since the program is written in an increasing order of complexity, and the student is made to participate actively while reading the
instructional material, the interest of the student is sustained.

(2) Connectivity among individual frames:

Before writing the programmed instruction, the rules by which the individual items are connected are determined. These rules are provided by models such as linear, branching, and several modifications of these. The relative merits of these models are discussed in the "Literature Review."

Linear programs present material in a sequential manner, going from simple to complex items. The reader must follow the programmed sequence regardless of the response to the questions following the material presented. A slightly modified version of linear programs is used in this system since (1) the subject matter has an ascending order of complexity and (2) there is a perceptible difference among the intellectual abilities of the readers. Some of the variations of this model from the linear program are (1) instead of the constructed response type test items used in pure linear programs, multiple-choice type of test items are written because, for the subject matter chosen, both recall and recognition skills are to be tested and the multiple-choice type does this very effectively; (2) The individual frames are bigger than those usually found in linear programs since the target group has a strong background in
calculus and this program is designed only to help them relearn what they have already learned.

(3) **Sequence of presentation of frames**

The order or arrangement of the sequence of items is the next thing to be considered. There are several methods, such as the mathetics system, the RULEG system, etc. Mathetics system is a "training system which gives a set of procedures to diagnose a problem and based on the diagnosis, a decision is made on the content and style of programming."

In the RULEG system, the material is presented in the form of rules (RU) and examples (EG). The rules and examples are logically sequenced. The relative merits of these sequencing procedures are discussed in the "literature review."

A slightly modified version of the RULEG system is followed in writing the item sequence in this program. The reason for using the RULEG system is that it is the most appropriate of the available models for a mathematics course. This is because all the concepts in mathematics can be stated in the forms of rules and a step by step procedure of the solution of the problem can be illustrated. Using this system the development of a frame is simple. In each sub topic, all the principal rules governing a concept are listed. Each rule is illustrated by an example. Then the major rules are divided into sub rules and wherever necessary, the sub rules are also illustrated with examples.
The rules and sub rules are logically arranged in the increasing order of complexity. At the end of a few rules, a self-test of multiple-choice type is given. The correctness of the responses made by the learner is immediately known since the answers are given in the program, thus providing positive reinforcement. The reader thus actively participates through the entire program.

A few sample frames are first written and shown to a small sample of students and their criticisms noted. At different stages of the program, feedback from the students is obtained and the appropriate alterations made to the program. The accuracy and relevancy of the material is checked by giving the program to a mathematics professor to be edited. He ensures that the program covers the material specified in the objectives of the program, that it acts as a good refresher guide, that the examples are appropriate and meaningful, and that the self-tests are appropriate. The program is again edited by a person who is proficient in English and has a sound background in calculus. This is to ensure the continuity of the material presented, the grammatical consistency in the language and the interest generated by the content of the program. All the relevant changes recommended are incorporated and a final draft of the program is made.
(4) **Medium of presentation**:

A print medium is chosen as the best medium for presenting the instructional material because (1) it is inexpensive (2) copies of the program can be made with ease (3) only those parts of the program that are needed can be given to each individual and (4) revision of the program can be done with ease and without much expense. A detailed description of the different media is given in the literature review.

**Use of the programmed material**

The test score of each student is analysed. All unanswered or incorrectly answered items are listed. Because each item belongs to a particular cell in the table of specifications, and because each cell represents a particular subject area and a particular learning level, it is easy to discern the deficient areas for each student. Since each student has problems in different subject areas, each of them is given different sections of the programmed material.

This review material is given to each student immediately following the evaluation of the pre-test and an appropriate amount of time is given to review the material.
Step 4: Evaluation of the solution

The effectiveness of the corrective action is evaluated by administering another test, called a post-test. The pre-test and post-test are designed to be approximately equivalent in difficulty level. This is achieved by using the table of specifications created earlier. The pre-test and post-test are designed by randomly picking an equal number of questions from the same cells. Since all the questions in a cell are of the same difficulty, the two tests are essentially interchangeable.

After the post-test is administered, the results are analysed. All comments and criticisms made by the students regarding the style, structure and content of the tests and the programmed material are carefully documented. If they indicate that the objectives of the program have been achieved, then the program is left as it is. Otherwise, based on the nature of problem, be it in the assumption of the level of knowledge of the learners, the objectives of the system, the model chosen, or the sequencing of the items, the program is revised.

Statistical analyses are performed on the results of the pilot test to determine whether (1) the pre-test and the post-test are equivalent forms (2) the tests are too difficult or too simple (3) the content of the tests are relevant to the students (4) the students are in favor of
the test format and (4) the programmed material helps in improving their performance.

Based on the results of the tests, and criticisms made by the examinees, the necessary changes are incorporated in the tests and the programmed material, and the process of testing and evaluating is repeated. This iterative procedure is repeated until an effective system is developed.
CHAPTER 5

RESULTS AND DISCUSSION

The previous chapter discussed the development of design criteria for choosing a test format and a corrective action for testing and rectifying students' deficiencies in basic calculus. A system design approach which involved problem identification, determination of the goals for the corrective action, design of the correction action and an evaluation of the corrective action, was discussed.

Problem identification was done by means of a test. The test was constructed from a question bank based on the table of specifications. The appropriateness of the test was therefore dependent upon the questions in the question bank. Several iterations were required to develop a question bank appropriate for this study. Initially, a sample question bank was designed and the questions given to a small group of students so as to determine whether or not they met the objectives of this investigation. Based on these results, the question bank was revised and tested on different group of students. Following this procedure, a question bank was designed which could achieve the objectives of this investigation. In a similar fashion, the programmed instruction was also tested on a sample group of students, and all the suggested relevant changes, regarding
the style, content and complexity of the material were incorporated. This updated programmed instruction was further revised following a review by people proficient in the field of mathematics. The effectiveness of this revised programmed instruction was evaluated by means of a test constructed from the question bank. The following sections describe the design of the question bank and programmed instruction.

Question Bank

The question bank consists of a series of questions concerning the following topics: limits and functions, the derivative, series, and integration; within each of the following levels of learning: knowledge, comprehension, application, and analysis. These topics were divided into subtopics, shown in the table of specifications (see appendix A).

The question bank contains several questions for each appropriate cell in the table of specifications. A level of difficulty or importance was attached to each cell. A subjective determination of the importance of each cell was made after discussions with several engineering faculty. Some cells contain no questions due to their complexity with respect to the level of learning or the subject area involved. For instance, questions in application with regard to increasing/decreasing functions would be
difficult at the undergraduate level of education. Hence no questions were designed for that cell. Another reason for leaving some cells empty was that it was difficult to design multiple-choice questions for them. For instance, multiple-choice questions at the application level of learning for partial differentiation would be difficult to design, with regard to the selection of appropriate distracters. Approximately sixty questions were designed for this sample question bank, at least two questions for each required cell in the specifications matrix. Some questions were taken from the Advancement Placement Examination in calculus conducted by the Educational Testing Service, while others were designed according to the methodology in the previous chapter. A sample question from the question bank is given below:

**Example**

Consider the cell corresponding to the "other indeterminate forms" of the type $1^{-}$ and the "application" area. The following question was designed.

\[
\lim_{{x \to 0}} (1 + x)^{\frac{0.2}{x}} =
\]

A. $e^{0.2}$  
B. 1  
C. $\infty$  
D. 0.2

This question requires that the student identify this as an indeterminate form and recall the correct solution or
methodology. This involves a knowledge of L'Hôpital's rule and the ability to compute the limit of a function using this rule. Thus, it is a true application problem.

The problem statement is of the incomplete statement form. The distracters are chosen based on mistakes commonly committed by students. In this case, common sense and logic are used to determine the distracters. Distracter A is the right choice. $\lim_{x \to 0} \frac{1}{x}$ is commonly calculated as 0 or $\infty$, therefore distracters B and C are given. Distracter D occurs as a result of an algebraic error commonly committed in the course of solving problems of the above type.

After the sample question bank was designed, the appropriateness of the questions in the question bank was tested. This involved checking the level of difficulty of the questions, their ability to discriminate between students with strong and weak backgrounds and the appropriateness of the distracters. This was done by means of a test.

**Sample Test**

Two sample tests, test 1 and test 2, were constructed by selecting questions from the question bank. The following steps were followed in constructing the tests: The following subject areas were included (1) limits and functions (2) the derivative (3) series and (4) integration. The levels of learning were assigned the
following priorities: (1) application (2) comprehension (3) analysis and (4) knowledge. This ranking was based on the assumption that the examinees had a strong background in calculus.

The number of questions selected in each subject area was based on the priorities given to the levels of learning. The application area, which has the highest priority, was considered first, and a number of questions was selected from each of the major subject areas. Among the major subject areas, the order of priority was (1) integration, (2) the derivative, (3) limits and functions, and (4) series. These priorities were based on the number of subtopics to be tested in each area and the extent to which each subject area is used in the introductory engineering courses. The number of questions selected for each area decreased accordingly. In a similar manner, questions were selected from the remaining three levels of learning.

Test 1 consisted of 30 questions. The number of questions in the four levels of learning were 13, 12, 6 and 5 respectively and in the four major topics were 12, 13, 9, and 4 respectively. Test 2 consisted of 24 questions. The figures corresponding to the four levels of learning and the four major topics were 11, 8, 6, 6 and 9, 7, 6, 2 respectively. Because some questions test more than one level of learning and some test more than one subject area the totals of the preceding numbers exceed 30 and 24.
respectively. Test 2 was constructed by eliminating 6 questions from test 1, that is, every question in test 2 was on test 1. This was done because the test was administered to undergraduate students and some questions were too difficult.

Test 1 was administered to 15 graduate students and Test 2, to 44 undergraduate students. In both cases, a maximum time of one hour was given to finish the test. The tests were scored manually using the conventional number right formula for evaluating a test. A statistical analysis was performed on the results of these tests, however it should be noted that only the results of those students who also took the post-test were considered for statistical analyses.

Tables 1 and 2 (column 2) give a summary of these statistics. The average score and the median were slightly below 50% for Test 1 and below 30% for Test 2. Tables 3 and 4 show the percent of correct responses chosen by the respective groups in each level of learning for Tests 1 and 2 respectively. The scores decreased as the level of learning progressed from knowledge to analysis. Higher scores were obtained in the knowledge and comprehension areas i.e., the lower learning levels, than in the application and analysis areas, the higher learning levels. This indicates that (1) there is an increasing complexity in the levels of learning and (2) the students were more
Sample size: 9

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<th>Post-test</th>
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<td>18.56</td>
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<tr>
<td>median</td>
<td>13.00</td>
<td>18.00</td>
</tr>
<tr>
<td>range</td>
<td>7 - 27</td>
<td>10 - 25</td>
</tr>
<tr>
<td>standard deviation</td>
<td>6.46</td>
<td>5.14</td>
</tr>
</tbody>
</table>

TABLE 1: TEST 1
Sample size: 6

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<th>Post-test</th>
</tr>
</thead>
<tbody>
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<td>average</td>
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<td>7.00</td>
</tr>
<tr>
<td>median</td>
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<tr>
<td>range</td>
<td>5 - 11</td>
<td>3 - 9</td>
</tr>
<tr>
<td>standard deviation</td>
<td>2.21</td>
<td>2.00</td>
</tr>
</tbody>
</table>

TABLE 2 : TEST 2
proficient in the simple computational skills than in the application of concepts to novel situations. The variation in scores between the two tests was expected since test 1 was administered to graduate students and test 2 was administered to undergraduate students.

An analysis of the results of these tests showed that (1) the problem statements were unambiguous, as indicated by the examinees, (2) the distracters were very effective in misleading the students with superficial knowledge, as shown by the low number of distracters left unmarked by anyone in the group, and (3) the items discriminated between the strong and weak students, as shown by the wide range of scores obtained. A serious drawback of this test was that it was too difficult. This became clear through the results of the test and the remarks made by the examinees. The examinees complained that (1) in some cases, even though they were able to recall the methods to be used, they could not apply them due to the complexity of the problems, (2) in some other cases, they were unable to understand the problem itself and (3) in a very few cases they were unable to recall the formulas, rules and methods studied earlier. This indicated that the examinees faced more problems at the higher learning levels than at the lower learning levels.

In summary, the results of these tests demonstrated that the items were valid in terms of the stem (problem statement) construction and the selection of the
distracters. In other words, the questions were direct and the distracters were appropriate for a majority of the items. But they also showed that the items were too difficult and that too much emphasis was placed on the higher levels of learning. Since the objective of this investigation was to determine the basic computational skills of the students, it was concluded that the questions had to be simplified.

Sample Programmed Instruction

The design of the programmed instruction began with the selection of the subject areas. The four subject areas discussed earlier were chosen. The scope of the programmed instruction was determined by making assumptions about the prior knowledge level of the target population.

An estimate of the prior knowledge level of the target group was known from tests 1 and 2. These tests illustrated that the students were weak in the higher learning levels and also, to a lesser extent, in the lower learning levels. In addition, the results of the tests indicated the specific areas in which many students had difficulty. For instance, problems on implicit differentiation, reversal of double integrals, continuous and discontinuous functions etc. were incorrectly answered by a majority of the students, and problems on partial differentiation, higher order differentiation, and L'Hôpital's rule were missed by only a
few students. Based on these results, differential emphasis was placed on the different sub topics while designing the programmed instruction.

The programmed instruction was designed using a modified linear program with a modified RULEG sequencing approach. Each major subject area was divided into several sub topics. For instance, Limits was divided into definition and properties of limits, one-sided limits, L'Hpital's rule and indeterminate forms of types 1-, \( \infty \), and 0°. For each of these sub topics, the related definitions, formulas and rules, appropriate examples illustrating the stated rules, and a short self-test to test the understanding of these rules were written. Several calculus books were referenced so as to state the theorems, rules and generalizations in as simple terms as possible. All of the above text material was presented in modules, called frames. Two sample frames are shown in the following two pages:
1.3 Indeterminate Forms

We introduce some very important methods of evaluating complex limits in this section.

If \( \lim_{x \to a} f(x) \) is of the form \( 0/0 \) or \( \infty/\infty \), then it is called an indeterminate form. Such limits should be evaluated based on the following rule.

"Differentiate the numerator and the denominator separately, and take the limit of the resulting fraction. If the resulting fraction has no limit or doesn't become infinite, then repeat the differentiation" This is called L'Hôpital's rule.

REMEMBER to stop differentiating as soon as the resulting fraction has a limit or becomes infinite, whichever comes first.
Example #2  

Find \( \lim_{{x \to 0}} \frac{e^x - x - 1}{x^2} \)

Solution  
The given limit is of the form 0/0. Hence use L'Hôpital's rule. The resulting limit is

\[
\lim_{{x \to 0}} \frac{e^x - 1}{2x}
\]

This is again of the form 0/0. Hence use L'Hôpital's rule again. The limit now becomes

\[
\lim_{{x \to 0}} \frac{e^x}{2}
\]

Hence the limit is equal to 1/2.

NOTE:  
1. L'Hôpital's rule is valid for one-sided limits also.
2. This rule applies only to indeterminate forms of the type 0/0 and \(\pm/\pm\).
3. Indeterminate forms of the type 0.\(\infty\) and \(\pm-\infty\) should be converted to the forms 0/0 and \(\pm/\pm\) before applying L'Hôpital's rule.
These frames are followed by examples illustrating Note 3 above, and subsequently a self-test on these concepts. Thus for each sub topic, one or more frames were constructed. At the end of each major subject, solutions to all the problems in the self-tests were given.

After writing several sections of the programmed instruction, the sections were shown to a few students to check for the clarity and comprehensibility of the ideas expressed. Upon completion of the first draft of the programmed instruction, it was given to two graduate students to be edited. Once the suggested corrections were made, the programmed instruction was again edited by both a mathematics professor and an engineering professor. Thus, after several editings, the final copy of the programmed instruction was complete.

The programmed instruction consisted of 95 pages of material including several self-tests. Several copies were made and distributed among the students who had taken tests 1 and 2. Students were given approximately 10 days to review this material.

The next step was to test the effectiveness of the sample programmed instruction with regard to the variables mentioned earlier. This required retesting of the material by means of a post-test.
Sample Post Tests

Tests 1 and 2 which were administered earlier were called the pre-tests. Post tests 1 and 2, were designed using the same criteria as in the pre-tests. The formats of the post tests were essentially the same as that of the pre-tests. The tests were administered to the same group of students after they completion of the programmed instruction. Only 9 of 15 graduate students and 6 of 44 students who took the pre-test studied the programmed instruction and took the post-tests.

The results of the post-tests are tabulated in Table 1 (column 3) (see page 63). Post test 1 showed nearly a 25% increase in the average score and a decrease in variance when compared with pre-test 1. This occurred because the performance of the students in the bottom half improved a little and that of the students in the top half remained unchanged. In the post test 2, there was a decrease in the average score as given in Table 2 (column 3) (see page 64).

Tests of hypotheses were performed on the results of pre and post tests 1 and 2 to determine if any improvement in scores occurred (see appendix B). Since the pre- and post- tests were administered to the same groups of students, a paired-t test was done. The hypothesis test for test 1 showed that there was a significant difference between the average scores of the pre and post tests, indicating that there was an improvement in the performance
of the students. The hypothesis test for test 2 showed no significant difference between the pre and post tests (see appendix B).

Tables 3 and 4 show the scores in the different learning levels for the pre-tests and the post-tests. These are graphically depicted in figures 1 and 2. In test 1, for each level of learning, the values corresponding to the post-test were higher than those of the pre-test. However, in test 2, the post-test values were slightly lower than the pre-test values.

In conclusion, it appears that the programmed instruction improved the performance of the graduate students significantly, and did not affect the performance of the undergraduate students. But the improvement among graduate students or lack of improvement among undergraduate students could not be definitely attributed to the programmed instruction. This was because the equivalence of the pre-tests and post-tests was not established and, as a result, there was a possibility that the improvement or lack of improvement could be due to the post tests being more or less difficult than the pre tests. Additionally, no check was made to ensure that the students actually studied the programmed instruction before taking the post test.

The programmed instruction was criticized by the examinees on several points. One was that there was too much material in each frame, making the reading tiresome and
<table>
<thead>
<tr>
<th>Level of learning</th>
<th>% correct responses</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>pre-test</td>
<td>post-test</td>
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<td>68.52</td>
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<td>application</td>
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<td>analysis</td>
<td>38.89</td>
<td>47.62</td>
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**TABLE 3: TEST 1**
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<th>Level of learning</th>
<th>% correct responses</th>
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<td>Test A</td>
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<tr>
<td>knowledge</td>
<td>70.83</td>
</tr>
<tr>
<td>comprehension</td>
<td>50.00</td>
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<tr>
<td>application</td>
<td>35.19</td>
</tr>
<tr>
<td>analysis</td>
<td>19.44</td>
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</tbody>
</table>

TABLE 4: TEST 2
FIG. 1: PRETEST VS POSTTEST - TEST 1

PERCENT OF CORRECT RESPONSES

Knowledge Comprehension Application Analysis

LEVELS OF LEARNING

<table>
<thead>
<tr>
<th></th>
<th>pre-test</th>
<th>post-test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Knowledge</td>
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<td></td>
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<tr>
<td>Comprehension</td>
<td></td>
<td></td>
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<tr>
<td>Application</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Analysis</td>
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<td></td>
</tr>
</tbody>
</table>
FIG. 2: PRETEST VS POSTTEST - TEST 2

PERCENT OF CORRECT RESPONSES

Knowledge Comprehension Application Analysis

LEVELS OF LEARNING
uninteresting. Others were that the examples were too difficult to comprehend and that the self tests were too difficult. That is, the programmed instruction was directed towards teaching and improving the higher levels of learning, without paying enough attention to improving the lower learning levels. This was in direct contrast to the objective of this investigation.

In order to overcome these deficiencies, both the question bank and the programmed instruction were revised.

Revised Question Bank

The results of the tests 1 and 2 indicated that the distracters were appropriate as indicated by the high percent (80%) of distracters chosen by one or more students in these tests. Hence, the approach in choosing the distracters was validated. The next area of concern was the difficulty level of the questions.

The questions in the original question bank were reduced in difficulty level by modifying them to test only the basic knowledge, comprehension and application of the subject matter. No questions were designed in the analysis area. Every question was closely examined to verify that it was not violating the objectives defined earlier in the "Introduction." The number of questions in the question bank was also increased, so that there was a greater choice in the selection of questions.
Revised Programmed Instruction

The two main limitations of the programmed material were that there was too much material in each frame, and the examples and test questions were too difficult.

The first limitation was resolved after consulting with a human factors person. In a majority of cases, not more than one basic idea and not more than one example was illustrated in one frame. Most of the self-tests were presented on separate frames. The text material was centered properly and key words were either highlighted or underlined, so that the reader does not miss any key concepts.

The second limitation was overcome through several steps. First, the less important sub topics, such as, logarithmic differentiation, inverse trigonometric functions, and complex trigonometric integrals were deleted. Next, the illustrated examples for the remaining sub topics were simplified. The question bank served as a reliable guide for this purpose. Finally, the self-tests were developed in such a way, that all the questions could be answered after reading the programmed instruction. This was verified by testing the questions in the self-tests on a sample group of students before including in the programmed instruction.

Before the final copy of the programmed instruction was made, it was shown to several students and their comments
and criticisms carefully documented. All relevant changes were incorporated. Thus, after several revisions, the complexity and bulk of the programmed instruction were reduced to manageable proportions. The revised programmed instruction consisted of 44 pages.

FINAL TEST

a) Pre-test

Based on the revised question bank and programmed instruction, another test was designed. A slightly improved approach was used to design this test. To overcome the danger of designing two unequivalent forms of the pre-test and the post test, two forms of the test, test A and test B were constructed. One half of the students was given test A and the other half was given test B. A comparison of the means would show the equivalence of the two tests. The group which took test A as the pre-test, was given test B as the post test and vice versa. A comparison was made of the improvement in performance in either case.

The tests were designed using the table of specifications and the question bank. In the table of specifications, each major topic was divided into sub topics and represented on the rows of the matrix. In each sub topic, different priorities were assigned to the different learning levels. The sub topics were also ranked according to their importance in the introductory courses in
engineering. A reference number was given to each sub topic which was the same as the section number corresponding to that sub topic in the programmed instruction.

The test construction began by randomly picking questions from the highest priority level of learning for each sub topic. This was repeated until all the levels of learning were exhausted or the required number of questions were chosen, whichever occurred first. This eliminated the danger of missing any sub topic or any required level of learning or deviating from the objectives of the test. A systematic procedure for constructing a test from the table of specifications is illustrated in the user's manual.

The two test forms, A and B, were scored using the same procedures as before. Table 5 shows the summary of statistics for these tests. The average scores, standard deviations, reliability coefficients and standard errors of measurement for tests A and B did not show any significant difference. The t-test performed on this data showed no significant difference between the means of the two tests. Thus the equivalence of the two tests was established (see appendix B).

b) Individualized Instruction

The results of each student's test were used to determine the items missed or answered incorrectly. As an example, student X couldn't answer questions on L'Hôpital's rule, increasing and decreasing functions,
differential of a logarithmic function, higher order derivatives, area under a curve and the evaluation of a double integral on test A. The table of specifications gives the section numbers in the programmed material which discussed this material. The student's personalized programmed material contained only these sections.

Thus each student was given a different programmed instruction based on individual test results. Approximately a week's time was given to each student to review the programmed instruction.

c) Post-test

As already described, the examinees who took test A as the pre-test took test B as the post-test and vice-versa. The tests were evaluated and the results were analyzed.

The results of the post-test indicated that there was an increase of approximately fifty percent in the average score (see Tables 5 and 6). The standard deviation in the post-test scores was high. This probably occurred because there was no means of motivating the students to study the programmed instruction. Therefore, those who did study probably improved their scores much more than those who did not study. As a result, the range of scores was wider in the post-tests. Table 7 gives a comparison of statistics for both forms of the pre-tests combined and the corresponding post-tests. Tables 8 and 9 give the percent of correct responses for the four levels of learning for the
Sample size: 5

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<tr>
<th>Variables</th>
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<th>Test B</th>
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<td>standard deviation</td>
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<td>std. error of</td>
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<tr>
<td>measurement</td>
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**TABLE 5: FINAL TEST -- PRE-TEST**
Sample size : 5

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</thead>
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<td>range</td>
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<tr>
<td>standard deviation</td>
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</table>

TABLE 6 : FINAL TEST -- POST-TEST
Sample size: 10

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<th>Post-test</th>
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</thead>
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<tr>
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<td>15.50</td>
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<tr>
<td>range</td>
<td>7 - 18</td>
<td>8 - 23</td>
</tr>
<tr>
<td>standard deviation</td>
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<td>4.56</td>
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TABLE 7: FINAL TEST
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<thead>
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TABLE 8: FINAL TEST -- PRE-TEST
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<th>Level of learning</th>
<th>% correct responses</th>
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</thead>
<tbody>
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TABLE 9: FINAL TEST -- POST-TEST
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<td>26.67</td>
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</table>

**TABLE 10: FINAL TEST**
two forms of pre-tests and the post-tests. Table 10 summarizes these values for both the pre-tests A and B combined and the corresponding post-tests. These values are graphically depicted in Figure 3. Figures 4, 5, 6 and 7 depict a comparison of performances of each student with regard to each level of learning. An independent t-test performed on the results of the pre- and post-tests showed that the average score on the post-test was significantly greater than the average score on the pre-test (see appendix B).

**DISCUSSION**

These results indicate that equivalent test forms of the required level of difficulty can be constructed by using the question bank and the table of specifications. The appropriateness of the questions in the question bank was established by iteratively testing them on several groups of students. The tests constructed by selecting questions from this question bank are appropriate for testing the required objectives of this investigation.

The reliability coefficients for the two tests were calculated using the variance technique described in the literature review. The values of the reliability coefficients were close to the estimated values for a 25-item test (Ebel, 1971). The significance of this statistic was that it indicated the proportion of the total
FIG. 3: PRETEST VS POSTTEST - FINAL TEST

PERCENT OF CORRECT RESPONSES

Knowledge  Comprehension  Application  Analysis

LEVELS OF LEARNING

- pre-test
- post-test
FIG. 4: PRETEST VS POSTTEST - KNOWLEDGE

PERCENT OF CORRECT RESPONSES

0 20 40 60 80 100 120

STUDENT

pre-test
post-test
FIG. 5: PRETEST VS POSTTEST - COMPREHENSION

PERCENT OF CORRECT RESPONSES

0 20 40 60 80 100 120

1 2 3 4 5 6 7 8 9 10

STUDENT

pre-test
post-test
FIG. 6: PRETEST VS POSTTEST - APPLICATION

PERCENT OF CORRECT RESPONSES

STUDENT

pre-test
post-test
FIG. 7: PRETEST VS POSTTEST - ANALYSIS

PERCENT OF CORRECT RESPONSES

STUDENT
variability among the test scores which was not attributable to errors of measurement. Even though the reliability coefficient determined the consistency of the testing procedure, constancy of pupil characteristics, and consistency over different samples of items, a serious shortcoming of this statistic is that it is a function of the nature of the group tested, i.e., the same test given to a different group of students could give a different reliability coefficient. Hence not much importance was attached to that statistic. Another statistic, standard error of measurement, is valid only for large samples or for a population which follows the normal distribution. A Cramer-Von Mises test performed on the results of the final tests showed that the scores are normally distributed (see appendix B). The standard error of measurement, \((SE_m)\) also calculated from variance technique, was approximately the same for both tests and was less than 8% of the maximum score. The \(SE_m\) means that for about two-thirds of all people tested, the observed scores lie within one \(SE_m\) of the true scores, and that for 95% of all people tested, the observed scores will not be more than two \(SE_m\) s away from the true score. In other words, if there are a large number of comparable forms of tests, then about 68% of the scores obtained on these tests would fall within one \(SE_m\) of his true score and about 95% within two \(SE_m\) s of his true score. The practical significance of this statistic lies in
setting a passing grade. For instance, if the passing grade is set at 15 out of 30, and if the $SE_m$ is 2, then a range could be set for the passing grade, which would be 13 - 17. Then if X obtains a score of 12, it can be argued that this is a failing score, since the grade falls out of the acceptable range. But if X gets a score of 13, which is equal to the lower limit of the acceptable range of scores, then X could be given the benefit of doubt and held another test. This would give the student a fair chance to pass the test and not penalize him for the errors of measurement.

CONCLUSIONS

The question bank consists of questions which test the required objectives of this investigation. Several equivalent test forms and tests of varying difficulties can be developed by simply selecting questions from the pertinent cells of the specifications matrix.

The higher percentage of scores in the knowledge area as compared with those in the comprehension area, and similarly the higher scores in the comprehension area as compared with the application and analysis areas clearly establish a definite hierarchy in the complexities of learning levels. In addition, these scores indicate the depth of knowledge possessed by the examinees. They showed that the students are quite proficient in the knowledge and comprehension areas, i.e. in remembering and recalling basic
concepts and solving simple problems. The corresponding scores in the post-test show a marked improvement in all the levels of learning. The improvement in the application and analysis areas is high, showing that the programmed instruction is useful in teaching the student the selection and application of formulas, rules, theorems, principles and concepts.

It is also determined that one multiple-choice test is not applicable for all purposes and for all people i.e., based on the target population, objectives of the course, and the intent of testing, different types of multiple-choice tests are needed. In other words, a test designed for graduate students should emphasize the higher levels of learning rather than the lower levels of learning, whereas for undergraduate students, the reverse would be more appropriate. A test based on certain predetermined design principles and with the help of a detailed table of specifications, will not inadvertently miss any subject area or over-emphasize any one area and the purpose and specificity of the test will not be lost. This method of testing acts as a feedback to the instructor in helping him know in which sections of the course, he has to place his emphasis.

Individualized instruction is possible by giving each student the necessary topics in the programmed instruction. This method of instruction will help the students refresh
their skills in basic calculus ranging from simple computational skills to complex analytical skills without having to review a number of books. Thus, this is time efficient and result-oriented. The success of the programmed instructional material is shown by the level of improvement found in the performance of the students after reviewing this material, even though there is no way of determining how much of the material is actually read.

In conclusion, this system provides a lasting contribution to the Industrial and Systems Engineering program. Students in ISE need a sound background in calculus to be successful in many introductory level courses. With the help of this programmed instruction, the calculus skills of each student can be improved in a short period of time (approximately one week). Thus, in addition to benefiting the students, this would enable the professor to concentrate on the subject he is teaching, rather than teaching the basic calculus.

There are two limitations in this study. (1) The sample sizes chosen for the tests are very small. This is due to the unavailability of a sufficient number of students to act as subjects in this study. Hence, no strong conclusions can be made about the effectiveness of the system, even though the results showed consistent improvement in the students' performances. (2) A control group is not used to determine if there is any change in the
students' scores in the absence of any programmed instruction. (3) No check is made to ensure that the students studied the programmed instruction before taking the post-test. This may explain why some students showed little improvement.
CHAPTER 6

RECOMMENDATIONS

The following improvements are recommended.

1. Multiple-choice tests should be modified to include an answer justification section at the end of each item, so that the students have an option to justify their answers and earn partial credit.

2. To remove the element of subjectivity in choosing the content for the tests, a panel should be appointed to rate the subject areas. These ratings should be taken as standards.

3. Each question should be tested on several groups of students before it is included in the question bank.

4. To make any assumptions about the knowledge level or abilities of the target population, the diagnostic tests should be given to a homogeneous group of students and the size of the group should be large.

5. In order to motivate the students to take the tests and the programmed instruction seriously, the students should be evaluated on these tests.

6. To present both auditory and visual stimuli at the same time so as to hold the attention of the viewer for longer durations, the programmed instruction should be presented on a video, assisted by commentary.
7. An interactive video should be developed, so that the user can respond to the stimuli provided by the video and also receive feedback.

8. To generate interest among the examinees, the test should be written on a floppy disk and presented on a computer screen. The process of grading should be made automatic, through a computer software program, so that the examinee would know his results as soon as he finishes the test. The program should be written in such a way that the student can move through the test and change his answers any number of times.

9. The programmed instruction also should be presented on a computer screen. For this a graphics package should be developed to enable the complex mathematical symbols to be shown on the screen.


BACKGROUND

a) Levels of learning:

1. Basic calculus is divided into four major subject areas
   a. limits and functions
   b. series
   c. derivatives
   d. integrals

2. Each major area is broken into several minor topics as given in table of specifications-1.

3. A question bank is prepared for each topic, and in each topic, questions pertaining to different levels of learning, each with varying complexities are developed.

   There are four levels of learning: knowledge, comprehension, application and analysis. The level of difficulty progressively increases from knowledge to analysis.

   Knowledge is the retention of specific facts, terminology, definitions, theorems and mathematical structure. Questions in the knowledge area test the ability to remember and recall basic ideas.

   Example #1  \( \frac{d}{dx} (\sin x) = \)

   A. \( \cos x \)  \hspace{1cm} C. \( -\cos x \)
   B. \( \sin x \)  \hspace{1cm} D. \( \csc x \)

   Comprehension is the ability to solve simple problems, where the choice of method is known and the solution of the problem is obtained by mere application of the formulas.
Example #2  If \( f(x) = x^3 + x^2 \), then \( f'(1) \) is equal to

\[
\begin{align*}
A. \ 2 & \quad C. \ 9/2 \\
B. \ 5 & \quad D. \ 1/2
\end{align*}
\]

Application is the ability to select the right method for solving a given problem. It involves the correct interpretation of the problem, the recollection of several methods and formulas, and the use of the best method to solve the problem. No non-routine problems are given in this section.

NOTE: An application problem can be converted to a comprehension problem by providing as a hint, the method to be used in solving that problem.

Example #3  \( \lim_{x \to 0} \frac{1}{e^x - 1} - \frac{1}{x} = \)

\[
\begin{align*}
A. \ \infty & \quad C. \ 1/2 \\
B. \ 0 & \quad D. \ -1/2
\end{align*}
\]

Explanation: The student has to identify that the above problem is of the indeterminate form \( \infty - \infty \) and that the L'Hôpital's rule has to be used.

Analysis involves solving non-routine problems. Questions in this section are exceptions to the rules and generalizations. Trick questions are included in this section.
Example #4 \[ \int_{-1}^{1} \frac{3}{x^2} \, dx = \]

A. -6  
B. -3  
C. 0  
D. non-existent

Explanation: This problem is not a direct substitution of the limits, after the integral of the function is determined. There is a trick involved in this problem. The integral is discontinuous at 0 and hence no integral exists. This is not obvious from the problem statement.

5. Since the entire testing and corrective measures developed are focussed on students in the introductory courses in ISE, questions testing only the first three levels of learning are developed. Analysis, being very complex, is not included.

b) Table of specifications-1

6. The chief elements of this table are:

   (1) subject areas
   (2) levels of learning and
   (3) ranks of the levels of learning for each sub topic.

7. This table is a matrix with the subject areas denoted by the rows and the levels of learning by the columns. In each cell of the matrix, the priority number or rank of that cell with respect to the level of learning and the sub topic corresponding to it is given.

8. The major topics (limits & functions, etc) are also given different priorities based on their use in introductory ISE courses.
<table>
<thead>
<tr>
<th>Major Topics</th>
<th>Sub-Topics</th>
<th>Rank</th>
<th>Knowledge</th>
<th>Application</th>
</tr>
</thead>
<tbody>
<tr>
<td>Limits and Continuity</td>
<td>1.1 Limit and Continuity</td>
<td>3</td>
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<td>3</td>
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<tr>
<td>1.2 Limits of Functions</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>1.3 Indeterminate Forms</td>
<td>3</td>
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<td>1</td>
<td></td>
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<tr>
<td>1.4 Other Indeterminate Forms</td>
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<td>3</td>
<td>2</td>
<td></td>
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<tr>
<td>2.2 Continuity</td>
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<td>2.3 Increasing and Decreasing Functions</td>
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<td>3.1 Derivatives and Applications</td>
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<td>3</td>
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<tr>
<td>3.2 Trigonometric Functions</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.3 Logarithmic and Exponential Functions</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td></td>
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<tr>
<td>3.4 Implicit Differentiation</td>
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<td>3.5 Related Rates and Differentiation</td>
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<td>Power Series and Taylor Series</td>
<td>4.3 Power Series and Taylor Series</td>
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<td>5.2 Trigonometric Formulas</td>
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<td>5.4 The Fundamental Theorem of Calculus</td>
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<td>5.5 Integrals Involving Trigonometric Functions</td>
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<td>5.11 Integrals Involving Inverse Hyperbolic Functions</td>
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<td>1</td>
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<td>5.12 Double Integrals</td>
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<td>1</td>
<td>2</td>
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<td>Major Topical Area</td>
<td>Subtopics</td>
<td>Notes</td>
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<td>3.1 Limit and Rate of Change</td>
<td>3.2 Continuity and One-Sided Limits</td>
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<td>4.1 Power Rule</td>
<td>4.2 Chain Rule</td>
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<td>5.1 Fundamental Theorem</td>
<td>5.2 Trigonometric Functions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Applications</td>
<td>6.1 Optimization</td>
<td>6.2 Conclusions</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table of Specifications - 2**
9. Likewise, the sub topics in each major topic are also ranked. The sub topics in each major subject area are grouped into two levels of importance. The more important sub topics are marked with an asterisk and again ranked as 1, 2, 3 etc., based on the extent of their importance.

c) Table of specifications-2

10. Table 2 is similar to Table 1, except that in place of the ranks in the cells, the question numbers in the question bank, relative to that category are listed.

11. Questions testing a particular level of learning, implicitly test all the levels of learning below it. For instance, questions testing comprehension also test knowledge, and questions testing application, test comprehension and knowledge. In other words, to solve a problem testing any one level of learning, a mastery in all the levels below that level, in that particular sub topic, is essential.

TEST PREPARATION

Step 1a: Choose the major subject area(s) to be tested (See Table of specifications-1 for a list of subject areas and the sub topics).

1b: Choose the length of the test "L" (number of items on the test)(See Rules 1a and 1b).
RULE 1a: At least one question should be asked from each sub topic. Hence the length of the test should be at least equal to the number of sub topics in the subject area(s) chosen.

RULE 1b: A minimum of two minutes should be allowed for each question.

RULE 2: In any sub topic, choose the first question from among those in the first ranked level of learning, the second from the second, the third from the third, the fourth from the first and so forth.

Step 2: Divide the total number of questions on the test into two parts -
(A) a multiple of the total number of sub topics (A)
(B) the remaining fraction (B)

CASE 1: A IS NOT EQUAL TO ZERO; B IS NOT EQUAL TO ZERO

STAGE 1:

Step 3: Start with the highest priority level of learning for the first sub topic in the chosen subject area and follow the order of sub topics given (See Table of specifications-1 for the priorities given to each cell).

Step 4: Choose a question at random from the question numbers listed in the cell corresponding to the chosen level of learning and sub topic (please see Rule 3)

RULE 3: If any cell is empty (i.e., contains no questions in the question bank, as shown in the table of specifications-2), then choose a question from the first ranked level of learning, instead.

Condition 1: All the sub topics in the subject area(s) and at the learning level chosen have been exhausted.
IF CONDITION 1 IS SATISFIED, THEN

Check condition 2

ELSE

Step 5: Choose the next sub topic in the table of specifications-1 and GO TO Step 4.

END IF

Condition 2: The number of questions selected is equal to A.

IF CONDITION 2 IS SATISFIED, THEN

Go To Step 6

ELSE

Check condition 3

END IF

Condition 3: All the levels of learning have been exhausted.

IF CONDITION 3 IS SATISFIED, THEN

Go To Step 3 and Repeat

ELSE

Choose The Next Ranked Level Of Learning

Go To Step 4 and Start With The First Sub Topic

END IF

Step 6: Choose the major subject area with the highest priority (See table of specifications-1).

Step 7: In the chosen major subject area, choose the sub topic with the highest weight attached to it.
Step 8: The order of choosing the levels of learning is unaltered. That is, if the second level of learning is finished by the time a number of questions are chosen, then continue with the third level of learning.

Step 9: Choose a question at random from the question numbers listed in the cell corresponding to the chosen level of learning and sub topic (please see Rule 3)

Rule 3: If any cell is empty (i.e., contains no questions in the question bank, as shown in the table of specifications-2), then choose a question from the first ranked level of learning, instead.

End

Condition 4: The number of questions chosen is B.

Condition 5: The number of sub topics in the chosen area has been exhausted.

Stage 2:

If condition 4 is satisfied, then

Go to Step 15

Else

If condition 5 is satisfied, then

Step 10: Choose the next lower ranked major subject area

Step 11: In the chosen major subject area, choose the sub topic with the highest weight attached to it.

Step 12: Choose a question at random from the question numbers listed in the cell corresponding to the chosen level of learning and sub topic (please see Rule 3)
RULE 3: If any cell is empty (i.e., contains no questions in the question bank, as shown in the table of specifications-2), then choose a question from the first ranked level of learning, instead.

GO TO STAGE 2

ELSE

Step 13: Choose the next lower ranked sub topic in the same major subject area

Step 14: Choose a question at random from the question numbers listed in the cell corresponding to the chosen level of learning and sub topic (please see Rule 3)

RULE 3: If any cell is empty (i.e., contains no questions in the question bank, as shown in the table of specifications-2), then choose a question from the first ranked level of learning, instead.

GO TO STAGE 2

END IF

END IF

Step 15: End the test and randomize the questions selected (so that all the questions of the same subject area are not in one place). This is only to break the monotony of the test.

END

CASE 2: \( A \) IS NOT EQUAL TO ZERO; \( B \) IS EQUAL TO ZERO

Follow CASE 1 until the condition 2 is satisfied and

Step 16: End the test and randomize the questions selected (so that all the questions of the same subject area are not in one place). This is only to break the monotony of the test.

END
HYPOTHESIS TESTS
PAIRED T-TEST FOR TEST 1 SCORES

\[ H_0 : \mu_1 = \mu_2 \]
\[ H_1 : \mu_1 < \mu_2 \]

<table>
<thead>
<tr>
<th>Student</th>
<th>Pre-test score</th>
<th>Post-test score</th>
<th>Difference ( d_i )</th>
<th>( d_i^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>13</td>
<td>17</td>
<td>-4</td>
<td>16</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>14</td>
<td>-3</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>23</td>
<td>-7</td>
<td>49</td>
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<tr>
<td>4</td>
<td>7</td>
<td>10</td>
<td>-3</td>
<td>9</td>
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<td>5</td>
<td>10</td>
<td>18</td>
<td>-8</td>
<td>64</td>
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<td>6</td>
<td>27</td>
<td>24</td>
<td>3</td>
<td>9</td>
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<td>7</td>
<td>24</td>
<td>25</td>
<td>-1</td>
<td>1</td>
</tr>
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<td>8</td>
<td>10</td>
<td>13</td>
<td>-3</td>
<td>9</td>
</tr>
<tr>
<td>9</td>
<td>19</td>
<td>23</td>
<td>-4</td>
<td>16</td>
</tr>
</tbody>
</table>

\[ \sum d_i = -30 \quad \sum d_i^2 = 182 \]

\[ d_{av} = -3.33 \]

\[ S_d^2 = \frac{\sum d_i^2 - (\sum d_i)^2 / n}{n - 1} = 10.25 \]

\[ t_0 = \frac{d_{av}}{S_d / \sqrt{n}} = -3.12 \]

Let \( \alpha = 0.05 \)

\[ -t_{0.05,8} = -1.86 \]

Since \( t_0 < -1.86 \), we reject the hypothesis that the two means are equal. The alternative hypothesis that the mean of the pre-test is less than the mean of the post-test is true.
PAIRED T-TEST FOR TEST 2 SCORES

\[ H_0 : \mu_1 = \mu_2 \]
\[ H_1 : \mu_1 < \mu_2 \]

<table>
<thead>
<tr>
<th>Student</th>
<th>Pre-test score</th>
<th>Post-test score</th>
<th>Difference</th>
<th>( d_i^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>7.36</td>
<td>3.64</td>
<td>13.249</td>
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<tr>
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<td>8</td>
<td>9.47</td>
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<td>0.53</td>
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<td>6</td>
<td>5</td>
<td>3.16</td>
<td>1.84</td>
<td>3.386</td>
</tr>
</tbody>
</table>

\[ \sum d_i = 1.82 \quad \sum d_i^2 = 22.775 \]

\[ d_{av} = 0.303 \]

\[ S_d^2 = \frac{\sum d_i^2 - (\sum d_i)^2 / n}{n - 1} = 4.442 \]

\[ t_0 = \frac{d_{av}}{S_d / \sqrt{n}} = 0.2465 \]

Let \( \alpha = 0.05 \)

\[ t_{0.05,5} = 2.015 \]

Since \( t_0 < 2.015 \), we cannot reject the hypothesis that the two means are equal.
HYPOTHESIS TEST FOR VARIANCE OF TWO MEANS
FOR THE FINAL TEST

\[
\begin{align*}
H_0 : \sigma_1 &= \sigma_2 \\
H_1 : \sigma_1 &\neq \sigma_2
\end{align*}
\]

<table>
<thead>
<tr>
<th>Student</th>
<th>Test A score</th>
<th>Test B score</th>
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<tbody>
<tr>
<td>1</td>
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<td>11</td>
</tr>
<tr>
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<tr>
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<td>8</td>
</tr>
<tr>
<td>5</td>
<td>13</td>
<td>11</td>
</tr>
</tbody>
</table>

\[
X_{A_{av}} = 10.8 \quad \sigma_A = 2.86 \\
X_{B_{av}} = 11.6 \quad \sigma_B = 3.38
\]

\[
F_0 = \frac{(2.86)^2}{(3.38)^2} = 0.716
\]

Let \( \alpha = 0.05 \)

\[
F_{0.025,5,5} = 7.15 \quad \text{and} \quad F_{0.975,5,5} = (7.15)^{-1} = 0.14
\]

Since \( F_0 < F_{0.025,5,5} \) and \( > F_{0.975,5,5} \), we cannot reject the hypothesis that the two variances are equal.
INDEPENDENT T-TEST FOR THE FINAL TEST SCORES (PRE-TEST)

\[ H_0 : \mu_1 = \mu_2 \]
\[ H_1 : \mu_1 \neq \mu_2 \]

\[ X_{Aav} = 10.8 \]
\[ \bar{X}_A = 2.86 \]
\[ n_A = 5 \]

\[ X_{Bav} = 11.6 \]
\[ \bar{X}_B = 3.38 \]
\[ n_B = 5 \]

\[
sp^2 = \frac{(n_A-1)(\sigma_A)^2 + (n_B-1)(\sigma_B)^2}{n_A + n_B - 2} = 9.802
\]

\[
t_0 = \frac{X_{Aav} - X_{Bav}}{sp \sqrt{(1/n_A + 1/n_B)}} = -0.4041
\]

Let \( \alpha = 0.05 \)

\[ t_{0.025,10} = 2.228 \quad \text{and} \quad -t_{0.025,10} = -2.228 \]

Since \( t_0 < t_{0.025,10} \) and \( > -t_{0.025,10} \), we cannot reject the hypothesis that the two means are equal.

\( \beta \) error

Let \( \beta = 2.90 \quad \alpha = 0.05 \quad \text{and} \quad \Delta = 6\% \)

\[
d = \Delta / (2 \sigma^-) = 1.03
\]

\[ n = 5 \quad n^* = 2n - 1 = 9 \]

Therefore, \( \beta \) (from OC curves) = 20%

That is, there is 20% probability that the difference in means between the two tests will not be detected.
FINAL TEST -- PRETEST A

CALCULATION OF RELIABILITY COEFFICIENT AND STANDARD ERROR OF MEASUREMENT

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<td>--</td>
<td>1</td>
<td>63</td>
</tr>
</tbody>
</table>

Rows: Students ; Columns: Items ; X: Correct response

\[ \Sigma p_i = \Sigma t_i = 54 \]
\[ \Sigma t_i^2 = 624 \quad (\Sigma t_i)^2 = 2916 \quad \Sigma p_i^2 = 168 \]
\[ n = 25 \quad k = 5 \]
\[ t_{av} = 10.8 \quad \sigma = 2.86 \]

Sum of squares "among students" = \( SS_s = 1.632 \)
Sum of squares "among items" = \( SS_i = 10.272 \)
Sum of squares "total" = \( SS_t = 30.672 \)

<table>
<thead>
<tr>
<th>Source of Variance</th>
<th>d.f.</th>
<th>Sum of squares</th>
<th>Variance = ( SS/d.f. )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>124</td>
<td>30.672</td>
<td>0.24735</td>
</tr>
<tr>
<td>Items</td>
<td>24</td>
<td>10.272</td>
<td>0.428</td>
</tr>
<tr>
<td>Students</td>
<td>4</td>
<td>1.632</td>
<td>0.408</td>
</tr>
<tr>
<td>Remainder</td>
<td>96</td>
<td>18.768</td>
<td>0.1955</td>
</tr>
</tbody>
</table>

0.408-0.1955

Coefficient of reliability = \( r_{tt} = \frac{0.408}{0.408} = 52.08 \% \)

Standard error of measurement = \( SE_m = \sqrt{\frac{18.768}{4}} \)

= 2.166
CRAMER-VON MISES TEST FOR THE FINAL TEST -- PRE-TEST A

\[ H_0: F(x) \text{ is a normal distribution} \]
\[ H_1: F(x) \text{ is not a normal distribution for at least one value of } x \]

<table>
<thead>
<tr>
<th>i</th>
<th>X(i)</th>
<th>F(X(i))</th>
<th>F(X(i)) - \frac{2i-1}{2n}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7</td>
<td>0.092</td>
<td>-0.008</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>0.264</td>
<td>-0.036</td>
</tr>
<tr>
<td>3</td>
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<td>0.3897</td>
<td>-0.1103</td>
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<td>0.77935</td>
<td>0.0794</td>
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<tr>
<td>5</td>
<td>15</td>
<td>0.92922</td>
<td>0.0292</td>
</tr>
</tbody>
</table>

\[
T_3 = \frac{1}{12n} + \sum_{i=1}^{n} \left[ F(X(i)) - \frac{2i-1}{2n} \right]^2
\]

\[
T_3 = 0.03735 \quad \text{(since } n = 5)\]

\[
T_3 \text{ as given by Anderson and Darling } = 0.461, \text{ at } \alpha = 0.05
\]

Since \( T_3 \) (calculated) is less than 0.461, the null hypothesis that the distribution of the scores is normal is accepted. The critical value of \( \alpha \) is 0.95 as given in the table by Anderson and Darling. Hence we strongly conclude that the null hypothesis is true.
FINAL TEST -- PRETEST B

CALCULATION OF RELIABILITY COEFFICIENT AND STANDARD ERROR OF MEASUREMENT

<table>
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<td>58</td>
</tr>
</tbody>
</table>

Rows: Students ; Columns: Items ; X: Correct response

\[ \sum pi = \sum ti = 58 \]
\[ \sum ti^2 = 730 \]
\[ (\sum ti)^2 = 3364 \]
\[ \sum pi^2 = 178 \]

\[ n = 25 \quad k = 5 \]
\[ t_{av} = 11.6 \quad \sigma = 3.38 \]

Sum of squares "among students" = \( SS_s = 2.288 \)
Sum of squares "among items" = \( SS_i = 8.688 \)
Sum of squares "total" = \( SS_t = 31.088 \)

<table>
<thead>
<tr>
<th>Source of Variance</th>
<th>d.f.</th>
<th>Sum of squares</th>
<th>Variance = SS/d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>124</td>
<td>31.088</td>
<td>0.25071</td>
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<td>24</td>
<td>8.688</td>
<td>0.362</td>
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<td>0.572</td>
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<tr>
<td>Remainder</td>
<td>120</td>
<td>20.112</td>
<td>0.2095</td>
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</tbody>
</table>

0.572-0.2095

Coefficient of reliability = \( r_{tt} = \frac{0.572}{0.2095} = 63.37 \% \)

\[ 0.572 \]

Standard error of measurement = \( SE_m = \sqrt{20.112 / 4} \)

\[ = 2.24 \]
CRAMER-VON MISES TEST FOR THE FINAL TEST -- PRE-TEST B

\[ H_0 : F(x) \text{ is a normal distribution} \]
\[ H_1 : F(x) \text{ is not a normal distribution for at least one value of } x \]

\[ \begin{array}{cccc}
  i & X(i) & F(X(i)) & F(X(i)) - \frac{2i-1}{2n} \\
  \hline
  1 & 8 & 0.13136 & 0.03136 \\
  2 & 10 & 0.31918 & 0.01918 \\
  3 & 11 & 0.42858 & -0.07142 \\
  4 & 11 & 0.42858 & -0.27142 \\
  5 & 18 & 0.97062 & 0.07062 \\
\end{array} \]

\[ T_3 = \frac{1}{12n} + \sum_{i=1}^{n} \left( F(X(i)) - \frac{2i-1}{2n} \right)^2 \]

\[ = 0.0851 \quad (\text{since } n = 5) \]

\[ T_3 \text{ as given by Anderson and Darling} = 0.461, \text{ at } \alpha = 0.05 \]

Since \( T_3 \) (calculated) is less than 0.461, the null hypothesis that the distribution of the scores is normal is accepted. The critical value of \( \alpha \) is 0.67 as given in the table by Anderson and Darling. Hence we strongly conclude that the null hypothesis is true.
PAIRED T-TEST FOR FINAL TEST SCORES

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 < \mu_2$$

<table>
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<tr>
<th>Student</th>
<th>Pre-test</th>
<th>Post-test</th>
<th>Difference</th>
<th>$d_i^2$</th>
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<td>$d_i$</td>
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</table>

$$\sum d_i = -50 \quad \sum d_i^2 = 290$$

$$d_{av} = -5.0$$

$$S_d^2 = \frac{\sum d_i^2 - (\sum d_i)^2}{n-1} = 4.44$$

$$t_0 = \frac{d_{av}}{S_d / \sqrt{n}} = -7.50$$

Let $\alpha = 0.05$

$$-t_{0.05,8} = -1.86$$

Since $t_0 < -1.86$, we reject the hypothesis that the two means are equal. The alternative hypothesis that the mean of the pre-test is less than the mean of the post-test is true.
PROGRAMMED INSTRUCTION
# TABLE OF CONTENTS

## 1. LIMITS
- 1.1 Definition ............................................. 1
- 1.2 Trigonometric Functions ............................ 3
- 1.3 One-sided Limits ..................................... 4
- TEST ....................................................... 6
- 1.4 Indeterminate Forms ............................... 7
- 1.5 Other Indeterminate Forms ....................... 10
- TEST ....................................................... 11

## 2. FUNCTIONS
- 2.1 Continuous Function ............................... 12
- 2.2 Discontinuity of a function .................... 13
- TEST ....................................................... 15
- 2.3 Discontinuity in an Interval .................... 16
- TEST ....................................................... 18
- 2.4 Increasing and Decreasing Functions .......... 19
- TEST ....................................................... 22

ANSWERS ....................................................... 23
1. LIMITS

The definition of a limit is very complex. It has to be studied from different points of view before the meaning becomes clear. Simply speaking, as \( x \) gets closer and closer to "a" (but does not become equal to a), then if a function \( f(x) \) gets closer and closer to some number "L", we say that the limit of \( f(x) \) as \( x \) approaches a, equals L.

1.1 Definition of a limit: Let \( f \) be a function that is defined on an open interval containing a, except possibly at "a" itself, and let "L" be a real number.

\[
\lim_{x \to a} f(x) = L
\]

means that for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \), such that if

\[ 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \varepsilon \]

Understand the above definition well. There is no need to memorize it.
Let us quickly state some of the important theorems on limits.

If \( f \) and \( g \) are two functions, with
\[
\lim_{x \to a} f(x) = L_1 \quad \text{and} \quad \lim_{x \to a} g(x) = L_2,
\]
then

1. \[
\lim_{x \to a} [f(x) + g(x)] = L_1 + L_2 \quad \text{(limit of a sum)}
\]

2. \[
\lim_{x \to a} [f(x) g(x)] = L_1 L_2 \quad \text{(limit of a product)}
\]

3. \[
\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2} \quad \text{(limit of a quotient) \quad \text{if } L_2 \neq 0}
\]

4. limit of a composite function

Suppose \( f \) and \( g \) are functions, \( a \) and \( b \) are numbers, \( f(b) \) is defined and
\[
\lim_{x \to b} f(x) = f(b) \quad \text{and} \quad \lim_{x \to a} g(x) = b
\]
then
\[
\lim_{x \to a} f[g(x)] = f(b)
\]

5. If a function \( f \) has a limit as \( x \) approaches \( a \), then
\[
\lim_{x \to a} n \cdot f(x) = n \cdot \lim_{x \to a} f(x)
\]
provided either \( n \) is an odd positive integer or \( n \) is an even positive integer and \( \lim_{x \to a} f(x) > 0 \)

**Example #1** Find \( \lim_{x \to 5} \sqrt[3]{3x^2 - 4x + 9} \)

**Solution** From the above theorem,
\[
\lim_{x \to 5} \sqrt[3]{3x^2 - 4x + 9} = \sqrt[3]{\lim_{x \to 5} (3x^2 - 4x + 9)}
\]
\[
= \sqrt[3]{64} = 4
\]
1.2 Trigonometric Functions

1. \( \lim_{{t \to 0}} \frac{\sin t}{t} = 1 \)

2. \( \lim_{{t \to 0}} \frac{1 - \cos t}{t} = 0 \)

Example #2: Find \( \lim_{{t \to 0}} \frac{1 - \cos t}{\sin t} \)

Solution:

\[
\lim_{{t \to 0}} \frac{1 - \cos t}{\sin t} = \lim_{{t \to 0}} \frac{1 - \cos t}{t} \cdot \frac{t}{\sin t} = \lim_{{t \to 0}} \frac{1 - \cos t}{t} \cdot \lim_{{t \to 0}} \frac{t}{\sin t} = (0)(1) = 0
\]

(since \( \lim_{{t \to 0}} \frac{t}{\sin t} = \lim_{{t \to 0}} \frac{1}{(\sin t)/t} = 1/1 = 1 \))
1.3 One-sided Limits

If \( f(x) \to L \) as \( x \to a^+ \), \( L \) is the one-sided limit from the right of \( f \). \( a^+ \) indicates that \( x \) tends to 'a' through values larger than \( a \).

If \( f(x) \to L \) as \( x \to a^- \), \( L \) is the one-sided limit from the left of \( f \). \( a^- \) indicates that \( x \) tends to 'a' through values smaller than \( a \).

The usual notations are

\[
\lim_{x \to a^+} f(x) = L \quad \text{and} \quad \lim_{x \to a^-} f(x) = L
\]

The figure below should make things clear.

```
left hand limit | right hand limit
--------------|------------------
---------------|------------------
               | a
```

Note: The limit of \( f \) exists if and only if
- both one-sided limits exist and
- both have the same value.
Example #3  
If \( f(x) = \frac{|x|}{x} \) find \( \lim_{x \to 0} f(x) \)

Solution  
Let us first find the one-sided limits of \( f \).

If \( x > 0 \), then \( f(x) = \frac{x}{x} = 1 \)

\[ \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1 \]

If \( x < 0 \), then \( f(x) = \frac{-x}{x} = -1 \)

\[ \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -1 = -1 \]

As the two one-sided limits are not equal, the limit doesn't exist.
1. \( \lim_{{x \to 0}} \left[ \frac{\sin 5x}{2x} + \frac{\tan x}{2x} + \frac{1 - \cos x}{3x} \right] \) = 
   a. \( \infty \)  
   b. 3  
   c. 4  
   d. non-existent

2. For the limit \( \lim_{{x \to 4}} \frac{|x-4|}{x-4} \) which one of the following is true?
   a. The left-hand limit is 1
   b. The right-hand limit is -1
   c. The left-hand limit is -1 and right-hand limit is 1
   d. The left-hand and the right-hand limits are both 0
1.4 Indeterminate Forms

We introduce some very important methods of evaluating complex limits in this section.

If \( \lim_{x \to a} f(x) \) is of the form \( 0/0 \) or \( \infty/\infty \), then it is called an indeterminate form. Such limits should be evaluated based on this rule.

"Differentiate the numerator and the denominator separately, and take the limit of the resulting fraction. If the resulting fraction has no limit or doesn't become infinite, then repeat the differentiation" This is called L'Hôpital's rule. This is the main theorem on indeterminate forms.

REMEMBER to stop differentiating as soon as the resulting fraction has a limit or becomes infinite, whichever comes first.
Example #4  
Find \( \lim_{x \to 0} \frac{e^x + e^{-x} - 2}{1 - \cos 2x} \)

Solution  The given limit is of the form 0/0. Hence use L'Hôpital's rule. Then the limit is

\[
\lim_{x \to 0} \frac{e^x - e^{-x}}{2 \sin 2x}
\]

This is again of the form 0/0. Hence use L'Hôpital's rule again. The limit now becomes

\[
\lim_{x \to 0} \frac{e^x + e^{-x}}{4 \cos 2x}
\]

Hence the limit is equal to 1/2.

NOTE:  \( \frac{d}{dx} (\sin x) = \cos x \);  \( \frac{d}{dx} (\cos x) = -\sin x \)

NOTE:  
1. L'Hôpital's rule is valid for one-sided limits also.
2. This rule applies only to indeterminate forms of the type 0/0 and \( \infty/\infty \).
3. Indeterminate forms of the type 0.\( \infty \) and \( \infty - \infty \) should be converted to the forms 0/0 and \( \infty/\infty \) before applying L'Hôpital's rule.
Example #5  Find \( \lim_{x \to 0} x^2 \ln x \)

Solution  This is of the form \( 0 \cdot \infty \). Hence rewrite it as
\[
\lim_{x \to 0} \frac{\ln x}{1/x^2} = \frac{\infty}{\infty}
\]

Applying L'Hôpital's rule, the above limit becomes
\[
\lim_{x \to 0} \frac{1/x}{-2/x^3} = \lim_{x \to 0} \frac{1}{-2x^2}
\]

Caution  It is sometimes wiser to simplify an expression before blindly applying L'Hôpital's rule. For instance, in the above expression, L'Hôpital's rule need not be used further, if it is simplified.

The above expression is equivalent to
\[
\lim_{x \to 0} -x^2/2 = 0
\]

Example #6  Find \( \lim_{x \to 0} \frac{1}{e^x-1} - \frac{1}{x} \)

Solution  This is of the form \( \infty - \infty \).

Rewrite it as
\[
\lim_{x \to 0} \frac{x-e^x+1}{xe^x-x} = \frac{0}{0}
\]

This is of the form \( 0/0 \). Hence using L'Hôpital's rule, we get the value of the limit as \(-1/2\).
1.5 Other Indeterminate Forms

Expressions such as \( f(x)^{g(x)} \) give rise to indeterminate forms like \( 0^0, \infty^0 \) and \( 1^\infty \). Note that \( 1^\infty \) is not \( 1 \).

Example #7  Find \( \lim_{x \to 0} (1 + 3x)^{\frac{1}{2x}} \)

Solution  This is of the form \( 1^\infty \)

a. Put \( y = (1 + 3x)^{\frac{1}{2x}} \)

b. \( \ln y = (\frac{1}{2x}) \ln(1 + 3x) \)

c. The right-hand expression is of the form \( 0/0 \). Hence use L'Hôpital's rule.

\[
\lim_{x \to 0} \ln y = \lim_{x \to 0} \ln(1 + 3x) = \lim_{x \to 0} \frac{3}{2x} = 3/2
\]

d. Therefore, \( \lim_{x \to 0} (1 + 3x)^{\frac{1}{2x}} = \lim_{x \to 0} y = e^{3/2} \)

Guidelines for the above rule.

1. Let \( y = f(x)^{g(x)} \)

2. Take logarithms: \( \ln y = g(x) \ln f(x) \)

3. Find \( \lim_{x \to c} \ln y \). Use L'Hôpital's rule where needed.

4. If \( \lim_{x \to c} \ln y = L \), then \( \lim_{x \to c} y = e^L \)
3. \( \lim_{x \to \infty} x \sin \left( \frac{1}{x} \right) = \) 
   a. 0  b. 1  c. \( \infty \)  d. non-existent

4. \( \lim_{x \to 0^-} x \sin x = \) 
   a. 0  b. 1  c. \( \infty \)  d. non-existent

5. \( \lim_{x \to 0} \frac{1}{x} - \frac{1}{x^2} = \) 
   a. 0  b. \( \infty \)  c. \( -\infty \)  d. 1
2. FUNCTIONS

2.1 Continuous Function

A continuous function can be defined in very simple terms as a function whose graph can be drawn without lifting the pencil from the paper.

Mathematically,

A function \( f(x) \) is said to be continuous for \( x = a \), if the limiting value of the function when \( x \) approaches \( a \) is, the value assigned to the function for \( x = a \).

Symbolically,

If \( \lim_{x \to a} f(x) = f(a) \) and \( f(a) \) is defined

then \( f(x) \) is continuous for \( x = a \).

Example #1 Consider the function \( f(x) = \frac{x^2-4}{x-2} \)

Is this continuous at \( x = 1 \).

Solution \( \lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2-4}{x-2} = \lim_{x \to 1} \frac{(x+2)(x-2)}{x-2} = \lim_{x \to 1} (x+2) = 3 \)

That is, \( \lim_{x = 1} f(x) = f(1) \)

Hence the function \( f \) is continuous for \( x = 1 \).
2.2 Discontinuity of a function

A function is discontinuous at \( x = a \), if either one of the following conditions is true.

1. **The function is not defined for** \( x = a \).

**Example #2** Let \( f(x) = \frac{x^2+4}{x^2-9} \). Is this continuous at \( x = 3 \).

**Solution** For \( x = 3 \), \( x^2-9 = 0 \)
Hence the function \( f(x) \) is not defined at \( x = 3 \).
Therefore \( f \) is discontinuous at \( x = 3 \).

2. **The function is defined for** \( x = a \), **but has no limit as** \( x \to a \).

**Example #3** Let \( F(x) = \begin{cases} 2x+1 & -1 \leq x \leq 1 \\ (1/2)x^2 - 3 & 1 < x < 4 \end{cases} \). Is this continuous at \( x = 1 \).

**Solution** \( F(1) = 2\times1 + 1 = 3 \)
Hence the function is defined at \( x = 1 \).

The left-hand limit of \( f \) is \( 2\times1+1 = 3 \).
The right-hand limit of \( f \) is \( (1/2)\times1^2-3 = -5/2 \).

Since the one-sided limits of \( f \) are not equal, the limit does not exist for \( x = 1 \).

Hence \( f \) is discontinuous at \( x = 1 \).
3. The function is defined for \( x = a \) and has a limit for \( x \to a \), but the limit is not equal to the value of the function. That is, 

\[
\lim_{x \to a} f(x) \text{ is not equal to } f(a).
\]

**Example #4** Let \( f(x) = \frac{2x^2 - 2x}{x-1} \) and \( f(1) = 1 \). Is this continuous at \( x = 1 \)?

**Solution** Since \( f(1) = 1 \), the function \( f \) is defined for \( x = 1 \).

\[
\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{2x^2 - 2x}{x-1} = \lim_{x \to 1} \frac{(2x)(x-1)}{x-1} = \lim_{x \to 1} 2x = 2.
\]

Although the function is defined and the limit exists for \( x \to a \), this is a discontinuous function, since

\[
\lim_{x \to 1} f(x) \text{ is not equal to } f(1).
\]
6. Is $\sin x$ continuous at $x = 0$.
   a. YES  b. NO

7. The function $f(x) = \frac{x^2 - x - 2}{x^2 - 2x}$ is discontinuous at $x = 2$.
   Which of the following are true about this function.
   
   i. $f(2)$ is not defined.
   ii. $\lim_{x \to 2} f(x)$ does not exist.
   iii. $\lim_{x \to 2} f(x)$ is not equal to $f(2)$.

   a. i only
   b. ii only
   c. i and iii only
   d. i and ii only
   e. i, ii and iii
2.3 Discontinuity in an Interval

Let us make sure that we know all about open and closed intervals. Suppose \( x = (a, b] \). The left bracket is called an open bracket and the right bracket is called a closed bracket. The expression means that \( x \) can take all values between \( a \) and \( b \) including \( b \) but not \( a \).

Now continuity in an interval can be defined as follows: A function \( f(x) \) is said to be continuous in an interval when it is continuous for all values of \( x \) in this interval.

If in addition, \( \lim_{x \to a^-} f(x) = f(a) \) and \( \lim_{x \to b^-} f(x) = f(b) \),

then \( f \) is continuous on the closed interval \([a,b]\).

In the above statement, the first limit is called the right-hand limit and the second limit is called the left-hand limit.

If a function \( f \) has either a right-hand or a left-hand limit, we say \( f \) is continuous from the right at \( a \) or \( f \) is continuous from the left at \( b \), respectively.
Example #5  Consider the function \( f(x) = \sqrt{9-x^2} \)
Is it continuous in the closed interval \([-3,3]\).

Solution  Let \( c \) be a value in the interval \((-3,3)\)
That is, \(-3 < c < 3\)

\[
\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{9-x^2} = \sqrt{9-c^2}
\]

Hence by definition of continuity at a point, the function 
f is continuous at \( x = c \). Since we have a closed interval, 
we need to check the continuity at the points -3 and 3 also.

\[
\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9-x^2} = \sqrt{9-9} = 0
\]

Hence \( f \) is continuous from the right at \( x = -3 \).

\[
\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{9-x^2} = \sqrt{9-9} = 0
\]

Hence \( f \) is continuous from the left at \( x = 3 \).
Therefore \( f \) is continuous on \([-3,3]\)

Example #6  Discuss the continuity of \( f \) if

\[
f(x) = \frac{\sqrt{x^2-9}}{x-4}
\]

Solution  \( f(x) \) is undefined at \( x = 4 \), since the 
denominator becomes zero at that point.

\( f(x) \) is undefined for \(-3 < x < 3\) also since the radicand
\( x^2-9 \) becomes negative in this interval.

So the intervals we should consider are
\((-\infty,-3], [3,4) \) and \((4,\infty)\)

The continuity within these intervals can be determined as
shown in the previous example.

Note again that \([3,4)\) means that \( x \) can take the value 3 and
all the values between 3 and 4 but not 4.
8. The function
\[ f(x) = \frac{\sqrt{x^2 - 9} \sqrt{25 - x^2}}{x-4} \]
is continuous in which of the following intervals?

i. \((-\infty, 0)\)  ii. \([-5, -3]\)  iii. \([-3, 3]\)

iv. \([3, 4]\)  v. \((4, 5]\)

a. i only
b. ii and iii only
c. ii and iv only
d. iii and v only
e. ii, iv and v only
2.4 Increasing and Decreasing Functions

**Definition:** Given $x_2 > x_1$ and both $x_1$ and $x_2$ are in the interval $I$, then a function $f$ is said to be

i) increasing on the interval $I$, if $f(x_2) \geq f(x_1)$

ii) decreasing on the interval $I$, if $f(x_2) \leq f(x_1)$

To put it in simpler words,

As $x$ increases, if the graph rises, the function is increasing; if the graph falls, the function is decreasing.

The increasing or decreasing nature of any function can be very easily determined by a test called the **first derivative test**.

**Note:** If you don't remember differential calculus, stop here and finish the section on "THE DERIVATIVE" and come back to this section.

**First Derivative Test:** Let $f$ be a function that is continuous on a closed interval $[a,b]$ and differentiable on the open interval $(a,b)$.

i. if $f'(x) \geq 0$ for all $x$ in $(a,b)$, then $f$ is increasing on $[a,b]$.

ii. if $f'(x) \leq 0$ for all $x$ in $(a,b)$, then $f$ is decreasing on $[a,b]$. 
Example #7  If \( f(x) = x^3 + x^2 - 5x - 5 \) find the intervals in which \( f \) is increasing and decreasing.

Solution  \[ f'(x) = 3x^2 + 2x - 5 = (3x+5)(x-1) \]

\( f'(x) \) becomes zero at 
\[ x = -5/3, 1 \]

These are called critical values of the function.

Hence consider the intervals, 
\[ (-\infty, -5/3), (-5/3, 1) \text{ and } (1, \infty) \]

For any \( x \) in the intervals \((-\infty, -5/3)\) and \((1, \infty)\) 
\[ f'(x) > 0. \]
Hence \( f \) is increasing on these intervals.

For any \( x \) in the interval \((-5/3, 1)\) 
\[ f'(x) < 0. \]
Hence \( f \) is decreasing on this interval.

Here is one last example, before we end the section on functions.
Example #8  Let \( f(x) = \frac{1}{2}(x - \sin x) \). Is \( f \) increasing or decreasing on \([0, 2\pi]\)?

Solution  \( f'(x) = \frac{1}{2} - \cos x \)

The critical values of the function are at \( x = \pi/3, 5\pi/3 \) (obtained by solving \( f'(x) = 0 \))

Hence consider the intervals
\([0, \pi/3], [\pi/3, 5\pi/3], \text{ and } [5\pi/3, 2\pi] \)

For any \( x \) in the intervals \([0, \pi/3]\) and \([5\pi/3, 2\pi]\)
\( f'(x) < 0 \).
Hence \( f \) is decreasing on these intervals.

For any \( x \) in the interval \([\pi/3, 5\pi/3]\)
\( f'(x) > 0 \).
Hence \( f \) is increasing on this interval.

If you are wondering how to find out if \( f'(x) \) is > or < 0 in any interval, here is how it is.

Take a value in the interval for \( x \) and substitute in \( f'(x) \).

For example,
\( \pi/2 \) is a value in the interval \([\pi/3, 5\pi/3]\)
\( f'(x) \) at \( x = \pi/2 = \frac{1}{2} - \cos(\pi/2) \)
\[ = \frac{1}{2} > 0 \]
9. If \( f(x) = 2\sin x + \cos 2x \), the critical numbers of \( f \) in the interval \([0, 2\pi]\) are

(Hint: \( \sin \pi/6 = 1/2; \cos \pi/6 = \sqrt{3}/2; \sin \pi/2 = 1; \cos \pi/2 = 1 \))

a. \( \pi/6 \) and \( 5\pi/6 \) only
b. \( \pi/2 \) and \( 3\pi/2 \) only
c. \( \pi/6, 5\pi/6, \pi/2, 3\pi/2 \) only
d. \( \pi/6, 5\pi/6, \pi/2, 3\pi/2 \) and \( 2\pi \)

10. If \( f(x) = x^4 + 2x^3 \) which of the following is/are true about the function.

a. increasing for \( x \leq -3/2 \)  
   b. decreasing for \( x \geq -3/2 \)

b. decreasing for \( x \leq -3/2 \)  
   d. increasing for \( x \geq -3/2 \)


ANSWERS

1. b
2. c
3. b
4. d
5. c
6. a
7. c
8. e
9. c
10. c, d
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Definition</td>
<td>1-4</td>
</tr>
<tr>
<td>3.2</td>
<td>Trigonometric Functions</td>
<td>5-6</td>
</tr>
<tr>
<td>3.3</td>
<td>Inverse Trigonometric Functions</td>
<td>7-8</td>
</tr>
<tr>
<td>3.4</td>
<td>Logarithm and Exponential Functions</td>
<td>9-11</td>
</tr>
<tr>
<td>3.5</td>
<td>Logarithmic Differentiation</td>
<td>12-13</td>
</tr>
<tr>
<td>3.6</td>
<td>Implicit Differentiation</td>
<td>14-15</td>
</tr>
<tr>
<td>3.7</td>
<td>Partial Differentiation</td>
<td>16-17</td>
</tr>
<tr>
<td>3.8</td>
<td>Higher Order Derivatives</td>
<td>18-19</td>
</tr>
<tr>
<td></td>
<td>ANSWERS</td>
<td>20</td>
</tr>
</tbody>
</table>
3. THE DERIVATIVE

3.1 Definition: The derivative of a function is the limit of the ratio of the increment of the function to the increment of the independent variable, when the latter increment varies and approaches zero as a limit.

Symbolically,

\[ \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \]

The derivative can be denoted as \( \frac{dy}{dx} \), \( y' \) or \( f'(x) \).

Theorem 1: The value of the derivative at any point of a curve is equal to the slope of the tangent line to the curve at that point.

Example #1: Find the slopes of the tangents to the parabola, \( y = x^2 \) at the vertex and at the point \( x = 1/2 \).

Solution: From the above theorem, the slope of the tangent at any point is the derivative of the curve at that point.

We have \( \frac{dy}{dx} = 2x \)
At the vertex, \( x = 0 \)
Hence \( \frac{dy}{dx} = 0 \).

That is, the slope of the tangent at the vertex is zero. Similarly, at \( x = 1/2 \), the slope is 1.
We now give some important formulas, which you should memorize.

1. $\frac{d}{dx} (k) = 0$ \hspace{1cm} \text{k is a constant.}$

2. $\frac{d}{dx} (u + v + w) = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}$ \hspace{1cm} \text{(sum rule)}$

3. $\frac{d}{dx} (ku) = k \frac{du}{dx}$

4. $\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}$ \hspace{1cm} \text{(product rule)}$

5. $\frac{d}{dx} (u_1u_2 \ldots u_n) = (u_1u_2 \ldots u_n) \frac{du_1}{dx} +$ \hspace{1cm}$ (u_1u_2 \ldots u_n) \frac{du_2}{dx} + \ldots$ \hspace{1cm}$ + (u_1u_2 \ldots u_{n-1}) \frac{du_n}{dx}$

6. $\frac{d}{dx} (u^n) = n u^{n-1} \frac{du}{dx}$ \hspace{1cm} \text{(chain rule)}$

\text{Caution: Do not forget to differentiate } u. \text{ u is a function of } x.$

7. $\frac{d}{dx} (x^n) = n x^{n-1}$ \hspace{1cm} \text{(power rule)}$

8. $\frac{d}{dx} \left( \frac{u}{v} \right) = \left( \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \right)$ \hspace{1cm} \text{(quotient rule)}$

Formulas 4 and 8 can also be written as

$(uv)' = uv' + vu'$ \hspace{1cm} \text{and}$

$(u/v)' = \frac{vu' - uv'}{v^2}$ \hspace{1cm} \text{respectively}$
Example #2  If \( y = (x^2 - 3)^5 \) find \( \frac{dy}{dx} \).

Solution  Let \( u = x^2 - 3 \) and \( n = 5 \)

From formula (6),
\[
\frac{dy}{dx} = 5(x^2-3)^4 \frac{d}{dx}(x^2-3)
\]
\[
= 5(x^2-3)^4 \cdot 2x
\]
\[
= 10x(x^2-3)^4
\]

Example #3  If \( y = \sqrt{\frac{1-cx}{1+cx}} \) find \( \frac{dy}{dx} \).

Solution  Let \( u = (1-cx)^{1/2} \) and \( v = (1+cx)^{1/2} \)

From formula (8),
\[
\frac{dy}{dx} = \frac{(1+cx)^{1/2} \frac{d}{dx}(1-cx)^{1/2} - (1-cx)^{1/2} \frac{d}{dx}(1+cx)^{1/2}}{(1+cx)}
\]
\[
= \frac{(1+cx)^{1/2} (1/2)(1-cx)^{-1/2}(-c) - (1-cx)^{1/2} (1/2)(1+cx)^{-1/2}c}{(1+cx)}
\]

Simplifying, we get
\[
\frac{dy}{dx} = \frac{-c}{(1+cx)\sqrt{(1-c^2x^2)}}
\]

Note that \( \frac{d}{dx} (1-cx)^{1/2} \) is found using the chain rule.
TEST

11. If \( y = x^2 \sqrt[3]{1+x^3} \), then \( \frac{dy}{dx} \) at \( x=1/2 \) is

a. 38/3
b. 20
c. 2/3
d. 8

12. If \( y = \frac{\sqrt[3]{1+2x}}{\sqrt[3]{1+3x}} \), then to find \( \frac{dy}{dx} \), which of the following formulas need to be used?

i. \( \frac{d}{dx} (u^n) \)
ii. \( \frac{d}{dx} (u/v) \)
iii. \( \frac{d}{dx} (kx) \)
iv. \( \frac{d}{dx} (uv) \)

a. ii only
b. i and ii only
c. i, ii and iii only
d. all the above
3.2 Trigonometric Functions

1. d/dx (sin x) = cos x
2. d/dx (cos x) = -sin x
3. d/dx (tan x) = sec^2 x
4. d/dx (cot x) = -csc^2 x
5. d/dx (sec x) = sec x tan x
6. d/dx (csc x) = -csc x cot x

Example #4  If \( Y = x^2 \sec^2(3x) \), find \( Y' \).

Solution \( Y' = x^2 \cdot 2\sec(3x) \cdot \frac{d}{dx} (\sec3x) + \sec^23x \cdot (3x^2) \)  
(product rule and chain rule)

\[
= 2x^2 \cdot \sec3x \cdot \sec3x \cdot \tan3x \cdot 3 + 3x^2 \cdot \sec^23x \\
= 3x^2 \cdot \sec^23x \cdot (2x\tan3x+1)
\]
13. The derivative of \( f(x) \), when \( f(x) = \sin^3 x \) is \( \frac{\sqrt{\cos x}}{\cos x} \)

a. \( \frac{(\sqrt{\cos x}) 3 \sin^2 x - \sin^3 x (1/2)(\cos x)^{-1/2})}{\cos x} \)

b. \( \frac{(\sqrt{\cos x}) \cos^3 x - \sin^3 x \sqrt{\sin x})}{\cos x} \)

c. \( \frac{(\sqrt{\cos x}) 3 \sin^2 x \cos x + \sin^3 x (1/2)(\cos x)^{-1/2} \sin x)}{\cos x} \)
3.3 Inverse Trigonometric Functions

The following formulas are given just to give a completeness to the program. There is no need to memorize them since we will show how to derive them.

1. \( \frac{d}{dx} (\text{arc sin} x) = \frac{1}{\sqrt{1 - x^2}} \)

2. \( \frac{d}{dx} (\text{arc cos} x) = -\frac{1}{\sqrt{1 - x^2}} \)

3. \( \frac{d}{dx} (\text{arc tan} x) = \frac{1}{1 + x^2} \)

4. \( \frac{d}{dx} (\text{arc cot} x) = -\frac{1}{1 + x^2} \)

5. \( \frac{d}{dx} (\text{arc sec} x) = \frac{1}{x \sqrt{x^2 - 1}} \) if \( |x| < 1 \)

6. \( \frac{d}{dx} (\text{arc csc} x) = -\frac{1}{x \sqrt{x^2 - 1}} \) if \( |x| < 1 \)
Example #5  To find dy/dx, given $y = \arcsin x$.

Solution  The above equation is equivalent to $\sin y = x$

Differentiating this equation, we get

$$\cos y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

Note that all the above formulas can be derived on similar lines.

Example #6  To find dy/dx, given $y = \arctan x^2$.

Solution  The above equation is equivalent to $\tan y = x^2$

Differentiating this equation, we get

$$\sec^2 y \frac{dy}{dx} = 2x$$
$$\frac{dy}{dx} = \frac{2x}{\sec^2 y} = \frac{2x}{(1 + \tan^2 y)}$$

(since $\sec^2 y - \tan^2 y = 1$)
$$= \frac{2x}{1 + (x^2)^2}$$
$$= \frac{2x}{1 + x^4}$$

The same result can be obtained using formula (3).

$$\frac{dy}{dx} = \frac{1}{1 + (x^2)^2} \cdot 2x$$
$$= \frac{2x}{1 + x^4}$$

TEST

14. If $y = \arccot 3$ then $dy/dx =$

a. $\pi/3$  b. $\pi/6$  c. $\pi/2$  d. $\pi$
3.4 Logarithm and Exponential Functions

There are only three basic formulas to be remembered in this section. However, they can be manipulated to arrive at several other related formulas.

1. \( \frac{d}{dx} (\ln x) = \frac{1}{x} \)
2. \( \frac{d}{dx} (e^x) = e^x \)
3. \( \frac{d}{dx} (a^x) = a^x \ln a \)

The related formulas are obtained by considering a function \( u \) in place of \( x \).

4. \( \frac{d}{dx} (\log_a u) = \frac{1}{u \ln a} \frac{du}{dx} \)
5. \( \frac{d}{dx} (e^u) = e^u \frac{du}{dx} \)
6. \( \frac{d}{dx} (a^u) = a^u \ln a \frac{du}{dx} \)
Example #7  Given $Y = \ln(x^2 + 5)$, find $Y'$.

Solution  
Given $y = (1/2) \ln(x^2+5)$, then 

$$Y' = (1/2) \frac{1}{x^2+5} 2x$$
$$= \frac{x}{x^2+5}$$

Example #8  Given $Y = x^2e^b$, where $b = -x^2$ find $Y'$.

Solution  

$$Y' = x^2 \frac{d}{dx}(e^b) + e^b \frac{d}{dx}(x^2)$$
$$= x^2 e^b \frac{d}{dx}(b) + 2x e^b$$

(Chain rule)
$$= x^2 e^b - 2x + 2x e^b$$
$$= 2x e^b (1-x^2)$$

Example #9  Given $Y = 2^b$, where $b = x^2$ find $Y'$.

Solution  
This is of the form $a^u$ and not of the form $a^x$.

Hence $Y' = 2^b \ln 2 \frac{d}{dx}(b)$
$$= 2b \ln 2 \cdot 2x$$
15. Given $f(x) = e^{3x} \ln(x^2 + 3)$, $f'(x) =$
   a. $e^{3x} \left( \frac{1}{x^2 + 3} \right) + \ln(x^2 + 3) \ e^{3x}$
   b. $e^{3x} \left( \frac{2x}{x^2 + 3} \right) + \ln(x^2 + 3) \ e^{3x}$
   c. $e^{3x} \left( \frac{2x}{x^2 + 3} \right) + \ln(x^2 + 3) \ e^{3x} \ 3$

16. Given $f(x) = 4^{-2x}$, $f'(x) =$
   a. $4^{-2x} \ln 4$
   b. $4^{-2x}$
   c. $4^{-2x} \ (-2) \ln 4$
   d. $-2x \ 4^{-2x-1}$
3.5 Logarithmic Differentiation

Sometimes we encounter expressions of the form $u(x)^v(x)$, where $u$ and $v$ are both functions of $x$. For such functions, we use a method called logarithmic differentiation.

Example #10 If $f(x) = x^x$, find $f'(x)$.

Solution Rewrite the above equation as $y = x^x$.

Taking logarithms, we have $\ln y = x \ln x$.

Differentiating, we get

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} + \ln x$$

$$\frac{dy}{dx} = y \left(1 + \ln x\right)$$

$$= x^x \left(1 + \ln x\right)$$

Caution: The above function is not of the form $a^x$.

This method can be extended to other types of problems, to save a lot of time and labor.

Example #11 Given $f(x) = \frac{x^a(3x+2)^{1/3}}{(2x-3)^3}$, $x$ not equal to $3/2$, 0, $-2/3$. Find $f'(x)$.

Solution Isn't this ugly. In such cases, we take logarithms.

$$\ln f(x) = \ln(x^a) + \ln (3x+2)^{1/3} - \ln (2x-3)^3$$

$$= 2 \ln x + (1/3) \ln(3x+2) - 3 \ln(2x-3)$$

Differentiating, we get

$$(1/f(x)) f'(x) = 2 \left(1/x\right) + (1/3)\left(1/(3x+2)\right) - 3\left(1/(2x-3)\right)$$

$$= 2/x + 1/(3x+2) - 6/(2x-3)$$

Therefore,

$$f'(x) = \frac{x^a(3x+2)^{1/3}}{(2x-3)^3} \left[2/x + 1/(3x+2) - 6/(2x-3)\right]$$
17. If \( y = (\ln x)^x \), \( y' = \)

a. \((\ln x)^x \ln(\ln x)\)

b. \((\ln x)^x [x (1/\ln x) + \ln x]\)

c. \((\ln x)^x (1/\ln x + \ln(\ln x))\)

d. non-existent

18. If \( y = \frac{j(x+2)}{j(x+3)} \), then to find \( y' \) the best approach is to use

a. product and quotient rules

b. implicit differentiation

c. logarithmic differentiation

d. trigonometric substitution
3.6 Implicit Differentiation

Consider the following functions.

\[ x^2 - 2xy + 3y^2 - 7 = 0, \quad x^3y^2 - 4 = 0, \quad y^3 - x - 5 = 0. \]

These are of the form \( f(x,y) = 0 \). These functions are said to be defined implicitly by the equation. This is one of the types of differentiation where errors are committed very often.

Example #12 Find \( y' \) if \( y^2 + x^2 = 1 \)

Solution Do not rewrite this equation as \( y = \sqrt{1-x^2} \).

Differentiate the equation as it is.

\[ 2y\ y' + 2x \]

Therefore,

\[ y' = -2x/2y = -x/y \]

Example #13 Find \( y' \) if \( y = x \sin y \)

Solution This is an implicit function, since \( y \) is a function of \( x \) and \( y \).

Therefore,

\[ y' = x \cos y \ y' + (\sin y) \]

\[ y'(1-x \cos y) = \sin y \]

\[ y' = \frac{\sin y}{(1-x \cos y)} \]

Caution: The common error is to write the derivative of \( y \) as

\[ y' = x \cos y + \sin y \]

Note: \( \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \)
19. If \( x^2 - xy + y^2 = 1 \) then \( \frac{dx}{dy} = \)

a. \( \frac{y-2x}{2y-x} \)

b. \( \frac{2y-x}{y-2x} \)

c. \( \frac{-2x}{2y-x} \)

d. \( \frac{2y-x}{-2x} \)

20. If \( \cos (x-y) = y \) then \( \frac{dy}{dx} = \)

a. \( \frac{-\sin(x-y)}{1-\sin(x-y)} \)

b. \( -\sin(x-y) \)

c. \( \frac{-(1-\sin(x-y))}{\sin(x-y)} \)

d. none of the above
3.7 Partial Differentiation

Let $z$ be a function of two variables, $z = f(x,y)$. If $y$ is held fixed, $z$ becomes a function of $x$ alone, and its derivative can be found. The result is called the partial derivative of $z$ with respect to $x$ and is denoted by $f_x$ or $z_x$.

**Example #14** If $u = \sin(2x+3y) + \cos(y-2z)$, find $u_x$, $u_y$, $u_z$.

**Solution** Treating $y$ and $z$ as constants, we have

$u_x = 2 \cos(2x+3y)$

Treating $z$ and $x$ as constants, we have

$u_y = 3 \cos(2x+3y) - \sin(y-2z)$

Treating $x$ and $y$ as constants, we have

$u_z = 2 \sin(y-2z)$

It is very important to note which are the independent variables. Note particularly that $u_x$ is not equal to $1/x$.

**Example #15** If $z = x/y + y/x$, find $z_x$ and $z_y$.

**Solution** Keeping $y$ constant,

$z_x = (1/y)1 + y (-1/x^2) = 1/y - y/x^2$

Keeping $x$ constant,

$z_y = x (-1/y^2) + (1/x)1 = -x/y^2 + 1/x$
21. If \( w = xyz e^{xyz} \), then \( w_x = \)

a. \( yz e^{xyz} \) yz

b. \( xy e^{xyz} \) (xyz+1)

c. \( xz e^{xyz} \) (xyz+1)

d. \( yz e^{xyz} \) (xyz+1)
3.8 Higher Order Derivatives

The derivative with respect to \( x \) of \( \frac{dy}{dx} \) is called the second derivative of \( y \) with respect to \( x \), or the derivative of second order. The derivative of the second derivative is called the third derivative of \( y \) with respect to \( x \). Similarly, we may speak of the fourth, fifth, ..., \( n \)th derivative of \( y \) with respect to \( x \).

The notation is \( \frac{d}{dx}(y) \), \( \frac{d^2}{dx^2}(y) \), ..., \( \frac{d^n}{dx^n}(y) \)

The notation \( y', y'', y''', ..., y^{(n)} \) is also popular.

This differentiation involves no new formulas. All the previous formulas learnt are applicable here.

**Example #16** If \( y = x^4 + x \), find the third derivative of \( y \).

**Solution** \( \frac{dy}{dx} = 4x^3 + 1 \)

Differentiating the above expression again, we get

\[
\frac{d^2y}{dx^2} = 12x^2 \quad \text{which is the second derivative of} \quad y.
\]

\[
\frac{d^3y}{dx^3} = 24x \quad \text{which is the required third derivative of} \quad y.
\]

**Example #17** If \( y = x^2 + xy \), find \( \frac{d^2y}{dx^2} \).

**Solution** In other words, the second derivative of \( y \) with respect to \( x \) is to be found.

\[
\frac{dy}{dx} = 2x + x\left(\frac{dy}{dx}\right) + y
\]

\[
\frac{dy}{dx} (1-x) = 2x + y
\]

\[
\frac{dy}{dx} = \frac{2x + y}{1 - x}
\]

Differentiating again, we have

\[
\frac{d^2y}{dx^2} = \frac{(1-x)(2+\frac{dy}{dx}) - (2x+y)(-1)}{(1-x)^2}
\]

Substituting for \( \frac{dy}{dx} \), we have

\[
\frac{d^2y}{dx^2} = \frac{(1-x)(2+(2x+y)/(1-x)) + (2x+y)}{(1-x)^2}
\]

Simplifying the above expression,

\[
\frac{d^2y}{dx^2} = \frac{2(1+x+y)}{(1-x)^2}
\]
22. If \( y = \sin 3x \), then \( \frac{d^3y}{dx^3} = \)

a. \(-27 \cos 3x\)

b. \(-\cos 3x\)

c. \(-9 \sin 3x\)

d. \(27 \sin 3x\)
ANSWERS

11. b
12. c
13. c
14. b
15. c
16. c
17. c
18. c
19. b
20. a
21. d
22. a
# TABLE OF CONTENTS

## 4. SERIES

4.1 Power Series .............................................. 1
4.2 Power Series Representation of Functions .......... 1
4.3 Taylor's Series ............................................. 3
4.4 Maclaurin's Series ........................................... 4

TEST ................................................................. 5

## 5. MEAN VALUE THEOREM ........................................ 6

5.1 Mean Value Theorem for Definite Integrals .......... 8

TEST ................................................................. 9

ANSWERS ............................................................ 10
4. SERIES

Since this is only a refresher course, let us look at only a few important series.

4.1 Power Series

If x is a variable and $a_0, a_1, \ldots, a_n$ are constants, a series of the form

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots = \sum_{n=0}^{\infty} a_n x^n$$

is called a power series in x.

4.2 Power Series Representation of Functions

A power series $\sum a_n x^n$ may be used to define a function $f$. For each $x$ in the interval of convergence, we let $f(x)$ equal the sum of the series, that is,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

If a function $f$ is defined in this way, we say that $\sum a_n x^n$ is a power series representation for $f(x)$. 
Since the actual technique of finding the power series representation for various functions is outside the scope of this program, we list the expansions of some standard series.

1. \[(1+x)^m = \sum_{n=0}^{m} \binom{m}{n} x^n \]
2. \[\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots \]
3. \[\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \ldots \]
4. \[\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \]
5. \[\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots \]
6. \[\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \]
7. \[\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \]
8. \[e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \]
9. \[e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots \]
10. \[\cosh x = \frac{(e^x + e^{-x})}{2} \]
11. \[\sinh x = \frac{(e^x - e^{-x})}{2} \]
4.3 Taylor's Series

The power series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

represents the function $f(x)$ for those, and only those values of $x$, for which the remainder approaches zero as $n \to \infty$.

The remainder is $R_n = \frac{f^{(n+1)}(\delta)}{(n+1)!}(x-a)^{n+1}$

(\delta between $a$ and $x$)

Writing $x-a = h$, we get the Taylor's series

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \ldots + \frac{f^{(n)}(a)}{n!}h^n + \ldots$$

a power series in $h$. 
4.4 Maclaurin's Series

This series is a special case of Taylor's series, namely, when \( a = 0 \).

**Example #1** Find the Taylor's series for \( \sin x \) in powers of \( x-\pi/6 \).

**Solution** We have \( x-\pi/6 = h \), that is, \( a = \pi/6 \)

\[
\begin{align*}
    f(\pi/6) &= \sin(\pi/6) = 1/2 \\
    f'(\pi/6) &= \cos(\pi/6) = \sqrt{3}/2 \\
    f''(\pi/6) &= -1/2 \\
    f'''(\pi/6) &= -\sqrt{3}/2
\end{align*}
\]

This pattern repeats itself. Substituting in the above formulas for Taylor's series

\[
\sin x = 1/2 + (\sqrt{3}/2)(x-\pi/6) - 1/(2*2!)(x-\pi/6)^2 - \ldots.
\]

The Maclaurin series for \( \sin x \) is obtained by putting \( a = 0 \).

**Example #2** Find \( \sin(0.1) \)

**Solution** Put \( x = 0.1 \) in the Maclaurin series for \( \sin x \)

\[
\sin(0.1) = 0.1 - (0.1)^3/3! + (0.1)^5/5! - \ldots.
\]
23. $1 - (0.2)^2/2! + (0.2)^4/4! - ...$ is a Maclaurin expansion for which one of the following.

a. $\sin(0.2)$

b. $\sinh(0.2)$

c. $\cosh(0.2)$

d. $\cos(0.2)$

e. $\ln(1.2)$
5. MEAN VALUE THEOREM

If a function $f$ is continuous on a closed interval $[a,b]$ and is differentiable on the open interval $(a,b)$, then there exists a number $c$ in $(a,b)$ such that

$$f(b) - f(a) = f'(c)(b-a)$$

This is also known as the Theorem of the Mean.

Example #1  Prove that the function $f$ defined by

$$f(x) = x^3 - 8x - 5$$

satisfies the hypothesis of the Mean Value Theorem on the interval $[1,4]$.

Solution  Since $f$ is a polynomial function, it is continuous and differentiable for all real numbers.

In particular, it is continuous on $[1,4]$ and differentiable on the open interval $(1,4)$.

Refer to example #15 in the section Limits and Functions to find out how the continuity of a function is determined in an interval.

Thus the hypothesis of the Mean Value Theorem is satisfied.
Example #2  For the previous problem, find a number c in the interval (1,4) that satisfies the conclusion of the theorem.

Solution  According to the Mean Value Theorem, there exists a number c in the interval (1,4) such that

\[ f(4) - f(1) = f'(c)(4-1) \]

Since \( f'(x) = 3x^2 - 8 \), this is equivalent to

\[ 27 - (-12) = (3c^2 - 8)(3) \]

Solving, we get \( c = \pm \sqrt{7} \)

Hence the desired number in the interval (1,4) is \( \sqrt{7} \).
5.1 Mean Value Theorem for Definite Integrals

If \( f \) is continuous on a closed interval \([a, b]\), then there is a number \( z \) in the open interval \((a, b)\) such that

\[
\int_{a}^{b} f(x) \, dx = f(z)(b-a)
\]

**Example #3** If \( \int_{0}^{3} [4 - (x^2/4)] \, dx = 39/4 \)

Find a number that satisfies the conclusion of the Mean Value Theorem.

**Solution** We have \( f(x) = 4 - (x^2/4) \)

Therefore,

\[
\int_{0}^{3} [4 - (x^2/4)] \, dx = (4 - (z^2/4))(3 - 0)
\]

Equivalently,

\[
39/4 = (16 - z^2)(3/4)
\]

\[
z^2 = 3
\]

Therefore, \( z = \sqrt{3} \) satisfies the conclusion of the Mean Value Theorem.

**Note:** The Mean Value Theorem is useful in solving several theorems in calculus, most importantly the Fundamental Theorem of Calculus.
24. The number that satisfies the conclusion of the mean value theorem for \( \int_0^3 3x^2 \, dx = 27 \) is

a. 3  

b. \( \sqrt{3} \)  

c. 0  

d. 1
ANSWERS

23. d
24. b
# TABLE OF CONTENTS

6. INTEGRATION ................................................................. 1
   6.1 Fundamental Properties of Integrals .................................. 1
   6.2 Trigonometric Formulas .................................................. 2
   6.3 Method of Substitution .................................................... 3
       TEST ................................................................. 5
   6.4 Definite Integral ........................................................... 6
   6.5 Theorems on Integration .................................................. 6
   6.6 Properties of the Definite Integral .................................... 7
   6.7 Fundamental Theorem of Calculus ....................................... 7
   6.8 Method of Substitution Revisited ....................................... 9
       TEST ................................................................. 10
   6.9 Integration by Parts ....................................................... 11
       TEST ................................................................. 13
   6.10 Trigonometric Integrals .................................................. 14
   6.11 Guidelines for evaluating integrals of the form \( \int \sin^m x \cos^n x \, dx \) ......................................................... 15
   6.12 Guidelines for evaluating integrals of the form \( \int \tan^m x \sec^n x \, dx \) ......................................................... 17
       TEST ................................................................. 19
   6.13 Trigonometric Substitution .............................................. 20
       TEST ................................................................. 22
   6.14 Partial Fractions .......................................................... 23
       TEST ................................................................. 26
   6.15 Area ........................................................................... 27
       TEST ................................................................. 29
   6.16 Infinite Limits of Integration ............................................ 30
   6.17 Integrals with Discontinuous Integrands .............................. 30
   6.18 Double Integrals ............................................................ 32
       TEST ................................................................. 34

ANSWERS ............................................................... 35
6. INTEGRATION

In general,

\[ \int f(x) \, dx = F(x) + C \]

means that \( F(x) + C \) is a function whose differential is \( f(x) \, dx \). \( F(x) + C \) is called an integral of \( f(x) \). \( C \) is called the constant of integration. \( f(x) \) is called the integrand.

6.1 Fundamental Properties of Integrals

1. \( \int k \, du = k \int du = ku \quad (k \text{ is a constant}) \)

2. \( \int (du + dv + \ldots + dw) = \int du + \int dv + \ldots + \int dw \)
   \[ = u + v + \ldots + w + C \]

3a. \( \int u^n \, du = \frac{(u^{n+1})}{n+1} + C \quad (n \text{ is not equal to } -1) \)

3b. \( \int u^n \, du = \ln u + C \quad (n = 1) \)

Example #1 Evaluate \( \int (4 + \frac{1}{t^3} + \frac{1}{t} + 3t) \, dt \)

Solution The given integral is equivalent to

\[ \int 4 \, dt + \int \frac{1}{t^3} \, dt + \int \frac{1}{t} \, dt + \int t^{1/5} \, dt \]

\[ = 4t + \frac{t^{-3+1}}{-3+1} + \ln t + \frac{t^{6/5}}{6/5} \]

\[ = 4t - \frac{1}{2}t^2 + \ln t + \frac{(5/6) \, t^{6/5}}{6/5} \]
4. \( \int e^u \, du = e^u + C \)
5. \( \int a^u \, du = \frac{a^u}{\ln a} + C \)

6.2 Trigonometric Formulas

1. \( \int \sin u \, du = -\cos u + C \)
2. \( \int \cos u \, du = \sin u + C \)
3. \( \int \sec^2 u \, du = \tan u + C \)
4. \( \int \csc^2 u \, du = -\cot u + C \)
5. \( \int \sec u \tan u \, du = \sec u + C \)
6. \( \int \csc u \cot u \, du = -\csc u + C \)
7. \( \int \sec u \, du = \ln |\sec u + \tan u| + C \)
8. \( \int \csc u \, du = -\ln |\csc u + \cot u| + C \)
We now describe some of the different methods of integration.

6.3 Method of Substitution

Consider the integral \[ \int (4-7x)^{16} \, dx \]
Remember that the integral is not of the form \( u^n \)

Let \( u = 4-7x \)
Then,
\[ du = -7dx \]
The given integral becomes \[ \int u^{16} \cdot -1/7 \, du \]

\[ = (-1/7) u^{16}/16 + C \]
\[ = (-1/112) (4-7x)^{16} + C \]

So by making a change of variable, the above integral could be converted to a known form. This technique is known as method of substitution.

Example #2  Find \[ \int x \sin (x^2) \, dx \]
Solution  Let \( u = x^2 \), then \( du = 2x \, dx \)
Therefore the integral is \[ \int \sin u \cdot 1/2 \, du \]

\[ = (-1/2) \cos u + C \]
\[ = (-1/2) \cos x^2 + C \]
Example #3  Find $\int x \sqrt{7-6x^2} \, dx$

Solution  Let $u = 7-6x^2$, then $du = -12x \, dx$
Therefore the integral is $\int u (-1/12) \, du$

$$= (-1/12) \frac{u^{3/2}}{(3/2)} + C$$

$$= (-1/18) (7-6x^2)^{3/2} + C$$

Note that the above integral is of the form $\int u^n \, du$ where $n = 1/2$

Example #4  Evaluate $\int (\cos \sqrt{x})/\sqrt{x} \, dx$

Solution  Let $u = \sqrt{x}$, then $du = 1/(2 \sqrt{x}) \, dx$
The integral is $\int (\cos u) 2 \, du$

$$= 2 \sin u + C$$

$$= 2 \sin \sqrt{x} + C$$

Remark: This method proceeds by trial and error. If one substitution fails, another, maybe entirely different substitution has to be tried.
TEST

25. The best substitution that can be used in solving the integral \( \int (1/x) \sin (\ln x) \, dx \) is to let \( u = \)

a. \( 1/x \)
b. \( \ln x \)
c. \( \sin (\ln x) \)
d. \( x \)

26. \( \int \sec^2 x \tan x \, dx = \)

(Hint: \( d/dx (\tan x) = \sec^2 x \); \( d/dx (\sec x) = \sec x \tan x \))

a. \( \sec (\tan x) + C \)
b. \( \sec x \tan x + C \)
c. \( (1/5) \sec^5 t + C \)
d. none of the above
6.4 Definite Integral

Let $f$ be a function that is defined on a closed interval $[a,b]$. The definite integral of $f$ from $a$ to $b$, denoted by

$$\int_a^b f(x) \, dx$$

is defined as the numerical measure of the area bounded by the curve $y = f(x)$, the $x$-axis and the ordinates of the curve at $x = a$ and $x = b$. This definition presupposes that these lines bound an area, that is, the curve does not rise or fall to infinity and does not cross the $x$-axis, and both $a$ and $b$ are finite.

6.5 Theorems on Integration

1. $\int_c^d f(x) \, dx = -\int_d^c f(x) \, dx$

   That is, interchanging the limits of integration changes the sign of the integral.

2. If $f(a)$ exists, then $\int_a^a f(x) \, dx = 0$

3. If $f$ is continuous on $[a,b]$, then $f$ is integrable on $[a,b]$.

4. If $f(x)$ is defined and increasing (or at least nondecreasing) on the closed interval $a \leq x \leq b$, then it is integrable there.
6.6 Properties of the Definite Integral

1. \( \int_{a}^{b} k \, dx = k \, (b-a) \)

2. If \( a < c < b \), and if \( f \) is integrable on both \([a,c]\) and \([c,b]\) then \( f \) is integrable on \([a,b]\) and

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx
\]

6.7 Fundamental Theorem of Integral Calculus

This theorem is used to find the definite integral without using limits of sums. Due to its importance in evaluating definite integrals, and because it exhibits the connection between differentiation and integration, the theorem is aptly called The Fundamental Theorem of Integral Calculus.

**Statement:** Suppose \( f \) is continuous on a closed interval \([a,b]\).

**Part I** If the function \( G \) is defined by

\[
G(x) = \int_{a}^{x} f(t) \, dt
\]
for all \( x \) in \([a,b]\), then \( G \) is an antiderivative of \( f \) on \([a,b]\).

**Part II** If \( F \) is any antiderivative of \( f \), then

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]

In other words, integration and differentiation are the inverses of each other.
Example #5  Evaluate $\int_{1}^{2} x^2 \, dx$

Solution  The antiderivative of $x^2$ is

$$\frac{x^3}{3} + C, \text{ where } C \text{ is a constant.}$$

$$\frac{x^3}{3} + C \text{ at } x = 2 \text{ is } \frac{8}{3} + C$$
$$\frac{x^3}{3} + C \text{ at } x = 1 \text{ is } \frac{8}{3} + C$$

The difference is $\frac{8}{3} - \frac{1}{3} = \frac{7}{3}$

Then $\int_{1}^{2} x^2 \, dx = \frac{7}{3}$

Note that the constant of integration is canceled which is always the case in evaluating definite integrals.

The chief purpose of this example is to show that there is no constant of integration in the evaluation of definite integrals, since it gets canceled.
6.8 Method of Substitution Revisited

In evaluating definite integrals using the method of substitution, it is sometimes quicker and easier to change the limits of integration.

Example #6 Evaluate \[ \int_{-2}^{0} x \sqrt{2x^2 + 1} \, dx \]

Solution Let \( u = 2x^2 + 1, \quad du = 4x \, dx \)

To change the limits of integration,

- if \( x = -2, \quad u = 9 \)
- if \( x = 0, \quad u = 1 \)

Therefore,

\[
\int_{-2}^{0} x \sqrt{2x^2 + 1} \, dx = \int_{9}^{1} u^{1/4} \, du
\]

\[
= (1/4) \left[ u^{3/2} \right]_{9}^{1}
\]

\[
= (1/6) (1^{3/2} - 9^{3/2})
\]

\[
= (1/6) (-26)
\]

\[
= -13/3
\]
TEST

27. \( \int_{\pi/2}^{0} \tan \left( x/2 \right) \, dx \) is equivalent to

a. \( \int_{\pi/2}^{0} \tan u \, du \)

b. \( \int_{\pi/4}^{0} \tan u \, du \)

c. \( 1/2 \int_{\pi/2}^{0} \tan u \, du \)

d. \( 2 \int_{\pi/4}^{0} \tan u \, du \)
6.9 Integration by Parts

The formula for the differential of a product is
\[ d(uv) = u \, dv + v \, du \]

By integrating both sides, we obtain
\[ uv = \int u \, dv + \int v \, du \]
or
\[ \int u \, dv = uv - \int v \, du \]

Remark: In the above formula for integration by parts, a proper choice of \(dv\) is crucial. A trial and error method should be adopted in choosing \(u\) and \(dv\).

Example #7 Find \( \int \ln x \, dx \)

Solution Let \( u = \ln x \) and \( dv = dx \)

Then, \( du = \frac{1}{x} \, dx \) and \( v = x \)

Integrating by parts using the above formula, we get
\[ \int \ln x \, dx = (\ln x) \, x - \int x \left( \frac{1}{x} \right) \, dx \]
\[ = x \ln x - x + C \]
Example #8  Find $\int e^x \sin x \, dx$

Solution  Let $u = e^x$ and $dv = \sin x \, dx$

Then, $du = e^x \, dx$ and $v = -\cos x$

Integrating by parts, we get

$$\int e^x \sin x \, dx = e^x (-\cos x) + \int \cos x \, e^x \, dx$$

This is a case where we need to apply integration by parts more than once.

Consider the integral on the right.

Let $u = e^x$ and $dv = \cos x \, dx$

then $du = e^x \, dx$ and $v = \sin x$

Applying integration by parts to the integral on the right, we have

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int \sin x \, e^x \, dx$$

That is,

$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$$

$$\int e^x \sin x \, dx = (1/2) e^x (\sin x - \cos x) + C$$

Remark: If we had chosen $u = \cos x$ and $dv = e^x$ in the second integral, we would have ended up with the equation

$$\int e^x \sin x \, dx = \int e^x \sin x \, dx$$

So enough care must be exercised in choosing $u$ and $dv$.  

28. \( \int x^2 e^{3x} \, dx = \)

a. \( e^{3x} \left( \frac{1}{3} x^2 - \frac{2}{9} x + \frac{2}{27} \right) + C \)

b. \( x e^{3x} (3x + 2) + C \)

c. \( \frac{(x^2e^{3x})}{3} - \frac{2}{3} x e^{3x} + C \)

d. \( \frac{(x^2e^{3x})}{3} - \frac{2}{9} x e^{3x} + \frac{2}{27} + C \)
6.10 Trigonometric Integrals

We consider integrals of the type \( \int \sin^n x \, dx \)

If \( n \) is an odd positive integer,

\[
\int \sin^n x \, dx = \int \sin^{n-1} x \sin x \, dx
\]

Since \( n-1 \) is even, we may use the identity

\( \sin^2 x = 1 - \cos^2 x \)

to obtain an integrable form.

**Example #9** Evaluate \( \int \sin^5 x \, dx \)

**Solution** The given integral is equivalent to

\[
\int \sin^4 x \sin x \, dx = \int (1 - \cos^2 x)^2 \sin x \, dx
\]

Let \( u = \cos x \), then \( du = -\sin x \, dx \)

Thus the given integral is equivalent to

\[
- \int (1 - u^2)^2 \, du
\]

\[
= - \int (1 - 2u^2 + u^4) \, du = - (u - 2u^3/3 + u^5/5) + C
\]

\[
= - (\cos x - (2\cos^3 x)/3 + (\cos^5 x)/5) + C
\]

**NOTE:**

1. A similar technique may be employed for odd powers of \( \cos x \).

2. If the integrand is \( \sin^n x \) or \( \cos^n x \) and \( n \) is even, then the half-angle formulas

\[
\sin^2 x = (1-\cos 2x)/2 \quad \text{or} \quad \cos^2 x = (1+\cos 2x)/2
\]

can be used to simplify the integrand.
6.11 Guidelines for evaluating integrals of the form
\[ \int \sin^m x \cos^n x \, dx \]

1. If both \( m \) and \( n \) are even integers, use half-angle formulas for \( \sin^2 x \) and \( \cos^2 x \) to reduce the exponents by one-half.

2. If \( n \) is an odd integer, write the integral as
\[ \int \sin^m x \cos^n x \, dx = \int \sin^m x \cos^{n-1} x \cos x \, dx \]
and express \( \cos^{n-1} x \) in terms of \( \sin x \) by using the trigonometric identity \( \cos^2 x = 1 - \sin^2 x \). The substitution \( u = \sin x \) then leads to an integrand that can be handled easily.

3. If \( m \) is an odd integer, write the integral as
\[ \int \sin^m x \cos^n x \, dx = \int \sin^{m-1} x \cos^n x \sin x \, dx \]
and express \( \sin^{m-1} x \) in terms of \( \cos x \) by using the trigonometric identity \( \sin^2 x = 1 - \cos^2 x \). Use the substitution \( u = \cos x \) to evaluate the resulting integral.
Example #10 Evaluate \[ \int \sin^a x \cos^{-a} x \, dx \]

Solution Since \( m \) is an odd integer, the given integral becomes

\[
\int \sin^a x \cos^{-a} x \sin x \, dx
\]

\[= \int (1 - \cos^2 x) \cos^{-a} x \sin x \, dx \]

Let \( u = \cos x \), then \( du = -\sin x \, dx \)

Thus the integral is

\[= - \int \left( u^{-a} - u^{-a} \right) \, du + C \]

\[= -(u^{-a}/-4 + u^{-2}/-2) + C \]

\[= (\cos^{-a}x)/4 - (\cos^{-2}x)/2 + C \]
6.12 Guidelines for evaluating integrals of the form
\[ \int \tan^m x \sec^n x \, dx \]

1. If \( n \) is an even integer, write the integral as
\[ \int \tan^m x \sec^{n-2} x \sec^2 x \, dx \]
and express \( \sec^{n-2} x \) in terms of \( \tan x \) by using the trigonometric identity \( \sec^2 x = 1 + \tan^2 x \). The substitution \( u = \tan x \) leads to a simple integral.

2. If \( m \) is an odd integer, write the integral as
\[ \int \tan^{m-1} x \sec^{n-1} x \sec x \tan x \, dx \]
Since \( m - 1 \) is even, \( \tan^{m-1} x \) may be expressed in terms of \( \sec x \) by means of the identity \( \tan^2 x = \sec^2 x - 1 \). The substitution \( u = \sec x \) then leads to a form that is readily integrable.

3. If \( n \) is odd and \( m \) is even, then another method such as integration by parts should be used.
Example #11 Find \( \int \tan^a x \frac{dx}{\sec x} \)

Solution Since \( m \) is an odd integer, the integral is equivalent to

\[
\int \tan^2 x \sec^{-4/3} x \tan x \sec x \, dx
\]

Let \( u = \sec x \), then \( du = \sec x \tan x \, dx \)

Thus the integral is

\[
= \int (u^2 - 1) u^{-4/3} \, du
\]

\[
= \int (u^{2/3} - u^{-4/3}) \, du
\]

\[
= \left( \frac{3}{5} \right) u^{5/3} + 3 u^{-1/3} + C
\]

\[
= \left( \frac{3}{5} \right) (\sec x)^{5/3} + 3 (\sec x)^{-1/3} + C
\]
29. The evaluation of the integral \[ \int \frac{\sec^4 u}{\tan u} \, du \]
is done by using/rewriting the integral as

a. \[ \int \tan^{-3/2} u \sec^3 u \sec u \tan u \, du \]
b. integration by parts

c. \[ \int \tan^{-1/2} u \sec^2 u \sec^2 u \, du \]
d. partial fractions
6.13 Trigonometric Substitution

If an integrand contains one of the expressions \( \sqrt{a^2-x^2} \), \( \sqrt{a^2+x^2} \) or \( \sqrt{x^2-a^2} \), where \( a > 0 \), the radical sign can be eliminated by either of the two methods given below:

Method 1: Use the trigonometric substitution listed in the following table.

<table>
<thead>
<tr>
<th>Given expression</th>
<th>Trigonometric Substitution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{a^2-x^2} )</td>
<td>( x = a \sin u )</td>
</tr>
<tr>
<td>( \sqrt{a^2+x^2} )</td>
<td>( x = a \tan u )</td>
</tr>
<tr>
<td>( \sqrt{x^2-a^2} )</td>
<td>( x = a \sec u )</td>
</tr>
</tbody>
</table>

To save us from memorizing the above formulas, there is an alternate method.

Method 2: Express each of the radicands given above as one of the sides of a right angle triangle. For instance, if the integrand contains the expression \( \sqrt{a^2-x^2} \), then represent it as the length of one side of a triangle as shown below -

The lengths of the other two sides can be easily obtained by using the **Pythagoras Theorem**. The integrand is then substituted by the equivalent trigonometric identity.

The following example solved using both these methods will help understand them better.
Example # 12  Evaluate \( \int \frac{1}{x^2 \sqrt{16-x^2}} \) \( dx \)

Solution  Method 1

The radicand is of the form \( \sqrt{a^2-x^2} \)

Hence let \( x = a \sin u = 4 \sin u \)

\( J(16-x^2) = J(16-16 \sin^2 u) = 4 \cos u \)

Since \( x = 4 \sin u \), \( dx = 4 \cos u \) \( du \)

Therefore the integral is

\[ \int \frac{4 \cos u}{(16 \sin^2 u)(4 \cos u)} \] \( du \)

\[ = \frac{1}{16} \int \csc^2 u \) \( du \)

\[ = -(1/16) \cot u + C \]

Method 2

The following relations are obtained from the figure shown opposite.

\[ x = 4 \sin u \]

\( dx = 4 \cos u \) \( du \)

\( J(16-x^2) = 4 \cos u \)

Therefore the integral is

\[ \int \frac{4 \cos u}{(4 \cos u)(4^2 \sin^2 u)} \] \( du \)

\[ = (1/16) \int \csc^2 u \] \( du \)

\[ = -(1/16) \cot u + C \]
30. \[ \int \frac{1}{(x^2-1)^{3/2}} \, dx = \]
(There can be more than one answer)

a. \( \frac{1}{\tan^3 u} + C \)

b. \( \sec u \tan^2 u + C \)

c. \( -\coth u + C \)

d. \( -\frac{1}{\sin u} + C \)
6.14 Partial Fractions

The sum $F_1 + F_2 + \ldots + F_K$ is called the partial fraction decomposition of $f(x)/g(x)$ and each $F_i$ is called a partial fraction.

Guidelines for finding partial fraction decomposition of $f(x)/g(x)$

1. If the degree of $f(x)$ is not lower than the degree of $g(x)$, use long division to obtain the proper form.

2. Express $g(x)$ as a product of linear factors $px+q$ or irreducible quadratic factors $ax^2+bx+c$, and collect repeated factors so that $g(x)$ is a product of different factors of the form $(px+q)^m$ or $(ax^2+bx+c)^n$, where $m$ and $n$ are nonnegative integers.

3. Apply the following rules.
   Rule 1. For each factor of the form $(px+q)^m$ where $m \geq 1$, the partial fraction decomposition contains a sum of $m$ partial fractions of the form
   $$\frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \ldots + \frac{A_m}{(px+q)^m}$$
   where each $A_i$ is a real number.

   Rule 2. For each factor of the form $(ax^2+bx+c)^n$ where $n \geq 1$ and $ax^2+bx+c$ is irreducible, the partial fraction decomposition contains a sum of $n$ partial fractions of the form
   $$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$
   where each $A_i$ and $B_i$ is a real number.
Example #13 Evaluate \( \int \frac{4x^2+13x-9}{x^3+2x^2-3x} \, dx \)

Solution \( \frac{4x^2+13x-9}{x^3+2x^2-3x} = \frac{4x^2+13x-9}{x(x+3)(x-1)} \)

That is,

\[
4x^2+13x-9 = A(x+3)(x-1) + Bx(x-1) + Cx(x+3)
\]

Let \( x = 0 \), then \(-9 = -3A\) or \( A = 3 \)

Let \( x = 1 \), then \( 8 = 4C \) or \( C = 2 \)

Let \( x = -3 \), then \(-12 = 12B \) or \( B = -1 \)

The partial fraction decomposition is

\[
\frac{4x^2+13x-9}{x(x+3)(x-1)} = \frac{3}{x} + \frac{-1}{(x+3)} + \frac{2}{(x-1)}
\]

The integral is

\[
\int \frac{3}{x} \, dx - \int \frac{1}{(x+3)} \, dx + \int \frac{2}{(x-1)} \, dx
\]

\[
= 3 \ln |x| - \ln |x+3| + 2 \ln |x-1| + C
\]

\[
= \ln \left| \frac{(x^3)(x-1)^2}{(x+3)} \right| + C
\]
Example #14  Express \( \frac{5x^3-3x^2+7x-3}{(x^2+1)^2} \) as partial fractions

Solution  Applying Rule 2, with \( n = 2 \)

\[
\frac{5x^3-3x^2+7x-3}{(x^2+1)^2} = \frac{(Ax+B)}{x^2+1} + \frac{(Cx+D)}{(x^2+1)^2}
\]

Therefore,

\[
(5x^3-3x^2+7x-3) = (Ax+B)(x^2+1) + (Cx+D)
\]

\[
= Ax^3 + Bx^2 + (A+C)x + (B+D)
\]

Comparing the coefficients of \( x^3 \) and \( x^2 \),
we obtain

\[
A = 5 \quad \text{and} \quad B = -3
\]

Similarly

\[
A + C = 7 \quad \text{and} \quad B + D = -3
\]

Solving, we get

\[
C = 2 \quad \text{and} \quad D = 0
\]

Therefore,

\[
\frac{5x^3-3x^2+7x-3}{(x^2+1)^2} = \frac{(5x-3)}{x^2+1} + \frac{2x}{(x^2+1)^2}
\]
TEST

31. \[ \int \frac{dx}{x^2 + x - 2} = \]

a. \( \frac{1}{3} \ln \left| \frac{x-1}{x+2} \right| + C \)

b. \( \frac{1}{3} \ln \left| \frac{x+2}{x-1} \right| + C \)

c. \( \frac{1}{3} \ln \left| (x-1)(x+2) \right| + C \)

d. \( (\ln |x-1|)(\ln |x+2|) + C \)
6.15 Area

The definite integral is useful in finding the area of a region bounded by one or more graphs and the coordinate axes.

If \( f \) and \( g \) are continuous and \( f(x) \geq g(x) \) for all \( x \) in \([a, b]\), then the area \( A \) of the region bounded by the graphs of \( f, g, x = a, \) and \( x = b \) is

\[
A = \int_{a}^{b} [f(x) - g(x)] \, dx
\]

**Example #15** Find the area of the region bounded by the lines \( x=1, \) \( x=2, \) y=3x and the curve \( y=x^2 \).

**Solution** We draw a sketch as shown in the figure for the above graphs.

![Graph showing the region bounded by the lines and curves](image)

The area is given by

\[
A = \int_{1}^{2} (3x - x^2) \, dx
\]

Note the sketch helps us to find out which graph is above the other.

Therefore,

\[
A = \frac{3x^2}{2} - \frac{x^3}{3} \bigg|_{1}^{2} = 13/6
\]
Example #16  Find the area bounded by $y^2 = x - 1$ and $y = x - 3$.

Solution  The graphs are sketched below.

They intersect at $(2,-1)$ and $(5,2)$
(The points of intersection are obtained by solving $y^2+1 = y+3$)

A close look at the graph will reveal that it is easier to integrate with respect to $y$ than with respect to $x$.

The equations to be considered are $x = y^2+1$ and $x = y+3$ \hspace{1cm} (y is between $-1$ & 2)

The area is equal to
\[
\int_{-1}^{2} [(y+3) - (y^2+1)] \, dy
\]
\[
= 13/3
\]

As an exercise, try to find the area for the above graph by integrating with respect to $x$. 
TEST

32. The area bounded by the curves $y = x$ and $y = x^3$ is
   a. $1/4$   b. $0$   c. $1/2$   d. $1$

33. The area bounded by the graphs of the equations $y = x^2$ and $y = \sqrt{x}$ is
   a. $1/3$   b. $2/3$   c. $1$   d. $1/6$
6.16 Infinite Limits of Integration

**Definition**

i. If $f$ is continuous on $[a, \infty]$, then

$$\int_{a}^{\infty} f(x) \, dx = \lim_{t \to \infty} \int_{a}^{t} f(x) \, dx$$

ii. If $f$ is continuous on $(-\infty, a]$, then

$$\int_{-\infty}^{a} f(x) \, dx = \lim_{t \to -\infty} \int_{t}^{a} f(x) \, dx$$

The above expressions are called improper integrals. Improper integrals may have two infinite limits of integration.

Let $f$ be continuous for all $x$. If $a$ is any real number, then

$$\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{a} f(x) \, dx + \int_{a}^{\infty} f(x) \, dx$$

6.17 Integrals with Discontinuous Integrands

If a function $f$ has an infinite discontinuity at some number in the interval it may still be possible to assign a value to the integral.

If $f$ has a discontinuity at a number $c$ in the open interval $(a, b)$ but is continuous elsewhere in $[a, b]$, then

$$\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx$$

provided both of the integrals on the right converge.
Example #17  Determine \( \int_{0}^{4} \frac{1}{(x-3)^2} \, dx \)

Solution  If the Fundamental theorem of calculus is directly applied, we obtain
\[
\left. \frac{-1}{x-3} \right|_{0}^{4} = \frac{-1}{3} - \frac{-1}{0} = -4/3
\]
This is obviously wrong, since the integrand is never negative. Also, at \( x = 3 \), the integrand becomes infinite.

Hence we need to use the above definition. We get
\[
\int_{0}^{4} \frac{1}{(x-3)^2} \, dx = \int_{0}^{3} \frac{1}{(x-3)^2} \, dx + \int_{3}^{4} \frac{1}{(x-3)^2} \, dx
\]
\[
\int_{0}^{3} \frac{1}{(x-3)^2} \, dx = \left. -\frac{1}{x-3} \right|_{0}^{3} = -1 - (-\infty) = -\infty
\]
Since this integral diverges, the limit does not exist.
6.18 Double Integrals

**Definition:** The definite double integral

\[ \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x,y) \, dy \, dx \]

may be interpreted as that portion of the volume of a right cylinder which is included between the plane XOY and the surface

\[ z = f(x,y), \]

the base of the cylinder being the area in the XOY plane bounded by the curves

\[ y = u_1, \quad y = u_2, \quad x = a_1, \quad x = a_2. \]

**Definition:**

\[ \int_{a}^{b} \int_{c}^{d} f(x,y) \, dx \, dy = \int_{a}^{b} \left[ \int_{c}^{d} f(x,y) \, dy \right] dx \]

This is called an iterated (double) integral.

**Example #18** Evaluate \( \int_{1}^{3} \int_{\pi/6}^{\pi} 2y \cos x \, dx \, dy \)

**Solution** By the above definition, the integral equals

\[
\int_{1}^{3} \int_{\pi/6}^{\pi} 2y \cos x \, dx \, dy \\
= \int_{1}^{3} 2y \sin x \bigg|_{\pi/6}^{\pi} \, dy \\
= \int_{1}^{3} (2y \sin y^2 - y) \, dy \\
= -\cos y^2 - 1/2 \, y^2 \bigg|_{1}^{3} \\
= (-\cos 9 - 9/2) - (-\cos 1 - 1/2) \\
= \cos 1 - \cos 9 - 4
Sometimes it is extremely difficult or even impossible to evaluate a given double integral. However by reversing the order of integration from dy dx to dx dy or vice versa, it may be possible to evaluate the double integral.

We illustrate below, the procedure of reversing the order of integration.

Example #19 Given \[ \int_{0}^{4} \int_{-y}^{2} y \cos x^2 \, dx \, dy \]
reverse the order of integration.

Solution: The graph of the equations is sketched below.

The given order of integration is dx dy. Hence we call the region R of type II. The left hand and right hand boundaries are the graphs of \( x=-y \) and \( x=2 \), respectively and \( 0 \leq y \leq 4 \).

If the order of integration is dy dx, then the region R is of type I. Then the lower and upper boundaries are given by \( y=0 \) and \( y=x^2 \) respectively, where \( 0 \leq x \leq 2 \).

Hence \[ \int_{0}^{4} \int_{-y}^{2} y \cos x^2 \, dx \, dy = \int_{0}^{2} \int_{0}^{x^2} y \cos x^2 \, dy \, dx \]
34. \( \int_2^1 \int_x^{1-x} x^2 y \, dy \, dx \) = __________

35. Given \( \int_0^2 \int_0^{2x} \sin y \, dy \, dx \), on reversing the order of integration we get which one of the following:

a. \( \int_0^2 \int_0^{2x} \sin y \, dx \, dy \)  
b. \( \int_0^2 \int_0^{\frac{y}{2}} \sin y \, dx \, dy \)  
c. \( \int_0^2 \int_0^{\frac{y}{2}} \sin y \, dy \, dx \)  
d. \( \int_0^2 \int_0^{2x} \sin y \, dx \, dy \)
ANSWERS

25.  b
26.  a
27.  d
28.  a
29.  c
30.  c, d
31.  a
32.  c
33.  a
34.  163/120
35.  b
TESTS
INSTRUCTIONS FOR THE PRE-TEST IN CALCULUS (Grad students)

PLEASE READ THE INSTRUCTIONS CAREFULLY

Time : 60 min  Max. marks : 30

CALCULATORS, SLIDE RULES AND REFERENCE MATERIALS ARE NOT ALLOWED TO BE USED DURING THE TEST

1. This test contains 30 multiple-choice questions. It is intended to test your knowledge and application of basic concepts studied in calculus 263 A, B and C courses.

2. Answer all the questions on the test paper itself by circling your response as shown below.
   a. b. c. d. e.

3. There is only one correct response for each question.

4. For every incorrect response, 1/4th of a point will be taken off. However, it will be to your advantage to answer such questions for which you can eliminate one or more answer choices as wrong.

5. Since all questions carry equal points, do not spend too much time on any one question.

GOOD LUCK !!

NAME : ____________________________________________

PHONE : ________________
1. \[ \int_{a}^{b} f(x) \, dx = F(b) - F(a), \] where \( f(x) \) is a continuous function in the interval \([a,b]\) and \( F(x) \) is the indefinite integral of \( f(x) \)." is a statement of which one of the following:

A. The Fundamental Theorem of Calculus  
B. Taylor's formula  
C. Maclaurin's formula  
D. Mean Value Theorem

2. \( \lim_{x \to 0} (1 + x)^{\frac{1}{\ln x}} = \)

<table>
<thead>
<tr>
<th>Option</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. ( e^{0.2} )</td>
<td>( e^{0.2} )</td>
</tr>
<tr>
<td>B. 1</td>
<td>1</td>
</tr>
<tr>
<td>C. ( \infty )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>D. e</td>
<td>e</td>
</tr>
</tbody>
</table>

3. \( \lim_{n \to \infty} \frac{5n}{e^{2n}} = \)

<table>
<thead>
<tr>
<th>Option</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. ( \infty )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>B. 0</td>
<td>0</td>
</tr>
<tr>
<td>C. 2.5</td>
<td>2.5</td>
</tr>
<tr>
<td>D. non-existent</td>
<td>non-existent</td>
</tr>
</tbody>
</table>

4. If \( f(x) = \frac{f(x^2 - 9)}{(x-4)} \), then \( f \) is continuous in which of the following intervals?

i. \((\infty, \infty)\) ii. \((\infty, -3]\) iii. \([3, 4)\) iv. \((4, \infty)\)

<table>
<thead>
<tr>
<th>Option</th>
<th>Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. i only</td>
<td>i only</td>
</tr>
<tr>
<td>B. ii and iii only</td>
<td>ii and iii only</td>
</tr>
<tr>
<td>C. ii, iii and iv only</td>
<td>ii, iii and iv only</td>
</tr>
<tr>
<td>D. i, iii and iv only</td>
<td>i, iii and iv only</td>
</tr>
</tbody>
</table>

5. Which one of the following functions shows that the statement "If a function is continuous at \( x = 0 \), then it is differentiable at \( x = 0 \)" is false?

<table>
<thead>
<tr>
<th>Option</th>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. ( f(x) = x^{-4/3} )</td>
<td>( x^{-4/3} )</td>
</tr>
<tr>
<td>B. ( f(x) = x^{-2/3} )</td>
<td>( x^{-2/3} )</td>
</tr>
<tr>
<td>C. ( f(x) = x^{1/3} )</td>
<td>( x^{1/3} )</td>
</tr>
<tr>
<td>D. ( f(x) = x^{4/3} )</td>
<td>( x^{4/3} )</td>
</tr>
<tr>
<td>E. ( f(x) = x^3 )</td>
<td>( x^3 )</td>
</tr>
</tbody>
</table>

6. If \( f(2) = 3, \ f'(2) = -1, \ g(2) = -5 \) and \( g'(2) = 2 \), then \( (fg)'(2) - (f/g)'(2) \) is

<table>
<thead>
<tr>
<th>Option</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. ( 72/5 )</td>
<td>( 72/5 )</td>
</tr>
<tr>
<td>B. (-11/5 )</td>
<td>(-11/5 )</td>
</tr>
<tr>
<td>C. ( 276/25 )</td>
<td>( 276/25 )</td>
</tr>
<tr>
<td>D. ( 322/25 )</td>
<td>( 322/25 )</td>
</tr>
</tbody>
</table>
7. If \( y = \sin^a (4x) \) then \( y' \) is

A. \( 12 \sin^2 (4x) \)  
B. \( 3 \sin^2 (4x) \cos^2 (4x) \)  
C. \( 3 \sin^2 (4x) \cos (4x) \)  
D. \( 12 \sin^2 (4x) \cos (4x) \)

8. If \( Jx + Jy = 1 \), then \( \frac{dx}{dy} = \)

A. \( -J(x/y) \)  
B. \( -1/24y \)  
C. \( -J(y/x) \)  
D. undefined

9. \( \int a^x \, dx = \)

A. \( \frac{a^x}{\ln a} \)  
B. \( a^x \)  
C. \( \frac{a^{x+1}}{x+1} \)  
D. non-existent

10. \( \int \frac{x^2}{e^b} \, dx \), where \( b = x^a \), is

A. \( (-1/3) \ln e^b + C \)  
B. \( e^{-b}/3 + C \)  
C. \( -1/(3e^b) + C \)  
D. \( (1/3) \ln e^b + C \)  
E. \( x^a/(3e^b) + C \)

11. The area of the region between the graph of \( y = 4x^a + 2 \) and the x-axis from \( x = 1 \) to \( x = 2 \) is

A. 36  
B. 23  
C. 20  
D. 17  
E. 9

12. If \( f(x) = e^x \), which one of the following is equal to \( f'(e) \)?

A. \( \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \)  
B. \( \lim_{h \to 0} \frac{e^{x-h} - e^x}{h} \)  
C. \( \lim_{h \to 0} e^{x-h} - e \)  
D. \( \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \)  
E. \( \lim_{h \to 0} \frac{e^{x+h} - 1}{h} \)

13. \( \frac{d}{dx} (\ln e^{2x}) = \)

A. \( 1/e^{2x} \)  
B. \( 2/e^{2x} \)  
C. 2x  
D. 1  
E. 2
14. If \( y = x^2 \sin y \), then \( y' = \)

A. \( x^2 \cos y \)
B. \( x^2 \cos y + 2x \sin y \)
C. \( (2x \sin y)/(1 - x^2 \cos y) \)
D. \( (2x \sin y + x^2 \cos y)/(1 - x^2 \cos y) \)

15. If \( f(x) = \frac{\ln x}{x} \), for \( x > 0 \), which one of the following is true?

A. \( f \) is increasing for all \( x > 0 \)
B. \( f \) is increasing for all \( x > 1 \)
C. \( f \) is decreasing for all \( x \) between 0 and 1
D. \( f \) is decreasing for all \( x > e \)
E. \( f \) is decreasing for all \( x \) between 1 and \( e \)

16. The function \( f(x) = \begin{cases} \frac{x^2-1}{x^2} & \text{for } x > -3 \\ \frac{x^2+x}{1} & \text{for } x \leq -3 \end{cases} \)
is discontinuous at which of the following points

i) -1       ii) 0       iii) -3

A. i) only
B. i) and ii) only
C. i), ii) and iii)
D. ii) only

17. \( \int \ln x \, dx = \)

A. \( x \ln x - x + C \)
B. \( \frac{1}{x} + C \)
C. \( e^x + C \)
D. non-existent

18. \( \int \frac{xe^b + 1}{e^b + 2x} \, dx \), where \( b = x^2 \), is of the form

A. \( \int u^n \, du \)
B. \( \int (1/u) \, du \)
C. \( \int e^u \, du \)
D. none of the above

19. \( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots \) is a power series representation of

A. \( \ln (1+x) \)
B. \( e^x \)
C. \( e^{-x} \)
D. \( \sinh x \)
20. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ is the Taylor series about zero for which one of the following functions?

A. $\sin x$  D. $e^x$
B. $\cos x$  E. $\ln (1+x)$
C. $e^{-x}$

21. Let $f$ be a function given by $f(x) = x^3 - 3x^2$. What are all the values of $c$ that satisfy the conclusion of the Mean Value Theorem of differential calculus on the closed interval [0,3].

A. 0 only  D. 3 only
B. 0 and 3  E. 2 only
C. 2 and 3

22. Which one of the following is not true about L'Hôpital's rule?

A. It can be used on the indeterminate forms $0/0$ and $\infty/\infty$
B. It is used for finding the derivative of a function
C. It can also be used on the indeterminate form $\infty - \infty$
D. None of the above

23. If $f(x) = (2x + 1)^4$, then the 4th derivative of $f(x)$ at $x = 0$ is

A. 0  D. 240
B. 24  E. 384
C. 48

24. If $y = 3^{x^x}$, then $y' =$

A. $(3^{x^x} \ln 3)/(2 \sqrt{x})$  C. $3^{x^x} \ln 3$
B. $(3/2) \frac{1}{x}^{1/2}$  D. $(3^{x^x})/(2 \sqrt{x})$

25. Given $\int_{0}^{4} \int_{y}^{\infty} y \cos x^3 \, dx \, dy$, the resulting integral after reversing the order of integration is

A. $\int_{0}^{4} \int_{y}^{\infty} y \cos x^3 \, dy \, dx$  C. $\int_{0}^{4} \int_{y}^{\infty} y \cos x^3 \, dy \, dx$
B. $\int_{0}^{2} \int_{y}^{\infty} x^2 \cos x^3 \, dy \, dx$  D. $\int_{0}^{4} \int_{y}^{\infty} y \cos x^3 \, dx \, dy$
26. If \( \int_{-1}^{1} e^{-b} \, dx = k \), where \( b = x^2 \), then \( \int_{-1}^{1} e^{-b} \, dx = \)

A. \(-2k\)  
B. \(k/2\)  
C. \(-k/2\)  
D. \(-k\)  
E. \(2k\)

27. \( \int_{0}^{1} x e^{-x} \, dx = \)

A. \(1 - 2e\)  
B. \(-1\)  
C. \(1\)  
D. \(1 - 2e^{-1}\)  
E. \(2e - 1\)

28. \( \int \frac{dx}{(x-1)(x+2)} = \)

A. \(\frac{1}{3} \ln \left| \frac{x-1}{x+2} \right| + C\)  
B. \(\frac{1}{3} \ln \left| \frac{x+2}{x-1} \right| + C\)  
C. \(\frac{1}{3} \ln \left| (x-1)(x+2) \right| + C\)  
D. \((\ln |x-1|)(\ln |x+2|) + C\)  
E. \(\ln \left| (x-1)(x+2)^2 \right| + C\)

29. The curves \( y = f(x) \) and \( y = g(x) \) shown in the figure below intersect at the point \((a,b)\). The area of the shaded region enclosed by these curves and the line \( x = -1 \) is given by

A. \( \int_{a}^{b} [f(x) - g(x)] \, dx + \int_{0}^{-1} [f(x) + g(x)] \, dx \)  
B. \( \int_{b}^{-1} g(x) \, dx + \int_{c}^{b} f(x) \, dx \)  
C. \( \int_{-1}^{c} [f(x) - g(x)] \, dx \)  
D. \( \int_{-1}^{a} [f(x) - g(x)] \, dx \)
30. All functions \( f \) defined on the xy-plane such that 
\[
f_x = 2x + y \quad \text{and} \quad f_y = x + 2y
\]
(where \( f_x \) and \( f_y \) are the partial derivatives of \( f \) with respect to \( x \) and \( y \) respectively) 
are given by \( f(x,y) = \)

A. \( x^2 + xy + y^2 + C \)  
B. \( x^2 - xy + y^2 + C \)  
C. \( x^2 - xy - y^2 + C \)  
D. \( x^2 + 2xy + y^2 + C \)  
E. \( x^2 - 2xy + y^2 + C \)
ANSWERS

1. A
2. A
3. B
4. C
5. C
6. C
7. D
8. A
9. A
10. B
11. D
12. D
13. E
14. B
15. D
16. C
17. A
18. B
19. B
20. C
21. E
22. B
23. E
24. A
25. C
26. B
27. D
28. A
29. D
30. A
INSTRUCTIONS FOR THE POST-TEST IN CALCULUS (Grad students)

PLEASE READ THE INSTRUCTIONS CAREFULLY

Time : 60 min

Max. marks : 30

CALCULATORS, SLIDE RULES AND REFERENCE MATERIALS ARE NOT ALLOWED TO BE USED DURING THE TEST

1. This test contains 30 multiple-choice questions. It is intended to test your knowledge and application of basic concepts studied in calculus 263 A, B and C courses.

2. Answer all the questions on the test paper itself by circling your response as shown below.

   a.   b.   c.   d.   e.

3. There is only one correct response for each question.

4. For every incorrect response, 1/4th of a point will be taken off. However, it will be to your advantage to answer such questions for which you can eliminate one or more answer choices as wrong.

5. Since all questions carry equal points, do not spend too much time on any one question.

GOOD LUCK !!
1. \[ \lim_{x \to 0} \frac{x + 1 - e^x}{x^3} = \]
   A. \(\infty\)  
   B. \(-\infty\)  
   C. 1  
   D. 0  
   E. none of the above

2. If \( y = \log(\log x) \), \( \frac{dy}{dx} = \)
   A. \(\frac{1}{\log x}\)  
   B. \(\frac{1}{x \log x}\)  
   C. \(\log x\)  
   D. 1

3. If \( h(x) = f(x) - g(x) \), \( f'(x) = -g(x) \) and \( g'(x) = f(x) \) then \( h'(x) = \)
   A. 0  
   B. 1  
   C. \(-4 f(x) g(x)\)  
   D. \([-g(x)]^2 - [f(x)]^2\)  
   E. \(-2 [-g(x) + f(x)]\)

4. \[ \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x} \, dx = \]
   A. \(\ln \sqrt{2}\)  
   B. \(\ln (\pi/4)\)  
   C. \(\ln \sqrt{3}\)  
   D. \(\ln (\sqrt{3}/2)\)  
   E. \(\ln 2\)

5. \[ \int_0^1 (4 - x^2)^{-3/2} \, dx = \]
   A. \((2 - \sqrt{3})/3\)  
   B. \((2\sqrt{3} - 3)/4\)  
   C. \(1/\sqrt{3}\)  
   D. \(1/4\sqrt{3}\)  
   E. \(\sqrt{3}/2\)

6. At \( x = 0 \), which one of the following is true of the function \( f \) defined by \( f(x) = x^2 + e^{-2x} \)?
   A. \(f\) is increasing  
   B. \(f\) is decreasing  
   C. \(f\) is discontinuous  
   D. none of the above
7. If \( f(x) = \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} \) \( x \) not equal to 2
and if \( f(2) = k \)
and if \( f \) is continuous at \( x = 2 \), then \( k = \)

A. 0  D. 1
B. 1/6  E. 7/5
C. 1/3

8. If the substitution \( u = x/2 \) is made, the integral
\[
\int_{2}^{4} \frac{1 - (x/2)^2}{x} \, dx =
\]

A. \( \int_{2}^{4} \frac{1-u^2}{u} \, du \)
B. \( \int_{2}^{4} \frac{1-u^2}{u} \, du \)
C. \( \int_{2}^{4} \frac{1-u^2}{u} \, du \)
D. \( \int_{0}^{1} \frac{1-u^2}{4u} \, du \)
E. \( \int_{0}^{1} \frac{1-u^2}{2u} \, du \)

9. \( \lim_{x \to 0} (x \csc x) \) is

A. \( -\infty \)  D. 1
B. \( -1 \)  E. \( \infty \)
C. 0

10. \( \lim_{x \to \infty} (1 + 5e^x)^{1/x} = \)

A. 0  D. \( e^5 \)
B. 1  E. non-existent
C. \( e \)

11. If \( \sqrt{x} + \sqrt{y} = 1 \), then \( \frac{dy}{dx} \) is

A. \( -\sqrt{x/y} \)  C. \( -\sqrt{y/x} \)
B. \( -1/(2 \sqrt{y}) \)  D. undefined

12. If \( \frac{d}{dx} \left( \frac{x+1}{3x-2} \right) = -5 \) at \( x = 1 \), then
\( \frac{d}{dx} \left[ \left( \frac{x+1}{3x-2} \right)^2 \right] \) at \( x = 1 \) is

A. 20  C. -10
B. 25  D. none of the above
13. \[ \int \frac{x^2}{e^b} \, dx \] where \( b = x^a \) is equal to

A. \( \frac{-1}{3} \ln b + C \)  
B. \( \frac{-1}{3e^b} + C \)  
C. \( -\frac{e^b}{3} + C \)  
D. \( \frac{1}{3} \ln e^b + C \)  
E. \( \frac{x^a}{(3e^b)} + C \)

14. Given \( \int_0^4 \int_y^2 y \cos x^a \, dx \, dy \), the resulting integral after reversing the order of integration is

A. \( \int_0^2 \int_y^4 y \cos x^a \, dy \, dx \)  
B. \( \int_0^2 \int_0^y x^2 y \cos x^a \, dy \, dx \)  
C. \( \int_0^2 \int_0^y y \cos x^a \, dy \, dx \)  
D. \( \int_0^2 \int_y^4 y \cos x^a \, dx \, dy \)

15. \( \sum_{n=0}^{\infty} (-1)^n x^n \) is a power series representation of \( n! \)

A. \( \frac{1}{1+x} \)  
B. \( \frac{1}{(1+x)^2} \)  
C. \( \sin x \)  
D. \( \cos x \)

16. The Mean Value Theorem guarantees the existence of a special point on the graph of \( y = Jx \) between \( (0,0) \) and \( (4,2) \). The coordinates of this point are

A. \( (2,1) \)  
B. \( (1,1) \)  
C. \( (2,\sqrt{2}) \)  
D. \( (1/2, 1/\sqrt{2}) \)  
E. none of the above

17. The coefficient of \( x^a \) in the Taylor series for \( e^{ax} \) about \( x = 0 \) is

A. \( 1/6 \)  
B. \( 1/3 \)  
C. \( 1/2 \)  
D. \( 3/2 \)  
E. \( 9/2 \)
18. \( f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & \text{for } x \text{ not equal to } 2 \\ 4 & \text{for } x \text{ equal to } 2 \end{cases} \)

is continuous at \( x = 2 \), because of which of the following reasons:

i. \( f(2) \) is defined
ii. \( \lim_{x \to 2} f(x) \) exists
iii. \( \lim_{x \to 2} f(x) = f(2) \)

A. i only
B. i and ii only
C. i and iii only
D. all of the above

19. If \( f \) is a function such that \( \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = 0 \), then which one of the following must be true.

A. the limit of \( f(x) \) as \( x \) approaches 2 does not exist
B. \( f \) is not defined at \( x = 2 \)
C. the derivative of \( f \) at \( x = 2 \) is 0
D. \( f \) is continuous at \( x = 0 \)
E. \( f(2) = 0 \)

20. If \( y = x^2 + 2 \) and \( u = 2x - 1 \), then \( \frac{dy}{du} = \)

A. \( \frac{(2x^2 - 2x + 4)}{(2x - 1)^2} \)
B. \( 6x^2 - 2x - 4 \)
C. \( x^2 \)
D. \( x \)
E. \( \frac{1}{x} \)

21. If \( f(x) = (x^2 + 1)^{\frac{1}{3}} \cdot x^3 \), then \( f'(1) = \)

A. \( -\frac{1}{2} \ln (8e) \)
B. \( -\ln (8e) \)
C. \( -\frac{3}{2} \ln 2 \)
D. \( -\frac{1}{2} \)
E. \( \frac{1}{8} \)

22. Which one of the following is a characteristic of \( \lim_{x \to 0} \frac{\lfloor x \rfloor}{x} \)?

A. the limit is 1
B. the limit is 0
C. the left-hand limit is -1 and the right-hand limit is 1
D. the left-hand limit is 1 and the right-hand limit is -1
23. \[ 
\int_{-1}^{1} \int_{1}^{4} (2x+6x^2y) \, dx \, dy = 
\]
A. 234  
B. 144  
C. 378  
D. 141

24. \[ 
\int_{0}^{1} x e^{-x} \, dx = 
\]
A. 1-2e  
B. -1  
C. 1  
D. 1-2e^{-1}  
E. 2e-1

25. \[ 
\int \frac{\ln x}{x} \, dx \]
A. \[ \int u^n \, du \]  
B. \[ \int \frac{1}{u} \, du \]  
C. \[ \int e^u \, du \]  
D. none of the above

26. \[ 
\lim_{x \to 0} \frac{\sec x - x}{\csc x + x} = 
\]
A. 0  
B. \( \infty \)  
C. \( -\infty \)  
D. doesn't exist

27. \[ 
\frac{d}{dx} \left[ \ln \left( \frac{1}{1-x} \right) \right] = 
\]
A. \( \frac{1}{1-x} \)  
B. \( \frac{1}{x-1} \)  
C. \( \frac{1}{x} \)  
D. \( x-1 \)  
E. \( (1-x)^2 \)

28. If \( y = a^u \), then \( y' = \)
A. \( a \, du/dx \)  
B. \( a^u \ln a \)  
C. \( a^u \ln a \, du/dx \)  
D. \( a^u \, du/dx \)

29. The area bounded by the graphs \( y = x \), \( y = 3x \) and \( x+y = 4 \) is
A. \( \frac{7}{2} \)  
B. 2  
C. \( \frac{11}{2} \)  
D. 4
30. Which one of the following statements is not true?

A. If a function \( f \) is differentiable at \( a \), then \( f \) is continuous at \( a \).
B. Integration and differentiation are the inverses of each other.
C. In the method of integration by parts, wrong choice of \( u \) and \( dv \) does not lead to a solution.
D. L'Hôpital's rule can be applied to forms of the type \( 1^0 \).

***
1. B 16. B
2. A 17. E
3. C 18. D
5. D 20. D
6. C 21. A
7. B 22. C
8. A 23. A
9. D 24. D
10. C 25. A
11. C 26. A
12. D 27. A
15. B 30. D
INSTRUCTIONS FOR TEST A IN CALCULUS (Undergrads)

PLEASE READ THE INSTRUCTIONS CAREFULLY

Time : 50 min  Max. marks : 24

CALCULATORS, SLIDE RULES AND REFERENCE MATERIALS ARE NOT ALLOWED TO BE USED DURING THE TEST

1. This test contains 24 multiple-choice questions. It is intended to test your knowledge and application of basic concepts studied in calculus 263 A, B and C courses.

2. Answer all the questions on the test paper itself by circling your response as shown below.
   a.  b.  c.  d.  e.

3. There is only one correct response for each question.

4. For every incorrect response, 1/4th of a point will be taken off. However, it will be to your advantage to answer such questions for which you can eliminate one or more answer choices as wrong.

5. Since all questions carry equal points, do not spend too much time on any one question.

GOOD LUCK !!

NAME : __________________________________________

PHONE : __________________________
1. "\( \int_{a}^{b} f(x) \, dx = F(b) - F(a) \), where \( f(x) \) is a continuous function in the interval \([a,b]\) and \( F(x) \) is the indefinite integral of \( f(x) \)." is a statement of which one of the following:

A. The Fundamental Theorem of Calculus
B. Taylor's formula
C. Maclaurin's formula
D. Mean Value Theorem

2. If \( f(x) = e^x \), which one of the following is equal to \( f'(e) \) ?

A. \( \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \)
B. \( \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \)
C. \( \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \)
D. \( \lim_{h \to 0} \frac{e^{x+1} - e^{x}}{h} \)
E. \( \lim_{h \to 0} \frac{e^{x+1} - e^{x}}{h} \)

3. \( \int \frac{xe^b + 1}{e^b + 2x} \, dx \), where \( b = x^2 \), is of the form

A. \( \int u^n \, du \)
B. \( \int (1/u) \, du \)
C. \( \int e^u \, du \)
D. none of the above

4. \( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots \) is a power series representation of

A. \( \ln (1+x) \)
B. \( e^x \)
C. \( e^{-x} \)
D. \( \sinh x \)

5. If \( f(x) = (2x + 1)^4 \), then the 4th derivative of \( f(x) \) at \( x = 0 \) is

A. 0
B. 24
C. 48
D. 240
E. 384
6. \( \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \) is the Taylor series about zero for which one of the following functions?

A. \( \sin x \)  
B. \( \cos x \)  
C. \( e^{-x} \)  
D. \( e^x \)  
E. \( \ln (1+x) \)

7. \( \lim_{n \to \infty} \frac{5n}{e^{2n}} = \)

A. \( \infty \)  
B. 0  
C. 2.5  
D. non-existant

8. The area of the region between the graph of \( y = 4x^2 + 2 \) and the x-axis from \( x = 1 \) to \( x = 2 \) is

A. 36  
B. 23  
C. 20  
D. 17  
E. 9

9. \( \frac{d}{dx} (\ln e^{2x}) = \)

A. \( \frac{1}{e^{2x}} \)  
B. \( \frac{2}{e^{2x}} \)  
C. \( 2x \)  
D. 1  
E. 2

10. Which one of the following is not true about L'Hôpital's rule?

A. It can be used on the indeterminate forms \( 0/0 \) and \( \infty/\infty \)  
B. It is used for finding the derivative of a function  
C. It can also be used on the indeterminate form \( \infty - \infty \) after some manipulation of the function  
D. None of the above

11. If \( \int_{-1}^{1} e^{-b} \, dx = k \), where \( b = x^2 \), then \( \int_{-1}^{0} e^{-b} \, dx = \)

A. \(-2k\)  
B. \(k/2\)  
C. \(-k/2\)  
D. \(-k\)  
E. 2k
12. If \( f(2) = 3, \ f'(2) = -1, \ g(2) = -5 \) and \( g'(2) = 2 \), then \( (fg)'(2) - (f/g)'(2) \) is

A. \( \frac{72}{5} \)  
B. \( -\frac{11}{5} \)  
C. \( \frac{276}{25} \)  
D. \( \frac{322}{25} \)

13. If \( y = \sin^2(4x) \) then \( y' \) is

A. \( 12 \sin^2(4x) \)  
B. \( 3 \sin^2(4x) \cos^2(4x) \)  
C. \( 3 \sin^2(4x) \cos(4x) \)  
D. \( 12 \sin^2(4x) \cos(4x) \)

14. If \( f(x) = \frac{\sqrt{x^2-9}}{x-4} \), then \( f \) is continuous in which of the following intervals

i. \((-\infty, \infty)\)  
ii. \((-\infty,-3]\)  
iii. \([3,4)\)  
iv. \((4,\infty)\)

A. i only  
B. ii and iii only  
C. ii, iii and iv only  
D. i, iii and iv only

15. Given \( \int_0^4 \int_y y \cos x^5 \, dx \, dy \), the resulting integral after reversing the order of integration is

A. \( \int_y^2 \int_0^4 y \cos x^5 \, dx \, dy \)  
B. \( \int_y^2 \int_0^x y \cos x^5 \, dx \, dy \)  
C. \( \int_0^2 \int_y^4 y \cos x^5 \, dx \, dy \)  
D. \( \int_0^2 \int_y^4 y \cos x^5 \, dx \, dy \)

16. \( \int_0^1 x e^{-x} \, dx = \)

A. \( 1 - 2e \)  
B. \(-1 \)  
C. \( 1 \)  
D. \( 1 - 2e^{-1} \)  
E. \( 2e - 1 \)

17. \( \int \frac{dx}{(x-1)(x+2)} = \)

A. \( \frac{1}{3} \ln \frac{|x-1|}{|x+2|} + C \)  
B. \( \frac{1}{3} \ln \frac{|x+2|}{|x-1|} + C \)  
C. \( \frac{1}{3} \ln \frac{|(x-1)(x+2)|}{|x-1|} + C \)  
D. \( \ln \frac{|x-1|}{|x+2|} \)  
E. \( \ln \frac{|(x-1)(x+2)^2|}{|x-1|} + C \)
18. All functions \( f \) defined on the xy-plane such that
\[
\begin{align*}
    f_x &= 2x + y \\
    f_y &= x + 2y
\end{align*}
\]
(where \( f_x \) and \( f_y \) are the partial derivatives of \( f \) with respect to \( x \) and \( y \) respectively)
are given by \( f(x,y) = \)

A. \( x^2 + xy + y^2 + C \)  
B. \( x^2 - xy + y^2 + C \)  
C. \( x^2 - xy - y^2 + C \)  
D. \( x^2 + 2xy + y^2 + C \)  
E. \( x^2 - 2xy + y^2 + C \)

19. \( \lim_{x \to 0} (1 + x)^{\frac{1}{b \ln x}} = \)

A. \( e^{\cdot 2} \)  
B. 1  
C. \( \infty \)  
D. \( e \)

20. \( \int \frac{x^2}{e^b} \, dx \), where \( b = x^a \), is

A. \( \frac{-1}{3} \ln e^b + C \)  
B. \( -e^b/3 + C \)  
C. \( -1/(3e^b) + C \)  
D. \( (1/3) \ln e^b + C \)  
E. \( x^a/(3e^b) + C \)

21. If \( y = x^2 \sin y \), then \( y' = \)

A. \( x^2 \cos y \)  
B. \( x^2 \cos y + 2x \sin y \)  
C. \( (2x \sin y)/(1 - x^2 \cos y) \)  
D. \( (2x \sin y + x^2 \cos y)/(1 - x^2 \cos y) \)

22. If \( f(x) = \ln x \), for \( x > 0 \), which one of the following is true?

A. \( f \) is increasing for all \( x > 0 \)  
B. \( f \) is increasing for all \( x > 1 \)  
C. \( f \) is decreasing for all \( x \) between 0 and 1  
D. \( f \) is decreasing for all \( x > e \)  
E. \( f \) is decreasing for all \( x \) between 1 and \( e \)

23. \( \int \ln x \, dx = \)

A. \( x \ln x - x + C \)  
B. \( 1/x + C \)  
C. \( e^x + C \)  
D. non-existent

24. If \( y = 3^{x^x} \), then \( y' = \)

A. \( (3^{x^x} \ln 3)/(2 \sqrt{x}) \)  
B. \( (3/2) x^{-1/2} \)  
C. \( 3^{x^x} \ln 3 \)  
D. \( (3^{x^x})/(2 \sqrt{x}) \)
1. A
2. D
3. B
4. B
5. E
6. C
7. B
8. D
9. E
10. B
11. B
12. C
13. D
14. C
15. B
16. D
17. A
18. A
19. A
20. C
21. C
22. D
23. A
24. A
INSTRUCTIONS FOR THE TEST B IN CALCULUS (Undergrads)

PLEASE READ THE INSTRUCTIONS CAREFULLY

Time : 50 min Max. marks : 24

CALCULATORS, SLIDE RULES AND REFERENCE MATERIALS ARE NOT ALLOWED TO BE USED DURING THE TEST

1. This test contains 24 multiple-choice questions. It is intended to test your knowledge and application of basic concepts studied in calculus 263 A, B and C courses.

2. Answer all the questions on the test paper itself by circling your response as shown below.
   a. b. c. d. e.

3. There is only one correct response for each question.

4. For every incorrect response, 1/4th of a point will be taken off. However, it will be to your advantage to answer such questions for which you can eliminate one or more answer choices as wrong.

5. Since all questions carry equal points, do not spend too much time on any one question.

GOOD LUCK!!

NAME : ________________________________

PHONE : ______________________________
1. \( f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & \text{for } x \neq 2 \\ x - 2 & \text{for } x = 2 \end{cases} \)

is continuous at \( x = 2 \), because of which of the following reasons:

i. \( f(2) \) is defined

ii. \( \lim_{x \to 2} f(x) \) exists

iii. \( \lim_{x \to 2} f(x) = f(2) \)

A. i only  
B. i and ii only  
C. i and iii only  
D. all of the above

2. Which one of the following statements is not true?

A. If a function \( f \) is differentiable at \( a \), then \( f \) is continuous at \( a \).

B. Integration and differentiation are the inverses of each other.

C. In the method of integration by parts, wrong choice of \( u \) and \( dv \) does not lead to a solution.

D. L'Hôpital's rule can be applied to forms of the type \( \frac{0}{0} \).

3. \( \sum_{n=0}^{\infty} (-1)^{n+1} nx^{n-1} \) is a power series representation of

A. \( \frac{1}{1+x} \)  
B. \( \frac{1}{(1+x)^2} \)  
C. \( \sin x \)  
D. \( \cos x \)

4. If the substitution \( u = x/2 \) is made, the integral

\[ \int_{\frac{1}{2}}^{4} \frac{1-(x/2)^2}{x} \, dx = \]

A. \[ \int_{1}^{2} \frac{1-u^2}{u} \, du \]
B. \[ \int_{2}^{4} \frac{1-u^2}{u} \, du \]
C. \[ \int_{1}^{2} \frac{1-u^2}{2u} \, du \]

D. \[ \int_{1}^{4} \frac{1-u^2}{4u} \, du \]
E. \[ \int_{2}^{4} \frac{1-u^2}{2u} \, du \]
5. Which one of the following is a characteristic of
\[ \lim_{x \to 0} \frac{|x|}{x} \]
A. the limit is 1
B. the limit is 0
C. the left-hand limit is -1 and the right-hand limit is 1
D. the left-hand limit is 1 and the right-hand limit is -1

6. The coefficient of \( x^3 \) in the Taylor series for \( e^{3x} \) about \( x = 0 \) is
A. \( \frac{1}{6} \)
B. \( \frac{1}{3} \)
C. \( \frac{1}{2} \)
D. \( \frac{3}{2} \)
E. \( \frac{9}{2} \)

7. The Mean Value Theorem guarantees the existence of a special point on the graph of \( y = \sqrt{x} \) between (0,0) and (4,2). The coordinates of this point are
A. (2,1)
B. (1,1)
C. (2,\( \sqrt{2} \))
D. (1/2,1/\( \sqrt{2} \))
E. none of the above

8. \[ \lim_{x \to 0} \frac{x + 1 - e^x}{x^3} = \]
A. \( \infty \)
B. \( -\infty \)
C. 1
D. 0
E. none of the above

9. If \( y = \log \log x \), then \( \frac{dy}{dx} = \)
A. \( \frac{1}{x \log x} \)
B. \( \frac{1}{\log x} \)
C. \( \log x \)
D. 1

10. If \( y = x^2 + 2 \) and \( u = 2x - 1 \), then \( \frac{dy}{du} = \)
A. \( \frac{(2x^2 - 2x + 4)}{(2x - 1)^2} \)
B. \( 6x^2 - 2x - 4 \)
C. \( x^2 \)
D. \( x \)
E. \( \frac{1}{x} \)
11. \[ \int_{-1}^{2} \int_{1}^{4} (2x+6x^2y) \, dx \, dy = \]
   
   A. 234 \hspace{1cm} C. 378
   B. 144 \hspace{1cm} D. 141

12. The area bounded by the graphs \( y = x \), \( y = 3x \) and \( x+y = 4 \) is
   
   A. 7/2 \hspace{1cm} C. 11/2
   B. 2 \hspace{1cm} D. 4

13. If \( h(x) = f^2(x) - g^2(x) \), \( f'(x) = -g(x) \) and \( g'(x) = f(x) \) then \( h'(x) = \)
   
   A. 0 \hspace{1cm} D. \([ -g(x) ]^2 - [ f(x) ]^2 \)
   B. 1 \hspace{1cm} E. \(-2 [ -g(x) + f(x) ] \)
   C. \(-4 \, f(x) \, g(x) \)

14. If \( f(x) = \frac{f(2x+5) - f(x+7)}{x-2} \) \( x \) not equal to 2
   and if \( f(2) = k \)
   and if \( f \) is continuous at \( x = 2 \), then \( k = \)
   
   A. 0 \hspace{1cm} D. 1
   B. 1/6 \hspace{1cm} E. 7/5
   C. 1/3

15. If \( \frac{d}{dx} (x+1) = -5 \) at \( x = 1 \), then
   \( \frac{d}{dx} \left[ \frac{(x+1)}{(3x-2)} \right]^2 \) at \( x = 1 \) is
   
   A. 20 \hspace{1cm} C. -10
   B. 25 \hspace{1cm} D. none of the above
   C. 1

16. \[ \int_{0}^{1} xe^{-x} \, dx = \]
   
   A. 1-2e \hspace{1cm} D. 1-2e^{-1}
   B. -1 \hspace{1cm} E. 2e-1
   C. 1
17. Given \[ \int_0^4 \int_y^2 y \cos x \, dy \, dx \], the resulting integral after reversing the order of integration is:

A. \[ \int_0^2 \int_y^4 y \cos x \, dy \, dx \]
B. \[ \int_0^2 \int_0^{x^2} y \cos x \, dy \, dx \]
C. \[ \int_0^2 \int_0^4 y \cos x \, dy \, dx \]
D. \[ \int_2^0 \int_y^4 y \cos x \, dx \, dy \]

18. If \( \sqrt{x} + \sqrt{y} = 1 \), then \( \frac{dy}{dx} \) is:

A. \( -\sqrt{x/y} \)
B. \( -1/(2 \sqrt{y}) \)
C. \( -\sqrt{y/x} \)
D. undefined

19. If \( f(x) = (x^2 + 1)^{a-3x} \), then \( f'(1) = \)

A. \( (-1/2) \ln (8e) \)
B. \( -\ln (8e) \)
C. \( (-3/2) \ln 2 \)
D. undefined

20. \( \lim_{x \to \infty} (1 + 5e^x)^{1/x} = \)

A. 0
B. 1
C. \( e^5 \)
D. \( e^6 \)

21. At \( x = 0 \), which one of the following is true of the function \( f \) defined by \( f(x) = x^2 + e^{-2x} \)?

A. \( f \) is increasing
B. \( f \) is decreasing
C. \( f \) is discontinuous
D. none of the above

22. \( \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x} \, dx = \)

A. \( \ln \sqrt{2} \)
B. \( \ln (\pi/4) \)
C. \( \ln \sqrt{3} \)
D. \( \ln (\sqrt{3}/2) \)
E. \( \ln 2 \)
23. \[ \int \frac{x^2}{e^b} \, dx \] where \( b = x^a \) is equal to

A. \((-1/3) \ln b + C\)  
B. \(-1/(3e^b) + C\)  
C. \(-e^b/3 + C\)  
D. \((1/3) \ln e^b + C\)  
E. \(x^a/(3e^b) + C\)

24. \[ \lim_{x \to 0} (x \csc x) \] is

A. \(-\infty\)  
B. \(-1\)  
C. \(0\)  
D. \(1\)  
E. \(\infty\)

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INSTRUCTIONS FOR TEST A IN CALCULUS

PLEASE READ THE INSTRUCTIONS CAREFULLY

Time : 50 min
Max. marks : 25

CALCULATORS, SLIDE RULES AND REFERENCE MATERIALS ARE NOT ALLOWED TO BE USED DURING THE TEST

1. This test contains 25 multiple-choice questions. It is intended to test your knowledge and application of basic concepts studied in calculus 263 A, B and C courses.

2. Answer all the questions on the test paper itself by circling your response as shown below.
   a. b. c. d. e.

3. There is only one correct response for each question.

4. For every incorrect response, 1/4th of a point will be taken off. However, it will be to your advantage to answer such questions for which you can eliminate one or more answer choices as wrong.

5. Since all questions carry equal points, do not spend too much time on any one question.

GOOD LUCK !!

NAME : __________________________________________

PHONE : _________________________________________
1. Which one of the following is not true about L'Hôpital's rule?

A. It can be used to find the limits of indeterminate forms like $0/0$ and $\infty/\infty$
B. It can be used to find the derivative of a function
C. It can be used on the indeterminate form $\infty - \infty$
after some manipulation of the function
D. None of the above

2. $f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{for } x \text{ not equal to } 2 \\ 4 & \text{for } x \text{ equal to } 2 \end{cases}$
is continuous at $x = 2$, because of which of the following reasons:

i. $f(2)$ is defined
ii. $\lim_{x \to 2} f(x)$ exists
iii. $\lim_{x \to 2} f(x) = f(2)$

A. i only C. i and iii only
B. i and ii only D. all of the above

3. Which one of the following formulas is not true?

A. $\frac{d}{dx} (\sin x) = \cos x$
B. $\frac{d}{dx} (\cos^2 x) = \sin^2 x$
C. $\frac{d}{dx} (\sin 2x) = 2 \cos 2x$
D. $\frac{d}{dx} (\cos^3 3x) = -9 \cos^2 3x \sin 3x$

4. $\int \sin 2u \, du =$

A. $\cos 2u + C$ C. $-(\cos 2u)/2 + C$
B. $2 \cos 2u + C$ D. $(\cos 2u)/2 + C$

5. Which one of the following theorems is false?

A. If $f$ is continuous on $[a,b]$, then $f$ is integrable on $[a,b]$
B. If $f$ is defined and increasing or constant on the closed interval $a \leq x \leq b$, then it is integrable there
C. If $f(a)$ exists, then $\int_a^b f(x) \, dx = 0$
D. $\int_c^d f(x) \, dx = \int_c^d f(-x) \, dx$
6. "\[ \int_a^b f(x) \, dx = F(b) - F(a), \] where \( f(x) \) is a continuous function in the interval \([a, b]\) and \( F(x) \) is the indefinite integral of \( f(x) \)." is a statement of which one of the following:
   A. The Fundamental Theorem of Calculus
   B. Taylor's formula
   C. Maclaurin's formula
   D. Mean Value Theorem

7. Which one of the following is a characteristic of \( \lim \limits_{x \to 0} \frac{|x|}{x} \)?
   A. the limit is 1
   B. the limit is 0
   C. the left-hand limit is equal to -1 and the right-hand limit is equal to 1
   D. the left-hand limit is equal to 1 and the right-hand limit is equal to -1

8. At \( x = 0 \), which one of the following is true of the function \( f \) defined by \( f(x) = x^2 + e^{-2x} \)?
   A. \( f \) is increasing
   B. \( f \) is decreasing
   C. \( f \) is discontinuous
   D. none of the above

9. If \( f(x) = 3 \), \( f'(x) = -1 \), \( g(x) = -5 \) and \( g'(x) = 2 \), at \( x = 2 \), then \( \frac{d}{dx} [f(x)g(x) - f(x)/g(x)] \) at \( x = 2 \) is
   A. \( \frac{72}{5} \)
   B. \(-\frac{11}{5}\)
   C. \( \frac{276}{25} \)
   D. \( \frac{322}{25} \)

10. \( \frac{d}{dx} (\ln e^{2x}) =\)
    A. \( \frac{1}{e^{2x}} \)
    B. \( \frac{2}{e^{2x}} \)
    C. \( 2x \)
    D. \( 1 \)
    E. \( 2 \)

11. \( \frac{d}{dx} \ln (x^2+2) =\)
    A. \( \frac{1}{x^2+2} \)
    B. \( x^2+2 \)
    C. \( \frac{2x}{x^2+2} \)
    D. \( \frac{2x+2}{x^2+2} \)
12. If $x^2 + y^2 = 1$, then $dy/dx$ is

A. $-(x/y)$
B. $-(1/2y)$
C. $-(y/x)$
D. undefined

13. All functions $f$ defined on the xy-plane such that $f_x = 2x + y$ and $f_y = x + 2y$ (where $f_x$ and $f_y$ are the partial derivatives of $f$ with respect to $x$ and $y$, respectively) are given by $f(x,y) =$

A. $x^2 + xy + y^2 + C$
B. $x^2 - xy + y^2 + C$
C. $x^2 - xy - y^2 + C$
D. $x^2 + 2xy + y^2 + C$
E. $x^2 - 2xy + y^2 + C$

14. If $f(x) = (2x + 1)^4$, then the 4th derivative of $f(x)$ at $x = 0$ is

A. 0
B. 24
C. 48
D. 240
E. 384

15. The curves $y = f(x)$ and $g(x)$ shown in the figure below intersect at the point $(a,b)$. The area of the shaded region enclosed by these curves and the line $x = -1$ is given by

A. $\int_{-1}^{a} (f(x) - g(x)) \, dx + \int_{-1}^{0} (f(x) + g(x)) \, dx$
B. $\int_{-1}^{b} g(x) \, dx + \int_{b}^{c} f(x) \, dx$
C. $\int_{-1}^{c} (f(x) - g(x)) \, dx$
D. $\int_{-1}^{a} (f(x) - g(x)) \, dx$
16. \[ \int_{-1}^{2} \int_{1}^{4} (2x+6x^2y) \, dx \, dy = \]

A. 234 \hspace{1cm} B. 144 \hspace{1cm} C. 378 \hspace{1cm} D. 141

17. \[ \lim_{n \to \infty} \frac{5n}{e^{2n}} = \]

A. \( \infty \) \hspace{1cm} B. 0 \hspace{1cm} C. 2.5 \hspace{1cm} D. non-existent

18. \[ \lim_{x \to 0} (1 + x)^{\frac{1}{2x}} = \]

A. \( e^{\ln x} \) \hspace{1cm} B. 1 \hspace{1cm} C. \( x \) \hspace{1cm} D. \( e \)

19. If \( y = \sin^2 (4x) \) then \( y' \) is

A. 12 \sin^2 (4x) \hspace{1cm} B. 3 \sin^2 (4x) \cos^2 (4x) \hspace{1cm} C. 3 \sin^2 (4x) \cos (4x) \hspace{1cm} D. 12 \sin^2 (4x) \cos (4x)

20. \[ \int \frac{1}{t} + e^{2t} + t^2 \, dt = \]

A. \(-1/t^2 + e^{2t}/2 + t^3/3 + C\) \hspace{1cm} B. \(-1/t^2 + e^{2t}/2 + t^3 + C\) \hspace{1cm} C. \( \ln t + e^{2t}/2 + 2t + C\) \hspace{1cm} D. \( \ln t + e^{2t}/2 + t^3/3 + C\)

21. \[ \int \frac{2e^{2x} + 3x^2}{e^{2x} + x^3} \, dx \text{ is of the form} \]

A. \( \int \frac{1}{u} \, du \) \hspace{1cm} B. \( \int e^u \, du \) \hspace{1cm} C. \( \int u^n \, du \) \hspace{1cm} D. none of the above

22. \[ \int_{0}^{1} x e^{-x} \, dx = \]

A. 1 - 2e \hspace{1cm} B. -1 \hspace{1cm} C. 1 \hspace{1cm} D. 1 - 2e^{-1} \hspace{1cm} E. 2e - 1
23. \[ \int x (2x + 3)^{10} \, dx = \]

A. \( x \frac{(2x+3)^{11}}{11} - \frac{(2x+3)^{11}}{11} + C \)
B. \( (2x+3)^{10} x \frac{(2x+3)}{11} + C \)
C. \( \frac{(2x+3)^{11}}{11} (22x-3) + C \)
D. \( \frac{(2x+3)^{11}}{22 \times 24} (26x+3) + C \)

24. Given \[ \int_0^4 \int_0^2 y \cos x^5 \, dx \, dy, \] the resulting integral after reversing the order of integration is

A. \[ \int_0^2 \int_y^4 y \cos x^5 \, dy \, dx \]
B. \[ \int_0^2 \int_0^{x^2} y \cos x^5 \, dy \, dx \]
C. \[ \int_0^2 \int_0^4 y \cos x^5 \, dy \, dx \]
D. \[ \int_2^4 \int_0^{y^2} y \cos x^5 \, dx \, dy \]

25. \[ \int_0^\infty \frac{1}{(x-4)^2} \, dx \]

A. \(-\frac{1}{4}\) C. \(\frac{1}{4}\)
B. does not exist D. \(-\frac{1}{192}\)

***
ANSWERS

1. B
2. C
3. B
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21. A
22. D
23. C
24. C
25. B
INSTRUCTIONS FOR TEST B IN CALCULUS

PLEASE READ THE INSTRUCTIONS CAREFULLY

Time: 50 min  Max. marks: 25

CALCULATORS, SLIDE RULES AND REFERENCE MATERIALS ARE NOT ALLOWED TO BE USED DURING THE TEST

1. This test contains 25 multiple-choice questions. It is intended to test your knowledge and application of basic concepts studied in calculus 263 A, B and C courses.

2. Answer all the questions on the test paper itself by circling your response as shown below.
   
   a.    b.    c.    d.    e.

3. There is only one correct response for each question.

4. For every incorrect response, 1/4th of a point will be taken off. However, it will be to your advantage to answer such questions for which you can eliminate one or more possible answers.

5. Since all questions carry equal points, do not spend too much time on any one question.

GOOD LUCK!!

NAME: _____________________________________________________________

PHONE: ___________________________________________________________
1. Which one of the following is true about L'Hôpital's rule?
   A. It is used to find the limit of a function of type \(0/0\) or \(\pm/\pm\)
   B. It is used to find the derivative of a function
   C. It is used to find the area enclosed between two curves
   D. It is used to find the limit of any function

2. The function \(f(x) = \frac{x^2 - x - 2}{x^2 - 2x}\) is discontinuous at \(x = 2\).
Which of the following are true about the function.
   i) \(f(2)\) is not defined
   ii) \(\lim_{x \to 2} f(x)\) does not exist
   iii) \(\lim_{x \to 2} f(x)\) is not equal to \(f(2)\)
   A. i only
   B. ii only
   C. i and iii only
   D. i and ii only
   E. i, ii and iii

3. If \(f(x) = e^x\), which one of the following is equal to \(f'(e)\)?
   A. \(\lim_{h \to 0} \frac{e^{x+h} - e^x}{h}\)
   B. \(\lim_{h \to 0} \frac{e^{x+h} - e^x}{h}\)
   C. \(\lim_{h \to 0} \frac{e^{x+h} - e}{h}\)
   D. \(\lim_{h \to 0} \frac{e^{x+h} - e^x}{h}\)
   E. \(\lim_{h \to 0} \frac{e^{x+h} - 1}{h}\)

4. \(\int \cos 2u \, du = \)
   A. \(\sin 2u + C\)
   B. \(2 \sin 2u + C\)
   C. \(-(\sin 2u)/2 + C\)
   D. \((\sin 2u)/2 + C\)

5. \(\int_{-3}^{5} f(x) \, dx + \int_{-3}^{5} f(x) \, dx = \)
   A. \(-\int_{1}^{-3} f(x) \, dx\)
   B. \(\int_{-3}^{1} f(x) \, dx\)
   C. \(\int_{-3}^{5} f(x) \, dx\)
   D. \(\int_{-3}^{6} f(x) \, dx\)
6. "\[ \int_a^b f(x) \, dx = F(b) - F(a), \] where \( f(x) \) is a continuous function in the interval \([a,b]\) and \( F(x) \) is the indefinite integral of \( f(x) \)." is a statement of which one of the following:

A. The Fundamental Theorem of Calculus  
B. Taylor's formula  
C. Maclaurin's formula  
D. Mean Value Theorem

7. \[ \lim_{x \to 4} \frac{|x-4|}{x-4} = \]

A. non-existent  
B. -1  
C. 1  
D. 0

8. At \( x = k \), the function \( f(x) \) is increasing, if

A. \( f'(x) = 0 \)  
B. \( f'(x) < 0 \)  
C. \( f'(x) > 0 \)  
D. \( f'(x) = f(x) \)

9. If \( f(x) = \sqrt{x} + x^4 \), then \( f'(1) \) is equal to

A. 2  
B. 5  
C. 9/2  
D. 1/2

10. If \( y = \log(\log x) \), \( \frac{dy}{dx} = \)

A. \( \frac{1}{\log x} \)  
B. \( \frac{1}{x \log x} \)  
C. \( \log x \)  
D. 1

11. If \( y = e^b \), where \( b = x^2 \), then \( \frac{dy}{dx} = \)

A. \( e^b \)  
B. \( e^b x^2 \)  
C. \( e^b \cdot 2x \)  
D. \( e^b / 2x \)

12. If \( xy + y^2 = 1 \), then \( \frac{dy}{dx} = \)

A. \( -(x+2y)/y \)  
B. \( -2y \)  
C. \( -y/(x+2y) \)  
D. 0
13. If \( f = x^3y^2 - 2x^2y + 3x \), then \( f_x = \) (\( f_x \) is the partial derivative of \( f \) with respect to \( x \))

A. \( 3x^2y^2 + 4xy + 3 \)  
B. \( 3x^2 - 4x + 3 \)  
C. \( 2x^3y - 2x^2 \)  
D. \( 3x^2y^2 - 4xy + 3 \)

14. If \( f(x) = 2x^2 \), then the 3rd derivative of \( f(x) \) at \( x = 1 \) is

A. 2  
B. 4  
C. 8  
D. 0

15. The area of the region between the graph of \( y = 4x^3 + 2 \) and the \( x \)-axis from \( x = 1 \) to \( x = 2 \) is

A. 36  
B. 23  
C. 20  
D. 17  
E. 9

16. \[ \int_{1}^{2} \int_{1-x}^{x} x^2 y \, dy \, dx = \]

A. \( \frac{165}{120} \)  
B. \( \frac{164}{120} \)  
C. \( \frac{163}{120} \)  
D. non-integrable

17. \( \lim_{x \to 0} \frac{x + 1 - e^x}{x^3} = \)

A. \( \infty \)  
B. \(-\infty\)  
C. 1  
D. 0  
E. none of the above

18. \( \lim_{x \to 0} \frac{1}{e^x - 1} - \frac{1}{x} = \)

A. \( \infty \)  
B. 0  
C. \( \frac{1}{2} \)  
D. \(-\frac{1}{2}\)

19. If \( \frac{d}{dx} (\cos x) = -\sin x \); \( \frac{d}{dx} (\sin x) = \cos x \); then \( \frac{d}{dx} (\cos^4(2x)) = \)

A. \( 8 \cos^3(2x) \)  
B. \(-4 \cos^3(2x) \sin^3(2x) \)  
C. \(-4 \cos^3(2x) \sin(2x) \)  
D. \(-8 \cos^3(2x) \sin^3(2x) \)
20. \[ \int x^{-\left(\frac{2}{3}\right)} \, dx = \]

A. \[ 3 x^{1/3} \]
B. \[ -\left(\frac{2}{3}\right) x^{-5/3} \]
C. \[ -\left(\frac{2}{3}\right) x^{1/3} \]
D. \[ (1/3) x^{2/3} \]

21. \[ \int (\ln x + e^{2x}) \left[ \frac{1}{x} + 2e^{2x} \right] \, dx \]

is of the form

A. \[ \int (\ln u + e^u) \, du \]
B. \[ \int u^n \, du \]
C. \[ \int \left(\frac{1}{u}\right) \, du \]
D. \[ \int (uv) \, du \]

22. \[ \int x^2 e^{3x} \, dx = \]

A. \[ e^{3x} \left[ \frac{1}{3} x^2 - \left(\frac{2}{9}\right) x + \left(\frac{2}{27}\right) \right] + C \]
B. \[ x e^{3x} \left(3x + 2\right) + C \]
C. \[ (x^2 e^{3x})/3 - \left(\frac{2}{3}\right) x e^{3x} + C \]
D. \[ (x^2 e^{3x})/3 - \left(\frac{2}{9}\right) x e^{3x} + (2/27) + C \]

23. In the evaluation of the integral \[ \int x e^{2x} \, dx, \] the best choice of \( u \) and \( dv \) respectively are

A. \( e^{2x} \) and \( x \, dx \)
B. \( x \) and \( e^{2x} \)
C. \( x e^{x} \) and \( e^{x} \, dx \)
D. either A or B

24. Given \[ \int_{0}^{1} \int_{0}^{2} \sin y \, dy \, dx, \] on reversing the order of integration, we get which one of the following?

A. \[ \int_{0}^{2} \int_{0}^{1} \sin y \, dy \, dx \]
B. \[ \int_{0}^{1} \int_{0}^{2} \sin y \, dy \, dx \]
C. \[ \int_{0}^{2} \int_{0}^{y/2} \sin y \, dy \, dx \]
D. \[ \int_{0}^{1} \int_{0}^{2} \sin y \, dx \, dy \]

25. \[ \int_{-1}^{1} \frac{3}{x^2} \, dx = \]

A. \(-6\)
B. \(-3\)
C. \(0\)
D. \(6\)
E. non-existent
### ANSWERS

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REVISED PROGRAMMED INSTRUCTION
TABLE OF CONTENTS

1. LIMITS .................................................. 1
   1.1 Definition ............................................. 1
   1.2 One-sided Limits ..................................... 3
       TEST .................................................. 4
   1.3 Indeterminate Forms .................................. 5
   1.4 Other Indeterminate Forms ......................... 8
       TEST .................................................. 9

2. FUNCTIONS ................................................. 10
   2.1 Continuous Function .................................. 10
   2.2 Discontinuity of a function ....................... 11
       TEST .................................................. 13
   2.3 Increasing and Decreasing Functions ............... 14
       TEST .................................................. 16

ANSWERS ................................................... 17
1. LIMITS

The definition of a limit is very complex. It has to be studied from different points of view before the meaning becomes clear. Simply speaking, as $x$ gets closer and closer to "$a$" (but does not become equal to $a$), then if a function $f(x)$ gets closer and closer to some number "$L$", we say that the limit of $f(x)$ as $x$ approaches $a$, equals $L$.

1.1 Definition of a limit: Let $f$ be a function that is defined on an open interval containing $a$, except possibly at "$a" itself, and let "$L" be a real number.

$$\lim_{x \to a} f(x) = L$$

means that for every $\epsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$

Understand the above definition well. There is no need to memorize it.
Let us quickly state some of the important theorems on limits.

If \( f \) and \( g \) are two functions, with
\[
\lim_{x \to a} f(x) = L_1 \quad \text{and} \quad \lim_{x \to a} g(x) = L_2,
\]
then

1. \( \lim_{x \to a} [f(x) + g(x)] = L_1 + L_2 \) (limit of a sum)

2. \( \lim_{x \to a} [f(x) \cdot g(x)] = L_1 L_2 \) (limit of a product)

3. \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2} \) \( L_2 \) is not zero (limit of a quotient)

4. limit of a composite function

Suppose \( f \) and \( g \) are functions, \( a \) and \( b \) are numbers, \( f(b) \) is defined and
\[
\lim_{x \to b} f(x) = f(b) \quad \text{and} \quad \lim_{x \to a} g(x) = b
\]
then
\[
\lim_{x \to a} f[g(x)] = f(b)
\]

5. If a function \( f \) has a limit as \( x \) approaches \( a \), then
\[
\lim_{x \to a} \, ^n f(x) = \, ^n \lim_{x \to a} f(x)
\]
provided either \( n \) is an odd positive integer or \( n \) is an even positive integer and \( \lim_{x \to a} f(x) > 0 \)
1.2 One-sided Limits

If \( f(x) \to L \) as \( x \to a^+ \). \( L \) is the one-sided limit from the right of \( f \). \( a^+ \) indicates that \( x \) tends to 'a' through values larger than \( a \).

If \( f(x) \to L \) as \( x \to a^- \). \( L \) is the one-sided limit from the left of \( f \). \( a^- \) indicates that \( x \) tends to 'a' through values smaller than \( a \).

The usual notations are

\[
\lim_{x \to a^+} f(x) = L \quad \text{and} \quad \lim_{x \to a^-} f(x) = L
\]

The figure below should make things clear.

left hand limit \( \quad \) right hand limit

\[
\text{---} \to \quad \text{<------}
\]

\[
\text{---------------------} \quad \text{---------------------}
\]

\( a \)

Note: The limit of \( f \) exists if and only if

- both one-sided limits exist and
- both have the same value.
Example #1  If \( f(x) = \frac{|x|}{x} \) find \( \lim_{x \to 0} f(x) \)

Solution  Let us first find the one-sided limits of \( f \).

If \( x > 0 \), then \( f(x) = \frac{x}{x} = 1 \)

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1
\]

If \( x < 0 \), then \( f(x) = \frac{-x}{x} = -1 \)

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -1 = -1
\]

As the two one-sided limits are not equal, the limit doesn't exist.

TEST

1. For the limit \( \lim_{x \to 4} \frac{|x-4|}{x-4} \)
   which one of the following is true?

   a. The left-hand limit is 1
   b. The right-hand limit is -1
   c. The left-hand limit is -1 and right-hand limit is 1
   d. The left-hand and the right-hand limits are both 0
1.3 Indeterminate Forms

We introduce some very important methods of evaluating complex limits in this section.

If \( \lim_{x \to a} f(x) \) is of the form \( 0/0 \) or \( \infty/\infty \), then it is called an indeterminate form. Such limits should be evaluated based on this rule.

"Differentiate the numerator and the denominator separately, and take the limit of the resulting fraction. If the resulting fraction has no limit or doesn't become infinite, then repeat the differentiation" This is called L'Hôpital's rule.

REMEMBER to stop differentiating as soon as the resulting fraction has a limit or becomes infinite, whichever comes first.
Example #2  
Find \( \lim_{x \to 0} \frac{e^x - x - 1}{x^2} \)

Solution  
The given limit is of the form 0/0. Hence use L'Hôpital's rule. Then the limit is

\[
\lim_{x \to 0} \frac{e^x - 1}{2x}
\]

This is again of the form 0/0. Hence use L'Hôpital's rule again. The limit now becomes

\[
\lim_{x \to 0} \frac{e^x}{2}
\]

Hence the limit is equal to 1/2.

NOTE:  
1. L'Hôpital's rule is valid for one-sided limits also.  
2. This rule applies only to indeterminate forms of the type 0/0 and \( \infty/\infty \).  
3. Indeterminate forms of the type 0.\( \infty \) and \( \infty-\infty \) should be converted to the forms 0/0 and \( \infty/\infty \) before applying L'Hôpital's rule.
Example #3  Find \( \lim_{x \to 0} x^2 \ln x \)

**Solution**  This is of the form \(0 \cdot \infty\). Hence rewrite it as

\[
\lim_{x \to 0} \frac{\ln x}{1/x^2}
\]

Applying L'Hôpital's rule, the above limit becomes

\[
\lim_{x \to 0} \frac{1/x}{(-2/x^3)}
\]

Caution  It is sometimes wiser to simplify an expression before blindly applying L'Hôpital's rule. For instance, in the above expression, L'Hôpital's rule need not be used further, if it is simplified.

The above expression is equivalent to

\[
\lim_{x \to 0} -x^2/2 = 0
\]

Example #4  Find \( \lim_{x \to 0} \frac{1}{e^x-1} - \frac{1}{x} \)

**Solution**  This is of the form \(-\infty\)

Rewrite it as

\[
\lim_{x \to 0} \frac{x-e^x+1}{xe^x-x}
\]

This is of the form \(0/0\). Hence using L'Hôpital's rule, we get the value of the limit as \(-1/2\).
1.4 Other Indeterminate Forms

Expressions such as $f(x)^{g(x)}$ give rise to indeterminate forms like $0^0$, $\infty^\infty$, and $1^-$. Note that $1^-$ is not 1.

Example #5 Find $\lim_{x \to 0} (1 + 3x)^{1/2x}$

Solution This is of the form $1^-$

a. Put $y = (1 + 3x)^{1/2x}$

b. $\ln y = (1/2x) \ln(1 + 3x)$

c. The right-hand expression is of the form $0/0$. Hence use L'Hôpital's rule.

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1 + 3x)}{2x}$$

$$= \lim_{x \to 0} \frac{3}{1 + 3x}$$

$$= \frac{3}{2}$$

d. Therefore, $\lim_{x \to 0} (1 + 3x)^{1/2x} = \lim_{x \to 0} y = e^{3/2}$

Guidelines for the above rule.

1. Let $y = f(x)^{g(x)}$

2. Take logarithms: $\ln y = g(x) \ln f(x)$

3. Find $\lim_{x \to c} \ln y$. Use L'Hôpital's rule where needed.

4. If $\lim_{x \to c} \ln y = L$, then $\lim_{x \to c} y = e^L$
TEST

2. \( \lim_{x \to \infty} \frac{e^{ax} - 1}{x^2} \) =
   
   a. 0  b. 1  c. \( \infty \)  d. non-existent

3. \( \lim_{x \to 0^+} x^\infty = \)
   
   a. 0  b. 1  c. \( \infty \)  d. non-existent

   (Hint: \( \lim_{x \to 0^+} (\ln x) = -\infty \))

4. \( \lim_{x \to 0} \frac{1 - \frac{1}{x^2}}{x} = \)
   
   a. 0  b. \( \infty \)  c. \( -\infty \)  d. 1
2. FUNCTIONS

2.1 Continuous Function

A continuous function can be defined in very simple terms as a function whose graph can be drawn without lifting the pencil from the paper.

Mathematically,

A function \( f(x) \) is said to be continuous for \( x = a \) if the limiting value of the function when \( x \) approaches \( a \) is, the value assigned to the function for \( x = a \).

Symbolically.

If \( \lim_{x \to a} f(x) = f(a) \) and \( f(a) \) is defined

then \( f(x) \) is continuous for \( x = a \).

**Example #1** Consider the function \( f(x) = \frac{x^2-4}{x-2} \)

Is this continuous at \( x = 1 \).

**Solution**

\[
\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2-4}{x-2} = \lim_{x \to 1} \frac{(x+2)(x-2)}{x-2} = \lim_{x \to 1} (x+2) = 3
\]

\[
f(1) = \frac{(1^2-4)}{(1-2)} = 3
\]

That is, \( \lim_{x \to 1} f(x) = f(1) \)

Hence the function \( f \) is continuous for \( x = 1 \).
2.2 Discontinuity of a function

A function is discontinuous at \( x = a \), if either one of the following conditions is true.

1. The function is not defined for \( x = a \).

Example #2 Let \( f(x) = \frac{x^2+4}{x^2-9} \). Is this continuous at \( x = 3 \).

Solution For \( x = 3 \), \( x^2-9 = 0 \).
Hence the function \( f(x) \) is not defined at \( x = 3 \).
Therefore \( f \) is discontinuous at \( x = 3 \).

2. The function is defined for \( x = a \), but has no limit as \( x \to a \).

Example #3 Let \( F(x) = \begin{cases} 2x+1 & -1 \leq x \leq 1 \\ (1/2)x^2 - 3 & 1 < x < 4 \end{cases} \). Is this continuous at \( x = 1 \).

Solution \( F(1) = 2*1 + 1 = 3 \)
Hence the function is defined at \( x = 1 \).

The left-hand limit of \( f \) is \( 2*1+1 = 3 \).
The right-hand limit of \( f \) is \( (1/2)*1^2-3 = -5/2 \).
Since the one-sided limits of \( f \) are not equal, the limit does not exist for \( x = 1 \).
Hence \( f \) is discontinuous at \( x = 1 \).
3. The function is defined for \( x = a \) and has a limit for \( x \to a \), but the limit is not equal to the value of the function. That is,

\[
\lim_{x \to a} f(x) \text{ is not equal to } f(a).
\]

Example #4  Let \( f(x) = \frac{2x^2 - 2x}{x - 1} \) for \( x \neq 1 \) and \( f(1) = 1 \). Is this continuous at \( x = 1 \)?

Solution  Since \( f(1) = 1 \), the function \( f \) is defined for \( x = 1 \).

\[
\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{2x^2 - 2x}{x - 1} = \lim_{x \to 1} \frac{2(x)(x-1)}{x-1} = \lim_{x \to 1} 2x = 2.
\]

Although the function is defined and the limit exists for \( x \to a \), this is a discontinuous function, since

\[
\lim_{x \to 1} f(x) \text{ is not equal to } f(1).
\]
5. Is \( f(x) = x^3 + 3x^2 + 3x + 1 \) continuous at \( x = -1 \).
   a. YES  
   b. NO  

6. The function \( f(x) = \frac{x^2 - x - 2}{x^2 - 2x} \) is discontinuous at \( x = 2 \).
   Which of the following are true about this function.
   
   i. \( f(2) \) is not defined.
   ii. \( \lim_{x \to 2} f(x) \) does not exist.
   iii. \( \lim_{x \to 2} f(x) \) is not equal to \( f(2) \).
   
   a. i only  
   b. ii only  
   c. i and iii only  
   d. i and ii only  
   e. i, ii and iii
2.3 Increasing and Decreasing Functions

**Definition:** Given \( x_2 > x_1 \) and both \( x_1 \) and \( x_2 \) are in the interval \( I \), then a function \( f \) is said to be

i) increasing on the interval \( I \), if \( f(x_2) \geq f(x_1) \)

ii) decreasing on the interval \( I \), if \( f(x_2) \leq f(x_1) \)

To put it in simpler words,

As \( x \) increases, if the graph rises, the function is increasing; if the graph falls, the function is decreasing.

The increasing or decreasing nature of any function can be very easily determined by a test called the **first derivative test**.

**Note:** If you don't remember differential calculus, stop here and finish the section on "THE DERIVATIVE" and come back to this section.

**First Derivative Test:** Let \( f \) be a function that is continuous on a closed interval \([a,b]\) and differentiable on the open interval \((a,b)\),

i. if \( f'(x) \geq 0 \) for all \( x \) in \((a,b)\), then \( f \) is increasing on \([a,b]\).

ii. if \( f'(x) \leq 0 \) for all \( x \) in \((a,b)\), then \( f \) is decreasing on \([a,b]\).
Example #5  If \( f(x) = x^3 + x^2 - 5x - 5 \) find the intervals in which \( f \) is increasing and decreasing.

Solution  \( f'(x) = 3x^2 + 2x - 5 = (3x+5)(x-1) \)

\( f'(x) \) becomes zero at \( x = -5/3, 1 \)

These are called \textit{critical values} of the function.

Hence consider the intervals,
\((-\infty, -5/3), (-5/3, 1)\) and \((1, \infty)\)

For any \( x \) in the intervals \((-\infty, -5/3)\) and \((1, \infty)\)
\( f'(x) > 0 \).
Hence \( f \) is increasing on these intervals.

For any \( x \) in the interval \((-5/3, 1)\)
\( f'(x) < 0 \).
Hence \( f \) is decreasing on this interval.
TEST

7. If \( f(x) = e^{2x} - 2x \), the critical number(s) of \( f \) is/are:

a. 1
b. 1 and 0
c. 0
d. 2

8. If \( f(x) = x^4 + 2x^3 \), which of the following is/are true about the function.
   (Hint: There can be more than one answer)
   a. increasing for \( x \leq -3/2 \)
   b. decreasing for \( x \geq -3/2 \)
   c. decreasing for \( x \leq -3/2 \)
   d. increasing for \( x \geq -3/2 \)
ANSWERS

1. c
2. c
3. b
4. c
5. a
6. c
7. c
8. c, d
# TABLE OF CONTENTS

3. THE DERIVATIVE ................................................................. 1
  3.1 Definition ...................................................................... 1
  TEST ................................................................................. 4
  3.2 Trigonometric Functions .................................................. 5
  TEST ................................................................................. 5
  3.3 Logarithm and Exponential Functions ................................. 6
  TEST ................................................................................. 7
  3.4 Implicit Differentiation ..................................................... 8
  TEST ................................................................................. 9
  3.5 Partial Differentiation ...................................................... 10
  TEST ............................................................................... 10
  3.6 Higher Order Derivatives .................................................. 11
  TEST ............................................................................... 12

ANSWERS ................................................................................. 13
3. THE DERIVATIVE

3.1 Definition: The derivative of a function is the limit of the ratio of the increment of the function to the increment of the independent variable, when the latter increment varies and approaches zero as a limit.

Symbolically,

\[ \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \]

The derivative can be denoted as \( \frac{dy}{dx} \), \( y' \) or \( f'(x) \).

Theorem 1: The value of the derivative at any point of a curve is equal to the slope of the tangent line to the curve at that point.
Given below are some important formulas, which you should know.

1. \( \frac{d}{dx} (k) = 0 \) \( k \) is a constant.

2. \( \frac{d}{dx} (u + v + w) = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} \) (sum rule)

3. \( \frac{d}{dx} (ku) = k \frac{du}{dx} \)

4. \( \frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx} \) (product rule)

5. \( \frac{d}{dx} (u_1u_2\ldots u_n) = (u_2u_3\ldots u_n)\frac{du_1}{dx} + (u_1u_3\ldots u_n)\frac{du_2}{dx} + \ldots + (u_1u_2\ldots u_{n-1})\frac{du_n}{dx} \)

6. \( \frac{d}{dx} (u^n) = n u^{n-1} \frac{du}{dx} \) (chain rule)

Caution: Do not forget to differentiate \( u \). \( u \) is a function of \( x \).

If \( y = f(u) \), \( u = g(v) \) and \( v = h(x) \), then

\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} \] (chain rule)

7. \( \frac{d}{dx} (x^n) = n x^{n-1} \) (power rule)

8. \( \frac{d}{dx} (u/v) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \) (quotient rule)

Formulas 4 and 8 can also be written as

\( (uv)' = uv' + vu' \) and

\( (u/v)' = \frac{vu' - uv'}{v^2} \) respectively.
Example #1 If $y = (x^2 - 3)^5$ find $\frac{dy}{dx}$.

Solution Let $u = x^2 - 3$ and $n = 5$

From formula (6),

$$\frac{dy}{dx} = 5(x^2-3)^4 \frac{d}{dx}(x^2-3)$$

$$= 5(x^2-3)^4 \cdot 2x$$

$$= 10x(x^2-3)^4$$

Example #2 If $y = \frac{\sqrt{1-cx}}{\sqrt{1+cx}}$ find $\frac{dy}{dx}$

Solution Let $u = (1-cx)^{1/2}$ and $v = (1+cx)^{1/2}$

From formula (8),

$$\frac{dy}{dx} = \frac{(1+cx)^{1/2} \frac{d}{dx}(1-cx)^{1/2} - (1-cx)^{1/2} \frac{d}{dx}(1+cx)^{1/2}}{(1+cx)}$$

$$= \frac{(1+cx)^{1/2}(1/2)(1-cx)^{-1/2}(-c) - (1-cx)^{1/2}(1/2)(1+cx)^{-1/2}c}{(1+cx)}$$

Simplifying, we get

$$\frac{dy}{dx} = -\frac{c}{(1+cx)\sqrt{1-c^2x^2}}$$

Note that $\frac{d}{dx} (1-cx)^{1/2}$ is found using the chain rule.
9. If \( y = x^a \cdot J(1+x^a) \), then \( \frac{dy}{dx} \) at \( x=2 \) is

a. 38/3  

b. 20  

c. 2/3  

d. 8

10. If \( y = \frac{J(1+2x)}{J(1+3x)} \), then to find \( \frac{dy}{dx} \), which of the following formulas need to be used?

i. \( \frac{d}{dx} (u^n) \)  

ii. \( \frac{d}{dx} (u/v) \)  

iii. \( \frac{d}{dx} (kx) \)  

iv. \( \frac{d}{dx} (uv) \)

a. ii only  

b. i and ii only  

c. i. ii. and iii only  

d. all the above
3.2 Trigonometric Functions

1. \( \frac{d}{dx} (\sin x) = \cos x \)
2. \( \frac{d}{dx} (\cos x) = -\sin x \)
3. \( \frac{d}{dx} (\tan x) = \sec^2 x \)
4. \( \frac{d}{dx} (\cot x) = -\csc^2 x \)
5. \( \frac{d}{dx} (\sec x) = \sec x \tan x \)
6. \( \frac{d}{dx} (\csc x) = -\csc x \cot x \)

**Example #3** If \( Y = x^3 \sin^2(3x) \), find \( Y' \).

**Solution**

\[
Y' = x^3 \frac{d}{dx} (\sin^2 3x) + \sin^2 3x \frac{d}{dx} (x^3)
\]

\[
= x^3 \cdot 2 \sin 3x \cos 3x \cdot 3 + \sin^2 3x \cdot 3x^2
\]

(product rule and chain rule)

\[
= 3x^2 \sin 3x \left(2x \cos 3x + \sin 3x\right)
\]

---

**TEST**

11. The derivative of \( f(x) \), when \( f(x) = \frac{\sin^2 x}{\cos x} \)

(Hint: Use quotient rule)

a. \( f'(x) = \frac{3 \sin^2 x - \sin^3 x (1/2)(\cos x)^{-1/2}}{\cos x} \)

b. \( f'(x) = \frac{\cos x \cdot \sin^3 x - \sin^3 x \cdot f'(\sin x)}{\cos x} \)

c. \( f'(x) = \frac{3 \sin^2 x \cos x + \sin^3 x (1/2)(\cos x)^{-1/2} \sin x}{\cos x} \)
3.3 Logarithm and Exponential Functions

There are only three basic formulas to be remembered in this section. However, they can be manipulated to arrive at several other related formulas.

1. \( \frac{d}{dx} (\ln x) = \frac{1}{x} \)

2. \( \frac{d}{dx} (e^x) = e^x \)

3. \( \frac{d}{dx} (a^x) = a^x \ln a \)

The related formulas are obtained by considering a function \( u \) in place of \( x \).

4. \( \frac{d}{dx} (\ln u) = \frac{1}{u} \frac{du}{dx} \)

5. \( \frac{d}{dx} (e^u) = e^u \frac{du}{dx} \)

6. \( \frac{d}{dx} (a^u) = a^u (\ln a) \frac{du}{dx} \)
Example #4 Given \( Y = \ln(x^2 + 5) \), find \( Y' \).

Solution \[
Y' = \frac{1}{x^2 + 5} \cdot 2x \\
= \frac{2x}{x^2 + 5}
\]

Example #5 Given \( Y = 2^b \), where \( b = x^2 \) find \( Y' \).

Solution This is of the form \( a^u \) and not of the form \( a^x \).

Hence \[
Y' = 2^b \ln 2 \cdot \frac{d}{dx}(b) \\
= 2^b \ln 2 \cdot 2x
\]

TEST

12. Given \( f(x) = e^{3x} \ln(x^2 + 3) \), \( f'(x) = \)

\begin{align*}
a. & \quad e^{3x} \left( \frac{1}{x^2 + 3} \right) + \ln(x^2 + 3) \cdot e^{3x} \\
b. & \quad e^{3x} \left( \frac{2x}{x^2 + 3} \right) + \ln(x^2 + 3) \cdot e^{3x} \\
c. & \quad e^{3x} \left( \frac{2x}{x^2 + 3} \right) + \ln(x^2 + 3) \cdot e^{3x} \cdot 3
\end{align*}

13. Given \( f(x) = 4^{-2x} \), \( f'(x) = \)

\begin{align*}
a. & \quad 4^{-2x} \ln 4 \\
b. & \quad 4^{-2x} \\
c. & \quad 4^{-2x} \cdot (-2) \ln 4 \\
d. & \quad -2x \cdot 4^{-2x-1}
\end{align*}
3.4 Implicit Differentiation

Consider the following functions.

\[ x^2 - 2xy + 3y^2 - 7 = 0, \quad x^3y^2 - 4 = 0, \quad y^3 - x - 5 = 0. \]

These are of the form \( f(x, y) = 0 \). These functions are said to be defined implicitly by the equation. This is one of the types of differentiation where errors are committed very often.

**Example #6** Find \( y' \) if \( y^2 + x^2 = 1 \)

**Solution** **Do not** rewrite this equation as \( y = \sqrt{1-x^2} \).

Differentiate the equation as it is.

\[ 2y \frac{dy}{dx} + 2x = 0 \]

Therefore,

\[ y' = -\frac{2x}{2y} = -\frac{x}{y} \]

**Example #7** Find \( y' \) if \( y = x e^y \)

**Solution** This is an implicit function, since \( y \) is a function of \( x \).

Therefore,

\[ y' = x e^y y' + e^y \]

\[ y'(1-x e^y) = e^y \]

\[ y' = \frac{e^y}{(1-x e^y)} \]

**Caution:** The common error is to write the derivative of \( y \) as

\[ y' = x e^y + e^y \]

**Note:** \( \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \)
14. If $x^2 - xy + y^2 = 1$ then $\frac{dx}{dy} =$

a. $\frac{y - 2x}{2y - x}$

b. $\frac{2y - x}{y - 2x}$

c. $\frac{-2x}{2y - x}$

d. $\frac{2y - x}{-2x}$

15. If $e^{xy} = y$ then $\frac{dy}{dx} =$

a. $\frac{e^{xy} y}{1 - x e^{xy}}$

b. $e^{xy} x$

c. $\frac{1 - x e^{xy}}{e^{xy} y}$

d. $e^{xy} (x + y)$
3.5 Partial Differentiation

Let $z$ be a function of two variables, $z = f(x,y)$. If $y$ is held fixed, $z$ becomes a function of $x$ alone, and its derivative can be found. The result is called the partial derivative of $z$ with respect to $x$ and is denoted by $f_x$ or $Z_x$.

**Example #8** If $u = \sin(2x+3y) + \cos(y-2z)$, find $U_x, U_y, U_z$.

**Solution** Treating $y$ and $z$ as constants, we have

$U_x = 2 \cos(2x+3y)$

Treating $z$ and $x$ as constants, we have

$U_y = 3 \cos(2x+3y) - \sin(y-2z)$

Treating $x$ and $y$ as constants, we have

$U_z = 2 \sin(y-2z)$

It is very important to note which are the independent variables. Note particularly that $U_x$ is not equal to $1/x_u$.

---

**TEST**

16. If $w = xyz \ e^{xyz}$, then $w_x =$

a. $yz \ e^{yz}$

b. $xy \ e^{xy}$

c. $xz \ e^{yz}$

d. $yz \ e^{xy}$
3.6 Higher Order Derivatives

The derivative with respect to \( x \) of \( \frac{dy}{dx} \) is called the second derivative of \( y \) with respect to \( x \), or the derivative of second order. The derivative of the second derivative is called the third derivative of \( y \) with respect to \( x \). Similarly, we may speak of the fourth, fifth .... nth derivative of \( y \) with respect to \( x \).

The notation is \( \frac{d}{dx} (y) \), \( \frac{d^2}{dx^2} (y) \), .... \( \frac{d^n}{dx^n} (y) \)

The notation \( y' \), \( y'' \), \( y''' \), ..... \( y^{(n)} \) is also popular.

This differentiation involves no new formulas. All the previous formulas learnt are applicable here.

Example #9 If \( y = x^4 + x \), find the third derivative of \( y \).

Solution \[ \frac{dy}{dx} = 4x^3 + 1 \]

Differentiating the above expression again, we get
\[ \frac{d^2y}{dx^2} = 12x^2 \] which is the second derivative of \( y \).
\[ \frac{d^3y}{dx^3} = 24x \] which is the required third derivative of \( y \).

Example #10 If \( y = x^2 + xy \), find \( \frac{d^2y}{dx^2} \).

Solution In other words, the second derivative of \( y \) with respect to \( x \) is to be found.

\[ \frac{dy}{dx} = 2x + x \left( \frac{dy}{dx} \right) + y \]
\[ (\frac{dy}{dx})(1-x) = 2x + y \]
\[ \frac{dy}{dx} = \frac{2x + y}{1-x} \]

Differentiating again, we have
\[ \frac{d^2y}{dx^2} = \frac{(1-x)(2+\frac{dy}{dx}) - (2x+y)(-1)}{(1-x)^2} \]

Substituting for \( \frac{dy}{dx} \), we have
\[ \frac{d^2y}{dx^2} = \frac{(1-x)(2+(2x+y)/(1-x)) + (2x+y)}{(1-x)^2} \]

Simplifying the above expression,
\[ \frac{d^2y}{dx^2} = \frac{2(1+x+y)}{(1-x)^2} \]
17. If $y = \sin 3x$, then $\frac{d^3y}{dx^3} =$

a. $-27 \cos 3x$

b. $-\cos 3x$

c. $-9 \sin 3x$

d. $27 \sin 3x$
ANSWERS

9. b
10. c
11. c
12. c
13. c
14. b
15. a
16. d
17. a
### TABLE OF CONTENTS

4. SERIES

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Power Series</td>
<td>1</td>
</tr>
<tr>
<td>4.2</td>
<td>Power Series Representation of Functions</td>
<td>1</td>
</tr>
<tr>
<td>4.3</td>
<td>Taylor's Series</td>
<td>3</td>
</tr>
<tr>
<td>4.4</td>
<td>Maclaurin's Series</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>TEST</td>
<td>5</td>
</tr>
<tr>
<td>4.5</td>
<td>MEAN VALUE THEOREM</td>
<td>6</td>
</tr>
<tr>
<td>4.6</td>
<td>Mean Value Theorem for Definite Integrals</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>TEST</td>
<td>9</td>
</tr>
</tbody>
</table>

ANSWERS ................................................................................. 10
4. SERIES

Since this is only a refresher course, let us look at only a few important series.

4.1 Power Series

If \( x \) is a variable and \( a_0, a_1, \ldots a_n \) are constants, a series of the form

\[
a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots = \sum_{n=0}^{\infty} a_nx^n
\]

is called a power series in \( x \).

4.2 Power Series Representation of Functions

A power series \( \sum a_nx^n \) may be used to define a function \( f \). For each \( x \) in the interval of convergence, we let \( f(x) \) equal the sum of the series. That is,

\[
f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots
\]

If a function \( f \) is defined in this way, we say that \( \sum a_nx^n \) is a power series representation for \( f(x) \).
Since the actual technique of finding the power series representation for various functions is outside the scope of this program, we list the expansions of some standard series.

1. \((1+x)^n = 1 + mx + m(m-1) x^2 + \ldots + m(m-1) \ldots (m-n+2) \frac{x^{n-1}}{2!} \cdot \frac{1}{(n-1)!} \)

2. \(\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + \ldots \)

3. \(\frac{1}{1+x^2} = 1 - 2x + 3x^2 - 4x^3 + \ldots + (-1)^{n-1}nx^{n-1} + \ldots \text{ if } |x| < 1\)

4. \(\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^{n-1}}{n+1} + \ldots \text{ if } |x| < 1\)

5. \(\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots - \frac{x^n}{n} - \ldots \text{ if } |x| < 1\)

6. \(\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots +(-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots \)

7. \(\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots +(-1)^n \frac{x^{2n}}{2n!} + \ldots \)

8. \(e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots \)
4.3 Taylor's Series

The power series

\[ f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n \]

represents the function \( f(x) \) for those, and only those values of \( x \), for which the remainder approaches zero as \( n \to \infty \).

The remainder is \( R_n = \frac{f^{(n+1)}(\delta)}{(n+1)!}(x-a)^{n+1} \) \( (\delta \text{ between } a \text{ and } x) \)

Writing \( x-a = h \), we get the Taylor's series

\[ f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \ldots + \frac{f^{(n)}(a)}{n!}h^n + \ldots \]

a power series in \( h \).
4.4 Maclaurin's Series

This series is a special case of Taylor's series, namely, when \( a = 0 \).

**Example #1**  Find the Taylor's series for \( \sin x \) in powers of \( x - \pi/6 \).

**Solution**  We have \( x - \pi/6 = h \), that is, \( a = \pi/6 \)

\[
\begin{align*}
  f(\pi/6) &= \sin(\pi/6) = 1/2 \\
  f'(\pi/6) &= \cos(\pi/6) = \sqrt{3}/2 \\
  f''(\pi/6) &= -1/2 \\
  f'''(\pi/6) &= -\sqrt{3}/2
\end{align*}
\]

This pattern repeats itself. Substituting in the above formulas for Taylor's series

\[
\sin x = 1/2 + (\sqrt{3}/2)(x-\pi/6) - 1/(2*2!)(x-\pi/6)^2 - \ldots.
\]

The Maclaurin series for \( \sin x \) is obtained by putting \( a = 0 \).

**Example #2**  Find \( \sin(0.1) \)

**Solution**  Put \( x = 0.1 \) in the Maclaurin series for \( \sin x \)

\[
\sin(0.1) = 0.1 - (0.1)^3/3! + (0.1)^5/5! - \ldots.
\]
18. $1-(0.2)^2/2!+(0.2)^4/4!-\ldots$ is a Maclaurin expansion for which one of the following.

a. $\sin(0.2)$

b. $\sinh(0.2)$

c. $\cosh(0.2)$

d. $\cos(0.2)$

e. $\ln(1.2)$
4.5 MEAN VALUE THEOREM

If a function f is continuous on a closed interval [a, b] and is differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

\[ f(b) - f(a) = f'(c)(b-a) \]

This is also known as the Theorem of the Mean.

Example #3

Prove that the function f defined by

\[ f(x) = x^3 - 8x - 5 \]

satisfies the hypothesis of the Mean Value Theorem on the interval [1, 4].

Solution

Since f is a polynomial function, it is continuous and differentiable for all real numbers.

In particular, it is continuous on [1, 4] and differentiable on the open interval (1, 4).

Refer to example #15 in the section Limits and Functions to find out how the continuity of a function is determined in an interval.

Thus the hypothesis of the Mean Value Theorem is satisfied.
Example #4  For the previous problem, find a number $c$ in the interval $(1, 4)$ that satisfies the conclusion of the theorem.

Solution  According to the Mean Value Theorem, there exists a number $c$ in the interval $(1, 4)$ such that

$$f(4) - f(1) = f'(c)(4 - 1)$$

Since $f'(x) = 3x^2 - 8$, this is equivalent to

$$27 - (-12) = (3c^2 - 8)(3)$$

Solving, we get $c = \pm \sqrt{7}$

Hence the desired number in the interval $(1, 4)$ is $\sqrt{7}$. 
4.6 Mean Value Theorem for Definite Integrals

If \( f \) is continuous on a closed interval \([a, b]\), then there is a number \( z \) in the open interval \((a, b)\) such that

\[
\int_{a}^{b} f(x) \, dx = f(z)(b-a)
\]

Example #5 If \( \int_{0}^{3} [4 - (x^2/4)] \, dx = 39/4 \)
Find a number that satisfies the conclusion of the Mean Value Theorem.

Solution We have \( f(x) = 4 - (x^2/4) \)

Therefore,

\[
\int_{0}^{3} (4 - (x^2/4)) \, dx = (4 - (z^2/4))(3 - 0)
\]

Equivalently,

\[
39/4 = (16 - z^2)(3/4)
\]

\[
z^2 = 3
\]

Therefore, \( z = \sqrt{3} \) satisfies the conclusion of the Mean Value Theorem.

Note: The Mean Value Theorem is useful in solving several theorems in calculus, most importantly the Fundamental Theorem of Calculus.
19. The number that satisfies the conclusion of the mean value theorem for \[ \int_{0}^{3} 3x^2 \, dx = 27 \] is

a. 3
b. \( \sqrt{3} \)
c. 0
d. 1
ANSWERS

18.  d
19.  b
**TABLE OF CONTENTS**

5. **INTEGRATION** .................................................. 1

5.1 Fundamental Properties of Integrals ...................... 1
5.2 Trigonometric Formulas ..................................... 2
5.3 Method of Substitution .................................... 3
    TEST ....................................................... 4
5.4 Definite Integral ........................................... 5
5.5 Theorems on Integration ................................... 5
5.6 Properties of the Definite Integral ...................... 6
5.7 Fundamental Theorem of Calculus ......................... 6
5.8 Integration by Parts ....................................... 7
    TEST ....................................................... 9
5.9 Area .......................................................... 10
    TEST ....................................................... 11
5.10 Infinite Limits of Integration ......................... 12
5.11 Integrals with Discontinuous Integrands ............... 12
5.12 Double Integrals ......................................... 14
    TEST ....................................................... 16

**ANSWERS** ..................................................... 17
5. INTEGRATION

In general,
\[ \int f(x) \, dx = F(x) + C \]

means that \( F(x) + C \) is a function whose differential is \( f(x) \, dx \). \( F(x) + C \) is called an integral of \( f(x) \). \( C \) is called the constant of integration. \( f(x) \) is called the integrand.

5.1 Fundamental Properties of Integrals

1. \( \int k \, du = k \int du = ku \) \hspace{1cm} (k is a constant)

2. \( \int (du + dv + \ldots + dw) = \int du + \int dv + \ldots + \int dw \)

   \[ = u + v + \ldots + w + C \]

3a. \( \int u^n \, du = \frac{u^{n+1}}{n+1} + C \) \hspace{1cm} (n is not equal to -1)

3b. \( \int \frac{1}{u} \, du = \ln u + C \)

Example #1 Evaluate \( \int (4 + 1/t^3 + 1/t + 5t) \, dt \)

Solution The given integral is equivalent to
\[
\int 4 \, dt + \int 1/t^3 \, dt + \int 1/t \, dt + \int t^{6/5} \, dt
\]
\[ = 4t - \frac{t^{-3+1}}{-3+1} + \ln t + \frac{t^{6/5}}{6/5} \]
\[ = 4t - \frac{1}{2}t^2 + \ln t + \frac{5}{6} t^{6/5} \]

.
4. \[ \int e^u \, du = e^u + C \]

5. \[ \int a^u \, du = \frac{a^u}{\ln a} + C \]

5.2 Trigonometric Formulas

1. \[ \int \sin u \, du = -\cos u + C \]

2. \[ \int \cos u \, du = \sin u + C \]

3. \[ \int \sec^2 u \, du = \tan u + C \]

4. \[ \int \csc^2 u \, du = -\cot u + C \]

5. \[ \int \sec u \tan u \, du = \sec u + C \]

6. \[ \int \csc u \cot u \, du = -\csc u + C \]

7. \[ \int \sec u \, du = \ln |\sec u + \tan u| + C \]

8. \[ \int \csc u \, du = -\ln |\csc u + \cot u| + C \]
We now describe some of the different methods of integration.

5.3 Method of Substitution

Consider the integral \( \int (4-7x)^{15} \, dx \)

Remember that the integral is not of the form \( u^n \)

Let \( u = 4-7x \)

Then, \( du = -7 \, dx \)

The given integral becomes \( \int u^{15} \cdot (-1/7) \, du \)

\[ = (-1/7) \frac{u^{16}}{16} + C \]

\[ = (-1/112) (4-7x)^{16} + C \]

So by making a change of variable, the above integral could be converted to a known form. This technique is known as method of substitution.

Example #2 Of what form is \( \int x \sqrt{7-6x^2} \, dx \)

Solution Let \( u = 7-6x^2 \), then \( du = -12x \, dx \)

Therefore the integral is \( \int u \frac{-1}{12} \, du \)

\[ = (-1/12) \int u \, du \]

That is, the integral is of the form \( \int u^n \, du \) where \( n = 1/2 \)
Example #3  Evaluate $\int \frac{\cos Jx}{Jx} \, dx$

Solution  Let $u = Jx$, then $du = 1/(2\sqrt{x}) \, dx$

The integral is $\int (\cos u) \, 2 \, du$

$= 2 \sin u + C = 2 \sin Jx + C$

Remark: This method proceeds by trial and error. If one substitution fails, another, maybe entirely different substitution has to be tried.

Rule of Thumb: If the Numerator is a derivative of the Denominator, then use the method of substitution. Assume the denominator is equal to $u$ and proceed as illustrated above.

TEST

20. The best substitution that can be used in solving the integral $\int (1/x) \sin (\ln x) \, dx$

is to let $u =$

a. $1/x$

b. $\ln x$

c. $\sin (\ln x)$

d. $x$
5.4 Definite Integral

Let \( f \) be a function that is defined on a closed interval \([a, b]\). The definite integral of \( f \) from \( a \) to \( b \), denoted by

\[
\int_{a}^{b} f(x) \, dx
\]

is defined as the numerical measure of the area bounded by the curve \( y = f(x) \), the x-axis and the ordinates of the curve at \( x = a \) and \( x = b \). This definition presupposes that these lines bound an area, that is, the curve does not rise or fall to infinity and does not cross the x-axis, and both \( a \) and \( b \) are finite.

5.5 Theorems on Integration

1. \[
\int_{c}^{d} f(x) \, dx = - \int_{c}^{d} f(x) \, dx
\]
   That is, interchanging the limits of integration changes the sign of the integral.

2. If \( f(a) \) exists, then \[
\int_{a}^{a} f(x) \, dx = 0
\]

3. If \( f \) is continuous on \([a, b]\), then \( f \) is integrable on \([a, b]\).

4. If \( f(x) \) is defined and increasing (or at least nondecreasing) on the closed interval \( a \leq x \leq b \), then it is integrable there.
5.6 Properties of the Definite Integral

1. \[ \int_{a}^{b} k \, dx = k \, (b-a) \]

2. If \( a < c < b \), and if \( f \) is integrable on both \([a,c]\) and \([c,b]\) then \( f \) is integrable on \([a,b]\) and
\[ \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx \]

5.7 Fundamental Theorem of Calculus

This theorem is used to find the definite integral without using limits of sums. Due to its importance in evaluating definite integrals, and because it exhibits the connection between differentiation and integration, the theorem is aptly called The Fundamental Theorem of Calculus.

**Statement:** Suppose \( f \) is continuous on a closed interval \([a,b]\)

**Part I** If the function \( G \) is defined by
\[ G(x) = \int_{a}^{b} f(x) \, dx \]
for all \( x \) in \([a,b]\), then \( G \) is an antiderivative of \( f \) on \([a,b]\).

**Part II** If \( F \) is any antiderivative of \( f \), then
\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) \]

In other words, integration and differentiation are the inverses of each other.
5.8 Integration by Parts

The formula for the differential of a product is
\[ d(uv) = u\,dv + v\,du \]

By integrating both sides, we obtain

\[ uv = \int u\,dv + \int v\,du \]

or

\[ \int u\,dv = uv - \int v\,du \]

**Remark:** In the above formula for integration by parts, a proper choice of \(dv\) is crucial. A trial and error method should be adopted in choosing \(u\) and \(dv\).

**Example #4** Find \( \int \ln x \, dx \)

**Solution** Let \( u = \ln x \) and \( dv = dx \)

Then, \( du = \frac{1}{x} \, dx \) and \( v = x \)

Integrating by parts using the above formula, we get

\[ \int \ln x \, dx = (\ln x)\,x - \int x \left(\frac{1}{x}\right) \, dx \]

\[ = x \ln x - x + C \]
Example #5 Find $\int x e^{2x} \, dx$

Solution Let $u = x$ and $dv = e^{2x} \, dx$
Then, $du = dx$ and $v = e^{2x}/2$

Integrating by parts, we get

$$\int x e^{2x} \, dx = x \frac{e^{2x}}{2} + \int \frac{e^{2x}}{2} \, dx$$
$$= x \frac{e^{2x}}{2} + \frac{e^{2x}}{4} + C$$

Remark: A choice of $u = e^{2x}$ and $dv = x \, dx$ would lead us to a more complicated integral than the one we started with.

So enough care must be exercised in choosing $u$ and $dv$. 

TEST

21. \[ \int x^2 e^{3x} \, dx = \]

a. \( e^{3x} \left[ \frac{1}{3} x^2 - \frac{2}{9} x + \frac{2}{27} \right] + C \)

b. \( x e^{3x} (3x + 2) + C \)

c. \( \frac{x^2 e^{3x}}{3} - \frac{2}{3} x e^{3x} + C \)

d. \( \frac{x^2 e^{3x}}{3} - \frac{2}{9} x e^{3x} + \frac{2}{27} + C \)
5.9 Area

The definite integral is useful in finding the area of a region bounded by one or more graphs and the coordinate axes.

If \( f \) and \( g \) are continuous and \( f(x) \geq g(x) \) for all \( x \) in \([a, b]\), then the area \( A \) of the region bounded by the graphs of \( f, g, x = a \), and \( x = b \) is

\[
A = \int_{a}^{b} [f(x) - g(x)] \, dx
\]

**Example #7** Find the area of the region bounded by the lines \( x=1 \), \( x=2 \), \( y=3x \) and the curve \( y=x^2 \).

**Solution** We draw a sketch as shown in the figure for the above graphs.

The area is given by

\[
A = \int_{1}^{2} (3x-x^2) \, dx
\]

Note the sketch helps us to find out which graph is above the other. Therefore,

\[
A = \frac{3x^2}{2} - \frac{x^3}{3} \bigg|_{1}^{2} = 13/6
\]
22. The area bounded by the curves $y = x$ and $y = x^3$ is
   a. $\frac{1}{4}$  b. 0  c. $\frac{1}{2}$  d. 1

23. The area bounded by the graphs of the equations $y = x^2$ and $y = \sqrt{x}$ is
   a. $\frac{1}{3}$  b. $\frac{2}{3}$  c. 1  d. $\frac{1}{6}$
5.10 Infinite Limits of Integration

**Definition**

i. If $f$ is continuous on $[a, \infty)$, then
$$\int_{a}^{\infty} f(x) \, dx = \lim_{t \to \infty} \int_{a}^{t} f(x) \, dx$$

ii. If $f$ is continuous on $(-\infty, a]$, then
$$\int_{a}^{\infty} f(x) \, dx = \lim_{t \to \infty} \int_{a}^{t} f(x) \, dx$$

The above expressions are called improper integrals. Improper integrals may have two infinite limits of integration.

Let $f$ be continuous for all $x$. If $a$ is any real number, then
$$\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{a} f(x) \, dx + \int_{a}^{\infty} f(x) \, dx$$

5.11 Integrals with Discontinuous Integrands

If a function $f$ has an infinite discontinuity at some number in the interval it may still be possible to assign a value to the integral.

If $f$ has a discontinuity at a number $c$ in the open interval $(a,b)$ but is continuous elsewhere in $[a,b]$, then
$$\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx$$

provided both of the integrals on the right converge.
Example #9 Determine \[ \int_{0}^{4} \frac{1}{(x-3)^2} \, dx \]

**Solution** If the Fundamental theorem of calculus is directly applied, we obtain

\[ \frac{-1}{x-3} \bigg|_{0}^{4} = \frac{-1}{4} - \frac{-1}{0} = \frac{-1}{0} \]

This is obviously wrong, since the integrand is never negative. Also, at \( x = 3 \), the integrand becomes infinite.

Hence we need to use the above definition. We get

\[
\int_{0}^{4} \frac{1}{(x-3)^2} \, dx = \int_{0}^{3} \frac{1}{(x-3)^2} \, dx + \int_{3}^{4} \frac{1}{(x-3)^2} \, dx
\]

\[
\int_{0}^{3} \frac{1}{(x-3)^2} \, dx = \frac{-1}{x-3} \bigg|_{0}^{3} = -\frac{1}{3} - (-\frac{1}{0}) = \infty
\]

Since this integral diverges, the limit does not exist.
5.12 Double Integrals

**Definition:** The definite double integral
\[
\int_{a_1}^{a_2} \int_{u_1}^{u_2} f(x,y) \, dy \, dx
\]
may be interpreted as that portion of the volume of a right cylinder which is included between the plane XOV and the surface
\[z = f(x,y),\]
the base of the cylinder being the area in the XOY plane bounded by the curves
\[y = u_1, \quad y = u_2, \quad x = a_1, \quad x = a_2.\]

**Definition:**
\[
\int_{a}^{b} \int_{c}^{d} f(x,y) \, dy \, dx = \int_{a}^{b} \left[ \int_{c}^{d} f(x,y) \, dy \right] \, dx
\]
This is called an **iterated** (double) integral.

**Example #10**
Evaluate \( \int_{0}^{2} \int_{x^2}^{2x} (x^3 + 4y) \, dy \, dx \)

**Solution**
By the above definition, the integral equals
\[
\int_{0}^{2} \left[ \int_{x^2}^{2x} (x^3 + 4y) \, dy \right] \, dx
\]
\[
= \int_{0}^{2} \left[ x^3y + 2y^2 \right]_{x^2}^{2x} \, dx
\]
\[
= \int_{0}^{2} (8x^2 - x^5) \, dx
\]
\[
= \left. \frac{8x^3}{3} - \frac{x^6}{6} \right|_{0}^{2}
\]
\[
= (64/3) - (64/6)
\]
\[
= 32/3
\]
Sometimes it is extremely difficult or even impossible to evaluate a given double integral. However by reversing the order of integration from $dy \, dx$ to $dx \, dy$ or vice versa, it may be possible to evaluate the double integral.

We illustrate below, the procedure of reversing the order of integration.

**Example #11** Given\[\int_0^4 \int_{2y}^4 y \cos x^5 \, dx \, dy\]
reverse the order of integration.

**Solution:** The graph of the equation is sketched below.

The given order of integration is $dx \, dy$. Hence we call the region $R$ of type II. The left hand and right hand boundaries are the graphs of $x=2y$ and $x=2$, respectively and $0 \leq y \leq 4$.

If the order of integration is $dy \, dx$, then the region $R$ is of type I. Then the lower and upper boundaries are given by $y=0$ and $y=x^2$ respectively, where $0 \leq x \leq 2$.

Hence \[\int_0^4 \int_{2y}^4 y \cos x^5 \, dx \, dy = \int_0^2 \int_0^{x^2} y \cos x^5 \, dy \, dx\]
TEST

24. \[ \int_2^1 \int_{x}^{2x} x^2 y \, dy \, dx = \] __________

25. Given \[ \int_0^1 \int_{2x}^{2} \sin y \, dy \, dx, \]
on reversing the order of integration we get which one of the following:

a. \[ \int_{2x}^{2} \int_0^1 \sin y \, dx \, dy \]

b. \[ \int_0^{y/2} \int_0^2 \sin y \, dx \, dy \]

c. \[ \int_0^{y/2} \int_0^2 \sin y \, dx \, dy \]

d. \[ \int_0^1 \int_{2x}^{2} \sin y \, dx \, dy \]
ANSWERS

20. b
21. a
22. a
23. a
24. $\frac{163}{120}$
25. b
QUESTION BANK
LIMITS
1. $\lim_{x \to 0} (1+x)^{1/x} = \phantom{a. \ e^{0.2}}$
   a. $e^{0.2}$
   *b. 1
   c. $\infty$
   d. e

2. $\lim_{x \to 0} (x+1-e^x)/x^a = \phantom{a. \infty}$
   a. $\infty$
   *b. $-\infty$
   c. 1
   d. 0
   e. none of the above

3. $\lim_{x \to 0} (x \csc x)$ is equal to
   a. $-\infty$
   b. -1
   c. 0
   *d. 1
   e. $\infty$

4. $\lim_{n \to \infty} r^n = 0$, if $|r|$ is
   a. greater than 1
   *b. less than 1
   c. equal to 1
   d. less than or equal to 1

5. $\lim_{n \to \infty} (1.01)^n$ is
   a. 0
   *b. $\infty$
   c. 1
   d. -1

6. $\lim_{n \to \infty} 5n/e^{2n}$ is equal to
   *a. 0
   b. $\infty$
   c. 2.5
   d. none of the above
7. \( \lim_{x \to \infty} (1+5e^x)^{1/x} \) is 
   a. 0 
   b. 1 
   c. \( e \) 
   d. \( e^5 \) 
   e. non-existent 

8. Which one of the following is a characteristic of 
   \( \lim_{x \to 0^\pm} \frac{|x|}{x} \) 
   a. the limit is 1 
   b. the limit is 0 
   c. the two one-sided limits are different 
   d. the left-hand limit is 1 and the right-hand limit is -1 

9. \( \lim_{x \to 0} \frac{1-\cos 2x}{\sin 3x} = \) 
   a. \( \frac{2}{3} \) 
   b. \( \frac{3}{2} \) 
   c. 0 
   d. 1 

10. \( \lim_{x \to 1} x^{1/(x-1)} = \) 
    a. 1 
    b. \( \infty \) 
    c. \( \frac{1}{e} \) 
    d. \( e \) 

11. \( \lim_{x \to \infty} \frac{x-1}{2x+\cos x} = \) 
    a. \( \frac{1}{2} \) 
    b. 0 
    c. \( \infty \) 
    d. doesn't exist 

12. Which one of the following is not true about L'Hôpital's rule? 
   a. It can be used on the indeterminate forms 0/0 and \( \infty/\infty \) 
   b. Wherever applicable, it always leads to a solution. 
   c. It can also be used on the indeterminate forms \( \infty-\infty \) after some manipulation of the expression 
   d. all of these
13. \[ \lim_{x \to 0} \frac{\sec x - x}{\csc x + x} = \]

*a. 0  
b. \(\infty\)  
c. \(-\infty\)  
d. doesn't exist

14. A limit does not exist for which one of the following?

a. \( \lim_{x \to 0} \frac{\sin x}{x} \)

*b. \( \lim_{x \to 3} \frac{|x - 3|}{x - 3} \)

c. \( \lim_{x \to 3} 5 \)

d. \( \lim_{x \to 2} \frac{2x^2 - 5x + 2}{5x^2 - 7x - 6} \)

15. \[ \lim_{x \to 0} \frac{\sin^2 2x}{\sin x} \]

a. 2  
b. \(\frac{1}{2}\)  
c. 0  
d. non-existent

16. If \( f(x) = e^x \), which one of the following is equal to \( f'(e) \)?

a. \( \lim_{h \to 0} \frac{e^{x+h} - h}{h} \)

*b. \( \lim_{h \to 0} \frac{(e^{x+h} - e^x)}{h} \)

c. \( \lim_{h \to 0} \frac{(e^{x+h} - e)}{h} \)

d. \( \lim_{h \to 0} \frac{(e^{x+h} - 1)}{h} \)
* e. \( \lim_{h \to 0} \frac{(e^{x+h} - e^x)}{h} \)

17. A limit does not exist for which one of the following:

a. \( \lim_{x \to 2} 8 \)

*b. \( \lim_{x \to 0} \frac{|x|}{x} \)

c. \( \lim_{x \to \infty} \frac{\sqrt{x^2 - 1}}{2x + 1} \)

d. \( \lim_{x \to 1} |x - 1| \)
18. \( \lim \frac{5-x}{x} \) does not exist because 
\[ x \to 5^+ \]

*a. the right hand limit does not exist
b. the left hand limit does not exist
c. the left hand and right hand limits are equal
d. the right hand limit is 0

19. A limit exists for which one of the following:

a. \( \lim \frac{5-x}{x} \) 
\[ x \to 5 \]
b. \( \lim \frac{|x-4|}{x-4} \) 
\[ x \to 4 \]
c. \( \lim \frac{1}{x} \) 
\[ x \to 0 \]
*d. none of the above

20. \( \lim \frac{1}{e^{x-1}} - \frac{1}{x} \) 
\[ x \to 0 \]

*a. \( \infty \)
b. 0
c. 1/2
*d. -1/2

21. \( \lim \frac{1}{x-1} - \frac{1}{\ln x} \) 
\[ x \to 1 \]

*a. \( \infty \)
b. 0
*c. -1/2
d. non-existent

22. \( \lim x^3 \frac{2^{-x}}{x} \) 
\[ x \to \infty \]

*a. \( \infty \)
b. 0
*c. 6
d. \( \frac{6}{(\ln 2)^3} \)
23. The graph below represents which one of the following limits:

\[ \lim_{x \to a^-} f(x) = L \]

a. \( \lim_{x \to a^-} f(x) = L \)

b. \( \lim_{x \to a^-} f(x) = L \)

c. \( \lim_{x \to a} f(x) = L \)

d. \( \lim_{x \to a^-} f(x) = -L \)

24. The figure below for \( f(x) \) represents which one of the following:

\[ y \]

\[ 1 \]

\[ -1 \]

a. limit of \( f(x) \) is 1

b. limit of \( f(x) \) is -1

c. limit of \( f(x) \) is 0

d. no limit exists for \( f(x) \)

25. A limit exists for a function, if

a. the left hand limit is equal to the negative of the right hand limit

* b. the left hand limit is equal to the right hand limit

c. one of the one-sided limits exist

d. both the one-sided limits are positive
26. For the figure below which one of the following is true?

\[ y = f(x) \]

- a. \( \lim_{x \to 1^-} f(x) = 4 \) and \( \lim_{x \to 1^-} f(x) = 3 \)
- b. \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} f(x) = 3 \)
- c. \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} f(x) = 4 \)
- d. \( \lim_{x \to 1^-} f(x) = 4 \) and \( \lim_{x \to 1^-} f(x) = 3 \)

27. \( \lim_{x \to 4} \frac{|x-4|}{x-4} \) =
- a. non-existent
- b. -1
- c. 1
- d. 0

28. Which one of the following is true about L'Hôpital's rule?

- a. It is used to find the limit of a function of type \( 0/0 \) or \( \infty/\infty \)
- b. It is used to find the derivative of a function
- c. It is used to find the area enclosed between two curves
- d. It is used to find the limit of any kind of function
29. If \( f(x)/g(x) \) has the indeterminate form \( 0/0 \) or \( \infty/\infty \) at \( x = c \), then \( \lim_{x \to c} f(x) \) according to L'Hôpital's rule is
\[
\begin{align*}
\text{a.} & \quad \lim_{x \to c} \frac{f'(x)}{g'(x)} \\
\text{b.} & \quad \lim_{x \to c} \frac{f(x)}{g(x)} \\
\text{c.} & \quad \lim_{x \to c} \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \\
\text{d.} & \quad 0 \text{ or } \infty
\end{align*}
\]

30. L'Hôpital's rule can't be applied for which one of the following?
\[
\begin{align*}
\text{a.} & \quad \lim_{x \to \infty} x^2 \quad e^x \\
\text{b.} & \quad \lim_{x \to 1} \frac{\ln x}{x-1} \\
\text{c.} & \quad \lim_{x \to 1} \frac{e^x - e}{\ln x} \\
\text{d.} & \quad \lim_{x \to 0} \frac{e^x + e^{-x}}{x^2}
\end{align*}
\]

31. Which one of the following is not an indeterminate form?
\[
\begin{align*}
\text{a.} & \quad 0 \cdot \infty \\
\text{b.} & \quad 0^0 \\
\text{c.} & \quad \infty^0 \\
\text{d.} & \quad 1^{-} \\
\text{e.} & \quad \text{none of the above}
\end{align*}
\]

32. Which one of the following is not an indeterminate form?
\[
\begin{align*}
\text{a.} & \quad \lim_{x \to 0^+} x \ln x \\
\text{b.} & \quad \lim_{x \to 0} (1/x)^2 \\
\text{c.} & \quad \lim_{x \to 1} \frac{1}{x-1} - \frac{1}{\ln x} \\
\text{d.} & \quad \lim_{x \to 1} \frac{(x-1)^{x-1}}{x-1}
\end{align*}
\]

33. If \( y \) is of the form \( 1^{-} \), then \( \ln y \) is of the form
\[
\begin{align*}
\text{a.} & \quad \infty \cdot \infty \\
\text{b.} & \quad 0 \cdot \infty \\
\text{c.} & \quad \infty \cdot 0 \\
\text{d.} & \quad \infty \cdot 1
\end{align*}
\]
34. \( \lim_{x \to 1} \frac{\log x}{x-1} = \)

a. \( \infty \)
b. 0
*c. 1
*d. doesn't exist

35. \( \lim_{x \to 0} x \log x = \)

*a. 0
b. -\( \infty \)
c. 1
d. -\( \infty \)
FUNCTIONS
1. \( f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{for } x \neq 2 \\ 4 & \text{for } x = 2 \end{cases} \)

is continuous at \( x = 2 \), because of which of the following reasons?

i. \( f(2) \) is defined

ii. \( \lim_{x \to a} f(x) \) exists

iii. \( \lim_{x \to a} f(x) = f(a) \)

a. i only

b. i and ii only

c. i, ii and iii

d. i and iii only

2. Which one of the following functions is not continuous at every real number?

a. sine and cosine

b. polynomial

c. rational

d. tangent

3. If \( f(x) = \frac{\ln x}{x} \) for \( x > 0 \), which of the following is true

a. \( f \) is increasing for all \( x > 0 \)

b. \( f \) is increasing for all \( x > 1 \)

c. \( f \) is decreasing for all \( x \) between 0 and 1

d. \( f \) is decreasing for all \( x \) between 1 and \( e \)

e. \( f \) is decreasing for all \( x > e \)

4. If \( \lim_{x \to a} f(x) = L \), where \( L \) is a real number, which of the following must be true?

a. \( f'(a) \) exists

b. \( f(x) \) is continuous at \( x = a \)

c. \( f(x) \) is defined at \( x = a \)

d. \( f(a) = L \)

e. none of the above

5. If \( f(x) = \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} \) for \( x \neq 2 \)

and if \( f(2) = k \) and if \( f \) is continuous at \( x = 2 \), then \( k = \)

a. 0

b. \( \frac{1}{6} \)

c. \( \frac{1}{3} \)

d. 1

e. \( \frac{7}{5} \)
6. At \( x = 0 \), which one of the following is true of the function \( f \) defined by \( f(x) = x^2 + e^{-2x} \)?

a. \( f \) is increasing
*b. \( f \) is decreasing
  
c. \( f \) is discontinuous
  
d. none of the above

7. If \( f \) is a function such that \( \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = 0 \), which one of the following must be true?

a. the limit of \( f(x) \) as \( x \) approaches 2 does not exist
b. \( f \) is not defined at \( x = 2 \)
  *c. the derivative of \( f \) at \( x = 2 \) is 0
  d. \( f \) is continuous at \( x = 0 \)
  e. \( f(2) = 0 \)

8. The function \( f(x) = \frac{x^2 - 1}{x^2 + x} \) for \( x > -3 \) is discontinuous at which of the following points?

i. \(-1\)  
ii. \(0\)  
iii. \(-3\)

a. i only
b. i and ii only
*c. i, ii and iii
  d. ii only

9. The function \( f(x) = \begin{cases} 2x^2 & -1 \leq x < 1 \\ 3 - x & 1 \leq x < 2 \end{cases} \) is discontinuous at which of the following points?

i. \(1\)  
ii. \(0\)  
iii. \(-1\)

a. i only
b. i and ii only
  c. i, ii and iii
  *d. iii only

10. If \( f'(x) \) is positive for all values of \( x \) in a certain interval, then \( f(x) \) is

a. decreasing throughout this interval
  *b. increasing throughout this interval
  c. constant throughout this interval
  d. none of the above
11. If \( f'(x) \) is negative for all values of \( x \) in a certain interval, then \( f(x) \) is

a. decreasing throughout this interval
b. increasing throughout this interval
c. constant throughout this interval
d. none of the above

12. The figure below is a graph for which one of the following?

![Graph](image)

a. increasing function
b. constant function
c. decreasing function
d. none of the above

13. The figure below is a graph for which one of the following?

![Graph](image)

a. increasing function
b. constant function
c. decreasing function
d. none of the above

14. At \( x = k \), the function \( f(x) \) is increasing, if

a. \( f'(x) = 0 \)
b. \( f'(x) < 0 \)
c. \( f'(x) > 0 \)
d. \( f'(x) = f(x) \)
THE DERIVATIVE
1. If \( f(x) = \sqrt{x} + x^4 \), then \( f'(1) \) is equal to

a. 2  
b. 5  
\*c. 9/2  
d. 2/9

2. If \( y = (x^2+1)/x^4 \), then \( y' = \)

\*a. \(-2x^2+4)/x^5\)  
b. \((6x^2+4)/x^5\)  
c. 2  
d. \((x^2+1)/x^5\)

3. If \( f(2)=3, f'(2)=-1, g(2)=-5 \), and \( g'(2)=2 \) then \((fg)'(2)-(f/g)'(2)\) is

a. 72/5  
b. -11/5  
\*c. 276/25  
d. -22/5

4. If \( y = \log \log x \), then \( dy/dx = \)

\*a. \(1/(x \log x)\)  
b. \(1/\log x\)  
c. \(\log x\)  
d. 1

5. If \( y = f(u), u = g(x) \) and the derivative \( dy/du \) and \( du/dx \) both exist, then the composite function defined by \( y = f(g(x)) \) has a derivative given by \( dy/dx = \)

a. \(f'(g(x))\)  
b. \(f(g'(x))\)  
\*c. \(f'(u)g'(x)\)  
d. none of the above

6. If \( y = \sin^3(4x) \) then \( y' = \)

a. \(3 \sin^2(4x)\)  
b. \(3 \sin^2(4x) \cos^2(4x)\)  
\*c. \(12 \sin^2(4x) \cos(4x)\)  
d. \(3 \sin^2(4x) \cos(4x)\)

7. If \( Jx + Jy = 1 \), then \( dx/dy = \)

\*a. \(-J(x/y)\)  
b. \(-1/J(2y)\)  
c. \(-J(y/x)\)  
d. undefined
8. \( \frac{d}{dx} (\ln e^{2x}) = \)
   a. \( \frac{1}{e^{2x}} \)
   b. \( e^{2x} \)
   c. \( 2x \)
   d. 1
   *e. 2

9. If \( h(x) = f^2(x) - g^2(x) \), \( f'(x) = -g(x) \), and \( g'(x) = f(x) \), then \( h'(x) = \)
   a. 0
   b. 1
   *c. \(-4 f(x) . g(x) \)
   d. \([-g(x)^2] - [f(x)]^2 \)
   e. \(-2 [-g(x) + f(x)] \)

10. If \( y = x^2 + 2 \) and \( u = 2x - 1 \), then \( \frac{dy}{du} = \)
    a. \( \frac{2x^2 - 2x + 4}{(2x - 1)^2} \)
    b. \( 6x^2 - 2x + 4 \)
    c. \( x^2 \)
    *d. \( x \)
    e. \( \frac{1}{x} \)

11. If \( y = \tan u \), \( u = \frac{v-1}{v} \) and \( v = \ln x \), then \( \frac{dy}{dx} \) at \( x=e \) is equal to
    a. 0
    b. \( \frac{1}{e} \)
    c. 1
    *d. \( \frac{2}{e} \)
    e. \( \sec^2 e \)

12. \( \frac{d}{dx} (\ln \frac{1}{1-x}) = \)
    *a. \( \frac{1}{1-x} \)
    b. \( \frac{1}{x-1} \)
    c. \( 1-x \)
    d. \( x-1 \)
    e. \( (1-x)^2 \)

13. If \( y = (\cos^2 x - \sin^2 x) \), then \( y' = \)
    a. \(-1 \)
    b. 0
    *c. \(-2 \sin (2x) \)
    d. \(-2 (\cos x + \sin x) \)
    e. \(2 (\cos x - \sin x) \)
14. If \( y = x^2 \sin y \), then \( y' = \)

a. \( x^2 \cos y \)
b. \( x^2 \cos y + 2x \sin y \)
* c. \( (2x \sin y)/(1-x^2 \cos y) \)
d. \( (2x \sin y + x^2 \cos y)/(1-x^2 \cos y) \)

15. If \( y = 6^x x^4 \), then \( y' = \)

a. \( 6(1/3)x^{5/3} \)
* b. \( 8 \ln x \)
c. \( 2x^{-7/3} \)
d. \( 8x^3 \)

16. If \( f(x) = (2x+1)^4 \) then the 4th derivative of \( f(x) \) at \( x = 0 \) is

a. 0
b. 24
c. 48
d. 240
* e. 384

17. If \( f(x) = \sin x \), then \( f^{100}(x) = \)

*a. \( \sin x \)
b. \( -\sin x \)
c. \( \cos x \)
d. \( -\cos x \)

18. If \( y = 3^x \), then \( y' = \)

a. \( 3^{1/2}x^{-1/2} \)
b. \( 3^x \ln 3 \)
* c. \( (3^x \ln 3)/(2 \ln x) \)
d. \( 3^x/(2 \ln x) \)

19. If \( \frac{d}{dx} [(x+1)/(3x-2)] = -5 \) at \( x = 1 \), then \( \frac{d}{dx} [(x+1)/(3x-2)]^2 \) at \( x = 1 \), is

a. 20
b. 25
c. -10
* d. none of the above
20. If \( f(x) = e^x \), which one of the following is equal to \( f'(e) \)

a. \( \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \)

b. \( \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \)

c. \( \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \)

d. \( \lim_{h \to 0} \frac{e^{x+h} - 1}{h} \)

e. \( \lim_{h \to 0} \frac{e^{x-h} - e^x}{h} \)

21. Which one of the following statements is true?

a. \( f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \)

b. \( f''(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \)

c. If \( f \) is differentiable at \( a \), then \( f \) is continuous at \( a \)

*d. If \( f \) is continuous at \( a \), then \( f \) is differentiable at \( a \)

22. \( \frac{d}{dx} [f(x) g(x) h(x)] = \)

*a. \( f(x) g(x) h'(x) + f(x) g'(x) h(x) + f'(x) g(x) h(x) \)

b. \( f(x) g(x) h'(x) + h(x) f'(x) g'(x) \)

c. \( f'(x) g'(x) h'(x) \)

d. \( f(x) g(x) h'(x) + f(x) g'(x) h(x) + f'(x) g(x) h(x) + f'(x) g'(x) h'(x) \)

23. If \( y = f(u) \), \( u = g(x) \), then the derivative of a composite function defined by \( y = f(g(x)) \) is given by \( \frac{dy}{dx} = \)

a. \( f'(u) \)

b. \( f'(g(x)) g(x) \)

*c. \( f'(g(x)) g'(x) \)

d. \( f'(g(x)) + g'(x) f'(x) \)

24. Which one of the following formulas is invalid.

a. If \( y = f(u) \), \( u = g(v) \), \( v = h(x) \). then
\[ \frac{dy}{dx} = \frac{(dy/du)}{(du/dv)} (dv/dx) \]

*b. \( \frac{d}{dx} [g(x)]^n = n [g(x)]^{n-1} \)

c. \( \frac{d}{dx} x^n = n x^{n-1} \)

d. none of the above

25. If \( y = \ln x \), then

*a. \( \frac{dy}{dx} = 1/x \)

b. \( \int y \, dx = 1/x + C \)

c. \( \frac{dy}{dx} = e^x \)

d. \( \int y \, dx = x \ln x + C \)
26. If $y = \ln x$, then

a. $y' = x$

*b. $\int \frac{1}{x} \, dx = y$

c. $\int y \, dx = e^x + C$

d. $\int y \, dx = \frac{1}{x} + C$

27. If $u = g(x)$, then

a. $\frac{d}{dx} e^u = e^u$

b. $\frac{d}{dx} a^u = a^u \ln a$

c. $\int e^u \, du = e^u / u + C$

*d. $\int a^u \, du = \frac{a^u}{\ln a} + C$

28. If $x^2 + y^2 = 1$, then $dy/dx =$

*a. $-(x/y)$

b. $-(y/x)$

c. $-(1/2y)$

d. undefined

29. If $xy + y^2 = 1$, then $dy/dx =$

a. $-(x+2y)/y$

b. $-2y$

*c. $-y/(x+2y)$

d. $0$

30. If $f = x^3y^2 - 2x^2y + 3x$, then $f_x =$

*a. $3x^2y^2 - 4xy + 3$

b. $3x^2 - 4x + 3$

c. $2x^3y - 2x^2$

d. $3x^2y^2 + 4xy + 3$

31. All functions $f$ defined on the xy-plane such that $f_x = 2x+y$ and $f_y = x+2y$ (where $f_x$ and $f_y$ are the partial derivatives of $f$ with respect to $x$ and $y$ respectively) are given by $f(x,y) =$

*a. $x^2 + xy + y^2 + C$

b. $x^2 - xy + y^2 + C$

c. $x^2 + 2xy + y^2 + C$

d. $x^2 - 2xy + y^2 + C$
32. If \( w = xyz e^{xyz} \), then \( w_x = \)

a. \((yz)^2 e^{xyz}\)
b. \(xy e^{xyz} (xyz + 1)\)
c. \(xz e^{xyz} (xyz + 1)\)
d. \(yz e^{xyz} (xyz + 1)\)

33. If \( y = \sin 3x \), then \(\frac{d^3y}{dx^3} = \)

*a. \(-27 \cos 3x\)
b. \(-\cos 3x\)
c. \(-9 \sin 3x\)
d. \(27 \sin 3x\)

34. If \( y = \sin (2x) \), then \(y''' = \)

a. \(2 \cos 2x\)
b. \(-8 \cos 2x\)
c. \(-4 \sin 2x\)
d. \(8 \cos 2x\)

35. Which one of the following formulas is not true?

a. \(\frac{d}{dx} (\sin x) = \cos x\)
*b. \(\frac{d}{dx} (\cos^2 x) = \sin^2 x\)
c. \(\frac{d}{dx} (\sin 2x) = 2 \cos 2x\)
d. \(\frac{d}{dx} (\cos^3 3x) = -9 \cos^2 3x \sin 3x\)

36. If \( f(x) = 3 \), \( f'(x) = -1 \), \( g(x) = -5 \) and \( g'(x) = 2 \), at \( x = 2 \), then \(\frac{d}{dx} [f(x) \cdot g(x) - f(x)/g(x)]\) at \( x = 2 \) is

a. \(\frac{72}{5}\)
b. \(-\frac{11}{5}\)
*c. \(\frac{276}{25}\)
d. \(\frac{322}{25}\)

37. \(\frac{d}{dx} \ln (x^2+2) = \)

a. \(\frac{1}{x^2+2}\)
b. \(x^2+2\)
*c. \(\frac{2x}{x^2+2}\)
d. \(\frac{2x+2}{x^2+2}\)

38. If \( y = e^b \), where \( b = x^2 \), then \(\frac{dy}{dx} = \)

a. \(e^b\)
b. \(e^b \cdot x^2\)
*c. \(e^b \cdot 2x\)
d. \(e^b / 2x\)
39. If \( f(x) = 2x^2 \), then the 3rd derivative of \( f(x) \) at \( x = 1 \) is

a. 2  
b. 4  
c. 8  
*d. 0  

40. If \( \frac{d}{dx} (\cos x) = -\sin x \); \( \frac{d}{dx} (\sin x) = \cos x \); then

\[ \frac{d}{dx} (\cos^2x) = \]

a. \( 8 \cos^3(2x) \)  
b. \(-4 \cos^3(2x) \sin^3(2x) \)  
c. \(-4 \cos^3(2x) \sin(2x) \)  
*d. \(-8 \cos^3(2x) \sin^3(2x) \)
SERIES
1. \(1 + x + x^2/2! + x^3/3! + \ldots + x^n/n! + \ldots\) is a power series representation of

* a. \(e^x\)
b. \(\ln (1+x)\)
c. \(e^{-x}\)
d. \(\sinh x\)

2. \(\sum_{n=0}^{\infty} (-1)^{n-1}nx^{n-1}\) is a power series representation of

\(n=0\)

a. \(1/(1+x)\)
* b. \(1/(1+x)^2\)
c. \(\sin x\)
d. \(\cos x\)

3. \(\sum_{n=0}^{\infty} (-1)^{n}x^n/n!\) is the Taylor series about zero for which one of the following functions?

a. \(\sin x\)
b. \(\cos x\)
c. \(e^x\)
* d. \(e^{-x}\)
e. \(\ln (1+x)\)

4. The coeff of \(x^n\) in the Taylor series for \(e^{ax}\) about \(x = 0\) is

a. \(1/6\)
b. \(1/3\)
c. \(1/2\)
d. \(3/2\)
* e. \(9/2\)

5. A power series in \(x\) of the form

\[\sum_{k=0}^{\infty} a_kx^k = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\]

is also called a/an

* a. Maclaurin series
b. Taylor series
c. Exponential series
d. Binomial series
6. A power series in \((x-a)\) of the form
\[
\sum_{k=0}^{\infty} a_k(x-a)^k = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots
\]
is also called a/an
a. Maclaurin series
* b. Taylor series
c. Exponential series
d. Binomial series

7. The Mean Value Theorem guarantees the existence of a special point on the graph of \(y = \sqrt{x}\) between (0,0) and (4,2), whose coordinates are
a. (2,1)
* b. (1,-1)
c. (2,\sqrt{2})
d. (1/2,1/\sqrt{2})
e. none of the above

8. Let \(f\) be a function given by \(f(x) = x^3 - 3x^2\). Which one of the following is/are the values of \(c\) that satisfy the conclusion of the Mean Value Theorem of differential calculus on the closed interval \([0,3]\)?
a. 0 only
* b. 2 only
c. 3 only
d. 0 and 3
e. 2 and 3

9. The number that satisfies the conclusion of the Mean Value Theorem for the function \(f(x) = (2x+3)/(3x+2)\) in the interval \((-1,0)\) is
a. \(-1/2\)
b. \((\sqrt{2}-2)/3\)
c. 0
* d. none of the above

10. If \(f(x) = x^3 - 8x - 5\), then the number \(c\) that satisfies the conclusion of the Mean Value Theorem on the interval \([1,4]\) is
*a. \(\sqrt{7}\)
b. \(\sqrt{(5/2)}\)
c. 21
d. 39
INTEGRATION
1. \[ \int_a^b f(x) \, dx = F(b) - F(a), \] where \( f(x) \) is a continuous function in the interval \([a, b]\) and \( F(x) \) is the indefinite integral of \( f(x) \) is a statement of

*a. Fundamental Theorem of Calculus  
b. Taylor's formula  
c. Maclaurin's formula  
d. Mean Value Theorem

2. \[ \int_2^8 \ln 5 \, dx = \]
   
a. 0  
b. \( \frac{1}{\ln 5} \)  
c. \( 6 \ln 5 \)  
d. \( \frac{6}{\ln 5} \)

3. \[ \int_{x^2}^a \frac{1}{x^2} \, dx = \]
   
a. \( 3x^\frac{1}{3} \)  
b. \( -\frac{2}{3}x^{-\frac{5}{3}} \)  
c. \( -\frac{2}{3}x^{\frac{1}{3}} \)  
d. \( \frac{1}{3}x^{\frac{1}{3}} \)

4. \[ \int a^x \, dx = \]
   
a. \( a^x/\ln a \)  
b. \( a^x \)  
c. \( a^{x-1}/(x+1) \)  
d. non-existent

5. \[ \int_0^8 \frac{dx}{J(1+x)} = \]
   
a. 1  
b. 3/2  
c. 2  
d. 4  
e. 6

6. \[ \int_0^1 J(x^2-2x+1) \, dx = \]
   
a. -1  
b. -1/2  
c. 1/2  
d. 1  
e. none of the above
7. If $\frac{dy}{dx} = \tan x$, then $y =$

a. $\frac{1}{2} \tan^2 x + C$

b. $\sec^2 x + C$

c. $\ln |\sec x| + C$

d. $\ln |\cos x| + C$

e. $\sec x \tan x + C$

8. $\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x} \, dx =$

*a. $\ln \sqrt{2}$

b. $\ln \pi/4$

c. $\ln \sqrt{3}$

d. $\ln (\sqrt{3}/2)$

e. $\ln e$

9. $\int x^2/e^b \, dx$, where $b = x^a$, is equal to

a. $-1/3 \ln e^b + C$

b. $-e^b/3 + C$

c. $-1/(3 e^b) + C$

d. $1/3 \ln e^b + C$

e. $x^a/(3 e^b) + C$

10. If $n$ is a non-negative integer, then $\int_0^1 x^n \, dx = \int_0^1 (1-x)^n \, dx$

for which one of the following:

a. no $n$

b. $n$ even only

c. $n$ odd only

d. non-zero $n$ only
* e. all $n$

11. $\int \sin (2x+3) \, dx =$

a. $1/2 \cos (2x+3) + C$

b. $\cos (2x+3) + C$

c. $-\cos (2x+3) + C$

d. $-1/2 \cos (2x+3) + C$

e. $-1/5 \cos (2x+3) + C$
12. The length of the graph of \( y = \tan x \) between \( x = a \) and \( x = b \), where \( 0 < a < b < \pi/2 \) is given by

\[
\begin{align*}
\text{a. } & \int_{a}^{b} \sqrt{1 + \tan^2 x} \, dx \\
\text{b. } & \int_{a}^{b} (x + \tan x) \, dx \\
\text{c. } & \int_{a}^{b} (1 + \sec^2 x) \, dx \\
\text{d. } & \int_{a}^{b} (1 + \tan^2 x) \, dx \\
\text{e. } & \int_{a}^{b} (1 + \sec^4 x) \, dx
\end{align*}
\]

13. The area of the region between the graph of \( y = 4x^2 + 2 \) and the x-axis from \( x=1 \) to \( x=2 \) is

\[
\begin{align*}
\text{a. } & 36 \\
\text{b. } & 23 \\
\text{c. } & 20 \\
\text{d. } & 17 \\
\text{e. } & 9
\end{align*}
\]

14. \[
\int \frac{dx}{\sqrt{25-x^2}} =
\]

\[
\begin{align*}
\text{a. } & \arcsin \frac{x}{5} + C \\
\text{b. } & \arcsin x + C \\
\text{c. } & \frac{1}{5} \arcsin \frac{x}{5} + C \\
\text{d. } & \sqrt{25-x^2} + C \\
\text{e. } & 2 \sqrt{25-x^2} + C
\end{align*}
\]

15. \[
\int_{-1}^{1} \frac{3}{x^2} \, dx =
\]

\[
\begin{align*}
\text{a. } & -6 \\
\text{b. } & -3 \\
\text{c. } & 0 \\
\text{d. } & 6 \\
\text{e. } & \text{non-existent}
\end{align*}
\]
16. If the substitution \( u = x/2 \) is made, the integral
\[
\int \frac{1-(x/2)^2}{x} \, dx =
\]
* a. \( \int \frac{1-u^2}{u} \, du \)
   b. \( \int \frac{1-u^2}{u} \, du \)
   c. \( \int \frac{1-u^2}{2u} \, du \)
   d. \( \int \frac{1-u^2}{4u} \, du \)
   e. \( \int \frac{1-u^2}{2u} \, du \)

17. \( \int \frac{\ln x}{x} \, dx \) is of the form

* a. \( \int u^n \, du \)
   b. \( \int \frac{1}{u} \, du \)
   c. \( \int e^u \, du \)
   d. none of the above

18. \( \int \frac{x e^b + 1}{e^b + 2x} \, dx \) (where \( b = x^2 \)) is of the form

a. \( \int u^n \, du \)
   * b. \( \int \frac{1}{u} \, du \)
   c. \( \int e^u \, du \)
   d. none of the above

19. \( \int \ln x \, dx =
\]
* a. \( x \ln x - x + C \)
   b. \( 1/x + C \)
   c. \( e^x + C \)
   d. non-existant
20. \( \ln 4 = \)
   a. \( \ln 3 + \ln 1 \)
   b. \( \ln \frac{8}{1} \ln 2 \)
   c. \( \int_{1}^{4} e^x \, dx \)
   d. \( \int_{1}^{4} \ln x \, dx \)
   *e. \( \int_{1}^{4} \frac{1}{t} \, dt \)

21. The slope of the line tangent to the graph of \( y = \ln(x/2) \)
   at \( x = 4 \) is
   a. 1/8
   *b. 1/4
   c. 1/2
   d. 1
   e. 4

22. If \( \int_{-1}^{1} \cos x \, dx = k \), then \( \int_{-1}^{0} \cos x \, dx = \)
   a. \(-2k\)
   b. \(-k\)
   c. \(-k/2\)
   *d. \(k/2\)
   e. 2k

23. If \( \int_{-1}^{0} \sin x \, dx = k/2 \), then \( \int_{-1}^{1} \sin x \, dx = \)
   a. \(-2k\)
   b. \(-k\)
   c. \(-k/2\)
   d. \(k/2\)
   *e. 0

24. \( \int_{0}^{1} x e^{-x} \, dx = \)
   a. 1-2e
   b. \(-1\)
   *c. 1-2e^{-1}
   d. 1
   e. 2e-1
25. \[ \int_0^1 \frac{x^2-1}{x+1} \, dx = \]
   a. 1/2
   b. 1
   c. 2
   d. 5/2
   e. \ln 3

26. If \[ \int_{-2}^2 (x^7+k) \, dx = 16, \] then k =
   a. -12
   b. -4
   c. 0
   *d. 4
   e. 12

27. \[ \int \tan 2x \, dx = \]
   a. -2 \ln |\cos (2x)| + C
   b. -1/2 \ln |\cos (2x)| + C
   *c. 1/2 \ln |\cos (2x)| + C
   d. 2 \ln |\cos (2x)| + C
   e. 1/2 \sec (2x) \tan (2x) + C

28. \[ \int_0^{\pi/3} \sin (3x) \, dx = \]
   a. -2
   b. -2/3
   c. 0
   *d. 2/3
   e. 2

29. \[ \int_0^4 1/(x-3)^2 \, dx = \]
   a. \infty
   b. -4/3
   c. 4/3
   d. 0

30. \[ \int_{-1}^1 \int_0^4 (2x+6x^2y) \, dx \, dy = \]
   a. 234
   b. 144
   c. 378
   d. 141
31. Let $R$ be the region bounded by the graphs of the equations $y = 3x$, $y = 3(3x-18)$ and $y = 0$. If $f$ is continuous on $R$, then the double integral $\iint_R f(x,y) \, dA$ in terms of iterated integrals is given by

* a. $\int_0^6 \int_0^9 f(x,y) \, dy \, dx + \int_0^6 \int_0^9 f(x,y) \, dy \, dx$

b. $\int_0^9 \int_0^6 f(x,y) \, dy \, dx$

c. $\int_0^9 \int_0^6 f(x,y) \, dy \, dx - \int_0^9 \int_0^6 f(x,y) \, dy \, dx$

d. none of the above

32. Given $\int_0^4 \int_y^2 y \cos x^2 \, dx \, dy$, the resulting integral after reversing the order of integration is

a. $\int_0^4 \int_0^2 y \cos x^2 \, dy \, dx$

b. $\int_0^4 \int_0^2 y \cos x^2 \, dy \, dx$

c. $\int_0^4 \int_0^2 x^2 y \cos x^2 \, dy \, dx$

d. $\int_0^4 \int_0^2 y \cos x^2 \, dx \, dy$

33. $\int x^r \, dx = \frac{1}{r+1} \cdot x^{r+1} + C$ is true for

a. all values of $r$

b. all positive values of $r$ only

* c. all values of $r$ except -1

d. all values of $r$ except 0

34. Which one of the following is false?

a. $\int a^x \, dx = \frac{a^x}{\ln a} + C$

b. $\int e^x \, dx = e^x + C$

* c. $\int \ln x \, dx = \frac{1}{x} + C$

d. $\int \sqrt{x} \, dx = \frac{2}{3} x^{3/2} + C$

35. The best substitution that can be used in solving the integral $\int \frac{1}{x} \sin (\ln x) \, dx$ is to let $u =$

a. $1/x$

* b. $\ln x$

c. $\sin (\ln x)$

d. $x$
36. Which one of the following theorems is false?

a. If \( f \) is continuous on \([a, b]\) then \( f \) is integrable on \([a, b]\)

b. If \( f \) is defined and increasing or constant on the closed interval \( a \leq x \leq b \), then it is integrable there.

c. If \( f(a) \) exists, then \( \int_a^c f(x) \, dx = 0 \)

d. \( \int_c^d f(x) \, dx = \int_d^c f(-x) \, dx \)

37. \( \int_1^5 f(x) \, dx + \int_{-3}^5 f(x) \, dx = \)

* a. \( \int_{-3}^1 f(x) \, dx \)

b. \( \int_5^5 f(x) \, dx \)

c. \( \int_{-3}^1 f(x) \, dx \)

d. \( \int_2^6 f(x) \, dx \)

38. \( \int_{-2}^6 f(x) \, dx - \int_{-2}^{-2} f(x) \, dx = \)

a. \( \int_{-2}^6 f(x) \, dx \)

* b. \( \int_{-2}^6 f(x) \, dx \)

c. \( \int_{-2}^6 f(x) \, dx \)

d. \( \int_{-2}^6 f(x) \, dx \)

39. \( \int x^2 e^{3x} \, dx = \)

* a. \( e^{3x} \left( \frac{1}{3} x^2 - \frac{2}{9} x + \frac{2}{27} \right) + C \)

b. \( x e^{3x} (3x + 2) + C \)

c. \( \frac{x^2 e^{3x}}{3} - \frac{2}{3} x e^{3x} + C \)

d. \( \frac{x^2 e^{3x}}{3} - \frac{2}{9} x e^{3x} + \frac{2}{27} + C \)

40. In the evaluation of the integral \( \int x e^{2x} \, dx \), the best choice of \( u \) and \( dv \) respectively are

a. \( e^{2x} \) and \( x \, dx \)

* b. \( x \) and \( e^{2x} \, dx \)

c. \( x e^x \) and \( e^x \, dx \)

d. either a or b
41. \( \int \sin 2u \, du = \)

a. \( \cos 2u + C \)
b. \( 2 \cos 2u + C \)
*c. \( -(\cos 2u)/2 + C \)
d. \( (\cos 2u)/2 + C \)

42. \( \int \cos 2u \, du = \)

a. \( \sin 2u + C \)
b. \( 2 \sin 2u + C \)
c. \( -(\sin 2u)/2 + C \)
*d. \( (\sin 2u)/2 + C \)

43. \( \int_1^5 f(x) \, dx + \int_{-3}^5 f(x) \, dx = \)

*a. \( -\int_{-3}^1 f(x) \, dx \)

b. \( \int_5^5 f(x) \, dx \)
c. \( \int_{-3}^1 f(x) \, dx \)
d. \( \int_2^6 f(x) \, dx \)

44. The curves \( y = f(x) \) and \( g(x) \) shown in the figure below intersect at the point \((a, b)\). The area of the shaded region enclosed by these curves and the line \( x = -1 \) is given by

\[
\int_{-1}^a (f(x) - g(x)) \, dx + \int_0^a (f(x) + g(x)) \, dx
\]

a. \( \int_{-1}^a (f(x) - g(x)) \, dx + \int_0^a (f(x) + g(x)) \, dx \)
b. \( \int_{-1}^b g(x) \, dx + \int_c^b f(x) \, dx \)
c. \( \int_{-1}^c (f(x) - g(x)) \, dx \)
*d. \( \int_{-1}^a (f(x) - g(x)) \, dx \)
45. \[ \int_2^1 \int_1^{1-x} x^2y \, dy \, dx = \]
   a. 165/120
   b. 164/120
   *c. 163/120
   d. non-integrable

46. \[ \int \frac{1}{t} + e^{zt} + t^2 \, dt = \]
   a. \(-1/t^2 + e^{zt}/2 + t^3/3 + C\)
   b. \(-1/t^2 + e^{zt}/2 + t^3 + C\)
   c. \(\ln t + e^{zt}/2 + 2t + C\)
   d. \(\ln t + e^{zt}/2 + t^3/3 + C\)

47. \[ \int x (2x + 3)^{10} \, dx = \]
   a. \(x (2x+3)^{11} - (2x+3)^{11} + C\)
   b. \((2x+3)^{10} x (2x+3) + C\)
   c. \((2x+3)^{11} (22x-3) + C\)
   d. \((2x+3)^{11} (26x+3) + C\)

48. \[ \int_{0}^{\infty} \frac{1}{(x-4)^2} \, dx \]
   a. \(-1/4\)
   *b. does not exist
   c. \(1/4\)
   d. \(-1/192\)

49. \[ \int x^{-\left(\frac{2}{3}\right)} \, dx = \]
   *a. \(3x^{1/3}\)
   b. \(-\left(\frac{2}{3}\right)x^{-\frac{5}{3}}\)
   c. \(-\left(\frac{2}{3}\right)x^{1/3}\)
   d. \(\left(\frac{1}{3}\right)x^{1/3}\)
50. \( \int (\ln x + e^{2x}) [(1/x) + 2e^{2x}] \, dx \) is of the form

* a. \( \int (\ln u + e^u) \, du \)

b. \( \int u^2 \, du \)

c. \( \int (1/u) \, du \)

d. \( \int (uv) \, du \)

51. In the evaluation of the integral \( \int x e^{2x} \, dx \), the best choice of \( u \) and \( dv \) respectively are

a. \( e^{2x} \) and \( x \, dx \)

*b. \( x \) and \( e^{2x} \)

c. \( x e^{2x} \) and \( e^{2x} \, dx \)

d. either a or b

52. Given \( \int_0^1 \int_0^{2x} \sin y \, dy \, dx \), on reversing the order of integration, we get which one of the following?

a. \( \int_0^2 \int_0^{\sin y} \, dx \, dy \)

*b. \( \int_0^2 \int_0^{\sin y/2} \, dx \, dy \)

c. \( \int_0^2 \int_0^{\sin y/2} \, dy \, dx \)

d. \( \int_0^1 \int_0^{2x} \sin y \, dx \, dy \)

53. The integral \( \int_0^\infty \frac{1}{(a-x)^2} \, dx \), which can be written as

\[ \int_a^\infty \frac{1}{(a-x)^2} \, dx + \int_0^\infty \frac{1}{(a-x)^2} \, dx \]

exists if

* a. both the right-hand side integrals converge

b. both the right-hand side integrals diverge

c. one of the two right-hand side integrals converge

d. both the right-hand side integrals are equal
54. \( \int_{0}^{4} \frac{1}{(4-x^2)} \, dx \) should be integrated by rewriting it as

a. \( \int_{0}^{4} \frac{1}{(4-x^2)} \, dx \)

*b. \( \int_{0}^{2} \frac{1}{(4-x^2)} \, dx + \int_{2}^{4} \frac{1}{(4-x^2)} \, dx \)

c. \( \int_{0}^{1} \frac{1}{(4-x^2)} \, dx + \int_{1}^{4} \frac{1}{(4-x^2)} \, dx \)

d. \( \int_{0}^{3} \frac{1}{(4-x^2)} \, dx + \int_{3}^{4} \frac{1}{(4-x^2)} \, dx \)