The Hybrid Method
of Network Analysis and
Topological Degree of Freedom

A Thesis Presented to
The Faculty of the College of Engineering and Technology
Ohio University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science

by
Shunquan Gao
March, 1981
October, 1981
ACKNOWLEDGMENTS

I wish to express my sincere gratitude to my advisor, Dr. Wai-Kai Chen, for his advice, guidance, invaluable comments and suggestions in the preparation of this thesis.
# Table of Contents

## Chapter I
### Introduction

## Chapter II
### Preliminaries

2.1 Notation

2.2 Preliminary definitions

## Chapter III
### The Complete Set of Variables and the Basic Set

3.1 The complete set of variables

3.2 The basic set

3.3 The hybrid rank

## Chapter IV
### Hybrid Analysis

4.1 The fundamental cutset and circuit matrices of a decomposition

4.2 The formulation of network equations

4.3 Examples

## Chapter V
### Decomposition by Inspection

5.1 The topological degree of freedom

5.2 Classification of graphs

5.3 Theorems about difs of subgraphs

5.4 Example by inspection

## Chapter VI
### Decomposition by Systematic Procedure

6.1 Absolutely non-sparse graphs and absolutely non-dense graphs

6.2 Criteria for testing absolute non-denseness or absolute non-sparseness

6.3 Harmonious decomposition
6.4 Systematic procedure for harmoniously decomposing a graph
6.5 Harmonious decomposition and complementary tree-pair
6.6 Harmonious decomposition and maximally distant tree-pair
6.7 Dyad D and minimum basic set
6.8 Dyad D and extremal tree
CHAPTER VII CONCLUSIONS
REFERENCES
CHAPTER 1

INTRODUCTION

The classical well-known methods of network analysis are the loop current method, the node potential method and its generalization—the cutset voltage method. Since 1957, when Bashkow introduced the A matrix [1], a new method of network analysis has been developed.

The proposed hybrid or mixed method is a natural extension. In fact the first step in network analysis is to select a set of independent variables of the network. Essentially the loop method selects the maximum number of independent currents as the independent variables, and the node and cutset methods select the maximum number of independent voltages as the independent variables. There are no reasons to select only one kind of network variables. If we select some voltages and some currents as a set of independent variables, it should lead to the hybrid method. Obviously the hybrid method of analysis includes the loop method, the node method, and the cutset method as its special cases.

The advantages of the hybrid method are numerous. First of all, because the hybrid method is more general, the minimum number of a complete set of independent variables
for the hybrid method cannot exceed the number of independent loop variables, the nodal variables, or the cutset variables. This means that the hybrid method will reduce the number of independent variables. Thus it will reduce the computational time in network analysis. Let us look at a very simple example of Fig. 1. The network can easily be simplified by using the equivalent

![Network Diagram](image)

**Fig. 1 A simple example**

transformations. If we impose a restriction on ourselves such that we cannot use any equivalent transformation, then we may choose, say, the voltage, \( V_1 \), of edge 1 and the current, \( I_5 \), of edge 5 as the set of independent variables. We need to formulate only two equilibrium equations (more complicated examples will be given later on). If we solve
the network by either the nodal method or the loop method, we need four equations.

Secondly, in computer-aided analysis the classical methods may cause a lack of generality and roundoff errors in computation. Since, in these methods, Ohm's law is expressed in a form involving either admittances or impedances for all the edges. The hybrid method considers as admittances certain edges (y edges) of the network and as impedances the other edges (z edges). Hence the hybrid method can satisfy these needs: each edge is describable in its most appropriate form. Because the difference in magnitude of the elements in the coefficient matrix of the hybrid method can be limited, the roundoff errors introduced in the solution process are reduced [2].

Finally, when state equations of a linear network or the equilibrium equations of a nonlinear network are formulated [3-5], the hybrid method is applicable.

It should be properly recognized that the hybrid method is fundamental in network analysis. However, at present it is still less well-known. One of reasons is that it has not yet received a systematic exposition. The existing literature is diverse and scattered [2,7-13]. Some of them are not even readable.
In applying the hybrid method, the main difficulty is how to choose the set of independent variables. To overcome this we introduce a graph classification in this thesis. We separate graphs into three classes: sparse, medium, and dense. This classification, comparatively speaking, is fairly intuitive. Then, based on this classification, we propose two methods to choose a set of independent variables which can easily and quickly be done by inspection or by computer. For this we introduce the concepts of absolute non-sparse subgraphs and absolute non-dense contracted graphs and advance new proofs of existing results. In the final part of this thesis, we expound the internal relations among the various results and provide a better insight into hybrid analysis.
CHAPTER II

PRELIMINARIES

2.1 NOTATION

The notation and terminology of this thesis will follow [6] (exceptions will be noted). By a graph \( G \) we mean an undirected graph. The rank and nullity of \( G \) will be denoted by \( r(G) \) (or \( r \)) and \( m(G) \) (or \( m \)). The rank and nullity of subgraph \( G_s \) will be denoted by \( r(G_s) \) and \( m(G_s) \). The numbers of nodes and edges of graph \( G \) will be denoted by \( n \) and \( b \). The numbers of nodes, edges, and components of subgraph \( G_s \) will be denoted by \( n(G_s) \), \( b(G_s) \), and \( p(G_s) \), respectively.

For a precise description of a graph \( G \{V,E\} \), both the node set \( V \) and the edge set \( E \), together with the incidence relationships, must be specified. For convenience, however, we shall describe \( G \) or any subgraph \( G_s \) of \( G \) by only its constituent edge set \( E \) or \( E_s \), whereas the nodes being implied.

Set-theoretic operations, union, intersection, difference, and ring sum are denoted by \( \cup \), \( \cap \), \( - \), and \( \oplus \), respectively. The complement of a subgraph \( G_s \) is denoted by \( G_s' \)

\[
G_s' = G - G_s
\]
The null set is denoted by \( \emptyset \). If \( S \) is a set of certain elements then \( |S| \) denotes the number of elements in \( S \). In addition, if \( G_s \) is a subgraph, then \( |G_s| \) denotes the number of edges of \( G_s \).

Let \( T, T', F, F', BS, \) and \( D \) denote tree, cotree, forest, coforest, basic set\(^{(1)}\), and dyad\(^{(1)}\), respectively. Let \( \{T\}, \{T'\}, \{F\}, \{F'\}, \{BS\}, \) and \( \{D\} \) denote the sets of trees, cotrees, forests, coforests, basic sets, and dyads of \( G \), respectively.

We denote the vector by letter overlaid with - and denote the matrix by a capital letter. For instance, a current vector is denoted by \( \bar{I} \), an impedance matrix is denoted by \( \bar{Z} \).

Notation not included here will be introduced later on as they are needed.

2.2 PRELIMINARY DEFINITIONS

For simplicity and without loss of generality, we assume the graph \( G \) to be connected\(^{(2)}\).

\(^{(1)}\)The definitions of basic set and dyad will be given later on.
\(^{(2)}\)The modifications to include disconnected graph are
DEFINITION 1: Removal of a set of edges $E_1 \subseteq E$ from a graph $G(V,E)$. By this we mean the operation that results in the subgraph $G_s(V_s, E - E_1)$, where $V_s$ denotes the set of all the nodes that are incident to the edges of the set $(E - E_1)$. For convenience, we shall denote $G(V_s, E - E_1)$ by $G - E_1$.

Note that even if $G$ is connected, $G - E_1$ may be unconnected.

DEFINITION 2: Contraction of a set of edges $E_1 \subseteq E$ from a graph $G(V,E)$. By this we mean the operation that first removes all the edges of $E_1$ from $G$, and then identifies all the pairs of end nodes of edges of $E_1$. For convenience we shall denote the derived graph by $G \odot E_1$.

Note the new symbol introduced here. While $G - E_1$ is a subgraph of $G$, $G \odot E_1$ is not. As an example of the two definitions, let $G$ be the graph given in Fig. 2(a), then $G - E_1$ and $G \odot E_2$ are shown in Fig. 2(b) and (c), respectively.

For convenience, the derived graph, $G \odot E_1$, is called a contracted graph of $G$, and the graph $G$ is called an expansion of the contracted graph $G \odot E_1$.

DEFINITION 3: Decomposition of $G$. Let $E_r$ be a subset of the straightforward.
Fig. 2 An example about removal and contraction.

$E_1 = \{e_6, e_7, e_8\}$, $E_2 = \{e_1, e_2, e_3, e_4, e_5, e_9, e_{10}, e_{11}\}$

edge set $E$ of a graph $G$. By decomposition we mean that for a chosen subset $E_r$, we derive two graphs such that

\[ G_c = G - E_r \]  
\[ G_r = G \ominus E_c \]  

where $E_c$ is the set of all the edges of $G_c$, i.e.

\[ E_c = E - E_r \]  

Then the ordered pair of $G_r$ and $G_c$ is denoted by $\{G_r; G_c\}$ and
is called a decomposition of \( G \). In addition, for convenience the \( G_r \) and \( G_c \) are called the first part and the second part of the decomposition.

**Note 1:** \( E_r \) in (1a) is just the set of all the edges of the contracted graph \( G_r \).

**Note 2:** Also we can derive \( G_r \) and \( G_c \) from the subset \( E_c \) of the edge set \( E \) (not from \( E_r \)).

**Note 3:** Obviously
\[
E_r \cup E_c = E \\
E_r \cap E_c = \emptyset
\]
i.e. the edge set \( E \) is partitioned into \( E_r \) and \( E_c \). However
\[
G_r \cup G_c \neq G \quad \text{(refer to Fig.2)}
\]

**Note 4:** If either \( G_r \) or \( G_c \) is given, the other is uniquely determined. Thus the notation \( \{ G_r ; \} \) or \( \{ ; G_c \} \) is sometimes used for the abbreviation.

**Note 5:** The subscripts \( r \) and \( c \) are mnemonics for removal and contraction. From \( G \), if \( G_r \) is removed we get \( G_c \), if \( G_c \) contracted we get \( G_r \).

As an example of the definition 3, consider the graph \( G \) given in Fig.2(a). Let
\[
E_r = \{ e_6, e_7, e_8 \}
or
\[ E_c = \{e_1, e_2, e_3, e_4, e_5, e_9, e_{10}, e_{11}\} \]
then the Gr and Gc are the graphs shown in Figs. 2(b) and 2(c).

From definition 3 we have two simple lemmas [9].

**Lemma 1:** The rank of a graph \( G \), \( r(G) \), equals the sum of ranks of its two decomposed parts, i.e.
\[ r(G) = r(Gr) + r(Gc) \] \( (2) \)

**Proof:** Let \( n_1 \) and \( n_2 \) be the numbers of nodes of Gr and Gc, respectively. Let \( p_2 \) be the component number of Gc. Gr can be derived from G by contracting Gc. Contracting Gc causes the number of nodes to be reduced by \( (n_2 - p_2) \), i.e.
\[ n_1 = n - (n_2 - p_2) \]
Thus
\[ n-1 = (n_1 - 1) + (n_2 - p_2) \]
Considering that G and Gr are connected, we have the expression (2) immediately.
Q.E.D.

**Lemma 2:** The nullity of a graph \( G \), \( m(G) \), equals the sum of nullities of its two decomposed parts, i.e.
\[ m(G) = m(Gr) + m(Gc) \] \( (3) \)

**Proof:** Let \( b_1 \) and \( b_2 \) be the numbers of edges of Gr and
\[ m(G) = b - r(G) \]
\[ = [b_1 + b_2] - [r(Gr) + r(Gc)] \]
\[ = [b_1 - r(Gr)] + [b_2 - r(Gc)] \]
\[ = m(Gr) + m(Gc) \]

Q.E.D.

For later use we introduce another definition.

**DEFINITION 4:** **Sectional subgraph** [6]. Let \( V_s \) be a subset of the node set of a graph \( G \). The **sectional subgraph**, defined by \( V_s \) and denoted by \( G[V_s] \), of \( G \) is the subgraph whose node set is \( V_s \), and whose edge set consists of all those edges in \( G \) connecting two nodes of \( V_s \).

Other definitions not included in this chapter will be introduced later on as they are needed.
CHAPTER III

THE COMPLETE SET OF VARIABLES AND THE BASIC SET

3.1 THE COMPLETE SET OF VARIABLES

Trees and cotrees in a graph are fundamental and important concepts. They play an important role in electrical network theory. There are many ways of defining trees and cotrees [14]. One of the definitions is a maximal circuitless subgraph of G for a tree and a maximal cutsetless subgraph of G for a cotree [15,16]. In these definitions, tree is defined by using the concept of circuit, whereas cotree is defined by using the concept of cutset.

Since a tree is circuitless, its branch voltage values can be assigned arbitrarily without violating the Kirchhoff voltage law. Since each link forms a fundamental circuit exclusively with some branches, from the Kirchhoff voltage law every link voltage can uniquely be determined by branch voltages. Therefore the branch voltages can determine all the edge voltages without invoking the element constitutive equations. This is essential in the node (cut set) analysis. Dually, the link currents of a cotree can uniquely determine all the edge currents without invoking the element constitutive equations, and this is essential in the loop
analysis. Since the branch voltages or the link currents can uniquely determine one of the two variables (voltage or current) in each edge of $G$ by using the Kirchhoff voltage law or the Kirchhoff current law, both are complete sets of independent variables.

Generally, we may define a complete set of variables for a graph in the following sense [3].

**DEFINITION 5:** Complete set of variables for a graph $G$. A set of edge voltages and currents is called a complete set of variables for a graph $G$ if they can be assigned arbitrarily without violating the Kirchhoff voltage law and the Kirchhoff current law and if they determine one of the two variables (voltage or current) in each edge of $G$ by using the Kirchhoff voltage law or the Kirchhoff current law and without invoking the element constitutive equations.

Such a complete set of variables is essential in the hybrid analysis.

### 3.2 THE BASIC SET

Like the tree in node (cutset) analysis or the cotree in loop analysis, we can introduce a definition of the basic set.
DEFINITION 6: Basic set. An ordered pair of edge sets, 
\((D_1, D_2)\), is a basic set, denoted by BS, of G, if all the 
currents in edges of \(D_1\) and all the voltages of edges of \(D_2\) 
form a complete set of variables for G.

Note 1: Definition 6 given above in terms of complete 
set of variables has a clear link to the Kirchhoff voltage 
law and the Kirchhoff current law in network analysis.

Note 2: \(D_2\) is a tree of G if and only if \(D_1\) is empty. 
\(D_1\) is a cotree of G if and only if \(D_2\) is empty.

Considering that \(G_c\) may be disconnected, we define the 
spanning forest for a disconnected graph.

DEFINITION 7: Spanning forest. A spanning subgraph of a 
graph G is said to be a spanning forest, if it has the same 
number of components as the graph G and contains no 
circuits, i.e. a maximal circuitless subgraph.

THEOREM 1: Let \((G_r,G_c)\) be a decomposition of G, \(T_{r'}\) be a 
cotree of \(G_r\), \(F_c\) be a spanning forest of \(G_c\). The ordered 
pair \((T_{r'},F_c)\) is a basic set of G. \((T_{r'},F_c)\) is called a 
basic set with respect to the decomposition \((G_r;G_c)\) of G.

Proof: To prove this we show that the edge currents of 
\(T_{r'}\) and the edge voltages of \(F_c\) form a complete set of
variables.

We first show that they can be assigned arbitrarily without violating the Kirchhoff current law and the Kirchhoff voltage law. Look at the edge currents of $Tr'$. If $Tr'$ contains no cutsets of $G$, then the edge currents can be assigned arbitrarily. One simple way to check whether an edge set $Es$ contains any cutset of $G$ is the following. Contract all the edges of $G$ except the edges in $Es$. Then the rank of the contracted graph is the number of independent cutsets contained in $Es$. Doing this for $Tr'$, we should contract the edges of $Tr$ and $Gc$. Consider the contraction of the edges in two steps: contract $Gc$ and then $Tr$. After contracting $Gc$ we get $Gr$. After contracting $Tr$, the tree of $Gr$, we get a contracted graph with zero rank. Thus, $Tr'$ contains no cutsets. Similarly $Pr$ contains no circuits, so its edge voltages can be assigned arbitrarily without violating the Kirchhoff voltage law.

On the other hand, edge currents of $Tr$ and edge voltages of $Fc'$ can be determined from the edge currents of $Tr'$ and the edge voltages of $Fc$, respectively, without invoking the element constitutive equations. Therefore one of the two variables in each edge of $G$ is determined. According to definition 6 they form a complete set of variables.

Q.E.D.
Note 1: Even with respect to a specific decomposition, the basic set, generally speaking, is not unique. Usually the number, |{BS}|, of basic sets for a graph far exceeds the number, |{T}|, of trees of the graph.

Note 2: It is easy to see that Tr U Pc is a tree of G, since each component of the forest Pc is jointed with tree Tr at only one different node and the resulting subgraph contains all the nodes of G.

Note 3: From note 2 it is easy to see that Tr' U Pc' is a cotree of G.

Note 4: The number, |BS|, of edges in a basic set is equal to the sum of m(Gr) and r(Gc), i.e.

\[ |BS| = m(Gr) + r(Gc) \]  

(4)

i.e. the |BS| depends upon the decomposition and does not depend upon the choice of cotrees of Gr and forests of Gc.

3.3 THE HYBRID RANK

Like the rank for a tree T of G, r(G)=|T|, we define a hybrid rank for a basic set.

DEFINITION 8: The number of edges of a basic set is called the hybrid rank of the basic set, and is denoted by h.
From definition 8 we have the following notes.

Note 1: The number of independent variables in hybrid analysis equals the hybrid rank of the corresponding basic set.

Note 2: Unlike the rank of a graph, the hybrid rank is not unique for a given graph. It depends upon the choice of the basic sets, or more precisely, it depends upon the choice of decompositions.

THEOREM 2 [9]: The hybrid rank of a basic set with respect to a decomposition \((Gr, Gc)\) is

\[
h = r(G) - [r(Gr) - m(Gr)]
\]

Similarl, from (3) we have

\[
m(Gr) = m(G) - m(Gc)
\]

and then substituting it into (4), we get (5b).

Q.E.D.

Theorem 2 implies that if we want to reduce the number of edges of a basic set, we should decompose \(G\) into \((Gr; Gc)\) such that the difference between the nullity and the rank of
the subgraph $G_c$ is increased or the difference between the rank and the nullity of the contracted graph $G_r$ is increased. Thus, we have

**COROLLARY 1:** The hybrid rank of a basic set with respect to a decomposition is less than the rank of the $G$ if and only if the rank of the first part $G_r$ of the decomposition is greater than the nullity of the same part, i.e.

$$ h < r(G) \quad \text{when} \quad r(G_r) > m(G_r) $$

**COROLLARY 2:** The hybrid rank of a basic set with respect to a decomposition is less than the nullity of the $G$ if and only if the nullity of the second part $G_c$ of the decomposition is greater than the rank of the same part, i.e.

$$ h < m(G) \quad \text{when} \quad m(G_c) > r(G_c) $$

Let $(G_r; G_c)$ be a decomposition. If a component of the $G_c$ is not a sectional subgraph of $G$, i.e. if there exists at least one edge in $G_r$, whose two end nodes are contained in a component of $G_c$, then we can move the edge from $G_r$ into $G_c$ to increase $m(G_c)$ and keep $r(G_c)$ unchanged. From (5b), we know that the above operation causes the hybrid rank $h$ to be reduced. Thus we have the following corollary.

**COROLLARY 3:** To attain the minimum hybrid rank, it is necessary that the second part of a decomposition be a
sectional subgraph or a union of sectional subgraphs of $G$. 
4.1 THE FUNDAMENTAL CUTSET AND CIRCUIT MATRICES OF A DECOMPOSITION

If it is suitable to express certain edges in admittance form, then these edges should belong to $G_c$; dually, if certain edges are suitable for impedance form, they belong to $G_r$.

If a decomposition of $G$ and its basic set have been chosen, then for the corresponding tree $T=T_r \cup T_c$, the fundamental cutset matrix can be written in the following partitioned form:

\[
Q_f = \begin{bmatrix}
Q_{f1} & U_r
\end{bmatrix}
\]

\[
T_r' \quad F_c' \quad T_r \quad F_c
\]

\[
= \begin{bmatrix}
Q_{rr} & Q_{rc} & U_{r1} & 0 \\
Q_{cr} & Q_{cc} & 0 & U_{r2}
\end{bmatrix}
\]  \hspace{1cm} (8)

and the fundamental circuit matrix:

\[
B_f = \begin{bmatrix}
U_m & B_{f12}
\end{bmatrix}
\]
where the columns have been arranged in the edge order as shown, and \( U_i \) is the identity matrix of order \( i \), and

\[
\begin{align*}
\Pi_1 &= \pi(G_r) \\
\Pi_2 &= \pi(G_c) \\
\mu_1 &= \mu(G_r) \\
\mu_2 &= \mu(G_c)
\end{align*}
\]

Obviously, every link of \( G_c \) forms a fundamental circuit exclusively with the branch edges of \( F_c \). Hence

\[
B_{cr} = [0]
\]

(10)

Because [6, p.53]

\[
Q_f = [-B_{f_{12}}, U r]
\]

\[
= \begin{bmatrix}
-B_{rr}, 0 & U r_1 & 0 \\
-B_{rc}, -B_{cc}, 0 & U r_2
\end{bmatrix}
\]

where the prime ' denotes the transposition operation on a matrix. Comparing the above expression with (8), we have

\[
\begin{align*}
B_{rr} &= -Q_{rr} \\
B_{rc} &= -Q_{cr} \\
B_{cc} &= -Q_{cc}
\end{align*}
\]

and

\[
Q_{cr} = [0]
\]

i.e.
\[
Q_f = \begin{bmatrix}
Q_{rr} & 0 & U_{r_1} & 0 \\
Q_{cr} & Q_{cc} & 0 & U_{r_2}
\end{bmatrix}
\]  \hspace{1cm} (11)

\[
B_f = \begin{bmatrix}
U_{m_1} & 0 & -Q_{rr} & -Q_{cr} \\
0 & U_{m_2} & 0 & -Q_{cc}
\end{bmatrix}
\]  \hspace{1cm} (12)

4.2 THE FORMULATION OF NETWORK EQUATIONS

We assume, without loss of generality, that if there exist sources in part \( G_r \), they are current sources in parallel with edges of \( G_r \), and if there exist sources in part \( G_c \), they are voltage sources in series with edges of \( G_c \). By using the Kirchhoff current law with respect to the first row of (11) and by using the Kirchhoff voltage law with respect to the second row of (12), we have

\[
Q_{rr}\bar{l}_r + \bar{\it r} = \bar{J}_r
\]

\[
\bar{V}_{lc} - Q_{cc}\bar{V}_{tc} = \bar{E}_c
\]

where \( \bar{l}_r \) is an \( m_1 \)-vector denoting the link currents of \( G_r \), \( \bar{\it r} \) is a \( r_1 \)-vector denoting the branch currents of \( G_r \), \( \bar{V}_{lc} \) is an \( m_2 \)-vector denoting the link voltages of \( G_c \), \( \bar{V}_{tc} \) is a \( r_2 \)-vector denoting the branch voltages of \( G_c \), \( \bar{J}_r \) is a \( r_1 \)-vector denoting the cut current-source of \( G_r \), \( \bar{E}_c^{(3)} \) is an \( m_2 \)-vector denoting the circuit voltage-source of \( G_c \). \( \bar{J}_r \) and \( \bar{E}_c \) are computed as follows:

\[\]  \[\]

\[^{(3)}\] Here \( \bar{E}_c \) is a source vector, not the set of edges of \( G_c \).
\[
\begin{align*}
\bar{J}_r &= \bar{Q}rr \bar{J}_l + \bar{J}_t \\
\bar{E}_c &= \bar{E}_l c - Qcc'\bar{E}_t \\
\end{align*}
\]

Obviously, we have

\[
\begin{align*}
\bar{V}_{tc} &= \bar{V}_{tc} \\
\bar{I}_{lr} &= \bar{I}_{lr} \\
\bar{V}_{lc} &= Qcc'\bar{V}_{tc} + \bar{E}_c \\
\bar{I}_{tr} &= -Qrr\bar{I}_{lr} + \bar{J}_r \\
\end{align*}
\]

Writing them in matrix form, we have

\[
\begin{bmatrix}
\bar{V}_{tc} \\
\bar{I}_{lr} \\
\bar{V}_{lc} \\
\bar{I}_{tr}
\end{bmatrix} =
\begin{bmatrix}
\bar{U}_{r_2} & 0 \\
0 & \bar{U}_{m_1} \\
Qcc' & 0 \\
0 & -Qrr
\end{bmatrix}
\begin{bmatrix}
\bar{V}_{tc} \\
\bar{I}_{lr} \\
\bar{V}_{lc} \\
\bar{I}_{tr}
\end{bmatrix} +
\begin{bmatrix}
\bar{O}_{r_2} \\
0 \\
\bar{E}_c \\
\bar{J}_r
\end{bmatrix}
\]

or

\[
\bar{X}_1 = \Lambda' \bar{X} + \bar{S}_1
\]

where

\[
\bar{X}_1 =
\begin{bmatrix}
\bar{V}_{tc} \\
\bar{I}_{lr} \\
\bar{V}_{lc} \\
\bar{I}_{tr}
\end{bmatrix}
\]

\[
\bar{X} =
\begin{bmatrix}
\bar{V}_{tc} \\
\bar{I}_{lr}
\end{bmatrix}
\]

\[
\Lambda =
\begin{bmatrix}
\bar{U}_{r_2} & 0 & Qcc & 0 \\
0 & \bar{U}_{m_1} & 0 & -Qrr
\end{bmatrix}
\]
\[ S_1 = \begin{bmatrix} \tilde{O}_{r_2} \\ \tilde{O}_{m_1} \\ \tilde{E}_c \\ \tilde{J}_r \end{bmatrix} \]  

where \( \tilde{O}_{r_2} \) and \( \tilde{O}_{m_1} \) are zero \( r_2 \)-vector and zero \( m_1 \)-vector, respectively.

Considering the Ohm's law in hybrid form, we have

\[
\begin{bmatrix} \bar{I}_{tc} \\ \bar{V}_{lr} \\ \bar{I}_{lc} \\ \bar{V}_{tr} \end{bmatrix} = \begin{bmatrix} \bar{V}_{tc} \\ \bar{I}_{lr} \\ \bar{V}_{lc} \\ \bar{I}_{tr} \end{bmatrix} = H\begin{bmatrix} \bar{I}_{tc} \\ \bar{V}_{lr} \\ \bar{I}_{lc} \\ \bar{V}_{tr} \end{bmatrix}
\]

or

\[ \bar{y} = \bar{H}\bar{x}, \]

where \( H \) is a hybrid immittance matrix expressing the constitutive relations of all the edges of \( G \), and

\[
\bar{y} = \begin{bmatrix} \bar{I}_{tc} \\ \bar{V}_{lr} \\ \bar{I}_{lc} \\ \bar{V}_{tr} \end{bmatrix}
\]

On the other hand, by using the Kirchhoff current law with respect to the second row of (11) and by using the Kirchhoff voltage law with respect to the first row of (12), we have

\[
\bar{I}_{tc} + Q_{cc}\bar{I}_{lc} + Q_{cr}\bar{I}_{lr} = \bar{J}_c
\]

\[
\bar{V}_{lr} - Q_{rr}\bar{V}_{tr} - Q_{cr}\bar{V}_{tc} = \bar{E}_r
\]
where

\[ \bar{J}_c = Q\bar{c}R\bar{r} \]  
\[ \bar{E}_r = -Q\bar{c}R\bar{r} \]  

Expression (22) can be rewritten as follows:

\[
\begin{bmatrix}
  \bar{U}_{r2} & 0 & Q\bar{c}c & 0 \\
  0 & \bar{U}_{m1} & 0 & -Qrr`
\end{bmatrix}
\begin{bmatrix}
  \bar{v}r \\
  \bar{v}l \\
  \bar{v}t
\end{bmatrix}
\begin{bmatrix}
  \bar{t}c \\
  \bar{v}c
\end{bmatrix}
= \begin{bmatrix}
  \bar{J}_c \\
  \bar{E}_r
\end{bmatrix}
\]  

or

\[ \Lambda \bar{y} + \Gamma \bar{x} = \bar{S}_2 \]  

where

\[ \Gamma = \begin{bmatrix}
  0 & Q\bar{c}r \\
  -Q\bar{c}r` & 0
\end{bmatrix} \]  

\[ \bar{S}_2 = \begin{bmatrix}
  \bar{J}_c \\
  \bar{E}_r
\end{bmatrix} \]  

Substituting (16) into (21), we have

\[ \bar{y} = \Lambda\Lambda' \bar{y} + \Lambda\bar{S}_1 \]  

Combining (28) and (25), we have the final expression

\[ (\Lambda\Lambda' + \Gamma) \bar{x} = \bar{S}_2 \]  

For convenience we rewrite the expression as follows:

\[ \Lambda = \begin{bmatrix}
  \bar{U}_{r2} & 0 & Q\bar{c}c & 0 \\
  0 & \bar{U}_{m1} & 0 & -Qrr`
\end{bmatrix} \]  

\[ \Gamma = \begin{bmatrix}
  0 & Q\bar{c}r \\
  -Q\bar{c}r` & 0
\end{bmatrix} \]
They can also be expressed in terms of submatrices of the fundamental circuit matrix as follows:

\[ \Lambda = \begin{bmatrix} U_{r_2} & 0 & -B_{cc'} & 0 \\ 0 & U_{m_1} & 0 & B_{rr} \end{bmatrix} \]

\[ \Gamma = \begin{bmatrix} 0 & -B_{rc'} \\ B_{rc'} & 0 \end{bmatrix} \]

\[ \bar{S}_1 = \begin{bmatrix} \bar{O}_{r_2} \\ \bar{O}_{m_1} \\ \bar{E}_{c} \\ \bar{J}_{r} \end{bmatrix} \]

\[ \bar{S}_2 = \begin{bmatrix} \bar{J}_{c} \\ \bar{E}_{r} \end{bmatrix} \]

\[ \bar{E}_{c} = \bar{E}_{lc} + B_{cc'\bar{E}_{tc}} \]

\[ \bar{J}_{r} = -B_{rr'\bar{J}_{lr}} + \bar{J}_{tr} \]

\[ \bar{J}_{c} = -B_{rc'\bar{J}_{lr}} \]

\[ \bar{E}_{r} = B_{rc'\bar{E}_{tc}} \]
By using (29), (30) or (31), we can systematically formulate the set of equations in hybrid analysis for any given linear network.

4.3 EXAMPLES

In order to illustrate the procedure clearly, we first consider a simple network given in Fig.3.

\[ E = \{ e_1, e_2 \} \]

\[ E_r = \{ e_3, e_4, e_5, e_6 \} \]

Fig. 3 An illustration of hybrid method

Suppose, owing to certain reasons, edge $e_3$ is expressed in impedance. We partition the edges as follows:

\[ E_c = \{ e_1, e_2 \} \]

\[ E_r = \{ e_3, e_4, e_5, e_6 \} \]
Choose

\[ F_c = e_1 \]
\[ T_r' = e_3 e_4 \]
\[ T = F_c \cup T_r = e_1 e_3 e_4 \]

Then

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 1 & 1 & -1 \\
0 & 0 & 1 & 0 & 0 & -1
\end{bmatrix}
\]

with

\[
B_{rr} = 
\begin{bmatrix}
0 & 0 \\
1 & 1
\end{bmatrix}
\]

\[
B_{rc} = 
\begin{bmatrix}
-1 \\
-1
\end{bmatrix}
\]

\[
B_{cr} = 
\begin{bmatrix}
0 & 0 \\
-1
\end{bmatrix}
\]

\[
B_{cc} = 
\begin{bmatrix}
-1
\end{bmatrix}
\]

From (31) we have

\[
\Lambda = 
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}
\]

\[
\Gamma = 
\begin{bmatrix}
0 & 1 & 1 \\
-1 & 0 & 0 \\
-1 & 0 & 0
\end{bmatrix}
\]
Because

\[
\begin{bmatrix}
  v_1 \\
  I_3 \\
  I_4 \\
  v_2 \\
  I_5 \\
  I_6
\end{bmatrix}
\]

the edge immittance matrix is

\[
H = \begin{bmatrix}
  g_1 & 0 & 0 & 0 & 0 & 0 \\
  0 & r_3 & 0 & 0 & 0 & 0 \\
  0 & 0 & r_4 & 0 & 0 & 0 \\
  0 & 0 & 0 & g_2 & 0 & 0 \\
  0 & 0 & 0 & 0 & r_5 & 0 \\
  0 & 0 & 0 & 0 & 0 & r_6
\end{bmatrix}
\]

The coefficient matrix is

\[
\Lambda H A' + \Gamma = \begin{bmatrix}
  g_1 + g_2 & 1 & 1 \\
  -1 & r_3 & 0 \\
  -1 & 0 & r_4 + r_5 + r_6
\end{bmatrix}
\]

The sources are given by

\[E_{1c} = E_2\]
\[ \text{Etc} = -E_1 \]
\[ \mathbf{j}_{lr} = \begin{bmatrix} -J_3 \\ 0 \end{bmatrix} \]
\[ \mathbf{j}_{tr} = \begin{bmatrix} J_5 \\ 0 \end{bmatrix} \]
\[ \mathbf{E}_c = E_2 + (-1)(-E_1) \]
\[ = E_1 + E_2 \]
\[ \mathbf{j}_r = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -J_3 \\ J_5 \end{bmatrix} = \begin{bmatrix} J_5 \\ 0 \end{bmatrix} \]
\[ \mathbf{S}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & E_1 + E_2 \\ 0 & 0 & 0 & J_5 & 0 \end{bmatrix} \]
\[ \mathbf{J}_c = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -J_3 \\ 0 \end{bmatrix} \]
\[ = -J_3 \]
\[ \mathbf{E}_r = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \begin{bmatrix} -E_1 \\ E_1 \end{bmatrix} = \begin{bmatrix} E_1 \\ E_1 \end{bmatrix} \]
Finally, we get the desired hybrid equation

\[
\begin{bmatrix}
(g_1 + g_2) & 1 & 1 \\
-1 & \gamma_3 & 0 \\
-1 & 0 & (\gamma_4 + \gamma_5 + \gamma_6)
\end{bmatrix}
\begin{bmatrix}
V_1 \\
I_3 \\
I_4
\end{bmatrix}
= \begin{bmatrix}
-J_3 - g_2 (E_1 + E_2) \\
E_1 \\
E_1 - \gamma_5 J_5
\end{bmatrix}
\]

Solving the equation we have

\[
\bar{\mathbf{x}} = \begin{bmatrix}
V_1 \\
I_3 \\
I_4
\end{bmatrix}
\]

\[
= \frac{1}{\Delta} \begin{bmatrix}
-E_1 (g_2 \gamma_3 \gamma_4 + \gamma_3 \gamma_4 \gamma_6) - E_2 g_2 \gamma_3 \gamma_4 - J_3 \gamma_3 \gamma_4 \gamma_6 + J_3 \gamma_3 \gamma_5 \\
E_1 g_2 \gamma_4 - E_2 g_2 \gamma_4 - J_3 \gamma_4 - J_5 \gamma_5 \\
E_1 g_1 \gamma_3 - E_2 g_2 \gamma_3 - J_3 \gamma_3 - J_5 \gamma_5 [1 + (g_1 + g_2) \gamma_3]
\end{bmatrix}
\]

where

\[
\Delta = (g_1 + g_2) \gamma_3 \gamma_4 + \gamma_3 + \gamma_4 \gamma_6
\]

\[
\gamma_4 \gamma_6 = \gamma_4 + \gamma_5 + \gamma_6
\]
In the above example we briefly illustrated the formulation of the set of equations in hybrid analysis. Now we turn to analyzing a practical example [18].

As the second example we consider the network shown in Fig. 4 consisting of three groups of nodes with a high density of edges in each group, interconnected by two ladder networks with relatively few edges between the nodes. It is abstracted from real power transmission networks, which are sometimes encountered.

Expressing the problem in Fig. 4 in terms of nodal voltage equations only results in forty-four variables, and using loop current equations only results in fifty variables. Partitioning, as shown by the broken lines (Gr) and the solid lines (Gc) in Fig. 4, reduces the number of variables in hybrid analysis to thirty-two consisting of three groups with six voltage variables each and two of seven current variables each.
Fig. 4 A practical example.
Note that the number of operations required to solve the system of linear equations is approximately proportional to the cube of the number of variables. Hence, for the second example the ratio of computation times required by the nodal analysis and by the hybrid analysis is about \((44/32)^3 = 2.6\). Similarly, the ratio of both required by the loop analysis and by the hybrid analysis is about \((50/32)^3 = 3.8\).

For short, we assume that all the branch admittances equal one, and only edges 1 and 94 have unit voltage-sources (upwards), respectively. In this example the edges are denoted only by the corresponding label numbers.

Choose

\[
\mathbf{Pc} = \{1, 2, 3, 4, 5, 6, 39, 40, 41, 42, 43, 44, 77, 78, 79, 80, 81, 82\}
\]

\[
\mathbf{Tr'} = \{19, 22, 25, 28, 31, 34, 37, 57, 60, 63, 66, 69, 72, 75\}
\]

The columns of the fundamental circuit matrix will be written in numerical order. For instance, the columns in the part of \(\mathbf{Pc}'\) are arranged in the order of \((7-18, 45-56, 83-94)\). Then, in (9) we have

\[
\mathbf{Br} = \begin{bmatrix}
\mathbf{Srr} & 0 \\
0 & \mathbf{Srr}
\end{bmatrix}
\]
where

\[
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
\end{bmatrix}
\]

\[
S_{rr} =
\begin{bmatrix}
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
B_{rc} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
B_{cc} =
\begin{bmatrix}
Scc & 0 & 0 \\
0 & Scc & 0 \\
0 & 0 & Scc \\
\end{bmatrix}
\]

where
\[
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
-1 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
-1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 
\end{bmatrix}
\]

From (31) we have

\[
\Lambda = \begin{bmatrix}
U_{1s} & 0 & -Bcc' & 0 \\
0 & U_{1s} & 0 & Brr
\end{bmatrix}
\]

\[
\Gamma = \begin{bmatrix}
0 & -Brc' \\
Brc & 0
\end{bmatrix}
\]

\[
H = U_{9\times9}
\]

\[
\Lambda H A^t = \begin{bmatrix}
U_{1s} & 0 & -Bcc' & 0 \\
0 & U_{1s} & 0 & Brr
\end{bmatrix}
\begin{bmatrix}
U_{1s} & 0 \\
0 & U_{1s}
\end{bmatrix}
\begin{bmatrix}
0 & -Bcc \\
- Bcc & 0 \\
0 & Brr'
\end{bmatrix}
\]

\[
= \begin{bmatrix}
U_{1s} & Bcc' & Bcc & 0 \\
0 & U_{1s} & BrrBrr'
\end{bmatrix}
\]
Finally

\[ A\bar{\mathbf{x}} = \bar{\mathbf{b}} \]

where

\[ \bar{\mathbf{x}} = \begin{bmatrix} \bar{\mathbf{v}}_{tc} \\ \bar{\mathbf{l}}_{lr} \end{bmatrix} \]

\[ \bar{\mathbf{b}}' = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
CHAPTER V

DECOMPOSITION BY INSPECTION

5.1 THE TOPOLOGICAL DEGREE OF FREEDOM

We have known that the number of independent variables of mixed analysis, i.e. the hybrid rank, varies depending on the choice of basic sets. In 1962 Amari [16] pointed out that hybrid rank could be made less than the minimum of rank and nullity for some networks. This minimum number of \( h \) is defined as the topological degree of freedom. It is uniquely determined from the topological structure of a graph. Formally, we have the following definition.

**DEFINITION 9:** The topological degree of freedom of a graph. The number \( d(G) \) given by

\[
d(G) = \min_i \{ h_i \}
\]

is called the topological degree of freedom of \( G \), where \( h_i \) is the hybrid rank of the \( i \)th basic set and the minimum is taken over all basic sets of \( G \). If the graph \( G \) to be considered is evident, we simply use the symbol \( d \).

**DEFINITION 10:** Minimum basic set. A basic set is called a minimum basic set if the number of its edges is minimal, i.e. it equals the topological degree of freedom of \( G \).
Note that the definition 9 is closely linked to the problem of formulating network equations in terms of minimum number of variables, but it does not tell us how to choose the set of variables and what properties accompanied. Hence in the remainder of this thesis, we shall elaborate how to get a minimum basic set.

5.2 CLASSIFICATION OF GRAPHS

For many graphs, the decompositions whose basic sets are minimal can be made by inspection. The inspection method enables us to get a better insight into the others.

Before presenting the method, it is necessary to introduce a classification of graphs, on which we were enlightened by Kajitani [19].

DEFINITION 11: A graph \( G \) or subgraph \( G_c \), is called a dense, medium, or sparse graph if the difference \( (m-r) \) or \( m(G_c)-r(G_c) \) is positive, zero, or negative, respectively.

For short, we denote the difference \( m(G_c)-r(G_c) \) by \( \text{dif}(G_c) \). Then

\[
\text{dif}(G_c) = m(G_c) - r(G_c) = b(G_c) - 2r(G_c) \tag{32}
\]

\[
= b(G_c) - 2n(G_c) + 2p_c \tag{33}
\]
The above difference is also called \( \text{dif} \).

It is rational that we use the above definition as the criterion of denseness of a graph. If two graphs have the same number of nodes, the graph that has more edges is denser. Conversely, if two graphs have the same number of edges, the graph that has more nodes is sparser. Therefore the concept of denseness of graph is fairly intuitive.

Using the notation of \( \text{dif}(\cdot) \), from (5) we have

\[
h = r + \text{dif}(\text{Gr})
\]

or

\[
h = m - \text{dif}(\text{Gc})
\]

For the sake of completeness, we define the \( \text{dif} \) of the null graph as

\[
\text{dif}(\phi) = 0
\]

In order to minimize \( h \), from (34) and (35) we should maximize \( \text{dif}(\text{Gc}) \) or minimize \( \text{dif}(\text{Gr}) \). From (32) it is easy to see that if in \( \text{Gr} \) there exists at least one edge whose both end nodes belong to a component of \( \text{Gc} \), then we can move the edge from \( \text{Gr} \) into \( \text{Gc} \) to increase its \( \text{dif} \) by one. Thus, corresponding to a minimum basic set, every component of \( \text{Gc} \) must be a sectional subgraph of \( G \).
5.3 THEOREMS ABOUT DIFS OF SUBGRAPHS

Before we present the inspection method, we introduce two useful theorems.

THEOREM 3: Let $G_{c_1}$ be the second part of a decomposition $\{G_{r_1}; G_{c_1}\}$. Let $G_c$ be the new second part of $\{G_{r}; G_{c}\}$, where $G_c$ is obtained by moving a subgraph $G_{c_2}$ from $G_{r_1}$ to the $G_{c_1}$. Then

$$\text{dif}(G_c) = \text{dif}(G_{c_1}) + \text{dif}(G_{c_2})$$

Here "moving" is meant that the edges of $G_c$ are the edges of both $G_{c_1}$ and $G_{c_2}$.

Proof: Let $b_2$, $b_2^1$, and $b_2^2$ be the numbers of edges of $G_{c}$, $G_{c_1}$, and $G_{c_2}$, respectively; let $r_1$, $r_2$, $r_{11}$, $r_{21}$, and $r_{22}$ be the ranks of $G_{r}$, $G_{c}$, $G_{r_1}$, $G_{c_1}$, and $G_{c_2}$, respectively.

Because the new first part $G_r$ is derived from $G_{r_1}$ by contracting $G_{c_2}$, we can consider $\{G_{r}; G_{c_2}\}$ as a decomposition of $G_{r_1}$. Therefore, from lemma 1 we have

$$r_{11} = r_1 + r_{22} \quad (38)$$

Similarly, because $\{G_{r_1}; G_{c_1}\}$ and $\{G_{r}; G_{c}\}$ are decompositions of $G$, from lemma 1 we have

$$r = r_{11} + r_{21} \quad (39)$$

$$r = r_1 + r_2 \quad (40)$$

Then, from (38)-(40) we have

$$r_2 = r_{21} + r_{22} \quad (41)$$
On the other hand

$$b_2 = b_{21} + b_{22}$$  \hspace{1cm} (42)

From (32), (41), and (42) we have

$$\text{dif}(Gc) = b_2 - 2r_2$$

$$= (b_{21} + b_{22}) - 2 (r_{21} + r_{22})$$

$$= (b_{21} - 2r_{21}) + (b_{22} - 2r_{22})$$

$$= \text{dif}(Gc_1) + \text{dif}(Gc_2)$$

Q.E.D.

Let $h$ and $h_1$ be the hybrid ranks with respect to the decompositions $\{G_r;Gc\}$ and $\{G_{r1};Gc_1\}$, respectively. Then we have

$$h = m - [\text{dif}(Gc_1) + \text{dif}(Gc_2)]$$

$$= [m - \text{dif}(Gc_1)] - \text{dif}(Gc_2)$$

$$= h_1 - \text{dif}(Gc_2)$$  \hspace{1cm} (43)

Thus, we have the following corollary.

COROLLARY 4: Let $\{G_{r1};Gc_1\}$ be a decomposition of $G$. If we redecompose $G$ by moving $Gc_2$ from $G_{r1}$ to $Gc_1$, we get a new decomposition $\{G_r;Gc\}$, Then the hybrid rank is reduced by $\text{dif}(Gc_2)$.

Based on theorem 3 and corollary 4 we may find an appropriate $Gc$ step by step. We first find a non-sparse elementary subgraph $Gc_1$ as an initial $Gc(0)$. We contract $Gc_1$ in $G$ to derive a simplified graph $G'$. We then search for another non-sparse subgraph $Gc_2$ in $G'$. Let $E_{c_1}$ and $E_{c_2}$ be
the sets of edges of $G_{c_1}$ and $G_{c_2}$. Then the resultant subgraph $G_{c(1)}$ of $G$ is composed of the edges of $G_{c_1}$ and $G_{c_2}$. The dif of $G_{c(1)}$ is the sum of the difs of $G_{c_1}$ and $G_{c_2}$. This process can be repeated until no further non-sparse subgraphs are contained in the simplified graph.

An example is used to outline our procedure.

**EXAMPLE:** Consider the graph given in Fig. 5[8]. First we take the subgraph of two parallel edges as $G_{c_1}$, i.e.

$$G_{c_1} = e_3e_6e_4e_{11}$$
$$\text{dif}(G_{c_1}) = 0$$
$$G' = G \ominus G_{c_1}$$

The graph $G'$ is shown in Fig. 5(b). From $G'$ we take all the multiple edges as $G_{c_2}$, i.e.

$$G_{c_2} = e_6e_2e_5e_6e_{12}$$
$$\text{dif}(G_{c_2}) = 1$$

After contracting $G_{c_2}$ we have $G''$ shown in Fig. 5(c). Since the self-loop $e_5$ in $G''$ is a dense subgraph, we pick $e_5$ as $G_{c_3}$, i.e.

$$G_{c_3} = e_5$$
$$\text{dif}(G_{c_3}) = 1$$

Contracting $e_5$ we have $G'''$ as shown in Fig. 5(d). Since there exist no non-sparse subgraphs in $G'''$, $G'''$ is the first part $G_r$ of a desired decomposition. The resultant $G_c$ is obtained by adding $G_{c_1}$, $G_{c_2}$, and $G_{c_3}$ together, as shown in
Fig. 5 Example for finding Gc step by step
Fig. 5(e). The dif of $G_c$ is

$$\text{dif}(G_c) = \text{dif}(G_{c_1}) + \text{dif}(G_{c_2}) + \text{dif}(G_{c_3})$$

$$= 2$$

Since

$$m(G) = 7$$

from (35) the number of independent variables is

$$h = 7 - 2 = 5$$

Our result is same as in [8].

Dually we have another theorem.

**Theorem 4:** Let $G_{r_1}$ be the first part of a decomposition $(G_{r_1}; G_{c_1})$. Let $G_r$ be the new first part of $(G_r; G_c)$, where $G_r$ is obtained by moving a contracted graph $G_{r_2}$ from $G_{c_1}$ into $G_{r_1}$. Then

$$\text{dif}(G_r) = \text{dif}(G_{r_1}) \cdot \text{dif}(G_{r_2})$$

(44)

where $G_{r_2}$ is the first part of a decomposition of $G_{c_1}$.

**Proof:** Let $b_1$, $b_{11}$, and $b_{12}$ be the numbers of edges of $G_r, G_{r_1}$, and $G_{r_2}$, respectively; let $r_1, r_2, r_{11}, r_{12},$ and $r_{21}$ be the ranks of $G_r, G_c, G_{r_1}, G_{r_2},$ and $G_{c_1}$, respectively.

Because the new second part $G_c$ is derived from $G_{c_1}$ by removing $G_{r_2}$, we can consider $(G_{r_2}; G_c)$ as a decomposition of $G_{c_1}$. Therefore, from lemma 1 we have

$$r_{21} = r_{12} \cdot r_2$$

(45)

Then, from (39), (40), and (45) we have
On the other hand
\[ b_1 = b_{11} + b_{12} \]  
(47)

From (32), (46) and (47), we have
\[ \text{dif}(Gr) = b_1 - 2r_1 \]
\[ = (b_1 - 2r_1) = (b_{11} + b_{12}) - 2(2r_{11} + r_{12}) \]
\[ = (b_{11} - 2r_{11}) + (b_{12} - 2r_{12}) \]
\[ = \text{dif}(Gr_1) + \text{dif}(Gr_2) \]

Q.E.D.

Let \( h \) and \( b_1 \) be the hybrid ranks with respect to \( \{Gr; Gc\} \) and \( \{Gr_1; Gc_1\} \), respectively. Then from (34) and (44) we have
\[ h = r + \text{dif}(Gr_1) + \text{dif}(Gr_2) \]
\[ = h + \text{dif}(Gr_2) \]

Thus, we have the following corollary.

COROLLARY 5: Let \( \{Gr_1; Gc_1\} \) be a decomposition of \( G \). If we redecompose \( G \) by moving \( Gr_2 \) from \( Gc_1 \) to \( Gr_1 \), we get a new decomposition \( \{Gr; Gc\} \). Then the hybrid rank is increased by \( \text{dif}(Gr_2) \).

In order to minimize \( \text{dif}(Gr) \), on the basis of theorem 4 and corollary 5, we may find \( Gr \) step by step. First we find a non-dense elementary subgraph \( Gr_1 \) as an initial \( Gr^{(0)} \). We remove \( Gr_1 \) from \( G \) to get a simplified graph \( G' \). We then search for another non-dense subgraph \( Gr_2 \) from \( G' \). Let \( Gr_1 \)
and $E_{r_2}$ be the sets composed of the edges of $Gr_1$ and $Gr_2$. Then the resultant graph $Gr(1)$ is composed of the edges of $E_{r_1}$ and $E_{r_2}$, and the dif of $Gr(1)$ is the sum of difs of $Gr_1$ and $Gr_2$. This process can be repeated until no further non-dense subgraph is contained in the simplified graph.

We recall that if one of $Gr$ and $Gc$ is given, another set is uniquely determined, i.e. $Gr$ and $Gc$ are interdependent. Therefore, there exists a simple relation between $\text{dif}(Gr)$ and $\text{dif}(Gc)$.

**Theorem 5:** If $Gr$ and $Gc$ are a decomposition of the graph $G$, then,

$$\text{dif}(G) = \text{dif}(Gr) + \text{dif}(Gc)$$  \hspace{1cm} (49)

**Proof:** From (5a) and (5b) we have

$$r(G) + \text{dif}(Gr) = m(G) - \text{dif}(Gc)$$

Hence

$$\text{dif}(G) = m(G) - r(G)$$

$$= \text{dif}(Gr) + \text{dif}(Gc)$$

Q.E.D.

From theorem 5 we know that we maximize $\text{dif}(Gc)$ when we minimize $\text{dif}(Gr)$ and vice versa. Therefore we may use both theorems 3 and 4 to get an appropriate decomposition.

It is useful to keep some non-dense or non-sparse
elementary graphs in mind. Self-loop, multiple and complete graph on 4 nodes are important non-sparse elementary graphs. If they occur in a graph $G$ as proper subgraphs, they can be included in the second part of an appropriate decomposition. Dually, series edges in $G$ and the subgraph shown on the right hand side of Fig. 6(b) are important non-dense elementary subgraphs of $G$. They can be included in the first part of an appropriate decomposition.

If the decomposition process of a graph $G$ can be completed step by step in the above way, on the basis of statements in the next chapter we can show that the hybrid rank corresponding to this decomposition is minimal.
5.4 Example by Inspection

Let us look at a more complex example originally given in [7]. It seems that most of the existing examples in literature can be appropriately decomposed step by step in the above way.

EXAMPLE: Consider the graph given in Fig. 7.

The subgraph composed of $e_1, e_2, e_3, e_4, e_5,$ and $e_6$ is the complete graph on four nodes. Let it belong to $G_c$. The multiple edges $\{e_{12}, e_{13}\}, \{e_{20}, e_{21}\}$, and $\{e_{24}, e_{25}, e_{26}\}$ also belong to $G_c$. On the other hand, since $\{e_{10}, e_{11}\}, \{e_{14}, e_{15}\}, \{e_{16}, e_{17}\}, \{e_{18}, e_{19}\}$, and $\{e_{22}, e_{23}\}$ are series edges, they belong to $G_r$.

Contracting the edges which belong to $G_c$ and removing the edges which belong to $G_r$, we get the simplified graph $G'$ shown in Fig. 7(b). The two components in Fig. 7(b) are non-sparse. They belong to $G_c$. Therefore, $G_c$ consists of $e_1-e_6$, $e_{12}-e_{13}$, $e_{20}-e_{21}$, and $e_{24}-e_{26}$, and $G_r$ consists of $e_{10}$, $e_{11}$, $e_{14}$, $e_{15}$, $e_{16}$, $e_{17}$, $e_{18}$, $e_{19}$, $e_{22}$, and $e_{23}$. Both parts of the decomposition are shown in Fig. 7(c) and (d).

As another example, we reconsider the network shown in Fig. 5. By inspection it can be appropriately decomposed immediately as presented in section 4.3.
Fig. 7 An example of a decomposition
Fig. 7 An example of a decomposition (continuation)
CHAPTER VI

DECOMPOSITION BY SYSTEMATIC PROCEDURE

6.1 ABSOLUTELY NON-SPARSE GRAPHS AND ABSOLUTELY NON-DENSE GRAPHS

On the basis of corollary 4, if a basic set with respect to \((\text{Gr};\text{Gc})\) is minimum, then the \text{Gr} consists of no dense subgraphs, because if there exists a dense subgraph \(\text{Gc}_2\), belonging to \text{Gr}, then we can redecompose it by moving \(\text{Gc}_2\) from the first part to the second part. In this way the hybrid rank \(h\) will be reduced. In other words, \text{Gr} must not be a supergraph(*) of a dense graph.

Similarly, from corollary 5, if a basic set with respect to \((\text{Gr};\text{Gc})\) is minimum, then the \text{Gc} contains no sparse contracted graphs, i.e. \text{Gc} must not be an expansion of a sparse graph.

DEFINITION 12: A graph is called an absolutely non-sparse graph if the graph contains no sparse contracted graphs. A graph is called an absolutely dense graph if the graph contains no sparse and medium contracted graphs.

(*) If a graph \(G_1\) is a subgraph of a graph \(G_2\), then the \(G_2\) is called a supergraph of the \(G_1\).
DEFINITION 13: A graph is called an absolutely non-dense graph if the graph contains no dense subgraphs. A graph is called an absolutely sparse graph if the graph contains no dense and medium subgraphs.

EXAMPLE: Look at Fig.5 on page 45 and Fig.7 on page 51 and 52 again.

The graph G of Fig.5(a) is not an absolutely non-sparse graph, since it contains G', which is a sparse contracted graph of G. G is not an absolutely non-dense graph either, since it contains the dense subgraph composed of \{e_1, e_2, e_3, e_4, e_5, e_6, e_11\}.

The Gc of Fig.5(d) is an absolutely sparse graph, since it contains no dense and no medium subgraphs.

The Gc of Fig.5(e) is an absolutely dense graph, since it contains no sparse and no medium contracted graphs.

The Gc of Fig.7(c) is an absolutely non-dense graph, since it contains no dense subgraph, but not an absolutely sparse graph, since it contains the medium subgraph composed of \{e_{10}, e_{11}\}.

The Gc of Fig.7(d) is an absolutely non-sparse graph, since it contains no sparse contracted graph, but not an
absolutely dense graph, since it contains the medium contracted graph composed of \( \{e_{12}, e_{13}\} \).

Up to the present we have known that an appropriate decomposition is such that its first part must be absolutely non-dense and its second part absolutely non-sparse. This is a necessary condition for obtaining a minimum hybrid rank with respect to the decomposition. Later on we shall show that it is also sufficient.

6.2 CRITERIA FOR TESTING ABSOLUTELY NON-DENSENESS OR NON-SPARSENESS

Now we investigate how to test denseness or sparseness of a graph. A tree-pair will be used.

For a tree-pair \((T_1, T_2)\), a common chord of \((T_1, T_2)\) is defined as an edge that is not contained in either \(T_1\) or \(T_2\). A common branch of \((T_1, T_2)\) is defined as an edge that is contained in both \(T_1\) and \(T_2\). We denote by \(L_0\) the set composed of all the common branches, and by \(L_0\) the set composed of all the common chords.

The diagram illustrating the relation between a tree-pair is shown in Fig. 8. In the diagram, the rectangle, left
circle, right circle, shuttle and shaded area stand for $G$, $T_1$, $T_2$, $T_0$, and $L_0$, respectively. From the diagram it is easy to see that the distance [6, p.374] of $(T_1, T_2)$ is the following.

\[
\text{distance of } (T_1, T_2) = |T_1| - |T_0| \\
= r - |T_0| 
\]  \hspace{1cm} (50)

or

\[
\text{distance of } (T_1, T_2) = |L_1| - |L_0| \\
= m - |L_0| 
\]  \hspace{1cm} (51)

From (50) and (51) we have the following lemma.

**LEMMA 3:** If a tree-pair $(T_1, T_2)$ has no common branches or no common chords, then $(T_1, T_2)$ is maximally distant [8].

**LEMMA 4:** The dif of a graph $G$ equals the number of common
chords minus the number of common branches with respect to an arbitrary tree-pair \((T_1, T_2)\), i.e.
\[ \text{dif}(G) = |L_0| - |T_0| \]

**Proof:** From (50) and (51) we have
\[ r - |T_0| = m - |L_0| \]
so
\[ \text{dif}(G) = |L_0| - |T_0| \]

Q.E.D.

**Lemma 5:** If a graph \(G\) contains no common branches with respect to a tree-pair \((T_1, T_2)\), then \(G\) is non-sparse.

**Proof:** Since there exists no common branch, the number of edges of \(G\), \(b\), is not less than \(2r\). From (32)
\[ \text{dif}(G) = b - 2r \geq 0 \]

Q.E.D.

**Lemma 6:** If a graph \(G\) contains no common chords with respect to a tree-pair \((T_1, T_2)\), then \(G\) is non-dense.

**Proof:** Since there exists no common chord, the number of edges of \(G\), \(b\), is not greater than \(2r\).
\[ \text{dif}(G) = b - 2r \leq 0 \]

Q.E.D.

From Lemma 3 we see that the \((T_1, T_2)\) in Lemma 5 or 6 is
maximally distant.

Stating lemmas 5 and 6 differently, we have the following lemmas.

**LEMMA 7**: If a graph $G$ is dense, then the $G$ contains at least one common chord for any tree-pair.

**LEMMA 8**: If a graph $G$ is sparse, then the $G$ contains at least one common branch for any tree-pair.

**LEMMA 9**: If there exist no common chords with respect to a tree-pair $(T_1, T_2)$ of $G$, then the $G$ is absolutely non-dense.

**Proof**: Since $G$ contains no common chord with respect to $(T_1, T_2)$ the graph $G$ is composed of only branches of $T_1$ and $T_2$. Obviously, any subgraph $G_c$ of $G$ is also composed of some of branches. Let

$Tc_1 = T_1 \cap G_c$

$Tc_2 = T_2 \cap G_c$

then

$G_c = Tc_1 \cup Tc_2$

Both $Tc_1$ and $Tc_2$ contain no circuit. They are subgraphs of two trees of $G_c$, i.e. $|Tc_1|$ and $|Tc_2|$ are not greater than the rank $r(G_c)$. Therefore any subgraph $G_c$ of $G$ is non-dense.

**Q.E.D.**
In order to obtain an absolutely dense subgraph (if there exists one) from a given graph $G$, we may use the following procedure.

**PROCEDURE FOR OBTAINING ABSOLUTELY DENSE SUBGRAPH**

1. Let $(T_1, T_2)$ be a tree-pair of $G$. From Lemma 9 if there exist no common chords, then $G$ is absolutely non-dense, i.e., $G$ contains no dense subgraph at all, stop searching for dense subgraph or else go to (2).

2. Let $e_1$ be a common chord of $(T_1, T_2)$ and let $L_1$ be the fundamental circuit of $G$ with respect to $T_1$ defined by $e_1$. If $L_1$ contains a common branch, say, $e_0 \in L_1$ is a common branch of $(T_1, T_2)$, then we remove $e_0$ from $T_1$ and add $e_1$ to the two-tree $(T_1-e_0)$ to get a new tree $T_1^{(1)}$. After doing this we have a new tree-pair $(T_1^{(1)}, T_2)$ and a smaller common chord set than before. Return to (1), and replace $T_1$ by $T_1^{(1)}$. If $L_1$ contains no common branches, go to (3).

3. Generate $L_2$ from $L_1$ such that $L_2$ is composed of all the fundamental circuits of $G$ with respect to $T_2$ defined by each edge of $L_1$. If a certain circuit defined by $e_j \in L_1$ contains common branch $e_0$, then we remove $e_0$ from $T_2$ and add $e_j$ to $(T_2-e_0)$ to get $T_2^{(1)}$. If $e_j = e_1$, $(T_1, T_2^{(1)})$ is a new tree-pair, the number of common chords is reduced by one. If $e_j \neq e_1$, i.e., $e_1$ is contained in $T_1$, we remove $e_j$ from $T_1$. 


and add $e_1$ to $(T_1-e_1)$ to get $T_1^{(1)}$. $(T_1^{(1)}, T_2^{(1)})$ is a new tree-pair, the number of common chords is also reduced by one. Return to (1), replace $(T_1, T_2)$ by new tree-pair. If $L_2$ contains no common branches, go to (4).

(4) Compare $L_2$ with $L_1$. If both are the same, then we call it $G_{C_1}$. $G_{C_1}$ is an absolutely dense subgraph of $G$ (it will be proved later on), or else replace $L_1$ by $L_2$, and return to (3).

Since $G$ is finite and the number of common chords decreases, the procedure terminates in a finite number of steps.

EXAMPLE: Find an absolutely dense subgraph for the graph $G$ of Fig.9(a).

(1) If the initial tree-pair $(T_1, T_2)$ is shown in Fig.9(b), then $e_{19}$ is a common chord.

(2) The fundamental circuit of $G$ with respect to $T_1$ defined by $e_{19}$ is

$$L_1 = \{e_{19}, e_{15}, e_{18}\}$$

$L_1$ contains no common branches.

(3) The subgraph composed of all the fundamental circuits of $G$ with respect to $T_2$ defined by $e_{19}$, $e_{15}$, and
Fig. 9 An example of constructing a harmonious decomposition

\[ L_2 = \{e_{19}, e_{17}, e_{18}, e_{14}, e_{16}, e_{15}\} \]

Since the fundamental circuit defined by \( e_{14} \) contains a common branch \( e_{16} \), we replace \( e_{14} \) in \( T_2 \) by \( e_{16} \) to get \( T_2^{(1)} \).

Since \( e_{14} \neq e_{16} \), we replace \( e_{14} \) in \( T_1 \) by \( e_{19} \) to get \( T_1^{(1)} \).

\( (T_1^{(1)}, T_2^{(1)}) \) is a new tree-pair shown in Fig. 9(c).

(4) Choose a new common chord with respect to
(T_1^{(1)}, T_2^{(1)}), say, e_{11}. Now
\[ L_1^{(1)} = \{e_{11}, e_{13}, e_{15}\} \]
\[ L_2^{(1)} = \{e_{11}, e_{10}, e_{18}, e_{13}, e_{17}, e_{16}, e_{15}\} \]
\[ L_3^{(1)} = \{e_{11}, e_{13}, e_{15}, e_{10}, e_{14}, e_{18}, e_{17}, e_{19}, e_{16}\} \]
\[ L_4^{(1)} = L_3^{(1)} \]
Thus we obtain an absolutely dense subgraph
\[ G_{c_{11}} = L_3^{(1)} \]
\[ G_{c_{11}} \] is shown in Fig. 9(d).

From the process of constructing \( G_{c_i} \) it is easy to see the following. If we denote the final tree-pair of the process by \( (T_1, T_2) \) and denote \( T_1 \cap G_{c_i} \) and \( T_2 \cap G_{c_i} \) by \( T_{11} \) and \( T_{21} \), then \( T_{11} \) and \( T_{21} \) are two edge-disjoint trees of \( G_{21} \). According to [11,12], \( G_{c_i} \) is said to have a complementary tree-pair. We have the following lemma with regard to \( G_{c_i} \).

**Lemma 10:** \( G_{c_i} \) of \( G \) is a dense subgraph of \( G \) and \( \text{dif}(G_{c_i}) = 1 \).

**Proof:** \( G_{c_i} \) is composed of a common chord and a complementary tree-pair \( (T_1, T_2) \). Complementary tree-pair contains no common branches, so
\[ b(G_{c_i}) = 2r(G_{c_i}) + 1 \]
\[ \text{dif}(G_{c_i}) = 1 > 0 \]
i.e. \( G_{c_i} \) is dense.

Q.E.D.

**Lemma 11:** Graph \( G_{c_i} \) contains no proper dense subgraph.
Proof: Let \((T_1, T_2)\) be a complementary tree-pair of
\(G_{c_1}\) and \(e_1\) be the common chord, i.e.
\[G_{c_1} = T_1 \cup T_2 \cup e_1\]
Let \(G_2\) be a dense subgraph of \(G_{c_1}\) and
\[T_{21} = G_2 \cap T_1\]
\[T_{22} = G_2 \cap T_2\]
then
\[|T_{21}| \leq r(G_2)
\[|T_{22}| \leq r(G_2)\]

Since \(G_2\) is dense, \(b(G_2)\) has to be greater than \(2r(G_2)\).
On the other hand
\[G_2 = T_{21} \cup T_{22} \cup (G_2 \cap e_1)\]
In order that \(b(G_2)\) is greater than \(2r(G_2)\), both \(|T_{21}|\) and
\(|T_{22}|\) must equal \(r(G_2)\), i.e. \(T_{21}\) and \(T_{22}\) form a
complementary tree-pair for \(G_2\), and \(e_1\) is also contained in
\(G_2\). This means that \(G_2\) must be \(G_{c_1}\). Thus there is no proper
dense subgraph in \(G_{c_1}\).

Q.E.D.

This lemma means that \(G_{c_1}\) is one of minimal dense
subgraphs in \(G\). From lemma 11 we have an important theorem.

THEOREM 6: Graph \(G_{c_1}\) is absolutely dense.

Proof: Any contracted graph \(G_1\) of \(G_{c_1}\) can be
considered as the first part of a decomposition \(\{G_1;G_2\}\) of
the $Gc_1$. From lemma 11, $G_2$ is non-dense, and from (49) and lemma 10

$$\text{dif}(G_1) = \text{dif}(Gc_1) - \text{dif}(G_2) \geq 1$$

i.e. any contracted graph of $Gc_1$ is dense. According to definition 10 the $Gc_1$ is absolutely dense.

Q.E.D.

Generally speaking, we have the following theorems and corollaries.

THEOREM 7: A graph $G$ contains a dense subgraph $Gc_1$ if and only if $G$ contains at least one common chord with respect to a maximally distant tree-pair $(T_1, T_2)$.

Proof: Sufficiency. If $G$ contains one common chord $e_1$ with respect to $(T_1, T_2)$, then we can construct the subgraph $Gc_1$ of $G$. According to lemma 10 $Gc_1$ is the dense subgraph of $G$.

Necessity. From lemma 9 it is obvious.

Q.E.D.

According to definition 13 we have

COROLLARY 6: If a graph $G$ contains no common chords with respect to a tree-pair $(T_1, T_2)$, then $G$ is absolutely non-dense.
It is not difficult to prove the following dual theorem and corollary. Since the proofs can be made with almost word-to-word substitution, such as common branch for common chord, fundamental cutset for fundamental circuit, cotree for tree, contraction for removal, etc., we omit their proofs.

**Theorem 8:** A graph $G$ contains a sparse contracted graph, if and only if its maximally distant tree-pair contains common branches.

**Corollary 7:** If there exists a tree-pair in $G$ that contains no common branches, then the $G$ is absolutely non-sparse.

### 6.3 Harmonious Decomposition

For convenience we define a special kind of decompositions as follows.

**Definition 14:** Harmonious decomposition. A decomposition $\{G_r; G_c\}$ is called a harmonious decomposition, if the $G_r$ is absolutely non-dense and $G_c$ is absolutely non-sparse.

We shall prove that the hybrid rank with respect to any harmonious decomposition of $G$ is always equal to the topological degree of freedom of $G$. In other words, any
harmonious decomposition of $G$ provides a base for minimum basic set. Before proving this proposition we introduce some lemmas.

**Lemma 12:** If $\{G; Gc\}$ is a harmonious decomposition of $G$, then each component of $Gc$ is a sectional subgraph of $G$.

**Proof:** Consider

$$Gr = G \sqcup Gc$$

Assume $Gc$ is not a sectional subgraph of $G$ or a union of sectional subgraphs of $G$. Then $Gr$ must contain at least one self-loop. Self-loop is dense. According to definition 1, $Gr$ is absolutely non-dense. It cannot contain a dense subgraph, a contradiction.

Q.E.D.

**Lemma 13:** If $\{Gr_1; Gc_1\}$ and $\{Gr_2; Gc_2\}$ are harmonious decompositions, and

$$Gcd_1 = Gc_1 \sqcup (Gc_2 \cap Gc_2)$$

then $Gcd_1$ is medium.

**Proof:** First we prove $Gcd_1$ is non-sparse. According to definition 14 $Gc_1$ is absolutely non-sparse. It contains no sparse contracted graph, so $Gcd_1$ is non-sparse.

Now we prove $Gcd_1$ is non-dense. Consider that $\{Gr_2; Gc_2\}$ is formed from $\{Gc_1 \cup Gc_2\}$ by moving $Gcd_1$ from second part...
to first part. Since $Gr_2$ is absolutely non-dense, $Gcd_1$ is non-dense. Therefore, $Gcd_1$ is medium.

Q.E.D.

This lemma means the difference between the second parts of different harmonious decompositions of $G$ is always medium.

Similarly we can show the difference between the first parts of different harmonious decompositions of $G$ is also medium.

**Theorem 9:** If $\{Gr_1; Gc_1\}$ and $\{Gr_2; Gc_2\}$ are two arbitrary but harmonious decompositions of $G$, then the diffs of the same parts are equal, i.e.

$$\text{dif}(Gr_1) = \text{dif}(Gr_2)$$

$$\text{dif}(Gc_1) = \text{dif}(Gc_2)$$

**Proof:** We first prove (53). On the basis of the lemma 13, the followings are medium

$$Gcd_1 = Gc_1 \ominus (Gc_1 \cap Gc_2)$$

$$Gcd_2 = Gc_2 \ominus (Gc_1 \cap Gc_2)$$

Thus

$$\text{dif}(Gcd_1) = 0$$

$$\text{dif}(Gcd_2) = 0$$

$\{Gcd_1; Gc_1 \cap Gc_2\}$ is a decomposition of $Gc_1$, and
\{(Gcd_2; Gc_3 \cap Gc_2)\} is a decomposition of Gc_2. From theorem 5
\[ \text{dif}(Gc_1) = \text{dif}(Gcd_1) + \text{dif}(Gc_1 \cap Gc_2) \]
\[ = \text{dif}(Gc_1 \cap Gc_2) \]
\[ \text{dif}(Gc_2) = \text{dif}(Gcd_2) + \text{dif}(Gc_1 \cap Gc_2) \]
\[ = \text{dif}(Gc_1 \cap Gc_2) \]
So we have (53).

Now we prove (52). By using (49) twice we have
\[ \text{dif}(Gr_2) + \text{dif}(Gc_2) = \text{dif}(Gr_1) + \text{dif}(Gc_1) \] (54)
From (53) and (54) the expression (52) follows.
Q.E.D.

From (53) and (35) we have the following corollary.

COROLLARY 8: The hybrid ranks with respect to all harmonious decompositions are equal.

THEOREM 10: Let \{(Gr; Gc)\} be a decomposition of a graph G. A cotree Tr' of Gr in union with a tree Tc of Gc is a minimum basic set, \{(Tr', Tc)\}, of G, if and only if the decomposition is harmonious.

Proof: Necessity. If \{(Gr; Gc)\} is not harmonious, then either Gr contains a dense subgraph, or Gc contains a sparse contracted graph (or both). First suppose Gr contains a dense subgraph Gr_1. If we contract Gr_1 in Gr and add Gr_1 to Gc, we obtain a new second part Gc^{(1)}. From theorem 3 we
have
\[ \text{dif}(G_{c(1)}) = \text{dif}(G_c) + \text{dif}(G_{r_1}) > \text{dif}(G_c) \]

From (35) the hybrid rank of the new basic set with respect to the new decomposition is less than the original one. This contradicts the assumption that the original basic set is minimum.

When \( G_c \) contains a sparse contracted graph, we can show a contradiction in a similar way. The details are omitted.

Sufficiency. Since the harmonious condition of \( \{G_1; G_2\} \) is necessary, and since the hybrid ranks with respect to all harmonious decompositions are the same, the sufficiency of the condition follows immediately.

\[ Q.E.D. \]

This theorem means that the hybrid rank with respect to any harmonious decomposition of \( G \) is equal to the topological degree of freedom of \( G \).

6.4 SYSTEMATIC PROCEDURE FOR CHOOSING A MINIMAL SET OF INDEPENDENT VARIABLES

On the basis of theorem 6 we can obtain a harmonious decomposition by using the following algorithm.
ALGORITHM FOR OBTAINING HARMONICUS DECOMPOSITION:

Step 0: Let $G^{(0)} = G$, $Gc^{(0)} = \phi$, $i=1$, and $CC^{(0)}$ be the set of common chords with respect to a tree-pair $(T_1, T_2)$.

Step 1: If $CC^{(i-1)} = \phi$, then let $Gc = Gc^{(i-1)}$ and $Gr = G^{(i-1)}$, and stop. Or else take an edge $e_i$ from $CC^{(i-1)}$, and try to construct a subgraph $Gc_i$ for $e_i$ by using the procedure outlined in the section 6.2. If finally there exists no subgraph $Gc_i$, return to the beginning of step 1; if there exists such a $Gc_i$, go to the next step.

Step 2: Let

$$Gc^{(i)} = Gc^{(i-1)} \cup Gc_i$$

$$G^{(i)} = G^{(i-1)} \oplus Gc_i$$

Set $i = i+1$, return to step 1.

When the algorithm is in progress, the common chord set is unceasingly reduced, so the algorithm is fulfilled in finite steps.

Note that we can take any tree-pair $(T_1, T_2)$ as starting point, even if $T_1$ and $T_2$ are same. In order to save time it is better to choose a pair of trees with large distance.

In comparison with the existing procedures, our procedure need not find a maximally distant tree-pair [8] or
a extremal tree [9], and it simplifies the given graph as early as possible. In addition, since our procedure is based on the classification of a graph, it, comparatively speaking, is intuitive.

In addition, our procedure and its theoretic foundation possess block structure. Therefore, for a large network, the graph can easily be torn to pieces, and then its harmonious decomposition can be discovered step by step.

**EXAMPLE:** Construct a harmonious decomposition for the graph of Fig. 10(a).

(1) From the example of section 6.2, we have known

\[ G_{c1} = L_1^{(1)} \]

\[ = \{ e_{11}, e_{13}, e_{15}, e_{10}, e_{14}, e_{16}, e_{17}, e_{19}, e_{16} \} \]

Let

\[ Gc^{(1)} = Gc_{c1} \]

\( Gc^{(1)} \) is shown in Fig. 10(c). Then

\[ G^{(1)} = Gc^{(1)} \]

The simplified graph \( G^{(1)} \) is shown in Fig. 10(d).

(2) For the self-loop \( e_{12} \) of \( G^{(1)} \) the fundamental circuit is itself, i.e.

\[ L_1^{(2)} = \{ e_{12} \} = L_2^{(2)} = Gc_{c12} \]

\[ Gc^{(2)} = Gc^{(1)} \cup e_{12} \]

\[ G^{(2)} = G^{(1)} \odot e_{12} \]
Fig. 10 An example of constructing a harmonious decomposition
(3) For the common chord \( e_a \) we try to construct the
\[ \text{Ge}_a. \]
\[ L_1^{(3)} = \{ e_6, e_6, e_7 \} \]
e_6 is a common branch, so replace \( T_1 \) by \((T_1 - e_a) \cup e_a\). The
new tree-pair is shown in Fig. 10(e).

(4) Since there exists no common chord in \( G^{(2)} \), so \( G^{(2)} \)
is absolutely non-dense. Therefore
\[ G_r = G^{(2)} \]
\[ G_c = G_c^{(2)} \]
\( G_r \) is shown in Fig. 10(e) and \( G_c \) in Fig. 10(f). The \( \{G_r, G_c\} \) is
a harmonious decomposition of \( G \).

Obviously, from the harmonious decomposition the
topological degree of freedom is
\[ d = m(Gr) + r(Gc) \]
\[ = 4 + 4 = 8 \]

6.5 HARMONIOUS DECOMPOSITION AND COMPLEMENTARY TREE-PAIR

When we use the procedure introduced in the last
section, every \( Gc_i \) contains a common chord and a
complementary tree-pair \((T_{1i}, T_{2i})\) of \( Gc_i \). Obviously, if \( GC \)
is connected, the union of all the \( T_{1i} \) is a tree of \( Gc \).
Similarly, the union of all \( T_{2i} \) is also a tree of \( Gc \). Both
are edge-disjoint, and form a complementary tree-pair of \( Gc \)
If $G_c$ is disconnected, then the union is a spanning forest. The two unions form a complementary spanning forest-pair. Thus, we have the following theorem.

**Theorem 11:** The second part of any harmonious decomposition contains a complementary tree (or forest) pair.

We have interesting theorems about complementary tree-pair.

**Theorem 12:** If a graph $G$ is a complementary tree-pair, the $G$ is both absolutely non-dense and non-sparse.

**Proof:** Since $G$ contains no common branches and no common chords with respect to the complementary tree-pair, from corollaries 6 and 7 $G$ is both absolutely non-dense and absolutely non-sparse.

Q.E.D.

**Theorem 13:** A graph $G$ is absolutely non-sparse if and only if $G$ contains a complementary tree-pair.

**Proof:** If $G$ contains a complementary tree-pair $(T_1, T_2)$, then $G$ contains no common branches with respect to

---

(s) From here on, for short we shall omit this appended note.
the \((T_1, T_2)\). From corollary 7 G is absolutely non-sparse.

Conversely, if G does not contain any complementary tree-pair, i.e. the maximally distant tree-pair of G contains common branches, then from theorem 8 G contains sparse contracted graph, i.e. G is not absolutely non-sparse.

Q.E.D.

If a tree of G contains no cutsets of G, we call it a cutsetless tree. Then we have the following theorem.

**THEOREM 14:** There exists a cutsetless tree in a graph G if and only if G contains a complementary tree-pair.

**Proof:** If G contains a complementary tree-pair, each one of the pair is cutsetless. Because if we contract the cotree, then the rank of the contracted graph is zero.

Conversely, if G has a cutsetless tree \(T_1\), containing no cutsets, \(T_1\) must be a subset of a cotree. This means that there exists another tree \(T_2\) in G, and that \(T_1 \cup T_2 = \emptyset\), i.e. \((T_1, T_2)\) is a complementary tree-pair.

Q.E.D.

On the basis of the theorem 13, we obtain a harmonious decomposition of a graph by searching for the maximum
subgraph of $G$ that contains a complementary tree-pair. This is the method used in [12].

6.6 HARMONIOUS DECOMPOSITION AND MAXIMALLY DISTANT TREE-PAIR

We construct $G_{c_1}$ with respect to an arbitrary tree-pair in the section 6.2. Now we consider the special case where the tree-pair is maximally distant.

**Lemma 14:** Any $G_{c_1}$ with respect to a maximally distant tree-pair $(T_1, T_2)$ contains no common branches.

**Proof:** We prove that $L_1$ contains no common branch. Because if $e_0 \in L_1$ defined by $e_1$ is a common branch of $(T_1, T_2)$, then we can remove $e_0$ from $T_1$ and add $e_1$ into $(T_1 - e_0)$ to obtain a new tree $T_1^{(1)}$. The tree-pair $(T_1^{(1)}, T_2)$ has longer distance than that of $(T_1, T_2)$. This contradicts the assumption.

Similarly, we can show any $L_n$ contains no common branches.

Q.E.D.

Let $T_0$ and $L_0$ be the common branch set and the common chord set with respect to maximally distant tree-pair $(T_1, T_2)$. Since
\[ Gr = G \otimes Gc \]

The contracted tree-pair, denoted by \((Tr_1, Tr_2)\), in \( Gr \) is still a tree-pair of \( Gr \). Because \( L_0 \) belongs to \( Gc \), there exists no common chords with respect to \((Tr_1, Tr_2)\) in \( Gr \). Hence, \((Tr_1, Tr_2)\) is maximally distant in \( Gr \).

The common branch set \( T_0 \) becomes the common branch set of \((Tr_1, Tr_2)\). \((Tr_1 - T_0)\) is a cctree of \( Gr \).

On the other hand, if we denote the difference between \( T_1 \) and \( T_2 \) by \( D_1 \), then
\[ D_1 = T_1 - T_0 \]

Let
\[ Dc_1 = Gc \cap D_1 \]

Since \( Gc \) contains no common branches, the above \( D_1 \) can be replaced by \( T_1 \). Hence
\[ Dc_1 = Gc \cap T_1 \]
i.e. \( Dc_1 \) is a tree or a spanning forest of \( Gc \).

Since \( Tr_1 \) is derived from \( T_1 \) by contracting \( Gc \) and \( Dc_1 \) is a tree of \( Gc \), we have
\[ Tr_1 = T_1 \otimes Dc_1 \]
The cctree \((Tr_1 - T_0)\) of \( Gr \) is equal to \((T_1 - T_0 \otimes Dc_1) = (D_1 \otimes Dc_1)\). Hence, \((D_1 \otimes Dc_1, Dc_1)\) is a minimum basic set of \( G \), i.e. we have the following theorem.

**Theorem 15:** The difference between a maximally distant tree-
pair of $G$ corresponds to a minimum basic set of the $G$.

Like $Gc_i$ we can dually construct $Gr_i$ by the following procedure. Let $(T_1, T_2)$ be a maximally distant tree-pair of $G$ and $T_0$ be the common branch set. Let $e_i \in T_0$ and let $C_i$ be the fundamental cutset of $G$ with respect to $T_i$ defined by $e_i$. Generate $C_2$ from $C_1$ such that $C_2$ is composed of all the fundamental cutsets of $G$ with respect to $T_2$ defined by each edge of $C_1$. Generally, $C_j$ can be generated from $C_{j-1}$ in the similar way. Because $G$ is finite, we can find minimum number $n$ satisfying

$$C_n = C_n$$

If we contract all the edges that do not belong to $C_n$, we construct the $Gr_i$.

It is not hard to prove the following two propositions about $Gr_i$. Their proofs are omitted.

**LEMMA 15:** Graph $Gr_i$ contains no proper sparse contracted graph. This means that $Gr_i$ is one of minimal sparse contracted graphs in $G$.

This is the dual of the lemma 11.

**THEOREM 16:** Graph $Gr_i$ is absolutely sparse. This is the dual of theorem 6.
The union of all \( G_{r_1} \), denoted by \( G_2 \), is a maximal absolutely sparse contracted graph of \( G \). The union of all \( G_{c_1} \), denoted by \( G_c \), is a maximal absolutely dense subgraph of \( G \). The derived graph \( G \oplus G_c - G_2 \), denoted by \( G_0 \), is a maximal absolutely non-dense and non-sparse derived graph of \( G \), i.e. the graph \( G_0 \) contains no dense subgraphs and no sparse contracted graphs. These \( G_0 \), \( G_c \) and \( G_2 \) correspond to the \( G_0 \), \( G_1 \), and \( G_2 \) of the principal partition of \( G \) in [8].

6.7 Dyad D and Minimum Basic Set

In this section we define a new kind of subgraphs for a graph \( G \), and investigate its properties and show that the subgraphs correspond to minimum basic sets.

DEFINITION 15: Dyad D. A maximal circuitless and cutsetless subgraph of graph \( G \) is called a dyad \( D \) of \( G \), where "maximal" means that the number of edges of the \( D \), is the largest among all the circuitless and cutsetless subgraphs of \( G \).

EXAMPLE : Look at Fig.1 on page 2 again. Obviously, any edge of \( G \) is a circuitless and cutsetless subgraph of \( G \), but it is not maximal. The subgraphs composed of one of \( \{e_1, e_2, e_3, e_4\} \) and one of \( \{e_5, e_6, e_7, e_8\} \) are dyads of \( G \), since each of them contains no circuits and no cutsets, and there exist no circuitless and cutsetless subgraphs with 3 edges.
Several properties immediately follow from the above definition.

**PROPERTY 1:** Dyad is a subtree (k-tree) or just a tree.

Because it contains no circuits.

**PROPERTY 2:** Dyad is a subcotree or just a cotree.

Because it contains no cutsets.

**PROPERTY 3:** The number of edges of a dyad $D$ of $G$, $|D|$, is not greater than either the rank or the nullity of $G$, i.e.

$$|D| \leq \min \{r, n\}$$

Because a dyad $D$ is a subgraph of both tree and cotree of $G$.

**THEOREM 17:** A subgraph of $G$ is a dyad $D$ of $G$ if and only if it is the difference of a maximally distant tree-pair $(T_1, T_2)$ of $G$.

**Proof:** Sufficiency. We show that the difference of any maximally distant tree-pair $(T_1, T_2)$ is a dyad $D$. Let $D$ denote the difference of $(T_1 - T_1 \cap T_2)$, i.e.

$$D_1 = T_1 \cap T_2$$

$D_1 \subseteq T_1$

$D_1 \subseteq T_2$.
D₁ is both circuitless and cutsetless. On the other hand, D₁ is maximal among this kind of subgraphs since it is the difference of a maximally distant tree-pair. Therefore, D₁ is a dyad D of G.

Necessity. If D is a dyad, being circuitless, there exists a tree T₁ as its supergraph. On the other hand, since D is cutsetless, there exists a cotree T₂ as its supergraph. The difference T₁ - T₁ ∩ T₂ is both circuitless and cutsetless and contains D. Because D is maximal, (T₁, T₂) must be maximally distant.

Q.E.D.

From the proof of theorem 17, we may deduce another theorem.

THEOREM 18: If a tree T₁ contains a dyad D and a cotree T₂ contains the same dyad D, then (T₁, T₂) is a maximally distant tree-pair.

Now we may state the theorem 15 in terms of dyad D.

THEOREM 19: A dyad D of graph G corresponds to a minimum basic set of G.

COROLLARY 9: The topological degree of freedom of a graph G, d, equals the number of edges of a dyad D of G.
6.8 Dyad D and Extremal Tree

In [9] the concept of an extremal tree was introduced and played a central role in obtaining their results. In this section we briefly present the relation between our dyad and their extremal tree.

**DEFINITION 16:** A tree $T_0$ of a graph $G$ is called an extremal tree if $T_0$ contains the minimum number of independent cutsets of $G$.

We recall the number of independent cutsets a subgraph $G_S$ of $G$ is equal to $r(G \ominus G_S')$. Therefore, an extremal tree $T_0$ is such a tree that

$$r(G \ominus T_0) \leq r(G \ominus T')$$

where $T$ is an arbitrary tree of $G$.

We have a theorem about the relation between a dyad and an extremal tree. Before presenting it we need the following lemma.

**LEMMA 16:** If a dyad $D$ of $G$ is subgraph of a tree $T_0$, then the $T_0$ contains $(r-d)$ independent cutsets of $G$.

**Proof:** Let $e_1, e_2, \ldots, e_{(r-d)}$ be the edges that belong to $T_0$ but not in $D$. Since $D$ is a maximal cutsetless and circuitless subgraph and each $e_j$ $[j=1,2,\ldots,(r-d)]$ does not
form a circuit exclusively with edges of $D$, each $e_j$ forms a cutset $C_j$ exclusively with edges of $D$. Obviously, $C_j$ \([j=1,2,\ldots,(r-d)]\) are independent, i.e. $T_0$ contains at least $(r-d)$ independent cutsets.

Now we show $T_0$ contains at most $(r-d)$ independent cutsets. Suppose $T_0$ contains $k$ \((k > r-d)\) independent cutsets of $G$. Then we can always eliminate $e_1$, $e_2$, $\ldots$, $e_{(r-d)}$ by performing ring sum operations on these $k$ independent cutsets to obtain a certain cutset composed only of some edges of $D$. This contradicts the fact that $D$ contains no cutsets.

Q.E.D.

THEOREM 20: A tree $T_0$ is an extremal tree if and only if the $T_0$ contains a dyad $D$.

Proof: From property 1 of a dyad there exists at least one tree $T_0$ containing $D$. From lemma 16 the number of independent cutsets in $T_0$ is $(r-d)$, where $d$ is the number of edges of $D$. Hence the minimum number of independent cutsets contained in any tree, $k$, is not greater than $(r-d)$, i.e.

$k \leq r-d$

On the other hand, if the number of independent cutsets of a tree $T_0$ is $k$, then the rank of the graph $G \oplus T_0'$ is $k$. From the graph $G \oplus T_0'$ we choose a tree, $T_{00}$, which is a
subgraph of $T_0$. Remove $T_{00}$ from $T_0$ we obtain a cutsetless and circuitless subgraph with $(r-k)$ edges. The number of edges of the resulting subgraph cannot exceed the number of edges of maximal cutsetless and circuitless subgraph $D$, i.e.

$$r-k \leq d$$

Hence $k=r-d$, i.e. the minimum number of independent cutsets contained in a tree is equal to $(r-d)$. Therefore a tree is extremal if and only if it contains a dyad.

Q.E.D.

COROLLARY 10: An extremal tree is one of the maximally distant trees.

Proof: This follows from the fact that each tree in a maximally distant tree-pair contains a dyad $D$ of $G$.

Q.E.D.

COROLLARY 11: The topological degree of freedom of graph $G$ is equal to the rank of $G$ minus the number of independent cutsets contained in an extremal tree.

Proof: From the proof of theorem 20 we know that the number of edges of a dyad $D$ of $G$ is equal to the rank minus the number of independent cutsets contained in an extremal tree, and from the corollary 8 the number of edges of $D$ equals the topological degree of freedom.

Q.E.D.
CHAPTER VII

CONCLUSIONS

The most important results presented in this thesis are summarized here.

A given network can be arbitrarily decomposed into two parts. In one part the edge currents are regarded as primary variables and the impedances as adequately characterizing the edges. In another part the dual quantities are true. Following the procedure presented in chapter 4 we can formulate the network hybrid equations from the fundamental circuit or cutset matrix. The nodal (or cutset) and the loop analyses can be viewed as the special cases of the hybrid analysis.

In order to investigate how to select a minimum complete set of independent variables, we first classify graphs into three classes: sparse, medium, and dense when the difference between the nullity and the rank is negative, zero, or positive, respectively. We then introduced two important new concepts: absolute non-denseness and absolute non-sparseness. Basing our analysis on the two new concepts we showed that the decomposition, with respect to which the set of independent variables is minimum, must be a harmonious decomposition, i.e. the first part and the second
part of the decomposition must be absolutely non-dense and absolutely non-sparse, respectively. Hence, only when a graph itself is absolutely non-dense or absolutely non-sparse, the loop or nodal (or cutset) analysis has minimum set of independent variables.

Furthermore, we derived interesting properties for both the absolutely non-dense graph and the absolutely non-sparse graph. These properties are simple, intuitive, and useful. We proved that the union of certain absolutely non-sparse graphs is still an absolutely non-sparse graph, i.e., the resultant graph still contains no sparse contracted graphs. On the other hand, for the absolutely non-dense graphs there exist the dual conclusions.

By applying the above-mentioned properties, we can decompose a given graph step by step. We pointed out that most of these graphs can be harmoniously decomposed by inspection. Even so, we presented a systematic procedure, which can be implemented on a computer. The advantage of our systematic procedure is that it can simplify the given graph as early as possible. The proposed methods to choose a complete set of independent variables can be done relatively easily or quickly. A practical example (Fig.4) was given.

In the final part of this thesis, we briefly showed the
relationships among our results and several existing results. The new concept, dyad, played an important role.

In addition, we hope that this thesis will provide a better insight into the hybrid method and facilitate the comprehension of the topological degree of freedom.
REFERENCES


