A CONSTITUTIVE LAW FOR LOESS AT ITS
NATURAL MOISTURE CONTENT AND
LOW-CONFINING PRESSURES/

A Thesis Presented to
The Faculty of the College of Engineering and Technology
Ohio University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science

by
Kevin M. Bral,

August, 1982

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C - \( J_1 \) value at center of the ellipse
D - constant
E - Young's modulus
\( E_1, E_2 \) - constants
F - general loading function of the stresses
G - shear modulus
\( J_1 \) - first invariant of the stress tensor
\( J_{2d} \) - second invariant of the stress deviator tensor
\( J_{3d} \) - third invariant of the stress deviator tensor
K - bulk modulus
P - pressure
R - eccentricity of the ellipse
\( R_1, R_2, R_3 \) - constants
\( S_{ij} \) - components of the stress deviator tensor
V - volume
\( V_s \) - velocity in the s-direction
\( V_t \) - total volume of the sample
W - constant
\( W_s \) - weight of soil solids
\( W_w \) - weight of water
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<td>hardening rule</td>
</tr>
<tr>
<td>Z</td>
<td>constant</td>
</tr>
<tr>
<td>b</td>
<td>$J_{2d}$ distance, measured at the center of the ellipse, from the $J_1$ axis to the failure surface</td>
</tr>
<tr>
<td>c</td>
<td>cohesion</td>
</tr>
<tr>
<td>$d\epsilon_{ij}$</td>
<td>components of the strain increment tensor</td>
</tr>
<tr>
<td>$d\sigma_{ij}$</td>
<td>components of the stress increment tensor</td>
</tr>
<tr>
<td>$d\lambda$</td>
<td>incremental constant</td>
</tr>
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<td>f</td>
<td>general loading function</td>
</tr>
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<td>$f_1$</td>
<td>failure surface in cap model</td>
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<td>loading surface in cap model</td>
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<td>g</td>
<td>plastic strain potential function</td>
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<td>r</td>
<td>radius</td>
</tr>
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<td>$a$</td>
<td>constant</td>
</tr>
<tr>
<td>$\beta$</td>
<td>constant</td>
</tr>
<tr>
<td>$\gamma_d$</td>
<td>dry density</td>
</tr>
<tr>
<td>$\delta_{ij}$</td>
<td>kronecker delta</td>
</tr>
<tr>
<td>$\epsilon_{ij}$</td>
<td>components of the strain tensor</td>
</tr>
<tr>
<td>$\epsilon_v$</td>
<td>volumetric strain</td>
</tr>
<tr>
<td>$\eta$</td>
<td>general parameter</td>
</tr>
<tr>
<td>$\theta$</td>
<td>angle between stress increment vector and plastic strain increment vector</td>
</tr>
</tbody>
</table>
\( \lambda \) - Lame\' constant

\( \nu \) - Poisson\'s ratio

\( \sigma_{ij} \) - components of the stress tensor

\( \sigma_m \) - mean stress

\( \sigma_n \) - normal stress

\( \sigma_o \) - yield stress in simple tension

\( \tau_{ij} \) - shearing stress components of the stress tensor

\( \tau \) - shearing stress on failure plane

\( \phi \) - angle of internal friction

\( \Phi \) - velocity potential function

\( \omega \) - water content

**Superscripts**

\( e \) - elastic

\( n \) - normal

\( p \) - plastic

\( T \) - total

\( t \) - tangent
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Chapter I

INTRODUCTION

Loess is a wind blown soil deposit which covers large areas of Central United States, Russia, Europe, and Asia. It consists of silt, fine sand, and clay particles ranging in size from 0.002 to 1.0 mm. in diameter. Generally the largest percentage of particles pass through the no. 100 sieve (0.150 mm.).

Loess is deposited in grasslands that border arid regions. The wind blown particles are caught by the grasses, with deposition keeping pace with the growth of new vegetation.

Loess is characterized as having a loose structure, cemented by clay and calcium carbonate from the former vegetative growth. In the dry condition, loess is fairly strong and has the ability to stand up, in large vertical cuts. Excess moisture softens the cementation and will cause collapse. (1, 2)

1.1) Previous Studies of the Mechanical Behavior of Loess

Little work has been reported in literature on the mechanical properties of loess and its performance as a foundation material. Following is a brief summary of this work.

Holtz and Gibbs (3) did an extensive study on driven piles in loess deposits in Nebraska. Recommendations were given concerning types of piles
and methods of driving for the best performance in loess soils. All analyses were done using pile load tests and driving record.

Clevenger (4) describes some engineering properties of Missouri River Basin and Denver area loess. Two case studies of residential building failure in the Denver area were cited. He indicates that loess at low moisture contents will normally support loads assigned to a silty clay. If the loess is exposed to moisture, unfavorable conditions result.

Peck and Ireland (5), in a discussion on Clevenger's paper, pointed out the low erosion resistance of loess. Designs based on plate load and standard penetration tests were discussed.

Zur and Wiseman (6) conducted a laboratory study on undisturbed samples of Negev loess. They concluded that the collapse behavior is dependent on shear stress as well as the saturation and mean stress level.

Benak (7) did an extensive investigation of the engineering properties of loess deposits in the Omaha, Nebraska - Council Bluffs, Iowa Region. Pile load tests, load-bearing tests and laboratory tests were performed. Results were compared to existing empirical formulas and he recommended a new design procedure for shallow foundations. Benak cites the need for a theoretical basis to describe the action of loess under load, so to fully substantiate the empirical relations for design.

Tsytovich, Abelev, Sidorghuk, and Polishchuk (8), at the Moscow Civil Engineering Institute, investigated stress and strain at the base of rigid plates on Russian loess. Natural water content and flooded conditions were
studied. Lateral earth coefficients as well as vertical strains were greater in the flooded loess. No analytical comparison was attempted.

Kanakov and Prokhorov (9), at the Gorki Civil Engineering Institute, studied the distribution of contact, vertical, and horizontal stresses under a pseudo-footing on undisturbed Russian loess. A comparison was made to the elastic solution in a semi-infinite mass. Different loess deposits were tested at the natural moisture content, a preliminary wetting, and a pressure wetting of $P = 0.2$ Mpa. Fair agreement was reached with the elastic solution for the case of initial wetting, but only in the upper zone of the soil mass. All other cases yielded marked discrepancies.

Sargand (10) analyzed a single driven pile in Nebraska loess utilizing a displacement finite element method. Fair agreement with measured field data was reached. He simulated the constitutive relation by using a piece-wise linear model. Sargand concluded there is a need for a better constitutive law for loess in order to improve the agreement between the analysis and the actual field data.

1.2) Scope of Present Investigation

The development of digital computers to their present state has provided geotechnical engineers with a new and powerful tool for analysis. The finite element method has gained increasing popularity in geomechanics. One example is the hybrid model developed by Sargand (11) to solve geomechanical problems. A more complete understanding of material behavior is the final
step in utilizing numerical methods to their fullest capacity.

The main objective of this study is to determine an accurate constitutive law, adaptable for numerical calculations, for Nebraska loess at its natural water content. The law will be determined from triaxial tests on undisturbed samples of loess obtained from Dr. Joseph V. Benak at the University of Nebraska in Omaha.
Chapter II

THE LOESS

Loess covers approximately 17 percent of the United States and Europe. Large areas of Russia and Siberia are covered as well as large areas of China. Loess is found in New Zealand and the plain regions of Argentina.

The loess used in this study was obtained from the campus of the University of Nebraska, in Omaha. Undisturbed samples taken from depths of 3 1/2 feet to 16 1/2 feet were tested. The results are used to develop a constitutive law for loess.

In this chapter a description of the engineering properties of the Omaha loess will be given. For a more detailed account of the Omaha - Council Bluffs loess, the reader is referred to Benak. (7)

2.1) Particle Size and Distribution

One hundred percent of the loess particles were found to pass the no. 100 sieve (0.150 mm.). A hydrometer analysis was performed to determine the distribution of these particles.

The results, Figure (2.1), show a uniform particle size ranging from 0.07 mm. to less than 0.002 mm., which is the upper limit for clay particles. The analysis also shows the loess contains about 7 percent clay.
<table>
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<th>Gravel</th>
<th>Sand</th>
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<tr>
<td>Coarse to medium</td>
<td>Fine</td>
</tr>
<tr>
<td>U.S. standard sieve sizes</td>
<td></td>
</tr>
<tr>
<td>No 4</td>
<td>No 10</td>
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Grain diameter, mm

Figure 2.1) Grain Size Distribution Curve
2.2) **General Properties**

The Atterberg limits are a convenient means to express the plastic type properties of a soil. They are defined by limits on different types of behavior, and are expressed as a water content. For a detailed description of Atterberg limits the reader is referred to (1,2).

Benak (7) reported the Atterberg limits for many samples of loess in the Omaha area. The values given here reflect an interpretation of his data.

<table>
<thead>
<tr>
<th>Range in (%)</th>
<th>Liquid Limit</th>
<th>Plasticity Index</th>
<th>Shrinkage Limit</th>
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<td></td>
<td>29 - 41</td>
<td>7 - 18</td>
<td>18 - 25</td>
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The water content of a soil is defined as the ratio of the weight of water to the weight of solids in a soil mass.

\[
\omega \% = \frac{W_w}{W_s} \times 100\%
\]  

(2.1)

The water content values for this study are shown for each triaxial test in Figures (2.3 - 2.8). These values represent the natural water content.

The dry unit weight is defined as

\[
\gamma_d = \frac{W_s}{V_t}
\]

(2.2)

The dry unit weights for this study are also shown for each triaxial test in Figures (2.3 - 2.8).
2.3) **Shearing Strength**

The shearing strength of a frictional material is frequently described by the empirical Mohr-Coulomb law:

\[
\tau = c + \sigma_n \tan \phi
\]  \hspace{1cm} (2.3)

For purely cohesive materials, \( \phi = 0 \) and, \( \tau = c \). Most soils shearing strength properties are expressed as \( c \) and \( \phi \) values.

A series of direct shear tests were performed on the loess samples. The Mohr-Coulomb plot Figure (2.2) is included to complete the discussion on general soil properties.

2.4) **Conventional Triaxial Compression**

The samples of loess were received, sealed in plastic bags, stored in cardboard tubes. They were pressed from a 3-inch Shelby tube which gave them a diameter of 2.9 inches.

The samples were trimmed to a diameter of 2.8 inches and a height of 5.6 inches. All samples were considered to have identical properties.

Stress-strain response was obtained from conventional consolidated-undrained (C - U) triaxial compression tests. In the C - U test, the sample was first consolidated at a confining pressure until primary consolidation was complete. Pore pressure was allowed to escape during this time. For the loess samples this took from 45 minutes to 4 hours.
Figure 2.2) Mohr-Coulomb Failure Envelope From Direct Shear Tests
After primary consolidation was completed, the load was applied using a strain controlled procedure. During this step, pore pressure was no longer allowed to dissipate. The strain rate was selected so that fifteen percent of axial strain was completed in 60 minutes. A complete treatment of the triaxial test procedure is given by Bishop and Henkel. (12)

The test results are shown in Figures (2.3-2.8). Loess, similar to other frictional materials, requires a higher stress for failure with increased confining pressure. The unloading portion of the test indicates that some elastic strains are recovered, but there is obviously substantial plastic deformation.

2.5) Hydrostatic Behavior

A hydrostatic pressure test was performed on the loess samples. In this test the three principal stresses were kept equal to one another by increasing only the confining pressure on the sample. The change in the total volume of the sample was recorded for each increase in pressure. Volumetric strains were then determined.

Unloading in the hydrostatic configuration resulted in the recovery of some elastic strains. These strains were subtracted from the total strains to yield the plastic behavior.

The test result indicating the total strain behavior is shown in Figure (2.9), plastic strain behavior is shown in Figure (4.7).
\( \omega = 23.9\% \)
\( \gamma = 82.8 \) \( \% \)
\( \sigma_3 = -0.14 \) \( \% \)

**Figure 2.3** Stress-Strain Response in C-U Triaxial Compression
$\omega = 23.4 \%$

$\gamma_d = 83.8 \, pc$f

$\sigma_3 = -0.5 \, tsf$

**Figure 2.4** Stress-Strain Response in C-U Triaxial Compression
\[ \omega = 21.2 \, \% \]
\[ \gamma = 8.42 \, \text{p.c.f} \]
\[ a = -1.5 \, \text{p.c.f} \]
Figure 2.6 Stress-Strain Response in C-U Triaxial Compression
\( \rho = 21.8 \% \)
\( \gamma = 85.8 \%\)
\( \sigma = -3.0 \%\)

Figure 2.7) Stress-Strain Response in C-U Triaxial Compression
\[ \omega = 21.9 \% \]
\[ \gamma_d = 84.3 \% \] 
\[ \sigma_3 = -4.0 \text{ psi} \]

Figure 2.8) Stress-Strain Response in C-U Triaxial Compression
Figure 2.9) Hydrostatic Response of Loess (Total Strains)

\[ \omega = 23.5\% \]
\[ \gamma = 91.2 \text{ PCf} \]
Chapter III

BASIC CONCEPTS IN PLASTICITY

A constitutive law describes a material's stress-strain behavior. This behavior can be elastic, plastic, or a combination of both elastic and plastic. An example of an elastic constitutive law is Hooke's law,

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2G \epsilon_{ij} \quad (3.1)$$

Loess does not behave as an elastic medium. This fact is demonstrated by observing Figures (2.3 - 2.8). Upon unloading, only a small portion of strains are recovered. Loess behavior must be then, described by an elasto-plastic model.

In the following sections, some concepts in elasticity and plasticity are reviewed.

3.1) Yield and Failure

Yield will be defined as the place in a material's behavior where exclusively elastic behavior ends. If, as in the case of loess, behavior does not usually start out exclusively elastic, then yield begins with loading. Once yield is reached, the material stays in the yield state until unloading occurs.

Failure is reached in a material when its behavior is no longer acceptable. This unacceptable behavior can take the form of a sudden decrease in
load carrying capability, or a limiting amount of strain. In the analysis of
metals, failure is often taken as being reached when yield occurs.

3.2) Some Well Known Yield Surfaces in Stress Space

The yield state in material behavior can be expressed as a function of
the stresses. The function represents a surface in space with the components
of the stress tensor being in the axes.

A general state of stress, $\sigma_{ij}$, has nine components.

$$\sigma = \begin{bmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_{zz}
\end{bmatrix}$$ (3.2)

Which may be reduced to six independent components, since $\tau_{ij} = \tau_{ji}$. Con-
ceptual difficulties arise trying to work in a six or nine dimensional space.

For the purpose of this manuscript, the principal values of the stress tensor
will be used as three orthogonal axes to form a three dimensional space. This
is known as the Haigh-Westergaard stress space. (13)

If all the states of stress with $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_m$ are
considered, an axis in stress space results. This axis makes equal angles
with the three axes of stress space. It is known as the hydrostatic or mean
stress axis, Figure (3.1a).

Now consider the first invariant of the stress tensor.

$$J_1 = \sigma_1 + \sigma_2 + \sigma_3$$ (3.3)
Figure 3.1a) Hydrostatic Axis in Stress Space

Figure 3.1b) J\textsubscript{1} Plane in Stress Space
\( J_1 \) = constant, represents a plane, in stress space, orthogonal to the hydrostatic axis. The \( J_1 \) plane will be located further out along the hydrostatic axis depending upon the value of this constant, Figure (3.1b).

A plane of particular importance in metals is the case,

\[
J_1 = 0
\]  

(3.4)

This locates the \( J_1 \) plane through the origin of stress space. On this plane the mean stress (\( \sigma_m \)) is equal to zero. This plane is referred to as the \( \pi \)-plane, Figure (3.2a).

Yield in metals generally does not exhibit a dependence on the mean stress. This type of yield surface will not change with respect to the hydrostatic axis and may be pictured two dimensionally in the \( \pi \)-plane.

The Von Mises yield criterion predicts yielding to occur when,

\[
J_{2d} = \frac{1}{3} \sigma_0^2
\]  

(3.5)

The Von Mises criterion plots a circle in the \( \pi \)-plane of radius,

\[
r = \sqrt{2/3} \sigma_0
\]  

(3.6)

The circle can be extended in stress space to form a circular cylinder, symmetric around the hydrostatic axis, Figures (3.2a, b).

The Tresca maximum shear criterion is another well known example of a yield surface. The Tresca criterion in its general form (13) predicts yielding to occur when,
Figure 3.2a) Von Mises Circle and Tresca Hexagon in the $\tau$-Plane

Figure 3.2b) Von Mises Cylinder in Stress Space
Only when the maximum and minimum values of the principal stresses are known in advance does the Tresca criterion reduce to,

\[ \sigma_1 - \sigma_3 = \sigma_0 \] (3.8)

The Tresca criterion can be shown to plot a regular hexagon, around the hydrostatic axis, inscribed in the Von Mises circle, (12), (13), Figures (3.2a, 3.3a).

3.3) Failure Surfaces in Soil Mechanics

The previous section was useful to introduce stress space and some concepts associated with it. However, loess was observed to have some permanent deformation from almost the onset of loading. Yield began with the loading. A stationary yield surface in stress space is of little use for loess. More detail will be given to this thought later. For now, some stationary surfaces useful in soil mechanics will be described. These are failure surfaces.

The mechanism governing the load carrying capability of a frictional soil is the friction between the individual soil grains. The friction will increase with increased mean stress resulting in a stiffer, more load resistant soil mass. In stress space the failure surface should then show dependency on the hydrostatic axis. It also follows that tensile behavior would be limited if friction governs the behavior. In loess, there is a cohesion between the soil grains, Figure (2.2). It is small enough, however, to consider loess
Figure 3.3a) Tresca Hexagon in Stress Space

Figure 3.3b) Mohr-Coulomb Surface in Stress Space
to have a tensile strength of zero.

A well-used failure criterion in soil mechanics is the Mohr-Coulomb law,

\[ \tau = c + \sigma_n \tan \phi \]  \hspace{1cm} (2.3)

In three dimensional stress space the equation becomes (15),

\[
\begin{align*}
\left[ (\sigma_1 - \sigma_2)^2 - (2c \cos \phi + (\sigma_1 + \sigma_2) \sin \phi)^2 \right] 
\times \\
\left[ (\sigma_2 - \sigma_3)^2 - (2c \cos \phi + (\sigma_2 + \sigma_3) \sin \phi)^2 \right] 
\times \\
\left[ (\sigma_3 - \sigma_1)^2 - (2c \cos \phi + (\sigma_1 + \sigma_3) \sin \phi)^2 \right] &= 0
\end{align*}
\]  \hspace{1cm} (3.9)

The surface is an irregular hexagonal cone, increasing in size along the hydrostatic axis, Figure (3.3b). The Mohr-Coulomb criterion works well, but the general form is seen to be complex.

A good approximation to the Mohr-Coulomb criterion was introduced by Drucker and Prager. (16) The Drucker-Prager criterion takes the form,

\( \sqrt{J_{2d}} - \alpha J_1 - \beta = 0 \) \hspace{1cm} (3.10)

This criterion is seen to be an extension of the Von Mises criterion (equation 3.5), and reduces to it for the case, \( \alpha = 0 \).

The Drucker-Prager surface forms a right circular cone, symmetric around the hydrostatic axis, Figure (3.4). The shape of this surface implies an isotropic material with no Bauschinger effect. This will be discussed in more detail in an upcoming section.
Figure 3.4) Drucker-Prager Failure Surface in Stress Space
3.4) **Yield in Soil Mechanics**

The discussion has been, until now, centered on the different forms that a failure surface may take. When a stress path reaches this surface, failure occurs. Material behavior under the failure surface is now discussed.

Much of the analysis for problems in geomechanics is still done by defining a failure surface and assuming linear elastic behavior under that surface. (1, 17) Loess behavior is observed to be far from linear elastic, (Figures 2.3 - 2.8).

This section begins the discussion of a model more suitable for the description of the stress-strain behavior of loess.

Material behavior was defined as yielding when the behavior was no longer exclusively elastic. That is, plastic deformation begins at the yield surface. Mathematically this concept may be described by a function $F$, defining the yield surface,

$$ F = F(\sigma_{ij}) $$  \hspace{1cm} (3.11)

The initial yield state will be defined to occur when,

$$ F(\sigma_{ij}) = 0 $$  \hspace{1cm} (3.12)

Elastic behavior will occur exclusively when,

$$ F(\sigma_{ij}) < 0 $$  \hspace{1cm} (3.13)
The case, $F > 0$, is not defined.

Loess was observed to yield from almost the onset of loading. Plastic deformation was observed until failure, unless unloading occurred. During unloading the elastic strains were recovered and upon reloading approximately the same strains were observed. When the level of stress reached the point of unloading, plastic deformation resumed as if unloading never occurred.

From the above discussion, two assumptions can be made about the yield surface.

1) The yield surface moves in stress space.

2) The state of stress remains on the yield surface unless unloading occurs, and then the yield surface does not move backwards.

For the yield surface to move outward in stress space, the yield function must be increasing. In order to maintain the original definition of yield equation (3.12) one can introduce,

$$X = X(\eta) \quad (3.14)$$

$X$ governs the formation of subsequent yield surfaces. The form of the yield function becomes,

$$f = F(\sigma_{ij}) + X(\eta) \quad (3.15)$$

With yield occurring when $f = 0$.

To examine the description of yield with equation (3.15), consider a body in equilibrium, loaded with a set of surface tractions that satisfy
equilibrium conditions. If \( f \), evaluated at a point on the stress path is,

\[
f < 0
\]

The behavior is elastic up to that point. When the point on the stress path is reached such that,

\[
f = 0
\]

plastic deformation begins, the state of stress is on the initial yield surface. 
If loading continues plastic deformation continues. \( X \) must increase in order to keep, \( f = 0 \). New yield surfaces are formed according to \( X \). These yield surfaces formed after initial yield will be named loading surfaces and their function named the loading function.

The condition that the function \( f \) remains equal to zero, or some other constant, during the formation of loading surfaces implies,

\[
f + df = 0 \quad (3.16a)
\]

or,

\[
df = 0 \quad (3.16b)
\]

This condition assures that the state of stress remains on the loading surface during loading. It is known as the consistancy condition. (18)

A special case of loading must be mentioned to complete the discussion and to insure continuity of the stress path. If the case,

\[
f = 0 \text{ and } dF = 0
\]
referring to equations (3.15, 3.11) occurs, the loading surface remains sta-
tionary but, the stress point may move on the loading surface. Plastic strains
will remain unchanged during this motion, defined as neutral loading.

The following cases cover all possible behavior.

- \(f < 0, \ \delta F < 0\) unloading
- \(f < 0, \ \delta F > 0\) elastic loading
- \(f = 0, \ \delta F = 0\) neutral loading
- \(f = 0, \ \delta F < 0\) onset of unloading
- \(f = 0, \ \delta F > 0\) loading

3.5) The Hardening Rule

The function \(X\) was introduced to control the formation of subsequent
loading surfaces. The formation of these surfaces, as loading progresses, is
the result of the need for an increase in stress to produce an increment of
strain after initial yield. This type of behavior is known as hardening. The
function \(X\) is known as the hardening rule.

There are three general forms used for the hardening rule to model
hardening behavior. The first form assumes the loading surface expands uni-
formly, without distortion. This is known as isotropic hardening and the gen-
eral form may be presented as,

\[ F(\sigma_{ij}) = X(\eta) \]  \hspace{1cm} (3.17)

with \(\eta\) depending on plastic strain history or other controlling parameter.
This form is seen to be in the same form as the loading function when it was
presented, Equation (3.15). Hence, isotropic hardening is the simplest form of hardening, Figure (3.5).

Certain materials exhibit behavior such that, when the yield point is moved out along a stress path in one direction, the yield point for the opposite stress path moves in. This effect is known as the Bauschinger effect. (14)

Isotropic hardening cannot account for the Bauschinger effect. A hardening model introduced by Prager (19, 20), can simulate some Bauschinger behavior. This model is known as kinematic hardening.

The kinematic hardening model assumes the loading surface moves in stress space as a rigid body. The general form will become,

\[ F(\sigma_{ij} - \sigma_{ij}^*) = X \]

(3.18)

With: \( \sigma_{ij}^* \) - co-ordinates of the center of the loading surface that change with plastic deformation.

\( X \) - constant in this model.

The kinematic model and the uniaxial stress-strain behavior it can duplicate are shown in Figure (3.6).

Combining the two above models results in the third general form of hardening, mixed hardening. Mixed hardening can best simulate the Bauschinger effect to different degrees. The general form will become,

\[ F(\sigma_{ij} - \sigma_{ij}^*) = X(\eta) \]

(3.19)

with the variables defined as before. For a more detailed account of hardening
Figure 3.5a) Stress Path Resulting with Isotropic Hardening

Figure 3.5b) Uniaxial Response of Above Isotropic Hardening
Figure 3.6a) Stress Path with Resulting Kinematic Hardening

Figure 3.6b) Uniaxial Response of Above Kinematic Hardening
and the hardening rule the reader is referred to, \((14, 18, 19, 20, 22)\).

In loess, the stress necessary to cause further plastic deformation was accompanied by a decrease in total volume of the sample. This can be explained by considering the individual soil grains and the friction between them. A decrease in total volume results in a stiffer structure capable of withstanding more load. Volumetric plastic strain appears to be the governing parameter for the function \(X\), one can then write,

\[
X = X(\epsilon_V^p)
\]

(3.20)

This relationship will be used in Chapter IV.

It should be mentioned, the above relation implies the use of a work-hardening hypotheses. The hardening behavior depends on the total plastic work done. The alternative being a strain-hardening hypothesis in which the hardening behavior depends on an equivalent plastic strain. \((13)\)

3.6) **The Flow Rule**

So far our main concern has been the loading function and its behavior in stress space. A relationship between stress and strain has yet to be determined.

To do this we will first assume that total strains consist of elastic and plastic parts. Decomposition should then be possible as follows.

\[
\varepsilon_{ij}^T = \varepsilon_{ij}^e + \varepsilon_{ij}^p
\]

(3.21)
Also, assume the elastic strains may be described by Hooke's law,

\[ \epsilon_{ij}^e = \frac{1 + \nu}{E} \sigma_{ij} - \delta_{ij} \frac{\nu}{E} \sigma_{kk} \]  

(3.22)

For the description of the plastic strains such a total strain theory will not work. Plastic strains are seen to be path dependent. An explanation of the meaning of path dependence is given in the appendix of (23).

As a consequence of the path dependence, the plastic strains must be calculated in increments along the stress path, accumulating them as loading progresses. This implies the next assumption, that of linearity between stress and strain increments.

For example,

If, \( d \epsilon_{ij}^{(1)} \) is caused by \( d \sigma_{ij}^{(1)} \)

and, \( d \epsilon_{ij}^{(2)} \) is caused by \( d \sigma_{ij}^{(2)} \)

then, \( (d \epsilon_{ij}^{(1)} + d \epsilon_{ij}^{(2)}) \) is caused by \( (d \sigma_{ij}^{(1)} + d \sigma_{ij}^{(2)}) \)

To determine the plastic strain increments, an analogy between plastic flow of the strains and the flow of an ideal fluid is used.

The velocity \( \mathbf{v}_S \) in idealized fluid flow (24) is, obtained, for any direction \( \mathbf{S} \), from the velocity potential function \( \Phi \) as,

\[ \mathbf{v}_S = \frac{\partial \Phi}{\partial \mathbf{S}} \]  

(3.23)

If the plastic strain increment is considered similar in behavior to a velocity (without the time derivative) and the direction considered in stress space, one
can write,

\[ d\varepsilon_{ij}^P = d\lambda \frac{\partial g}{\partial \sigma_{ij}} \]  

(3.24)

with \( g \) the plastic strain potential function and \( d\lambda \) a proportionality constant which may vary with the loading.

The potential function \( g \) represents the surface in stress space to which the plastic strain increment vector everywhere is normal. In line with the assumption of linearity between increments of stress and strain the potential function will be shown to be the loading function.

Consider a state of stress represented as a point on the loading surface, Figure (3.7). When an increment \( d\sigma_{ij} \) is applied, \( d\varepsilon_{ij}^T \) is observed. In line with equation (3.21),

\[ d\varepsilon_{ij}^T = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p \]  

(3.25)

According to the linearity assumption, \( d\sigma_{ij} \) may also be decomposed into two vectors.

\[ d\sigma_{ij} = d\sigma_{ij}^t + d\sigma_{ij}^n \]  

(3.26)

The decomposition is shown in Figure (3.7).

The tangent portion of \( d\sigma_{ij} \) constitutes the case of neutral loading, since \( f \) must equal zero and that portion of the response is elastic, (See Sec. 3.4). \( d\sigma_{ij}^t \) must therefore move on the loading surface without moving the surface in space. The \( d\sigma_{ij}^t \) vector must be tangent to the loading surface. This
Figure 3.7) Decomposition of $d\sigma_{ij}^T$
sets the direction for $d\sigma_{ij}^n$ as normal to the loading surface.

The elastic strains are described by Hooke's law and line up with the tangent stress vector. The plastic strain increments then must be normal to the loading surface.

Recall the potential function represents the surface on which the plastic strain increment vector everywhere is normal. Hence, $f = g$ and

$$d\epsilon_{ij}^p = d\lambda \frac{\partial f}{\partial \sigma_{ij}}$$  \hspace{1cm} (3.27)

Equation (3.27) is known as the associated flow rule. The potential function is associated with the loading function. If $f \neq g$, then equation (3.24) would be a non-associated flow rule. The assumption of associated flow is used in this manuscript. The case of non-associated flow will not be considered further.

3.7) Theoretical Considerations

The discussion so far has been general in the sense that no limitations have been placed on the loading function with a theoretical basis. The purpose of this section is to clarify this.

The behavior of the loess is first assumed to be stable. Stability is defined by Drucker (21) according to two postulates. They will be presented briefly.

Consider a body in equilibrium in which there is a state of stress $\sigma_{ij}$ and strain $\epsilon_{ij}^T$. Suppose an external agency applies additional surface tractions resulting in an increase in stress $d\sigma_{ij}$ and strain $d\epsilon_{ij}^T$. Now suppose the
external agency removes the tractions and the elastic strains, \( d\varepsilon_{ij}^e \), are released.

For a stable work-hardening material during the application of stresses,

\[
d\sigma_{ij}(d\varepsilon_{ij}^e + d\varepsilon_{ij}^p) > 0 \quad (3.28a)
\]

And over the cycle of application and removal of the stresses,

\[
d\sigma_{ij} d\varepsilon_{ij}^p \geq 0 \quad (3.28b)
\]

Or, energy over and above the elastic energy cannot be extracted from the body.

Stable and unstable behavior is shown for the uniaxial test in (18), Figure (3.8a).

The stability postulates pose a limitation on the shape of the loading surface. Consider a state of stress in a body in equilibrium, \( \sigma_{ij}^1 \). Now apply \( d\sigma_{ij} \) to \( \sigma_{ij}^1 \) to obtain \( \sigma_{ij}^2 \), ie,

\[
\sigma_{ij}^2 - \sigma_{ij}^1 = d\sigma_{ij}
\]

\( d\sigma_{ij} \) will cause \( d\varepsilon_{ij}^T \), part elastic part plastic. When \( d\sigma_{ij} \) is removed, the elastic strains are recovered and from equation (3.28b).

\[
(\sigma_{ij}^2 - \sigma_{ij}^1) d\varepsilon_{ij}^p \geq 0 \quad (3.29)
\]

Equation (3.29) represents the scalar product of two vectors, (Figure 3.8b). Equation (3.29) may be written as,
Figure 3.8a) Stable (a) and Unstable (b) Behavior in the Uniaxial Test

Figure 3.8b) $\theta$-Requirement for Convexity Condition
\[ d \sigma_{ij} d \epsilon_{ij}^p \cos \theta \geq 0 \]  \hspace{1cm} (3.30)

With \( \theta \) shown in Figure (3.8b). For this equation to be positive or zero,

\[ \frac{\pi}{2} \geq \theta \geq - \frac{\pi}{2} \]  \hspace{1cm} (3.31)

Since \( d \epsilon_{ij}^p \) is normal to the loading surface. This requirement on \( \theta \), with normality of \( d \epsilon_{ij}^p \), means the loading surface must be convex in stress space.

The question of uniqueness of solution now arises. Assume two solutions are possible, two different sets of stress and strain satisfy the proposed constitutive law. By applying the theorem of virtual work,

\[ \int_V (\sigma_{ij}^1 - \sigma_{ij}^2) (\epsilon_{ij}^1 - \epsilon_{ij}^2) \, dV = 0 \]  \hspace{1cm} (3.32)

With 1 and 2 corresponding to the two possible solutions.

If the integrand in equation (3.32) can be shown to be positive definite, uniqueness is established. It is shown in (18, 25) that by using a constitutive law derived from equation (3.27) for a stable material with the normality condition for plastic strains, equation (3.32) does not hold, and a unique solution is guaranteed.
Chapter IV

THE CAP MODEL

The constitutive model utilized for the mechanical description of loess is the cap model. It is based on an isotropic loading function and the associated flow rule.

4.1) Previous Studies of Cap Models

The cap model was introduced by Dimaggio and Sandler (26). They fit the model to McCormick Ranch sand. The reported results were in good agreement with experimental data.

Baron, Nelson and Sandler (27) compared a cap model to variable modulii and elastic-perfectly plastic models in ground shock studies. Acceptance agreement with field data was reached for all three models. The cap model was noted to be the only model investigated to satisfy all theoretical postulates, as well as fit all laboratory data, for the material under consideration.

Sandler, Dimaggio and Baladi (28) introduced a general method to fit the cap model to different geological materials. A partially saturated sandy clay was modeled. Very good agreement with experiment was reported.

Cap models were fitted to an artificial soil and Ottawa sand in the verification of a multi-axial testing device by Mould, (15). The effects of different stress paths on cap shape were determined. Reasonable agreement
between experimental data and the cap predictions were reported.

4.2) **General Description of the Cap Model**

If completely isotropic material behavior is assumed, the cap model can be represented in the stress invariant space of $J_1$ and $J_{2d}$ exclusively. The $J_1$ and $J_{2d}$ plane studied, can then be any plane coincident with the hydrostatic axis.

The cap model is represented by a stationary failure surface ($f_1$), and an elliptical yield surface ($f_2$), that translates according to the hardening rule $X(e^p_v)$. The general model is shown in Figure (4.1). The square root is used on $J_{2d}$ to keep the units consistent. The sign convention used will be positive (+) for tension and negative (-) for compression.

It should be emphasized that the model represents a surface in stress space. The surface can be visualized by revolving $f_1$ and $f_2$ around the hydrostatic axis. Figure (4.2)

4.3 **Cap Surfaces to Represent Loess**

Failure of loess, in the triaxial tests, was taken as, 15% axial strain, or as a decrease in load carrying capability with increasing axial strain. The latter condition indicates unstable behavior. The tests were performed at confining pressures in the range, 0.14 tsf to 4.0 tsf. This resulted with a $J_1$ range of -0.4 tsf to -17.6 tsf. In this range of $J_1$ values, a Drucker-Prager failure surface provided the best fit to experimental data.
Figure 4.1) General Cap Model Shown with Identifying Parameters
Figure 4.2) Cap Model in Stress Space
Failure is defined on the surface when,

\[ f_1 = 1 \]  

(4.2)

The choice of using one rather than zero, or another constant in equation (4.2), is one of personal preference.

The loading surface is represented by contours of constant volumetric plastic strain in the $\sqrt{J_{2d}} - J_1$ space. The best fit to the experimental data was obtained from half of an ellipse, with its eccentricity (R) changing as a function of C. $f_2$ takes the form,

\[ f_2 = \frac{1}{(Rb)^2} \left[ R^2 J_{2d} + (J_1 - C)^2 \right] \]  

(4.3)

with,

\[ R = R_1 + R_2 C + R_3 C^2 \]  

(4.4)

The ellipse moves in space according to the hardening rule, $X(\frac{\epsilon^p_v}{W})$, its form being,

\[ X = \frac{1}{D} \ln \left[ 1 + \frac{\epsilon^p_v}{W} \right] + Z \]  

(4.5)

$X$ is related to $f_2$ by,

\[ Rb = X - C \]  

(4.6)
The state of stress is on the loading surface when,

\[ f_2 = 1 \]  \hspace{1cm} (4.7)

In Chapter III, elastic strains were said to occur beneath the loading surface, or if the state of stress moves on the loading surface without setting it in motion. This elastic behavior was determined from the unloading-reloading portions of the triaxial tests. It is best described as,

\[ d \sigma_{ij} = \left( K - \frac{2}{3} G \right) \delta_{ij} d \epsilon^{e}_{kk} + 2G d \epsilon^{e}_{ij} \]  \hspace{1cm} (4.8a)

with,

\[ K = \text{Constant} \]  \hspace{1cm} (4.8b)

\[ G = \frac{E}{2(1 + \nu)} \]  \hspace{1cm} (4.8c)

\[ E = E_1 \left( \sqrt{3J_{2d}} + J_1 \right) + E_2 \]  \hspace{1cm} (4.8d)

\[ \nu = \nu_1 \left( \sqrt{3J_{2d}} + J_1 \right) + \nu_2 \]  \hspace{1cm} (4.8e)

The failure surface and yield surfaces for different values of \( \epsilon^{p}_v \) are shown in Figure (4.3).

4.4) The Cap Equations

The cap model is based on equation (3.27),

\[ d \epsilon^{p}_{ij} = d \lambda \left( \frac{\partial f}{\partial \sigma^{ij}} \right) \]  \hspace{1cm} (3.27)
Figure 4.3) Cap Shape Used for Loess
To find $d\lambda$, first use the consistancy condition equation (3.16) with a general form that can pertain to $f_1$ or $f_2$,

$$f = f(\sigma_{ij}, \varepsilon_{ii}^p)$$  (4.9)

$$df = 0$$  (3.16)

$$df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{\partial \varepsilon_{ii}^p} d\varepsilon_{kk}^p = 0$$  (4.10)

Using the decomposition assumption of equation (3.25),

$$d\varepsilon_{ij}^e = d\varepsilon_{ij}^T - d\varepsilon_{ij}^p$$  (4.11)

Substituting into equation (4.8a),

$$d\sigma_{ij} = (K - \frac{2}{3}G) \delta_{ij} (d\varepsilon_{kk}^T - d\varepsilon_{kk}^p) + 2G(d\varepsilon_{ij}^T - d\varepsilon_{ij}^p)$$  (4.12)

Substituting equation (3.27) into equation (4.12),

$$d\sigma_{ij} = (K - \frac{2}{3}G) \delta_{ij} (d\varepsilon_{kk}^T - d\lambda \frac{\partial f}{\partial \sigma_{kk}}) + 2G(d\varepsilon_{ij}^T - d\varepsilon_{ij}^p)$$  (4.13)

Now substituting equation (4.13) and equation (3.27) into equation (4.10) yields,

$$\frac{\partial f}{\partial \sigma_{ij}} \left[ (K - \frac{2}{3}G) \delta_{ij} (d\varepsilon_{kk}^T - d\lambda \frac{\partial f}{\partial \sigma_{kk}}) + 2G(d\varepsilon_{ij}^T - d\varepsilon_{ij}^p) \right] +$$

$$\left[ \frac{\partial f}{\partial \varepsilon_{ii}^p} d\lambda \frac{\partial f}{\partial \sigma_{kk}} \right] = 0$$  (4.14)

Solving for $d\lambda$, 

Knowing the total strain increments and the required loading function gradients, \( d\lambda \) can be computed. The plastic strain components may then be calculated from equation (3.27).

4.5) Gradients Required in Cap Equations

The gradients required in equation (4.15) will now be calculated. The loading function will be presented first, with respect to the stress \((\sigma_{ij})\),

\[
f_2 = f_2 (J_1, J_{2d})
\]

\[
\frac{\partial f_2}{\partial \sigma_{ij}} = \frac{\partial f_2}{\partial J_1} \frac{\partial J_1}{\partial \sigma_{ij}} + \frac{\partial f_2}{\partial J_{2d}} \frac{\partial J_{2d}}{\partial \sigma_{ij}}
\]

\[
f_2 = \frac{1}{(Rb)^2} \left[ R^2 J_{2d} + (J_1 - C)^2 \right]
\]

\[
\frac{\partial f_2}{\partial J_1} = \frac{2(J_1 - C)}{(Rb)^2}
\]

\[
\frac{\partial f_2}{\partial J_{2d}} = \frac{1}{b^2}
\]
\[ J_1 = \sigma_{ii} \]  
(4.20)

\[ \frac{\partial J_1}{\partial \sigma_{ij}} = \delta_{ij} \]  
(4.21)

\[ J_{2d} = \frac{1}{6} \left( (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right) \]  
(4.22)

\[ \frac{\partial J_{2d}}{\partial \sigma_{ij}} = \sigma_{ij} - \delta_{ij} \frac{\sigma_{kk}}{3} = S_{ij} \]  
(4.23)

Substituting equations (4.18-4.23) into equation (4.17) results in,

\[ \frac{\partial f_2}{\partial \sigma_{ij}} = \frac{1}{(Rb)^2} \left[ 2(J_1 - C) \delta_{ij} + R^2 S_{ij} \right] \]  
(4.24)

The strain gradient of \( f_2 \) is more complicated, with respect to the strain (\( \varepsilon_{ii}^p \)).

\[ f_2 = f_2 (R, b, C) \]  
(4.25)

\[ \frac{\partial f_2}{\partial \varepsilon_{ii}^p} = \frac{\partial f_2}{\partial R} \frac{\partial R}{\partial \varepsilon_{ii}^p} + \frac{\partial f_2}{\partial b} \frac{\partial b}{\partial \varepsilon_{ii}^p} + \frac{\partial f_2}{\partial C} \frac{\partial C}{\partial \varepsilon_{ii}^p} \]  
(4.26)

It will be necessary to obtain an expression for \( b \). From equation (4.1),

\[ f_1 = \sqrt{J_{2d}} + \alpha J_1 + \beta = 1 \]  
(4.1)

When

\[ J_1 = C, \ \sqrt{J_{2d}} = b, \] see Figure (4.1) and,
\[ b = 1 - \alpha C - \beta \]  

(4.27)

Also, from equation (4.6),

\[ b = \frac{X - C}{R} \]  

(4.6a)

Now one is in the position to write equation (4.26) as,

\[ \frac{\partial f_2}{\partial \epsilon_{ii}^p} = \frac{\partial f_2}{\partial R} \frac{\partial C}{\partial C} + \frac{\partial f_2}{\partial \epsilon_{ii}^p} \frac{\partial b}{\partial C} + \frac{\partial f_2}{\partial C} \frac{\partial C}{\partial C} \]  

(4.27)

\[ \frac{\partial f_2}{\partial \epsilon_{ii}^p} = \frac{\partial C}{\partial \epsilon_{ii}^p} \left[ \frac{\partial f_2}{\partial R} \frac{\partial R}{\partial C} + \frac{\partial f_2}{\partial \epsilon_{ii}^p} \frac{\partial b}{\partial C} + \frac{\partial f_2}{\partial C} \right] \]  

(4.28)

\[ \frac{\partial C}{\partial \epsilon_{ii}^p} \]  

must be evaluated. From equation (4.27)

\[ \frac{\partial b}{\partial \epsilon_{ii}^p} = \frac{\partial b}{\partial C} \frac{\partial C}{\partial \epsilon_{ii}^p} = -\alpha \frac{\partial C}{\partial \epsilon_{ii}^p} \]  

(4.29)

From equation (4.6a),

\[ \frac{\partial b}{\partial \epsilon_{ii}^p} = \frac{\partial b}{\partial X} \frac{\partial X}{\partial \epsilon_{ii}^p} + \frac{\partial b}{\partial C} \frac{\partial C}{\partial \epsilon_{ii}^p} + \frac{\partial b}{\partial R} \frac{\partial R}{\partial \epsilon_{ii}^p} \frac{\partial C}{\partial \epsilon_{ii}^p} \]  

Combining with equation (4.29),

\[ -\alpha \frac{\partial C}{\partial \epsilon_{ii}^p} = \frac{1}{R} \frac{\partial X}{\partial \epsilon_{ii}^p} - \frac{1}{R} \frac{\partial C}{\partial \epsilon_{ii}^p} - \left[ \frac{(X-C)(2R_3C+R_2)}{R^2} \right] \frac{\partial C}{\partial \epsilon_{ii}^p} \]  

(4.30)
Solving,

\[
\frac{\partial C}{\partial \varepsilon_{ii}} = \frac{\partial x}{\partial \varepsilon_{ii}} \left[ \frac{R}{R - \alpha R^2 + (X - C) (2R_3C + R_2)} \right] \quad (4.31)
\]

Recall equation (4.5),

\[
X = \frac{1}{D} \ln \left[ 1 + \frac{\varepsilon_v^p}{W} \right] + Z \quad (4.5)
\]

\[
\frac{\partial X}{\partial \varepsilon_{ii}} = \frac{1}{D(W + \varepsilon_v^p)} \quad (4.32)
\]

Substituting equation (4.32) into equation (4.31),

\[
\frac{\partial C}{\partial \varepsilon_{ii}} = \frac{1}{D(W + \varepsilon_v^p)} \left[ \frac{R}{R - \alpha R^2 + (X - C) (2R_3C + R_2)} \right] \quad (4.33)
\]

From equation (4.4),

\[
\frac{\partial R}{\partial C} = 2R_3C + R_2 \quad (4.34)
\]

From equation (4.3),

\[
\frac{\partial f_2}{\partial R} = \frac{-2 (J_1 - C)^2}{R^3 b^2} \quad (4.35)
\]

\[
\frac{\partial f_2}{\partial b} = -\frac{2}{b^3} \left[ J_{2d} + \frac{(J_1 - C)^2}{R^2} \right] \quad (4.36)
\]
\[
\frac{\partial f_2}{\partial C} = \frac{-2(J_1 - C)}{(Rb)^2}
\] (4.37)

And,

\[
\frac{\partial b}{\partial C} = -a
\] (4.38)

Substituting equations (4.33 - 4.38) into equation (4.28) yields,

\[
\frac{\partial f_2}{\partial \varepsilon_{ii}} = \left[ \frac{1}{D(W + \varepsilon_{11}^P)} \right] \left[ \frac{R - \alpha R^2}{(X - C)(2R_3 + R_2)} \right] \frac{-2(R_3 C + R_2)(J_1 - C)^2}{R^3 b^2} + \frac{a^2 b^3}{b^3} \left( J_{2d} + \frac{(J_1 - C)^2}{R^2} \right) - \frac{2(J_1 - C)}{(Rb)^2}
\] (4.39)

If plastic strains are computed from equation (3.27) using \( f_1 \) as the required function, dilatant strains at failure are predicted. These strains were not duplicated in the laboratory. Failure taken when the stress state reached \( f_1 \) provided the best correlation with experimental data. The gradients of \( f_1 \) will be presented next to complete the discussion, although these gradients were not used.

\[
f_1 = f_1 (J_1, J_{2d})
\] (4.40)

\[
\frac{\partial f_1}{\partial \varepsilon_{ij}} = \frac{\partial f_1}{\partial J_1} \frac{\partial J_1}{\partial \varepsilon_{ij}} + \frac{\partial f_1}{\partial J_{2d}} \frac{\partial J_{2d}}{\partial \varepsilon_{ij}}
\] (4.41)
Using equation (4.1),

\[
\frac{\partial f_1}{\partial J_1} = \alpha \tag{4.42}
\]

\[
\frac{\partial f_1}{\partial J_{2d}} = \frac{1}{2 \sqrt{J_{2d}}} \tag{4.43}
\]

Substituting equations (4.21, 4.23, 4.42, 4.43) into equation (4.41) yields,

\[
\frac{\partial f_1}{\partial \sigma_{ij}} = \frac{5_{ij}}{2 \sqrt{J_{2d}}} + \alpha \delta_{ij} \tag{4.44}
\]

The values of the constants used in equations (4.1 - 4.44) for loess are presented below.

\[
\begin{align*}
D &= 0.023 \text{ tsf}^{-1} & \alpha &= 0.158 \\
E_1 &= -31.21 & \beta &= 0.524 \text{ tsf} \\
E_2 &= 296.35 \text{ tsf} & \nu_1 &= 0.0132 \text{ tsf}^{-1} \\
K &= 394.6 \text{ tsf} & \nu_2 &= 0.375 \\
R_1 &= -0.279 & \\
R_2 &= -0.010 \text{ tsf}^{-1} & \\
R_3 &= -0.034 \text{ tsf}^{-2} & \\
W &= 0.15 & \\
Z &= -1.0 \text{ tsf}
\end{align*}
\]
4.6) **Procedure for Use of Cap Equations**

To use the cap equations the applied stress or strain must be divided into increments. The smaller these increments, the smaller the error from the linearity assumption. Since the calculation of a single increment is rather lengthy, the cap equations are best suited for computer implementation. Sandler and Rubin (29) presented an algorithm with a corresponding Fortran subroutine to perform general cap model calculations. The incorporation of the loess equations into this subroutine would be a rather complex task in itself. The majority of the equations used in this manuscript are of different form than Sandler's. Therefore, the cap model calculations were performed by hand. The following procedure is the one found best suited for the hand calculation of the loess cap equations.

1) A stress increment is applied, \( \sigma_{ij} \), strains are calculated assuming elastic behavior from,

\[
d \xi^e_{ij} = \frac{1 + \nu}{E} d\sigma_{ij} - \left[ \frac{\nu}{E} d\sigma_{kk} - \delta_{ij} \right]
\]

2) \( C \) is calculated by iterating,

\[
\frac{X - C}{1 - AC - \beta} = R_1 + R_2 C + R_3 C^2
\]

3) The proper function \( f_1 \) or \( f_2 \) is evaluated,

\[
f_1 \quad \text{if} \quad C > J_1
\]
\[ f_2 \quad \text{if} \quad C \leq J_1 \]

4) If the proper function is \( f_1 \),

If \( f_1 < 1 \) elastic behavior was the correct assumption, go to step 1

If \( f_1 \geq 1 \) failure has occurred, stop

5) If the proper function is \( f_2 \),

If \( f_2 < 1 \), elastic was the correct assumption, go to step 1

If \( f_2 > 1 \), plastic strains must have occurred. Increase value of \( \epsilon^p_v \) until \( f_2 = 1.00 \)

6) Compute the gradients for \( f_2 \), compute \( d\lambda \), compute \( d\epsilon^p_{ij} \), go to step 1.

At any time in the above procedure, if too large an increment is applied, the increment can be removed and subdivided into smaller increments. This may be necessary if the case \( f \gg 1 \) occurs.

The stress and strain increments are accumulated during the procedure. The accumulated values are used to calculate \( J_1, J_{2d}, X, b, \) and \( C \).

4.7) Verification of Loess Cap Model

The loess cap model was utilized to predict stress-strain response for the (C-U) triaxial compression test configuration. The predicted tests consolidation pressures were 1.0 tsf, 2.5 tsf, and 3.5 tsf. One unloading-reloading cycle was included in each prediction.
Two laboratory (C-U) triaxial compression tests were performed at consolidation pressures of 1.0 tsf, and 3.5 tsf. Sample limitations prevented the running of the 2.5 tsf test.

The results are presented in Figures (4.4 - 4.6). Excellent agreement between model and experiment was reached for the 1.0 tsf test. Acceptable agreement is shown by the 3.5 tsf test. The difference between the model and experiment in the 3.5 tsf test is believed to be due to sample irregularity.

4.8) **Fitting the Cap Model to Laboratory Data**

The cap model was shown to be accurate in predicting triaxial stress-strain behavior in loess. It should then be beneficial to potential readers of this manuscript to include a description of the procedure used to fit the cap model to laboratory data.

Mould (15) reported that the stress path chosen to obtain the cap model parameters, affects the shape of the cap. This indicates the isotropic assumption is not completely valid. Mould suggests utilizing as many different stress paths as possible to fit the cap. This will yield an average cap shape, more suitable for solving a general geotechnical problem.

The first step in developing a cap model for a material is to perform as many different laboratory strength tests on it as possible. The complete state of stress and strain, on the sample, must be known at all times during testing. Unloading-reloading cycles should be included in the tests.

Elastic behavior is obtained from the unloading-reloading path.
Figure 4.4) Verification of Loess Cap Model

\[ \omega = 23.5 \%
\]

\[ \gamma_d = 91.2\,\%\]

\[ q_3 = -1.0 + \$ \]

---

\[ - (q - q_3) \]

\[ (t + f) \]

\[ - \varepsilon_f \]
Figure 4.5) Verification of Loess Cap Model

$\sigma_j = -2.5 \, \text{tsf}$
\( \Theta \) EXPERIMENTAL
\( \omega = 21.3 \% \)
\( \gamma_d = 88.7 \, \text{pc} \)
MODEL PREDICTION
\( \sigma_3 = -3.5 \, \text{tsf} \)

\( -(\sigma_1 - \sigma_3) \)
\( (\text{tsf}) \)

**Figure 4.6** Verification of Loess Cap Model
Hooke's law can be applied directly or a non-linear elastic law can be used.

The hardening rule must be evaluated by a test independent of $J_{2d}$. The form of the hardening rule used was introduced by Dimaggio (26) as,

$$-3P = J_1 = X = \frac{1}{D} \ln \left[ 1 + \frac{\xi_v^p}{W} \right] + Z \quad (4.5)$$

It requires a hydrostatic pressure test to evaluate constants $D$, $W$, and $Z$. The equation is represented by an exponential curve in a plot of $P$ vs. $\xi_v^p$. The constant $W$ represents the $\xi_v^p$ value of the asymptote, $D$ controls the concavity, and $Z$ becomes the $P$ intercept.

Performing the hydrostatic pressure test accurately is difficult. For many materials the measured values of volumetric strain are smaller than the experimental corrections. The measured and assumed behavior of loess under hydrostatic pressure is shown in Figure (4.7)

Next the failure surface must be defined. Failure points are plotted in $J_1$, $J_{2d}$ space in the range of values tested. Drucker-Prager, Von Mises or any combination of the two may be used to obtain a best fit to the data. Smooth exponential failure surfaces are described in (26, 27, 28).

Elastic strains are now subtracted off each test. The distribution of volumetric plastic strains on each stress path is then determined. Volumetric plastic strains are accumulated from the origin of the hardening rule, along the $J_1$ axis, and up each stress path. Contours of constant $\xi_v^p$ are sketched in on the stress path diagram. The loading surfaces shape can then be determined.

If the $\xi_v^p$ contours reveal a concave or otherwise unsuitable loading
Figure 4.7) Hydrostatic Response of Loess (Plastic Strains)
surface, the hardening rule should be adjusted and another attempt made. The hardening rule is usually the most uncertain parameter and this should be varied to give a best total fit to the laboratory data.

Once reasonable $\epsilon_0$ contours are drawn, the aspect ratio of the ellipse can be measured. $R$ can be constant or can vary with $C$. The form of $f_2$ given by equation (4.3) should be suitable to describe most geological materials.

4.9) Conclusion and Recommendation for Future Work

The cap model presented satisfies Drucker's stability postulates and guarantees a unique solution. This should provide the user of the model with a substantial level of confidence in the cap model.

The model was shown to be capable of describing the triaxial compression, stress-strain behavior of loess at its natural moisture content. The experimental results correlate with the model predictions fairly well.

Much work is still left to be done in this area. Different stress paths should be used to check the generality of the loess model. Adjustment of the governing parameters could accompany this check.

Loess is known as a collapsible soil, especially when subjected to excess moisture. Its behavior under conditions of varying moisture content should be the next step in investigating the general stress-strain behavior of loess.
BIBLIOGRAPHY


