RINGS CHARACTERIZED BY PROPERTIES OF DIRECT SUMS OF
MODULES AND ON RINGS GENERATED BY UNITS

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Ashish K. Srivastava
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by

ASHISH K. SRIVASTAVA

has been approved for

the Department of Mathematics

and the College of Arts and Sciences by

Surender K. Jain

Distinguished Professor of Mathematics

Benjamin M. Ogles

Dean, College of Arts and Sciences
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RINGS CHARACTERIZED BY PROPERTIES OF DIRECT SUMS OF MODULES
AND ON RINGS GENERATED BY UNITS (90 pp.)

Director of Dissertation: Surender K. Jain

The study of rings over which the direct sums of modules have certain properties is a well recognized topic for research in Ring Theory and Homological Algebra. In this dissertation, the class of rings over which every essential extension of a direct sum of simple right modules is a direct sum of quasi-injective right modules is studied. It is shown that under this condition on a ring $R$, (i) $R$ must be directly finite, (ii) $R$ has bounded index of nilpotence if $R$ is also right non-singular, and (iii) $R$ is right noetherian when $R$ is semi-regular in the sense that $R/J(R)$ is a von-Neumann regular ring.

This dissertation initiates the study of rings having the property that each right ideal is a finite direct sum of quasi-injective right ideals. These rings have been named as right Nakayama-Fuller rings (in short, $NF$-rings). Prime right self-injective right $NF$- rings are shown to be simple artinian. Right artinian right non-singular right $NF$- rings are shown to be upper triangular block matrix rings over rings which are either zero rings or division rings. Examples are provided to show that the $NF$- rings are not left-right symmetric nor they are Morita invariant.

Carl Faith, Cailleau, Megibben and others have studied $\Sigma$-injective module $M$ in
the sense that every direct sum of copies of $M$ is injective. In this dissertation, a new
categorization for an injective module to be $\Sigma$-injective has been provided. This
leads to a new categorization of right noetherian rings which generalizes results of
Cartan-Eilenberg, Bass and Beidar et al.

Zelinsky proved that every element in the ring of linear transformations of a
vector space $V$ over a division ring $D$ is a sum of two units unless $\dim V = 1$ and
$D = \mathbb{Z}_2$. Zelinsky’s result has been extended to include all the previous known
results by proving that every element of a right self-injective ring $R$ is a sum of
two units if and only if $R$ has no factor ring isomorphic to $\mathbb{Z}_2$, thus answering a
long-standing question on a characterization of right self-injective ring generated by
units.

Approved:

Surender K. Jain

Distinguished Professor of Mathematics
Preface

This dissertation is an outcome of the author’s original research in the structure of rings via direct sums of modules having certain properties. The study of different classes of rings in terms of direct sum of modules has a long history. Probably, one of the most surprising results in Ring Theory and Homological algebra is the theorem due to Cartan-Eilenberg and Bass which states that a ring is right noetherian if and only if every direct sum of injective modules is injective. In the same vein, a ring is right q.f.d. (every quotient of the ring, as a right module has finite Goldie dimension) if and only if every direct sum of weakly-injective right modules is weakly-injective. The beauty of these results lies in the fact that they establish relationship between two seemingly unrelated concepts: noetherian property (or finiteness of Goldie dimension) and injectivity (or weak injectivity) via direct sum of modules. Motivated by these results, numerous authors have studied direct sum of modules satisfying certain conditions to characterize rings. Notables among these are Beidar, Cailletou, Clark, Faith, Fuller, Gaursaud, Goodearl, Huynh, Jain, Kurshan, Matlis, Megibben, Muller, Nakayama, Paap, Renault, Valette, Walker and Wisbauer.

Generalizing the result of Cartan-Eilenberg and Bass [4], Beidar and Ke [8] showed that each essential extension of the direct sum of injective modules is again a direct sum of injective modules if and only if the ring is right noetherian. While devoting Chapter 1 for basic definitions and notations, we discuss in Chapter 2
the rings with the property: (*) every essential extension of a direct sum of simple right $R$-modules is a direct sum of quasi-injective right $R$-modules. It is not known whether the rings with the property (*) are necessarily right noetherian. We prove that for a von Neumann regular ring $R$, the ring $R$ is noetherian if and only if it satisfies the property (*). This is later generalized to show that if $R$ is a semiregular ring with the property (*) then $R$ is right noetherian. A right self-injective ring with the property (*) is shown to be Quasi-Frobenius.

Nakayama [51] and Fuller [23] showed that over an artinian serial ring every module is a direct sum of uniserial quasi-injective modules. In Chapter 3, we investigate rings having the property that each right ideal is a finite direct sum of quasi-injective right ideals and call these rings as right Nakayama-Fuller rings (in short, NF-rings). Several classes of these rings have been studied in this chapter. We have shown that a prime right self-injective right NF-ring must be simple artinian. Right artinian right non-singular right NF-rings are upper triangular block matrix rings over rings which are either zero rings or division rings. We have also shown that the NF-ring is not left-right symmetric nor it is Morita invariant.

An injective module $M$ is called Σ-injective if every direct sum of copies of $M$ is injective. It is well-known that Σ-injective modules provide a great deal of information about the structure of rings. For example, a ring is right noetherian if and only if every injective module over such a ring is Σ-injective. If $R$ is an integral
domain, then the injective hull $E(R_R)$ of $R$ is $\Sigma$-injective if and only if $R$ is a right Ore domain [16]. Goursaud-Valette showed that if a ring $R$ admits a faithful $\Sigma$-injective module, then $R$ is a right Goldie ring [25]. In Chapter 4, we provide a new characterization for an injective module to be $\Sigma$-injective.

In Chapter 5, we investigate the rings generated by units. In 1954 Zelinsky [60] proved that every element in the ring of linear transformations of a vector space $V$ over a division ring $D$ is a sum of two units unless $\dim V = 1$ and $D = \mathbb{Z}_2$. As $\text{End}_D(V)$ is a right self-injective ring, Zelinsky’s result gives rise to the following question: For which class of right self-injective rings every element is a sum of two units? In this chapter, we give complete characterization of right self-injective rings in which each element is a sum of two units. The unit sum number of a ring $R$, denoted by $\text{usn}(R)$, is the least integer $n$, if it exists, such that every element of $R$ can be written as sum of exactly $n$ units of $R$. We have also given complete characterization of the unit sum number of right self-injective rings.
Dedicated

to

my family

with

Love, gratitude and deepest admiration
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Chapter 1

Preliminaries

All rings considered in this dissertation have identity element and all modules are right unital unless otherwise stated. The Jacobson radical of a ring $R$ is denoted by $J(R)$ and the maximal right ring of quotients is denoted by $Q_{\text{max}}(R)$. The socle of a right $R$-module $M$ is denoted by $Soc(M)$.

We shall give definitions of basic terms which will be frequently used throughout this dissertation. These may be found in any standard text of Ring Theory and Module Theory (e.g. [2], [45], [46], [59]).

**Definition 1.0.1.** A right $R$-module $E \supseteq M_R$ is called an essential extension of $M$ if every nonzero submodule of $E$ intersects $M$ nontrivially. $E$ is said to be a maximal essential extension of $M$ if no module properly containing $E$ can be an essential extension of $M$. 

If $E \supseteq M$ is an essential extension, we say that $M$ is an essential submodule of $E$, and write $M \subseteq_e E$.

**Definition 1.0.2.** A submodule $L$ of $M$ is called an essential closure of a submodule $N$ of $M$ if it is a maximal essential extension of $N$ in $M$.

**Definition 1.0.3.** A submodule $K$ of $M$ is called a complement if there exists a submodule $U$ of $M$ such that $K$ is maximal with respect to the property that $K \cap U = 0$.

**Definition 1.0.4.** A right $R$-module $M$ is called $N$-injective, if every $R$-homomorphism from a submodule $L$ of $N$ to $M$ can be lifted to an $R$-homomorphism from $N$ to $M$.

A right $R$-module $M$ is called an injective module if $M$ is $N$-injective for every right $R$-module $N$.

By Baer’s criterion, a right $R$-module $M$ is injective if and only if $M$ is $R_R$-injective.

For every right $R$-module $M$, there exists a minimal injective module containing $M$, which is unique upto isomorphism, called the injective hull (or injective envelope) of $M$. The injective hull of $M$ is denoted by $E(M)$. $E(M)$ is indeed a maximal essential extension of $M$.

A ring $R$ is called right self-injective if $R$ is injective as a right $R$-module.

**Definition 1.0.5.** A right $R$-module $M$ is called quasi-injective if $\text{Hom}_R(-, M)$ is
right exact on all short exact sequences of the form $0 \rightarrow K \rightarrow M \rightarrow L \rightarrow 0$.

Johnson and Wong [39] characterized quasi-injective modules as those that are fully invariant under endomorphism of their injective hulls.

In other words, a module $M$ is quasi-injective if it is $M$-injective.

**Definition 1.0.6.** A short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of right $R$-modules is pure if the induced sequence of abelian groups $0 \rightarrow \text{Hom}(E, A) \rightarrow \text{Hom}(E, B) \rightarrow \text{Hom}(E, C) \rightarrow 0$ is exact for every finitely presented right $R$-module $E$.

A right $R$-module $M$ is called pure-injective if the sequence $0 \rightarrow \text{Hom}(C, M) \rightarrow \text{Hom}(B, M) \rightarrow \text{Hom}(A, M) \rightarrow 0$ is exact for every pure exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$.

**Definition 1.0.7.** A module $M$ is called cotorsion if every short exact sequence $0 \rightarrow M \rightarrow E \rightarrow F \rightarrow 0$ with $F$ flat, splits.

By ([26]), if $M$ is a flat cotorsion right $R$-module and $S = \text{End}(M_R)$, then $S/J(S)$ is a regular, right self-injective ring.

**Definition 1.0.8.** A right $R$-module $M$ is called quasi-continuous if it satisfies the following two properties:

(i) Every submodule of $M$ is essential in a direct summand of $M$.

(ii) If $N_1$ and $N_2$ are direct summands of $M$ with $N_1 \cap N_2 = 0$ then $N_1 \oplus N_2$ is also
a direct summand of $M$.

Quasi-continuous modules are also known as $\pi$-injective modules in the literature.

**Definition 1.0.9.** A right $R$-module $M$ is called continuous if it satisfies the following two properties:

(i) Every submodule of $M$ is essential in a direct summand of $M$.

(ii) Every submodule of $M$ isomorphic to a direct summand of $M$ is itself a direct summand of $M$.

In general, we have the following implications.

Injective $\implies$ Quasi-injective $\implies$ Continuous $\implies$ Quasi-continuous

**Definition 1.0.10.** A right $R$-module $M$ is said to satisfy the exchange property if for every right $R$-module $A$ and any two direct sum decompositions $A = M' \oplus N = \oplus_{i \in I} A_i$ with $M' \cong M$, there exist submodules $B_i$ of $A_i$ such that $A = M' \oplus (\oplus_{i \in I} B_i)$.

If this hold only for $|I| < \infty$, then $M$ is said to satisfy the finite exchange property.

**Definition 1.0.11.** A right $R$-module $P$ is called a projective module if $P$ is a direct summand of a free module.

**Definition 1.0.12.** A module $M$ is called directly finite if $M$ is not isomorphic to any proper direct summand of itself.

**Definition 1.0.13.** A ring $R$ is called directly finite if $R$ is directly finite as an $R$-module, equivalently, $xy = 1$ implies $yx = 1$, for all $x, y \in R$. 
Definition 1.0.14. A ring $R$ is called von-Neumann regular if for every $x \in R$ there exists $y \in R$ such that $x = xyx$, equivalently, every principal right (left) ideal of $R$ is generated by an idempotent.

Definition 1.0.15. A regular ring is called abelian if all its idempotents are central.

Definition 1.0.16. An idempotent $e$ in a regular ring $R$ is called abelian (directly finite) idempotent if the ring $eRe$ is abelian (directly finite).

Definition 1.0.17. An idempotent $e$ in a regular right self-injective ring is called faithful idempotent if $0$ is the only central idempotent orthogonal to $e$, that is, $ef = 0$ implies $f = 0$, where $f$ is a central idempotent.

Definition 1.0.18. A regular right self-injective ring is said to be of Type I provided it contains a faithful abelian idempotent.

Definition 1.0.19. A regular right self-injective ring $R$ is said to be of Type II provided $R$ contains a faithful directly finite idempotent but $R$ contains no nonzero abelian idempotents.

Definition 1.0.20. A regular right self-injective ring is of Type III if it contains no nonzero directly finite idempotents.

Definition 1.0.21. A regular right self-injective ring is of (i) Type $I_f$ if $R$ is of Type I and is directly finite, (ii) Type $I_\infty$ if $R$ is of Type I and is purely infinite,
(iii) Type $II_f$ if $R$ is of Type $II$ and is directly finite, (iv) Type $II_\infty$ if $R$ is of Type $II$ and is purely infinite (see [24], pp. 111-115).

**Proposition 1.0.1.** If $R$ is a regular right self-injective ring of Type $I_f$ then $R \simeq \prod R_n$ where each $R_n$ is an $n \times n$ matrix ring over an abelian regular right self-injective ring (see [24], p.120).

**Definition 1.0.22.** The index of a nilpotent element $x$ in a ring $R$ is the least positive integer $n$ such that $x^n = 0$. The index of a two-sided ideal $J$ in $R$ is the supremum of the indices of all nilpotent elements of $J$. If this supremum is finite, then $J$ is said to have bounded index.

**Definition 1.0.23.** A ring $R$ is called a right $q$-ring if every right ideal of $R$ is quasi-injective (see [35], [7]).

**Definition 1.0.24.** A ring $R$ is called a right duo ring if every right ideal of $R$ is two-sided. If a ring is both right duo and left duo then it is called a duo ring.

It is known that a right self-injective right duo ring is a duo ring (see Remark 2.3, page 314, [7]).

**Definition 1.0.25.** A module is called uniserial if its submodules are linearly ordered with respect to inclusion.

**Definition 1.0.26.** A ring $R$ is called a right (left) serial ring if $R_R$ ($R_R$) is a direct
sum of uniserial modules. If a ring is both left as well as right serial ring then it is called a serial ring.

**Definition 1.0.27.** A right $R$-module $M$ is called linearly compact in the discrete topology if any finite solvable system \( \{ x \equiv x_a \pmod{I_a} : a \in A \} \) of congruences is solvable for any index set $A$, where $x_a \in M$ and $I_a$ is a submodule. A ring $R$ is called right linearly compact ring if $R_R$ is linearly compact.

**Definition 1.0.28.** A ring $R$ is called semiregular if $R/J(R)$ is von Neumann regular.

**Definition 1.0.29.** A ring $R$ is said to be a semilocal ring if $R/J(R)$ is semisimple artinian.

In a semilocal ring $R$ every set of orthogonal idempotents is finite, and $R$ has only finitely many simple modules up to isomorphism.

**Definition 1.0.30.** A ring $R$ is called semiperfect if $R/J(R)$ is semisimple artinian and idempotents modulo $J(R)$ can be lifted.

**Definition 1.0.31.** Two rings $R$ and $S$ are said to be Morita equivalent if there exists a category equivalence $F : \text{mod-}R \rightarrow \text{mod-}S$.

**Definition 1.0.32.** A ring theoretic property $\mathcal{P}$ is said to be Morita invariant if, whenever $R$ has the property $\mathcal{P}$, so does every ring Morita equivalent to $R$. 
Definition 1.0.33. Let $X$ be a finite partially ordered set and $R$ any ring. The incidence ring of $X$ with coefficients in $R$, denoted by $I(X, R)$, is the ring of functions \{ $f : X \times X \to R$ such that $f(x, y) = 0$ for each $x \nleq y$ \}; multiplication is given by $(fg)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)$. The ring $I(X, R)$ may also be viewed as the ring of square matrices with entries in $R$, whose rows and columns are indexed by $X$, where the $(x, y)$ entry is 0 whenever $x \nleq y$.

Given a cardinal $\alpha$ and a module $N$, we denote by $N^{(\alpha)}$ the direct sum of $\alpha$ copies of the module $N$.

Definition 1.0.34. A module $N$ is said to be $\Sigma$-injective provided that $N^{(\alpha)}$ is injective for any cardinal $\alpha$.

Definition 1.0.35. A module $M$ is said to be locally finite dimensional if any finitely generated submodule of $M$ has finite uniform (Goldie) dimension.

Definition 1.0.36. A module $M$ is said to be locally q.f.d. if any finitely generated submodule of any homomorphic image of $M$ has finite uniform (Goldie) dimension.

Definition 1.0.37. A ring $R$ is said to be right q.f.d. if every cyclic right $R$-module has finite uniform (Goldie) dimension, that is, every direct sum of submodules of a cyclic module has finite number of terms.

Definition 1.0.38. We say that Goldie dimension of $N$ with respect to $U$, $\text{G dim}_U(N)$, is less than or equal to $n$, if for any independent family \{ $V_j : j \in J$ \} of nonzero
submodules of $N$ such that each $V_j$ is isomorphic to a submodule of $U$, we have that $|\mathcal{J}| \leq n$.

**Definition 1.0.39.** The module $N$ is said to be q.f.d. relative to $U$ if for any factor module $\tilde{N}$ of $N$, $G \dim_U(\tilde{N}) < \infty$.

Note that if $V \subseteq e U$, then $G \dim_U(N) = G \dim_V(N)$ for all $N$.

**Definition 1.0.40.** Given two $R$-modules $M$ and $N$, the trace of $M$ in $N$ is defined as: $\text{Tr}_M(N) = \sum \{ f(M) : f \in \text{Hom}(M_R, N_R) \}$.

**Definition 1.0.41.** A ring $R$ is said to have $n$-sum property, for a positive integer $n$, if its every element can be written as a sum of exactly $n$ units of $R$.

The unit sum number of a ring $R$, denoted by $\text{usn}(R)$, is the least integer $n$, if it exists, such that $R$ has the $n$-sum property.

If $R$ has an element which is not a sum of units then we set $\text{usn}(R)$ to be $\infty$, and if every element of $R$ is a sum of units but $R$ does not have $n$-sum property for any $n$, then we set $\text{usn}(R) = \omega$.

**Definition 1.0.42.** The unit sum number of a module $M$, denoted by $\text{usn}(M)$, is the unit sum number of its endomorphism ring.
Chapter 2

Direct Sum of Simple Modules

It is well-known that right noetherian rings can be characterized in terms of direct sums of injective modules. Several results in this direction exist in literature as obtained by Bass [4], Papp [52], Kurshan [44], Goursaud-Valette [25], Beidar-Ke [8], Beidar-Jain [9], and other authors (see e.g. [46], [59]). On the other hand, there are several characterizations of right noetherian rings in terms of decomposition of injective modules too due to Matlis [47], Papp [52] and Faith-Walker [21].

**Theorem 2.0.1.** (see [46]) Let $R$ be a ring. Then the following statements are equivalent:

1. $R$ is right noetherian;

2. Every direct sum of injective right $R$-modules is injective;
Every countable direct sum of injective hulls of simple right $R$-modules is injective.


**Theorem 2.0.2.** Let $R$ be a ring. Then the following statements are equivalent:

1. $R$ is right noetherian;
2. Every essential extension of a direct sum of a family of injective right $R$-modules is a direct sum of injective modules;
3. Given a family \{\(S_i: i = 1, 2, \ldots\)\} of simple right $R$-modules, each essential extension of \(\bigoplus_{i=1}^{\infty} E(S_i)\) is a direct sum of injective modules;
4. For any family \{\(S_i: i = 1, 2, \ldots\)\} of simple right $R$-modules, there exists an infinite subset $I$ of natural numbers such that \(\bigoplus_{i \in I} E(S_i)\) is an injective module.

Later, in the same direction, Beidar and Jain [9] proved the following.

**Theorem 2.0.3.** A ring $R$ is right noetherian if and only if $R$ is right q.f.d. and every essential extension of a direct sum of injective hulls of simple modules is a direct sum of quasi-injective modules.

In the case of a commutative ring, using topological arguments Beidar and Jain [9] proved that every essential extension of a semisimple $R$-module is a direct sum of quasi-injective modules if and only if $R$ is an artinian principal ideal ring.

Kurshan [44] obtained the following result.
Theorem 2.0.4. A ring \( R \) is right noetherian if and only if \( R \) is right q.f.d. and has the property that every submodule of a cyclic \( R \)-module with simple essential socle is finitely generated.

Faith proved that a right \( R \)-module \( M \) is noetherian if and only if \( M \) is q.f.d. and satisfies the acc on subdirectly irreducible (or colocal) submodules ([18], [19]).

We study the class of rings \( R \) which satisfy the property:

\((*)\) Every essential extension of a direct sum of simple \( R \)-modules is a direct sum of quasi-injective \( R \)-modules.

Throughout this chapter, this property will be referred to as property \((*)\).

We begin with our key lemma.

Lemma 2.0.1. Let \( R \) be a ring which satisfies the property \((*)\). Let \( M \) be a finitely generated \( R \)-module. Then there exists a positive integer \( n \) such that for any simple \( R \)-module \( S \), we have

\[ \text{Gdim}_S(M) \leq n. \]

Proof. Suppose \( \text{Gdim}_S(M) = \infty \) for some simple submodule \( S \) of \( M \). Let \( C \) be a complement of \( Tr_S(M) \) in \( M \). Then \( Tr_S(M) \oplus C \subset_e M \). Factoring out by \( C \) we get that \( Tr_S(M) \) is essentially embeddable in \( M/C \). That means \( Tr_S(M) \cong B/C \) for some \( B/C \subset_e M/C \). This gives that \( Tr_S(M/C) = B/C \subset_e M/C \). Since \( M/C \) is also finitely generated, and \( \text{Gdim}_S(M/C) = \infty \) we may assume, without any loss of
generality, that $\text{Tr}_S(M) \subseteq eM$. Therefore, $\text{Soc}(M) = \text{Tr}_S(M)$ and $\text{Soc}(M) \subset_e M$. Hence, by (*) we get $M = \bigoplus_{k=1}^{\infty} Q_k$, where each $Q_k$ is quasi-injective. Since $M$ is finitely generated, we conclude that $M = \bigoplus_{k=1}^{n} Q_k$. Now, $\text{Soc}(M) = \bigoplus_{k=1}^{n} \text{Soc}(Q_k)$.

Thus, there exists an index $1 \leq k \leq n$ such that $G\dim_S(\text{Soc}(Q_k)) = \infty$. Since $Q_k$ is also finitely generated and quasi-injective, we may, without any loss of generality, further assume that $M$ is quasi-injective. Next, we choose an independent family $\{T_i : i \in I\}$ of submodules of $\text{Soc}(M)$ such that $\bigoplus_{i \in I} T_i = \text{Soc}(M)$. Clearly each $T_i$ is isomorphic to $S$. Let $\widehat{T}_i$ be an essential closure of $T_i$ in $M$, $i \in I$. Since $\{T_i : i \in I\}$ is an independent family of submodules of $M$, so is $\{\widehat{T}_i : i \in I\}$. Since $|I|$ is infinite, $\bigoplus_{i \in I} \widehat{T}_i$ is not finitely generated and so $\bigoplus_{i \in I} \widehat{T}_i \neq M$. Let $L$ be a maximal submodule of $M$ containing $\bigoplus_{i \in I} \widehat{T}_i$. Since $\text{Soc}(M) \subseteq eM$ and $\text{Soc}(M) = \bigoplus_{i \in I} T_i \subset \bigoplus_{i \in I} \widehat{T}_i \subset L \subset M$, we conclude that $\text{Soc}(M)$ is an essential submodule of $L$ and so (*) implies that $L = \bigoplus_{k \in \mathcal{K}} U_k$ where each $U_k$ is quasi-injective. We claim that $|\mathcal{K}| < \infty$.

Assume to the contrary that $|\mathcal{K}| = \infty$. Choose two infinite disjoint subsets $\mathcal{K}_1$ and $\mathcal{K}_2$ of $\mathcal{K}$ such that $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$. Set $V_j = \bigoplus_{k \in \mathcal{K}_j} U_k$, $j = 1, 2$. Then $L = V_1 \oplus V_2$.

For $j = 1, 2$, let $W_j$ be an essential closure of $V_j$ in $M$. Then each $W_j$ is a direct summand of $M$ and is $M$-injective since $M$ is quasi-injective. So $W_1 \oplus W_2$ is also a direct summand and $M$-injective. Now $\text{Soc}(M) \subseteq_e W_1 \oplus W_2 \subset_e M$, and therefore $M = W_1 \oplus W_2$. Hence, $M/L = (W_1 \oplus W_2)/(V_1 \oplus V_2) = W_1/V_1 \times W_2/V_2$. Since $M/L$
is a simple module, either $W_1 = V_1$ or $W_2 = V_2$. Since each $W_i$ is a direct summand of $M$, it is finitely generated. Thus, $W_i \neq V_i$ for $i = 1, 2$, a contradiction. Therefore $|\mathcal{K}| < \infty$.

Let $U_k \subset \hat{U}_k \subset M$, where $\hat{U}_k$ is an essential closure of $U_k$ in $M$ and $k \in \mathcal{K} = \{1, 2, \ldots, m\}$. Then $U_1 \oplus \ldots \oplus U_m \subset \hat{U}_1 \oplus \ldots \oplus \hat{U}_m \subset eM$. But, since $M$ is quasi-injective, $\hat{U}_1 \oplus \ldots \oplus \hat{U}_m$ is a direct summand of $M$. Therefore, $M = \hat{U}_1 \oplus \ldots \oplus \hat{U}_m$.

Next, $M/L = (\oplus_{k \in \mathcal{K}} \hat{U}_k)/(\oplus_{k \in \mathcal{K}} U_k) = \hat{U}_l/\bar{U}_l$ and so $\hat{U}_l/\bar{U}_l$ is a simple module. We claim that $G \dim_S(Soc(U_l)) = \infty$. Else, let $G \dim_S(Soc(U_l)) < \infty$. We now proceed to show that this leads to a contradiction. Let $\pi$ be the canonical projection of $L = \oplus_{k \in \mathcal{K}} U_k$ onto $U_l$. Choose independent family of simple submodules $P_1, P_2, \ldots, P_t$ of $U_l$ such that $P = \oplus_{j=1}^t P_j = Soc(U_l)$. From $Soc(M) \subset eL$, it follows by intersecting both sides with $U_l$ that $P = \oplus_{j=1}^t P_j = Soc(U_l) \subset eU_l$. Then for every $1 \leq j \leq t$, there exists a finite subset $\mathcal{I}_j \subset \mathcal{I}$ such that $P_j \subset \oplus_{s \in \mathcal{I}_j} T_s$. Let $\hat{P}_j$ be an essential closure of $P_j$ in $\oplus_{s \in \mathcal{I}_j} \hat{T}_s$. Since $\oplus_{s \in \mathcal{I}_j} \hat{T}_s$ is an $M$-injective submodule of $L$, so is $\hat{P}_j$. As $P_j \subset U_l$, $\pi|_{P_j}$ is monomorphism and so $\pi(\hat{P}_j) \cong \hat{P}_j$. Recalling that $P = Soc(U_l) \subset eU_l$, we now conclude that $\oplus_{j=1}^t \pi(\hat{P}_j) \subset eU_l$ and so $U_l$ is $M$-injective, a contradiction. Therefore, $G \dim_S(Soc(U_l)) = \infty$. 
Set \( \Lambda = \text{End}(\hat{U}_i), \Delta = \text{End}(\hat{U}_i/U_i) \) and \( \Omega = \text{End}(\text{soc}(\hat{U}_i)) \). Then \( \Lambda U_i = U_i \) and so each element \( \lambda \in \Lambda \) induces an endomorphism of the factor module \( \hat{U}_i/U_i \).

Therefore, there exists a ring homomorphism \( f : \Lambda \to \Delta \). Set \( I = \ker(f) \) and note that \( I \neq \Lambda \). Since \( \Delta \) is a division ring, \( \Lambda/I \) is a domain. Next, \( \lambda(\text{soc}(\hat{U}_i)) \subseteq \text{soc}(\hat{U}_i) \) and so the map \( g : \Lambda \to \Omega \), where \( g(\lambda) = \lambda|_{\text{soc}(\hat{U}_i)} \), is a homomorphism of rings. Since \( \hat{U}_i \) is quasi-injective, \( g \) is a surjective map. Now, \( J(\Lambda) = \{ \alpha \in \Lambda : \ker(\alpha) \subseteq \hat{U}_i \} \) (see [50], p.44). It can be shown that \( \ker(g) = J(\Lambda) \). Also, it is known that idempotents modulo \( J(\Lambda) \) can be lifted to \( \Lambda \) (see [50], p.48).

As shown above in the previous paragraph, \( \text{Gdim}_S(\text{soc}(\hat{U}_i)) = \infty \). So, \( \text{soc}(\hat{U}_i) \) is a direct sum of infinitely many modules each isomorphic to \( S \).

Therefore, there exist two isomorphic submodules \( L_1 \) and \( L_2 \) such that \( \text{soc}(\hat{U}_i) = L_1 \oplus L_2 \). Let \( e_i \in \Omega \) be the canonical projection of \( \text{soc}(\hat{U}_i) \) onto \( L_i, i = 1, 2 \).

Clearly, \( e_1e_2 = 0 = e_2e_1, \)
\( e_1 + e_2 = 1 \). Since \( \frac{\Lambda}{J(\Lambda)} \cong \Omega \) and \( g \) is surjective, there exist \( v_i \in \Lambda \) such that \( g(v_i) = e_i \)
for \( i = 1, 2 \). This gives \( v_i^2 - v_i \in J(\Lambda) \). This implies that there exist \( u_i^2 = u_i \in \Lambda, i = 1, 2 \) such that \( u_i - v_i \in J(\Lambda) = \ker(g) \). This gives that \( g(u_i) = g(v_i) = e_i \).

Since orthogonal idempotents can be lifted to orthogonal idempotents, \( u_1u_2 = 0 = u_2u_1 \).

Also, since \( e_1 + e_2 = 1 \), we have \( u_1 + u_2 = 1 \). By (Proposition 21.21, [45]), there exist \( c \in u_1u_2 \) and \( d \in u_2u_1 \) with \( cd = u_1, dc = u_2 \). Since \( \Lambda/I \) is a domain, either \( u_1 \in I \), or \( u_2 \in I \). In both cases \( c, d \in I \) and so \( u_1 = cd \in I \) and \( u_2 = dc \in I \), forcing \( 1 = u_1 + u_2 \in I \), a contradiction. Therefore, \( \text{Gdim}_S(M) \) can not be infinite.
So, for any finitely generated module $M$ and for any simple submodule $S$ we have $G \dim_S(M) < \infty$.

Next, we proceed to show that there exists a positive integer $n$ such that $G \dim_S(M) \leq n$ for each simple module $S$.

Else, suppose for any integer $n > 1$, there exists a simple submodule $S_n \subseteq M$ such that $G \dim_{S_n}(M) > n$. Then there exists a family of pairwise non-isomorphic simple modules $\{T_i: i = 1, 2, \ldots\}$ such that $G \dim_{T_i}(M) \geq i$. As explained in the beginning of the proof of this lemma we will, without any loss of generality, assume that $\text{Soc}(M) \subseteq eM$. By $(\ast)$ we know that $M = \bigoplus_{k=1}^{\infty} Q_k$ where each $Q_k$ is quasi-injective. Since $M$ is finitely generated, we conclude that $M = \bigoplus_{k=1}^{n} Q_k$. Clearly there exists an index $1 \leq l \leq n$ and an ascending sequence of indexes $i_1, i_2, \ldots, i_j, \ldots$ such that $G \dim_{T_{i_j}}(Q_l) \geq j$. Therefore there exists an independent family of simple submodules $\{S_{pq}: p = 1, 2, \ldots, q = 1, 2, \ldots, p\}$ of $Q_l$ such that $S_{ij} \cong S_{pq}$ if and only if $p = i$. Let $L$ be an essential closure of $\bigoplus_{i,j} S_{ij}$ in $Q_l$. Then $L$ is both a quasi-injective module and a direct summand of $Q_l$. Hence $L$ is a direct summand of $M$ and so $L$ is finitely generated. Let $\hat{S}_{ij}$ be an essential closure of $S_{ij}$ in $L$. Since $S_{ij} \cong S_{ik}$, $\hat{S}_{ij} \cong \hat{S}_{ik}$ for all $1 \leq j, k \leq i$ and $i = 1, 2, \ldots$. We now set $U_k = \bigoplus_{i=k}^{\infty} \hat{S}_{ik}$ and denote by $W_k$ an essential closure of $U_k$ in $L$. Note that $W_k$ is finitely generated. Therefore, $W_k \neq U_k$ for all $k = 1, 2, \ldots$. In particular, there exists a maximal submodule $P_1$ of $W_1$ containing $U_1$. Let $k > 1$ and let $W_k'$ be an essential closure of $\bigoplus_{i=k}^{\infty} \hat{S}_{i1}$ in $W_1$. So,
it follows that $W'_k \cong W_k$ for all $k = 2, 3, \ldots$. Note that $W_1 = (\oplus_{i=1}^{k-1} \hat{S}_{i1}) \oplus W'_k$. Since $U_1 = \oplus_{i=1}^{\infty} \hat{S}_{i1} \subset P_1$, we have $\hat{S}_{i1} \subset P_1$, and so we conclude from modular law that $P_1 = (\oplus_{i=1}^{k-1} \hat{S}_{i1}) \oplus P'_k$ where $P'_k = P_1 \cap W'_k$. Therefore, $W_1 / P_1 \cong W'_k / P'_k$ which implies $P'_k$ is a maximal submodule of $W'_k$ for all $k = 2, 3, \ldots$. Furthermore, as $W'_k \cong W_k$, there exists a maximal submodule $P'_k$ of $W'_k$ such that $W_k / P'_k \cong W'/P'_k \cong W_1 / P_1$.

Set $P = \oplus_{k=1}^{\infty} P_k$ and $S = W_1 / P_1$. Now, $L / P \supset \oplus_{k=1}^{\infty} W_k \cong W_1 / P_1 \times W_2 / P_2 \times \ldots$. This gives that $G \dim_S (L / P) = \infty$, which is not true because, as shown earlier, any finitely generated module has finite Goldie dimension with respect to any simple module. This completes the proof.

\[ \square \]

**Theorem 2.0.5.** Let $R$ be a ring which satisfies the property that for any finitely generated $R$-module $M$, there exists a positive integer $n$ such that $G \dim_S (M) \leq n$ for any simple $R$-module $S$. Let $K, L$ be $R$-modules with $K$ finitely generated and $K \subseteq eL$. Let $\Lambda = \text{End}(L)$. Then $\Lambda$ is directly finite.

**Proof.** Assume that $\Lambda$ is directly infinite. Then, there exist $x, y \in \Lambda$ such that $xy = 1$ and $yx \neq 1$. Set $e_{ij} = y^{i-1}x^{j-1} - y^ix^j$ for all $i, j = 1, 2, \ldots$. It can be easily checked that $\{e_{ij} : i, j = 1, 2, \ldots\}$ is an infinite set of nonzero matrix units. Let $n > 1$. Since $K \subseteq eL$, there exists a nonzero cyclic submodule $U_{n, 1}$ of $K$ such that $U_{n, 1} \subseteq e_{n^2, n^2}L \cap K$. Now we produce cyclic submodules $U_{n, i} \subseteq K$, where $U_{n, i} \cong U_{n, j}$ for all $i, j = 2, 3, \ldots, n$.

We now produce these cyclic submodules $U_{n, i}$ of $K$ by induction. Consider the
module $U_{n,1}$ defined in the previous paragraph. Choose $x_2 \in e_{n^2+1,n^2}U_{n,1} \cap K$. Then $x_2 = e_{n^2+1,n^2}x_1$, where $x_1 \in U_{n,1}$. Denote $U_{n,2} = x_2R$ and redefine $U_{n,1}$ by setting $U_{n,1} = x_1R$. Define the module homomorphism $\varphi : U_{n,1} \to U_{n,2}$ by $\varphi(x) = e_{n^2+1,n^2}x$. Clearly, this is an epimorphism. Suppose $e_{n^2+1,n^2}x = 0$. Then $e_{n^2+n^2}(e_{n^2+1,n^2}x) = 0$, which gives $e_{n^2+n^2}x = 0$ and hence $x = 0$. Therefore $\varphi$ is an isomorphism, and so $U_{n,1} \cong U_{n,2}$. Suppose now that we have defined cyclic submodules $U_{n,1} \cong U_{n,2} \cong \ldots \cong U_{n,j-1}$ in $K$, where $U_{n,i} = x_iR$, $i = 1, 2, \ldots, j - 1$. Next, we choose $x_j$ such that $x_j \in e_{n^2+j-1,n^2+j-2}U_{n,j-1} \cap K$ and write $x_j = e_{n^2+j-1,n^2+j-2}x_{j-1}r_{j-1}$ where $r_{j-1} \in R$. Let $x'_{j-1} = x_{j-1}r_{j-1}$, and set $U_{n,j} = x_jR$. Now redefine $U_{n,j-1} = x'_{j-1}R$ (which is contained in the previously constructed $U_{n,j-1}$). Then $U_{n,j-1} \cong U_{n,j}$ under the isomorphism that sends $x \in U_{n,j-1}$ to $e_{n^2+j-1,n^2+j-2}x$. We redefine preceding $U_{n,1}, U_{n,2}, \ldots, U_{n,j-2}$ accordingly so that they all remain isomorphic to each other and to $U_{n,j-1}$. Note that the family $\{U_{n,i} : n = 2, 3, \ldots, \ i = 1, 2, \ldots, n\}$ is independent. By our construction, $U_{n,i} \cong U_{n,j}$ for all $n = 2, 3, \ldots$ and $1 \leq i, j \leq n$. Therefore, there exist maximal submodules $V_{n,i}$ of $U_{n,i}$, $n = 2, 3, \ldots$ and $1 \leq i \leq n$, such that $U_{n,i}/V_{n,i} \cong U_{n,j}/V_{n,j}$ for all $n, i, j$. Set $V = \oplus_{n,i}V_{n,i}$, $\tilde{K} = K/V$ and $S_n = U_{n,1}/V_{n,1}$. Clearly, we have $\frac{K}{V} \supseteq \frac{\oplus U_{n,i}}{\oplus V_{n,i}} \cong \frac{U_{2,1}}{V_{2,1}} \times \frac{U_{2,2}}{V_{2,2}} \times \frac{U_{3,1}}{V_{3,1}} \times \ldots$. Thus, $G\dim_{S_n}(\tilde{K}) \geq n$ for all $n = 2, 3, \ldots$, which contradicts to the assumption. Therefore, $\Lambda$ is directly finite.

**Theorem 2.0.6.** Let $R$ be a ring which satisfies the property (*). Let $K, L$ be $R$-
modules with $K$ finitely generated and $K \subseteq eL$. Let $\Lambda = \text{End}(L)$. Then $\Lambda$ is directly finite.

**Proof.** This follows from Lemma 2.0.1 and Theorem 2.0.5. \qed

As a special case of the above result, we have the following lemma.

**Lemma 2.0.2.** Let $R$ be a ring which satisfies the property $(\ast)$. Then $Q_{\text{max}}^r(R)$ is directly finite and hence $R$ is directly finite.

**Proof.** It follows directly from the above theorem by taking $K = R$ and $L = Q_{\text{max}}^r(R)$. \qed

**Theorem 2.0.7.** Let $R$ be a ring which satisfies the property that for any finitely generated $R$-module $M$, there exists a positive integer $n$ such that $G\text{dim}_S(M) \leq n$ for any simple $R$-module $S$. Let $K$ be a finitely generated $R$-module with $L = E(K)$ and $\Lambda = \text{End}(L)$. Then the factor ring $\Lambda/J(\Lambda)$ is the direct product of a finite number of matrix rings over abelian regular right self-injective rings.

**Proof.** By ([56]), $\tilde{\Lambda} = \Lambda/J(\Lambda)$ is a von-Neumann regular, right self-injective ring. By Proposition 4, the ring $\Lambda$ is directly finite and hence so is the ring $\tilde{\Lambda}$. In view of (Theorem 10.22, [24]), $\tilde{\Lambda} = A_1 \times A_2$ where $A_1$ is of Type $I_f$ while $A_2$ is of Type $II_f$. Assume that $A_2 \neq 0$. Then by (Proposition 10.28, [24]) there exist idempotents $e_2', e_3', ..., \in A_2$ such that $(A_2)_{A_2} \cong n(e_n'A_2)$. In particular, for $n = 3$,
(A_2)_{A_2} \cong 3(e'_3 A_2), \text{ and so } A_2 = e_1 A_2 \oplus e_2 A_2 \oplus e_3 A_2 \text{ where } e_1, e_2, e_3 \in A_2 \subseteq \Lambda \text{ are nonzero orthogonal idempotents such that their sum is the identity of the ring } A_2. \text{ Clearly, } e_i \Lambda = e_i A_2, \text{ and } e_j A_2 = e_j \Lambda \text{ for all } 1 \leq i, j \leq 3 \text{ and so } e_i \Lambda \cong e_j \Lambda. \text{ Therefore, there exist orthogonal idempotents } u_1, u_2, u_3 \in \Lambda \text{ such that } u_i + J(\Lambda) = e_i \text{ for all } i = 1, 2, 3 \text{ (see Corollary 3.9, [50]). In view of (Proposition 21.21, [45]), } u_i \Lambda \cong u_j \Lambda \text{ for all } 1 \leq i, j \leq 3. \text{ Therefore, there exist nonzero cyclic submodules } U_{2i} \subseteq u_i L \cap K, \text{ } i = 1, 2, \text{ such that } U_{2i} \cong U_{22}. \text{ By (Corollary 10.9, [24]), } e_3 A_2 e_3 = \text{End}(e_3 A_2) \text{ is of Type II}_f \text{ and so as above there exist nonzero orthogonal idempotents } f_1, f_2, f_3, f_4 \in e_3 A_2 e_3 \text{ such that } f_i(e_3 A_2 e_3) \cong f_j(e_3 A_2 e_3) \text{ for all } 1 \leq i, j \leq 4. \text{ As before, we lift these orthogonal idempotents to orthogonal idempotents } v_1, v_2, v_3, v_4 \in u_3 \Lambda u_3 \text{ such that } v_i(u_3 \Lambda u_3) \cong v_j(u_3 \Lambda u_3) \text{ for all } i, j. \text{ By (Proposition 21.20, [45]), there exist } a \in v_i(u_3 \Lambda u_3)v_j \text{ and } b \in v_j(u_3 \Lambda u_3)v_i \text{ such that } v_i = ab \text{ and } v_j = ba. \text{ Then the mapping which sends } v_i x \text{ to } bv_i x, \text{ where } x \in \Lambda, \text{ gives isomorphism of } v_i \Lambda \text{ onto } v_j \Lambda. \text{ So, } v_i \Lambda \cong v_j \Lambda \text{ for all } i, j. \text{ Furthermore, we see that there exist nonzero cyclic submodule } U_{3i} \subseteq v_i L \cap K, \text{ } i = 1, 2, 3 \text{ such that } U_{3i} \cong U_{3j} \text{ for all } 1 \leq i, j \leq 3. \text{ Continuing in this fashion, we construct an independent family } \{U_{ij} : i = 2, 3, ..., ; 1 \leq j \leq i \} \text{ of nonzero cyclic submodules of } K \text{ such that } U_{ij} \cong U_{ik} \text{ for all } 1 \leq j, k \leq i; i = 2, 3, .... \text{ Therefore, there exist maximal submodules } V_{ij} \text{ of } U_{ij}, \text{ } 1 \leq j \leq i \text{ ; } i = 2, 3, ... \text{ such that } U_{ij}/V_{ij} \cong U_{ik}/V_{ik} \text{ for all } i, j, k. \text{ Setting } V = \oplus_{i,j} V_{ij}, K = K/V \text{ and } S_i = U_{i1}/V_{i1}, \text{ we get that } G\dim S_i(\bar{K}) \geq i \text{ for all } i = 2, 3, ... \text{ which contradicts}
to the assumption. Therefore, $A_2 = 0$ and $\tilde{\Lambda}$ is of Type $I_f$. Hence $\tilde{\Lambda} = \prod_{i=1}^{\infty} A_i$ where each $A_i$ is of Type $I_i$, that is, each $A_i$ is an $i \times i$ matrix ring over an abelian regular right self-injective ring (see Theorem 10.24, [24]). Now, we claim that this product must be a finite product. Suppose not, then for any positive integer $n$ there exists an index $m \geq n$ such that $A_m \neq 0$. Now, for any fixed $k$, we can easily write matrix units $\{e_{ij}^k : 1 \leq i, j \leq k\}$ which are $k \times k$ matrices. So we have an infinite family of nonzero matrix units $\{\{e_{ij}^k : 1 \leq i, j \leq k\} : k = 2, 3, \ldots\} \subseteq \tilde{\Lambda}$. Now, since $K \subseteq e L$, there exists a nonzero cyclic submodule $U_{k,1}$ of $K$ such that $U_{k,1} \subseteq e_{1,1}^k L \cap K$ and then starting with $U_{k,1}$, we construct an independent family $\{U_{k,i} : k = 2, 3, \ldots, 1 \leq i \leq k\}$ of cyclic submodules of $K$ such that $U_{k,i} \cong U_{k,j}$ for all $k, i, j$ (exactly as in the proof of Theorem 2.0.6). Therefore, there exist maximal submodules $V_{k,i}$ of $U_{k,i}$, $k = 2, 3, \ldots$ and $1 \leq i \leq k$, such that $U_{k,i}/V_{k,i} \cong U_{k,j}/V_{k,j}$ for all $k, i, j$. Setting $V = \bigoplus_{k,i} V_{k,i}$, $\tilde{K} = K/V$ and $S_k = U_{k,1}/V_{k,1}$, we get that $G\dim S_k(\tilde{K}) \geq k$ for all $k = 2, 3, \ldots$ which contradicts to the assumption. Therefore, there exists a positive integer $n$ such that $\tilde{\Lambda} = \prod_{i=1}^{n} A_i$. Thus, $\Lambda/J(\Lambda)$ is the direct product of a finite number of matrix rings over abelian regular right self-injective rings. \[\square\]

**Theorem 2.0.8.** Let $R$ be a ring which satisfies the property (*). Let $K$ be a finitely generated $R$-module with $L = E(K)$ and $\Lambda = \text{End}(L)$. Then the factor ring $\Lambda/J(\Lambda)$ is the direct product of a finite number of matrix rings over abelian regular right self-injective rings.
Proof. This follows from Lemma 2.0.1 and Theorem 2.0.7. □

As a consequence of the above theorem we have

Lemma 2.0.3. Let $R$ be a right nonsingular ring which satisfies property (*), then $Q_{\text{max}}^r(R)$ (and therefore $R$) has a bounded index of nilpotence.

Proof. In the above theorem by taking $K = R$ and $L = Q_{\text{max}}^r(R)$, we have that $Q_{\text{max}}^r(R)$ is a finite direct product of matrix rings over abelian regular right self-injective rings. Therefore, $Q_{\text{max}}^r(R)$ (and hence $R$) has a bounded index of nilpotence. □

We are interested in investigating the class of rings for which the property (*) implies that the ring is right noetherian. We know that it is true for commutative rings [9]. Under the assumption that the ring $R$ is right q.f.d., it was also shown in [9] that $R$ is right noetherian if and only if $R$ satisfies the property (*). Now, we proceed to show that a von Neumann regular ring is noetherian if and only if it satisfies the property (*). We first consider an abelian regular ring.

Proposition 2.0.2. Let $R$ be an abelian regular ring with the property (*). Then $R$ is right noetherian.

Proof. Assume $R$ is not right noetherian. Then $R$ has an infinite family of orthogonal central idempotents \{e_i : i = 1, 2, ...\}. Set $A_i = e_i R$. This gives an infinite family...
\{A_i : i = 1, 2, \ldots\} of nonzero two-sided ideals of \( R \) such that \( A_iA_j = A_i \cap A_j = 0 \) for all \( i \neq j \). Now, since \( R \) has no nonzero nil ideals, for each index \( i \geq 1 \) there exists a prime ideal \( P_i \) of \( R \) such that \( A_i \not\subseteq P_i \). Since \( R \) is right nonsingular, by Lemma 2.0.3, \( R \) has bonded index of nilpotence. We know that each prime homomorphic image of a von-Neumann regular ring with bounded index of nilpotence is simple artinian. Hence, each \( R/P_i \) is a simple artinian ring. Since \((A_i + P_i)/P_i \) is a nonzero ideal of the simple artinian ring \( R/P_i \), \( (A_i + P_i)/P_i = R/P_i \). As \( A_i/A_i \cap P_i \cong (A_i + P_i)/P_i \), we note that \( A_i/A_i \cap P_i \) is a simple artinian ring. Set \( B = \bigoplus_{i=1}^{\infty} (A_i \cap P_i) \). Then \( (A_i + B)/B \cong A_i/A_i \cap B = A_i/A_i \cap P_i \) and so \( S_i = (A_i + B)/B \) is a simple artinian ring and is an ideal of \( R/B \). Clearly, \( \{S_i : i = 1, 2, \ldots\} \) is an independent family of ideals of \( R/B \). Let \( K/B \) be a complement of \( \bigoplus_{i=1}^{\infty} S_i \) in \( R/B \). Then \( \bigoplus_{i=1}^{\infty} S_i \oplus K/B \subset_e R/B \). Note that \( R/B \) is a duo ring as it is abelian regular. Factoring out by \( K/B \), we obtain that \( \bigoplus_{i=1}^{\infty} S_i \) is essentially embeddable in \( R/K \). Without any loss of generality, we may assume that \( \bigoplus_{i=1}^{\infty} S_i \subset_e R/B \). Then (by [34], Theorem 4.1(2)) we have \( Q = Q_{\text{max}}(R/B) = \prod_{i=1}^{\infty} S_i \). Set \( \tilde{R} = R/B \). Now, by (*) \( \tilde{R} = e_1\tilde{R} \oplus \cdots \oplus e_k\tilde{R} \), where each \( e_i\tilde{R} \) is quasi-injective. Since each \( e_i \) is a central idempotent, each \( e_i\tilde{R} \) is \( e_j\tilde{R} \)-injective. Hence, \( e_1\tilde{R} \oplus \cdots \oplus e_k\tilde{R} \) is quasi-injective. Thus, \( \tilde{R} \) is a right self-injective duo ring. Therefore, \( R/B \) is a \( q \)-ring. Since \( R/B \) is a regular \( q \)-ring, \( R/B = S \oplus T \), where \( S \) is semisimple and \( T \) has zero socle (see Theorem 2.18, [35]). But \( R/B \) has essential socle. So \( T = 0 \) and hence \( R/B \) is semisimple, a contradiction. Therefore,
Proposition 2.0.3. If $R$ is a ring with the property (*) and $ReR = R$ then $eRe$ is also a ring with the property (*).

Proof. We know that if $ReR = R$, then $mod-R$ and $mod-eRe$ are Morita equivalent under the functors given by $\mathcal{F}: mod-R \rightarrow mod-eRe$, $\mathcal{G}: mod-eRe \rightarrow mod-R$ such that for any $M_R$, $\mathcal{F}(M) = Me$ and for any module $T$ over $eRe$, $\mathcal{G}(T) = T \otimes_{eRe} eR$.

Suppose $R$ is a ring with the property (*). Let $N$ be an essential extension of a direct sum of simple $eRe$-modules $\{S_i : i \in I\}$. This gives, $\oplus_i S_i \otimes_{eRe} eR \subset \oplus N \otimes_{eRe} eR$. By Morita equivalence each $S_i \otimes_{eRe} eR$ is a simple $R$-module. Thus, $N \otimes_{eRe} eR$ is an essential extension of a direct sum of simple $R$-modules. But since $R$ is a ring with (*), we have $N \otimes_{eRe} eR = \oplus_i A_i$, where $A_i$’s are quasi-injective $R$-modules. By Morita equivalence we get that each $A_i e$ is quasi-injective as an $eRe$-module. Then $N = NeRe = A_1 e \oplus ... \oplus A_n e$ is a direct sum of quasi-injective $eRe$-modules. Hence $eRe$ is a ring with the property (*). \qed

Proposition 2.0.4. If $M_n(R)$ is a ring with the property (*) then $R$ is also a ring with the property (*).

Proof. We have $R \cong e_{11} M_n(R) e_{11}$ and $M_n(R) e_{11} M_n(R) = M_n(R)$, where $e_{11}$ is a matrix unit. Therefore, the result follows from the Proposition 2.0.3. \qed
**Theorem 2.0.9.** Let $R$ be a regular ring with the property (*). Then $R$ is right noetherian.

*Proof.* We know that $R$ has bounded index of nilpotence hence each primitive factor ring of $R$ is artinian. Therefore, $R \cong M_n(S)$ for some abelian regular ring $S$ (see Theorem 7.14, [24]). By Proposition 2.0.4, $S$ has the property (*). Therefore, by Proposition 2.0.2, $S$ is right noetherian. Hence, $R$ is right noetherian. \[\Box\]

**Lemma 2.0.4.** Let $R$ be a semilocal ring with the property (*). Then $R$ is right noetherian.

*Proof.* We claim that $R_R$ is right q.f.d.

Consider any cyclic module $R/I$. Suppose there exists an infinite direct sum $A_1/I \oplus A_2/I \oplus \ldots \subset R/I$, where $\frac{A_i}{I} = \frac{a_i R + I}{I}$. Let $M_i/I$ be a maximal submodule of $A_i/I$ for each $i$, and set $M/I = \oplus M_i/I$. Then $\frac{A_i}{M_i} \times \frac{A_2}{M_2} \times \ldots \cong \frac{A_1 \oplus A_2 \oplus \ldots}{M_1 \oplus M_2 \oplus \ldots} \subset R/M$.

Each $A_i/M_i$ is a simple module. Set $S_i = A_i/M_i$. Since a semilocal ring has only finitely many simple modules up to isomorphism, $G \dim_S(R/M) = \infty$, for some $i$. This gives a contradiction to Lemma 2.0.1. Therefore, $R$ is right q.f.d. Hence, by (Theorem 2.2, [9]), $R$ is right noetherian. \[\Box\]

**Theorem 2.0.10.** Let $R$ be a semiregular ring with the property (*). Then $R$ is right noetherian.
Proof. $R/J(R)$ is a von Neumann regular ring with the property (*). Therefore, by Theorem 2.0.9, $R/J(R)$ is a right noetherian and hence a semisimple artinian ring. So, $R$ is a semilocal ring. Now, by Lemma 2.0.4, $R$ is right noetherian. \qed

**Corollary 2.0.1.** A right self-injective ring with the property (*) is Quasi-Frobenius.
Chapter 3

Direct Sum of Quasi-injective Right Ideals

Artinian serial rings introduced by Asano and Köthe is an interesting class of rings that are also known as generalized uniserial rings. Nakayama showed that every module over an artinian serial ring is a direct sum of uniserial modules [51]. Warfield showed that every finitely presented module over a serial ring is a direct sum of uniserial modules [58]. Later, Fuller proved that every indecomposable module over an artinian serial ring is quasi-injective [23]. Therefore, every right ideal in an artinian serial ring is a finite direct sum of quasi-injective right ideals. In this chapter we study rings having the property that each right ideal is a finite direct
sum of quasi-injective right ideals. Such rings will be called right Nakayama-Fuller rings (in short, NF-rings). A special class of these rings are those rings in which each right ideal is quasi-injective. Such rings are called right q-rings and these were studied by Beidar et al. [7], Byrd [6], Hill [27], Ivanov ([32], [33]) and Jain et al. [35].

We start with some basic lemmas.

**Lemma 3.0.5.** (Nakayama, [51]) Each module over an artinian serial ring is a direct sum of uniserial modules.

**Lemma 3.0.6.** (Fuller, [23]) Let $R$ be an artinian serial ring. Then each indecomposable right $R$-module is quasi-injective.

**Lemma 3.0.7.** (Jain et al., [35]) Let $R$ be a ring then the following are equivalent:

(i) Each right ideal of $R$ is quasi-injective, that is, $R$ is a right q-ring;

(ii) $R$ is right self-injective and each essential right ideal of $R$ is two-sided;

(iii) $R$ is right self-injective and each right ideal of $R$ is of the form $eT$, where $e$ is an idempotent and $T$ is a two-sided ideal.

**Lemma 3.0.8.** Let $R$ be a right self-injective, right NF-ring then each essential right ideal $E$ of $R$ is of the form $E = e_1T_1 \oplus e_2T_2 \oplus \ldots \oplus e_nT_n$ where $T_1, \ldots, T_n$ are two-sided ideals of $R$ and $e_1, \ldots, e_n$ are orthogonal idempotents with $e_1 + e_2 + \ldots + e_n = 1$.

**Proof.** By hypothesis, $E = E_1 \oplus E_2 \oplus \ldots \oplus E_n$, where $E_i$’s are quasi-injective right
ideals. This gives $R = \hat{E}_1 \oplus \ldots \oplus \hat{E}_n$, where $\hat{E}_i$ denotes injective hull of $E_i$. Write each $\hat{E}_i$ as $e_i R$ with $e_1 + e_2 + \ldots + e_n = 1$. Since each $E_i$ is quasi-injective, we have $e_i R e_i E_i = E_i$. This gives $e_i R E_i = E_i$. Denote each two-sided ideal $RE_i$ as $T_i$. So, we have $E = e_1 T_1 \oplus e_2 T_2 \oplus \ldots \oplus e_n T_n$ where $T_1, \ldots, T_n$ are two-sided ideals of $R$ and $e_1, \ldots, e_n$ are orthogonal idempotents with $e_1 + e_2 + \ldots + e_n = 1$.

**Lemma 3.0.9.** Let $R$ be a right NF-ring with no nontrivial idempotents. Then $R$ is a right $q$-ring.

**Proof.** Let $R$ be a right NF-ring with no nontrivial idempotents. We have, $R = E_1 \oplus \ldots \oplus E_k$ where each $E_i$ is quasi-injective right ideal. Write $E_i = e_i R$. Since $R$ has no nontrivial idempotents, $R = E_1$. Therefore, $R$ is right self-injective ring. Now, let $E$ be any essential right ideal of $R$. Then by Lemma 3.0.8, $E = e_1 T_1 \oplus e_2 T_2 \oplus \ldots \oplus e_n T_n$ where $T_1, \ldots, T_n$ are two-sided ideals of $R$ and $e_1, \ldots, e_n$ are orthogonal idempotents with $e_1 + e_2 + \ldots + e_n = 1$. But, again since there are no nontrivial idempotents, $E = T_1$. Thus, $R$ is a right self-injective ring in which each essential right ideal is two-sided. Therefore, $R$ is a right $q$-ring.

**Corollary 3.0.2.** Let $R$ be a local ring. Then $R$ is a right NF-ring if and only if $R$ is a right $q$-ring if and only if $R$ is a right self-injective duo ring.

**Proof.** This follows from the above lemma and the fact that a local right $q$-ring is a duo ring.
Proposition 3.0.5. If $R$ is a right NF-ring and $ReR = R$ then $eRe$ is also a right NF-ring.

Proof. We have Morita equivalence given by the functors $\mathcal{F}: \text{mod-}R \rightarrow \text{mod-}eRe$, $\mathcal{G}: \text{mod-}eRe \rightarrow \text{mod-}R$ such that for any $M_R$, $\mathcal{F}(M) = Me$ and for any module $T$ over $eRe$, $\mathcal{G}(T) = T \otimes_{eRe} eR$.

Suppose $R$ is a right NF-ring. Let $A$ be any right ideal of $eRe$. Then $AeR \cong A \otimes_{eRe} eR$ and $AeR$ is a right ideal of $R$. Therefore, $AeR = A_1 \oplus ... \oplus A_n$ where $A_i$'s are quasi-injective right ideals in $R$. By Morita equivalence we get that each $A_ie$ is quasi-injective as an $eRe$-module. Then $A = A_1e \oplus ... \oplus A_ne$ is a direct sum of quasi-injective right ideals. Hence $eRe$ is a right NF-ring. \hfill \square

Theorem 3.0.11. If $\mathbb{M}_n(R)$ is a right NF-ring, then $R$ is also a right NF-ring.

Proof. Consider the matrix unit $e_{11}$. Then we have $R \cong e_{11}\mathbb{M}_n(R)e_{11}$ and $\mathbb{M}_n(R)e_{11}\mathbb{M}_n(R) = \mathbb{M}_n(R)$. Therefore, the result follows from above proposition. \hfill \square

Later, we will show that if $R$ is a right NF-ring then $\mathbb{M}_n(R)$ need not be a right NF-ring.

Lemma 3.0.10. A simple ring $R$ is a right NF-ring if and only if $R$ is artinian.

Proof. Let $R$ be a simple, right NF-ring. Then $R = e_1R \oplus ... \oplus e_nR$ where each $e_iR$ is a quasi-injective right ideal. As $R$ is a simple ring, $Re_1R = R$. Therefore,
$R \cong (e_1R)^k/L \cong S$, where $S$ is a direct summand of $(e_1R)^k$ and $L$ is a submodule of $(e_1R)^k$. Because $e_1R$ is quasi-injective, $S_R$ and hence $R_R$ is quasi-injective. Thus, $R$ is right self-injective.

Now, let $E$ be an essential right ideal of $R$. By Lemma 3.0.8, we know that $E = e_1T_1 \oplus ... \oplus e_kT_k$, where $T_1, ..., T_n$ are two-sided ideals of $R$ and $e_1, ..., e_n$ are orthogonal idempotents with $e_1 + e_2 + ... + e_n = 1$. But since $R$ is a simple ring, $T_1 = ... = T_k = R$, so we have $E = R$. Thus, $R$ has no essential right ideal other than $R$ itself. Hence $R$ is artinian.

The converse is trivial.

Next, we give an example of a right self-injective, simple ring which is not a right NF-ring. This ring is also von Neumann regular.

**Example 3.0.1.** Let $S$ be an integral domain which is not a right Öre domain. Consider $R = Q_{\text{max}}^r(S)$. We know that $R$ is a simple, right self-injective, von Neumann regular ring. Clearly, $R$ is not a right NF-ring. Because if $R$ were right NF-ring then by Lemma 3.0.10, $R$ would be artinian and $S$ a right Öre domain, which is not true.

We remark that if $R$ is a von Neumann regular, right self-injective, right NF-ring then $R$ need not be artinian.
Example 3.0.2. Let $S$ be an infinite boolean ring and suppose $R = Q_{\text{max}}^r(S)$. Clearly $R$ is commutative, von Neumann regular, self-injective and hence a NF-ring. But, $R$ is not artinian.

Next, we consider a prime, right self-injective, right NF-ring.

Lemma 3.0.11. A prime, right self-injective, right NF-ring must be von Neumann regular.

Proof. We prove $Z(R_R) = 0$. If possible, let $x$ be a non-zero element in $Z(R_R)$. Then there exists an essential right ideal $E$ of $R$ such that $xE = 0$. Since by Lemma 3.0.8, $E = e_1T_1 \oplus e_2T_2 \oplus ... \oplus e_nT_n$ where $T_1, ..., T_n$ are two-sided ideals of $R$ and $e_1, ..., e_n$ are orthogonal idempotents with $e_1 + e_2 + ... + e_n = 1$ and $R = e_1R \oplus e_2R \oplus ... \oplus e_nR$, we have $xe_iT_i = 0$. But since $R$ is prime, $xe_i = 0$. Therefore, $x = 0$, a contradiction. Hence $R$ is right non-singular and therefore, $R$ must be von Neumann regular. 

Theorem 3.0.12. Let $R$ be a prime, right self-injective ring. Then $R$ is right NF-ring if and only if $R$ is artinian.

Proof. Let $R$ be a prime, right self-injective, right NF-ring. By above lemma $R$ is von Neumann regular. Since the two-sided ideals of $R$ are well-ordered (see [24], Proposition 8.5), $R$ has a unique maximal two-sided ideal, say $A$. Let $N/A$ be a maximal essential right ideal of $R/A$. Then $N$ is an essential maximal right ideal
of $R$. By Lemma 3.0.8, $N = e_1T_1 \oplus e_2T_2 \oplus ... \oplus e_nT_n$, where $T_1, ..., T_n$ are two-sided ideals of $R$ and $e_1, ..., e_n$ are orthogonal idempotents with $e_1 + e_2 + ... + e_n = 1$. Since $R/N$ is a simple module, we have $\frac{R}{N} \cong \frac{e_iR}{e_iT_i}$. Now, since $\frac{e_iR}{e_iT_i}$ is a direct summand of $\frac{R}{T_i}$, it is projective as $R/T_i$-module. As $R/N$ is an $R/T_i$-module and $T_i \subset A \subset N$, it follows that $R/N$ is a simple, projective $R/A$-module. Therefore, $R/A$ is simple ring with non-zero socle. Therefore, $R$ has bounded index of nilpotence (see [24], page 79). Hence $R$ is artinian.

The converse is obvious. \hfill \Box

**Corollary 3.0.3.** The ring of linear transformations, $R = End_F(V)$ is a right NF-ring if and only if vector space $V$ is finite-dimensional.

**Lemma 3.0.12.** Let $R$ be a von Neumann regular ring. Suppose every ring homomorphic image of $R$ is a right self-injective, right NF-ring, then $R$ is semisimple artinian.

**Proof.** Let $P$ be a prime ideal of $R$. Then $R/P$ is a prime, von Neumann regular, right self-injective, right NF-ring. By Theorem 3.0.12, $R/P$ is artinian. Since $R$ is a von Neumann regular, right self-injective ring and each primitive factor ring of $R$ is artinian, we have $R \cong \Pi_{i=1}^k M_{n_i}(S_i)$ where each $S_i$ is an abelian regular right self-injective ring (see [24], Theorem 7.20). Thus, each $S_i$ is a right self-injective duo ring.
Now, let \( C_i \) be any right ideal of \( S_i \). Since \( S_i \) is a duo ring, \( C_i \) is a two-sided ideal and we have \( \frac{M_{n_i}(S_i)}{M_{n_i}(C_i)} \cong M_{n_i}(S_i/C_i) \). Since each \( M_{n_i}(S_i) \) satisfies the property that homomorphic images are right self-injective, clearly \( M_{n_i}(S_i/C_i) \) is right self-injective and hence \( S_i/C_i \) is right self-injective. So, each cyclic right \( S_i \)-module is quasi-injective. Hence \( S_i \) is a finite direct sum of rings each of which is semisimple artinian or a rank 0 duo right linearly compact ring [43]. But, since \( S_i \) is a von Neumann regular ring, we may conclude that it must be semisimple artinian. Hence \( R \) is semisimple artinian.

Next, we show

**Proposition 3.0.6.** Let \( R \) be a right NF-ring and suppose every ring homomorphic image of \( R/J(R) \) is right self-injective, right NF-ring, then \( R \) is semiperfect.

**Proof.** By Lemma 3.0.12, \( R/J(R) \) is semisimple artinian. Now, let composition length of \( R/J(R) \) be \( n \), then \( R_R \) cannot be a direct sum of more than \( n \) submodules, so \( R_R \) is a finite direct sum of indecomposable right ideals, each of which is quasi-injective. Let \( A = eR \) be any indecomposable right summand of \( R_R \). As \( A = eR \) is quasi-injective, its ring of endomorphisms \( eRe \) is a local ring. Hence by [2], \( R \) is semiperfect.

We propose the following question;

Is every right self-injective, right NF-ring a directly finite ring?
In case of a right $q$-ring, we can give the positive answer but, in general, we do not know the answer.

**Proposition 3.0.7.** Every right $q$-ring is directly finite.

*Proof.* Let $R$ be a right $q$-ring. Then $R$ is a right self-injective ring in which each essential right ideal is two-sided. Clearly $R/J(R)$ is also a right self-injective ring.

Let $E/J(R)$ be an essential right ideal of $R/J(R)$. Then $E$ is an essential right ideal of $R$. Since $R$ is a right $q$-ring, $E$ is two-sided. Thus, each essential right ideal of $R/J(R)$ is two-sided and hence $R/J(R)$ is a right $q$-ring. Since $R/J(R)$ is a von Neumann regular right $q$-ring, $R/J(R) = S \oplus T$ where $S$ is semisimple artinian and $T$ has zero socle (see Theorem 2.18, [35]).

Since in a $q$-ring with zero socle, each maximal right ideal is two-sided, $T$ can be easily seen to be directly finite. Hence $R/J(R)$ is directly finite.

Let $a, b \in R$ with $ab = 1$. Then $\tilde{a} \tilde{b} = \tilde{1} = \tilde{b} \tilde{a}$. Hence $ba - 1 \in J(R)$. So, $ba \in U(R)$, therefore $bau = 1$ for some $u \in R$, which gives $au = a$ and hence $ba = 1$.

Therefore, $R$ must be directly finite. \qed

Clearly each right $q$-ring is a right NF-ring. However, there are numerous examples of right NF-rings that are not right $q$-rings.

**Lemma 3.0.13.** An artinian serial ring is a right NF-ring.
Proof. This follows from Lemma 3.0.5 and Lemma 3.0.6.

**Example 3.0.3.** Group ring \( \mathbb{Z}_3[S_3] \) is an artinian serial ring and therefore it is a right NF-ring. Denote \( \sigma = (1\ 2\ 3) \) and \( \tau = (1\ 2) \) in \( S_3 \). Then \( (1 + \sigma + \tau)R \) is an essential right ideal of \( R \) but not two-sided. Therefore, \( \mathbb{Z}_3[S_3] \) is not a right \( q \)-ring.

**Example 3.0.4.** The ring of upper triangular matrices \( \text{UTM}_n(D) \) over a division ring \( D \) is an artinian serial ring and hence a right Nakayama-Fuller ring, but not a \( q \)-ring as \( \text{UTM}_n(D) \) is not self-injective.

**Lemma 3.0.14.** Let \( R \) be an artinian serial ring which is not semisimple. Then \( \mathbb{M}_n(R) \) is a left and right NF-ring but not a left or right \( q \)-ring.

Proof. Clearly, \( \mathbb{M}_n(R) \) is artinian serial. Hence \( \mathbb{M}_n(R) \) is a left and right NF-ring. Since \( R \) is not semisimple, \( \mathbb{M}_n(R) \) is not a left or right \( q \)-ring [35].

**Example 3.0.5.** \( \mathbb{M}_2(\mathbb{Z}/4\mathbb{Z}) \) is a left and right Nakayama-Fuller ring, but not a left or right \( q \)-ring as \( \mathbb{Z}/4\mathbb{Z} \) is not semisimple. Also, to see a direct proof for the fact that \( \mathbb{M}_2(\mathbb{Z}/4\mathbb{Z}) \) is not a right \( q \)-ring, we may observe that \[ \begin{bmatrix} 2\mathbb{Z}/4\mathbb{Z} & 2\mathbb{Z}/4\mathbb{Z} \\ \mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} \end{bmatrix} \] is an essential right ideal of \( \mathbb{M}_2(\mathbb{Z}/4\mathbb{Z}) \) which is not a two-sided ideal.

Next, we give an example of an incidence ring which is a left NF-ring but not a right NF-ring.
Example 3.0.6. Let $X = \{1, 2, 3, 4\}$ be a partially ordered set with $1 < 2 < 3$ and $1 < 2 < 4$. Let $F$ be a field. The incidence ring is given as

$$R = I(X, F) = \begin{bmatrix} F & F & F & F \\ 0 & F & F & F \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{bmatrix}.$$ 

$R$ is a left artinian, left serial ring. Note that $\text{Soc}(R R) = F e_{11} + F e_{12} + F e_{13} + F e_{14}$. It is a folklore that $R R$ is non-singular. But for completeness, we prove this result. Suppose $Z(R R) \neq 0$. Then there exists $z \neq 0 \in \text{soc}(Z(R R))$. Clearly, $z = a_{11} e_{11} + a_{12} e_{12} + a_{13} e_{13} + a_{14} e_{14}$. Then $e_{11} z = z$, which implies $e_{11} \notin \text{l.ann}(z)$ and therefore, $F e_{11} \cap \text{l.ann}(z) = 0$. Hence, $\text{l.ann}(z)$ is not essential in $R R$, a contradiction. So, $R R$ is non-singular.

$R e_{11}$ is simple, so quasi-injective. As $R e_{11}$ is non-singular and quasi-injective, $\text{End}(R e_{11}) \cong \text{End}(E(R e_{11}))$. Hence any endomorphism of $E = E(R e_{11})$ is given as multiplication by some element of $F$. Next, we show that $R e_{22}$, $R e_{33}$, and $R e_{44}$ are quasi-injective. We have $R e_{33} \subseteq E$. So, $\sigma(R e_{33}) \subseteq R e_{33}$, $\forall \sigma \in \text{End}(E)$. Therefore, $R e_{33}$ is quasi-injective. Similarly, $R e_{44}$ is quasi-injective. For every left ideal $C \subseteq R e_{33}$ and $\forall \sigma \in \text{End}(E)$, we have $\sigma(C) \subseteq C$ as $R e_{33}$ is uniserial. Hence $C$ is quasi-injective. Now, $R e_{22} \cong R e_{23} \subseteq R e_{33}$. Therefore, $R e_{22}$ is quasi-injective. So, $R R$ is a direct sum of quasi-injectives.
Next, we show that any indecomposable left ideal is uniserial and quasi-injective.

Suppose there exists a left ideal \( A \neq 0 \) which is indecomposable and not uniform. Choose a left ideal \( A \) of smallest composition length. Let \( \pi_i : R \rightarrow Re_{ii} \) be projection. If for some \( i \), \( \pi_i(A) = Re_{ii} \), we get \( A = A' \oplus B' \) for some \( B' \), \( A' \cong Re_{ii} \).

This gives \( B' = 0 \) and \( A \cong Re_{ii} \), a contradiction as \( A \) is not uniform. Therefore, \( \pi_i(A) \subseteq J(R)e_{ii} \) for all \( i \), which gives \( A \subseteq \begin{bmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = B_2 \oplus B_3 \oplus B_4 \).

Let \( \pi'_i : J(R) \rightarrow B_i \). Suppose \( \pi'_2(A) \neq 0 \). Now, \( B_2 \) is minimal and \( B_2 \cong Re_{11} \) which is projective. So, \( A \cong B_2 \) which is uniform, a contradiction. Hence \( \pi'_2(A) = 0 \).

This gives that \( A \subseteq \begin{bmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = L. \)

Clearly, \( \text{l.ann}(L) = \begin{bmatrix} 0 & 0 & F & F \\ 0 & 0 & F & F \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{bmatrix} \)

Let us denote \( \text{l.ann}(L) \) by \( U \), then \( L \) is a left \( R/U \)-module.
\[
\begin{bmatrix}
F & F \\
0 & F
\end{bmatrix},
\]
which is artinian serial.

As \(A \subseteq L\), we note that \(A\) is a module over \[
\begin{bmatrix}
F & F \\
0 & F
\end{bmatrix},
\]
This implies that \(A\) is uniserial and hence uniform. Let \(B\) be any indecomposable left ideal in \(R\) then \(B\) is uniform. For some \(i\), \(\pi_i(\text{soc}(B)) \neq 0\) and so \(\pi_i\) is one-one on \(\text{soc}(B)\). As \(B\) is uniform, \(\pi_i|_B : B \rightarrow R_{ii}\) is one-one. We have \(B \cong \pi_i(B) \subseteq R_{ii}\). This gives that \(B\) is uniserial and quasi-injective. Now, let \(I\) be any left ideal of \(R\). Then \(I\) is a finite direct sum of indecomposable left ideals. Since we have already proved above that any indecomposable left ideal is quasi-injective, \(I\) is a finite direct sum of quasi-injective left ideals. Thus, \(R\) is a left NF-ring.

Now, we will show that \(R\) is not a right NF-ring. We have \(e_{11}R = F_{e_{11}}+F_{e_{12}}+F_{e_{13}}+F_{e_{14}}\). Note that \(F_{e_{13}}, F_{e_{14}}\) are minimal right ideals. Clearly, \(e_{11}R\) is not uniform and hence not quasi-injective. Therefore, \(R\) is not a right NF-ring.

**Remark 3.0.1.** The property of being NF-ring is not left-right symmetric.

The following example is analogous to the above example.

**Example 3.0.7.** Let \(X = \{1, 2, 3, 4\}\) be a partially ordered set with \(1 < 3 < 4\) and \(2 < 3 < 4\). Let \(F\) be a field. Then the incidence ring is given as
\[ R = I(X, F) = \begin{bmatrix}
F & 0 & F \\
0 & F & F \\
0 & 0 & F \\
0 & 0 & 0 & F
\end{bmatrix}. \]

By similar arguments as above, \( R \) is a right NF-ring but not a left NF-ring.

Let \( R \) be any right NF-ring and \( S = \mathbb{M}_n(R) \). Now \( R \cong e_{11}\mathbb{M}_n(R)e_{11} \) is right NF-ring under the equivalence functor \( \mathcal{F} : \text{mod-}\mathbb{M}_n(R) \to \text{mod-}e_{11}\mathbb{M}_n(R)e_{11} \), where \( \mathcal{F}(M) = Me_{11} \). In particular, \( \mathcal{F}(e_{11}\mathbb{M}_n(R)) = e_{11}\mathbb{M}_n(R)e_{11} \) gives that any right ideal contained in \( e_{11}\mathbb{M}_n(R) \) is a direct sum of finitely many quasi-injective modules. However, the ring \( \mathbb{M}_n(R) \) need not be a right NF-ring.

The example which follows shows that matrix ring over a right NF-ring need not be a right NF-ring.

**Example 3.0.8.** Let \( F \) be any field. Consider the ring \( R = F[x, y] \) where \( x^2 = 0 \) and \( y^2 = 0 \). Then \( R = F + Fx + Fy + Fxy \) is a commutative, local, artinian ring.

Here, \( J(R) = Fx + Fy + Fxy \).

Let \( z = a + bx + cy + dxy \in \text{ann}(J) \). This implies that \((a + bx + cy + dxy)x = 0\), which gives \( ax + cxy = 0 \), and so, \( a = 0 \) and \( c = 0 \). Similarly, \((a + bx + cy + dxy)y = 0\), gives \( a = 0 \) and \( b = 0 \). So, we have \( z = dxy \). Therefore, \( \text{soc}(R) = Fxy \). Thus, \( R \) is a commutative local artinian ring with simple socle and hence a self-injective ring.
([17], Page 217, Exercise 5). Therefore, \( R \) is a right NF-ring.

Let \( K = \text{soc}(R) \). Then \( R/K \) is of length 3 and \( J(R/K) = A \oplus B \) where \( A \) and \( B \) are simples.

Let \( T = M_2(R) \). Then \( T = e_{11}T \oplus e_{22}T \), where \( e_{11}T \cong e_{22}T \).

Note \( e_{11}M_2(R/K)/e_{11}M_2(J(R/K)) \) is simple. Therefore, \( e_{11}M_2(R/K) \) is local as \( M_2(R/K) \cong M_2(R/K)/M_2(J(R/K)) \)-module, and hence as \( T = M_2(R) \)-module. Observe that \( e_{11}M_2(R/K) \) has two minimal right ideals \( e_{11}A + e_{12}A \) and \( e_{11}B + e_{12}B \). By suitably factoring, we then obtain a local \( T \)-module, say, \( N_T \) with \( \text{length} \text{soc}(N_T) = 2 \). If \( S_T \) is a simple module, then \( N \) is embeddable in \( E(N) \cong E(S) \oplus E(S) \cong T \). Thus, \( N \) is a local, indecomposable, non-uniform module embeddable in \( T \) and so \( T \) contains a right ideal which is not quasi-injective. Therefore, \( T \) is not a right NF-ring.

**Remark 3.0.2.** The right NF-ring is not a Morita invariant property. In particular, for \( n > 1 \), the \( n \times n \) matrix ring over a commutative self-injective ring (\( q \)-ring) need not be right or left NF-ring.

Now, we will discuss the right artinian, right non-singular, right NF-rings. First, we state a well-known result.

**Lemma 3.0.15.** Let \( R \) be right artinian and let \( eR \) be a right non-singular indecomposable quasi-injective right ideal, where \( e \) is an idempotent in \( R \). Then \( eRe \) is a division ring.
Lemma 3.0.16. Let $R$ be a right artinian, right non-singular, right NF-ring. Let $e, f$ be any two indecomposable idempotents in $R$ such that $eRf \neq 0$. Let $D = eRe$ and $D' = fRf$.

(i) Then $eRf$ is a one-dimensional left vector space over $D$.

(ii) For any $0 \neq z \in eRf$, there exists embedding $\sigma : D' \rightarrow D$ such that for $fbf \in D'$, $zfbf = \sigma(fbf)z$.

(iii) If $R$ is also right serial, then $\sigma$ is an isomorphism.

Proof. (i) Consider any two non-zero elements $erf$ and $esf$ in $eRf$. As $erfR \cong fR \cong esfR$, we get an $R$-isomorphism $\sigma : erfR \rightarrow esfR$ such that $\sigma(erf) = esf$. Since $eR$ is quasi-injective, we extend $\sigma$ to $\eta : eR \rightarrow eR. Now \eta(e) = eue$. Then $\sigma(erf) = euerf$. Therefore, $esf = euerf$. Hence $eRf$ is one-dimensional left vector space over $D$.

(ii) Now $eRf = Dz$, for some $z \in eRf$. Therefore, given any $fbf \in fRf$, $zfbf = uz$, for some $u \in D$. This defines a monomorphism $\sigma : D' \rightarrow D$ such that $\sigma(fbf) = u$. Also, we have $\sigma(fbf)z = uz = zfbf$.

(iii) Consider any two non-zero elements $erf$ and $esf$ in $eRf$. As $eR$ is uniserial, we may suppose $esfR \subseteq erfR$. Then $esf = erfuf$ for some $fuf \in fRf$. Since $fRf$ is a division ring, it follows that $esfRf = erfRf$. Hence $eRf$ is one-dimensional over $D'$. Therefore, $eRf = Dz = zD'$. From which it is immediate that $\sigma$ is an isomorphism. \qed
Proposition 3.0.8. Let $R$ be a right artinian, right non-singular, right NF-ring.

(i) If $e, f$ are two indecomposable idempotents in $R$ such that $eRf \neq 0$, then for any $0 \neq z \in eRf$, $eRez = zfRf$.

(ii) If $R$ is an indecomposable ring, then $eRe \cong fRf$ for any two indecomposable idempotents $e$ and $f$.

Proof. (i) $Soc(eR) = wgR$ for some indecomposable idempotent $g \in R$, and $w = ewg$. Let $\mu : gR \rightarrow wgR$ be a non-zero $R$-homomorphism given by $\mu(gr) = wgr$, it is monic because $gR$ is uniform. Then $gR \cong wgR$. Thus, $Soc(eR) = eRgR$. By Lemma 3.0.16, $eRew = eRg$. This yields, $Soc(eR) = eRewR = eRgR$. Consider any non-zero $erg$, $esg \in eRg$. As $Soc(eR)$ is simple, we get $eRgR = ergR = esgR$, and so $eRgRg = ergRg = esgRg$, whence it is easy to verify that, $eRg = eRew = wgRg$ for any non-zero $w \in eRg$. So the induced monomorphism $\eta : gRg \rightarrow eRe$ given by $\eta(grg)w = wgrg$ is an isomorphism.

Consider an embedding $\lambda : fR \rightarrow eR$, where $\lambda(fr) = zfr$. Then $\lambda(Soc(fR)) = zfRgR$. Let $v \in fRg$ be such that $zv \neq 0$. Then $w = zv \in eRg$. Now, $v$ and $w$ induce isomorphisms $\sigma_1 : gRg \rightarrow fRf$, and $\sigma_2 : gRg \rightarrow eRe$ respectively. Furthermore, $\sigma : fRf \rightarrow eRe$ is a monomorphism induced by $z$. Because $\sigma_2 = \sigma\sigma_1$, $\sigma$ is also an isomorphism.

(ii) Let $S = \{e_1, \ldots, e_m\}$ be a basic orthogonal set of indecomposable idempotents in $R$. For any two distinct members $e, f \in S$, set $e \leq f$ if $eRf \neq 0$, equivalently,
if $fR$ embeds in $eR$. This is a partial ordering on $S$. As $R$ is indecomposable, $S$ is connected. We may take $e, f \in S$. There exists a path $e = e_1, e_2, \ldots, e_k = f$. By definition, for any $i < k$, $e_i Re_{i+1} \neq 0$. By (i), $e_i Re_i \cong e_{i+1} Re_{i+1}$. Hence, $eRe \cong fRf$.

If we assume that the ring is right serial in addition to being right artinian and right non-singular, we have the following characterization for the ring to be a right NF-ring.

First, recall that by Warfield [58], such a ring is right hereditary.

**Theorem 3.0.13.** Let $R$ be a right artinian, right non-singular, right serial ring. Then the following are equivalent:

(i) $R$ is a right NF-ring;

(ii) For any two indecomposable idempotents $e, f \in R$, if $eRf \neq 0$ then $eRf$ is one-dimensional left vector space over $eRe$ and one-dimensional right vector space over $fRf$.

**Proof.** (i)$\Rightarrow$(ii) follows by Proposition 3.0.8.

Conversely, suppose (ii) holds. Let $e$ be any indecomposable idempotent in $R$. Let $A$ be a non-zero right ideal contained in $eR$. As $R$ is right non-singular and right serial, $A \cong fR$ for some indecomposable idempotent $f \in R$. Let $\sigma : A \rightarrow eR$
be a non-zero $R$-homomorphism. Then $\sigma$ is a monomorphism, and $\sigma(A) \subseteq A$.

For some indecomposable idempotent $g \in R$, $\text{Soc}(eR) = eu g R$ for some $u \in R$, $\sigma(eu g) = eu v g$ for some $v \in R$. By (ii), $eu v g = e w e u g$, for some non-zero $e w e$. Let $\eta$ be the endomorphism of $eR$ given by left multiplication by $e w e$. If $\lambda = \eta|_A$, then $\sigma - \lambda$ being zero on $\text{Soc}(eR)$, is zero. Hence $\eta$ extends $\sigma$. So, $eR$ is quasi-injective. This also proves that $A$ is quasi-injective. In a right artinian right non-singular right serial ring, any right ideal is a finite direct sum of uniserial right ideals. As shown above any uniserial right ideal is quasi-injective. Hence $R$ is a right NF-ring.

**Theorem 3.0.14.** Let $R$ be an indecomposable right artinian, right non-singular, right NF-ring. Then

\[
R \cong \begin{bmatrix}
M_{n_1}(e_1 Re_1) & M_{n_1 \times n_2}(e_1 Re_2) & \cdots & M_{n_1 \times n_k}(e_1 Re_k) \\
0 & M_{n_2}(e_2 Re_2) & \cdots & M_{n_2 \times n_k}(e_2 Re_k) \\
0 & 0 & M_{n_3}(e_3 Re_3) & \cdots & M_{n_3 \times n_k}(e_3 Re_k) \\
& & \ddots & \ddots & \ddots \\
& & & 0 & 0 & \cdots & M_{n_k}(e_k Re_k)
\end{bmatrix}
\]

where $e_i Re_i$ is a division ring, $e_i Re_i \cong e_j Re_j$ for each $1 \leq i, j \leq k$ and $n_1, \ldots, n_k$ are any positive integers. Furthermore, if $e_i Re_i \neq 0$, then it is one-dimensional left vector space over $e_i Re_i$ and one-dimensional right vector space over $e_j Re_j$. 
Proof. Let $R$ be an indecomposable right artinian, right non-singular, right NF-ring. There exists an independent family $\mathcal{F} = \{e_iR : 1 \leq i \leq n\}$ of indecomposable right ideals such that $R = \bigoplus_{i=1}^{n} e_iR$. After renumbering, we may write $R = [e_1R] \oplus [e_2R] \oplus \ldots \oplus [e_kR]$, where for $1 \leq i \leq k$, $[e_iR]$ denotes the direct sum of those $e_jR$ that are isomorphic to $e_iR$. Let $[e_iR]$ be a direct sum of $n_i$ copies of $e_iR$. Consider $1 \leq i < j \leq k$. We arrange in such a way that $\text{length}(e_jR) \leq \text{length}(e_iR)$. Suppose $e_jRe_i \neq 0$, then we have an embedding of $e_iR$ into $e_jR$, hence $\text{length}(e_iR) \leq \text{length}(e_jR)$. But by assumption $\text{length}(e_jR) \leq \text{length}(e_iR)$, so $\text{length}(e_iR) = \text{length}(e_jR)$, we get $e_jR \cong e_iR$, which is contradiction. Hence $e_jRe_i = 0$ for $j > i$.

Thus, we have

$$R \cong \begin{bmatrix}
\mathbb{M}_{n_1}(e_1R_1) & \mathbb{M}_{n_1 \times n_2}(e_1R_2) & \ldots & \mathbb{M}_{n_1 \times n_k}(e_1R_k) \\
0 & \mathbb{M}_{n_2}(e_2R_2) & \ldots & \mathbb{M}_{n_2 \times n_k}(e_2R_k) \\
0 & 0 & \mathbb{M}_{n_3}(e_3R_3) & \ldots & \mathbb{M}_{n_3 \times n_k}(e_3R_k) \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & \ldots & \mathbb{M}_{n_k}(e_kR_k)
\end{bmatrix}$$

We have already seen earlier that each $e_iRe_i$ is a division ring, $e_iRe_i \cong e_jRe_j$ for each $1 \leq i, j \leq k$ and if $e_iRe_j \neq 0$ then it is a one-dimensional left vector space over $e_iRe_i$ as well as a one-dimensional right vector space over $e_jRe_j$ (see Lemma 3.0.15 and Proposition 3.0.8).
Let us call the following property as (*):

\((*)\) : For \(1 \leq i, j \leq k\) with \(i \neq j\) and primitive orthogonal idempotents \(e_i, e_j\), either \(e_iRe_j \neq 0\) or \(e_jRe_i \neq 0\). In other words, for a right non-singular ring either \(e_iR\) is embeddable in \(e_jR\) or \(e_jR\) is embeddable in \(e_iR\).

We remark that (*) holds if \(R\) is an indecomposable right non-singular serial ring.

Under the hypothesis (*) we have the following

**Theorem 3.0.15.** Let \(R\) be an indecomposable right artinian, right non-singular ring with the condition (*). Then \(R\) is a right NF-ring if and only if

\[
R \cong \begin{bmatrix}
M_{n_1}(D) & M_{n_1 \times n_2}(D) & \ldots & M_{n_1 \times n_k}(D) \\
0 & M_{n_2}(D) & \ldots & M_{n_2 \times n_k}(D) \\
0 & 0 & M_{n_3}(D) & \ldots & M_{n_3 \times n_k}(D) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \ldots & M_{n_k}(D)
\end{bmatrix}
\]

where \(D\) is a division ring and \(n_1, \ldots, n_k\) are any positive integers.

**Proof.** Let \(R\) be an indecomposable right artinian, right non-singular, right NF-ring. By the above theorem,
where each $e_i Re_i$ is a division ring and $e_i Re_i \cong e_j Re_j$ for each $1 \leq i, j \leq k$. Furthermore, by condition (*), we have $e_i Re_j \neq 0$ for each $1 \leq i < j \leq k$. Therefore, each $e_i Re_j$ is a one-dimensional left vector space over $e_i Re_i$ as well as a one-dimensional right vector space over $e_j Re_j$. Let us denote division ring $e_i Re_i$ by $D$. Then we have

\[
R \cong \begin{bmatrix}
M_{n_1} (D) & M_{n_1 \times n_2} (D) & \cdots & M_{n_1 \times n_k} (D) \\
0 & M_{n_2} (D) & \cdots & M_{n_2 \times n_k} (D) \\
0 & 0 & M_{n_3} (D) & \cdots & M_{n_3 \times n_k} (D) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & M_{n_k} (D)
\end{bmatrix}
\]

Conversely, suppose that
\[ R \cong \begin{bmatrix}
M_{n_1}(D) & M_{n_1 \times n_2}(D) & \ldots & M_{n_1 \times n_k}(D) \\
0 & M_{n_2}(D) & \ldots & M_{n_2 \times n_k}(D) \\
0 & 0 & M_{n_3}(D) & \ldots & M_{n_3 \times n_k}(D) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \ldots & M_{n_k}(D)
\end{bmatrix} \]

where \( D \) is a division ring and \( n_1, \ldots, n_k \) are positive integers.

Clearly, \( R \) is an indecomposable, right non-singular ring. By Eisenbud-Griffith, we know that \( R \) is an artinian serial ring. Therefore, \( R \) is a right NF-ring. \( \square \)
Chapter 4

Direct Sum of Injective Modules

Carl Faith introduced the concept of $\Sigma$-injectivity in [16] that has been a subject of study by several authors. Following characterizations are well-known for an injective module to be $\Sigma$-injective.

**Theorem 4.0.16.** *(Cailleau [12], Faith [16], and Megibben [48])* For an injective module $M_R$, the following are equivalent:

1. $M$ is $\Sigma$-injective;
2. $M$ is countably $\Sigma$-injective;
3. $R$ satisfies ACC on the set of right ideals $I$ of $R$ that are annihilators of subsets of $M$;
4. $M$ is a direct sum of indecomposable $\Sigma$-injective modules.
We first start with a key lemma.

**Lemma 4.0.17.** Let $M$ be an injective module and suppose there exists an infinite cardinal $\alpha$ such that every essential extension of $M^{(\alpha)}$ is a direct sum of injective modules. Then

(a) Given a direct sum $G = \bigoplus_{i \in \mathbb{N}} M_i$, $M_i \cong M$, and nonzero injective submodules $V_i$ of $M_i$, $i \in \mathbb{N}$, there exists an infinite subset $J \subseteq \mathbb{N}$ and nonzero injective submodules $V_j' \subseteq V_j$, $j \in J$, such that $\bigoplus_{j \in J} V_j'$ is injective.

In particular, if $\{V_i : i \in \mathbb{N}\}$ is an independent family of uniform injective submodules of $M$ then $\bigoplus_{j \in J} V_j$ is injective for some infinite subset $J \subseteq \mathbb{N}$.

(b) $R$ is right q.f.d. relative to $M$.

**Proof.** (a) Set $E = E(G)$. Since $V_i$ is an injective submodule of $M_i$, $M_i = V_i \oplus M_i'$ for some submodule $M_i' \subseteq M_i$. Therefore, $G = (\bigoplus_{i \in \mathbb{N}} V_i) \oplus (\bigoplus_{i \in \mathbb{N}} M_i')$. Let $H$ and $H'$ be essential closures of $\bigoplus_{i \in \mathbb{N}} V_i$ and $\bigoplus_{i \in \mathbb{N}} M_i'$ in $E$, respectively. Clearly, $E = H \oplus H'$. If $\bigoplus_{i \in \mathbb{N}} V_i = H$, then there is nothing to prove.

Consider now the case when $\bigoplus_{i \in \mathbb{N}} V_i \neq H$. Pick $x \in H \setminus \bigoplus_{i \in \mathbb{N}} V_i$. Let $Q$ be a submodule of $H$ maximal with respect to the properties that $\bigoplus_{i \in \mathbb{N}} V_i \subseteq Q$ and $x \notin Q$. Set $P = Q \oplus H'$ and note that $E/P = (H \oplus H')/(Q \oplus H') \cong H/Q$ is a subdirectly irreducible module.

Now, $G \subseteq_e E = H \oplus H'$ and $P = Q \oplus H' \subset_e H \oplus H'$. Therefore, $G \subseteq_e P$. Hence, by our assumption, $P = \bigoplus_{k \in K} W_k$, where each $W_k$ is a nonzero injective
module. Since $P \subsetneq E$ and $P \neq E$, $P$ is not injective and so $|\mathcal{K}| = \infty$.

We claim that for any finite subset $\mathcal{L}$ of $\mathcal{K}$ and for any positive integer $n$ there exists $i > n$ such that $V_i \cap (\oplus_{k \in \mathcal{L}} W_k)$ is not essential in $V_i$.

Suppose the above claim is not true. Then there exists a finite subset $\mathcal{L} \subseteq \mathcal{K}$ and an integer $n \geq 1$ such that $V_i \cap (\oplus_{k \in \mathcal{L}} W_k) \subsetneq V_i$ for all $i > n$. Let $A$ be an essential closure of $\oplus_{i > n} (V_i \cap (\oplus_{k \in \mathcal{L}} W_k))$ in $\oplus_{k \in \mathcal{L}} W_k$ which is injective and so $A$ is also injective.

We have $\oplus_{i \geq n+1} (V_i \cap (\oplus_{k \in \mathcal{L}} W_k) \subsetneq V_i \oplus V_2 \oplus \ldots \oplus V_n \oplus A \subset E = H \oplus H'$. Therefore, $(V_1 \oplus V_2 \oplus \ldots \oplus V_n \oplus i \geq n+1 (V_i \cap (\oplus_{k \in \mathcal{L}} W_k))) \cap H \subset (V_1 \oplus V_2 \oplus \ldots \oplus V_n \oplus A) \cap H \subset H$.

Since $V_i \cap (\oplus_{k \in \mathcal{L}} W_k) \subsetneq V_i$ for all $i > n$, we have $V_1 \oplus V_2 \oplus \ldots \oplus V_n \oplus i \geq n+1 (V_i \cap (\oplus_{k \in \mathcal{L}} W_k)) \subsetneq \oplus_{i \in \mathbb{N}} V_i \subsetneq H$. Setting $B = V_1 \oplus V_2 \oplus \ldots \oplus V_n \oplus A$, we obtain that $B \cap H = (V_1 \oplus V_2 \oplus \ldots \oplus V_n \oplus A) \cap H$ is an essential submodule of $H$. Furthermore, $B \cap H \subsetneq B$. For, if $0 \neq b \in B$ then $b = v_1 + \ldots + v_n + a$ where $v_i \in V_i$, $a \in A$. If $a = 0$, $b \in H$ and we are done. If $a \neq 0$ then there exists $r \in R$ such that $0 \neq ar \in \oplus_{i \geq n+1} (V_i \cap (\oplus_{k \in \mathcal{L}} W_k))$ and hence $ar \in \oplus_{i \in \mathbb{N}} V_i$. Thus $0 \neq br \in H$.

We know that $H'$ is a complement of $H$ in $E$. Now we claim that $H'$ is a complement of $B$ in $E$ as well. To show this we need to show that $H'$ is maximal submodule of $E$ with respect to the property that $H' \cap B = 0$. If $H' \cap B \neq 0$, then since $H \cap B \subsetneq B$, we have $(H' \cap B) \cap (H \cap B) \neq 0$. This implies $(H' \cap H) \cap B \neq 0,$
a contradiction because \( H' \cap H = 0 \). Therefore, \( H' \cap B = 0 \). Now to complete the proof, let \( L \supset H' \) such that \( L \cap B = 0 \). We first show that \( L \cap H = 0 \). If \( L \cap H \neq 0 \), then since \( L \cap H \) is a nonzero submodule of \( H \) and \( H \cap B \subset e \), we have \( (L \cap H) \cap (H \cap B) \neq 0 \), which gives \( (L \cap B) \cap H \neq 0 \), a contradiction. Thus \( L \cap H = 0 \). Since \( H' \subset L \) and \( H' \) is a complement of \( H \) in \( E \), we get \( H' = L \), proving that \( H' \) is a complement of \( B \) in \( E \).

Therefore, \( B \oplus H' \subseteq e \). But since both \( B \) and \( H' \) are injective, \( B \oplus H' \) is injective. Thus \( E = B \oplus H' = (V_1 \oplus V_2 \oplus ... \oplus V_n \oplus A) \oplus H' \subseteq Q + P + H' = P \), a contradiction because \( P \subset E \) and \( P \neq E \).

This proves that for any finite subset \( \mathcal{L} \) of \( \mathcal{K} \) and for any positive integer \( n \) there exists \( i > n \) such that \( V_i \cap (\bigoplus_{k \in \mathcal{L}} W_k) \) is not essential in \( V_i \).

We now proceed by induction to construct a sequence of submodules \( \{W'_{k_j} : j = 1, 2, ..., n, ...\} \) such that each \( W'_{k_j} \) is a nonzero injective submodule of \( W_{k_j} \) isomorphic to a submodule \( V'_{i_j} \) of \( V_{i_j} \), where \( k_1, k_2, ..., k_n, ... \) are distinct elements of \( \mathcal{K} \) and \( 1 \leq i_1 < i_2 < ... < i_n < ... \).

Let \( i_1 \geq 1 \) be arbitrary. Now \( V_{i_1} \subset \bigoplus_{k \in \mathcal{K}} W_k \) implies, there exists a nonzero submodule \( V'_{i_1} \) of \( V_{i_1} \) such that \( V'_{i_1} \) is isomorphic to a submodule \( W'_{k_1} \) of \( W_{k_1} \) for some \( k_1 \in \mathcal{K} \). Clearly, we may choose \( V'_{i_1} \) to be injective submodule of \( V_{i_1} \).

For \( n \geq 1 \), assume that we have a sequence \( \{W'_{k_j} : j = 1, 2, ..., n\} \) with the above stated property. By the fact proved above, there exists \( i_{n+1} > i_n \) such that
\[ X = V_{i+1} \cap (\oplus_{k \in \mathcal{K}_1} W_k) \] is not essential in \( V_{i+1} \), where \( \mathcal{K}_1 = \{k_1, k_2, \ldots, k_n\} \). Let \( X' \) be a complement of \( X \) in \( V_{i+1} \). Then \( X' \cap (\oplus_{k \in \mathcal{K}_1} W_k) = X' \cap X = 0 \). We have \( X' \subset V_{i+1} \subset (\oplus_{k \in \mathcal{K}_1} W_k) \oplus (\oplus_{k \in \mathcal{K}_2} W_k) \), where \( \mathcal{K}_2 = \mathcal{K} \setminus \mathcal{K}_1 \). Let \( \pi : (\oplus_{k \in \mathcal{K}_1} W_k) \oplus (\oplus_{k \in \mathcal{K}_2} W_k) \rightarrow \oplus_{k \in \mathcal{K}_2} W_k \) be the projection. Then for \( \pi |_{X'} : X' \rightarrow \oplus_{k \in \mathcal{K}_2} W_k \),\[ ker(\pi |_{X'}) = X' \cap (\oplus_{k \in \mathcal{K}_1} W_k) = 0. \] Therefore, \( X' \) is isomorphic to some submodule \( C \) of \( \oplus_{k \in \mathcal{K}_2} W_k \). So, the module \( C \) contains a nonzero submodule which is isomorphic to a submodule \( F \) of \( W_{n+1} \) for some \( k_{n+1} \in \mathcal{K}_2 \). Denote by \( W'_{k_{n+1}} \) an essential closure of \( F \) in \( W_{n+1} \). Since \( F \) is isomorphic to a submodule of the injective module \( V_{i+1} \), we conclude that \( W'_{k_{n+1}} \) is isomorphic to a submodule of \( V_{i+1} \) as well. Obviously the family \( \{W'_{k_j} : j = 1, 2, \ldots, n + 1\} \) satisfies the required property. This completes the induction argument.

Now set \( \mathcal{K}' = \{k_1, k_2, \ldots, k_n, \ldots\} \). Choose disjoint subsets \( \mathcal{K}'_1 \) and \( \mathcal{K}'_2 \) of \( \mathcal{K} \) such that \( \mathcal{K} = \mathcal{K}'_1 \cup \mathcal{K}'_2 \) and \( \mathcal{K}' \cap \mathcal{K}'_1 = \{k_1, k_3, \ldots, k_{2n+1}, \ldots\} \). Clearly, \( \mathcal{K}' \cap \mathcal{K}'_2 = \{k_2, k_4, \ldots, k_{2n}, \ldots\} \).

Now we claim that either \( \oplus_{k \in \mathcal{K}_1} W_k \) is injective or \( \oplus_{k \in \mathcal{K}_2} W_k \) is injective.

Set \( V = \oplus_{k \in \mathcal{K}_1} W_k \) and \( W = \oplus_{k \in \mathcal{K}_2} W_k \). We have \( P = V \oplus W \). Let \( \hat{V} \) and \( \hat{W} \) be essential closures of \( V \) and \( W \) respectively in \( E \). Clearly, both \( \hat{V} \) and \( \hat{W} \) are injective. Now, \( P = V \oplus W \subset \hat{V} \oplus \hat{W} \subset E \). Because \( P \subset E \), we obtain \( E = \hat{V} \oplus \hat{W} \). Therefore, \( E/P = (\hat{V} \oplus \hat{W})/(V \oplus W) \cong (\hat{V}/V) \times (\hat{W}/W) \). Since \( E/P \) is shown to be subdirectly irreducible in the beginning of the proof, we have either \( V = \hat{V} \) or
Thus, we may assume, without loss of generality, that the module $\bigoplus_{k \in \mathcal{K}}^\infty W_k$ is injective. Since $\bigoplus_{n=0}^\infty W_{k2n+1}$ is a direct summand of $\bigoplus_{k \in \mathcal{K}}^\infty W_k$, we get that $\bigoplus_{n=0}^\infty W_{k2n+1}$ is injective. Recalling that $\bigoplus_{n=0}^\infty V_{2n+1} \cong \bigoplus_{n=0}^\infty W_{k2n+1}$, we conclude that $\bigoplus_{n=0}^\infty V_{2n+1}$ is an injective module. This completes the proof.

(b) Assume to the contrary that there exists a cyclic right $R$-module $C$ with $G\dim_M(C) = \infty$. Then $C$ has an independent family $\{ C_i : i = 1, 2, \ldots \}$ of nonzero submodules such that each $C_i$ is isomorphic to a submodule of $M$. Set $D_i$ to be closure of $C_i$ in $M$. Then $\{ D_i : i = 1, 2, \ldots \}$ is a family of injective submodules of $M$. Therefore by (a), there exists an infinite subset $J \subseteq \{ 1, 2, \ldots \}$ and nonzero injective submodules $D_j' \subseteq D_j$, $j \in J$, such that $\bigoplus_{j \in J} D_j'$ is injective. Set $C_j' = C_j \cap D_j'$, $j \in J$ and note that $C_j' \neq 0$. Since $\bigoplus_{j \in J} D_j'$ is injective, the natural inclusion $C_j' \rightarrow \bigoplus_{j \in J} D_j'$ can be extended to a homomorphism $f : C \rightarrow \bigoplus_{j \in J} D_j'$. Because $C$ is cyclic, there exists a finite subset $\mathcal{K} \subseteq J$ such that $f(C) \subseteq \bigoplus_{k \in \mathcal{K}} D_k'$, and so $C_j' = f(C_j') \subseteq f(C) \cap D_j' = 0$ for all $j \notin \mathcal{K}$. Therefore, $C_j \cap D_j' = 0$ for all $j \notin \mathcal{K}$, which contradicts that $C_i \subseteq D_i$ for each $i$. Therefore, $R$ is right q.f.d. relative to $M$. \qed

In next theorem, we provide a new characterization for an injective module $M$ to be $\Sigma$-injective.
**Theorem 4.0.17.** Let $M_R$ be an injective module. Then the following statements are equivalent:

(a) $M$ is $\Sigma$-injective;

(b) There exists an infinite cardinal $\alpha$ such that every essential extension of $M^{(\alpha)}$ is a direct sum of injective modules.

**Proof.** (b) $\implies$ (a). Suppose that $M^{(\lambda)}$ is not injective for some infinite cardinal $\lambda$. Set $E = E(M^{(\lambda)})$, pick $x \in E \setminus M^{(\lambda)}$ and let $L = xR$. By Lemma 4.0.17 (b), $R$ is right q.f.d. relative to $M$. From this it follows that every nonzero cyclic and hence every nonzero submodule of $M$ contains a uniform submodule. Now, consider the set $S$ of independent families $(M_k)_{k \in K}$ of uniform injective modules $0 \neq M_k \subseteq M$. Suppose $S$ is partially ordered by $(M_k)_{k \in K} \leq (N_l)_{l \in L}$ if and only if $K \subseteq L$ and $M_k = N_k$ for $k \in K$. By Zorn’s lemma we get a maximal independent family $(M_i)_{i \in I}$ of uniform injective submodules. Clearly $\bigoplus_{i \in I} M_i \subseteq eM$, because otherwise we will get a contradiction to the maximality of this independent family of submodules. This yields that we have an independent family $\{W_i \mid i \in I\}$ of uniform injective submodules of $M^{(\lambda)}$ such that each $W_i$ is isomorphic to a submodule of $M$ and $\bigoplus_{i \in I} W_i \subseteq eM^{(\lambda)}$.

Now we proceed to show that there is a sequence of pairwise distinct elements $i_1, i_2, \ldots$ in $I$ and an independent family of direct summands $V_1, V_2, \ldots$ of $E$ such that $V_j \cong W_{i_j}$ and $\pi_{j-1}(L) \cap V_j \neq 0$ for all $j \geq 1$, where $I_0 = I$, and $I_j = I_{j-1} \setminus \{i_j\}$ for
\( i_j \in \mathcal{I} \). Set \( E_0 = E \), \( E_j \) as an essential closure of \( \oplus_{i \in \mathcal{I}} W_i \) in \( E_{j-1} \), \( \pi_0 = id_E \), and \( \pi_j \) as the projection of \( E \) onto \( E_j \) along \( V_1 \oplus ... \oplus V_j \).

Since \( \oplus_{i \in \mathcal{I}} W_i \subseteq e_\lambda M(\lambda) \subset e_\lambda E \) and \( L \) is a nonzero submodule of \( E \), we have \( L \cap (\oplus_{i \in \mathcal{I}} W_i) \neq 0 \). This implies \( L \) contains a nonzero submodule \( X_1 \) isomorphic to a submodule of some \( W_i \), say, \( W_{i_1} \). Then, \( E(X_1) \cong W_{i_1} \). Set \( V_1 = E(X_1) \). Clearly, \( L \cap V_1 \neq 0 \).

For \( n \geq 1 \), assume that we have a sequence \( \{V_j\} \), \( 1 \leq j \leq n \), of submodules of \( M \) with the above stated properties. Since \( x \notin M(\lambda) \), \( L = xR \notin \oplus_{i=1}^n V_i = \ker(\pi_n) \), and so \( \pi_n(L) \neq 0 \). Now \( \oplus_{i \in \mathcal{I}n} W_i \subset e_\lambda E_n \) and because \( \pi_n : E \rightarrow E_n \), we have \( \pi_n(L) \cap (\oplus_{i \in \mathcal{I}n} W_i) \neq 0 \). Now \( \pi_n(L) \cap (\oplus_{i \in \mathcal{I}n} W_i) \) contains a nonzero cyclic uniform submodule, say, \( C \). This implies, there exists a finite subset \( \mathcal{K} = \{k_1, k_2, ..., k_m\} \subseteq \mathcal{I}_n \) such that \( C \subseteq \oplus_{k \in \mathcal{K}} W_k \). Let \( V_{n+1} \) be the essential closure of \( C \) in \( \oplus_{k \in \mathcal{K}} W_k \). Since \( \oplus_{k \in \mathcal{K}} W_k \) is injective, \( V_{n+1} \) is injective. So, \( \oplus_{k \in \mathcal{K}} W_k = V_{n+1} \oplus D \) for some submodule \( D \) of \( \oplus_{k \in \mathcal{K}} W_k \). Since \( V_{n+1} \) is injective, it has the exchange property. Therefore, \( \oplus_{k \in \mathcal{K}} W_k = V_{n+1} \oplus (\oplus_{k \in \mathcal{K}} W_k') \) for some submodules \( W_k' \) of \( W_k \). Since \( W_k' \) are injective and each \( W_k \) is indecomposable, either \( W_k' = 0 \) or \( W_k' = W_k \). We recall that \( V_{n+1} \) is uniform because it is the closure of uniform module \( C \). Comparing the Goldie dimension on each side of \( \oplus_{k \in \mathcal{K}} W_k = V_{n+1} \oplus (\oplus_{k \in \mathcal{K}} W_k') \), we get that there exists exactly one index \( k_t \in \mathcal{K} \) such that \( W_{k_t}' = 0 \), and for all \( k(\neq k_t) \in \mathcal{K} \), \( W_k' = W_k \). So, \( \oplus_{k \in \mathcal{K}} W_k = V_{n+1} \oplus (\oplus_{k \in \mathcal{K} \setminus \{k_t\}} W_k) \). This yields \( V_{n+1} \cong (\oplus_{k \in \mathcal{K}} W_k)/(\oplus_{k \in \mathcal{K} \setminus \{k_t\}} W_k) \cong \)
$W_{k_1}$. Setting $i_{n+1} = k_1$, we have $V_{n+1} \cong W_{i_{n+1}}$. Note that $\pi_n(L \cap V_{n+1}) \subseteq \pi_n(L) \cap \pi_n(V_{n+1})$. Since $\pi_n$ is identity on $E_n$ and $V_{n+1} \subset E_n$, $\pi_n(L) \cap \pi_n(V_{n+1}) = \pi_n(L) \cap V_{n+1}$.

Also, as $\ker(\pi_n) = V_1 \oplus \ldots \oplus V_n$, $\pi_n(L \cap V_{n+1}) \neq 0$. Therefore, $\pi_n(L) \cap V_{n+1} \neq 0$.

Thus, we have obtained a sequence of submodules $\{V_j\}$, $j = 1, 2, \ldots, n+1$, with the required properties. This completes the induction argument.

Now we claim that there exists a properly ascending chain $N_0 \subset N_1 \subset \ldots \subset N_j \subset \ldots$ of submodules of $L$ such that $N_0 = 0$ and $E(N_j/N_{j-1}) \cong V_j$ for all $j \geq 1$.

Set $N_1 = L \cap V_1$. Since $V_1$ is a uniform injective module, $E(N_1/N_0) \cong V_1$.

By construction of the family $\{V_j\}$, $\pi_1(L) \cap V_2 \neq 0$ and since $\pi_1$ is onto, there exists a uniquely determined submodule $N_2$ of $L$ such that $\pi_1(L) \cap V_2 = \pi_1(N_2)$.

Since $\pi_1(N_1) = 0$ but $\pi_1(N_2) \neq 0$, we obtain $N_2 \supset N_1$. Next, by isomorphism theorem, $\pi_1(N_2) \cong N_2/N_2 \cap V_1$. Now $N_2 \cap V_1 = N_2 \cap V_1 \cap L = N_2 \cap N_1 = N_1$. So, $N_2/N_1 \cong \pi_1(N_2) = \pi_1(L) \cap V_2$ and hence $E(N_2/N_1) \cong V_2$. Because $\pi_2$ is onto, there exists a uniquely determined submodule $N_3$ of $L$ such that $\pi_2(N_3) = \pi_2(L) \cap V_3$. Note that $\pi_2\pi_1 = \pi_2$. Since $\ker(\pi_2) = V_1 \oplus V_2$, $\pi_2(N_2) = \pi_2(\pi_1(N_2)) = \pi_2(\pi_1(L) \cap V_2) = 0$ but $\pi_2(N_3) \neq 0$, we obtain $N_3 \supset N_2$. Now, $\pi_2(N_3) = \pi_2(\pi_1(N_3)) \cong \pi_1(N_3)/\pi_1(N_3) \cap (V_1 \oplus V_2)$.

The natural map $\varphi : \pi_1(N_3) \longrightarrow N_3/N_1$ given by $\pi_1(n_3) \longmapsto n_3 + N_1$ for $n_3 \in N_3$ is well-defined because if $n_3 \in \ker(\pi_1|_{N_3}) = V_1 \cap N_3 = N_1$, then $n_3 \in N_1$. This is clearly an isomorphism. Furthermore, the restriction of $\varphi$ to $\pi_1(N_3) \cap (V_1 \oplus V_2)$ is isomorphism onto $N_2/N_1$. For, if $\pi_1(n_3) \in V_1 \oplus V_2$ then $\pi_1(n_3) = v_1 + v_2$ for some
$v_1 \in V_1$ and $v_2 \in V_2$. This gives $\pi_1(n_3) = \pi_1(v_2) = v_2$ and hence $\pi_1(n_3) \in \pi_1(L) \cap V_2 = \pi_1(N_2)$. So, the restriction of $\varphi$ to $\pi_1(N_3) \cap (V_1 \oplus V_2)$ sends $\pi_1(n_3) \mapsto n_2 + N_1$, $n_3 \in N_3$, $n_2 \in N_2$ where $\pi_1(n_2) = \pi_1(n_3)$. Clearly, this is also well-defined. For, if $\pi_1(n_3) = 0$, then $\pi_1(n_2) = 0$ and this gives $n_2 \in V_1 \cap N_2 = N_1$. Therefore, $\pi_2(N_3) \cong \frac{\pi_1(N_3)}{\pi_1(N_3) \cap (V_1 \oplus V_2)} \cong \frac{N_3/N_1}{N_2/N_1} \cong N_3/N_2$. Now, $N_3/N_2 \cong \pi_2(N_3) = \pi_2(L) \cap V_3$ and hence $E(N_3/N_2) \cong V_3$.

Continuing in this fashion, we create a properly ascending chain $N_0 \subset N_1 \subset \ldots \subset N_j \subset \ldots$ of submodules of $L$ such that $N_0 = 0$ and $E(N_j/N_{j-1}) \cong V_j$ for all $j \geq 1$.

Since $\{V_j\}$, $j \in \mathbb{N}$ is an independent family of uniform injective modules isomorphic to a submodule of $M$, by above lemma, there exists an infinite subset $\mathcal{J} \subseteq \mathbb{N}$ such that $\oplus_{j \in \mathcal{J}} V_j$ and hence $\oplus_{j \in \mathcal{J}} E(N_j/N_{j-1})$ is injective. Set $N = \cup_{i \in \mathbb{N}} N_i$. Given $j \in \mathcal{J}$, let $\alpha_j : N \rightarrow E(N_j/N_{j-1})$ be the canonical mapping. Let $\alpha : N \rightarrow \oplus_{j \in \mathcal{J}} E(N_j/N_{j-1})$ be defined by $\alpha(x) = \{\alpha_j(x)\}_{j \in \mathcal{J}}$ for all $x \in N$. Since $\oplus_{j \in \mathcal{J}} E(N_j/N_{j-1})$ is injective, we may extend $\alpha$ to $\alpha^* : L \rightarrow \oplus_{j \in \mathcal{J}} E(N_j/N_{j-1})$. As $L$ is finitely generated, there exists a finite subset $\mathcal{K} \subseteq \mathcal{J}$ such that $\alpha^*(L) \subseteq \oplus_{k \in \mathcal{K}} E(N_k/N_{k-1})$. For $j \in \mathcal{J} \setminus \mathcal{K}$ and $x \in N_j$ we have $\alpha_j(x) = x + N_{j-1} = 0$, showing that $N_{j-1} = N_j$, a contradiction.

Therefore, $M^{(\lambda)}$ is injective for any cardinal $\lambda$ and hence $M$ is $\Sigma$-injective.

(a) $\implies$ (b) is obvious.
This completes the proof of Theorem 4.0.17.

As a consequence of Theorem 4.0.17, we have the following characterization for a right noetherian ring.

**Theorem 4.0.18.** Let $R$ be a ring. Then the following are equivalent:

(i) $R$ is right noetherian;

(ii) For each injective module $M_R$, there exists an infinite cardinal $\alpha$ such that every essential extension of $M^{(\alpha)}$ is a direct sum of injective modules.

**Proof.** (i) $\Rightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (i) follows from Theorem 4.0.17 and by Faith-Walker [21] that a ring $R$ is right noetherian if and only if every injective right $R$-module is $\Sigma$-injective.

**Remark 4.0.3.** The above result generalizes a result of Beidar-Ke [8] which states that a ring $R$ is right noetherian if and only if every essential extension of a direct sum of injective right $R$-modules is again a direct sum of injective right $R$-modules.

Note that [8] indeed generalizes a result of Bass [4] that a ring is right noetherian if and only if every direct sum of injective modules is injective.
In 1954 Zelinsky [60] proved that every element in the ring of linear transformations of a vector space $V$ over a division ring $D$ is a sum of two units unless $\text{dim } V = 1$ and $D = \mathbb{Z}_2$. As $\text{End}_D(V)$ is a (von-Neumann) regular ring, the Zelinsky’s result generated a lot of interest in the following question: ‘Which regular rings have the property that every element is a sum of (two) units?’.

Clearly, a ring $R$, having $\mathbb{Z}_2$ as a factor ring, cannot have every element as sum of two units. In 1958, Skornyakov [55] conjectured that in all regular rings (which do not have $\mathbb{Z}_2$ as a factor ring) every element is sum of units. It is easy to see that if $R$ is a unit regular ring with 2 invertible, then every element can be written as sum of two units (see [14]). This follows from the fact that in any ring with 2
invertible, any idempotent $e$ can be written as sum of two units as $e = (1 + e) - 1$, where $(1 + e)^{-1} = 1 - e/2$, and every element of a unit regular ring is a product of a unit and an idempotent. The conjecture of Skornyakov was settled by Bergman (see [27]) in negative who produced a directly-finite, regular ring $R$ with 2 invertible, in which not all elements are sum of units.

As $End_D(V)$ is also a right self-injective ring, a natural question that arises from Zelinsky’s result is: ‘In which right self-injective rings every element is sum of two units?’. In this direction Raphael ([53], Proposition 11) proved that if in a right self-injective ring $R$ every idempotent is sum of two units, then every element can be written as sum of even number of units. Recently, Vámó in [57] proved that every element of a right self-injective ring $R$ is sum of two units if $R$ has no non-zero corner ring which is Boolean. But the condition that no corner ring is Boolean is not necessary as there are many right self-injective rings, having non-zero corner Boolean rings, whose every element is sum of two units. For example, if we take $S$ to be a self-injective Boolean ring, then every element of $R = M_n(S)$, $n > 1$, is sum of two units although $R$ has non-zero Boolean corner rings.

The following result characterizes the right self-injective rings with the property that every element is sum of two units.

**Theorem 5.0.19.** For a right self-injective ring $R$, the following conditions are equivalent:
(1) Every element of $R$ is sum of two units;

(2) Identity of $R$ is sum of two units;

(3) $R$ has no factor ring isomorphic to $\mathbb{Z}_2$.

Proof. The implications $(1) \implies (2) \implies (3)$ are obvious.

Now, we proceed to show $(3) \implies (1)$.

By [56] we know that $R/J(R)$ is regular, right self-injective ring. Also, every element of $R$ is sum of two units if and only if every element $R/J(R)$ is sum of two units, therefore we may assume that $R$ is regular also. If $R$ is purely infinite, then $R_R \cong R^2_R$, and so $R \cong M_2(R)$. By ([3], Corollary 2.6), every $A \in M_2(R)$ admits a diagonal reduction, i.e. there exist invertible matrices $P$ and $Q$ in $M_2(R)$ such that $PAQ$ is a diagonal matrix, say $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Then $PAQ = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & b \end{pmatrix}$ is sum of two units and so $A$ is a sum of two units. Thus, every element of $M_2(R)$ and hence every element of $R$ is a sum of two units. Now, by ([24], Proposition 10.21), $R \cong S \times T$, where $S$ is purely infinite and $T$ is directly finite. So, in view of above, we may assume that $R$ is directly finite. Since, $R$ is a directly finite, regular, right self-injective ring, $R \cong R_1 \times R_2$ where $R_1$ is Type $I_f$ and $R_2$ is Type $II_f$ ([24], Theorem 10.22).

First, we show that every element of $R_1$ is sum of two units. Since by ([24],
Theorem 10.24), $R_1 \cong \Pi M_n(S_i)$ where each $S_i$ is an abelian regular right self-injective ring, it is enough to show that each element of $M_n(S_i)$ is sum of two units. But, if $n > 1$, then by above discussion we are through. So, it is enough to show that every element in an abelian regular ring $S_i$, which has no factor isomorphic to $\mathbb{Z}_2$, is sum of two units. Let $a \in S_i$. Suppose, to the contrary, that $a$ is not sum of two units. Let $\Omega = \{I : I$ is an ideal of $S_i$ and $a + I$ is not sum of two units in $S_i/I\}$. Clearly, $\Omega$ is non-empty and it can be easily checked that $\Omega$ is inductive (for example see [22], Theorem 2). So, by Zorn's Lemma, $\Omega$ has a maximal element, say, $I$. Clearly, $S_i/I$ is an indecomposable ring and hence has no central idempotent. But, as $S_i/I$ is abelian regular, $S_i/I$ must be a division ring. Since, $a + I$ is not sum of two units in $S_i/I$, it follows that $S_i/I \cong \mathbb{Z}_2$, a contradiction. Thus, each element of $R_1$ is sum of two units.

Finally, we show that every element of $R_2$ is sum of two units. Since $R_2$ is Type $II_f$, it has no non-zero abelian idempotents. Therefore, by ([24], Proposition 10.28) there exists an idempotent $e_2 \in R_2$ such that $(R_2)_{R_2} \cong 2(e_2R_2)$ and so $R_2 \cong M_2(e_2R_2e_2)$. By [3], we conclude that every element of $M_2(e_2R_2e_2)$ and hence every element of $R_2$ is sum of units. Therefore, each element of $R$ is sum of units. This completes the proof.

Note that $End_D(V)$ where $V$ is a right vector space over division ring $D$ is a regular, right self-injective ring. As a consequence of our above theorem, we get the
following celebrated result of Zelinsky [60].

**Corollary 5.0.4.** Every element of $\text{End}_D(V)$ is sum of two units, except when $\dim(V_D) = 1$ and $D = \mathbb{Z}_2$.

**Proof.** $\text{End}_D(V)$ is a regular, right self-injective ring. In view of above theorem, it suffices to show that identity of $\text{End}_D(V)$ is sum of two units, except when $\dim(V_D) = 1$ and $D = \mathbb{Z}_2$.

If $\dim(V_D) = 1$ and $D \neq \mathbb{Z}_2$ then we can choose $a(\neq 0, 1) \in \mathbb{Z}_2$. Clearly, $1 = a + (1 - a)$ where both $a$ and $(1 - a)$ are units.

If $\dim(V_D) = 2$, we have
\[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}
= \begin{pmatrix}
1 & 1 \\
1 & 0 \\
\end{pmatrix} + \begin{pmatrix}
0 & -1 \\
-1 & 1 \\
\end{pmatrix}.
\]

If $\dim(V_D) = 3$, we have
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0 \\
\end{pmatrix} + \begin{pmatrix}
0 & -1 & -1 \\
-1 & 0 & 0 \\
-1 & 0 & 1 \\
\end{pmatrix}.
\]

If $\dim(V_D)$ is more than 3, decompose $V$ into a direct sum of two and three dimensional subspaces and proceed as above. \qed

As every right self-injective ring is an exchange right quasi-continuous ring, the following result generalizes Theorem 5.0.19.

**Proposition 5.0.9.** Let $M_S$ be a quasi-continuous module with finite exchange property and $R = \text{End}_S(M)$. If no factor ring of $R$ is isomorphic to $\mathbb{Z}_2$, then every
element of $R$ is sum of two units.

Proof. Let $\Delta = \{ f \in R : \ker f \subseteq e \cdot M \}$. Then $\Delta$ is an ideal of $R$. By ([50], Cor. 3.13), $\tilde{R} = R/\Delta \cong R_1 \times R_2$, where $R_1$ is regular, right self-injective and $R_2$ is an exchange ring with no non-zero nilpotent element. We have already shown in Theorem 5.0.19 that each element of $R_1$ is sum of two units. Since, $R_2$ has no non-zero nilpotent element, each idempotent in $R_2$ is central. Now, if any element $a \in R_2$ is not sum of two units, then as in the proof of Theorem 5.0.19, we find an ideal $I$ of $R_2$ such that $x = a + I \in R_2/I$ is not sum of two units in $R_2/I$ and $R_2/I$ has no central idempotent. This implies that $R_2/I$ is an exchange ring without any non-trivial idempotent, and hence it must be local. If $S = R_2/I$ then $x + J(S)$ is not sum of two units in $S/J(S)$, which is a division ring. Therefore, $S/J(S) \cong \mathbb{Z}_2$, a contradiction. Hence, every element of $R_2$ is also sum of two units. Therefore, every element of $\tilde{R}$ is sum of two units. Next, we observe that $\Delta \subseteq J(R)$. Suppose to the contrary that $\Delta \nsubseteq J(R)$, then $\Delta$ contains a nonzero idempotent, say $e$. But as $\ker(e) \subseteq e \cdot M$, $\ker(e) = M$ and so $e = 0$, a contradiction. Thus $\Delta \subseteq J(R)$. Therefore, we may conclude that every element of $R$ is sum of two units. This completes the proof.

Remark 5.0.4. As continuous module is quasi-continuous and also has exchange property ([50], Theorem 3.24), it follows that in the endomorphism ring of a continuous (and hence also of injective and quasi-injective) module, every element is sum
of two units provided no factor of endomorphism ring is isomorphic to $\mathbb{Z}_2$.

**Corollary 5.0.5.** Every element of endomorphism ring of a flat cotorsion (in particular, pure injective) module is sum of two units if no factor of endomorphism ring is isomorphic to $\mathbb{Z}_2$.

Next, we show an interesting application of our result for group rings.

**Proposition 5.0.10.** If $R$ is a right self-injective ring and $G$ a locally finite group, then every element of $RG$ is sum of two units unless $R$ has a factor ring isomorphic to $\mathbb{Z}_2$.

**Proof.** Let $\alpha$ be any arbitrary element of $RG$ then $\alpha = r_0 + r_1 g_1 + r_2 g_2 + \ldots + r_n g_n$. Let $H = \langle g_1, \ldots, g_n \rangle$ be the subgroup generated by $g_1, \ldots, g_n$. Since $G$ is locally finite, $H$ must be finite. Clearly, $\alpha \in RH$. Now, since $R$ is right self-injective and $H$ is a finite group, the group ring $RH$ is right self-injective. Note that if $R$ has no factor ring isomorphic to $\mathbb{Z}_2$ then the group ring $RH$ also has no factor ring isomorphic to $\mathbb{Z}_2$. Therefore, by Theorem 5.0.19, $\alpha = u_1 + u_2$ where $u_1, u_2 \in RH$ are units. But then $u_1, u_2$ will be units in $RG$ also. Hence, every element of $RG$ is sum of two units. \qed

Now, we rephrase our results in terms of unit sum numbers.
Lemma 5.0.18. For any nonzero regular right self-injective ring $R$, $\text{usn}(M_2(R)) = 2$. In particular, for any purely infinite regular right self-injective ring $R$, $\text{usn}(R) = 2$.

Proof. By ([3], Corollary 2.6) every $A \in M_2(R)$ admits a diagonal reduction i.e., there exist invertible matrices $P$ and $Q$ in $M_2(R)$ such that $PAQ$ is a diagonal matrix. Clearly $PAQ$ and so $A$ can be written as a sum of two units.

If $R$ is purely infinite regular right self-injective ring, then $R_R \cong (R \oplus R)_R$ implying that, as rings, $R \cong M_2(R)$ and so the result follows from above. \qed

Using Lemma 5.0.18 we get the following

Lemma 5.0.19. A nonzero regular right self-injective ring $R$ has a ring decomposition $R = S \times T$ where $\text{usn}(S) = 1$ or $2$ and $T$ is an abelian regular right self-injective ring.

Proof. By ([24], Proposition 10.21) $R \cong R_1 \times R_2$, where $R_1$ is either zero or purely infinite and $R_2$ is directly finite. By Lemma 5.0.18, $\text{usn}(R_1) = 1$ or $2$. By ([24], Theorem 10.22) $R_2 \cong R_3 \times R_4$, where $R_3$ is Type $I_f$ and $R_4$ is Type $II_f$. Further, by ([24], Theorem 10.24), $R_3 \cong \prod M_n(S_i)$, where each $S_i$ is an abelian regular right self-injective ring. Also, as $R_4$ has no nonzero abelian idempotents, there exists an idempotent $e \in R_4$ such that $(R_4)_{R_4} \cong (eR_4 \oplus eR_4)_{R_4}$ (see [24], Proposition 10.28), implying that, as rings, $R_4 \cong M_2(eR_4e)$. Thus $R \cong R_1 \times (\prod M_n(S_i)) \times M_2(eR_4e)$. 


We let $S = R_1 \times (\prod_{n>1} M_n(S_i)) \times M_2(eR_4e)$ and $T = \prod S_i$. By Lemma 5.0.18 $\text{usn}(S) = 1$ or $2$.

\begin{proof}

Lemma 5.0.20. ([57], Proposition 19) Let $M_R$ be a nonsingular injective module. Then $\text{End}(M_R)$ is a Boolean ring if and only if the identity endomorphism of no direct summand of $M$ is a sum of two units.

Our main theorem 5.0.19 can be rephrased as

Lemma 5.0.21. For a nonzero right self-injective ring $R$, the following conditions are equivalent:

(1) $\text{usn}(R) = 2$;

(2) The identity of $R$ is a sum of two units;

(3) $R$ has no factor ring isomorphic to $\mathbb{Z}_2$.

Lemma 5.0.22. Let $T$ be an abelian regular right self-injective ring. Then $T = T_1 \times T_2$ where $\text{usn}(T_1) = 1$ or $2$ and $T_2$ is either zero or a Boolean ring.

Proof. In view of Lemma 5.0.21 it is enough to show that $T$ has a ring decomposition $T = T_1 \times T_2$, where $T_2$ is a Boolean ring and the identity in $T_1$ is a sum of two units in $T_1$. We shall prove this using a standard Zorn’s lemma argument as used in ([57], Lemma 17). Let $T$ be the set of all pairs of the form $(A, u)$ where $A$ is a submodule of $T_T$ and $u$ is an automorphism of $A$ such that $I_A - u$ is also an automorphism of $A$, where $I_A$ is the identity automorphism of $A$. Clearly $(0, 0) \in T$. Then $T$ has
an obvious partial order i.e., \((A, u) \leq (A', u')\) if \(A \subseteq A'\) and \(u'\) agrees with \(u\) on \(A\). Also \(\mathcal{T}\) is easily seen to be inductive and so by Zorn’s lemma \(\mathcal{T}\) has a maximal element \((T_1, v)\) say. If \(T_1\) is not injective, then \(v\) extends to an automorphism \(v'\) of an injective hull \(E(T_1)\) of \(T_1\) in \(T\), and so \((E(T_1), v')\) \(\in\) \(\mathcal{T}\) violating the maximality of \((T_1, v)\). Thus \(T_1\) is injective and so \(T = T_1 \oplus T_2\) for some submodule \(T_2\) of \(T\).

As \((T_1, v)\) is a maximal element of \(\mathcal{T}\), it is clear that the identity endomorphism of no direct summand of \(T_2\) is a sum of two units. So, by Lemma 5.0.20, \(\text{End}_T(T_2)\) is a Boolean ring. As every idempotent in \(T\) is central, the module decomposition \(T_T = T_1 \oplus T_2\) gives us a ring decomposition \(T = T_1 \times T_2\) with \(\text{End}_T(T_2) \cong T_2\) a Boolean ring. Also the identity in \(T_1\) is a sum of two units in \(T_1\).

We know that a right self-injective ring is right continuous (see [24]). The following example shows that Lemma 5.0.22 is not true even for commutative regular continuous rings. Let \(F\) be a field with \(\mathbb{Z}_2\) its proper subfield. Set \(F_n = F\) and \(K_n = \mathbb{Z}_2\) for each positive integer \(n\). Let

\[
R = \{(x_n)_n \in \prod N F_n : x_n \in K_n\}.
\]

Then \(R\) is a commutative continuous ring (see [24], Example 13.8) which clearly is regular. It is easy to see that the identity of \(R\) is not a sum of two units. An element \((x_n)_n\) of \(R\) is an idempotent precisely when every component is either 0 or 1 and so it is clear that \(eR\) is not a Boolean ring for any idempotent \(e \in R\). Also the element
(1, 0, 1, 0, ....) is not a sum of units in \( R \) implying that \( \text{usn}(R) = \infty \).

The following result characterizes the regular right self-injective rings with various unit sum numbers.

**Theorem 5.0.20.** The unit sum number of a nonzero regular right self-injective ring \( R \) is 2, \( \omega \) or \( \infty \). Moreover,

1. \( \text{usn}(R) = 2 \) if and only if \( R \) has no (nonzero) Boolean ring as a ring direct summand.
2. \( \text{usn}(R) = \omega \) if and only if \( R \) has \( \mathbb{Z}_2 \), but no Boolean ring with more than two elements, as a ring direct summand. Moreover, in this case every non-invertible element of \( R \) is a sum of either two or three units.
3. \( \text{usn}(R) = \infty \) if and only if \( R \) has a Boolean ring with more than two elements as a ring direct summand.

**Proof.** In view of Lemma 5.0.19 and Lemma 5.0.22, \( R = R_1 \times B \) where \( \text{usn}(R_1) = 1 \) or 2 and \( B \) is a Boolean ring. It is clear that the unit sum number of a nonzero Boolean ring is \( \infty \) unless it is isomorphic to \( \mathbb{Z}_2 \), in which case the unit sum number is \( \omega \). So (1) is immediate. If \( \text{usn}(R) \neq 2 \), then \( B \neq 0 \). Clearly, \( \text{usn}(R) = \omega \) if and only if \( B \cong \mathbb{Z}_2 \) and \( \text{usn}(R) = \infty \) if and only if \( B \) has more than two elements. Also if \( \text{usn}(R) = \omega \), then \( R \cong R_1 \times \mathbb{Z}_2 \) and it is clear than any non-invertible element of \( R \) is a sum of either two or three units. \( \square \)
In ([29], Question E, page 192) Henriksen asked if there is a regular ring, with every element sum of units, in which there are non-invertible elements that are not sum of two units. The following example answers this question.

**Example 5.0.9.** Let $S$ be a nonzero regular right self-injective ring which does not have a factor ring isomorphic to $\mathbb{Z}_2$. For instance, take $S$ to be any field other than $\mathbb{Z}_2$. Let $R = S \times \mathbb{Z}_2$. Clearly $R$ is a regular right self-injective ring and, by Theorem 5.0.20, $\text{usn}(R) = \omega$. But the element $(0, 1)$ of $R$ is a non-unit which can’t be written as a sum of two units.

**Theorem 5.0.21.** The unit sum number of a nonzero right self-injective ring $R$ is $2, \omega$ or $\infty$. Moreover,

1. $\text{usn}(R) = 2$ if and only if $R$ has no factor ring isomorphic to $\mathbb{Z}_2$.

2. $\text{usn}(R) = \omega$ if and only if $R$ has a factor ring isomorphic to $\mathbb{Z}_2$, but has no factor ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. In this case every non-invertible element of $R$ is a sum of either two or three units.

3. $\text{usn}(R) = \infty$ if and only if $R$ has a factor ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

**Proof.** We know that $R/J(R)$ is a nonzero regular right self-injective ring (see [56]) and also it is clear that $\text{usn}(R) = \text{usn}(R/J(R))$. So the result follows from Theorem 5.0.20 and the fact that any Boolean ring with more than two elements has a factor ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. \qed
OPEN QUESTIONS

Question 1 Let $R$ be a ring with the property that every essential extension of a direct sum of simple right $R$-modules is a direct sum of quasi-injective right $R$-modules. Is $R$ necessarily right noetherian?

Question 2 Is every right self-injective right NF-ring a directly-finite ring?

Question 3 Let $R$ be a regular ring with $\text{usn}(R) \neq \infty$. Is every non-invertible element a sum of two or three units?

Question 4 Let $R$ be a regular ring in which 2 is invertible such that $\text{usn}(R) \neq \infty$. Is every non-invertible element of $R$ a sum of two units?

Question 5 Characterize unit regular rings whose every element is a sum of two units.
Bibliography


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