AN ANALYSIS OF PLASMA CURRENT AND HORIZONTAL
PLASMA POSITION FEEDBACK CONTROL SYSTEM OF
AN ISX TOKAMAK POWER REACTOR

A Thesis presented to
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In Partial Fulfillment
of the Requirements for the Degree
Master of Science

By
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Ohio University Library
ABSTRACT

This study is an investigation and design of a feedback control system to control the plasma current and its horizontal position of a Tokamak power reactor machine under operational conditions. The task is to keep these parameters as close as possible to their desired values.

First section of this thesis will introduce the problem in general and will give an orientation of plasma and control currents. The dynamics of the system is described in detail in section II. The state variable technique has been used for the analysis of the system. The state equations and output equations have been derived in section III. Section IV is devoted to optimization of the problem and a suitable control law is found for the system inputs. The solution of optimization problem of this section requires solving an algebraic Riccati equation of the order $n$ in general, which results \( \frac{n(n + 1)}{2} \) non-linear equations with \( \frac{n(n + 1)}{2} \) unknowns. This is indeed a difficult problem for $n > 3$. An alternative new method is used to find the Riccati matrix. This is explained in detail in appendix I. Feedback circuits and power supplies are discussed.
in sections V and VI respectively.

Finally a CSMP simulation of the model is given in section VII to determine the validity of the system.
ACKNOWLEDGMENTS

The author would like to express his appreciation to the members of his graduate committee, Dr. Harold F. Klock, Dr. Brian Manhire, Dr. Harry M. Kaneshige and especially his advisor Dr. G.V.S. Raju for his numerous and significant contributions to this study. He also wishes to express his gratitude to Tom Mullins for his assistance in the computer work.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>iii</td>
</tr>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>v</td>
</tr>
<tr>
<td>LIST OF ILLUSTRATIONS</td>
<td>viii</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>x</td>
</tr>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. SYSTEM DYNAMICS</td>
<td>9</td>
</tr>
<tr>
<td>III. STATE VARIABLE TECHNIQUE</td>
<td>18</td>
</tr>
<tr>
<td>IV. PLASMA CURRENT AND POSITION OPTIMIZATION</td>
<td>25</td>
</tr>
<tr>
<td>A. Quadratic Criteria</td>
<td>25</td>
</tr>
<tr>
<td>B. Cost Function</td>
<td>27</td>
</tr>
<tr>
<td>C. Tracking Problem</td>
<td>30</td>
</tr>
<tr>
<td>D. Riccati Equation</td>
<td>31</td>
</tr>
<tr>
<td>V. FEEDBACK CIRCUITS</td>
<td>42</td>
</tr>
<tr>
<td>VI. POWER SUPPLY MODEL</td>
<td>47</td>
</tr>
<tr>
<td>A. Inner Supply</td>
<td>47</td>
</tr>
<tr>
<td>B. Outer Supply</td>
<td>48</td>
</tr>
<tr>
<td>VII. SIMULATION RESULTS</td>
<td>51</td>
</tr>
<tr>
<td>APPENDICES</td>
<td>61</td>
</tr>
<tr>
<td>I. A SCHUR METHOD FOR SOLVING ALGEBRAIC</td>
<td>62</td>
</tr>
<tr>
<td>RICCATI EQUATION</td>
<td>62</td>
</tr>
<tr>
<td>II. THE TRANSFER FUNCTION OF THE BLOCK</td>
<td>108</td>
</tr>
<tr>
<td>CONTAINING PLASMA AND COIL EQUATIONS</td>
<td>108</td>
</tr>
</tbody>
</table>
TABLE OF CONTENTS (cont.)

Page

BIBLIOGRAPHY................................................................. 115
## LIST OF ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I-1</td>
<td>Nuclear Fusion Reactors of the Future</td>
<td>6</td>
</tr>
<tr>
<td>I-2</td>
<td>Open Configuration for Magnetic Confinement of Plasma</td>
<td>7</td>
</tr>
<tr>
<td>I-3</td>
<td>Closed Configuration for the Confinement of Plasma</td>
<td>7</td>
</tr>
<tr>
<td>I-4</td>
<td>Orientation of Plasma and Control Currents</td>
<td>8</td>
</tr>
<tr>
<td>II-1</td>
<td>Approximations to $f_i$'s Functions</td>
<td>17</td>
</tr>
<tr>
<td>III-1</td>
<td>Plasma and Coil Equations</td>
<td>18</td>
</tr>
<tr>
<td>IV-1</td>
<td>Plasma and Coil Currents Block</td>
<td>25</td>
</tr>
<tr>
<td>IV-2</td>
<td>The Structure of the Approximate Optimal Time-Invariant System When the Desired Output is a Constant $Z$</td>
<td>40</td>
</tr>
<tr>
<td>IV-3</td>
<td>The Block Diagram of the Entire System</td>
<td>41</td>
</tr>
<tr>
<td>V-1</td>
<td>Feedback Circuit Block Diagram</td>
<td>42</td>
</tr>
<tr>
<td>V-2</td>
<td>Feedback Circuits Block Diagram</td>
<td>46</td>
</tr>
<tr>
<td>VI-1</td>
<td>The Behavior of the Voltage and Current in the Current-Controlled Power Supply Model</td>
<td>50</td>
</tr>
<tr>
<td>VII-1</td>
<td>Plasma Current Response for $a + 5 \text{ kamps.}$ Perturbation (Desired Value $= -100 \text{ kamps.}$)</td>
<td>53</td>
</tr>
<tr>
<td>VII-2</td>
<td>Plasma Current Response for $a - 5 \text{ kamps.}$ Perturbation (Desired Value $= -100 \text{ kamps.}$)</td>
<td>54</td>
</tr>
</tbody>
</table>
## LIST OF ILLUSTRATIONS (cont.)

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>VII-3</td>
<td>Plasma Current Response for a + 5 kamps.</td>
<td>55</td>
</tr>
<tr>
<td></td>
<td>Perturbation (Desired Value = -112 kamps.)</td>
<td></td>
</tr>
<tr>
<td>VII-4</td>
<td>Plasma Current Response for a - 5 kamps.</td>
<td>56</td>
</tr>
<tr>
<td></td>
<td>Perturbation (Desired Value = -112 kamps.)</td>
<td></td>
</tr>
<tr>
<td>VII-5</td>
<td>Plasma Position Response for a + 0.03 meters</td>
<td>57</td>
</tr>
<tr>
<td></td>
<td>Perturbation (Desired Value = 0.95 meters)</td>
<td></td>
</tr>
<tr>
<td>VII-6</td>
<td>Plasma Position Response for a - 0.03 meters</td>
<td>58</td>
</tr>
<tr>
<td></td>
<td>Perturbation (Desired Value = 0.95 meters)</td>
<td></td>
</tr>
<tr>
<td>VII-7</td>
<td>Plasma Position Response for a + 0.03 meters</td>
<td>59</td>
</tr>
<tr>
<td></td>
<td>Perturbation (Desired Value = 1.03 meters)</td>
<td></td>
</tr>
<tr>
<td>VII-8</td>
<td>Plasma Position Response for a - 0.03 meters</td>
<td>60</td>
</tr>
<tr>
<td></td>
<td>Perturbation (Desired Value = 1.03 meters)</td>
<td></td>
</tr>
<tr>
<td>A-II-1</td>
<td>Block Diagram of the System</td>
<td>114</td>
</tr>
</tbody>
</table>
LIST OF TABLES

Table       Page
II-1  Values of L's                       14
II-2  Values of L's                       14
II-3  Values of R                         15
V-1  Coefficients of the error functions e_2 and e_3  45
VII-1 Specifications of the Plasma Currents and Position Simulations  52
I. INTRODUCTION

Prospects for generating of electric power from controlled nuclear fusion - the harnessing of energy released during the combination of very light atomic nuclei to form somewhat heavier ones - are meeting with renewed enthusiasm as a result of significant progress in this area since 1968. Nevertheless, the technology still is far from commercial reality, whereas the use of controlled nuclear fission- involving energy released in the splitting of very heavy atomic nuclei into lighter ones - has been accepted commercially for some time. In the U.S., the goal for a practical demonstration of fusion energy is the end of century. In the meantime the emphasis in the nation's fusion program has been shifting from research to engineering and technology [1]. Figure I-1 shows nuclear fusion reactors of future based on the Tokamak concept of magnetic confinement may well resemble this conceptual design from Oak Ridge National Laboratory [2].

Fusion power reactors need an input energy to establish the fuel conditions necessary for appreciable nuclear energy release. This input energy heats the fusion fuel
to high temperatures ($10^7$-$10^8$ K). At such temperature, the fuel is in the form of an ionized gas. This ionized gas, or plasma has the ability of conducting electricity. J.D. Lawson, a British scientist has shown that the product of the plasma density and the plasma confinement time required for fusion-energy breakeven depends on the fusion fuel. Investigation in this area show that Deuterium-Tritium mixture requires the lowest fusion temperature.

Plasma can be confined by magnetic fields which provide magnetic pressure that counteract the kinetic pressure of the plasma. Magnetic confinement schemes include open- and closed geometry types. An open configuration is shown in Figure I-2 [1]. The magnetic field lines leave the confinement region and the plasma is able to escape through the ends of the system. In the closed configuration, Figure I-3 [1], the field lines are contained in a toroidal volume and the plasma can escape only by moving across field lines.

The main question in fusion power concept is the availability of fuel. Deuterium is an isotope of hydrogen which is found from water and can be produced easily. Tritium, however, is not so abundant and it will be necessary to breed this isotope through interactions of neutrons with Lithium in Deuterium-Tritium fusion reactors. The
worldwide reserves of Lithium are about 10 million tonnes or about 5 billion MW per year of electric energy which could support a global population of 10 billion for 500 years.

Among the various methods of generating power from nuclear fusion, Tokamak machine is more feasible. There are two sets of pulsed-power supplies in this machine, one to control the plasma current and the other to supply power to injectors of beams of neutral particles. In this type of reactors, the intrinsic ohmic heating in the plasma will not be sufficient to raise its temperature to ignition. High-power neutral-beam injectors are under development at the Lawrence Laboratories at Berkeley and Livermore, California and also at Oak Ridge National Laboratory (ORNL).

A Tokamak machine includes a toroidal chamber with minor and major radii of 1.5 and 6 meters respectively. Superconducting windings are used for the toroidal induction coils to obtain the high value of magnetic induction for the confinement of plasma. Induction coils are huge, D-shaped coils with inside diameters of 7 and 10 meters and an induction of about 8 tesla.

There is a 0.8 meter-thick stainless steel blanket surrounding the plasma to convert the kinetic energy of
the fusion neutrons into heat.

Considering the environmental and safety aspects, the potential biological impact of a fusion reactor is expected to be at least ten times less than that associated with advanced fission reactors. A fundamental concern in accident analysis for fission reactor is the possibility, in the event of a loss-of-coolant accident, that heat generated after a shutdown of the reactor might lead to vaporization and disposal of radioactive material. With fusion reactors, removal of this afterheat is expected to pose less of a problem.

As for wastes, the primary source of radioactive waste from fusion reactors will be the activated structural material of the blanket that have a finite useful lifetime. In contrast, fission products such as $^{90}\text{Sr}$ and $^{137}\text{Cs}$ from nuclear fission must be stored for centuries to decay to nontoxic levels and products such as $^{99}\text{Tc}$ and some other actinides require thousands of centuries to decay to nontoxic levels.

The great engineering and environmental problems associated with Deuterium-Tritium fusion power reactors recently inspired a group of experts to re-examine the neutronless fusion reactor. One of the possible fuel cycles is the
Proton-boron-11 with both fuel practically inexhaustible and cheap [3]. Much of the power output for this concept emerges as moderately hard electromagnetic radiation (X-rays) and the power of charged particles.

A Tokamak device as a whole is an electric system consisting of power supplies, coil sets, eddy current, etc. The behavior of such system could be modeled by treating the various components as electrical circuits [4].

The ISX-B device is a Tokamak with major radius of 93 cm. The rectangular vacuum vessel has coils mounted around the outside to control the plasma.

There are five major parameters that we will be considering in our plasma control problem, four currents and the horizontal position of plasma. Figure 1-4 shows the relative orientation of the currents mentioned above [5]. These parameters are:

1) Plasma current, $I_1$ which is a negative current.
2) Inner ohmic heating current, $I_2$.
3) Outer ohmic heating current, $I_3$.
4) Symmetrical shell (eddy) current, $I_5$.
5) Horizontal position of plasma, $R_1$. 
Figure I-1 Nuclear Fusion Reactor of Future
(courtesy of IEEE Spectrum)
Figure I-2 Open configuration for magnetic confinement of plasma.

Figure I-3 Closed configuration for the confinement of plasma.
Figure I-4 Orientation of plasma and control currents
The dynamics of the system are given in this section and explained in detail. The numerical values are listed in the tables at the end of this section.

First, the current equations are given. These equations are taken from an ISX-B program paper by Charlton, Swain and Neilson [4].

1. Plasma current, $I_1$.

$$L_{11} \dot{I}_1 + L_{12} \dot{I}_2 + L_{13} \dot{I}_3 + L_{15} \dot{I}_5 + \left[ L_{11} \dot{I}_{10} + L_{12} \dot{I}_{20} + L_{13} \dot{I}_{30} + L_{15} \dot{I}_{50} \right] R_I + R_1 I_1 = 0 \quad (II-1)$$

2. Inner ohmic heating current, $I_2$.

$$L_{12} \dot{I}_1 + L_{22} \dot{I}_2 + L_{23} \dot{I}_3 + L_{25} \dot{I}_5 + L_{12} \dot{I}_{10} R_I + R_2 I_2 = V_2 \quad (II-2)$$

3. Outer ohmic heating current, $I_3$.

$$L_{13} \dot{I}_1 + L_{23} \dot{I}_2 + L_{33} \dot{I}_3 + L_{35} \dot{I}_5 + L_{13} \dot{I}_{10} R_I + R_3 I_3 = V_3 \quad (II-3)$$

4. Symmetrical shell (eddy) current, $I_5$.

$$L_{15} \dot{I}_1 + L_{25} \dot{I}_2 + L_{35} \dot{I}_3 + L_{55} \dot{I}_5 + L_{15} \dot{I}_{10} R_I + R_5 I_5 = 0 \quad (II-4)$$

Values for $L$'s, $\dot{L}$'s and $R$'s are given by D. Swain and are listed in Tables II-1, II-2 and II-3. [5]
(5) Plasma equilibrium equation:

\[ R_1 = - \frac{I_1 (2.1 + \beta_p)}{K_1 B_v} \]  

(II-5)

where \( R_1 \) is the horizontal position and is a function of \( I_1, \beta_p \) and the vertical field \( B_v \). The vertical field \( B_v \) has contribution from all currents and can be expressed as below [6].

\[ B_v = \sum f_i(R_1)I_i \quad i = 1,2,3,5 \]  

(II-6)

where \( f_i \)'s are functions of both horizontal \( (R_1) \) and vertical \( (Z) \) displacements. We have limited our discussion only to horizontal displacement in this thesis. This relationship is shown in Figure II-1. Using this figure we can write \( f_i \)'s as:

\[ f_1 = -(8.85 - 7.65 R_1) \quad \text{for} \quad R_1 \leq 0.85 \text{ m} \]  

(II-7)

\[ = -(5.54 - 3.75 R_1) \quad \text{for} \quad R_1 > 0.85 \text{ m} \]

\[ f_2 = -(107.16 - 94.7 R_1) \quad \text{for} \quad R_1 \leq 0.85 \text{ m} \]  

(II-8)

\[ = -(62.08 - 41.67 R_1) \quad \text{for} \quad R_1 > 0.85 \text{ m} \]

\[ f_3 = 34.35 R_1 \]  

(II-9)

\[ f_5 = -3.32 \]  

(II-10)

For our analysis we will consider \( f_i \)'s as:
Also for the analysis purposes, we will linearize equations (II-7) through (II-10) to obtain an equation of the form:

\[
R_I = R_{I0} + \sum g_i (I_i - I_{i0}) + \int R(t)
\]  

(II-11)

where:

\(R_{I0}\) is the initial value of plasma position or its desired value, \(I_{i0}\) is the initial value of the \(i\)-th current. \(g_i\)'s are given as below:

\[
g_1 = - \frac{k}{B_{vo}} + \frac{K I_{10} f_1}{B_{vo}^2} = - \frac{K}{B_{vo}} \left[1 - \frac{f_1 I_{10}}{B_{vo}}\right] 
\]  

(II-12)

\[
g_{2,3,5} = \frac{K I_{10}}{B_{vo}^2} f_i \quad i = 2,3,5
\]  

(II-13)

The constant \(K_1\) in equation (II-5) is given as:

\[
K_1 = \frac{4 \pi R_I}{\mu_0}
\]  

(II-14)

where \(\mu_0 = 10^{-7} \text{ h/m}\).

Substituting equation (II-14) in (II-5) we will have:
We will call \( \frac{\mu_0 (2.1 + \beta_p)}{4\pi} \frac{I_1}{B_v} \) to be a new constant namely \( K \), used in equations (II-12) and (II-13).

\( g_1, g_2, g_3 \) and \( g_5 \) have been calculated for different initial values of \( I_1, \beta_p \) and \( R_{I0} \) and the results of three test cases are summarized in Table II-4. We will consider the numerical values of the first case for building our system.

Substituting the values of \( g \)'s in equation (II-11) and considering the initial values of \( R_{I0} \) and \( \delta R \) equal to 0.95 meter and 0.03 meter respectively, we will get:

\[
R_I = -0.0027 I_1 + 0.105 I_2 - 0.143 I_3 + 0.0145 I_5 + 1.0046
\]

We should also substitute the values of \( L \)'s, \( \dot{L} \)'s and \( R \)'s in equations (II-1) to (II-4) to get the currents equations. These calculations have been made and the result is the set of five differential equations below:

\[
254 \dot{I}_1 + 1514 \dot{I}_2 + 2012 \dot{I}_3 + 0.13 \dot{I}_5 - 116.12 \dot{R}_I = 0
\]
\[
1514 \dot{I}_1 + 9120 \dot{I}_2 + 12050 \dot{I}_3 + 3.68 \dot{I}_5 + 1580 \dot{R}_I + 370 I_2 = V_2
\]
\begin{align*}
2012 \dot{I}_1 + 12050 \dot{I}_2 + 16210 \dot{I}_3 - 4.59 \dot{I}_5 - 1590 \dot{R}_I \\
+ 1330 \dot{I}_3 &= V_3 \\
0.13 \dot{I}_1 + 3.68 \dot{I}_2 - 4.59 \dot{I}_3 + 0.549 \dot{I}_5 + 190 \dot{R}_I \\
+ 76 \dot{I}_5 &= 0
\end{align*}

\begin{align*}
-0.0027 \dot{I}_1 + 0.105 \dot{I}_2 - 0.143 \dot{I}_3 + 0.0145 \dot{I}_5 - \dot{R}_I &= -1.0046
\end{align*}

(II-16)

where \( I_1 \), the plasma current, is in kamps and \( R_I \), the plasma position, is in meters. \( V_2 \) and \( V_3 \) are the power supplies and are defined in Volts. They control \( I_1 \) and \( R_I \) and will be discussed in a later section.
<table>
<thead>
<tr>
<th>Table II-1</th>
<th>Values of L's</th>
</tr>
</thead>
<tbody>
<tr>
<td>L's</td>
<td>Values ($\mu h$)</td>
</tr>
<tr>
<td>$L_{11}$</td>
<td>254.000</td>
</tr>
<tr>
<td>$L_{12}$</td>
<td>1514.000</td>
</tr>
<tr>
<td>$L_{13}$</td>
<td>2012.000</td>
</tr>
<tr>
<td>$L_{15}$</td>
<td>0.130</td>
</tr>
<tr>
<td>$L_{22}$</td>
<td>9120.000</td>
</tr>
<tr>
<td>$L_{23}$</td>
<td>12050.000</td>
</tr>
<tr>
<td>$L_{25}$</td>
<td>3.680</td>
</tr>
<tr>
<td>$L_{33}$</td>
<td>16210.000</td>
</tr>
<tr>
<td>$L_{35}$</td>
<td>-4.590</td>
</tr>
<tr>
<td>$L_{55}$</td>
<td>0.549</td>
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<table>
<thead>
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<th>Table II-2</th>
<th>Values of $L'$s</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L'$s</td>
<td>Values ($\mu h/m$)</td>
</tr>
<tr>
<td>$L'_{11}$</td>
<td>1.2</td>
</tr>
<tr>
<td>$L'_{12}$</td>
<td>-15.8</td>
</tr>
<tr>
<td>$L'_{13}$</td>
<td>15.9</td>
</tr>
<tr>
<td>$L'_{15}$</td>
<td>-1.9</td>
</tr>
</tbody>
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Table II-3 Values of R (ohms)

<table>
<thead>
<tr>
<th>R</th>
<th>Value (ohms)</th>
</tr>
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<tbody>
<tr>
<td>R₁</td>
<td>0.0-0.14</td>
</tr>
<tr>
<td>R₂</td>
<td>3.7 x 10⁻⁴</td>
</tr>
<tr>
<td>R₃</td>
<td>1.3 x 10⁻³</td>
</tr>
<tr>
<td>R₅</td>
<td>7.6 x 10⁻⁵</td>
</tr>
</tbody>
</table>

* R₁ will be considered 0.0, for this analysis.
<table>
<thead>
<tr>
<th>$I_{10}$ (KA)</th>
<th>$I_{20}$ (KA)</th>
<th>$I_{50}$ (KA)</th>
<th>$I_{30}$ (KA)</th>
<th>$R_{10}$ (M)</th>
<th>$B_{vo}$</th>
<th>$\beta_p$</th>
<th>$g_1$ (M/KA)</th>
<th>$g_2$</th>
<th>$g_3$</th>
<th>$g_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-100</td>
<td>7.00</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.95</td>
<td>1.21</td>
<td>270</td>
<td>0.105</td>
<td>-0.143</td>
<td>0.0145</td>
</tr>
<tr>
<td>-100</td>
<td>4.93</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.96</td>
<td>2.40</td>
<td>370</td>
<td>0.077</td>
<td>-0.104</td>
<td>0.0105</td>
</tr>
<tr>
<td>-112</td>
<td>1.92</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>1.03</td>
<td>4.90</td>
<td>591</td>
<td>0.052</td>
<td>-0.071</td>
<td>0.0070</td>
</tr>
</tbody>
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Table II-4: Numerical values of $g_1$, $g_2$, $g_3$, and $g_5$ for different test cases.
Figure II-1  Approximations to $f_i$ functions.

$f$'s are in gauss/kA.

$R$ is in meter.
III STATE VARIABLE TECHNIQUE

Five equations were derived from the basic information and characteristics of the system in last section. In this section we will find the state and the output equations of the system. Figure III-1 shows plasma and coil equations block which is considered to be the main part of the system.

\[ V_2 \quad \text{Plasma & coil equations} \quad I_1 \]
\[ V_3 \quad \text{R}_I \]

**Figure III-1** Plasma and coil equations block.

\( V_2 \) and \( V_3 \) are the power supplies that control the outputs \( I_1 \) and \( R_I \) respectively.

The general state form of the system can be written as:

\[
\dot{X} = AX + BU
\]  \hspace{1cm} (III-1)

and the output equation:

\[
Y = CX
\]  \hspace{1cm} (III-2)

where for our system:

- \( A \) is a \((5\times5)\) constant matrix.
- \( B \) is a \((5\times2)\) constant matrix.
C is a (2x5) constant matrix.

X is a vector of five states $X_1, X_2, X_3, X_4$ and $X_5$ that represent $I_1, I_2, I_3, I_5$ and $R_I$ respectively.

U is a vector of 2 inputs namely $V_2$ and $V_3$.

Y is the output of the system that we are concerned with.

\[
\begin{bmatrix}
X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5
\end{bmatrix} =
\begin{bmatrix}
I_1 \\ I_2 \\ I_3 \\ I_5 \\ R_I
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \\ \dot{X}_5
\end{bmatrix} =
\begin{bmatrix}
\dot{I}_1 \\ \dot{I}_2 \\ \dot{I}_3 \\ \dot{I}_5 \\ \dot{R}_I
\end{bmatrix}
\]

\[
U =
\begin{bmatrix}
V_2 \\ V_3
\end{bmatrix}
\]

\[
Y =
\begin{bmatrix}
I_1 \\ R_I
\end{bmatrix}
\]

In this section we will start with the set of five equations of (II-16). These are all differential equations except for the last one. Taking the derivative of both sides of this equation yields:

\[-0.0027 \dot{I}_1 + 0.105 \dot{I}_2 - 0.143 \dot{I}_3 + 0.0145 \dot{I}_5 - \dot{R}_I = 0\]
Therefore we have five linear equations with five unknowns $I_1, I_2, I_3, I_5$ and $R_I$. We will apply the kramer method to find these unknowns as functions of $I_1, I_2, I_3, I_5, R_I, V_2$ and $V_3$.

\[
\begin{align*}
254I_1 + 1514I_2 + 2012I_3 + 0.13I_5 - 116.12R_I &= 0 \\
1514I_1 + 9120I_2 + 12050I_3 + 3.68I_5 + 1580R_I &= V_2 - 370I_2 \\
2012I_1 + 12050I_2 + 16210I_3 - 4.59I_5 - 1590R_I &= V_3 - 1330I_3 \\
0.13I_1 + 3.68I_2 - 4.59I_3 + 0.549I_5 + 190R_I &= -76I_5 \\
-0.0027I_1 + 0.105I_2 - 0.143I_3 + 0.0145I_5 &= R_I = 0
\end{align*}
\]

(III-3)

$\Delta$, the determinant of the system will be:

\[
\Delta = \begin{vmatrix}
254 & 1514 & 2012 & 0.13 & -116.12 \\
1514 & 9120 & 12050 & 3.68 & 1580 \\
2012 & 12050 & 16210 & -4.59 & -1590 \\
0.13 & 3.68 & -4.59 & 0.549 & 190 \\
-0.0027 & 0.105 & -0.143 & 0.0145 & -1
\end{vmatrix}
\]

(III-4)

A Fortran program has been used to calculate the determinant of this matrix.

\[
\Delta = -9096236
\]

The values for $I_1, I_2, I_3, I_5$ and $R_I$ are:
\[ \begin{align*}
\dot{I}_1 &= \frac{0 \quad 1514 \quad 2012 \quad 0.13 \quad -116.12}{(V_2 - 370 \quad I_2) \quad 9120 \quad 12050 \quad 3.68 \quad 1580} \\
&\quad \quad \quad \quad (V_3 - 1330I_3) \quad 12050 \quad 16210 \quad -4.59 \quad -1590 \\
&\quad \quad \quad -76 \quad I_5 \quad 3.68 \quad -4.59 \quad 0.549 \quad 190 \\
&\quad \quad \quad 0 \quad 0.105 \quad -0.143 \quad 0.0145 \quad -1 \\
\end{align*} \]

\[ \Delta \]

\[ \begin{align*}
\dot{I}_2 &= \frac{254 \quad 0 \quad 2012 \quad 0.13 \quad -116.12}{1514 \quad (V_2 - 370 \quad I_2) \quad 12050 \quad 3.68 \quad 1580} \\
&\quad \quad \quad \quad 2012 \quad (V_3 - 1330I_3) \quad 16210 \quad -4.59 \quad -1590 \\
&\quad \quad \quad 0.13 \quad -76 \quad I_5 \quad -4.59 \quad 0.549 \quad 190 \\
&\quad \quad \quad -0.0027 \quad 0 \quad -0.143 \quad 0.0145 \quad -1 \\
\end{align*} \]

\[ \Delta \]

\[ \begin{align*}
\dot{I}_3 &= \frac{254 \quad 1514 \quad 0 \quad 0.13 \quad -116.12}{1514 \quad 9120 \quad (V_2 - 370 \quad I_2) \quad 3.68 \quad 1580} \\
&\quad \quad \quad \quad 2012 \quad 12050 \quad (V_3 - 1330I_3) \quad -4.59 \quad -1590 \\
&\quad \quad \quad 0.13 \quad 3.68 \quad -76 \quad I_5 \quad 0.549 \quad 190 \\
&\quad \quad \quad -0.0027 \quad 0.105 \quad 0 \quad 0.0145 \quad -1 \\
\end{align*} \]

\[ \Delta \]
The determinants in equations (III-5) through (III-9) can not be solved by a computer program since they have elements such as $I_2$, $I_3$, $I_5$, $V_2$ and $V_3$ in them. They have been solved on the basis of "paper and pencil" and the results are given below:

$$I_1 = 18.776 \, I_2 + 19.44 \, I_3 - 38.82 \, I_5 - 0.05 \, V_2 - 0.0146 \, V_3$$

$$I_2 = -7.53 \, I_2 + 11.37 \, I_3 + 19.80 \, I_5 + 0.02 \, V_2 - 0.008 \, V_3$$

$$I_3 = 3.29 \, I_2 - 11 \, I_3 - 10.25 \, I_5 -0.0089 \, V_2 + 0.008 \, V_3$$

$$I_5 = 87.7 \, I_2 -184.75 \, I_3 - 267.4 \, I_5 - 0.237 \, V_2 + 0.139 \, V_3$$
\[
\begin{align*}
\dot{R}_1 &= -0.04 \ I_2 + 0.0347 \ I_3 - 0.233 \ I_5 + 0.0001 \ V_2 - 0.000026 \ V_3 \\
&= (III-10)
\end{align*}
\]

Equations (III-10) now can be put in the state form of (III-1).

\[
\begin{bmatrix}
\dot{I}_1 \\
\dot{I}_2 \\
\dot{I}_3 \\
\dot{I}_5 \\
\dot{R}_1
\end{bmatrix} =
\begin{bmatrix}
0 & 18.77 & 19.44 & -38.82 & 0 \\
0 & -7.53 & 11.37 & 19.80 & 0 \\
0 & 3.29 & -11 & -10.25 & 0 \\
0 & 87.7 & -184.75 & -267.4 & 0 \\
0 & -0.04 & 0.0347 & -0.233 & 0
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
I_5 \\
R_1
\end{bmatrix} +
\begin{bmatrix}
-0.05 \\
0.02 \\
0.0089 \\
-0.237 \\
0.0001
\end{bmatrix}
\begin{bmatrix}
V_2 \\
V_3
\end{bmatrix}
\]

where:

\[
A =
\begin{bmatrix}
0 & 18.77 & 19.44 & -38.82 & 0 \\
0 & -7.53 & 11.37 & 19.80 & 0 \\
0 & 3.29 & -11 & -10.25 & 0 \\
0 & 87.7 & -184.75 & -267.4 & 0 \\
0 & -0.04 & 0.0347 & -0.233 & 0
\end{bmatrix}
\]

\[
B =
\begin{bmatrix}
-0.05 & -0.0146 \\
0.02 & -0.008 \\
0.0089 & 0.008 \\
-0.237 & 0.139 \\
0.0001 & -0.000026
\end{bmatrix}
\]
The output $Y$ that we are concerned with is:

$$
Y = \begin{bmatrix}
I_1 \\
R_1
\end{bmatrix} = \begin{bmatrix}
X_1 \\
X_5
\end{bmatrix}
$$

where again, $I_1$ is the plasma current and $R_1$ is the plasma position.

The matrix $C$ of equation (III-2) then will be:

$$
C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

In this section we put the set of five equations (II-16) in the state form and found the constant matrices $A$, $B$ and $C$ of equations (III-1) and (III-2). These matrices will be used in Section IV for optimization designs and the feedback systems.
IV PLASMA CURRENT AND POSITION OPTIMIZATION PROBLEM

The quadratic criteria is used for an optimal design of our system. The dynamic equations of the system were given in section II and they were put in the state variable form in section III.

\[
\dot{X} = AX + BU
\]

\[
Y = CX
\]

The main goal in this section is to find an optimal input, \(U\) such that the output of the system shown in Figure IV-1 be "close" to a desired value.

\[
U = \begin{bmatrix} V_2 \\ V_3 \end{bmatrix}
\]

Figure IV-1 Plasma and Coil Currents Block.

QUADRATIC CRITERIA

The objective is to control the system IV-1 so that the output vector \(Y(t)\) is "near" the vector \(Z(t)\) where \(Z(t)\) is the desired output. We may then define an error
Therefore generally speaking, the control objective is:

Find a control law $U(t)$ so that the error $e(t)$ is "small".

The most general form of the cost function that is considered in control problems is:

$$J(u) = \frac{1}{2} \left[ Z(T) - Y(T) \right] , F \left[ Z(T) - Y(T) \right] +$$

$$\frac{1}{2} \int_{t_0}^{T} \left[ <[Z(t) - Y(t)], Q(t)[Z(t) - Y(t)]> + <U(t), R(t)U(t)> \right] dt$$

$T$ is the terminal time and is usually specified. In our problem $T = \infty$, since our system is a time invariant system and matrices $A$, $B$ and $C$ in equations (IV-1) are constants. $F$ is a constant $(mxm)$ positive semidefinite matrix$, Q(t)$ is an $(mxm)$ positive semidefinite matrix and $R(t)$ is an $(rxr)$ positive definite matrix.

Let us consider the first part of the equation (IV-3)

$$\frac{1}{2} <e(t), F e(t)>$$

This term which is called terminal cost, guarantees that the error $e(t)$ at the terminal time $T$ is small. Since

---

1. For a definition of positive semidefinite, positive definite refer to any optimal control book such as references 7 and 8.
our terminal time is \( T = \infty \), we can set \( F = 0 \) and rely upon the rest of the cost function to guarantee that the terminal error is not excessively large. According to Athans\(^7\), \( F \) and \( Q \) matrices cannot both be zero. If we set \( Q = F = 0 \) in equation (IV-3), then the only \( U \) that minimize the cost function will be \( U(t) = 0 \) which is trivial. To exclude this case, we will consider \( Q \neq 0 \). A model of the cost function for a Tokamak machine is given by R. Gran, M. Rossi and F. Sobierajski\(^9\) as below:

\[
J = \int_0^\infty \left( w_1 \Delta R_1^2 + w_2 \Delta \mathbf{i_1}^2 + V_2^2 + w_3 V_3^2 \right) dt \quad (IV-4)
\]

where \( w_1, w_2 \) and \( w_3 \) are some coefficients to be calculated. \( V_2 \) and \( V_3 \) are the power supply voltages and \( F = 0 \).

To find \( w_1, w_2 \) and \( w_3 \) we expand the terms in equation (IV-3), considering that the notation \( \langle X, Y \rangle = X^T \cdot Y \) where \( X \) and \( Y \) are vectors and \( X^T \) is the transpose of the vector \( X \).

\[
J(u) = \frac{1}{2} \int_0^\infty \left[ \langle e(t), Q e(t) \rangle + \langle U(t), R U(t) \rangle \right] dt \quad (IV-5)
\]

e\( (t) \), the error function in equation (IV-5) is actually:

\[
e(t) = \begin{bmatrix}
\Delta \mathbf{i_1} \\
\Delta R_1
\end{bmatrix}
\quad (IV-6)
\]
Substituting equation (IV-6) in the first term of the equation (IV-5) yields:

\[
\langle e(t), Q e(t) \rangle = \langle \begin{bmatrix} \Delta I_1 \\ \Delta R_I \end{bmatrix}, \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} \Delta I_1 \\ \Delta R_I \end{bmatrix} \rangle 
\]

\[
= \begin{bmatrix} \Delta I_1 & \Delta R_I \end{bmatrix} \begin{bmatrix} Q_{11} \Delta I_1 + Q_{12} \Delta R_I \\ Q_{21} \Delta I_1 + Q_{22} \Delta R_I \end{bmatrix}
\]

\[
= Q_{11} \Delta I_1^2 + Q_{12} \Delta I_1 \Delta R_I + Q_{21} \Delta I_1 \Delta R_I + Q_{22} \Delta R_I^2
\]

(IV-7)

The second term of equation (IV-5) is:

\[
\langle U(t), R U(t) \rangle = \langle \begin{bmatrix} V_2 \\ V_3 \end{bmatrix}, \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} V_2 \\ V_3 \end{bmatrix} \rangle
\]

\[
= \begin{bmatrix} V_2 & V_3 \end{bmatrix} \begin{bmatrix} R_{11} V_2 + R_{12} V_3 \\ R_{21} V_2 + R_{22} V_3 \end{bmatrix}
\]

\[
= R_{11} V_2^2 + R_{12} V_2 V_3 + R_{21} V_2 V_3 + R_{22} V_3^2
\]

(IV-8)

We can substitute equations (IV-7) and (IV-8) in equation (IV-5) and get:

\[
J(u) = \frac{1}{2} \int_0^\infty \left[ Q_{11} \Delta I_1^2 + (Q_{12} + Q_{21}) \Delta I_1 \Delta R_I + Q_{22} \Delta R_I^2 \\
+ R_{11} V_2^2 + (R_{11} + R_{21}) V_2 V_3 + R_{22} V_3^2 \right] dt
\]

(IV-9)
Now, if we compare equations (IV-9) and (IV-4) we can calculate the coefficients $w_1$, $w_2$ and $w_3$:

$$w_1 = Q_{22}$$
$$w_2 = Q_{11}$$
$$w_3 = R_{22}$$

We also conclude that:

$$Q_{12} + Q_{21} = 0$$
$$R_{12} + R_{21} = 0$$
$$R_{11} = 1$$

Matrices $Q$ and $R$ are considered to be symmetric. See for example page 783 of reference [7]. Therefore:

$$Q_{12} = Q_{21} = 0$$
$$R_{12} = R_{21} = 0$$

For simplicity of calculations we shall consider:

$$R_{22} = 1$$
$$Q_{11} = Q_{22} = 1$$

These assumptions of course are arbitrary and depend to the specifications of a particular system. However, $R$ and $Q$ have to be chosen such that $R$ to be a positive definite matrix and $Q$ a positive semidefinite matrix. The $Q$ and $R$ matrices that we will consider for our analysis therefore will be:

$$R = Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(IV-10)
The theory that we will be using to solve the output-regulator problem of (IV-1) is actually true in the case of $T$, the terminal time, being fixed and specified. A theory for the case $T = \infty$ is not available at this time. However according to Athans [7] for engineering problems, such as the one we are concerned, could be approximated by this theory.

By some investigation and realization we find our problem to be a Tracking problem [10]. The difference between a tracking problem and an output-regulator is that in the first one we are trying to compare the system output with a desired value, in other words to track the output pattern, while in the later one, the inputs to the system are zeros.

Suppose that the vector $Z(t)$ is the desired output pattern and has the same dimension of the output $Y(t)$. Our objective is to control the system in such a way to make the output $Y(t)$ be near $Z(t)$ without excessive control energy expenditure.

The error function was given in equation (IV-2). Substituting for $Y(t)$ by the second equation of (IV-1) we get:

$$e(t) = Z(t) - C X(t)$$
Z(t) is a constant and is given to us as:

\[ Z(t) = Z = Y = \begin{bmatrix} -100 & KA \\ 0.95 \text{ Meter} \end{bmatrix} \quad (IV-11) \]

Then there exist a unique control law \( U(t) \) and is given as below [7]:

\[ U(t) = R^{-1} B^T \begin{bmatrix} g(t) - K(t) X(t) \end{bmatrix} \quad (IV-12) \]

where the \((n \times n)\) real, symmetric and positive definite matrix \( K(t) \) is the solution of Riccati-type matrix differential equation:

\[
\dot{K}(t) = - K(t) A(t) - A^T(t) K(t) + \\
K(t) B(t) R^{-1}(t) B^T(t) K(t) - C^T(t) Q(t) C(t)
\]

(IV-13)

We know that as \( T \to \infty \), the "gain" matrix \( K(t) \) of the Riccati equation (IV-13) tends to the constant positive definite matrix \( K \). Therefore \( K \) will be the solution of the algebraic matrix equation:

\[-K A - A^T K + K B R^{-1} B^T K - C^T Q C = 0 \quad (IV-14)\]

We have calculated \( A, B, C, R \) and \( Q \) matrices of equation (IV-14).
A is a (5x5) matrix.

B is a (5x2) matrix.

C is a (2x5) matrix.

R is a (2x2) matrix.

Q is a (2x2) matrix.

K is a (5x5) matrix.

\[
K = \begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55}
\end{bmatrix}
\]  

Even though equation (IV-14) is an algebraic equation, it is still very difficult to solve for \( K \). This equation results in solving 15 nonlinear equations with 15 unknowns. Nonlinear equations in contrast to the linear equations that could be programmed easily on a digital computer, cannot be programmed easily. The other problem with nonlinear equations is that there is not a unique solution for them and each solution has to be checked and chosen such that the matrix \( K \) be a positive definite matrix.

The "paper and pencil" solution of equation (IV-14) for \( n > 3 \) in general (\( n = 5 \) in our case) is almost hopeless. An alternative solution was introduced recently by Alan J.
Laub [11]. This method which is called a Schur method was modified and used to find the matrix K. This has been explained in detail in Appendix I. The main Fortran program and all the subroutines are also included in that Appendix. The solution of K matrix is given here:

\[
K = \begin{bmatrix}
6.2714 & 29.2594 & 54.1076 & -0.8108 & -7.6385 \\
29.2594 & 159.6670 & 246.0733 & -1.7773 & -135.2679 \\
54.1076 & 246.0733 & 484.3711 & -8.4462 & 29.0088 \\
-0.8108 & -1.7773 & -8.4462 & 0.3190 & -10.7080 \\
-7.6385 & -135.2679 & 29.0088 & -10.7080 & 790.7151
\end{bmatrix}
\]

The control law \( U \) given in equation (IV-12) for our time-invariant system where the K matrix is a constant is:

\[
U(t) = R^{-1}B^T \left[ g(t) - KX(t) \right] \tag{IV-16}
\]

Using this control law the system equation can be written as below: See reference [7] page 802 for mathematical detail.

\[
\dot{X}(t) \equiv (A - BR^{-1}B^TK)X(t) + BR^{-1}B^Tg(t) \tag{IV-17}
\]

where \( g(t) \) is given by:

\[
\dot{g}(t) \equiv (A - BR^{-1}B^TK)^TX(t) - C^TQZ \tag{IV-18}
\]

Let \( G, S \) and \( W \) be the matrices:
G = A - B R^{-1} B^T K \quad \text{(IV-19)}
S = B R^{-1} B^T \quad \text{(IV-20)}
W = C^T Q \quad \text{(IV-21)}

Then the equations (IV-17) and (IV-18) will be:

\dot{X} \simeq G X(t) + S g(t) \quad \text{(IV-22)}

\dot{g}(t) \simeq -G^T g(t) - W Z \quad \text{(IV-23)}

The solution of equation (IV-23) is derived and discussed by Athans [7], page 803, for time-invariant systems and is given as:

\begin{align*}
g(t) &\simeq -(G^T)^{-1} W Z \simeq g \\
\end{align*} \quad \text{(IV-24)}

where g is a constant vector. In this case, equation (IV-22) can be rewritten as:

\dot{X} \simeq G X(t) + S g \quad \text{(IV-25)}

G, S and W can be calculated from equations (IV-19), (IV-20) and (IV-21). g then can be calculated from equation (IV-24). This value then can be substituted in equation (IV-25) for g. At this time we will be able to solve equation (IV-25) to find X. The values of X then can be substituted in equation (IV-16) to find the optimal input vector U. These calculations have been made and the results are summarized below:
G of equation (IV-19) is:

\[
\begin{bmatrix}
0.047152 & 18.986708 & 19.865729 & -38.827297 & 0.026349 \\
-0.018933 & -7.616396 & 11.198903 & 19.802962 & -0.011701 \\
-0.008364 & 3.251315 & -11.075464 & -10.248718 & -0.004242 \\
0.224596 & 88.722909 & -182.719892 & -267.435235 & 0.142290 \\
-0.000095 & -0.040433 & 0.033845 & -0.232985 & -0.000057
\end{bmatrix}
\]

W of equation (IV-21) is:

\[
W = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}
\]

g of equation (IV-24) is:

\[
g = \begin{bmatrix}
-120506.0457 \\
-1647280.115 \\
-30153.36912 \\
-110673.5915 \\
8433799.8550
\end{bmatrix}
\]

S of equation (IV-20) is:
Then equation (IV-25), \( \dot{X} = G\dot{X}(t) + Sg \) will be:

\[
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4 \\
X_5
\end{bmatrix} = \begin{bmatrix}
4.71 \times 10^{-2} & 1.89 \times 10^1 & 1.98 \times 10^1 & -3.88 \times 10^1 & 2.63 \times 10^{-2} \\
-1.89 \times 10^{-2} & -7.61 \times 10^0 & 1.12 \times 10^1 & 1.98 \times 10^1 & -1.17 \times 10^{-2} \\
-8.36 \times 10^{-3} & 3.25 \times 10^0 & -1.11 \times 10^1 & -1.02 \times 10^1 & -4.24 \times 10^{-3} \\
2.24 \times 10^{-1} & 8.87 \times 10^0 & -1.83 \times 10^2 & -2.67 \times 10^2 & 1.42 \times 10^{-1} \\
-9.50 \times 10^{-5} & -4.04 \times 10^{-2} & 3.38 \times 10^{-2} & -2.33 \times 10^{-1} & -5.70 \times 10^{-5}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4 \\
X_5
\end{bmatrix}
\]  

\( (IV-28) \)

\[
+ \begin{bmatrix}
1.92 \times 10^1 \\
4.97 \times 10^0 \\
-8.29 \times 10^0 \\
-9.84 \times 10^1 \\
5.00 \times 10^{-3}
\end{bmatrix}
\]  

\( (IV-29) \)

Equation (IV-29) can be solved for \( X_1, X_2, X_3, X_4 \) and \( X_5 \) that represent \( I_1, I_2, I_3, I_5 \) and \( R_I \) respectively.

A CSMP simulation program has been used to plot the response of the system for \( I_1 \) and \( R_I \). This is given in
Section VII. Analytical expressions can also be calculated for use in equation (IV-16) to find the control law $U(t)$. This is rather difficult to obtain. The procedure is to find $X$ from equation (IV-30) given below:

$$X = \Phi(t) X_0 + \int_0^t \Phi(t-\tau) D \, d\tau$$

(IV-30)

where:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$X_0 = \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \\ x_{40} \\ x_{50} \end{bmatrix}$$

$$\Phi(t) = e^G$$

(IV-31)

$$D = S \, g$$

$S, \, g$ and $G$ are given in equations (IV-28), (IV-27) and (IV-26) respectively.
Different methods are available to find $\Phi(t)$ such as Sylvester Interpolation, Laplace transform etc. See for example references [12], [13] or [14].

The method that is recommended here is to find $\Phi(t)$ from equation (IV-32) given below:

$$\Phi(t) = L^{-1} \left[ SI - G \right]^{-1} \quad (IV-32)$$

where $I$ is the identity matrix of order (5x5) and $L^{-1}$ denotes the Laplace inverse.

Now we can present the block diagram for the optimal control system as proposed by Athans and Falb[7]. This block diagram is shown in Figure IV-2. This structure as it is mentioned before, is an approximate optimal time invariant system. Developing a theory for the case of $T = \infty$, would be a future research subject. Nevertheless, the indicated approximation system of Figure IV-2 can be used for this engineering problem. All the elements in the blocks of Figure IV-2 have been calculated.

Figure (IV-3) shows the block diagram of the entire control system[5]. As it can be seen there are four different blocks, Plasma and Coil equation, Feedback Circuits, Inner Supply and Outer Supply.

The block containing plasma and coil equations was
described in this section. The transfer function of this block is derived and discussed in Appendix II. The feedback circuits and the power supplies will be discussed in Sections V and VI respectively.
Figure IV-2  The structure of the approximate optimal time invariant system when the desired output is a constant vector $Z$. 
Figure IV-3 The Block Diagram of the Entire Control System.
V FEEDBACK CIRCUITS

The block diagram of the entire system is given in Figure (IV-3) of Section four. One of the blocks of this figure, the feedback circuits block, will be discussed in this section and the transfer function will be derived. Figure (V-1) shows this block and specifies the inputs and the outputs.

![Feedback Circuits Block Diagram](image)

As it can be seen, two sets of inputs \([I_1(t), R_1(t)]\) and \([I_{10}, R_{10}]\) are combined and the outputs \(e_2\) and \(e_3\) are produced. The error functions \(e_2\) and \(e_3\) are given in several papers, see for example [4] or [6] and they have the general form of:

\[
e_2 = K_R I_1 \left[ \Delta R_I + \mathcal{C}_R I_1 \Delta R_I + \mathcal{C}_{I_1} I_1 \Delta I_1 \right] \quad \text{(V-1)}
\]

\[
e_3 = K_R O \left[ \Delta R_I + \mathcal{C}_R O \Delta R_I + \mathcal{C}_{I_1} O \Delta I_1 \right] + K_I I_1 \left[ \Delta I_1 + \mathcal{C}_{I_1} \Delta I_1 \right] \quad \text{(V-2)}
\]

As a first try, some numerical values are considered
for the coefficients of equations (V-1) and (V-2). These values are listed in Table (V-1). We substitute these values in equations (V-1) and (V-2) and we get:

\[ e_2 = 4(\Delta I_1 + 0.005 \Delta I_1) \]  \hspace{1cm} (V-3)

\[ e_3 = 60(\Delta R_I + 0.005 \Delta R_I) \]  \hspace{1cm} (V-4)

where:

\[ \Delta I_1 = I_1 - I_{10} \]
\[ \Delta R_I = R_I - R_{I0} \]

Therefore:

\[ e_2 = 0.02 \dot{I}_1 + 4 I_1 + 400 \]  \hspace{1cm} (V-5)

\[ e_3 = 0.3 \dot{R}_I + 60 R_I - 57 \]  \hspace{1cm} (V-6)

We have considered the initial values of \( I_{10} \) and \( R_{I0} \) as:

\[ I_{10} = -100 \text{ KAmpl.} \]
\[ R_{I0} = 0.95 \text{ m} \]

To find the transfer function of the feedback circuits we will take the Laplace transform of equations (V-5) and (V-6).

\[ E_2(s) = (4 + 0.02 s)I_1(s) + \frac{400}{s} \]  \hspace{1cm} (V-7)

\[ E_3(s) = (60 + 0.3 s)R_I(s) - \frac{57}{s} \]  \hspace{1cm} (V-8)
Equations (V-7) and (V-8) can be put in the form:

\[
\begin{bmatrix}
E_2(s) \\
E_3(s)
\end{bmatrix} = \begin{bmatrix}
(4 + 0.02s) & 0 \\
0 & (60 + 0.3s)
\end{bmatrix} \begin{bmatrix}
I_1(s) \\
R_1(s)
\end{bmatrix} + \begin{bmatrix}
\frac{400}{s} \\
\frac{-57}{s}
\end{bmatrix}
\]

(V-9)

Equations (V-7) and (V-8) have been used to draw the block diagram of the feedback circuits. This is shown in Figure (V-2).
TABLE (V-1)
Coefficients of the error functions $e_2$ and $e_3$.

<table>
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<tr>
<th>Coefficients</th>
<th>Value</th>
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<tr>
<td>$K^O_{R}$</td>
<td>60.0 kamp/m</td>
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<tr>
<td>$\tau^I_{I}$</td>
<td>5.0 msec.</td>
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<tr>
<td>$\tau^O_{R}$</td>
<td>5.0 msec.</td>
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</tbody>
</table>
Figure (V-2) Feedback Circuits Block Diagram.
VI POWER SUPPLY MODEL

There are basically two power supplies in the Tokamak control system as shown in Figure (IV-3) of Section four. They are Inner and Outer power supplies. Their role is to control $I_1$ and $R_1$, the plasma current and its position. In this section we will briefly discuss these supplies.

**Inner Supply**

It is assumed the inner supply to be current-regulated \([4]\) and \([6]\) . The model used in these papers, utilizes a control signal \(I_{C2}\) which satisfies:

\[
\tau_2 I_{C2} = K_2 e_2 \tag{VI-1}
\]

where $e_2$ is the error signal and was discussed in section five. $K_2$ is the constant of the supply and is considered to be 1. $\tau_2 = 5$ msec. Figure VI-1 shows $V_2$ as a function of $I_2$. The output voltage $V_2$ depends linearly on the difference between the demanded current $I_{C2}$ and the actual current $I_2$.

\[
V_2 = R_{\text{eff2}} (I_{C2} - I_2) \tag{VI-2}
\]

where:

\[
R_{\text{eff2}} = 0.2 \text{ ohms}.
\]
Outer Supply

Similarly we have the output voltage $V_3$ which is described as:

$$V_3 = \text{R}_{\text{eff}3} (I_{C3} - I_3) \quad \text{(VI-3)}$$

where:

$$\text{R}_{\text{eff}3} = 0.05 \text{ ohms}.$$ 

and the control signal $I_{C3}$ can be calculated from

$$\zeta_3 I_{C3} = K_3 e_3 \quad \text{(VI-4)}$$

where

$$\zeta_3 = 5 \text{ msec}.$$ 

$$K_3 = 1$$

There are some numerical values given by different people for the power supply voltages. The one in the reference [6] states that the power supply chosen for the inner control winding can not be negative. Its maximum voltage is +72 volts. It is set to 0 if $I_2 \geq I_{C2}$ and for the case $I_2 < I_{C2}$, it will be set at the smaller of $72 - 0.002 I_2$ or $1000(\frac{I_{C2} - I_2}{I_{C2}})$.

The power supply chosen for the outer winding is
positive with some capability for negative excursions. Its maximum positive voltage is +130 volts and it is set to \(-0.0028(I_3 - I_{C3})\) whenever \(I_3 \geq I_{C3}\). For the case \(I_3 < I_{C3}\), \(V_3\) will be set at the smaller of 130 volts or \(0.056(I_{C3} - I_3)\).
The behavior of the voltage and current in the current-controlled power supply model.

Figure VI-1 The behavior of the voltage and current in the current-controlled power supply model.
VII SIMULATION RESULTS

A CSMP simulation of our system for $X_1$ and $X_5$ which represent $I_1$, the Plasma current, and $R_1$ the plasma horizontal position, has been performed and the results are given in this section.

As the plots show, for any small step perturbation the system corrects its error and tracks the desired value.

The specifications of these simulations are summarized in Table (VII-1). For simplicity, an approximate model has been used and the higher order exponential terms have been omitted. The approximate system seems to be stable. However the response of the actual system might need some compensation. This would be an area for future research.

The behavior of $X_2$, $X_3$ and $X_4$ which represent $I_2$, $I_3$ and $I_5$ respectively has not been considered throughout this thesis, since the control of plasma current and its position was our only concern. However a similar procedure can be used to control all other currents. Here are some future research topics:

1. Controlling other currents of the system.
2. Controlling the Vertical plasma position.
3. Controlling the response time of the system.
### Table (VII-1) Specifications of the plasma current and position simulations.

<table>
<thead>
<tr>
<th>Figure</th>
<th>Response</th>
<th>Desired output value</th>
<th>Perturbation value</th>
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<td>VII-4</td>
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<td>VII-6</td>
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<td>VII-8</td>
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**Figure (VII-1)** Plasma current response for a +5 ka perturbation. The desired value is -100 kamps.
Figure (VII-2) Plasma current response for a - 5 ka perturbation. The desired value is -100 kamps.
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<td>2.5000E+01</td>
<td>-1.1133E+02</td>
<td>+</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure (VII-3)** Plasma current response for a + 5 ka perturbation. The desired value is -112 kamps.
Figure (VII-4) Plasma current response for a -5 ka perturbation. The desired value is -112 kamps.
Figure (VII-5) Plasma position response for a +0.03 m perturbation. The desired value is 0.95 meters.
Figure (VII-6) Plasma position response for a perturbation. The desired value is 0.95 meters.
Figure (VII-7) Plasma position response for a + 0.03 m perturbation. The desired value is 1.03 meters.
Figure (VII-8) Plasma position response for a $-0.03$ m perturbation. The desired value is 1.03 meters.
APPENDIX I

A SCHUR METHOD FOR SOLVING ALGEBRAIC RICCATI EQUATION

The algebraic form of the Riccati equation can be presented as:

\[ K A + A^T K + C^T Q C - K B R^{-1} B^T K = 0 \]  \hspace{1cm} (A-I-1)

where \( A \) and \( K \) are (nxn) matrices, \( B \) is an (nxm) matrix, \( C \) is a (pxn) matrix. \( Q \) and \( R \) are considered to be symmetric matrices. Therefore \( C^T Q C \) and \( B R^{-1} B^T \) will be symmetric matrices. Let:

\[ C^T Q C = H \]
\[ B R^{-1} B^T = G \]

then:

\[ K A + A^T K + H - K G K = 0 \]  \hspace{1cm} (A-I-2)

\( A, H \) and \( G \) are all matrices of order (nxn) in general. \( K \) therefore will be an (nxn), symmetric matrix. To find the elements of matrix \( K \) we have to solve \( \frac{n(n + 1)}{2} \) nonlinear equations.

**EXAMPLE 1**

Find the Riccati matrix \( K \) if \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \), \( B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( C = \begin{bmatrix} 0 & 1 \end{bmatrix} \) and \( Q = R = 1. \)
SOLUTION

Substituting A, B, C, Q and R matrices in equation (A-I-1) yields:

\[
\begin{bmatrix}
K_{11} & K_{12} \\
K_{12} & K_{22}
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
K_{11} & K_{12} \\
K_{12} & K_{22}
\end{bmatrix}
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\begin{bmatrix}
K_{11} & K_{12} \\
K_{12} & K_{22}
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 1
\end{bmatrix}
\]

\[
- \begin{bmatrix}
K_{11} & K_{12} \\
K_{12} & K_{22}
\end{bmatrix}
\begin{bmatrix}
0 \\
1
\end{bmatrix}
(1)
\begin{bmatrix}
0 & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
K_{11} & K_{12} \\
K_{12} & K_{22}
\end{bmatrix}
= 0
\]

or:

\[
\begin{bmatrix}
( - K_{12}^2 ) & (K_{11} - K_{12} K_{22} ) \\
(K_{11} - K_{12} K_{22} ) & (2K_{12} + 1 - K_{22} )
\end{bmatrix}
= 0
\]

or:

\[
- K_{12}^2 = 0
\]

\[
K_{11} - K_{12} K_{22} = 0
\]

\[
2K_{12} + 1 - K_{22}^2 = 0
\]

(A-i-3)

Solving the three nonlinear equations of (A-I-3) we get:

\[
K = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\quad \text{and} \quad K = \begin{bmatrix}
0 & 0 \\
0 & -1
\end{bmatrix}
\]

The acceptable choice for $K_{22}$ is +1, since $K_{22} = -1$ does not make the K matrix positive semidefinite.

In the above example we had to solve three nonlinear equations which in this case were simple. For higher order
systems the solution is very complicated.

The nature of nonlinear equations is that they do not have a unique solution and even if we find a set of solution, we have to pick the one that makes the K matrix nonnegative definite and that requires some extra calculation.

The Schur method described by Laub [11] automatically produces the desired solution for the K matrix, that is, the nonnegative definite solution. The program, however, is capable of producing other solutions as well. The method is a variant of the classical eigenvector approach to Riccati equations as can be seen in papers such as [15] and [16], for example.

We will state the following theorems to support the method:

1. Let A to be an (nxn) matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$. Then there exists a unitary similarity transformation $U$ such that $U^H A U$ is upper triangular with diagonal elements $\lambda_1, \ldots, \lambda_n$ in that order. $H$ denotes the conjugate transpose.

2. Let A to be an (nxn) matrix. Then there exists an orthogonal similarity transformation $U$ such that $U^T A U$ is quasi-triangular. Moreover, $U$ can be chosen so that
the 2x2 and 1x1 diagonal blocks appear in any desired order.

In Riccati equation of (A-I-2) it is assumed that 
\[ G = G^T \geq 0, \quad H = H^T \geq 0. \]
We can form a matrix \( Z \) of order (2nx2n) such that:

\[
Z = \begin{bmatrix}
A & -G \\
-H & -A
\end{bmatrix}
\]

Then our assumptions guarantee that \( Z \) has no pure imaginary eigenvalues\[17\]. Therefore by theorem 2 we can find an orthogonal transformation \( U \) of order (2nx2n) which puts \( Z \) in the Real Schur Form

\[
U^T Z U = S = \begin{bmatrix}
S_{11} & S_{12} \\
0 & S_{22}
\end{bmatrix}
\]

It is possible to arrange \( S \) such that the real parts of the spectrum of \( S_{11} \) are negative while the real parts of the spectrum of \( S_{22} \) are positive. We can then partition the transformation \( U \) into four (nxn) blocks.

\[
U = \begin{bmatrix}
U_{11} & U_{12} \\
U_{21} & U_{22}
\end{bmatrix}
\]

Then we can use a theorem that states:
$U_{11}$ is invertible and $U_{11}^T K = U_{21}^T [18]$.

Therefore there are two steps to the Schur vector approach. First, reduction of a $2n \times 2n$ matrix to an ordered real Schur form and second, is the solution of an $n$-th order linear matrix equation.

The Double Francis QR Algorithm [19], although finds the eigenvalues of a matrix, it does not guarantee any special order for the eigenvalues on the diagonal of the Schur form. Stewart [20] has published Fortran subroutines for calculating and ordering the Real Schur Form of a real upper Hessenberg matrix. Therefore it is required to put the $Z$ matrix in upper Hessenberg form first. The Upper Hessenberg Form is completely described and discussed in several papers. See [21] for example.

Stewart's software HQR3 consists of four subroutines as follow:

1. Subroutine HQR3. (main subroutine)
2. Subroutine EXCHNG.
3. Subroutine QRSTEP.
4. Subroutine SPLIT.

To use HQR3 we have to use another software package called EISPACK [22] and [23]. We will use four subroutines of this package to prepare our system for HQR3. These subroutines are listed below:
1. BALANC
2. ORTHES
3. ORTRAN
4. BALBAK

Subroutine BALANC simply balances a real general matrix. This is actually a normalization process [24]. Subroutine ORTHES, reduces a balanced matrix to Upper Hessenberg form using orthogonal transformation [21]. Subroutine ORTRAN accumulates the transformations from the Hessenberg reduction [25]. Subroutine BALBAK back transforms the orthogonal matrix to a nonsingular matrix corresponding to the original matrix.

Up to this point we have found the orthogonal transformation $U$ that can be partitioned as:

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

The second step to be implemented is the solution of an $n$-th order linear matrix equation:

$$K U_{11} = U_{21} \quad (A-I-7)$$

We have used subroutine DECOMP and SOLVE [26] to solve equation (A-I-7).

A summary of the Schur method algorithm is given below:
1. Form matrix \( Z = \begin{bmatrix} A & -G \\ -H & -A^T \end{bmatrix} \).

2. Use subroutine BALANC to balance \( Z \).

3. Use subroutine ORTHES to reduce the balanced matrix to upper Hessenberg form using orthogonal transformations.

4. Use subroutine ORTRAN to accumulate the transformations from Hessenberg reduction.

5. Use subroutine HQR3 to determine an ordered real Schur form from the Hessenberg matrix.

6. Use subroutine BALBAK to back transform the orthogonal matrix to a nonsingular matrix corresponding to the original matrix.

7. Partition matrix \( U \) into four (nxn) matrices \( U_{11} \), \( U_{12} \), \( U_{21} \) and \( U_{22} \).

8. Solve the \( n \) linear equations \( K U_{11} = U_{21} \) to find the Riccati matrix \( K \).

Example 2

Find the \( K \) matrix of equation \((A-I-1)\) where:
The matrix $Z$ is:

$$Z = \begin{bmatrix} A & -G \\ -H & -A^T \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 \end{bmatrix}$$

and the matrix

$$U = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{5}}{10} & -\frac{3\sqrt{5}}{10} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{5}}{10} & -\frac{3\sqrt{5}}{10} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{3\sqrt{5}}{10} & \frac{\sqrt{5}}{10} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{3\sqrt{5}}{10} & \frac{\sqrt{5}}{10} & \frac{1}{2} \end{bmatrix}$$

is an orthogonal matrix which reduces $Z$ to the real Schur form of

$$S = U^T Z U = \begin{bmatrix} -1 & 0 & 1 & -\frac{1}{2} \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Partitioning matrix $U$ into four $2\times2$ matrices results:
The main Fortran program and all the subroutines are given at the end of this appendix. These subroutines were used to find the $K$ matrix of section IV on an IBM-370 machine. The total CPU time was 16 seconds, which is rather low comparing to some other methods for solving the algebraic Riccati equations.

\[
U_{11} = \begin{bmatrix}
\frac{1}{2} & -\frac{\sqrt{5}}{10} \\
-\frac{1}{2} & -\frac{\sqrt{5}}{10}
\end{bmatrix}
\]

\[
U_{21} = \begin{bmatrix}
\frac{1}{2} & -\frac{3\sqrt{5}}{10} \\
-\frac{1}{2} & -\frac{3\sqrt{5}}{10}
\end{bmatrix}
\]

and finally:

\[
\begin{bmatrix}
K_{11} & K_{12} \\
K_{12} & K_{22}
\end{bmatrix} = \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}
\]
Main program

C ***
C *** THIS PROGRAM SOLVES A RICCATI EQUATION OF THE ORDER
C *** 100 OR LESS. A,B,C,R AND Q MATRICES ARE GIVEN.
C *** FIRST WE FORM THE Z MATRIX.
C ***
DIMENSION A (10,10)
DIMENSION AM (5,5), BM (5,2), CM (2,5), QM (2,2), RM (2,2)
DIMENSION AMT (5,5), EMT (2,5), CMT (5,2), RMINV (2,2)
DIMENSION GM1 (2,5), GM (5,5), QC (2,5), HM (5,5)
DIMENSION P (10), SCALE (10), V (10,10), ER (10), EI (10), TYPE (10)
DIMENSION UCNE (5,5), UTHWO (5,5), UONEI (5,5), X (5,5)
DOUBLE PRECISION UONEX (5,10), CONST, ZERO
COMMON UONEX, ISNFLG, IDIM, J2, J1, ZERO, IPIVOT
C ***
C *** READ MATRICES A, B, C, Q AND R.
C ***
1101 READ (5,1101) ((AM(I,J), J=1,5), I=1,5)
1102 FORMAT (5P10.3)
1103 READ (5,1102) ((BM(I,J), J=1,2), I=1,5)
1104 FORMAT (2F10.3)
1105 READ (5,1103) ((CM(I,J), J=1,5), I=1,2)
1106 FORMAT (5F5.1)
1107 READ (5,1104) ((QM(I,J), J=1,2), I=1,2)
1108 FORMAT (2F5.1)
C ***
C *** FIND THE TRANSPORSE OF A, B AND C MATRICES.
C ***
1110 CONTINUE
1111 CONTINUE
1112 CONTINUE
1113 CONTINUE
1114 CONTINUE
C ***
C *** FIND THE INVERSE OF MATRIX R.
C ***
RMDET=AM(1,1)*RM(2,2)-RM(1,2)*RM(2,1)
RMINV(1,1) = RM(2,2) / BMDET
RMINV(1,2) = -RM(1,2) / BMDET
RMINV(2,1) = -RM(2,1) / BMDET
RMINV(2,2) = RM(1,1) / BMDET

C ***
C *** BUILD MATRIX 2.
C ***

DO 1120 I=1,2
DO 1120 J=1,5
T=0.0
DO 1119 L=1,2
1119 T=T+RMINV(I,L) * BM(I,J)
1120 GM1(I,J) = T
DO 1122 I=1,5
DO 1122 J=1,5
T=0.0
DO 1121 L=1,2
1121 T=T+BMT(I,L) * GM1(L,J)
1122 GM(I,J) = T
DO 1124 I=1,2
DO 1124 J=1,5
T=0.0
DO 1123 L=1,2
1123 T=T+CMT(I,L) * GM(L,J)
1124 QC(I,J) = T
DO 1126 I=1,5
DO 1126 J=1,5
T=0.0
DO 1125 L=1,2
1125 T=T+CMT(I,L) * QC(L,J)
1126 HM(I,J) = T
DO 1130 I=1,5
DO 1130 J=1,5
1130 A(I,J) = AM(I,J)
DO 1131 I=1,5
DO 1131 J=6,10
1131 A(I,J) = -GM(I,J-5)
DO 1132 I=6,10
DO 1132 J=1,5
1132 A(I,J) = -HM(I-5,J)
DO 1133 I=6,10
DO 1133 J=6,10
1133 A(I,J) = -AMT(I-5,J-5)
WRITE(6,789)
789 FORMAT (1H1, // // A, 10X, 'MATRIX 2 : ', //)
WRITE(6,1140) (A(I,J), J=1,10), I=1,10
1140 FORMAT (7X, 10F7.2)
READ (5, 130) KKNA, KKN
130 FORMAT (I2)
NA=KKNA
N=K\times K

C *** BALANCE MATRIX Z.
C ***

CALL BALANC (NA, N, A, NLOW, NUP, SCALE)
WRITE (6, 60)

60 FORMAT (//'//, 10X, 'MATRIX Z AFTER BEING BALANCED: ',//)
WRITE (6, 2) ((A (I, J), J=1, N), I=1, NA)

WRITE (6, 3) (SCALE (I), I=1, NA)

3 FORMAT (//'//, 10X, 'SCALE (I) = ',//, 10X, 5F7.2,
     *10X, 5F7.2)
WRITE (6, 4) NLOW, NUP

4 FORMAT (//'//, 20X, 'NLOW= ', I2, 10X, 'NUP= ', I2)

CALL ORTHES (NA, N, NLCW, NUP, A, P)
WRITE (6, 61)

61 FORMAT (1H1, /////, 10X, 'MATRIX Z IN HESSENBERG FORM: ',/) //
WRITE (6, 10) ((A (I, J), J=1, N), I=1, NA)

10 FORMAT (7X, 10F7.2)
WRITE (6, 11) (P (N), N=1, NA)

11 FORMAT (//'//, 10X, 'P (N) = ',//, 10X, 5 (1X, F7.2),
     *10X, 5 (1X, F7.2))
WRITE (6, 12) NLOW, NUP

12 FORMAT (//'//, 20X, 'NLOW= ', I2, 10X, 'NUP= ', I2)

CALL ORTRAN (NA, N, NLCW, NUP, A, P, V)
WRITE (6, 63)

63 FORMAT (10X, 'MATRIX Z AFTER ORTRAN: ',//)
WRITE (6, 20) ((A (I, J), J=1, N), I=1, NA)

20 FORMAT (7X, 10F7.2)
WRITE (6, 21) (P (I), I=1, NA)

21 FORMAT (1H1, /////, 10X, 'P (N) = ',//, 10X, 5 (1X, F7.2), 10X, 5 (1X, F7.2), //)
WRITE (6, 4) NLOW, NUP

C *** FIND EIGENVALUES OF Z.
C *** FIND THE ORTHOGONAL TRANSFORMATION U.
C ***

CALL HQR3 (A, V, N, NLCW, NUP, 1.6E-10, ER, EI, TYPE, NA, NA)
WRITE (6, 70)

70 FORMAT (//'//, 10X, 10X, 'THE ORTHOGONAL MATRIX U: ',//)
WRITE (6, 30) ((A (I, J), J=1, N), I=1, NA)

30 FORMAT (7X, 10F7.1)
WRITE (6, 80)

80 FORMAT (//'//, 10X, 'THE ORTHOGONAL MATRIX U: ',//)
WRITE (6, 30) ((V (I, J), J=1, N), I=1, NA)
WRITE (6, 31) (ER (N), N=1, NA)
WRITE (6, 32) (EI (N), N=1, NA)

31 FORMAT (1H1, /////, 10X, 'EI (N) = ',//, 10X, 5F8.1,
C     REBALANCE THE MATRICES Z AND U.
C     CALL BALBAK(NA,N,KLCW,NUP,SCALE,N,V)
      WRITE(6,110)
      FORMAT(1H1,1H,10X,'RSF FORM OF MATRIX Z AFTER REEALANCING :',//)
      WRITE(6,111) ((A(I,J),J=1,N),I=1,NA)
      FORMAT(7X,10F7.1)
      WRITE(6,112)
      FORMAT(1H1,1H,10X,'ORTHOGONAL MATRIX U AFTER REEALANCING :',//)
      WRITE(6,113) ((V(I,J),J=1,N),I=1,NA)
      FORMAT(7X,10F7.2)
      NNN=U/2
      III=N/2
      DO 504 I=1,III
      DO 503 J=1,III
      503 UONE(I,J)=V(I,J)
      504 CONTINUE
      III=III+1
      KKK=III-1
      DO 506 I=III,N
      DO 505 J=1,KKK
      505 UTWO(I-NNN,J)=V(I,J)
      506 CONTINUE
      WRITE(6,507)
      FORMAT(1H1,1H,'MATRICE UONE :',//)
      WRITE(6,508) ((UONE(I,J),J=1,NNN),I=1,NNN)
      FORMAT(10X,5F10.3)
      WRITE(6,509)
      FORMAT(1H1,1H,'MATRICE UTWO :',//)
      WRITE(6,510) ((UTWO(I,J),J=1,NNN),I=1,NNN)
      FORMAT(10X,5F10.3)
C     FIND THE INVERSE OF MATRIX UONE.
C     ZERO= .000001
      IDIM=N/2
      ISPLG=0
      J1=IDIM+1
      J2=IDIM*2
      DO 603 I=1,IDIM
      DO 603 J=1,J2
      603 UONEX(I,J)=0.0
      DO 604 N=1,IDIM
      J3=N*IDIM
UONEX(N,J3)=1.0
DO 599 I=1,IDIM
DO 598 J=1,IDIM
598 UONEX(I,J)=UONE(I,J)
599 CONTINUE
DO 701 IPIVOT=1,IDIM
IF((DABS(UONEX(IPIVCT,IPIVOT))) .LE.ZERO) CALL SWAPBW
IF(ISOONFLG) 698,699,698
599 DO 702 IROW=1,IDIM
IF(IROW.EQ.IPIVCT) GO TO 702
IF((DABS(UONEX(IROW,IPIVOT))) .LE.ZERO) GO TO 702
CONST=-(UONEX(IROW,IPIVOT)/UONEX(IPIVOT,IPIVCT))
DO 700 ICOL=1,J2
700 UONEX(IROW,ICOL)=UCNEX(IROW,ICOL)+CONST*UONEX(IPIVOT,ICOL)
702 CONTINUE
DO 705 IROW=1,IDIM
CONST=UCNEX(IROW,1ICOW)
DO 705 ICOL=J1,J2
UONEX(IROW,ICOL)=UCNEX(IROW,ICOL)/CONST
705 CONTINUE
WRITE(6,901)
901 FORMAT('THE INVERTED MATRIX UONE :',/)
DO 903 I=1,IDIM
DO 902 J=J1,J2
902 UONEI(I,J-IDIM)=UONE(I,J)
903 CONTINUE
WRITE(6,904) ((UONEI(I,J),J=1,IDIM),I=1,IDIM)
904 FORMAT(10X,5F10.3)
WRITE(6,1005)
1005 FORMAT(1H1,///,10X,'RICCATI MATRIX K :',/)
GO TO 952
998 WRITE(6,613)
613 FORMAT(1H0,10X,'THE MATRIX IS SINGULAR :',/)
952 CONTINUE
C ***
C *** FIND THE UPPER TRIANGULAR ELEMENTS OF
C *** RICCATI MATRIX K.
C ***
DO 1001 I=1,IDIM
DO 1001 J=1,IDIM
T=0.
DO 1000 L=1,IDIM
1000 T=T+UTWO(I,L)*UCNEI(L,J)
1001 X(I,J)=T
DO 1004 I=1,IDIM
DO 1003 J=1,IDIM
IF(I.LE.J) WRITE(6,1002) I,J,X(I,J)
1002 FORMAT(//,10X,'K',211,*,F11.4)
1003 CONTINUE
1004 CONTINUE
STCP
END
Subroutine SWAPRW

SUBROUTINE SWAPRW

C ***
C *** THIS SUBROUTINE IS A PART OF THE PROGRAM
C *** TO FIND THE INVERSE OF A MATRIX.
C ***

DOUBLE PRECISION FIXIT, UONEX(5, 10), ZERO
COMMON UONEX, ISNFLG, IDIM, J2, J1, ZERO, IPIVOT

IF (IPIVOT .EQ. IDIM) GO TO 1
IROWSW = IPIVOT + 1

3 IF ((DABS(UONEX(IROWSW, IPIVOT))) .LE. ZERO) GO TO 2
DO 4 ICCLSW = 1, J2
FIXIT = UONEX(IPIVOT, ICOLSW)
UONEX(IPIVOT, ICOLSW) = UONEX(IROWSW, ICOLSW)
UONEX(IROWSW, ICCLSW) = FIXIT

4 CONTINUE
RETURN

1 ISNFLG = 1
RETURN

2 IF (IROWSW .EQ. IDIM) GO TO 5
IROWSW = IRCWSW + 1
GO TO 3

5 ISNFLG = 1
RETURN
END
Subroutine BALANC

SUBROUTINE BALANC(NM,N,A,LOW,IGH,SCALE)
INTEGER I,J,K,L,M,N,JJ,NM,IGH,LOW,IEXC
REAL A(NM,N),SCALE(N)
REAL C,P,G,R,S,B2,RADIX
REAL ABS
LOGICAL NOCONV

C *** RADIX IS A MACHINE DEPENDENT PARAMETER SPECIFYING
C *** THE BASE OF THE MACHINE FLOATING POINT REPRESENTATION.
C ***
RADIX = 16.
B2=RADIX*RADIX
K=1
L=N
GO TO 100

C *** IN-LINE PRECEDURE FOR ROW AND COLUMN EXCHANGE.
C ***
20 SCALE(M) = J
   IF(J.EQ.M) GO TO 50
   DO 30 I = 1,L
       F=A(I,J)
       A(I,J)=A(I,M)
       A(I,M)=F
   CONTINUE
   DO 40 I=K,N
       F=A(J,I)
       A(J,I)=A(M,I)
       A(M,I)=F
   CONTINUE
50 GO TO (80,130),IEXC

C *** SEARCH FOR ROWS ISOLATING AN EIGENVALUE AND
C *** PUSH THEM DOWN.
C ***
80 IF (L.EQ.1) GO TO 280
    L=L-1
C *** FOR J=L STEP -1 UNTIL 1 DO
C ***
100 DO 120 JJ=1,L
      J=L+1-JJ
      DO 110 I=1,L
      IF (I.EQ.J) GO TO 110
      IF (A(J,I)<SE.0.0) GO TO 120
   CONTINUE
   M=L
   IEXC=1
   GO TO 20
120 CONTINUE
GO TO 140
C ***
C *** SEARCH FOR COLUMNS ISOLATING AN EIGENVALUE AND
C *** PUSH THEM LEFT.
C ***
130  K=K+1
140  DO 170  J=K,L
     DO 150  I=K,L
     IF(L.EQ.J) GC TO 150
     IF(A(I,J).NE.0.0) GC TO 170
150  CONTINUE
   M=K
   IMEX = 2
GO TO 20
170  CONTINUE
C ***
C *** NOW BALANCE THE SUBMATRIX IN ROWS K TO L.
C ***
180  DO 180  I=K,L
     SCALE(I)=1.0
C ***
C *** ITERATIVE LOOP FOR NCRM REDUCTION.
C ***
190  NOCONV=.FALSE.
     DO 270  I=K,L
     C=0.0
     R=0.0
     DO 200  J=K,L
     IF(J.EQ.I) GO TO 200
     C=C+ABS(A(J,1))
     R=R+ABS(A(I,J))
200  CONTINUE
C ***
C *** GUARD AGAINST ZERO C OR R DUE TO UNDERFLOW.
C ***
     IF(C.EQ.0.0 .OR. R.EQ.0.0) GC TO 270
     G=R/RADIX
     F=1.0
     S=C+R
210  IF(C.GE.G) GC TO 220
     F=F*RADIX
     C=C*B2
     GO TO 210
220  G=R/RADIX
230  IF(C.LT.G) GC TO 240
     F=F/RADIX
     C=C/B2
     GO TO 230
C ***
C *** NOW BALANCE.
C ***

240 IF ((C+R)/F .GE. 0.95*5) GO TO 270

   G=1.0/F
   SCALE(I)=SCALE(I)*F
   NOCONV=.TRUE.
   DO 250 J=K,N

250   A(I,J)=A(I,J)*G
   DO 260 J=1,L

260   A(J,I)=A(J,I)*F
   CONTINUE
   IF (NOCONV) GO TO 190

280   LOW=K
   IGH=L
   RETURN
   END
Subroutine ORTHES

SUBROUTINE ORTHES (NM, N, LOW, IGH, A, ORT)
INTEGER I, J, M, N, II, JJ, IA, KP, NM, IGH, KP1, LOW
REAL A (NM, N), ORT (IGH)
REAL F, G, H, SCALE
REAL SQRT, ABS, SIGN
LA = IGH - 1
KP1 = LOW + 1
IF (LA.LT. KP1) GO TO 200
DO 180 M = KP1, LA
H = 0.0
ORT (M) = 0.0
SCALE = 0.0

C ***
C *** SCALE COLUMN (ALGOL TOL THEN NEEDED)
C ***
DO 90 I = M, IGH
90 SCALE = SCALE + ABS (A (I, M - 1))
IF (SCALE .EQ. 0.0) GO TO 180
MP = M + IGH

C ***
C *** FOR I = IGH STEP -1 UNTIL M DO
C ***
DO 100 II = M, IGH
I = MP - II
ORT (I) = A (I, M - 1) / SCALE
H = H + ORT (I) * ORT (I)
100 CONTINUE
G = -SIGN (SQRT (H), ORT (M))
H = H - ORT (M) * G
ORT (M) = ORT (M) + G

C ***
C *** FORM (I - (U*UT)/H) * A
C ***
DO 130 J = M, N
F = 0.0

C ***
C *** FOR I = IGH STEP -1 UNTIL M DO
C ***
DO 110 II = M, IGH
I = MP - II
F = F + ORT (I) * A (I, J)
110 CONTINUE
F = F / H
DO 120 I = M, IGH
120 A (I, J) = A (I, J) - F * ORT (I)
130 CONTINUE

C ***
C *** FORM (I - (U*UT)/H) * A*(I - (U*UT)/H)
C ***
DO 160 I = 1, IGH
F=0.0

C *** FOR J = IGH STEP -1 UNTIL M DO
C ***

DO 140 JJ=M,IGH
  J=M-P-JJ
  F=F+ORT(J)*A(I,J)
140 CONTINUE
  F=F/H
DO 150 J=M,IGH
150 A(I,J)=A(I,J)-F*ORT(J)
160 CONTINUE
  ORT(M)=SCALE*ORT(M)
  A(M,M-1)=SCALE*G
180 CONTINUE
200 RETURN
END
Subroutine ORTRAN

Subroutine ORTRAN(NM, N, LOW, IGH, A, ORT, Z)
INTEGER I, J, N, KL, MM, MP, NM, IGH, LOW, MP1
REAL A(NM, IGH), ORT(IGH), Z(NM, N)
REAL G
C ***
C *** INITIALIZE Z TO IDENTITY MATRIX
C ***
DO 30 I=1, N
DO 30 J=1, N
30 Z(I, J) = 0.0
Z(I, I) = 1.0
CONTINUE
KL = IGH - LOW - 1
IF(KL .LT. 1) GO TO 200
C ***
C *** FOR MP = IGH - 1 STEP -1 UNTIL LOW + 1 DO
C ***
DO 140 MM = 1, KL
MP = IGH - MM
IF(A(MP, MP-1) .EQ. 0.0) GO TO 140
MP1 = MP + 1
DO 100 I = MP1, IGH
100 ORT(I) = A(I, MP - 1)
DO 130 J = MP, IGH
G = 0.0
DO 110 J = MP, IGH
G = G + ORT(J) * Z(I, J)
110 CONTINUE
C ***
C *** DIVISOR BELOW IS NEGATIVE OF H FORMED IN ORTHES.
C ***
C ***
C *** DOUBLE DIVISION AVOIDS POSSIBLE UNDERFLOW.
G = (G / ORT(MP)) / A(MP, MP - 1)
DO 120 I = MP, IGH
120 Z(I, J) = Z(I, J) + G * ORT(I)
CONTINUE
140 CONTINUE
200 RETURN
END
Subroutine HQR3

```fortran
SUBROUTINE HQR3(A, V, N, NLOW, NUP, EPS, ER, EI, TYPE, NA, NV)
INTEGER N, NA, NLOW, NUP, NV, TYPE(N)
REAL A(NA, N), EI(N), ER(N), EPS, V(NV, N)
C ** HQR3 REDUCES THE UPPER HESSIENBERG MATRIX A TO QUASI-
C ** TRIANGULAR FORM BY UNITARY SIMILARITY TRANSFORMATIONS.
C ** THE EIGENVALUES OF A, WHICH ARE CONTAINED IN THE 1*1
C ** AND 2*2 DIAGONAL BLOCKS OF THE REDUCED MATRIX, ARE
C ** ORDERED IN DESCENDING ORDER OF THE VALUES ALONG THE
C ** DIAGONAL. THE TRANSFORMATIONS ARE ACCUMULATED IN THE
C ** ARRAY V. HQR3 REPLIES THE SUBROUTINES EXCHG,
C ** QRSTEP AND SPLIT. THE PARAMETERS IN THE CALLING
C ** SEQUENCE ARE (STANDARD PARAMETERS ARE ALTERED BY THE
C ** SUBROUTINE)
C *** *A
AN ARRAY THAT INITIALLY CONTAINS THE N*N
UPPER HESSIANBERG MATRIX TO BE REDUCED. ON
RETURN A CONTAINS THE REDUCED QUASI-
TRIANGULAR MATRIX.
C *** *V
AN ARRAY THAT CONTAINS A MATRIX INTO WHICH
THE REDUCED TRANSFORMATIONS ARE TO BE
MULTIPLIED.
C *** N
THE ORDER OF THE MATRIX A AND V.
C *** NLOW A(NLOW, NLOW-1) AND A(NUP+1, NUP) ARE
ASSUMED TO BE ZERO, AND ONLY ROWS NLOW
THROUGH NUP AND COLUMNS NLOW THROUGH
NUP ARE TRANSFORMED, RESULTING IN THE
CALCULATION OF EIGENVALUES NLOW
THROUGH NUP.
C *** EPS
A CONVERGENCE CRITERION.
C *** *ER
AN ARRAY THAT ON THE RETURN CONTAINS THE REAL
PARTS OF THE EIGENVALUES.
C *** *EI
AN ARRAY THAT ON THE RETURN CONTAINS THE
IMAGINARY PARTS OF THE EIGENVALUES.
C *** *TYPE
AN INTEGER ARRAY WHOSE I-TH ENTRY IS
0 IF THE I-TH EIGENVALUE IS REAL,
1 IF THE I-TH EIGENVALUE IS COMPLEX
WITH POSITIVE IMAGINARY PART,
2 IF THE I-TH EIGENVALUE IS COMPLEX
WITH NEGATIVE IMAGINARY PART,
-1 IF THE I-TH EIGENVALUE WAS NOT
CALCULATED SUCCESSFULLY.
C *** NA THE FIRST DIMENSION OF THE ARRAY A.
C *** NV THE FIRST DIMENSION OF THE ARRAY V.
C ** THE CONVERGENCE CRITERION EPS IS USED TO DETERMINE
C ** WHEN A SUBDIAGONAL ELEMENT OF A IS NEGLIGIBLE.
C ** SPECIFICALLY A(I+1, I) IS REGARDED AS NEGLIGIBLE
C ** IF
C ** ABS(A(I+1, I)) .LT. EPS*(ABS(A(I, I)) + ABS(A(I+1, I+1))).
C ** THIS MEANS THAT THE FINAL MATRIX REDUCED BY THE
C ** PROGRAM WILL BE EXACTLY SIMILAR TO A+E WHERE E IS
C ** OF ORDER EPS*NOEM(A), FOR ANY REASONABLY BALANCED NORM
```

C *** SUCH AS THE ROW-SUM NORM.
C *** INTERNAL VARIABLES
C ***
INTEGER I, IT, L, NU, NI, NU
REAL E1, E2, P, Q, R, S, T, W, X, Y, Z
LOGICAL FAIL
C ***
C *** INITIALIZE
C ***
DC 10 I=NLCW, NUP
TYPE (I) = -1
10 CONTINUE
T=0.
C ***
C *** MAIN LOOP. FIND AND ORDER EIGENVALUES.
C ***
NU=NUP
20 IF (NU .LT. NLOW) GC TO 240
IT=0
C ***
C *** QR LOOP. FIND NEGLIGIBLE ELEMENTS AND PERFORM
C *** QR STEP.
C ***
30 CONTINUE
C *** SEARCH PACK FOR NEGLIGIBLE ELEMENTS.
C ***
L=NU
40 CONTINUE
IF (L.EQ. NLOW) GC TO 50
IF (ABS (A(L, L-1)) .LT. EPS * (ABS (A(L-1, L-1)) + ABS (A(L, L))))
*GO TO 50
L=L-1
GO TO 40
50 CONTINUE
C ***
C *** TEST TO SEE IF AN EIGENVALUE OR A 2*2 BLOCK
C *** HAS BEEN FOUND.
C ***
X=A(NU, NU)
IF (L .EQ. NU) GO TO 160
Y=A(NU-1, NU-1)
W=A(NU, NU-1)*A(NU-1, NU)
IF (L .EQ. NU-1) GO TO 100
C ***
C *** TEST ITERATION COUNT, IF IT IS 30 QUIT. IF
C *** IT IS 10 OR 20 SET UP AN AD-HOC SHIFT.
C ***
IF (IT .EQ. 30) GO TO 240
IF (IT.NE.10 .AND. IT.NE.20) GO TO 70
**C *** AD-HOC SHIFT.**

**C *** T=T+X**

DO 60 I=NL,NU

A(I,I)=A(I,I)-X

**60 CONTINUE**

S=ABS(A(NU,NU-1)) + ABS(A(NU-1,NU-2))

X=0.75*S

Y=X

W=-0.4375*S**2

**70 CONTINUE**

IT=IT+1

**C *** LOOK FOR TWO CONSECUTIVE SMALL SUB-DIAGONAL ELEMENTS.**

**C *****

NL=NU-2

**80 CONTINUE**

Z=A(NL,NL)

R=X-Z

S=Y-Z

P=(R*S-W)/A(NL+1,NL)+A(NL,NL+1)

Q=A(NL+1,NL+1)-Z-R-S

R=A(NL+2,NL+1)

S=ABS(P)+ABS(Q)+ABS(R)

P=P/S

Q=Q/S

R=R/S

IF(NL .EQ. L) GO TO 90

IF(ABS(A(NL,NL-1)) * (ABS(Q)+ABS(R)) .LE.*EPS*ABS(P) * (ABS(A(NL-1,NL-1)) + ABS(Z) + ABS(A(NL+1,NL+1))))

*GO TO 90

NL=NL-1

GO TO 80

**90 CONTINUE**

**C *** PERFORM A QR STEP BETWEEN NL AND NU.**

**C *** CALL QRSSTEP(A,V,P,Q,R,NL,NU,N,NA,NV)**

GO TO 30

**C *** 2*2 BLOCK FOUND.**

**C *****

100 IF (NU .NE. LOW+1) A(NU-1,NU-2) = 0.

A(NU,NU)=A(NU,NU)+1

A(NU-1,NU-1)=A(NU-1,NU-1)+T

TYPE(NU)=0

TYPE(NU-1)=0

MU=NU
C *** LOOP TO POSITION 2*2 BLOCK.
C ***
110 CONTINUE
   NL=MU-1
C ***
C *** ATTEMPT TO SPLIT THE BLOCK INTO TWO REAL EIGENVALUES.
C ***
C *** CALL SPLIT (A,V,N,NL,E1,E2,NA,NV)
C ***
C *** IF THE SPLIT WAS SUCCESSFUL, GO AND ORDER THE REAL EIGENVALUES.
C ***
   IF (A (MU,MU-1) .EQ. 0.) GO TO 170
C ***
C *** TEST TO SEE IF THE BLOCK IS PROPERLY POSITIONED, AND IF NOT EXCHANGE IT.
C ***
   IF (MU .EQ. NUP) GO TO 230
   IF (A (MU,EQ. NUP-1) .EQ. 0.0) GC TO 130
   IF (A (MU+2,MU+1) .EQ. 0.0) GC TO 130
C ***
C *** THE NEXT BLOCK IS 2*2.
C ***
   IF (A (MU-1,MU-1) .GE. A (MU,MU) - A (MU-1,MU) * A (MU,MU-1)  
   * A (MU+1,MU+1) .GE. A (MU+2,MU+2) - A (MU+1,MU+2) *  
   A (MU+2,MU+1)) * GO TO 230
   CALL EXCHNG (A,V,N,NL,2,2,EPS,FAIL,NA,NV)
   IF (.NOT. FAIL) GO TO 120
   TYPE (NL)=-1
   TYPE (NL+1)=-1
   TYPE (NL+2)=-1
   TYPE (NL+3)=-1
   GO TO 240
120 CONTINUE
   MU=MU+2
   GO TO 150
130 CONTINUE
C ***
C *** THE NEXT BLOCK IS 1*1.
C ***
   IF (A (MU-1,MU-1) .GE. A (MU,MU) - A (MU-1,MU) * A (MU,MU-1)  
   * A (MU+1,MU+1) .GE. A (MU+1,MU+1) ** 2)
   * GO TO 230
   CALL EXCHNG (A,V,N,NL,2,1,EPS,FAIL,NA,NV)
   IF (.NOT. FAIL) GO TO 140
   TYPE (NL)=-1
   TYPE (NL+1)=-1
TYPE(NL+2)=-1
GO TO 240
140 CONTINUE
MU=MU+1
150 CONTINUE
GO TO 110
C ***
C *** SINGLE EIGENVALUE FOUND.
C ***
160 NL=0
A(NU,NU)=A(NU,NU)*T
IF(NU .NE. NLOW) A(NU,NU-1)=0.0
TYPE(NU)=0
MU=NU
C ***
C *** LOOP TO POSITION ONE OR TWO REAL EIGENVALUES.
C ***
170 CONTINUE
C ***
C *** POSITION THE EIGENVALUE LOCATED AT A(NL,NL).
C ***
180 CONTINUE
IF(MU .EQ. NUP) GO TO 220
IF(MU .EQ. NUP-1) GO TO 200
IF(A(MU+2,MU+1) .EQ. 0.) GO TO 200
C ***
C *** THE NEXT BLOCK IS 2*2.
C ***
IF(A(MU,MU)**2 .GE. 
*A(MU+1,MU+1)*A(MU+2,MU+2)-A(MU+1,MU+2)*A(MU+2,MU+1))
*GO TO 230
CALL EXCHNG(A,V,N,MU,1,2,EPS,FAIL,NA,NV)
IF(.NOT. FAIL) GO TO 190
TYPE(NU)=-1
TYPE(MU+1)=-1
TYPE(MU+2)=-1
GO TO 240
190 CONTINUE
MU=MU+2
GO TO 210
200 CONTINUE
C ***
C *** THE NEXT BLOCK IS 1*1.
C ***
IF((A(MU,MU)) .LE. (A(MU+1,MU+1)))
*GO TO 220
CALL EXCHNG(A,V,N,MU,1,1,EPS,FAIL,NA,NV)
MU=MU+1
210 CONTINUE
GO TO 180
CONTINUE
MU=NL
NL=0
IF (MU . NE. 0) GO TO 170
C ***
C *** GC BACK AND GET THE NEXT EIGENVALUE.
C ***
230 CONTINUE
NU=LU-1
GO TO 20
C ***
C *** ALL THE EIGENVALUES HAVE BEEN FOUND AND ORDERED.
C *** COMPUTE THEIR VALUES AND TYPE.
C ***
240 IF (NU . LT. NLOW) GO TO 260
DO 250 I=1,NU
    A (I,I)=A (I,I)+T
250 CONTINUE
260 CONTINUE
NU=NUP
270 CONTINUE
IF (TYPE (NU) . NE. -1) GO TO 280
NU=NU-1
GO TO 310
280 CONTINUE
IF (NU . EQ. NLOW) GO TO 290
IF (A (NU,NU-1) . EQ. 0.) GO TO 290
C ***
C *** 2*2 BLOCK
C ***
CALL SPLIT (A,V,N,NU-1,E1,E2,NA,NV)
IF (A (NU,NU-1) . EQ. 0.) GO TO 290
ER (NU)=E1
EI (NU-1)=E2
ER (NU-1)=ER (NU)
EI (NU)=-EI (NU-1)
TYPE (NU-1)=1
TYPE (NU)=2
NU=NU-2
GO TO 300
290 CONTINUE
C ***
C *** SINGLE RCCT.
C ***
ER (NU)=A (NU,NU)
EI (NU)=A (NU,NU)
EI (NU)=0.
NU=NU-1
300 CONTINUE
310 CONTINUE
IF (NU .GE. NLOW) GO TO 270
RETURN
END
Subroutine EXCHNG

INTEGER B1,B2,L,NA,NV
REAL A(NA,N),EP$,V(NV,N)
LOGICAL FAIL

C ***
C *** GIVEN THE UPPER HESSENEEBR MATRIX A WITH CONSECUTIVE
C *** STARTING AT A(L,L), EXCHNG PRODUCES A UNITARY
C *** SIMILARITY TRANSFORMATION THAT EXCHANGES THE BLOCKS
C *** ALONG WITH THEIR EIGENVALUES. THE TRANSFORMATION
C *** IS ACCUMULATED IN V. EXCHNG REQUIRES THE SUBROUTINE
C *** QRSTEP. THE PARAMETERS IN THE CALLING SEQUENCE ARE
C *** (STARRED PARAMETERS ARE ALTERED BY THE SUBROUTINE)
C ***
C *** *A THE MATRIX WHOSE BLOCKS ARE TO BE
C *** INTERCHANGED.
C *** *V THE ARRAY INTO WHICH THE TRANSFORMATIONS
C *** ARE TO BE ACCUMULATED.
C *** N THE ORDER OF THE MATRIX A.
C *** L THE POSITION OF BLOCKS.
C *** B1 THE SIZE OF THE FIRST BLOCK.
C *** B2 THE SIZE OF THE SECOND BLOCK.
C *** EPS A CONVERGENCE CRITERION.
C *** *FAIL A LOGICAL VARIABLE WHICH IS FALSE ON A
C *** NORMAL RETURN. IF THIRTY ITERATIONS WERE
C *** PERFORMED WITHOUT CONVERGENCE, FAIL IS SET
C *** TO TRUE AND THE ELEMENT:
C *** A(L+B2,L+B2-1) CANNOT BE ASSUMED ZERO.
C *** NA THE FIRST DIMENSION OF THE ARRAY A.
C *** NV THE FIRST DIMENSION OF THE ARRAY V.
C ***
C *** INTERNAL VARIABLES.
C ***
INTEGER I,IT,J,L1,M
FAIL=.FALSE.
IF(B1 .EQ. 2) GO TO 40
IF(B2 .EQ. 2) GO TO 10
C ***
C *** INTERCHANGE 1*1 AND 1*1 BLOCKS.
C ***
L1=L+1
Q=A(L+1,L+1)-A(L,L)
P=A(L,L+1)
R=AMAX1(P,Q)
IF(R .EQ. 0.) RETURN
P=P/R
Q=C/R
R=SQt(R**2+Q**2)
P=P/R
\[ W = A(L+1,L) * A(L,L+1) \]
\[ P = 1. \]
\[ Q = 1. \]
\[ R = 1. \]

CALL QRSTEP(A, V, P, Q, R, L, M, N, NA, NV)

IT = 0

50
IT = IT + 1
IF (IT .LE. 30) GO TO 60
FAIL = .TRUE.
RETURN

60
CONTINUE

Z = A(L, L)
R = X - Z
S = Y - Z
P = (R * S - W) / A(L+1, L) + A(L, L+1)
Q = A(L+1, L+1) - Z - B - S
R = A(L+2, L+1)
S = ABS(P) + ABS(Q) + ABS(R)
P = P / S
Q = Q / S
R = R / S

CALL QRSTEP(A, V, P, Q, R, L, M, N, NA, NV)
IF (ABS(A(M-1, M-2)) .GT. EPS * (ABS(A(M-1, M-1)) + ABS(A(M-2, M-2))))
GO TO 50
A(M-1, M-2) = 0.
RETURN

END
Subroutine QRSTEP

SUBROUTINE QRSTEP (A, V, P, Q, R, NL, NU, NA, NV)
INTEGER N, NA, NL, NU, N
REAL A (NA, N), P, Q, R, V (NV, N)
C ***
C *** QRSTEP PERFORMS ONE IMPLICIT QR STEP ON THE
C *** UPPER HESSENBERT MATRIX A. THE SHIFT IS DETERMINED
C *** BY THE NUMBERS E, Q, AND R, AND THE STEP IS APPLIED TO
C *** ROWS AND COLUMNS NL THROUGH NU. THE TRANSFORMATIONS
C *** ARE ACCUMULATED IN V. THE PARAMETERS IN THE CALLING
C *** SEQUENCE ARE (STARTED PARAMETERS ARE ALTERED BY THE
C *** SUBROUTINE)
C ***
C *** *A THE UPPER HESSENBERT MATRIX ON WHICH THE
C *** *V THE ARRAY IN WHICH THE TRANSFORMATIONS
C *** *P *Q PARAMETERS THAT DETERMINE THE SHIFT.
C *** *R
C *** NL THE LOWER LIMIT OF THE STEP.
C *** NU THE UPPER LIMIT OF THE STEP.
C *** N THE ORDER OF THE MATRIX A.
C *** NA THE FIRST DIMENSION OF THE ARRAY A.
C *** NV THE FIRST DIMENSION OF THE ARRAY V.
C ***
C *** INTERNAL VARIABLES
C ***
INTEGER I, J, K, NL2, NL3, NUM1
REAL S, X, Y, Z
LOGICAL LAST
NL2=NL+2
DO 10 I=NL2,NU
A(I, I-2) = 0.
10 CONTINUE
IF (NL2 .EQ. NU) GO TO 30
NL3=NL+3
DO 20 I=NL3,NU
A(I, I-3) = 0.
20 CONTINUE
30 CONTINUE
NUM1=NU-1
DO 130 K=NL,NUM1
C ***
C *** DETERMINE THE TRANSFORMATIONS.
C ***
LAST = K .EQ. NUM1
IF (K .EQ. NL) GO TO 40
P=A(K, K-1)
Q=A(K+1, K-1)
R=0.
IF (.NOT. LAST) E=A(K+2,K-1)
X=ABS(P)+ABS(Q)+ABS(R)
IF (X .EQ. 0.) GO TO 130
P=P/X
Q=Q/X
R=R/X
CONTINUE
S=SQRT(E**2+Q**2+R**2)
IF (P .LT. 0.) S=-S
IF (K .EQ. NL) GO TO 50
A(K,K-1)=-S*X
GO TO 60
CONTINUE
IF (NL .NE. 1) A(K,K-1)=-A(K,K-1)
CONTINUE
P=P+S
X=P/S
Y=Q/S
Z=R/S
Q=Q/P
R=R/P
C *** PREMULTIPLY
C ***
DO 30 J=K,N
P=A(K,J)+Q*A(K+1,J)
IF (LAST) GO TO 70
P=P+P*A(K+2,J)
A(K+2,J)=A(K+2,J)-P*Z
CONTINUE
A(K+1,J)=A(K+1,J)-P*Y
A(K,J)=A(K,J)-P*X
CONTINUE
C ***
C *** POSTMULTIPLY.
C ***
J=MINO(K+3,NU)
DO 100 I=1,J
P=X*A(I,K)+Y*A(I,K+1)
IF (LAST) GO TO 90
P=P+Z*A(I,K+2)
A(I,K+2)=A(I,K+2)-P*R
CONTINUE
A(I,K+1)=A(I,K+1)-P*Q
A(I,K)=A(I,K)-P
CONTINUE
C ***
C *** ACCUMULATE THE TRANSFORMATION IN V.
C ***
DO 120 I=1,N
\[ P = X \cdot V(I,K) + Y \cdot V(I,K+1) \]
\[ \text{IF (LAST) GO TO 110} \]
\[ P = P + Z \cdot V(I,K+2) \]
\[ V(I,K+2) = V(I,K+2) - E \cdot B \]

110 CONTINUE

V(I,K+1) = V(I,K+1) - E \cdot Q
V(I,K) = V(I,K) - P

120 CONTINUE

130 CONTINUE
RETURN
END
Subroutine SPLIT

SUBROUTINE SPLIT (A, V, N, L, E1, E2, NA, NV)
INTEGER L, N, NA, NV
REAL A(NA, N), V(NV, N)
C ***
C *** GIVEN THE UPPER HESSENBERG MATRIX A WITH A 2*2 BLOCK
C *** STARTING AT A(L, L), SPLIT DETERMINES IF THE
C *** CORRESPONDING EIGENVALUES ARE REAL OR COMPLEX. IF THEY
C *** ARE REAL, A ROTATION IS DETERMINED THAT REDUCED THE
C *** BLOCK TO UPPER TRIANGULAR FORM WITH THE EIGENVALUES
C *** OF LARGEST ABSOLUTE VALUE APPEARING FIRST. THE
C *** ROTATING IS ACCUMULATED IN V. THE EIGENVALUES (REAL
C *** OR COMPLEX) ARE RETURNED IN E1 AND E2. THE PARAMETERS
C *** IN THE CALLING SEQUENCE ARE (STARRED PARAMETERS ARE
C *** ALTERED BY THE SUBROUTINE)
C *** *A
C *** *V
C *** N
C *** L
C *** *E1
C *** *E2
C *** NA
C *** NV
C *** INTERNAL VARIABLES.
INTEGER I, J, L1
REAL P, Q, R, T, U, W, X, Y, Z
X = A(L+1, L+1)
Y = A(L, L)
W = A(L, L+1) * A(L+1, L)
P = (Y-X) / 2.
Q = P**2 + W
IF (Q .GE. 0.) GO TO 5
C ***
C *** COMPLEX EIGENVALUES
C ***
E1 = P + X
E2 = SQRT (-Q)
RETURN
5 CONTINUE
C ***
C *** TWO REAL EIGENVALUES. SET UP TRANSFORMATION.
C ***
Z = SQRT (Q)
IF (P .LT. 0.) GO TO 10
Z = P + Z
C ***
GO TO 20
10 CONTINUE
    Z = P - Z
20 CONTINUE
    IF (Z .EQ. 0.) GO TO 30
    R = -W/Z
    GO TO 40
30 CONTINUE
    R = 0.
40 CONTINUE
    IF (ABS(X+Z) .GE. ABS(X+R)) Z = R
        Y = Y - X - Z
        X = -Z
        T = A(L, L+1)
        U = A(L+1, L)
        IF (ABS(Y) + ABS(U) .LE. ABS(T) + ABS(X)) GO TO 60
    Q = U
        P = Y
    GO TO 70
60 CONTINUE
    Q = X
        P = T
70 CONTINUE
    R = SQRT(P**2 + Q**2)
    IF (R .GT. 0.) GC TO 80
    E1 = A(L, L)
    E2 = A(L+1, L+1)
    A(L+1, L) = 0.
    RETURN
80 CONTINUE
    P = P/R
    Q = Q/R
C ***
C *** PREMULTIPLY.
C ***
    DO 90 J = L, N
        Z = A(L, J)
        A(L, J) = F*Z + C*A(L+1, J)
        A(L+1, J) = P*A(L+1, J) - Q*Z
    90 CONTINUE
C ***
C *** POSTMULTIPLY.
C ***
    L1 = L + 1
    DO 100 I = 1, L1
        Z = A(I, L)
        A(I, L) = P*Z + Q*A(I, L+1)
        A(I, L+1) = F*A(I, L+1) - Q*Z
    100 CONTINUE
C ***
C *** ACCUMULATE THE TRANSFORMATION IN V.
C ***
DO 110 I = 1, N
  Z = V(I, L)
  V(I, L) = P*Z + Q*V(I, L+1)
  V(I, L+1) = P*V(I, L+1) - C*Z
110 CONTINUE
  A(L+1, L) = 0.
  E1 = A(L, L)
  E2 = A(L+1, L+1)
RETURN
END
Subroutine BALBAK

SUBROUTINE BALBAK(NM,N,LOW,IGH,SCALE,M,Z)
INTEGER I,J,K,M,N,II,NM,IGH,LOW
REAL SCALE(N),Z(NM,N)
REAL S
IF (M .EQ. 0) GC TO 200
IF (IGH .EQ. LOW) GC TO 120
DO 110 I=LCW,IGH
S=SCALE(I)

C *** LEFT HAND EIGENVECTORS ARE BACK TRANSFORMED
C *** IF THE FOREGOING STATEMENT IS REPLACED BY
C *** S=1.0/SCALE(I).
C ***
DO 100 J=1,M
100 Z(I,J)=Z(I,J)*S
CONTINUE

C *** FOR I = LOW-1 STEP -1 UNTIL 1, IGH+1 STEP 1
C *** UNTIL N DO.
C ***
120 DO 140 II=1,N
I=II
IF (I .GE. LOW .AND. I .LE. IGH) GO TO 140
IF (I .LT. LOW) I=LCW-II
K=SCALE(I)
IF (K .EQ. I) GC TO 140
DO 130 J=1,M
S=Z(I,J)
Z(I,J)=Z(K,J)
Z(K,J)=S
130 CONTINUE
140 CONTINUE
200 RETURN
END
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SCALE(I):

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NLOW= 1 NUP=10
### Matrix Z in Hessenberg Form:

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-16.00 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & -47.30 & 150.01 & 81.13 & 53.76 & 8.17 & -0.03 & 0.02 & -0.13 & -0.14 \\
0.0 & -0.01 & 235.97 & 135.22 & 89.13 & 13.67 & -0.10 & -0.04 & -0.04 & -0.06 \\
0.0 & 0.00 & -0.00 & -0.35 & 5.81 & 42.25 & -18.41 & -21.53 & -0.30 & -2.40 \\
-16.00 & 0.00 & 0.00 & 0.00 & 0.00 & 42.44 & -274.57 & 119.92 & 140.35 & 1.38 & 15.03 \\
0.0 & -18.77 & 14.09 & 0.32 & -2.79 & -0.03 & -4.19 & -1.57 & 0.05 & -0.24 \\
0.0 & -19.44 & -213.28 & -0.09 & -36.34 & -0.00 & 0.59 & -0.65 & 0.03 & -0.10 \\
0.0 & 38.82 & -99.99 & 0.11 & -21.51 & -0.03 & -0.15 & -0.15 & -0.50 & 0.44 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.00 & 0.05 & -0.10 & -0.54 & 0.50
\end{array}
\]

### P(N):

\[
\begin{array}{ccccccc}
-0.00 & 16.00 & 47.30 & -235.99 & 0.35 \\
-45.53 & 0.04 & -1.16 & 0.26 & -0.47
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NLOW= 1  
NUP=10

### Matrix Z After ORTRAN:

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0.0 & -47.30 & 150.01 & 81.13 & 53.76 & 8.17 & -0.03 & 0.02 & -0.13 & -0.14 \\
0.0 & -0.01 & 235.97 & 135.22 & 89.13 & 13.67 & -0.10 & -0.04 & -0.04 & -0.06 \\
0.0 & 0.00 & -0.00 & -0.35 & 5.81 & 42.25 & -18.41 & -21.53 & -0.30 & -2.40 \\
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0.0 & -18.77 & 14.09 & 0.32 & -2.79 & -0.03 & -4.19 & -1.57 & 0.05 & -0.24 \\
0.0 & -19.44 & -213.28 & -0.09 & -36.34 & -0.00 & 0.59 & -0.65 & 0.03 & -0.10 \\
0.0 & 38.82 & -99.99 & 0.11 & -21.51 & -0.03 & -0.15 & -0.15 & -0.50 & 0.44 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.00 & 0.05 & -0.10 & -0.54 & 0.50
\end{array}
\]
$P(N) :$

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$NLOW = 1 \quad NUP = 10$

**The RSF Form of $Z = U(T) * Z * U$**

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**The Orthogonal Matrix $U$**

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**MATRIX UONE:**

\[
\begin{array}{cccccc}
-0.140 & 0.682 & 3.936 & -4.655 & -0.473 \\
0.074 & -0.042 & -0.138 & -0.053 & 0.023 \\
-0.039 & -0.053 & -0.370 & 0.549 & 0.041 \\
-0.996 & -0.002 & 0.002 & -0.022 & 0.000 \\
0.001 & 0.001 & 0.028 & -0.075 & -0.001 \\
\end{array}
\]

**MATRIX UTWO:**

\[
\begin{array}{cccccc}
-0.000 & 0.175 & 0.405 & -0.456 & -0.059 \\
-0.000 & 0.212 & -0.776 & -0.572 & -0.034 \\
-0.000 & 0.957 & 0.055 & 0.252 & 0.023 \\
0.000 & -0.046 & -0.114 & 0.010 & 0.004 \\
-0.000 & 0.000 & 0.022 & -0.083 & 0.996 \\
\end{array}
\]

**THE INVERTED MATRIX UONE:**

\[
\begin{array}{cccccc}
-0.007 & 0.043 & -0.098 & -0.996 & 0.0 \\
58.939 & 271.204 & 525.510 & -8.622 & -1.475 \\
-17.520 & -101.988 & -141.735 & 0.320 & 122.346 \\
-5.407 & -31.621 & -43.992 & 0.110 & 22.661 \\
-9.734 & -146.577 & 11.091 & -10.556 & 793.142 \\
\end{array}
\]
RICCATI MATRIX $K$:

| $K_{11}$ | 6.2714 |
| $K_{12}$ | 29.2594 |
| $K_{13}$ | 54.1076 |
| $K_{14}$ | -0.8108 |
| $K_{15}$ | -7.6385 |
| $K_{22}$ | 159.6670 |
| $K_{23}$ | 246.0733 |
| $K_{24}$ | -1.7773 |
| $K_{25}$ | -135.2679 |
| $K_{33}$ | 484.3711 |
| $K_{34}$ | -8.4462 |
| $K_{35}$ | 29.0088 |
| $K_{44}$ | 0.3190 |
| $K_{45}$ | -10.7080 |
| $K_{55}$ | 790.7151 |
APPENDIX II

THE TRANSFER FUNCTION OF THE BLOCK CONTAINING PLASMA AND COIL EQUATIONS.

For designing a device to control the plasma current and its position, as a first try one might think of a direct and classical method such as Root locus analysis of the system and compensation to solve the problem. This requires knowing the transfer function of the plant. Calculating this transfer function might seem a simple process since the input and the output of the system is given. However the mathematical calculations are very difficult and time consuming for a fifth order system such as ours. Since this calculations were already made prior to time of using the state variable technique, we included as an appendix here for its possible use in future research work.

The dynamic equations of the system are given in Section II. They are:

\[254 \frac{d}{dt} I_1 + 1514 \frac{d}{dt} I_2 + 2012 \frac{d}{dt} I_3 + 0.13 \frac{d}{dt} I_5 - 116.12 \frac{d}{dt} R_I = 0\]

\[1514 \frac{d}{dt} I_1 + 9120 \frac{d}{dt} I_2 + 12050 \frac{d}{dt} I_3 + 3.68 \frac{d}{dt} I_5 + 1580 \frac{d}{dt} R_I + 370 I_2 = V_2\]

\[2012 \frac{d}{dt} I_1 + 12050 \frac{d}{dt} I_2 + 16210 \frac{d}{dt} I_3 - 4.59 \frac{d}{dt} I_5 - 1590 \frac{d}{dt} R_I + 1330 I_3 = V_3\]
We are trying to find $I_1$ and $R_I$ in terms of $V_2$ and $V_3$. We will write equations (A-II-1) in the Laplace form.

$$0.13 \ I_1 + 3.68 \ I_2 - 4.59 \ I_3 + 0.549 \ I_5 + 190 \ R_I + 76I_5 = 0$$

$$-0.002I_1 + 0.105 \ I_2 - 0.143I_3 + 0.0145I_5 - R_I = 0$$

(A-II-1)

Now we have five linear equations with five unknowns $I_1(s), I_2(s), I_3(s), I_5(s)$ and $R_I(s)$. $V_2(s)$ and $V_3(s)$ are the inputs to the system.

We will use the Kramer method to solve for $I_1$ and $R_I$.

$$254 \ I_1(s) + 1514 \ I_2(s) + 2012 \ I_3(s) + 0.13 \ I_5(s) - 116.12R_I(s) = 0$$

$$1514I_1(s) + (9120+ \frac{370}{s})I_2(s) + 12050I_3(s) + 3.68I_5(s) + 1580R_I(s) = \frac{V_2}{s}$$

$$2012I_1(s) + 12050I_2(s) + (16210+ \frac{1330}{s})I_3(s) - 4.59I_5(s) - 1590R_I(s) = \frac{V_3}{s}$$

$$0.13I_1(s) + 3.68I_2(s) - 4.59I_3(s) + (0.549+ \frac{76}{s})I_5(s) + 190R_I(s) = 0$$

$$-0.0027I_1(s) + 0.105I_2(s) - 0.143I_3(s) + 0.0145I_5(s) - R_I(s) = 0$$

(A-II-2)
\[
\begin{array}{cccccc}
0 & 1514 & 2012 & 0.13 & -116.12 \\
\frac{V_2}{s} (9120+ \frac{370}{s}) & 12050 & 3.68 & 1580 \\
\frac{V_3}{s} & 12050 & (16210+ \frac{1330}{s})-4.59 & -1590 \\
0 & 3.68 & -4.59 & (0.549+ \frac{76}{s}) & 190 \\
0 & 0.105 & -0.143 & 0.0145 & -1 \\
\end{array}
\]

\[I_1 = \frac{254}{s} \]

\[
\begin{array}{cccccc}
254 & 1514 & 2012 & 0.13 & -116.12 \\
1514 (9120+ \frac{370}{s}) & 12050 & 3.68 & 1580 \\
2012 & 12050 & (16210+ \frac{1330}{s})-4.59 & -1590 \\
0.13 & 3.68 & -4.59 & (0.549+ \frac{76}{s}) & 190 \\
-0.0027 & 0.105 & -0.143 & 0.0145 & -1 \\
\end{array}
\]

(A-II-3)
\[
\begin{array}{cccccc}
254 & 1514 & 2012 & 0.13 & 0 \\
1514 & \left(9120 + \frac{370}{s}\right) & 12050 & 3.68 & \frac{V_2}{s} \\
2012 & 12050 & \left(16210 + \frac{1330}{s}\right) - 4.59 & \frac{V_3}{s} \\
0.13 & 3.68 & -4.59 & \left(0.549 + \frac{76}{s}\right) & 0 \\
-0.0027 & 0.105 & -0.143 & 0.0145 & 0 \\
\end{array}
\]

\[
R_I = \frac{254}{1514} \left(9120 + \frac{370}{s}\right) \left(16210 + \frac{1330}{s}\right) - 4.59 \left(0.549 + \frac{76}{s}\right)
\]

\[
\begin{array}{cccccc}
254 & 1514 & 2012 & 0.13 & -116.12 \\
1514 & \left(9120 + \frac{370}{s}\right) & 12050 & 3.68 & 1580 \\
2012 & 12050 & \left(16210 + \frac{1330}{s}\right) - 4.59 & -1590 \\
0.13 & 3.68 & -4.59 & \left(0.549 + \frac{76}{s}\right) & 190 \\
-0.0027 & 0.105 & -0.143 & 0.0145 & -1 \\
\end{array}
\]

(A-II-4)
The determinants in equations (A-II-3) and (A-II-4) can not be solved by a computer since there are parameters such as \(s, V_2, \) and \(V_3\). A hand calculation was made and the results are given here:

\[
I_1 = \frac{(s + 3.08)(s + 106.8)}{(s + 0.95)(s + 3.91)(s + 281)} V_2(s) \tag{A-II-5}
\]

\[
+ \frac{(s + 0.67)(s + 636)}{(s + 0.95)(s + 3.91)(s + 281)} V_3(s) \tag{A-II-4}
\]

\[
R_I = \frac{(s+8)(s+6.16 \pm j0.7)(s+772)}{(s + 0.95)(s + 3.9)(s + 7)(s + 7.07)(s+281)} V_2(s) \tag{A-II-5}
\]

\[
+ \frac{(s+0.67)(s+6.16)(s+7 \pm j0.11)(s+636)}{(s+0.95)(s+3.91)(s+6.17)(s+7)(s+7.07)(s+281)} V_3(s) \tag{A-II-5}
\]

Our system has two inputs and two outputs. Their relationship can be rewritten as:

\[
I_1 = G_{11} V_2 + G_{12} V_3 \tag{A-II-6}
\]

\[
R_I = G_{21} V_2 + G_{22} V_3 \tag{A-II-7}
\]

where \(G_{11}, G_{12}, G_{21} \) and \(G_{22}\) are:
We can combine equations (A-11-6) and (A-11-7) to find the overall transfer function:

\[
G_{11} = \frac{(s + 3.08)(s + 106.8)}{(s + 0.95)(s + 3.91)(s + 281)}
\]

\[
G_{12} = \frac{(s + 0.67)(s + 636)}{(s + 0.95)(s + 3.91)(s + 281)}
\]

\[
G_{21} = \frac{(s+8)(s+6.16 \pm j0.7)(s+772)}{(s+0.95)(s+3.9)(s+7)(s+7.07)(s+281)}
\]

\[
G_{22} = \frac{(s+0.67)(s+6.16)(s+7 \pm j0.11)(s+636)}{(s+0.95)(s+3.91)(s+6.17)(s+7)(s+7.07)(s+281)}
\]

We can combine equations (A-II-6) and (A-II-7) to find the overall transfer function:

\[
\begin{bmatrix}
I_1 \\
R_I
\end{bmatrix} =
\begin{bmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{bmatrix}
\begin{bmatrix}
V_2 \\
V_3
\end{bmatrix}
\]

and therefore \( G \) will be:

\[
G =
\begin{bmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{bmatrix}
\]

Figure (A-II-1) shows the interrelation of \( G_{11}, G_{12}, G_{21} \) and \( G_{22} \).
Figure (A-II-1)  Block Diagram of the System.
BIBLIOGRAPHY


