A SIMPLE ALGORITHM FOR DESIGNING CONTROL SYSTEMS AND ITS APPLICATIONS IN ROBOTICS

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Chapter 1

Introduction

1.1 Background

Automatic control has played a vital role in the advancement of engineering and science. Automatic control produces the means for obtaining the optimal performance of dynamic systems, thus improving productivity and relieving the drudgery of many routine repetitive manual operations.

Automatic control is very important not only for space-vehicle systems, missile-guidance systems and robotic systems, but also for modern manufacturing processes—such as the numerical control of machine tools in the manufacturing industries, in the design of vehicles in the automobile industries and in controlling pressure, temperature, humidity, viscosity and flow in the chemical industries.

Much research has been conducted on the theories and methods of design and control of automatic systems in recent years, especially in the field of modern control theory. In the next section, this will be discussed briefly.

1.2 Literature Review

Control systems may be classified as linear or nonlinear. Their designs can be based on either classical control theory or modern control theory.
1.2.1 Classical Control Theory


The essence of designing control systems by classical control theory is in how to design a controller and choose the parameters which allow the poles and zeros of the system equation to be located at the desired positions. Therefore, the desired performance, stability and transient and steady-state responses of the system can be achieved.

Classical control theory is simple, and it is successfully used in manufacturing and industrial processes. However, classical control theory is not suitable for controlling multiple input multiple output (MIMO) time-varying nonlinear and uncertain systems.

1.2.2 Modern Control Theory

Modern control theory is used for multiple-input multiple-output systems (Hang, C. C. et al, 1993, Lin, C. F., 1994, Marino, R. and Tomei, P., 1995). Based on time-domain analysis and synthesis using state variables, the designer can cope with the increased complexity of modern plants and the stringent requirements placed on
accuracy, weight and cost—particularly for military, space and industrial applications. Modern control theory employs mathematical formulations of problems and applies mathematical theory to design systems. The designer is able to produce a control system that is optimal with respect to the performance index considered. The pole placement design, estimator design and optimal controller design are the proposed techniques of modern control theory. The principal mathematical tool of modern control theory is linear algebra, and the analysis is performed in the time domain.

There are many similarities between modern and classical control theories when designing control systems. The primary similarity between the two is the method used to choose the parameters and the mathematical model of the controller. The roots of the controlled system can then be placed at various positions, so that the desired responses of the controlled system can be obtained.

Modern control theory is successfully used in modern industrial controls. However, it is complicated, and it cannot be used in time-varying systems. Because the analysis and controller design is based on accurate mathematical model, it is sensitive to small inaccuracies in the system model.

In recent years, much research has been carried out on new control theories—such as digital control, adaptive control, fuzzy control, knowledge-based control and neural network-based control. The main control principles of those methods are described below.
1.2.2.1 Digital Control Systems

Digital control systems (Phillips, C. L. and Nagle, H. T., 1990, Astron. K. J. and Wittenmark, B., 1997) are naturally controlled by a digital computer. The digital control systems have gained popularity and importance in all industries due to the advances made in digital computers. Among the advantages of digital control are the increased flexibility of the control programs and the decision-making of digital systems. The program which characterizes a digital controller can be modified to accommodate design changes or adaptive performances without any variations on the hardware. Therefore, digital control systems are widely used in all industries.

The design of digital systems is similar to that of the classical and modern control theories. By properly choosing the parameters and the mathematical model of the controller, the roots of the controlled system can then be placed at various positions, so that the desired responses of the controlled system can be obtained.

1.2.2.2 Adaptive Control Systems

Adaptive control systems can be divided into Model Reference Adaptive Systems (MRAS) and Self-tuning Regulator (STR) (Ciliz, M. K. and Narendra, K. S., 1996, Ghanadan, R. and Blankenship, G. L., 1996, Hang, C. C. et al, 1993). The Model Reference Adaptive system is based on the concept that the performance of the system is specified by a model, and the parameters of the controller are adjusted according to the errors between the model and the plant. The basic principle of the MRAS is illustrated in Figure 1.1. The desired performance is expressed in terms of a reference model that
yields the desired response to the command signal. For example, the feedback loop consists of the plant and the controller. The error \( e(t) = y(t) - y_m(t) \) is the difference between the output and the model and is used to adjust the parameters of the controller through the outer loop. Hence, by necessity the inner loop is required to operate at a rate much faster than the outer loop.

The model-following problem can be solved by using the pole placement design. The system performance is specified by a mathematical model for the desired response. The parameters in the controller are adjusted to achieve \( y(t) = y_m(t) \) for a given input signal. As such, optimization techniques are the basic tools in MRAS design.

![Control block diagram of MRAS](image)

The Self-Tuning Regulator (STR) method is based on a two-step procedure. The first step is to identify the plant unknown parameters using some estimation techniques. The second step is to design the controller on-line using the estimated unknown parameters. This approach is shown in Figure 1.2, in which the estimated parameters are
treated as if they are the true values. There are numerous techniques for parameter estimations—such as least squares, maximum likelihood and Kalman filtering. There are also many controller design techniques that can be applied, including minimum variance, pole placement and model following. Naturally, different combinations of estimation techniques and controller design techniques lead to regulators with different properties.

![Figure 1.2 Control block diagram of STR](image)

1.2.2.3 **Fuzzy Control Systems**

Fuzzy control systems are rule-based systems, in which a set of so-called fuzzy rules represents a control decision mechanism to adjust the effects of a certain cause coming from the systems (Gupta, M. M. and Sinha, N. K., 1996, Lin, C. F., 1994). The aim of fuzzy control systems is normally to substitute for or replace a skilled human operator with a fuzzy rule-based system.

A fuzzy controller typically takes the form of a set of IF-THEN rules whose antecedents (IF part) and consequents (THEN part) are themselves membership functions, which have a value between 0 and 1. An alternative way of implementing a fuzzy control regime, which is similar to a conventional control law, is to use a standard
crisp logic controller—such as a PID—and the fuzzy IF-THEN rules to tune the gains of the conventional controller.

1.2.2.4 Knowledge-Based Control or Expert Control Systems

A knowledge-based control system or expert control system seeks to incorporate knowledge about control system design and practical operation to automate tasks normally performed by experienced control engineers (Gupta, M. M. and Sinha, N. K., 1996, Lin, C. F., 1994). Although knowledge-based control may be developed using conventional programming techniques, the use of the expert system technique is advantageous, at least in the development stage.

The expert control system, which consists of the expert controller and plant, is shown in Figure 1.3. The controller has the knowledge base that consists of rules that characterize strategies concerning how to control the plant. The inference engine is

![Figure 1.3 Control block diagram of expert control system](image)
designed to simulate the control expert's decision-making process in collecting reference inputs and plant outputs and reasoning about what command input to generate for the plant.

The expert controller must be designed so that it can coordinate the use of the plant outputs and inputs to decide what plant command input to generate so that the closed loop specifications are met. After the expert controller has been designed and the full closed loop system model of the expert control system has been specified, the dynamic properties of the expert control system must be analyzed carefully to ensure that the system performs within the desired specifications and properties.

One of the most popular types of expert systems today is the rule-based expert system. It represents knowledge in the form of IF-THEN rules.

1.2.2.5 Neural Network-Based Control Systems

In recent years, neural networks have received much attention in control system engineering (Gupta, M. M. and Sinha, N. K., 1996, Cheng, Y. C. and Chen, B. S., 1997). A neural network is, simply, a network of neural-like fundamental processing elements. Different types of network architectures, as well as fundamental processing elements, have been developed. One particular type of neural network is the multi-layered feed forward network with the adaline-with-a-sigmoid as the fundamental processing element. For this network, the fundamental processing element for a four-input situation is shown in Figure 1.4. The key property of this architecture that makes it useful is that, given a set of multivariate nonlinear functional relationships, the network can be used to learn
and emulate the function that represents the input-output relation. For the purposes of designing a controller for nonlinear dynamic systems, the most important property is the ability of a neural network to provide a realization of an arbitrary, unknown, nonlinear map using only a collected set of data and an appropriate learning strategy.

The list of the methods is not complete. Some of them (Cheng, Y. C. and Chen, B. S., 1997, Ciliz, M. K. and Narendra, K. S., 1996) are ongoing research subjects. However, most of them are based on a rather involved mathematical model and are too complicated to be widely used in industries.

![Block diagram of neural network](image)

Figure 1.4 Block diagram of neural network

### 1.3 Objective

Usually the control system desired by the classical control theory or the modern control theory is designed through the following steps (Lin, C. F., 1994, Marino, R. and Tomei, P., 1995, Phillips, C. L. and Nagle, H. T., 1990). First, the mathematical model of the plant is derived and analyzed. Second, the controller's mathematical model is designed, and the stability, transient and steady responses of the whole system are analyzed in order to get the required response. In the analysis process, the control signal
is determined based on the error signal. the system differential equations are treated as an initial value problem and the system differential equations are analyzed in the whole control process. If the results are not satisfactory, the controller is redesigned until the desired response is obtained. This approach has been used successfully in the past. However, the classical control theory cannot be used in the control of MIMO systems and time-varying systems, and modern control theory employs complicated mathematics. Unfortunately, it is too complicated to be widely used in industries.

In order to simplify the design and analysis process, this thesis puts forward a new simple algorithm for designing control systems. The novelty of the algorithm is as follows. First, instead of analyzing and controlling the system in the whole control time, the algorithm divides the control process into small steps; that is, it discretizes the control time into small intervals. When the sampling frequency is high enough, the difference equations that describe the controlled plant can be simplified greatly, and some of the uncertainty can be neglected. Second, the control signal is determined based on the desired output, not on the error signal. Third, the simplified equation of the controlled plant is chosen as the controller equation. Thus, the analysis and design process can be simplified and an accurate result can be obtained.

1.4 Limitations

The algorithm can be used for single-input single-output (SISO) linear and nonlinear systems, as well as for multiple-input multiple-output (MIMO) linear and nonlinear systems. However, there are some limitations.
First, for SISO linear systems, the roots of the characteristic equation should be distinct. The sampling time must be small enough so that the coefficients of the highest order derivative of the output and control are one order greater than the other terms of the output and control in the system equation, respectively.

Second, for SISO nonlinear systems, after the nonlinear systems are linearized, the linearized systems must satisfy the conditions of the SISO linear systems.

Third, for MIMO linear and nonlinear systems, the algorithm can only be used for the systems in which every control signal can be expressed explicitly. The sampling time must be high enough so that the condition of SISO systems can be satisfied.

Last, the algorithm cannot be used to control the nonlinear systems in which time delay, backlash, relay and dead zone nonlinearities are contained.

1.5 Organization of Thesis

This thesis is organized as follows. Chapter 2 introduces a new algorithm for designing and controlling SISO linear systems. The control principle is presented, and the mathematical analysis is carried out. Simulation examples are given, and the new algorithm is compared with conventional methods.

Chapter 3 illustrates that the same algorithm may be used to control the coupled MIMO linear systems.

In Chapter 4 the new algorithm is developed to control SISO nonlinear systems. First, the difference equation of the nonlinear system is derived. Second, the nonlinear
equation is linearized and simplified. Last, the control principle is presented, and examples from current literatures are utilized to verify the algorithm.

In Chapter 5 the new algorithm is utilized in controlling MIMO nonlinear systems and is illustrated by three examples from robot control. The simulation results of the new algorithm are compared with that of digital high-gain PD control method, adaptive control using multiple models and switching method and nonlinear adaptive tracking $H^s$ control design via neural network method.

Chapter 6 presents the conclusions.

All the simulations in the thesis are carried out by using the software of MATLAB (MathWorks, Inc., 1990).
Chapter 2
Control of SISO Linear Systems

2.1 Introduction

Consider the control block diagram (Figure 2.1) of the conventional method, where \( r \) and \( y \) are the input signal and measured process output, \( c \) is a control signal, and \( C \) and \( G \) denote the controller and plant, respectively. The controlled plant of single input single output (SISO) linear system has the general form of the Laplace transfer function.

\[
G(s) = \frac{Y}{U} = \frac{b_1s^m + b_2s^{m-1} + \ldots + b_{m+1}}{s^n + a_1s^{n-1} + \ldots + a_n}
\]  

(2.1)

where \( n \) and \( m \) are the highest powers of the denominator and the numerator, respectively.

![Figure 2.1 Control block diagram of the conventional method](image)

2.2 Control Principle

2.2.1 Case 1: \((n-m)=1\)

If \((n-m)=1\), and the poles of \( G(s) \) are distinct, by factoring the polynomials, this same function could also be expressed as
\[ G(s) = \sum_{i=1}^{n} \frac{d_i}{s + p_i} \]  
(2.2)

or it can be expressed as sum of second-order equations

\[ G(s) = \frac{Y}{U} = \sum_{j=1}^{\frac{n}{2}} \left( \frac{X_j}{U} \right) = \sum_{j=1}^{\frac{n}{2}} \left( \frac{k_j s + l_j}{s^2 + g_j s + h_j} \right) \]  
(2.3)

where \( X_j (j=1,2,\ldots,n/2) \) and \( U \) are the Laplace transform of the outputs and control signal of the second-order systems, respectively. \( g_j, h_j, k_j, l_j (j=1,2,\ldots,n/2) \) are all constants. If \( n \) is an odd number, \( G(s) \) can consist of \((n-1)/2\) second-order systems and one first-order system.

### 2.2.1.1 Derivation of Difference Equation

When Equation (2.3) is expressed in time domain by the inverse Laplace transform, it becomes

\[ y(t) = \sum_{j=1}^{\frac{n}{2}} x_j(t) \]  
(2.4.1)

\[ x_j^{(2)} + g_j x_j' + h_j x_j = k_j u + l_j u \]  
(2.4.2)

where \( x_j \) and \( u \) are the output and the control signal of the second-order system, respectively.

If the plant is controlled by computer, and the system input \( r \) is a unit step and the system desired response \( y \) is the thinner curve, (see Figure 2.2), the whole process control time can be discretized into \( n \) equal steps or intervals with points \( t_0, t_1, t_2, \ldots, t_n \) and the interval between each point or sampling time \( T \) is defined as \( T = (t_n - t_0)/n \).
In an arbitrary interval \((t_{i-1}, t_i)\), Equation (2.4.2) can be approximated in the form of a difference equation as

\[
x_{i,j} - \frac{2x_{i-1,j} + x_{i-2,j}}{T^2} + g_j \frac{x_{i,j} - x_{i-2,j}}{2T} + h_j x_{i,j} = k_j \frac{u_i - u_{i-2}}{2T} + l_j u_i
\]  

(2.5)

and \(x_{i,j}\) can be written as:

\[
x_{i,j} = \frac{k_j \frac{T}{2} (u_i - u_{i-2}) + l_j u_i T^2 + 2x_{i-1,j} - x_{i-2,j} (1 - g_j \frac{T}{2})}{1 + g_j \frac{T}{2} + h_j T^2}
\]  

(2.6)

where \(x_{k,j}\) \((k=i, i-1, i-2)\) is the output of the second-order system at time \(t_k\), and \(u_k\) \((k=i, i-1, i-2)\) is the control signal at time \(t_k\).

### 2.2.1.2 Simplification

When the sampling frequency increases, the sampling time \(T\) will be small. If the following condition is satisfied,

\[
\max(g_j T/2, l_j T^2, h_j T^2) < 0.1 \quad (j=1, 2, \ldots, n/2)
\]  

(2.7)
the $T$ and above terms can be neglected. However, $k_j T/2$ cannot be neglected; otherwise, there will be no control signal. Then Equation (2.6) may be written as

$$ x_{i, j} = k_j \frac{T}{2} (u_i - u_{i-2}) + 2x_{i-1, j} - x_{i-2,j} \quad (2.8) $$

In an arbitrary interval $(t_{i-1}, t_i)$, according to Equations (2.3), (2.4) and (2.8) the mathematical model of the controlled plant can be expressed as

$$ y_i = f(u_i, x_{i,j}) = \sum_{j=1}^{\frac{n}{2}} x_{i,j} = b_1 \frac{T}{2} (u_i - u_{i-2}) + 2y_{i-1} - y_{i-2} $$

or

$$ u_i = u_{i-2} + \frac{2(y_i - 2y_{i-1} + y_{i-2})}{b_1 T} \quad (2.9) $$

where

$$ b_1 = \sum_{j=1}^{\frac{n}{2}} k_j $$

$y_i$ is the desired plant output at time $t_i$. Equation (2.9) is a recursive difference equation. Thus, when the sampling time is small enough to satisfy (2.7), the mathematical model of the controlled plant can be described with only one unknown parameter.

2.2.1.3 Control Principle

In the interval $(t_{i-1}, t_i)$, the mathematical model of the controller is chosen to be the same as Equation (2.9). The control process can be considered as the one in which $u_i$
is the input, initial states are $y_{i-1}$, $y_{i-2}$ and $u_{i-2}$ and desired output is $y_i$. If the initial states, plant output $y_{i-1}$ at time $t_{i-1}$ and the desired output $y_i$ at time $t_i$ are known, the control signal $u_i$ at time $t_i$ can be calculated from Equation (2.9). When the control $u_i$ at time $t_i$ is used to control the plant, the desired plant output $y_i$ at time $t_i$ can be achieved. Therefore, Equation (2.9) can be chosen as the mathematical model of the controller, and from that equation the control signal can be calculated for controlling the plant.

The plant desired output at time $t_{i+1}$ can be determined from the following equation:

$$y_i = y_{i-1} + (r_i - y_{i-1}) \alpha$$

(2.10)

where $\alpha$ is a coefficient that determines the output response speed and overshoot and $r_i$ is the input of the system at time $t_i$, which is also the required output of the plant at time $t_i$.

The plant can be controlled continuously by repeating the same procedure in the next time interval $(t_i, t_{i+1})$, etc.

In short, the control steps are as follows:

1. The control process is discretized first into a number of time steps.
2. In the interval $(t_{i-1}, t_i)$, the difference equations of the controlled plant are derived for a typical time step.
3. The desired output at time $t_{i+1}$ is calculated from Equation (2.10).
4. The control signal is found from equation (2.9).
5. The control signal is sent to the plant and the output at time $t_{i+1}$ is obtained.
6. The procedure is repeated in the next time step, until the end of the process.
The control block diagram of the new algorithm is shown in Figure 2.3. Comparing Figure 2.1 with Figure 2.3, the main differences between the conventional method and the new algorithm are as follows:

1. In the conventional method, the control time is analyzed as a whole process and the differential equations of the system are analyzed and solved as an initial value problem. In the new approach, the control time is discretized, and the differential equations are analyzed and solved as a boundary value problem.

2. In the conventional method, various methods were developed to obtain the accurate mathematical model of controlled plant. In the new algorithm, attention is paid to simplify the model with fewer unknown parameters.

3. In the conventional method, the control signal is found based on the error signal and the controller function. The control signal \( u \) is calculated by error signal \( e \) multiplied by the controller function \( C \). In the new algorithm, the control signal \( u_i \) is found based on the desired output \( y_i \) and the controller function.
2.2.2 Case 2: \((n-m)>1\)

If \((n-m)=2\), Equation (2.3) becomes

\[
G(s) = \frac{Y}{U} = \sum_{j=1}^{n} \left( \frac{k_j}{s^2 + g_js + h_j} \right)
\]

As described in the case of \((n-m)=1\), after we take inverse Laplace transform with Equation (2.11) and express it by the difference equation, then simplify it, Equation (2.11) can be approximated as

\[
y_i = f(u_i, x_{i,j}) = \sum_{j=1}^{n} x_{i,j} = b_1T^2u_i + 2y_{i-1} - y_{i-2}
\]

or

\[
u_i = (y_i - 2y_{i-1} + y_{i-2})/(b_1T^2)
\]

Equation (2.12) has only one unknown parameter, \(b_1\).

For a general case of \((n-m)=j\) \((j=3,4,...,n)\), and if the differential equation of the controlled plant is

\[
y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y + a_n = b_1u^{(m)} + b_2u^{(m-1)} + \cdots + b_{m+1}
\]

\((n-m)=j\)

where \(a_i\) \((i=1, 2, \ldots, n)\), \(b_j\) \((j=1, 2, \ldots, m+1)\) are constants.

According to Newton backward difference polynomial formula (Hoffman, J. D., 1992),

\[
y_i^{(n)} = \frac{1}{T^n} (\nabla^n y_i) + O(T)
\]

(2.14)
where \( \frac{1}{T^n} (\nabla^n y_i) \) is the numerical difference of \( y_i^{(n)} \) and \( O(T) \) is the error of the derivative, Equation (2.13) can be written as
\[
\frac{1}{T^n} \nabla^n y_i + a_1 \frac{1}{T^{n-1}} \nabla^{n-1} y_i + \ldots + a_{n-1} y_i + a_n = b_1 \frac{1}{T^m} \nabla^m u_i + b_2 \frac{1}{T^{m-1}} \nabla^{m-1} u_i + \ldots + b_{m+1} + O(T) \tag{2.15}
\]

If both sides of Equation (2.15) are multiplied by \( T^n \), it becomes
\[
\nabla^n y_i + a_1 T \nabla^{n-1} y_i + \ldots + a_{n-1} T^n y_i + a_n T^n = b_1 T^{n-m} \nabla^m u_i + b_2 T^{t-m+1} \nabla^{m-1} u_i + \ldots + b_{m+1} T^n + O(T^{n-1}) \tag{2.16}
\]

When the sampling frequency increases, the sampling time \( T \) may be made small. If the following conditions are satisfied,
\[
a_1 T >> \max(a_2 T^2, \ldots, a_{n-1} T^{n-1}, a_n T^n) \tag{2.17-1}
\]
\[
b_1 T^{n-m} >> \max(b_2 T^{n-m-1}, \ldots, b_{m+1} T^n) \tag{2.17-2}
\]

neglecting the \( T^2 \) and above terms, Equation (2.16) becomes
\[
\nabla^n y_i = -a_1 T \nabla^{n-1} y_i + b_1 T^{n-m} \nabla^m u_i
\]
or
\[
y_i = f(y_{i-1}, y_{i-2}, \ldots, y_{i-n}, u_i, u_{i-1}, \ldots, u_{i-m}) \tag{2.18}
\]

Therefore, it is clear that when the sampling time is small enough to satisfy (2.17), the mathematical model of the controlled plant can be described with only two unknown parameters, \( a_1 \) and \( b_1 \).
When the model of the controlled plant is described by the state-variable model

\[
\dot{X}(t) = AX(t) + Bu(t) \tag{2.19-1}
\]

\[
y(t) = CX(t) + Du(t) \tag{2.19-2}
\]

Equation (2.19) can be changed into the same form as Equation (2.13) by the general method (Phillips, L. and Harbor, R. D., 1996) and then be simplified to have the form of Equation (2.18).

Therefore, whatever the linear system, if the sampling time is satisfied with (2.17), it can be described by a difference equation with only one or two unknown parameters, and the system can be controlled by the method described in Section 2.2.1.

2.3 Simulation

To verify the results of the theoretical analysis and to compare the responses of the new method with the conventional method, two simulation examples are given in this section.

2.3.1 Example 1, \((n-m)=1\)

When the plant transfer function is as follows:

\[
G(s) = \frac{15s^3 + 22s^2 + 16s + 17}{s^4 + 10s^3 + 23s^2 + 20s + 24} \tag{2.20}
\]

if the conventional method (Phillips, L. and Harbor, R. D., 1996) is used to control the plant, the simulation diagram and the simulation result are shown in Figures 2.4 and 2.5,
respectively. If the plant is controlled by the new algorithm, Figure 2.3, and when the model of the controller has the form of Equation (2.9), the simulation has the results shown in Figure 2.6. The results show that the plant is stable whether it is controlled by the new algorithm or by the conventional method.

Figure 2.4 Simulation diagram of the conventional method

However, if the parameters of the controlled plant change very much—say ten times their initial value for simplicity—the simulation diagram and the result with the conventional method are shown in Figures 2.7 and 2.8. It is obvious that the system is unstable. The simulation result implementing the new algorithm is shown in Figure 2.9, displaying very positive results. From Figures 2.8 and 2.9, it is clear that when the
parameters of the control model and controlled plant differ considerably, the new algorithm still possesses good performance robustness. It can also be seen that when the plant is unstable, without compensating, the plant is still controllable and has good stability robustness.

![Figure 2.6 Simulation result of the new algorithm](image)

**Figure 2.6** Simulation result of the new algorithm

![Figure 2.7 Simulation diagram of the conventional method when plant parameters change ten times their initial value](image)

**Figure 2.7** Simulation diagram of the conventional method when plant parameters change ten times their initial value
When the parameters of the controlled plant are arbitrarily changed--from those in Figure 2.7 to Figure 2.10--and the order of the plant transfer function are also changed but the controller model is still the same as described by Equation (2.9), the simulation results for the conventional method are shown in Figure 2.11. The simulation results for the new algorithm, Figure 2.12, indicate that the outcome of the new algorithm is superior. The settling time is just 0.02 second, and the steady-state error is 0.0002. The system controlled by the conventional method is unstable (Fig 2.11).
Figure 2.10  Simulation diagram of the conventional method when plant parameters and its order have changed

Figure 2.11 Simulation result of the conventional method when plant parameters and order have changed

Figure 2.12 Simulation result of the new algorithm when plant parameters and order have changed
2.3.2 Example 2, \((n-m)=2\)

If the model of the controlled plant is arbitrarily chosen, as in Figure 2.13, the simulation results for the conventional method (Phillips, L. and Harbor, R. D., 1996) and the new algorithm are shown in Figures 2.14 and 2.15. It is clear that the new algorithm is superior. The system controlled by the conventional method is unstable. The same system controlled by the new algorithm has fast response speed and good steady state accuracy.

\[
\begin{align*}
\text{Step Input} & \xrightarrow{\text{Sum}} \frac{125s^3+23s^2+566s+39}{s^5+233s^4-3933s^3+221s+43} \xrightarrow{\text{Transfer Fcn}} \text{Graph} \\
& \xrightarrow{\text{Gain}}
\end{align*}
\]

Figure 2.13 Simulation diagram of Example 2 with the conventional method

![Simulation diagram of Example 2 with the conventional method](image)

Figure 2.14 Simulation result of Example 2 with the conventional method

![Simulation result of Example 2 with the conventional method](image)
2.4 Summary

This chapter has presented an algorithm based on the finite difference method for control of SISO linear systems. It is different from the conventional method (Phillips, L. and Harbor, R. D., 1996) in that it discretizes the control process into a number of time steps, and the desired process response is controlled and achieved in each step. The algorithm is one of the digital predictive controls. However, a comparison of Figure 2.1 with Figure 2.3 shows that both the control principles and the control results are different.

The illustrative examples show that the algorithm is practical and simple and has a good transient and steady-state response. If the sampling time is small enough for a $n$th-order nonlinear system, the model of the controlled plant can be simplified and described by a difference equation with only two unknown parameters. Furthermore,
even if the model of the controlled plant and the controller differ greatly—for example, when the parameters of the controlled plant change ten times of their initial values or the order of the plant have changed—if it is controlled by the conventional method, the plant will be unstable. But if it is controlled by the new algorithm, the plant can still be controlled with good steady and transient properties and good performance and stability robustness.

The disadvantage of the new algorithm is that the sampling frequency must be high, which requires fast data processing and increases the cost of the hardware of the control system—such as the computer, A/D and D/A converters.
Chapter 3

Control of Coupled MIMO Linear Systems

3.1 Introduction

Consider the control block diagram of the plant which is multiple-input multiple-output (MIMO) linear system, Figure 3.1, where $U$ and $Y$ are the control signal and measured output, $R$ is input, $E$ is the error signal and $C, K$ denote controller and state feedback vector.

\[
\begin{align*}
U' &= AX(t) + BU(t) \\
Y(t) &= CX(t) + DU(t)
\end{align*}
\]

Figure 3.1 Control block diagram of the conventional method

Generally, the controlled plant can be described in state-space as

\[
\begin{align*}
X'(t) &= AX(t) + BU(t) \\
Y(t) &= CX(t) + DU(t)
\end{align*}
\]

where

\[
X(t) = [x_1, x_2, ..., x_n]^T, \quad X'(t) = [x'_1, x'_2, ..., x'_n]^T.
\]
For partly-coupled linear systems, the matrices of $A$, $B$, $C$ and $D$ for two inputs, two outputs and second order system, for example, can be described as

$$
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
a_{21} & a_{22} & a_{23} & a_{24} \\
0 & 0 & 0 & 1 \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}, \quad
B = \begin{bmatrix}
b_{21} & 0 \\
0 & 0 \\
0 & b_{42}
\end{bmatrix},
$$

$$
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}, \quad
D = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
$$

The differential equations can be written as the first-order system

$$
x_1' = x_2
$$

$$
x_2' = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + b_{21}u_1
$$

$$
x_3' = x_4
$$

$$
x_4' = a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 + b_{42}u_2
$$

$$
y_1 = x_1
$$

$$
y_2 = x_3
$$

### 3.2 Control Principle

#### 3.2.1 Derivation of Difference Equation

If the system input is a unit step, and the system desired response is the thinner curve shown in Figure 2.2, the whole control time can be discretized into $n$ equal steps with points $t_0, t_1, t_2, \ldots, t_n$, and the interval between each point or sampling time is
\[ T = \frac{t_u - t_0}{n} \]  \hspace{1cm} (3.3)

In an arbitrary interval \((t_i, t_j)\), according to backward difference formula (Hoffman, J. D., 1992), Equations (3.2) can be approximated in the form of difference equations as

\[ \frac{(x_{1,i} - x_{1,i-1})}{T} = x_{2,i} \]  \hspace{1cm} (3.4-1)

\[ \frac{x_{2,i} - x_{2,i-1}}{T} = a_{21} x_{1,i} + a_{22} x_{2,i} + a_{23} x_{3,i} + a_{24} x_{4,i} + b_{21} u_{1,i} \]  \hspace{1cm} (3.4-2)

\[ \frac{(x_{3,i} - x_{3,i-1})}{T} = x_{4,i} \]  \hspace{1cm} (3.4-3)

\[ \frac{x_{4,i} - x_{4,i-1}}{T} = a_{41} x_{1,i} + a_{42} x_{2,i} + a_{43} x_{3,i} + a_{44} x_{4,i} + b_{42} u_{2,i} \]  \hspace{1cm} (3.4-4)

### 3.2.2 Simplification

Substituting Equations (3.4-1) and (3.4-3) into (3.4-2), and multiplied by \(T^2\), Equation (3.4-2) becomes

\[ x_{1,i} - 2 x_{1,i-1} + x_{1,i-2} = a_{21} x_{1,i} T^2 + a_{22} (x_{1,i} - x_{1,i-1})T + a_{23} x_{3,i} T^2 + a_{24} (x_{3,i} - x_{3,i-1})T + b_{21} u_{1,i} T^2 \]  \hspace{1cm} (3.5)

When the sampling frequency increases, such that the following condition is satisfied:
the $T$ and higher order terms of Equation (3.5) can be neglected, because they are much smaller than the terms of the left side of the equal sign of Equation (3.5). Then Equation (3.5) becomes

$$x_{1,d} - 2x_{1,d-1} + x_{1,d-2} = b_{21}u_{1,d}T^2$$

(3.7-1)

If the same procedure is applied to Equation (3.4-4), it becomes

$$x_{3,d} - 2x_{3,d-1} + x_{3,d-2} = b_{42}u_{2,d}T^2$$

(3.7-2)

Therefore, Equation (3.3-2), which has ten unknown parameters, can be simplified to have only two unknown parameters.

In (3.6), 0.1 can be chosen as 0.01 or 0.001. The smaller the number is, the smaller the $T$ must be, and the smaller the terms of the $T$ and higher order in Equation (3.5). Hence, those terms are more insignificant, and the simplified Equation (3.7) is more accurate. However, the term $b_{21}T^2$, $b_{42}T^2$ in Equation (3.7) cannot be neglected: otherwise, there will be no control signals.

From Equations (3.2.5) and (3.2.6), the outputs become

$$y_{1,d} = x_{1,d}$$

(3.8-1)

$$y_{2,d} = x_{3,d}$$

(3.8-2)

Generally, if matrix $A$ is $n \times n$, and the controlled plant consists of $j$ inputs, $j$ outputs second-order subsystems, the matrices of $A$, $B$, $C$ and $D$ can be written as
By repeating the same analysis as above, the following conclusion can be reached:

If the following condition is satisfied,

\[
\max (|a_{2,1}|T^2, |a_{2,2}|T, |a_{2,3}|T^2, |a_{2,4}|T, \ldots, |a_{2,n-1}|T^2, |a_{2,n}|T, \\
|a_{4,1}|T^2, |a_{4,2}|T, |a_{4,3}|T^2, |a_{4,4}|T, \ldots, |a_{4,n-1}|T^2, |a_{4,n}|T, \\
\ldots, |a_{n,1}|T^2, |a_{n,2}|T, |a_{n,3}|T^2, |a_{n,4}|T, \ldots, \\
|a_{n,n-1}|T^2, |a_{n,n}|T, ) < 0.1 \\
(n=2j)
\]

(3.9)

the \((2j+1)j\) parameters of the equations of the controlled plant can be simplified to the difference equations with only \(j\) parameters. The equations are
\[ x_{k,i} - 2x_{k,i-1} + x_{k,i-2} = b_{k-1,(k+1)/2} u_{(k+1)/2,i} T^2 \]

or

\[ u_{(k+1)/2,i} = \frac{(x_{k,i} - 2x_{k,i-1} + x_{k,i-2})}{b_{k-1,(k+1)/2} T^2} \] (3.10-1)

\[ y_{(k+1)/2,i} = x_{k,i} \quad (k=1,3,5,...,n-1) \] (3.10-2)

(3.9) can be easily satisfied by decreasing the sampling time \( T \). Therefore, the equations of the controlled plant can be simplified greatly, and from Equation (3.10-1) the control signal can easily be determined.

### 3.2.3 Control Principle

If the mathematical model of the controller in the interval \( (t_{i-1}, t_i) \) is chosen to be the same as the one from Equation (3.10-1), the control process can be considered where control signals are \( U_i (U_i = [u_{1,i}, u_{2,i}, ..., u_{j,i}]^T) \), initial states are \( X_{i-1} (X_{i-1} = [x_{1,i-1}, x_{3,i-1}, ..., x_{n-1,i-1}]^T) \), \( X_{i-2} \) and the desired outputs are \( Y_i (Y_i = X_{i}) \). If the initial states \( X_{i-1} \), \( X_{i-2} \), and the desired output \( Y_i \) at time \( t_i \) are known, the control signal \( U_i \) at time \( t_i \) can be calculated by Equation (3.10). The control \( U_i \) at time \( t_i \) can be sent out to control the plant, and the desired plant output \( Y_i \) at time \( t_i \) can be achieved. Therefore, the simplified model of the controlled plant--Equation (3.10)--can be chosen as the mathematics model of the controller, and from this equation the control signal can be calculated.

The plant desired output at time \( t_i \) could be found from the following equation:

\[ y_{l,i} = y_{l,i-1} + (r_{l,i} - y_{l,i-1}) \alpha_i \quad l=1,2,...,j \] (3.11)
where \( \alpha_i \) is a coefficient which determines the output response speed and overshoot. \( r_i \) is the input of the system at time \( t_i \), which is also the required output of the plant at time \( t_i \).

The process is repeated in the next time step. This way the plant can be controlled continuously.

In short, the control steps are as follows:

1. The control process is discretized first into a number of time steps.
2. In the interval \((t_{i-1}, t_i)\), the difference equations of the controlled plant are derived for a typical time step.
3. The desired output at time \( t_i \) is calculated from Equation (3.11).
4. The control signal is found from Equation (3.10).
5. The control signal is sent to the plant and the output at time \( t_{i+1} \) is obtained.
6. The procedure is repeated in the next time step until the end of the process.

The control block diagram of the new algorithm is shown in Figure 3.2. Comparing the control block diagram of SISO linear system--Figure 2.3--with that of MIMO linear system--Figure 3.2--the main difference between those two figures is that all the single signals in Figure 2.3 are replaced by the multiple signals in Figure 3.2. The main control principles are the same.

![Figure 3.2 Control block diagram of the new algorithm](image)
3.3 Simulation

In order to verify the theoretical analysis results and illustrate the design procedure, simulations are presented in this section.

The matrices $A$, $B$, $C$ and $D$ of two inputs and two outputs controlled plant are arbitrarily chosen as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -500 & -15 & -0.5 & -10 \\ 0 & 0 & 0 & 1 \\ -600 & -10 & -40 & -10 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}^T,$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

When the plant is designed in state space with the approach of pole assignment using full state feedback (Phillips, L. and Harbor, R. D., 1996), the unit step response of the system is shown in Figure 3.3. If it is controlled by the new algorithm using the same procedure discussed in Section 3.2, the differential equation of the controlled plant can be simplified as

$$x_{1,i} - 2x_{1,i-1} + x_{1,i-2} = 0.5 u_{1,i} T^2$$

$$x_{3,i} - 2x_{3,i-1} + x_{3,i-2} = 10 u_{2,i} T^2$$

or
\[
\begin{align*}
\dot{u}_{1,i} &= \frac{(x_{1,i} - 2x_{1,i-1} + x_{1,i-2})}{0.5 T^2} \\
\dot{u}_{2,i} &= \frac{(x_{3,i} - 2x_{3,i-1} + x_{3,i-2})}{10 T^2}
\end{align*}
\] (3.12-1, 3.12-2)

If Equation (3.12)--the simplified model of the controlled plant--is chosen as the controller and if the coefficients in Equation (3.11) are chosen as \(a_i=0.1 \ (i=1,2)\) and sampling time \(T\) is chosen as 0.01 second, the simulation result, Figure 3.4, shows that the response is much better than that in (Phillips, L. and Harbor, R. D., 1996). The response speed of the new algorithm is about six times faster than that of the conventional method. Also, the response is smoother.

Figure 3.3 Response of the conventional method in Phillips, L. and Harbor, R. D., (1996)
Figure 3.4 Response of the new algorithm

Next, consider the case when matrix $A$ is changed into

$$
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-400 & -15 & -0.5 & -10 \\
0 & 0 & 0 & 1 \\
-600 & -10 & -40 & -10 \\
\end{bmatrix}
$$

that is, only one parameter $a_{21}$ has been changed, while matrices $B$, $C$ and $D$ remain the same. The simulation results for the conventional method are shown in Figure 3.5. It is clear that the system is unstable, and the conventional method fails. If the plant is controlled by the new algorithm and the controller is still kept unchanged (Equation
(3.12)), the simulation result in Figure 3.6 indicates that the system still has good performance.

Figure 3.5 Response of the conventional method of Phillips, L. and Harbor, R. D., (1996) when parameter $a_{21}$ is changed

Figure 3.6 Response of the new algorithm when parameter $a_{21}$ is changed
Finally, consider the case when the parameters in the matrix \( A \) are arbitrarily changed into

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
4 & 5 & 5 & 10 \\
0 & 0 & 0 & 1 \\
6 & 10 & 4 & 10
\end{bmatrix}
\]

and matrix \( B, C \) and \( D \) do not change. Using the same controller Equation (3.12), the simulation result of the new algorithm is shown in Figure 3.7. It is obvious that the plant still can be controlled successfully.

In summary, when the plant is designed and controlled by the conventional method, good results can be obtained initially, but when one or more parameters are changed, the results become unsatisfactory. However, when the plant is controlled by the new algorithm, superior performance can still be obtained.

Figure 3.7 Response of the new algorithm when matrix \( A \) is changed
3.4 Summary

This chapter presents a simple algorithm for control of coupled MIMO linear systems. The novel ideas of the new algorithm are as follows. First, instead of the accurate model of the plant being developed, the simplified model with fewer unknown parameters is derived. Second, the control signal is found based on the desired output, not on the error signal. Third, the system is analyzed and solved as a boundary value problem, not as an initial value problem.

Simulation examples have shown that the algorithm is practical and simple and has a good dynamic and steady-state response. If some conditions are satisfied, for a \( j \) inputs, \( j \) outputs system with \((2j+1)j\) unknown parameters (the number of inputs and outputs may be different), the model of the controlled plant can be simplified to a difference equation with only \( j \) unknown parameters. Furthermore, even if the model of controlled plant and the controller differ considerably and when the plant is unstable, it can still be controlled with good steady- and transient- state properties.

In this chapter, a special kind of partly coupled MIMO system is studied. The equation is similar to that of linearized robotic manipulators. Hence, if the algorithm presented in this chapter is developed, it can be used for controlling robotic manipulators.
Chapter 4

Control of SISO Nonlinear Systems

4.1 Introduction

Consider the control block diagram of the conventional method shown in Figure 2.1, \( r(t) \) is the input signal or the desired output, \( y(t) \) is the measured process output, \( e(t) = r(t) - y(t) \) is the error and \( u(t) \) is the control signal. \( C \) and \( G \) denote the controller and the plant.

Let us assume that the controlled plant may be described by the following single-input single-output \( n \)-th order nonlinear differential equation

\[
y^{(n)} = f(t; y, y', \ldots, y^{(n-1)}; u, u', \ldots, u^{(m)})
\]  

Equation (4.1) can also be expressed in the state-space form. First, introduce \( x_1 = y \), \( u_1 = u \), then

\[
x_1' = x_2
\]  
\[
x_2' = x_3
\]  
\[
\ldots
\]  
\[
x_{n-1}' = x_n
\]  
\[
x_n' = f(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_{m+1})
\]  
\[
u_1' = u_2
\]  
\[
\ldots
\]
\[ u_{m+1} = u_m \]  

(4.2-n+m)

where \( x_j \) \((j=1,2,\ldots,n)\) are state variables.

For the conventional method, the calculation of the control signal \( u_1 \) is based on the error signal; i.e., the error is multiplied by the controller function \( u_1 = eC \), and the controller design is based on the whole system--the controller and the plant--in the whole control time.

### 4.2 Control Principle

#### 4.2.1 Derivation of Difference Equation

Both the input \( r \) and the system desired response \( y \) may be discretized into \( n \) equal steps \( t_0, t_1, t_2, \ldots, t_n \) with the sampling time \( T=(t_n-t_0)/n \), as shown in Figure 2.2. The input \( r \) in Figure 2.2 is, for convenience, chosen to be the unit step function; otherwise, it may be an arbitrary function.

In a typical interval \((t_{i-1}, t_i), i=1,2,\ldots,n\), we introduce the notation for the function

\[ x_j(t_i) = x_{j,i} \]

and its approximated derivative

\[ x_j'(t_i) = x'_{j,i} = (x_{j,i} - x_{j,i-1}) / T + O(T) \]

such that Equations (4.2) can be written as the difference equations

\[ x'_{1,i} = (x_{1,i} - x_{1,i-1}) / T + O(T) = x_{2,i} \]  

(4.3-1)

\[ x'_{2,i} = (x_{2,i} - x_{2,i-1}) / T + O(T) = x_{3,i} \]  

(4.3-2)

\[ \ldots \]
\[
X'_{n-1,i} = \left( x_{n-1,i} - x_{n-1,i-1} \right) / T + O(T) = x_{n,i}
\]  
\[\text{(4.3-n-1)}\]

\[
x'_{n,i} = \left( x_{n,i} - x_{n,i-1} \right) / T + O(T) = f(x_{1,i}, x_{2,i}, \ldots, x_{n,i}; u_{1,i}, u_{2,i}, \ldots, u_{m+1,i})
\]  
\[\text{(4.3-n)}\]

\[
u'_{1,i} = \left( u_{1,i} - u_{1,i-1} \right) / T + O(T) = u_{2,i}
\]  
\[\text{(4.3-n+1)}\]

\[
u'_{m,i} = \left( u_{m,i} - u_{m,i-1} \right) / T + O(T) = u_{m+1,i}
\]  
\[\text{(4.3-n+m)}\]

while

\[
y_i = x_{1,i},
\]

\[
u_i = u_{1,i}.
\]

For example, let the nonlinear system be described by the following equation

(Shyu, K. K. and Lin, C. Y., 1996) which will be repeatedly used throughout this chapter:

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\times
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\times
\begin{bmatrix}
DX + u
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\]  
\[X' = A \times X + B \times (DX + u) \]  
\[\text{(4.4-1)}\]

where

\[
D = \begin{bmatrix}
0.2 \sin(x_1) & -0.5 \sin(x_2) & \sin(x_3)
\end{bmatrix}
\]  
\[\text{(4.4-2)}\]

\[
X = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
\]

and the input is bounded, such that \( |u| \leq 10 \).

When expressed in a form of the difference equations, the three equations in (4.4-1) become

\[
\frac{x_{1,i} - x_{1,i-1}}{T} = x_{2,i}
\]  
\[\text{(4.5-1)}\]
4.2.2 Linearization

If the interval \((t_{i-1}, t_i)\) is short enough, all the variables can be considered constants, as shown in Figure 2.2, and Equation (4.3-n) can be linearized and approximated in the form

\[
\frac{x_{3,j} - x_{3,i-1}}{T} = -0.2x_{1,j} + 45x_{2,j} - 9x_{3,j} + 0.2 \sin(x_{1,i})x_{1,j} - 0.5x_{2,j} + \sin(x_{3,i})x_{3,i} + u_i
\]  

(4.5-3)

\[
\frac{x_{2,j} - x_{2,i-1}}{T} = x_{3,i}
\]  

(4.5-2)

\[
\frac{x_{3,j} - x_{3,i-1}}{T} = a_j + b k (k=1,2, \ldots, n+1) and b k (k=1,2, \ldots, m+1) are constants. Notice that the total number of unknown coefficients in (4.6) is \(n+m+2\).

The linearization process is illustrated through the following examples

1. \((x_{j,i})^n \rightarrow (x_{j,i})^{n-1} x_{j,i} = a_1 x_{j,i}\)

where constant \(a_1 = (x_{j,i})^{n-1}\)

2. \((x_{j,i})^n (x_{k,i})^l \rightarrow (x_{j,i})^n (x_{k,i})^{l-1} x_{k,i} = b_1 x_{k,i}\)

where constant \(b_1 = (x_{j,i})^n (x_{k,i})^{l-1}\)

or

\[
(x_{j,i})^n (x_{k,i})^l \rightarrow (x_{k,i})^l (x_{j,i})^{n-1} x_{j,i} = b_2 x_{j,i}
\]
where constant \( b_2 = (x_{k,i})^{\prime}\left(x_{j,i}\right)^{n-1} \), depending on which term, \( x_{j,i} \) or \( x_{k,i} \), is needed in Equation (4.6).

For illustration, if this linerization process is applied to our example, Equation (4.5) becomes

\[
\frac{x_{1,i} - x_{1,i-1}}{T} = x_{2,i} \quad (4.7-1)
\]

\[
\frac{x_{2,i} - x_{2,i-1}}{T} = x_{3,i} \quad (4.7-2)
\]

\[
\frac{x_{3,i} - x_{3,i-1}}{T} = a_1 x_{1,i} + a_2 x_{2,i} + a_3 x_{3,i} + b_1 u_i \quad (4.7-3)
\]

where

\[
a_1 = -0.2 x_{1,i} + 0.2 \sin(x_{1,i})
\]

\[
a_2 = 4.5 - 0.5
\]

\[
a_3 = \sin(x_{3,i}) - 9
\]

\[
b_1 = 1.
\]

### 4.2.3 Simplification

Substituting all other equations in (4.3)--except (4.3-n)--into Equation (4.6) and using notation for the backward differences (Hoffman, J. D., 1992)

\[
\nabla x_{1,i} = x_{1,i} - x_{1,i-1}
\]
\[ \nabla^2 x_{1,i} = \nabla x_{1,i} - \nabla x_{1,i-1} = x_{1,i} - 2x_{1,i-1} + x_{1,i-2} \]

......

\[ \nabla^n x_{1,i} = \nabla^{n-1} x_{1,i} - \nabla^{n-1} x_{1,i-1} = f_i(x_{1,i}, x_{1,i-1}, x_{1,i-2}, \ldots, x_{1,i-n}) \]

after multiplication by \( T^n \). Equation (4.6) can be written as

\[ \nabla^n x_{1,i} + O(T) = a_1 T^n + a_2 T^n x_{1,i} + a_3 T^{n-1} \nabla x_{1,i} + \ldots + a_{n+1} T^n x_{1,i} \]

\[ + b_1 T^n u_{1,i} + b_2 T^n \nabla u_{1,i} + \ldots + b_{m+1} T^n u_{1,i} + O(T) \quad (4.8) \]

For a large enough sampling frequency, i.e., for small enough \( T \) such that the following conditions are satisfied,

\[ |a_{n+1} T| >> \max(|a_1 T^n|, |a_2 T^n|, |a_3 T^{n-1}|, \ldots, |a_n T^2|) \quad (4.9-1) \]

\[ |b_{m+1} T^{n-m}| >> \max(|b_1 T^n|, |b_2 T^{n-1}|, \ldots, |b_m T^{n-m+1}|) \quad (4.9-2) \]

and after neglecting the \( T^2 \) and higher-order terms while retaining only the lowest order terms in \( a's \) and \( b's \), Equation (4.8) can be simplified to

\[ \nabla^n x_{1,i} + O(T) = a_{n+1} T^n x_{1,i} + b_{m+1} T^n u_{1,i} + O(T) \quad (4.10) \]

The lowest order terms of \( a_{n+1} \) and \( b_{m+1} \) in (4.8) may be a different order, but they both must be included; otherwise, the control signal \( u_{1,i} \) will be lost.

Specifically, when \( n=1 \), Equation (4.8) becomes

\[ \nabla x_{1,i} + O(T) = a_1 T + a_2 T x_{1,i} + b_1 T u_{1,i} \]

And if \( \max(|a_1 T|, |a_2 T|) << 1 \), it will be simplified as

\[ \nabla x_{1,i} + O(T) = b_1 T u_{1,i} \]

with only one unknown parameter \( b_1 \).
The linear algebraic Equation (4.10) may easily be solved for \( u_{1,i} \). In general, the solution has the form

\[
 u_{1,i} = g(x_{1,i}, x_{1,i-1}, \ldots, x_{1,i-n}; u_{1,i-1}, u_{1,i-2}, \ldots, u_{1,i-m}) = g(\cdot)
\]  

(4.11)

To conclude, when the sampling time is small enough to satisfy Equation (4.9), the discretized mathematical model (4.6) of the controlled plant with \( n+m+2 \) unknown parameters has been simplified and is described by only one or two unknown parameters \( a_{n+1} \) and \( b_{m+1} \), i.e., the coefficients that multiply \( x_{n,i} \) and \( u_{m+1,i} \) in Equation (4.6).

For illustration, we now apply the simplified procedure to the same example of Equation (4.7). After substitution of Equations (4.7-1), (4.7-2) and using the notation of backward difference, Equation (4.7-3) becomes

\[
 \begin{align*}
 \frac{x_{3,i} - x_{3,i-1}}{T} &= \left( \frac{x_{2,i} - x_{2,i-1}}{T} - \frac{x_{2,i-1} - x_{2,i-2}}{T} \right) / T \\
 &= \frac{x_{1,i} - 3x_{1,i-1} + 3x_{1,i-2} - x_{1,i-3}}{T^3} = \frac{1}{T^3} \nabla^3 x_{1,i} \\
 &= a_1 x_{1,i} + a_2 x_{2,i} + a_3 x_{3,i} + b_1 u_i \\
 &= a_1 x_{1,i} + a_2 \frac{x_{1,i} - x_{1,i-1}}{T} + a_3 \frac{x_{1,i} - 2x_{1,i-1} + x_{1,i-2}}{T^2} + b_1 u_i \\
 &= a_1 x_{1,i} + a_2 \frac{1}{T} \nabla x_{1,i} + a_3 \frac{1}{T^2} \nabla^2 x_{1,i} + b_1 u_i
\end{align*}
\]

(4.12)

After multiplication by \( T^3 \) and neglecting the \( T^2 \) and higher-order terms, Equation (4.12) becomes
\[ \nabla^3 x_{1,i} = x_{1,i} - 3x_{1,i-1} + 3x_{1,i-2} - x_{1,i-3} \]

\[ = a_3 T (x_{1,i} - 2x_{1,i-1} + x_{1,i-2}) + T^3 b_1 u_i \]

\[ = a_3 T \nabla^2 x_{1,i} + T^3 b_1 u_i \]

such that

\[ u_i = \{x_{1,i} - 3x_{1,i-1} + 3x_{1,i-2} - x_{1,i-3} - a_3 T (x_{1,i} - 2x_{1,i-1} - x_{1,i-2})\} / (b_1 T^3) \]

\[ = \{(1 - a_3 T)x_{1,i} + (2a_3 T - 3)x_{1,i-1} + (3 - a_3 T)x_{1,i-2} - x_{1,i-3}\} / (b_1 T^3) \]

(4.13)

### 4.2.4 Control Principle

We have just shown that if the mathematical model of the controller is chosen to be the same as Equation (4.11), and if the initial states \( u_{1,j} \) \((j=i-1,i-2,...,i-m)\), \( x_{1,k} \) \((k=i-1,i-2,...,i-n)\) and the desired output \( y_i \) are known, the control signal \( u_{1,i} \) can be calculated as the only unknown from Equation (4.11).

Then the desired output \( y_i \) can be found from the desired output equation

\[ y_i = y_{i-1} + (r_i - y_{i-1}) \alpha \]

(4.14)

where \( r_i \) is the input of the system at time \( t_i \), and \( \alpha \) is a yet undetermined coefficient.

Equation (4.14) shows that the desired output \( y_i \) at time \( t_i \) is equal to the measured output \( y_{i-1} \) at time \( t_{i-1} \), plus some small portion of the output error at time \( t_{i-1} \); i.e., at every new step, the desired output increases for a certain amount \( \alpha \) of the output error of the previous step, (see Figure 2.2). The value of \( \alpha \) determines the output response speed and overshoot. For example, if \( \alpha = 1 \), i.e., if the desired output increases 100% of the error of
the previous step. The plant will be unstable. If it is too small, i.e., $\alpha < 1$ as, for example, $\alpha = 0.001$, the desired output at the new step increases 0.1% of the output error in the previous step, and the response speed of the plant will be too slow. According to our experience, when $0.1 < \alpha < 0.2$, both the transient and steady response of the controlled plant will be satisfactory.

Based on the above, in the interval $(t_{i-1}, t_i)$, the control process can be considered where the control signal is $u_{1,i}$. Initial states are $u_{1,j}$ ($j = i-1, i-2, ..., i-m$), $x_{1,k}$ ($k = i-1, i-2, ..., i-n$) and the desired output is $y_i$. With the control $u_{1,i}$ at time $t_i$, the desired plant output $y_i$ at time $t_i$ can be achieved. Therefore, Equation (4.11) can be chosen as the mathematical model of the controller, and that equation provides the control of the plant. Thus, the control signal is calculated based on the plant-desired output—not on the error signal—such as in the conventional method.

The control process can also be viewed as the following problem. In the interval $(t_{i-1}, t_i)$, given the $n$-th order equation with $n$ boundary conditions found at the previous points ($k = i-1, i-2, ..., i-n$), as well as at the current point $i$, determine the control signal at the current point $i$. Then the control signal at time step $i$, $u_{1,i}$, is determined from Equation (4.11) and is based on $u_{1,j}$ ($j = i-1, i-2, ..., i-m$) and $x_{1,k}$ ($k = i-1, i-2, ..., i-n$) from the previous steps, with the desired output $y_i$ from the current step. This is another difference between the new algorithm and the conventional method—i.e., the control process is treated as the boundary value problem, not as the initial value problem.

The plant can be controlled continuously by repeating the process in the next interval $(t_i, t_{i+1})$. 
In short, the main control steps are the same as that of linear systems. They will be repeated again as follows:

1. The control process is discretized first into a number of time steps.
2. In the interval \( (t_{i-1}, t_i) \), the difference equations of the controlled plant are derived for a typical time step.
3. The desired output at time \( t_i \) is calculated from Equation (4.14).
4. The control signal is found from Equation (4.11).
5. The control signal is sent to the plant and the output at time \( t_i \) is obtained.
6. The procedure is repeated in the next time step, until the end of the process.

The control block diagram of the new algorithm is shown in Figure 4.1. Comparing Figure 2.1 with Figure 4.1, the main differences of the new algorithm from the conventional algorithm may be described as follows:

1. In the conventional method, the control time is analyzed as a whole process, and differential equations of the system are analyzed and solved as an initial value problem. In the present approach, the control time is discretized, and the differential equations are analyzed and solved as a boundary value problem.

2. In the conventional digital control method, the stability, transient and steady response of the whole system (including the controller and the plant) are analyzed as a whole control process, whereas in the new algorithm, the plant difference equation is analyzed in a small interval.

3. In the conventional approach, the control signal is found based on the error signal and controller function. The control signal \( u_{1,i} \) is calculated from the error signal \( e \).
multiplied by \( C \). In the new algorithm, the control signal is calculated from the desired output, real output and the plant model; i.e., from Equation (4.11) (see Figure 4.1). At any instant \( t_i \), according to the desired output \( y_i \), the corresponding control signal can be determined from the controller Equation (4.11).

![Control block diagram of the new algorithm](image)

**Figure 4.1** Control block diagram of the new algorithm

### 4.3 Simulations

#### 4.3.1 Example 1

Consider again the example of Section 4.2 (Shyu, K. K. and Lin, C. Y., 1996) and described by Equation (4.4). The output \( y \) is \( x_1 \) and the simplified difference equation of controller is (4.13); i.e.,

\[
u_i = \frac{(1 - a_3 T)x_{1,i} + (2a_3 T - 3)x_{1,i-1} + (3 - a_3 T)x_{1,i-2} - x_{1,i-3}}{b_1 T^3}
\]

If \( T=0.01 \), \( a \) in Equation (4.14) is taken to be equal to 0.1, and initial states are \( X(0) = \begin{bmatrix} 0 & 0.68 & 0 \end{bmatrix}^T \). The simulation results are shown in Figure 4.2. For comparison, the simulation results of Shyu, K. K. and Lin, C. Y., (1996) are shown in Figure 4.3. In Shyu, K. K. and Lin, C. Y., in order to reduce settling time and also maintain the sliding motion in a larger bound, a complicated adaptive sliding model is proposed.
From Figures 4.2 and 4.3, we see that the performance of the new algorithm is better, the setting time is only half of that in Shyu, K. K. and Lin, C. Y., (1996), while the design method is much simpler.

Next, consider the case in which the system equations are changed to

\[
X' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 200 & 50 & -9 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (DX + u)
\]

\[
D = \begin{bmatrix} 20 \sin(x_1) & 50 \sin(x_2) & \sin(x_3) \end{bmatrix}
\]

i.e., some of the parameters are increased in magnitude one hundred times with respect to their initial values. The controller equation is still the same as Equation (4.13) and the control signal is not bounded. The simulation result is shown in Figure 4.4 for the unit step input. It is clear that even when there are great differences between the model of the controller and controlled plant, the system still possesses good performance robustness.

![Figure 4.2 Response of the new algorithm](image)
Figure 4.3 Response of the method of Shyu, K. K. and Lin, C. Y., (1996)

Figure 4.4 Response of the new algorithm with changed parameters
4.3.2 Example 2.

Consider the inverted pendulum on a cart assuming point masses and a massless rod (Teel. A. R., 1996),

\[ x^{(2)} = \frac{1}{M/m + \sin^2(\theta)} \left( \frac{u}{m} + (\theta')^2 I \sin(\theta) - g \sin(\theta) \cos(\theta) \right) \] (4.15-1)

\[ \theta^{(2)} = \frac{1}{l} \frac{1}{M/m + \sin^2(\theta)} \left( -\frac{u}{m} \cos(\theta) - (\theta')^2 l \cos(\theta) \sin(\theta) + \frac{m + M}{m} g \sin(\theta) \right) \] (4.15-2)

where \( x \) represents the position of the cart, \( \theta \) represents the angle of the pendulum with \( \theta = 0 \) representing the inverted position, \( M \) is the mass of the cart, \( m \) is the mass at the end of rod whose length is \( l \) and \( u \) is the control input force acting on the cart.

If the equations in (4.15) are expressed as difference equations, the first-order system is

\[ x_{1,i} = x_i \] (4.16-1)

\[ \frac{x_{1,i} - x_{1,i-1}}{T} = x_{2,i} \] (4.16-2)

\[ \frac{x_{2,i} - x_{2,i-1}}{T} = \frac{1}{(M/m + \sin^2(\theta_{3,i-1}))} \left( \frac{u_i}{m} + x_{4,i-1} l \sin(x_{3,i-1}) - g \sin(x_{3,i-1}) \cos(x_{3,i-1}) \right) \] (4.16-3)

\[ x_{3,i} = \theta_i \] (4.16-4)

\[ \frac{x_{3,i} - x_{3,i-1}}{T} = x_{4,i} \] (4.16-5)
\[
\frac{x_{4,i} - x_{4,i-1}}{T} = \frac{1}{l} \left( \frac{1}{M/m} + \sin^2(x_{3,i-1}) \right) \left( -\frac{u_i}{m} \cos(x_{3,i-1}) - x_{4,i-1} l \cos(x_{3,i-1}) \sin(x_{3,i-1}) \right.
\]

\[
+ \left. \frac{m + M}{m} g \sin(x_{3,i-1}) \right)
\]

(4.16-6)

Because we control the angle of the pendulum only, following the procedure described in Section 4.2, Equation (4.16-6) can be simplified to the single controller equation

\[
u_i = -m \left( \frac{lM}{mT} (x_{4,i} - x_{4,i-1}) \right)
\]

If \(m=2\) kg, \(M=6\) kg, \(l=1\) m, \(g=9.81\) m/s\(^2\), \(\alpha=0.1\), \(T=0.001\) second and with unit step input, the simulation result is shown in Figure 4.5. It can be seen that the new algorithm has a good transient and steady response.

In Teel, A. R., (1996), a rather involved nonlinear small gain theorem is presented that provides a formalism for analyzing the behavior of the system, and an iterative procedure has been derived for controlling the system in a general nonlinear feed forward form. The method in Teel, A. R., (1996) is very complicated.

![Figure 4.5](image-url)
4.4 Summary

This chapter presents a simple control algorithm for single-input single-output nonlinear systems. The desired process response is achieved by analyzing and controlling the system in every step. The algorithm may be viewed as a digital predictive control.

The novel ideas of the new algorithm are as follows. First, instead of the accurate model of the plant being developed, the simplified model with only two unknown parameters is derived. Second, the control signal is based on the desired output, not on the error signal. Third, the system is analyzed and solved as the boundary value problem, rather than the initial value problem.

The examples show that the algorithm is practical and simple and has a good transient and steady response. If the sampling frequency is high enough, the \( n \)th-order nonlinear equation of the controlled plant can be simplified to the difference equation with only one or two unknown parameters. Furthermore, when the model of controlled plant and the controller greatly differ, the plant can still be controlled with good steady and transient properties.

The disadvantage of the new algorithm for control of nonlinear systems is the same as that of linear systems.

A problem of dynamic control of the plant has been reduced to a careful monitoring and the appropriate adjustment of the parameter \( \alpha \). This requires more investigations to be done. It should be possible to extend the algorithm to the multiple-input multiple-output systems. This is a subject of the next chapter.
Chapter 5

Control of MIMO Nonlinear Systems and Its Applications in Robotics

5.1 Introduction

In order to show that the same approach may be extended to multiple-input
multiple-output systems (MIMO), we solve the control problem of an \( n \) degree-of-
freedom manipulator described by the following equation of motion:

\[
M(q)\ddot{q} + V(q, \dot{q}) \dot{q} + g(q) = u(t)
\]  

(5.1)

where

\[
M(q) = \begin{bmatrix}
m_{11}(q) & m_{12}(q) & \cdots & m_{1n}(q) \\
m_{21}(q) & m_{22}(q) & \cdots & m_{2n}(q) \\
\vdots & \vdots & \ddots & \vdots \\
m_{n1}(q) & m_{n2}(q) & \cdots & m_{nn}(q)
\end{bmatrix}
\]

\[
V(q, \dot{q}) = \begin{bmatrix}
v_{11}(q, \dot{q}) & v_{12}(q, \dot{q}) & \cdots & v_{1n}(q, \dot{q}) \\
v_{21}(q, \dot{q}) & v_{22}(q, \dot{q}) & \cdots & v_{2n}(q, \dot{q}) \\
\vdots & \vdots & \ddots & \vdots \\
v_{n1}(q, \dot{q}) & v_{n2}(q, \dot{q}) & \cdots & v_{nn}(q, \dot{q})
\end{bmatrix}
\]

\[
g(q) = [g_1(q), g_2(q), \ldots, g_n(q)]^T,
\]

\[
u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^T.
\]

\( q \) is the \( n \times 1 \) joint angle vector, \( u \) is the \( n \times 1 \) control torque vector, \( M(q) \) is the \( n \times n \)
inertia matrix. \( \mathbf{v}(q) \) is the \( n \times n \) matrix representing the centrifugal and Coriolis terms and \( g(q) \) is the \( n \times 1 \) vector of gravity force term.

Equation (5.1) is a \( n \) inputs \( n \) outputs nonlinear system, and it can also be written as

\[
m_{j,1}(q)\ddot{q}_1 + m_{j,2}(q)\ddot{q}_2 + \ldots + m_{j,n}(q)\ddot{q}_n + v_{j,1}(q,\dot{q})\dot{q}_1 + v_{j,2}(q,\dot{q})\dot{q}_2 + \ldots + v_{j,n}(q,\dot{q})\dot{q}_n + g_j(q) = u_j(t) \quad j=1,2,\ldots,n
\]

Generally, robotic manipulators are controlled by computers. The digital control block diagram of the conventional method (Phillips, C. L. and Nagle, H. T., 1990) is shown in Figure 5.1, where \( R \) is input, \( E \) is the error signal, \( u \) is the control signal, \( D \) represents the digital controller, \( \text{ZOH} \) represents zero-order hold and \( G \) represents the controlled plant, i.e., robotic manipulator.

![Figure 5.1 Digital control block diagram of the conventional method](image)

### 5.2 Control Principle

#### 5.2.1 Linearization and Derivation of Difference Equation

If the whole control time is divided into \( n \) equal steps with points \( t_0, t_1, t_2, \ldots, t_n \) and the sampling time
\[ T = \frac{t_n - t_0}{n} \]  

(5.3)

the system input \( r_j \) and the system desired response \( q_j \) can be discretized, as shown in Figure 2.2. When the data hold is a zero-order hold, or if sampling time is small enough, all the variables \( q, \dot{q}, R \) can be considered constants in an arbitrary interval \((t_{i-1}, t_i)\). Therefore, \( m_{j,i}(q_i) \), \( v_{j,2}(q_i, \dot{q}_i) \), and \( g_j(q_i) \) are all scalars, and Equation (5.2) becomes linear. If the notation of the backward difference formula (Hoffman, J. D., 1992) is used, Equation (5.2) can be described as the difference equations

\[
m_{j,1}(q_i) \frac{\nabla^2 q_{i}}{T^2} + m_{j,2}(q_i) \frac{\nabla^2 q_{2,i}}{T^2} + \cdots + m_{j,n}(q_i) \frac{\nabla^2 q_{n,i}}{T^2} \\
+ v_{j,1}(q_i, \dot{q}_i) \frac{\nabla q_{i}}{T} + v_{j,2}(q_i, \dot{q}_i) \frac{\nabla q_{2,i}}{T} + \cdots + v_{j,n}(q_i, \dot{q}_i) \frac{\nabla q_{n,i}}{T} \\
+ g_j(q_i) = u_{j,i} \tag{5.4-1}
\]

where

\[ u_{j,i} = u_j(t_i) \tag{5.4-2} \]

\[ q_{j,i} = q_j(t_i) \tag{5.4-3} \]

\[
\frac{\nabla q_{j,i}}{T} = \dot{q}_j(t_i) = (q_{j,i} - q_{j,i-2})/T + O(T^2) \tag{5.4-4} 
\]

\[
\frac{\nabla^2 q_{j,i}}{T^2} = \ddot{q}_j(t_i) = (q_{j,i} - 2q_{j,i-1} + q_{j,i-2})/T^2 + O(T^2) \tag{5.4-5} 
\]

\[ q_i = [q_{1,i}, q_{2,i}, \ldots, q_{n,i}]^T \]

\[ \dot{q}_i = [\dot{q}_{1,i}, \dot{q}_{2,i}, \ldots, \dot{q}_{n,i}]^T \]

\[ \ddot{q}_i = [\ddot{q}_{1,i}, \ddot{q}_{2,i}, \ldots, \ddot{q}_{n,i}]^T \quad j=1,2,\ldots,n \]
Substituting Equations (5.4-4) and (5.4-5) into (5.4-1) and multiplying both sides of Equation (5.4-1) by $T^2$, (5.4-1) becomes

\[ m_{j,1}(q_i)(q_{i,i} - 2q_{i,i-1} + q_{i,i-2}) + m_{j,2}(q_i)(q_{i,2,i} - 2q_{i,2,i-1} + q_{i,2,i-2}) + \ldots + m_{j,n}(q_i)(q_{n,i} - 2q_{n,i-1} + q_{n,i-2}) + T\gamma_{j,1}(q_i, \dot{q}_i)(q_{1,i} - q_{1,i-2}) + T\gamma_{j,2}(q_i, \dot{q}_i)(q_{2,i} - q_{2,i-2}) + \ldots + T\gamma_{j,n}(q_i, \dot{q}_i)(q_{n,i} - q_{n,i-2}) + T^2 g_j(q_i) + T^2 O(T^2) = T^2 u_j \]

(5.5)

Equation (5.5) is the so-called ‘exact model’ of robot manipulators expressed in the form of the difference equation and it is an approximation of Equation (5.4-1). It can be used as the controller equation.

5.2.2 Simplification

When the sampling frequency increases and the following conditions

\[ \min\{ |m_{j,1}(q_i)q_{1,i}|, |m_{j,1}(q_i)q_{1,i-1}|, |m_{j,2}(q_i)q_{2,i}|, |m_{j,2}(q_i)q_{2,i-1}|, \ldots, |m_{j,n}(q_i)q_{n,i}|, |m_{j,n}(q_i)q_{n,i-1}| \} \]

are satisfied--i.e., the terms of the left side of the greater sign of (5.6) are much greater than that of the right side of (5.6)--the $T$ and higher order terms of Equation (5.5) can be

\[ \max\{ T|v_{j,1}(q_i, \dot{q}_i)q_{1,i}|, T|v_{j,1}(q_i, \dot{q}_i)q_{1,i-1}|, T|v_{j,2}(q_i, \dot{q}_i)q_{2,i}|, \ldots, T|v_{j,n}(q_i, \dot{q}_i)q_{n,i}|, T|v_{j,n}(q_i, \dot{q}_i)q_{n,i-1}|, T^2|g_j(q_i)| \}

j=1,2,\ldots,n

(5.6)
neglected. However, the terms $T^{-i,j} (j=1,2,...,n)$ cannot be neglected; otherwise, there will be no control signal. Hence, Equation (5.5) can be simplified as

$$m_{j,1}(\mathbf{q})(q_{i,j} - 2q_{i,j-1} + q_{i,j-2}) + m_{j,2}(\mathbf{q})(q_{i,j} - 2q_{i,j-1} + q_{i,j-2})$$

$$+ ... + m_{j,n}(\mathbf{q})(q_{n,i} - 2q_{n,i-1} + q_{n,i-2}) + O(T) = T^2 u_{j,i}$$

or

$$u_{j,i} = m_{j,1}(\mathbf{q})(q_{i,j} - 2q_{i,j-1} + q_{i,j-2})/T^2 + m_{j,2}(\mathbf{q})(q_{i,j} - 2q_{i,j-1} + q_{i,j-2})/T^2$$

$$+ ... + m_{j,n}(\mathbf{q})(q_{n,i} - 2q_{n,i-1} + q_{n,i-2})/T^2$$

(5.7)

(5.6) can be easily realized by sufficiently decreasing the sampling time $T$. Therefore, Equations (5.5), which has $n(2n+1)$ unknown parameters, can be simplified to Equation (5.7), with only $n^2$ unknown parameters, and the control signal $u_{j,i}$ can be determined from Equation (5.7). Thus, Equation (5.7) can be used as the simplified model of robot manipulators.

### 5.2.3 Control Principle

In the interval $(t_{i-1}, t_i)$, the mathematical model of the controller is chosen to be the same as Equation (5.7). The control process can be considered as one in which control signals are $u$, initial states are $\mathbf{q}_{i-1}$, $\mathbf{q}_{i-2}$ and desired outputs are $\mathbf{q}_i$. If the initial states $\mathbf{q}_{i-1}$, $\mathbf{q}_{i-2}$, and the desired output $\mathbf{q}_i$ at time $t_i$ are known, the control signal $u$, at time $t_i$ can be determined from Equation (5.7). If the control $u$, at time $t_i$ is used to control the plant, the desired plant output $\mathbf{q}_i$ at time $t_i$ can be achieved. Therefore, Equation (5.7) can
be used as the mathematical model for the controller, and the control signal can be calculated from this equation.

The plant desired output $q_{j,i}$ at time $t_i$ can be determined according to the desired output equation

$$q_{j,i} = q_{j,i-1} + (r_{j,i} - q_{j,i-1}) \alpha_j$$

where $r_i$ is both the input of the system and the required output of the plant at time $t_i$. $\alpha_j$ are coefficients which decide the output response speed and overshoot. According to simulations, all the values of $\alpha_j$ can be chosen in the range of $\alpha_j = 0.1 - 0.3$.

The plant can be controlled continuously by repeating the above procedure in subsequent time steps.

The digital control block diagram is shown in Figure 5.2. The main difference between the conventional method and the new algorithm is that the digital controller $D$ of the method consists of the desired output equation (5.8) and the simplified model of controlled plant, i.e., robotic manipulator, Equation (5.7). In other words, the control signal $u_i$ is determined according to the desired output $q_i$, not the error signal $E$, and the simplified model of a robotic manipulator.

![Figure 5.2 Digital control block diagram of the new algorithm](image-url)
5.3 Examples of Applications in Robotics

5.3.1 Example 1

Consider a two-link manipulator described in Figure 5.3. Let the parameters for the equation of motion (5.1) be the same as that in Wu, S. T., (1997), i.e.,

\[
\begin{align*}
M(q) &= \begin{bmatrix} 3.34 + 2.4 \cos q_2 & 0.97 + 1.2 \cos q_2 \\ 0.97 + 1.2 \cos q_2 & 0.97 \end{bmatrix} \\
V(q, \dot{q}) &= \begin{bmatrix} -1.2 \dot{q}_2 \sin q_2 & -1.2 (\dot{q}_1 + \dot{q}_2) \sin q_2 \\ 1.2 \dot{q}_1 \sin q_2 & 0 \end{bmatrix} \\
g(q) &= \begin{bmatrix} 24.5 \cos q_1 + 11.76 \cos (q_1 + q_2) \\ 11.76 \cos (q_1 + q_2) \end{bmatrix}
\end{align*}
\]

Figure 5.3 The two-link robotic manipulator

From the equation of motion (5.1) the exact control torque applied to link 1 and link 2 \(u_1\) and \(u_2\) can be expressed as
\[
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
= \begin{bmatrix}
3.34 + 2.4 \cos q_2 & 0.97 + 1.2 \cos q_2 \\
0.97 + 1.2 \cos q_2 & 0.97
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2
\end{bmatrix}
+ \begin{bmatrix}
-1.2 \dot{q}_2 \sin q_2 & -1.2 (\dot{q}_1 + \dot{q}_2) \sin q_2 \\
1.2 \dot{q}_1 \sin q_2 & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{q}_1 \\
\ddot{q}_2
\end{bmatrix}
+ \begin{bmatrix}
24.5 \cos q_1 + 11.76 \cos (q_1 + q_2) \\
11.76 \cos (q_1 + q_2)
\end{bmatrix}
\]
\[\text{(5.9)}\]

If the exact model—Equation (5.9)—is chosen as the controller equation, the simulation result is shown in Figure 5.4. The sampling time \( T \) is 0.01 second, the step inputs \( r_1(t) = 3, r_2(t) = 1 \) and the steady output of joint angles are \( q_1 = 3.02464 \) and \( q_2 = 0.96813 \).

If the sampling time \( T \) is 0.001 second, the simulation result is shown in Figure 5.5, and the steady output of joint angles are \( q_1 = 3.00025 \) and \( q_2 = 0.99968 \). Hence, by decreasing the sampling time \( T \), the steady state error can be decreased, and response speed can be increased.

![Figure 5.4 Angle response with exact model and \( T = 0.01 \)](image-url)
Following the procedure discussed in Section 5.2, Equation (5.9) can be simplified as

\[
m_{l_1}(q_{1,l} - 2q_{1,l-1} + q_{1,l-2}) / T^2 + m_{l_2}(q_{2,l} - 2q_{2,l-1} + q_{2,l-2}) / T^2 = u_{l,l} \quad (5.10-1)
\]

\[
m_{2_1}(q_{1,l} - 2q_{1,l-1} + q_{1,l-2}) / T^2 + m_{2_2}(q_{2,l} - 2q_{2,l-1} + q_{2,l-2}) / T^2 = u_{2,l} \quad (5.10-2)
\]

where

\[
m_{l_1} = 3.34 + 2.4 \cos q_2,
\]

\[
m_{l_2} = 0.97 + 1.2 \cos q_2,
\]

\[
m_{2_1} = 0.97 + 1.2 \cos q_2,
\]

\[
m_{2_2} = 0.97
\]

or
If the sampling time $T$ is 0.01 second, the inputs are

$$r_1(t) = 3, \quad r_2(t) = 1$$

The controller equations are (5.11) and (5.8). The simulation results for joint angles $q_1$ and $q_2$ are shown in Figure 5.6. The steady-state output of joint angles are $q_1=3.01732$ and $q_2=0.97512$. Figures 5.7 and 5.8 illustrate the simulation results of joint angles $q_1$, $q_2$, and the errors $e_1$, $e_2$, when the sampling time $T$ is 0.001 second and other parameters remain unchanged. The steady-state output of joint angles are $q_1=3.00018$ and $q_2=0.99975$, and the errors are $e_1=-0.00018$ and $e_2=0.00025$.

$$u_{1,i} = m_{11} \nabla^2 q_{1,i} / T^2 + m_{12} \nabla^2 q_{2,i} / T^2$$

(5.11-1)

$$u_{2,i} = m_{21} \nabla^2 q_{1,i} / T^2 + m_{22} \nabla^2 q_{2,i} / T^2$$

(5.11-2)

Figure 5.6 Angle response with simplified model and $T=0.01$
By comparing the simulation results of the exact model—Figures 5.4 and 5.5—with that of the simplified model—Figures 5.6 to 5.8—it is clear that if the parameters are the same, the simplified model performs as well as the exact model.

Figure 5.7 Angle response with simplified model and $T=0.001$

Figure 5.8 Angle errors with simplified model and $T=0.001$
The simulation results for the joint angles $q_1$ and $q_2$ and errors $e_1$ and $e_2$ with the method from Wu, S. T., (1997) are shown in Figures 5.9, 5.10. It is obvious that the response of the new algorithm is better than that of the method in Wu, S. T., (1997), because the settling time and tracking errors of the new algorithm are smaller.

Figure 5.9 Angle response with the method of Wu, S. T., (1997) and $T=0.001$

Figure 5.10 Angle errors with the method of Wu, S. T., (1997) and $T=0.001$
5.3.2 Example 2

Let us consider an example of a two-link manipulator with parameter perturbations, friction forces $f(q)$ and bounded disturbances $u_d(t)$. This example is modeled by the following differential equation (Cheng, Y. C. and Chen, B. S., 1997):

$$ M(q)\ddot{q} + V(q, \dot{q})\dot{q} + f(q) + g(q) + u_d(t) = u(t) $$

(5.12)

where

$$ M(q) = \begin{bmatrix} (m_1 + m_2)l_1^2 & m_2 l_1 l_2 (s_1 s_2 + c_1 c_2) \\ m_2 l_1 l_2 (s_1 s_2 + c_1 c_2) & m_2 l_2^2 \end{bmatrix} $$

$$ V(q, \dot{q}) = m_2 l_1 l_2 (c_1 s_2 - s_1 c_2) \begin{bmatrix} 0 & -\dot{q}_2 \\ -\dot{q}_1 & 0 \end{bmatrix} $$

$$ f(q) = [0.1 \dot{q}_1, 0.1 \dot{q}_2] $$

$$ g(q) = [- (m_1 + m_2) l_1 g s_1, -m_2 l_2 g s_2]^T $$

$$ u_d(t) = [u_{d1}(t), u_{d2}(t)]^T $$

The parameters for the equation of motion (5.12) are

- link mass $m_1 = 1\, kg$, $m_2 = 10\, kg$
- link length $l_1 = 1\, m$, $l_2 = 1\, m$
- initial joint angle $q_1(0) = -2\, (rad)$
  $q_2(0) = -2\, (rad)$
- initial joint speed $\dot{q}_1(0) = 0\, (rad / s)$
  $\dot{q}_2(0) = 0\, (rad / s)$
gravity \[ g = 9.8 \left( \frac{m}{s^2} \right) \]

and the shorthand notations \( c_1 = \cos(q_1), \ c_2 = \cos(q_2), \ s_1 = \sin(q_1), \ s_2 = \sin(q_2) \) are used.

Suppose the parameters \( m_1 \) and \( m_2 \) are perturbed in the following form:

\[ \Delta m_1 = 0.1 \sin t \quad \text{and} \quad \Delta m_2 = \sin t \]

The disturbances \( u_{d1} \) and \( u_{d2} \) are square waves with periods of \( 2\pi \).

\[ u_{d1} = u_{d2} = \begin{cases} 2 & 0 \leq t < \pi \\ 0 & \pi \leq t < 2\pi \end{cases} \]

Adopting the above parameters, the equation of motion (5.12) for two-link rigid robotic manipulator with parameter perturbations, frictional forces and bounded disturbances will have the form

\[
\begin{bmatrix}
11 + 1.1 \sin(t) & (10 + \sin(t))(s_1 s_2 + c_1 c_2) \\
(10 + \sin(t))(s_1 s_2 + c_1 c_2) & 10 + \sin(t)
\end{bmatrix}
\begin{bmatrix}
\ddot{q}_1 \\
\ddot{q}_2
\end{bmatrix}
+ (10 + \sin(t))(c_1 s_2 - s_1 c_2)
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2
\end{bmatrix}
\begin{bmatrix}
0.1 \dot{q}_1 \\
0.1 \dot{q}_2
\end{bmatrix}
+ \begin{bmatrix}
-(11 + 1.1 \sin(t)) gs_1 \\
-(10 + \sin(t)) gs_2
\end{bmatrix}
- \begin{bmatrix}
u_{d1} \\
u_{d2}
\end{bmatrix}
= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\]

Equation (5.13) can be simplified to have the same form as Equation (5.7), that is,

\[
m_{11}(q_{1,i} - 2q_{1,i-1} + q_{1,i-2}) + m_{12}(q_{2,i} - 2q_{2,i-1} + q_{2,i-2}) = T^2 u_{1,i}
\]

\[
m_{21}(q_{1,i} - 2q_{1,i-1} + q_{1,i-2}) + m_{22}(q_{2,i} - 2q_{2,i-1} + q_{2,i-2}) = T^2 u_{2,i}
\]
where

\[ m_{11} = 11 + 1.1 \sin(t), \quad m_{12} = (10 + \sin(t))(s_1 s_2 + c_1 c_2) \, , \]

\[ m_{21} = (10 + \sin(t))(s_1 s_2 + c_1 c_2), \quad m_{22} = 10 + \sin(t) \]

Therefore, the two-link robotic manipulator with twelve unknown parameters can be simplified and described by only four parameters. Furthermore, the uncertain parameters—such as friction forces and bounded disturbances—can be omitted, which makes it easy to design and control the robotic manipulator.

From Equation (5.14), the control signals can be determined from the following equations:

\[
\begin{align*}
    u_{1,i} &= m_{11} \ddot{q}_{1,i} + m_{12} \ddot{q}_{2,i} \\
    u_{2,i} &= m_{21} \ddot{q}_{1,i} + m_{22} \ddot{q}_{2,i}
\end{align*}
\]  

(5.15-1)

(5.15-2)

If the sampling time \( T \) is 0.01 second, the inputs are

\[ r_1(t) = r_2(t) = 2 \cos(t) + 1 \]

and the controller equations are (5.15). The simulation results for joint angles \( q_1 \) and \( q_2 \) are shown in Figures 5.11 and 5.12. The simulation results for the joint angles \( q_1, q_2 \) with the method of Cheng, Y. C. and Chen, B. S., (1997) are shown in Figures 5.13 and 5.14. It is clear that the response of the new algorithm is better than that of Cheng, Y. C. and Chen, B. S., (1997), because the actual outputs and the desired outputs are difficult to distinguish from the simulation diagrams (Figures 5.11 and 5.12). If the simulation results for angles \( q_1, q_2 \) are zoomed (Figures 5.15 and 5.16), it can be seen that the settling times are less than 0.1 second. Besides, the new algorithm is simpler than that of Cheng,
Y. C. and Chen, B. S., (1997), in which complicated neural networks and adaptive control are used.

Figure 5.11 Response of joint angle $q_1$ by the new algorithm

Figure 5.12 Response of joint angle $q_2$ by the new algorithm
Figure 5.13 Response of joint angle $q_1$ with the method of Cheng, Y. C. and Chen, B. S., (1997)

Figure 5.14 Response of joint angle $q_2$ with the method of Cheng, Y. C. and Chen, B. S., (1997)
Figure 5.15 Zoomed diagram of the response of joint angle $q_1$ by the new algorithm

Figure 5.16 Zoomed diagram of the response of joint angle $q_2$ by the new algorithm
The response of joint angle speeds $\dot{q}_1, \dot{q}_2$ by the new algorithm are shown in Figures 17 and 18. The angle speeds are large due to high initial response speed. However, these speeds can be decreased by choosing a small value $\alpha$.

Figure 5.17 Response of angle speed $\dot{q}_1$ by the new algorithm

Figure 5.18 Response of angle speed $\dot{q}_2$ by the new algorithm
In conclusion, according to the simulations, it is obvious that the manipulator controlled by the new algorithm has better performance.

5.3.3 Example 3

In this example, we will consider the case when the payload of the controlled robot is changed during the control process. The two-link manipulator is described by the following equation (Ciliz, M. K. and Narendra, K. S., 1996):

\[
\begin{bmatrix}
  m_{11} & m_{12} \\
  m_{21} & m_{22}
\end{bmatrix}
\begin{bmatrix}
  \ddot{q}_1 \\
  \ddot{q}_2
\end{bmatrix}
+ \begin{bmatrix}
  h \dot{q}_2 & h(\dot{q}_1 + \dot{q}_2) \\
  -h \dot{q}_1 & 0
\end{bmatrix}
\begin{bmatrix}
  \dot{q}_1 \\
  \dot{q}_2
\end{bmatrix}
= \begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}
\]  

(5.16)

where \( h = -m_2 l_1 l_2 \sin(q_2) \)

\[
m_{11} = m_1 l_{c1}^2 + I_1 + m_2 (l_1^2 + l_2^2 + 2l_1 l_2 \cos(q_2)) + I_2
\]

\[
m_{12} = m_2 l_1 l_2 \cos(q_2) + m_2 l_2^2 + I_2
\]

\[
m_{21} = m_{12}
\]

\[
m_{22} = m_2 l_2^2 + I_2
\]

link mass \( m_1 = 0.7 \text{ kg}, m_2 = 0.5 \text{ kg} \)

link length \( l_1 = 0.3 \text{ m}, l_2 = 0.3 \text{ m} \)

the center of mass for link 1 \( l_{c1} = 0.15 \text{ m} \)

the center of mass for link 2 \( l_{c2} = 0.15 \text{ m} \)

the moment of inertia for link 1 \( I_1 = 0.14 \text{ kg m}^2 \)

the moment of inertia for link 2 \( I_2 = 0.1 \text{ kg m}^2 \)
initial joint angle $q_1(0) = -2 (rad)$

$q_2(0) = -2 (rad)$

initial joint speed $\dot{q}_1(0) = 0 (rad / s)$

$\dot{q}_2(0) = 0 (rad / s)$

gravity $g = 9.8 (m / s^2)$

and $u_1, u_2$ are control torque applied to links 1 and 2. Following the procedure discussed in Section 5.2, Equation (16) can be simplified to have the form of Equation (5.7).

$$m_1(q_{1,i} - 2q_{1,i-1} + q_{2,i-1}) + m_2(q_{2,i} - 2q_{2,i-1} + q_{2,i-2}) = T^2u_{1,i}, \quad (5.17-1)$$

$$m_2(q_{2,i} - 2q_{1,i-1} + q_{2,i-2}) + m_2(q_{2,i} - 2q_{2,i-1} + q_{2,i-2}) = T^2u_{2,i}, \quad (5.17-2)$$

Therefore, the two-link robotic manipulator with eight unknown parameters can be simplified and described by only four parameters, and most of the uncertain parameters--such as friction forces and bounded disturbances--can be omitted, which makes it easy to design and control the robot manipulator.

From Equation (5.17), the control signals can be determined by the following equations:

$$u_{1,i} = m_1\ddot{q}_{1,i} + m_{12}\ddot{q}_{2,i}, \quad (5.18-1)$$

$$u_{2,i} = m_{21}\ddot{q}_{1,i} + m_{22}\ddot{q}_{2,i}, \quad (5.18-2)$$

where

$$\ddot{q}_{1,i} = (q_{1,i} - 2q_{1,i-1} + q_{1,i-2}) / T^2$$

$$\ddot{q}_{2,i} = (q_{2,i} - 2q_{2,i-1} + q_{2,i-2}) / T^2$$
The mass $m_2$ of link 2 is assumed to change when the manipulator picks a load with a different mass. Hence, the parameters $m_i$ ($i=1,2; j=1,2$) will change too. When the robot is unloaded, the minimum mass of link 2 is equal to 0.5 kg; when the robot picks a maximum load 1.5 kg, the maximum mass of link 2 is equal to 2 kg. The average value of the maximum mass and minimum mass of link 2 is chosen as the corresponding parameter of the controller: i.e., $m_2=(2+0.5)/2=1.25$ kg.

If the inputs are

$$r_1(t) = \frac{\pi}{4} + 2(1 - \cos(3t))$$

$$r_2(t) = \frac{\pi}{6} + (1 - \cos(5t)),$$

when the manipulator is unloaded, i.e., $m_2=0.5$ kg, the simulation results of the tracking errors and responses of joint angles $q_1$ and $q_2$ are shown in Figures 5.19 and 5.20.

![Figure 5.19](image.png)

Figure 5.19 The tracking errors $e_1, e_2$ of joint angles $q_1, q_2$ with the new algorithm when the robot is unloaded
When the manipulator is loaded, i.e., $m_2=2\, kg$, and the corresponding parameter of the controller is still 1.25 $kg$, the simulation results of the tracking errors and responses of joint angles $q_1$ and $q_2$ are shown in Figures 5.21 and 5.22. As it is seen, the simulation results are very good.

Figure 5.20 The responses of joint angles $q_1$, $q_2$ with the new algorithm when the robot is unloaded

Figure 5.21 The tracking errors $e_1$, $e_2$ of joint angles $q_1$, $q_2$ with the new algorithm when the robot is loaded
Next, let us consider the case when the payload will be changed during the control process. At the beginning, if the robot is unloaded, i.e., $m_2=0.5$ kg and at $t=1$ second, it picks up a 1.5 kg load, i.e., $m_2=2$ kg; then at 2 seconds, it is unloaded again. The simulation results of the tracking errors and responses of joint angles $q_1$ and $q_2$ during the process is shown in Figure 5.23. It can be seen that when the load is changed, the tracking errors of the joint angles $q_1$ and $q_2$ remain almost the same.

In Ciliz, M. K. and Narendra, K. S., (1996), a similar control process is studied. That is, at the beginning, the robot is unloaded, i.e., $m_2=0.5$ kg, and at 4 seconds, it picks up a 1.5 kg load, i.e., $m_2=2$ kg. The simulation result of the tracking errors of the joint angle $q_1$ with the method in Ciliz, M. K. and Narendra, K. S., (1996) is shown in Figure 5.24. It is obvious that when the load is changed, the tracking errors are also increased. Besides, a complicated adaptive control method using multiple models and switching is utilized. Therefore, the new algorithm is simpler, and the robot controlled by the new algorithm shows better performance.
Figure 5.23 The tracking errors of joint angles $q_1$, $q_2$ with the new algorithm when the load is changed.

Figure 5.24 The tracking errors of joint angles $q_1$ with the algorithm of Ciliz, M. K. and Narendra, K. S., (1996), when the load is changed.
5.4 Summary

This chapter presents a new algorithm for control of robot manipulators described as an \( n \) input \( n \) output nonlinear system. By discretizing the control time, the differential equations of the controlled robot can be expressed by difference equations. If Equation (5.6) is satisfied, for an \( n \) input \( n \) output nonlinear system with \( (2n+1)n \) unknown parameters, it can be simplified with only \( n^2 \) unknown parameters, and all the uncertain parameters--such as frictional forces and disturbances--can be neglected. Therefore, the design and control procedure can be simplified greatly, and the control precision can be increased.

The simplified model is used for control of robot manipulators with parameter perturbations, friction forces, bounded disturbances or varying payload. Simulations show that the robot manipulators controlled by the new algorithm have a good performance. Moreover, the control performances of simplified models and the exact model are compared. With computer simulations it is shown that the algorithm is practical and simple, and the simplified model performed as well as that of the exact model.

Furthermore, the simulation results of the new algorithm are compared with those of digital high-gain PD control method (Wu, S. T., 1997), neural networks and adaptive control method (Cheng, Y. C. and Chen, B. S., 1997) and adaptive control method using multiple models and switching (Ciliz, M. K. and Narendra, K. S., 1996). It is shown that the new algorithm is simpler, and the robot manipulator controlled by the new algorithm has a better transient and steady-state response.
Chapter 6

Conclusions

6.1 Conclusions

This thesis presents a new algorithm for the design and control of SISO linear systems, MIMO coupled linear systems, SISO nonlinear systems and MIMO nonlinear systems. It is a combination of the finite difference method and digital predictive control method. However, it is different from conventional methods in that it discretizes the whole control time into many small steps or intervals. The desired whole process response may be achieved by designing and controlling the plant in a small interval.

The novel ideas of the new algorithm are as follows. First, instead of developing the accurate plant model, the simplified model with fewer unknown parameters is derived, and the simplified model is used as the controller equation. Second, the control signal is based on the desired output, not on the error signal. Third, the system is analyzed and solved as the boundary value problem, rather than the initial value problem. If some conditions are satisfied, the equation of the controlled plant can be simplified greatly, which makes it easy to design and control the plant.

Furthermore, a number of simulation examples solved illustrate that the algorithm is practical and simple and has a good transient and steady-state response.
The following conclusions are reached:

1. For a SISO \( n \)th-order linear system, the model of the controlled plant can be simplified to have only one or two unknown parameters and be described by a difference equation. When the parameters of the controlled plant have changed ten times of their initial values, or when the order of the plant has changed, if the plant is controlled by the conventional method, it fails. However, if the plant is controlled by the new algorithm, it can still be controlled with good steady- and transient-state response and with good performance and stability robustness.

2. For partly coupled MIMO linear systems with \( j \) inputs, \( j \) outputs and with \((2j+1)j\) unknown parameters, the model of the controlled plant can be simplified to a difference equation with only \( j \) unknown parameters. Besides, when the model of controlled plant and the controller differ considerably--and even when the plant is unstable--it can still be controlled with good steady and transient properties.

3. For SISO nonlinear systems, the \( n \)th-order nonlinear equation of the controlled plant can be easily linearized and simplified to the difference equation with only one or two unknown parameters. Furthermore, when the model of the controlled plant and the controller differ considerably, the plant can still be controlled successfully.

4. For MIMO systems of robot manipulators, an \( n \) input \( n \) output nonlinear system with \((2n+1)n\) unknown parameters can be simplified to have only \( n^2 \) unknown parameters, and all the uncertain parameters--such as frictional forces and disturbances--can be neglected. Therefore, the design and control procedure can be simplified greatly, and the control precision can be increased.
The simplified model is used for the control of robot manipulators with parameter perturbations, friction forces, bounded disturbances and varying payload. Simulations show that the robot manipulators controlled by the new algorithm have a good performance. Besides, the control performances of simplified models and the exact model are compared. Computer simulations have verified that the algorithm is practical and simple, and the simplified model performs virtually as well as the exact model.

Last, the simulation results of the new algorithm are compared with that of digital high-gain PD control method [26], adaptive control using multiple models and switching method [4] and nonlinear adaptive tracking $H_\infty$ control design via neural networks method [3]. It is shown that the new algorithm is simple, and the robot manipulator controlled by the new algorithm has a better transient and steady response.

The new algorithm can be used in single input single output, multiple input multiple output systems (both linear and nonlinear) and time-varying systems. The main mathematical tool is the difference equation, and it is analyzed in time domain.

The disadvantage of the new algorithm is that the sampling frequency must be high, which requires fast data processing and increases the cost of the hardware of the control system--such as the computer or an A/D and D/A converter. However, with the development of computer technology, the main frequency of a small computer has reached a hundred million times per second, and the cost of computers have decreased considerably. Hence, it is not a serious problem.
6.2 Future Recommendations

This thesis has proposed a simple algorithm for designing control systems. Using computer simulations, the results of theoretical analysis have been verified. However, the stability analysis, which is very important for control systems, is not presented in the thesis. The system stability is verified only by simulations. The theoretical analysis of the stability is a task for further research.

Second, the stability, transient and steady responses are heavily dependent on the parameter \( \alpha \) in the desired output equation. How to carefully monitor and dynamically adjust the parameter \( \alpha \) is another topic worthy of investigation.

Third, because of the presence of noises in the real time control, before the algorithm can be used in the control of industrial processes, experiments are needed to verify the results of theoretical analysis and simulations.

Last, further research is needed for controlling the flexible structure systems and the nonlinear systems in which time delay, backlash, relay and dead zone nonlinearities are contained.
Bibliography


