A Unified Approach to Orthogonally Multiplexed Communication
Using Wavelet Bases and Digital Filter Banks

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Chapter 1

Introduction

1.1 General Background

Orthogonally multiplexed communication is a modulation format that places independent quadrature amplitude modulated (QAM) symbols on orthogonal signals [1]-[5]. These orthogonal signals are typically subcarriers with properly spaced center frequencies. Hence this signaling technique tends to be referred to as multicarrier modulation (MCM) or, less frequently, multitone modulation [1]. A square root raised cosine pulse shape is typically used for the data. With MCM, the data bandwidth is divided into uniform bands. This allows variable coding and equalization strategies in each band tailored to the channel distortions within the particular band [1][5]. Indeed, this modulation has yielded superior performance on distorted channels as compared to single rate systems [1]. In addition, this was achieved with reduced complexity using an efficient fast Fourier transform (FFT) based implementation [2] and has lead to a prototype development [3]. Applications of this technique have primarily been for bandwidth efficiency on telephone grade channels. A very recent paper appeared applying MCM to spread spectrum [4]. A direct sequence spread spectrum signal was
mixed with multiple subcarriers in an analog fashion. The chip pulse employed was non
return to zero (NRZ).

Orthonormal wavelets are a recent advance in the representation of finite energy
signals in $L^2(R)$ [6]-[10]. This contrasts the traditional Fourier expansion which can only
represent signals in $L^2(0,2\pi)$ with complex exponentials as basis functions [8]. The basis
functions in the wavelet expansion are obtained by scaling and translating a single
function called the wavelet function. The wavelet function is necessarily bandpass. The
scaling feature represents the signal of interest as a superposition of long-duration signals
with narrow bandwidth and short-duration signals with wide bandwidth. This also
contrasts a Fourier-based approach which is restricted by the uncertainty principle to
achieving resolution in time or frequency but not both domains. Consequently, wavelets
have been successfully applied to the analysis and representation of nonstationary
transient signals, in particular, for speech and image compression and edge detection
[6][10][11].

Multiresolution approximations (MRA), introduced by Mallat [6][7], took the
theory of wavelets to a new level of development by producing a framework not only for
understanding wavelets but also for constructing a basis. For dyadic scaling and integer
translations, an MRA is a sequence of embedded subspaces of $L^2(R)$, denoted $V_{2^j}$,
whose elements are approximations, at a given resolution, to signals in $L^2(R)$. Resolu-
tion can be loosely thought of as bandwidth where fine resolution implies wide
bandwidth and coarse resolution implies narrow bandwidth. The detail lost when
changing from an approximation of fine resolution to a coarse resolution forms the
orthogonal complement of the coarse resolution subspace in the finer resolution subspace.
That is, let $W_{2^j}$, the space of detail signals at a resolution $2^j$, be the orthogonal
complement of $V_{2^j}$ in $V_{2^{j+1}}$. It was the detail signals which were of most interest in the
early image processing applications of wavelets. Mallat then showed that scaling a
function by $2^j$ and taking all integer shifts constituted an orthonormal basis for $V_{2^j}$. This function was called the scaling function and is inherently lowpass. Further, scaling the wavelet function by $2^j$ and taking its integer shifts formed an orthonormal basis for $W_{2^j}$. Finally, taking all dyadic scales and integer shifts of the wavelet function produced an orthonormal basis for $L^2(R)$.

In the development of MRAs, a recursive algorithm surfaced for computing the series expansion coefficients. Namely, filtering the expansion coefficients of an approximation at a particular resolution with digital lowpass and highpass filters and decimating their outputs produced the coefficients of the approximation and detail signals, respectively, at a coarser resolution [6]. The recursion continued by operating on the decimated lowpass output. The algorithm came to be known as the discrete wavelet transform (DWT). The inverse was readily obtained using expanders and related digital lowpass and highpass filters [6]. This not only made wavelet processing practical but produced a fundamental connection to a major branch of signal processing called multirate systems [16][17][20]. Namely, the DWT was seen as a nonuniform analysis filter bank and the IDWT was the corresponding nonuniform synthesis filter bank. The nonuniformity refers to the nonuniform channel partitioning of the signal bandwidth by the filter bank. In the DWT case, the center frequencies in the channels follow an octave spacing. Of interest in the theory of filter banks is the notion of perfect reconstruction [20]. That is, if a digital signal is applied to an analysis bank which creates aliasing, amplitude and phase distortions, does there exist a synthesis bank which will cancel these distortions and produce the original signal to within possibly a fixed integer delay. This was indeed possible and it was found that the same condition for perfect reconstruction, the so-called power complementary property [20], along with an admissibility condition, were necessary and sufficient conditions for the orthonormality of the MRA basis[6][19].
This allowed certain filter design procedures from filter bank theory to be used in wavelet applications.

Research in filter banks has also produced M-channel uniform banks where the bandwidth in each channel is equal. Since some applications require the use of uniform banks, wavelet theory has been extended to include these conditions. The wavelets associated with these banks are called M-band wavelets [19] which are a special case of wavelet packets [9]. M-band wavelets are sometimes referred to as multiplicity-M wavelets [19]. In contrast to the single wavelet corresponding to the scaling function in the dyadic case, for M-band wavelets there are M-1 wavelet functions.

1.2 Contributions of this Research

In this research, we propose to use orthonormal wavelet basis functions as the orthogonal signals on which to place the QAM data symbols. Specifically, the purpose of this research is the development of a general, unified approach to orthogonally multiplexed communications using wavelet bases and of the implementation of these systems using perfect reconstruction digital filter banks. We then extend these waveforms to spread spectrum applications for the effective mitigation of additive interference. Contributions of this research are:

1) Fundamental development of multiscale modulation (MSM) as a new method to achieve a nonuniform partitioning of the data bandwidth in orthogonally multiplexed communications.

2) Fundamental development of M-band wavelet modulation (MWM) as a general approach to orthogonally multiplexed communications requiring a uniform partitioning of the data bandwidth.

3) Development of efficient digital implementations for MSM and MWM using nonuniform and uniform digital filter banks.

4) Derivation of a new even-order 2-channel FIR filter bank for use in tree-
structured filter banks requiring linear phase.

5) Derivation and solution of the delay equalization problem which occurs when a
tree-structured FIR analysis/synthesis filter bank is converted to a
synthesis/analysis filter bank as required in the communication application.

6) Established connection between Meyer basis and square root Nyquist pulse
design resulting in the derivation of square root raised cosine wavelet; proof
that the Meyer scaling function is a valid M-band scaling function; derivation
of a bound on the rolloff parameter as a function of multiplicity M; and a linear
phase FIR filter design for new even order filter bank.

7) Development of a unifying framework for spread spectrum communication
using wavelet basis functions.

8) Development of an optimum filter for use in multichannel spread spectrum
receivers based on a maximum signal-to-noise ratio criteria.

9) Derivation of theoretical performance in additive interference channels
including a comparative study between optimum, uniform and excision filter
weights.

1.3 Outline of this Dissertation

Chapter 2. This chapter presents the fundamental theory of multiscale
modulation (MSM). This development includes the derivation of the waveform, tiling
diagram, power spectral density, bandwidth efficiency and performance in additive white
Gaussian noise. Additionally, a digital filter bank implementation is introduced which
makes use of the IDWT and DWT.

Chapter 3. This chapter presents the fundamental theory of M-band wavelet
modulation. This development includes the derivation of the waveform, tiling diagram,
power spectral density, bandwidth efficiency and performance in additive white Gaussian
noise. Additionally, a digital filter bank implementation is introduced which makes use of uniform synthesis and analysis filter banks.

Chapter 4. This chapter presents a detailed development of efficient implementations of the new modulation formats based on digital filter banks. A number of new results of general interest to the signal processing community are obtained. In particular, a new even order FIR filter bank is presented. Also, a number of practical issues resulting from the use of nonzero group delay FIR implementations are identified and solved.

Chapter 5. This chapter discusses approaches to pulse shape and filter design. As a result important connections are made between MRA bases and classical communication waveforms. In particular, the relation between the Meyer basis and the square root Nyquist pulse is determined. This allows us to derive a number of generalizations and new results concerning the Meyer basis. Additionally, we use the Meyer scaling function to design linear phase FIR filters for use in the new filter bank structure developed in Chapter 4.

Chapter 6. This chapter presents a unified framework for spread spectrum signal design. In particular, dimensionality in time is obtained with classical direct sequence using a scaling function chip pulse, dimensionality in time-frequency is obtained with the new multiscale modulation and dimensionality in frequency is obtained with M-band wavelet modulation. An optimum filter structure is developed for the effective mitigation of additive interference. A comparative study is then conducted which evaluates the optimum, uniform and excision filter weights with respect to the three spread spectrum waveforms under different interference conditions.

Chapter 7. This chapter provides concluding remarks in regard to the research performed in this dissertation. In addition, a number of future research areas are outlined.
Chapter 2

Multiscale Modulation

2.1 Introduction

The purpose of this chapter is to develop the fundamental theory of orthogonally multiplexed communication using orthonormal dyadic wavelet basis functions as the orthogonal signals on which the QAM sequences are placed. This development includes the derivation of the waveform, power spectral density, tiling diagram, bandwidth efficiency and performance in additive white Gaussian noise. As will be seen, dyadic wavelets provide a nonuniform decomposition of the time-frequency plane. This allows for greater effectiveness against channel impairments. We will exploit this feature in Chapter 6 where the waveform will be applied to spread spectrum applications. In addition, efficient digital filter banks exist for the implementation of these systems which we introduce in this chapter and study in detail in Chapter 4.

Most of the work in applying wavelets to communications has been for orthogonal communications [27]. This bandwidth inefficient format is fundamentally different from the formats being developed in this dissertation. An approach recently put forth using dyadic wavelets in an orthogonally multiplexed manner is fractal modulation.
MSM, the format proposed in this dissertation for dyadic wavelets, has this format as a special case. In addition, the general development presented here, in particular the concept of a super-symbol, avoids the practical problems encountered with fractal modulation. Namely, fractal modulation suffers a significant buffering problem and as a result can only consider short finite sequences. Also treated as a special case is quadrature-quadrature phase shift keying (Q^2-PSK) modulation [15]. Wavelet functions exist which are superior in bandwidth containment to the sinusoidal MSK pulses originally proposed for Q^2-PSK. Indeed, in Chapter 5, we will find that the familiar square root raised cosine Nyquist pulse gives rise to valid wavelet functions. Additionally, the digital filter bank implementation is seen to be superior to the analog approach of conventional Q^2-PSK.

We begin this chapter with a review of essential results needed for the understanding and development of Multiscale Modulation with particular emphasis on multiresolution approximations (MRA). The MRA framework for constructing orthonormal wavelet bases is largely due to Mallat. Proofs of major results can be found in references [6]-[8]. Notation will follow the development of Mallat.

### 2.2 Essential Results On Orthonormal Dyadic Wavelets

**Definition 2.1** An MRA is a sequence of closed subspaces of $L^2(\mathbb{R})$, denoted $V_{2^j}$, satisfying

\begin{align}
1. \forall j \in \mathbb{Z}, \quad V_{2^j} \subset V_{2^{j+1}} \\
2. \forall j \in \mathbb{Z}, \quad f(x) \in V_{2^j} \iff f(2x) \in V_{2^{j+1}} \\
3. \bigcup_{j \in \mathbb{Z}} V_{2^j} \text{ is dense in } L^2(\mathbb{R}) \\
4. \bigcap_{j \in \mathbb{Z}} V_{2^j} = \{\Theta\} \\
5. \exists g \in V_1 \text{ s.t. } g(x-n), \forall n \in \mathbb{Z}, \text{ is a Riesz basis for } V_1
\end{align}

where:

$\mathbb{Z}$ is the set of integers
R is the set of real numbers

$L^2(R)$ is the vector space of finite energy signals defined on $R$

$\Theta$ is the zero vector in $L^2(R)$

A Riesz basis is also referred to as an unconditional basis [8].

**Theorem 2.2** Let $(V_{2^j})_{j \in \mathbb{Z}}$ be an MRA, then $\exists! \phi \in V_1$ s.t. $\sqrt{2^j} \phi(2^j x - n), \forall n \in \mathbb{Z}$, is an orthonormal basis for $V_{2^j}$. $\phi$ is called the scaling function.

Since $\phi \in V_1 \subset V_2$, the scaling function can be written as a series expansion in the basis functions for $V_2$, namely,

$$\phi(x) = \sqrt{2} \sum_{n \in I} h_n \phi(2x - n) \tag{2.2}$$

where $I$ is the index set and taken to be $Z$ unless otherwise noted. From (2.2), $h_n$ is seen to be the inner product of $\phi(x)$ with $\sqrt{2} \phi(2x - n)$. This sequence plays an important role in the discrete wavelet transform (DWT). The Fourier transform of (2.2) is

$$\Phi(f) = \frac{1}{\sqrt{2}} H\left(\frac{f}{2}\right) \Phi\left(\frac{f}{2}\right) \tag{2.3}$$

where $H$ is the discrete-time Fourier transform [37] of $h_n$ and must satisfy $H(0) = \sqrt{2}$. This constraint on $H$ is called the admissibility condition.

**Definition 2.3** Let $W_{2^j}$ be the orthogonal complement of $V_{2^j}$ in $V_{2^{j+1}}$, that is, $V_{2^{j+1}} = V_{2^j} \oplus W_{2^j}$ and $V_{2^j} \perp W_{2^j}$.

**Theorem 2.4** Let $(V_{2^j})_{j \in \mathbb{Z}}$ be an MRA with scaling function $\phi$, then $\sqrt{2^j} \psi(2^j x - n), \forall n \in \mathbb{Z}$, is an orthonormal basis for $W_{2^j}$ where $\psi$ satisfies

$$\psi(x) = \sqrt{2} \sum_{n \in I} g_n \phi(2x - n) \tag{2.4}$$
with \( g_n = (-1)^n h_{1-n} \). Moreover, \( \sqrt{2^j} \psi(2^j x - n), \forall n, j \in \mathbb{Z}^2 \), is an orthonormal basis for \( L^2(\mathbb{R}) \). \( \psi \) is called the wavelet function.

**Remark 2.5** From Definition 2.3 and Theorems 2.2 and 2.4, we see that the scaling and wavelet functions are orthogonal at different coarse scales. Additionally, the wavelet function is orthogonal to itself at different scales by Theorem 2.4. But, the scaling function is not orthogonal to itself at different scales due to the embedded vector space property of Definition 2.1.

The expression (2.4) can be viewed as a series expansion in the basis functions for \( V_2 \) since \( \psi \in W_1 \subset V_2 \). Thus, \( g_n \) is the inner product of \( \psi(x) \) with \( \sqrt{2} \phi(2x-n) \). This sequence also plays an important role in the DWT. The Fourier transform of (2.4) is

\[
\Psi(f) = \frac{1}{\sqrt{2}} G\left( \frac{f}{2} \right) \Phi\left( \frac{f}{2} \right)
\]

where \( G \) is the discrete-time Fourier transform of \( g_n \).

**Theorem 2.6** Let \( h_n \) and \( g_n \) be the series coefficients of the scaling and wavelet functions in an MRA then their transforms satisfy

\[
|H(f)|^2 + |G(f)|^2 = 2, \quad \forall f \in \mathbb{R}
\]

The relation in (2.6) is commonly referred to as the power complementary property.

**Theorem 2.7** Let \( h_n \) and \( g_n \) be the series coefficients of the scaling and wavelet functions in an MRA. Let \( f \in L^2(\mathbb{R}) \) then \( \forall j \in \mathbb{Z} \)

\[
a_{2^{jn}}(n) = \sum_{k \in \mathbb{E}} h(n - 2k) a_{2^j}(k) + \sum_{k \in \mathbb{E}} g(n - 2k) d_{2^j}(k)
\]

where
Equation (2.7) defines one stage of the inverse DWT while equations (2.10) and (2.11) define one stage of the DWT.

The signal processing operations associated with equation (2.7) and with equations (2.10) and (2.11) are illustrated in Figures 2.1 and 2.2, respectively. These structures will be studied in greater detail in Chapter 4.

\[
a_{2^j}(n) = \sqrt{2^j} \int_{-\infty}^{\infty} f(x) \phi(2^j x - n) \, dx \quad (2.8)
\]

and

\[
d_{2^j}(n) = \sqrt{2^j} \int_{-\infty}^{\infty} f(x) \psi(2^j x - n) \, dx \quad (2.9)
\]

Moreover,

\[
a_{2^j}(n) = \sum_{k \in \mathbb{I}} h(k-2n)a_{2^{j+1}}(k) \quad (2.10)
\]

and

\[
d_{2^j}(n) = \sum_{k \in \mathbb{I}} g(k-2n)a_{2^{j+1}}(k) \quad (2.11)
\]
2.3 Fundamental Theory Of Multiscale Modulation (MSM)

2.3.1 Waveform Development

In general [12], a quadrature amplitude modulation (QAM) digital signal is defined as

\[ m(t) = \sqrt{E} \sum_{k \in \mathbb{C}} d_k \phi \left( \frac{t}{T} - k \right) \]  

(2.12)

where: \( \phi \) is the pulse shape

\( E \) is the average symbol energy

\( T \) is the symbol duration

\( d_k \) are complex-valued QAM symbols

The data symbols are assumed to be identically distributed, zero mean and unit variance. Let \( \phi \in V_1 \) be a scaling function in an MRA, then the signal defined by (2.12) is free of intersymbol interference (ISI) by Theorem 2.2 and the orthonormality of \( \phi \). Hence, MRAs provide a broad new class of pulse shapes for use in conventional data communications. Assuming the data symbols are independent, the power spectral density of this modulation format is [13]
\[ S_{QAM}(f) = E|\Phi(f \cdot T)|^2 \]  

(2.13)

where: \( \Phi(t) \overset{F.T.}{\longrightarrow} \Phi(f) \)

By Definitions 2.1 and 2.3, the \( V_1 \) vector space can be decomposed into the direct sum

\[ V_1 = V_{2^{-\nu \cdot 0}} \oplus W_{2^{-\nu \cdot 0}} \oplus \cdots \oplus W_{2^{-1}} \]  

(2.14)

where \( J \) is a positive integer. It follows that \( \phi \) can be expanded in the basis functions for the subspaces on the right hand side of (2.14). We now rewrite (2.12) as the multidimensional signal

\[ m(t) = \sqrt{2^{-(J-1)}} \frac{E}{T} \sum_{j \in I} a_j^0 \phi \left( 2^{-(J-1)} \frac{T}{T} - j \right) + \sum_{n=1}^{J-1} \sqrt{2^{-(J-n)}} \frac{E}{T} \sum_{j \in I} a_j^n \psi \left( 2^{-(J-n)} \frac{T}{T} - j \right) \]

(2.15)

where: \( a_j^n \) are complex-valued QAM symbols for \( n = 0, \ldots, J-1 \)

Since data has been placed at different scales, (2.15) will be referred to as multiscale modulation (MSM). By Remark 2.5, orthonormality of the individual basis functions prevents ISI. Again by Remark 2.5, their mutual orthogonality prevents interference across scales. As special cases, we note (2.12) is obtained for \( J = 1 \), a general form of quadrature-quadrature phase shift keying (Q2PSK) modulation is obtained for \( J = 2 \) and fractal modulation is obtained when \( a_j^0 = 0 \) and \( a_j^n = a_j^m \) for \( n \neq m \). In definition (2.15), the parameter \( T \) represents the equivalent single full rate symbol period. In as much, the symbol period on the scaling pulse is \( 2^{J-1} T \) while the symbol period on the \( n \)th wavelet pulse is \( 2^{J-n} T \). For the particular case of Q2PSK, \( T/2 \) corresponds to the bit period.
2.3.2 Power Spectral Density

To derive the power spectral density of MSM, we need the following lemma.

**Lemma 2.8** Let $\phi$ and $\psi$ be scaling and wavelet functions in an MRA, then their Fourier transforms satisfy

$$\left| \Phi \left( \frac{f}{2} \right) \right|^2 = |\Phi(f)|^2 + |\Psi(f)|^2, \quad \forall f \in \mathbb{R} \quad (2.16)$$

Proof: Using (2.3) and (2.5), we have

$$|\Phi(f)|^2 + |\Psi(f)|^2 = \frac{1}{2} \left| H \left( \frac{f}{2} \right) \right|^2 \left| \Phi \left( \frac{f}{2} \right) \right|^2 + \frac{1}{2} \left| G \left( \frac{f}{2} \right) \right|^2 \left| \Phi \left( \frac{f}{2} \right) \right|^2 \quad (2.17)$$

Factoring out the term $\left| \Phi \left( \frac{f}{2} \right) \right|^2$ yields

$$|\Phi(f)|^2 + |\Psi(f)|^2 = \frac{1}{2} \left( \left| H \left( \frac{f}{2} \right) \right|^2 + \left| G \left( \frac{f}{2} \right) \right|^2 \right) \left| \Phi \left( \frac{f}{2} \right) \right|^2 \quad (2.18)$$

Applying Theorem 2.6 to (2.18) proves the lemma.

Now, assuming the symbols at different scales are independent and identically distributed with zero mean and unit variance, the power spectral density for MSM is given by

$$S_{MSM}(f) = E \left| \Phi \left( fT2^{J-1} \right) \right|^2 + E \sum_{n=1}^{J-1} \left| \Psi \left( fT2^{J-n} \right) \right|^2 \quad (2.19)$$

To simplify (2.19), apply Lemma 2.8 to the term involving the scaling function and the $n=1$ term of the series. This produces a twice rate term involving the scaling function.
We can repeat the application of Lemma 2.8 with this new term and the $n=2$ term of the series. Since the series is finite, we can continually repeat application of Lemma 2.8 yielding

$$S_{\text{MSM}}(f) = E[\Phi(fT)]^2$$

(2.20)

which is identical to the power spectral density of QAM as given by (2.13). Consequently, this waveform can be used on existing channels which have been designed around conventional QAM signals. This allows interoperability with systems where redesign would not be cost effective as in the case of existing satellite systems. We also note that (2.20) is only dependent on the scaling function and not on the wavelet function.

2.3.3 Tiling Diagram

We see from the power spectral density calculation, in particular (2.19) and (2.20), that dimensionality has been obtained by decomposing the time-frequency plane. A useful mechanism for visualizing this decomposition is the tiling diagram [10]. The tiling diagram for MSM is illustrated in Figure 2.3. The tiling diagram indicates the concentration of energy in the time-frequency plane of each symbol. Since the uncertainty principle prevents simultaneous limitation in time and frequency, the indicated concentration is in a mean square sense [32]. It is evident that the modulation format is composed of long, low frequency pulses as well as short, high frequency pulses. Hence, the decomposition is nonuniform in time and frequency. In Chapter 3, we will introduce a waveform based on M-band wavelets which achieves a uniform decomposition. The ability to decompose the time-frequency plane enables time and frequency selective processing for greater effectiveness against channel impairments. We will consider additive interference channels in Chapter 6. In designing for these applications, the notion of a super-symbol is particularly noteworthy. The duration of a
super-symbol is the symbol time of the sequence having the scaling function pulse shape. For example, two super-symbols are shown in Figure 2.3. The number of symbols in a super-symbol is $2^{J-1}$.

2.3.4 Bandwidth Efficiency

To determine the bandwidth efficiency of MSM, let $a_j^n$ be a member of a $2^Kn$ QAM constellation. The bandwidth efficiency, $\rho$, is defined by the ratio of bit rate to required bandwidth with units of bps/Hz. Then using the bandwidth of $\Phi$ indicated in (2.20), the efficiency becomes
\[
\rho_{\text{MSM}} = \frac{2^{-(J-1)} K_0 + \sum_{n=1}^{J-1} 2^{-(J-n)} K_n}{1 + \beta}
\]

(2.21)

where: \( \beta \) is the percent excess bandwidth from Nyquist signaling.

Typically, \( \beta \in [0, 1] \). Simplifying (2.21) yields

\[
\rho_{\text{MSM}} = \frac{2^{-(J-1)} K_0 + \sum_{n=1}^{J-1} 2^{-(J-n)} K_n}{1 + \beta}
\]

(2.22)

When \( K_n = K \) at each scale, the bandwidth efficiency of MSM is the same as a single rate system as defined by (2.12) assuming the same pulse shape, \( \phi \), is used in the comparison, namely, \( \frac{K}{1 + \beta} \) bps/Hz. Thus, theoretical bandwidth efficiency does not increase when employing orthogonal pulse shapes. In practice, however, different multidimensional schemes may be more or less robust to bandlimiting effects which will have a direct impact on achievable efficiencies. For example, consider a four dimensional scheme, namely, MSM with \( J = 2 \). The scaling function is lowpass while the wavelet function is bandpass. The two pulse shapes in Q2PSK also have these general characteristics. Since most distortion in linear bandlimited channels occurs at band edge, the lowpass channel will pass virtually unscathed with most distortion occurring in the bandpass channel. This suggests that the multidimensional system will suffer less from ISI and achieve better efficiency. This has been shown for Q2PSK with practical channel filtering[15].

### 2.3.5 Performance In Additive White Gaussian Noise

The optimum receiver in AWGN for MSM consists of a bank of filters matched to each pulse shape in (2.15) [12]. Effectively, the receiver is simply projecting the received signal onto the basis functions defining MSM. Taking the inner product of (2.15) with
each basis function defining a pulse, we see that the average symbol energy in each
dimension is the same. In particular, the symbol energy is $E$. Further, the projection of
AWGN on an orthonormal basis yields uncorrelated zero mean Gaussian random
variables with variance $\frac{N_0}{2}$ [24]. To determine the average symbol error probability in
AWGN, let $P_{sym}^n$ be the symbol error probability in AWGN associated with the $a^n$
($n=0,...,J-1$) sequence at the $n^{th}$ scale having average energy $E$. Again, let $a^n_j$ be a
member of a $2^{K_n}$ QAM constellation. Since there is no ISI or cross channel interference,
the average probability of a symbol error in MSM is

$$
\overline{P}_{sym} = \sum_{n=0}^{J-1} \Pr(a^n)P_{sym}^n
$$

(2.23)

Let $\Pr(a^n)$ be the relative frequency of $a^n$ in a super-symbol, that is,

$$
\Pr(a^n) = \begin{cases} 
2^{-(J-1)}, & n = 0 \\
2^{-(J-n)}, & 1 \leq n \leq J-1 
\end{cases}
$$

(2.24)

Using (2.24) in (2.23) yields

$$
\overline{P}_{sym} = 2^{-(J-1)}P_{sym}^0 + \sum_{n=1}^{J-1} 2^{-(J-n)}P_{sym}^n
$$

(2.25)

which simplifies to $P_{sym}$ when $P_{sym}^n = P_{sym}$ for all $n$. From (2.24), we see that the finer
scale data sequences have the most impact on error probability. This feature may be
exploited in coding applications combating channel distortions, for example, the
bandlimited channel just mentioned.

To evaluate (2.25), we need the symbol error probabilities at each scale. For
arbitrary constellations, these calculations can be difficult. As a result, it is common in
practice to resort to approximations and bounds. For reference, we provide the expressions for two prevalent constellations. Specifically, the symbol error probability for multiple phase shift keying (PSK) is well approximated by [30]

\[
P_{\text{sym}}^n = \begin{cases} 
Q\left(\sqrt{2}\gamma_b\right), & K_n = 1 \\
2Q\left(\sqrt{2}\gamma_b K_n \sin\left(\frac{\pi}{2K_n}\right)\right), & K_n \geq 2
\end{cases}
\]  (2.26)

where \(Q()\) is the Gaussian tail probability, \(\gamma_b\) is \(\frac{E_b}{N_0}\) and \(E_b = \frac{E_c}{K_n}\) is the bit energy. A tight upper bound for rectangular QAM is [12]

\[
P_{\text{sym}}^n \leq 4Q\left(\frac{3}{2K_n - 1}\gamma_b K_n\right)
\]  (2.27)

For arbitrary QAM constellations, reference [31] provides a procedure for computing the symbol error probabilities.

We note finally in this section, that some applications may warrant a different average symbol energy in each channel. In this case, the definition in (2.15) requires only a slight modification. But if this is desired, the power spectral density calculation of Section 2.3.2 and the bandwidth efficiency calculation of Section 2.3.4 must be revisited. For the applications considered in this dissertation, the same constellation is used at each scale, therefore, an equal energy assumption is appropriate.

2.4 Implementation

As noted, the optimum receiver in AWGN for MSM consists of a bank of filters matched to each pulse shape in (2.15). Alternatively, a much more efficient and
Figure 2.4. Transceiver Processing For Multiscale Modulation

equivalent realization makes use of the discrete wavelet transform. The transceiver processing is illustrated in Figure 2.4. Throughout this dissertation, we will assume carrier and symbol synchronization which allows us to consider lowpass equivalent signals [12]. At the transmitter, a data source is demultiplexed into \( J \) symbol streams operating at dyadic sub-rates of the source. An IDWT is computed generating a sequence at the source rate. This sequence is transmitted across the AWGN channel using a pulse shape defined by \( \phi \). The received signal is match filtered and sampled. Taking the DWT of this sequence produces soft decisions of the transmitted symbols which are forwarded to the appropriate data sink possibly for further processing, for example, Viterbi decoding when coding is being employed.

To see in further detail how the transforms are able to replace much of the analog processing, begin with the transmitter and note because of (2.14), (2.15) can be written as

\[
m(t) = \sqrt{\frac{E}{T}} \sum_{k=1}^{T} x_k \phi \left( \frac{t - k}{T} \right)
\]  

(2.28)
where $x_k$ is a complex sequence determined by the data symbols in (2.15). Further note, this sequence will not in general be elements of a traditional QAM constellation so that the signal in (2.28) is subtly different from (2.12). To obtain the relationship between $x_k$ and the data symbols we apply Theorem 2.7, in particular equation (2.7), in a recursive manner similar to the derivation of the power spectral density. Namely, combine the symbols on the scaling function pulse with the symbols on the $n=1$ wavelet pulse in accordance with (2.7). Then apply equation (2.7) again to this new sequence and the symbols on the $n=2$ wavelet function pulse. Since the series is finite, repeating this process will produce $x_k$. The signal processing is illustrated in Figure 2.5 and corresponds to a nonuniform synthesis filter bank.

![Figure 2.5. Detailed Illustration Of The Transmitter IDWT](image)

Now, because of (2.28), a scaling function pulse shape is applied to the $x_k$ sequence for transmission across the channel. This function is illustrated in greater detail in Figure 2.6. Mathematically, the output of the impulse generator is
Figure 2.6 Pulse Generator

\[ x(t) = \frac{1}{\sqrt{T}} \sum_{k \in \ell} x_k \delta \left( \frac{t}{T} - k \right) \]  \hspace{1cm} (2.29)

where \( \delta \) is the Dirac delta function. The pulse generator output can now be written as

\[ m(t) = \frac{1}{\sqrt{T}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{T}} \sum_{k \in \ell} x_k \delta \left( \frac{\tau}{T} - k \right) \frac{1}{\sqrt{T}} \phi \left( \frac{t-\tau}{T} \right) d\tau \]  \hspace{1cm} (2.30)

Note that the filter is scaled by \( \frac{1}{\sqrt{T}} \) to insure orthonormality. Making the variable substitution \( \tau = uT \), (2.30) becomes

\[ m(t) = \frac{1}{\sqrt{T}} \sum_{k \in \ell} x_k \int_{-\infty}^{\infty} \delta(u-k) \phi \left( \frac{t-u}{T} \right) du \]  \hspace{1cm} (2.31)

Applying the distribution property of Dirac delta functions [32], (2.31) reduces to (2.28).

At the receiver, the output of the match filter is given by

\[ r(t) = \frac{1}{\sqrt{T}} \int_{-\infty}^{\infty} \sum_{p \in \ell} x_p \phi \left( \frac{\tau-p}{T} \right) \frac{1}{\sqrt{T}} \phi \left( \frac{t-\tau}{T} \right) d\tau + \tilde{n}(t) \]  \hspace{1cm} (2.32)
where \( \tilde{n}(t) \) is the filtered AWGN. Sampling (2.32) at \( kT \) yields

\[
r_k = \frac{\sqrt{E}}{T} \sum_{p \in \mathbb{I}} x_p \int_{-\infty}^{\infty} \phi\left(\frac{\tau - p}{T}\right) \phi\left(\frac{\tau - k}{T}\right) d\tau + n_k
\]  

(2.33)

where \( n_k \) is the sampled noise sequence. Making the variable substitution \( \tau = uT \) and using the orthonormal property of \( \phi \), (2.33) reduces to

\[
r_k = \sqrt{E} x_k + n_k
\]  

(2.34)

We can now recursively apply equations (2.10) and (2.11) of Theorem 2.7 to the sequence \( r_k \) producing estimates of the data symbol. The signal processing is illustrated in Figure 2.7 and corresponds to a nonuniform analysis filter bank. In the absence of noise, we will recover the transmitted data symbols precisely. It is interesting to note from Figures 2.5 and 2.7 that the digital filters in the DWT are matched to the digital filters in the IDWT. Finally, it is shown in Section 3.4 how the analog pulse shaping and match filters can be replaced with digital processing.

![Figure 2.7. Detailed Illustration Of The Receiver DWT](image)

Figure 2.7. Detailed Illustration Of The Receiver DWT
Chapter 3

M-Band Wavelet Modulation

3.1 Introduction

In Chapter 2, we used dyadic orthonormal wavelets to define a communication waveform which achieves a nonuniform partitioning of the data bandwidth. Since some applications require a uniform partitioning of the bandwidth, for example the mitigation of a stationary narrowband interference, we develop in this chapter a new modulation format which achieves a uniform decomposition based on M-band wavelet basis functions. In as much, this waveform possesses many of the advantages which multicarrier modulation has demonstrated but with the additional advantages attributed to wavelet theory and processing. Thus, the purpose of this chapter is the fundamental theory of orthogonally multiplexed communication using orthonormal M-band wavelet basis functions. As was done in Chapter 2, this development includes the derivation of the waveform, power spectral density, tiling diagram, bandwidth efficiency and performance in additive white Gaussian noise. In addition, an efficient digital filter bank implementation will be introduced with a detailed study being deferred to Chapter 4 so that uniform and nonuniform banks can be treated in a unified manner.
In some ways, M-band wavelets are a natural generalization of dyadic wavelets. Consequently, most of the presentation in this chapter parallels the development in Chapter 2 but with less depth. An important difference exists though in the typical application of M-band wavelets as is the case for this dissertation. Namely, M is usually large (>100) so that one is not interested in multiple resolution decompositions. That is to say that the signal is generally decomposed once into M uniform subbands. In contrast, with dyadic wavelets, the lowpass components are repeatedly decomposed which gives rise to the nonuniform decomposition that was seen in Chapter 2. One significance of this is that many of the deeper mathematical results regarding MRAs are not pertinent for our development in this chapter. With this in mind, we begin with a review of the results which are relevant to the development of M-band Wavelet Modulation. This review is based on references [19] and [29].

3.2 Essential Results On Orthonormal M-band Wavelets

Definition 3.1 For the integer $M>2$ and $\forall j \in \mathbb{Z}$, an M-band wavelet system consists of a sequence of closed subspaces denoted, $V_{M_j}$ and $W_{M_j}^l (l = 1, \ldots, M-1)$, satisfying

1. $V_{M_{j+1}} = V_{M_j} \oplus W_{M_j}^1 \oplus \cdots \oplus W_{M_j}^{M-1}$
2. $V_{M_{j+1}} \perp W_{M_{j+1}}^1 \perp \cdots \perp W_{M_{j+1}}^{M-1}$
3. $\exists \phi \in V_1 \ s.t. \ \phi(x-n), \ \forall n \in \mathbb{Z}$, is an orthonormal basis for $V_1$
4. For $l = 1, \ldots, M-1, \exists \psi_l \in W_1^l \ s.t. \ \psi_l(x-n), \ \forall n \in \mathbb{Z}$, is an orthonormal basis for $W_1^l$

Theorem 3.2 Let $V_{M_j}$ be the sequence of subspaces in an M-band wavelet system as defined in Definition 3.1. Then $\forall j \in \mathbb{Z}$, $\sqrt{M_j} \phi(M_j x - n), \forall n \in \mathbb{Z}$, is an orthonormal basis for $V_{M_j}$. $\phi$ is called an M-band scaling function.

Analogous to (2.2), the M-band scaling function has the following series expansion
\[ \phi(x) = \sqrt{M} \sum_{n \in I} h(n) \phi(Mx - n) \] (3.2)

The Fourier transform of (3.2) is

\[ \Phi(f) = \frac{1}{\sqrt{M}} H \left( \frac{f}{M} \right) \Phi \left( \frac{f}{M} \right) \] (3.3)

where \( H \) is the discrete-time Fourier transform of \( h(n) \) and must satisfy \( H(0) = \sqrt{M} \).

**Theorem 3.3** Let \( W^l_{Mj} (l = 1, \ldots, M-1) \) be the sequence of subspaces in an M-band wavelet system as defined in Definition 3.1. Then \( \forall j \in \mathbb{Z}, \sqrt{M^j} \psi_l(M^j x - n), \forall n \in \mathbb{Z}, \) is an orthonormal basis for \( W^l_{Mj} \). \( \psi_l \) is called the \( l \)th M-band wavelet function.

For \( l = 1, \ldots, M-1 \), the analogous series expansions for the M-band wavelet functions are

\[ \psi_l(x) = \sqrt{M} \sum_{n \in I} g_l(n) \phi(Mx - n) \] (3.4)

The Fourier transform of (3.4) is

\[ \Psi_l(f) = \frac{1}{\sqrt{M}} G_l \left( \frac{f}{M} \right) \Phi \left( \frac{f}{M} \right) \] (3.5)

where \( G_l \) is the discrete-time Fourier transform of \( g_l(n) \).

An important distinction between dyadic and M-band wavelets is that as indicated in Theorem 2.4, the dyadic wavelet is determined from the dyadic scaling function. In contrast, as noted in [29], there is not a unique set of M-band wavelet functions associated with the M-band scaling function.
Theorem 3.4 Let \( h(n) \) and \( g_l(n) \) \((l = 1, \ldots, M-1)\) be the series coefficients of the M-band scaling and wavelet functions. Then their transforms satisfy

\[
|H(f)|^2 + \sum_{l=1}^{M-1} |G_l(f)|^2 = M, \quad \forall f \in R
\]

(3.6)

The relation in (3.6) is the general power complementary property.

3.3 Fundamental Theory of M-band Wavelet Modulation

3.3.1 Waveform Development

We start with the general QAM signal defined in (2.12) which we repeat here for convenience, namely,

\[
m(t) = \sqrt{\frac{E}{T}} \sum_{k} d_k \phi \left( \frac{t}{T} - k \right)
\]

(3.7)

where: \( \phi \) is the pulse shape

- \( E \) is the average symbol energy
- \( T \) is the symbol duration
- \( d_k \) are complex-valued QAM symbols

Let \( \phi \in V_1 \) be an M-band scaling function for some \( M > 2 \). By Definition 3.1, the \( V_1 \) vector space can be decomposed into the direct sum

\[
V_1 = V_{1 \frac{M}{M}} \oplus W_{1 \frac{M}{M}} \oplus \cdots \oplus W_{1 \frac{M-1}{M}}
\]

(3.8)

It follows that \( \phi \) can be expanded in the basis functions for the subspaces on the right hand side of (3.8). We now rewrite (3.7) as the multidimensional signal
\[ m(t) = \sqrt{\frac{E}{MT}} \sum_{j=1}^{M-1} a_j^p \phi \left( \frac{t}{MT} - j \right) + \sqrt{\frac{E}{MT}} \sum_{n=1}^{M-1} \sum_{j=1}^{L} a_j^n \psi_n \left( \frac{t}{MT} - j \right) \]  

(3.9)

where: \( a_j^n \) are complex-valued QAM symbols for \( n = 0, \ldots, M-1 \)

The communication signal defined by (3.9) will be referred to as M-band Wavelet Modulation (MWM). By Theorems 3.2 and 3.3, the waveform defined in (3.9) is free of ISI. Also using Definition 3.1, the waveform is free of cross channel interference. In contrast to MSM, the symbol period on each pulse in (3.9) is equal and has a value of MT.

### 3.3.2 Power Spectral Density

To derive the power spectral density of MWM, we need the following lemma.

**Lemma 3.5** Let \( \phi \) and \( \psi_n \) (\( n = 1, \ldots, M-1 \)) be scaling and wavelet functions in an M-band wavelet system for some \( M > 2 \), then their Fourier transforms satisfy

\[
\left| \Phi \left( \frac{f}{M} \right) \right|^2 = \left| \Phi (f) \right|^2 + \sum_{n=1}^{M-1} \left| \Psi_n (f) \right|^2, \quad \forall f \in \mathbb{R} 
\]  

(3.10)

Proof: Using (3.3) and (3.5), we have

\[
\left| \Phi (f) \right|^2 + \sum_{n=1}^{M-1} \left| \Psi_n (f) \right|^2 = \frac{1}{M} \left| H \left( \frac{f}{M} \right) \right|^2 \left| \Phi \left( \frac{f}{M} \right) \right|^2 + \frac{1}{M} \sum_{n=1}^{M-1} \left| G_n \left( \frac{f}{M} \right) \right|^2 \left| \Phi \left( \frac{f}{M} \right) \right|^2 
\]  

(3.10)

Factoring out the term \( \left| \Phi \left( \frac{f}{M} \right) \right|^2 \) yields

\[
\left| \Phi (f) \right|^2 + \sum_{n=1}^{M-1} \left| \Psi_n (f) \right|^2 = \frac{1}{M} \left( \left| H \left( \frac{f}{M} \right) \right|^2 + \sum_{n=1}^{M-1} \left| G_n \left( \frac{f}{M} \right) \right|^2 \right) \left| \Phi \left( \frac{f}{M} \right) \right|^2 
\]  

(3.11)
Applying Theorem 3.4 to (3.11) proves the lemma.

Now, assuming the symbols on each pulse shape are independent and identically distributed with zero mean and unit variance, the power spectral density for MWM is given by

\[
S_{MWM}(f) = E|\Phi(fTM)|^2 + E \sum_{n=1}^{M-1} |\Psi_n(fTM)|^2
\]  
(3.12)

Applying Lemma 3.5 to (3.12) yields

\[
S_{MWM}(f) = E|\Phi(fT)|^2
\]  
(3.13)

which is identical in form to the power spectral density of MSM and QAM as given by (2.13). Hence, MWM and MSM will have comparable bandwidth requirements assuming the dyadic scaling function is also an M-band scaling function. In Chapter 5, we will see that the Meyer scaling function possesses this property.

### 3.3.3 Tiling Diagram

The tiling diagram for MWM is illustrated in Figure 3.1. The figure clearly indicates the uniform decomposition of the data bandwidth. Namely, each pulse has effectively the same pulse duration in a mean square sense [32]. In addition, their effective bandwidth is also the same. Using the super symbol definition of Chapter 2, the number of symbols in an MWM super symbol is $M$. So when $M=2^{J-1}$, we see that the data bandwidth for MWM and MSM has been decomposed into the same number of components but with different orientations with respect to the time-frequency plane. In Chapter 6, we will compare this difference when mitigating additive interference in spread spectrum systems. We will use the super symbol concept to define the spreading.
### 3.3.4 Bandwidth Efficiency

To determine the bandwidth efficiency of MWM, let $a_j^n$ be a member of a $2^{K_n}$ QAM constellation. Using the bandwidth of $\Phi$ indicated in (3.13), the bandwidth efficiency is then given by

$$\rho_{MWM} = \frac{\frac{1}{MT} \sum_{n=0}^{M-1} K_n}{\frac{1+\beta}{T}}$$

where: $\beta$ is the percent excess bandwidth from Nyquist signaling

Typically, $\beta \in [0,1]$. Simplifying (3.14) yields

![Figure 3.1. Tiling Diagram For M-band Wavelet Modulation](image-url)
When $K_n = K$, MWM and MSM will have the same bandwidth efficiency.

3.3.5 Performance In Additive White Gaussian Noise

The optimum receiver in AWGN for MWM is a bank of filters matched to each pulse shape in (3.9). As with MSM, the receiver is effectively projecting the received signal upon the basis functions defining MWM. Taking the inner product of (3.9) with each basis function defining a pulse, we again see that the average energy in each dimension is $E$. Further, the projection of AWGN on an orthonormal basis yields uncorrelated zero mean Gaussian random variables with variance $\frac{N_0}{2}$ [24]. Now to determine the average symbol error probability AWGN, let $P_{sym}^n$ be the symbol error probability in AWGN associated with the $a^n$ (n=0,...,M-1) sequence having an average energy of $E$. Again, let $a_j^n$ be a member of a $2^K_n$ QAM constellation. Since there is no ISI or cross channel interference, the average probability of a symbol error in MWM is then

$$P_{sym} = \frac{1}{M} \sum_{n=0}^{M-1} K_n \frac{1}{1 + \beta}$$

(3.15)

Since the relative frequency of $a^n$ in an MWM super-symbol is the same, the data sequences are equiprobable. Thus, the average symbol error probability becomes

$$\overline{P_{sym}} = \frac{1}{M} \sum_{n=0}^{M-1} P_{sym}^n$$

(3.16)
which simplifies to \( P_{\text{sym}} \) when \( P_{\text{sym}}^n = P_{\text{sym}} \) for all \( n \). For specific evaluation of (3.17), we can use (2.26) or (2.27) for the symbol error probabilities of typical QAM signals.

### 3.4 Implementation

As noted, the optimum receiver in AWGN for MWM is a bank of filters matched to each pulse shape in (3.9). Also, as was seen with MSM, a much more efficient realization is obtained with digital filter banks. The transceiver processing is illustrated in Figure 3.2. At the transmitter, a data source is demultiplexed into \( M \) symbol streams operating at one \( M^{\text{th}} \) the source rate. A uniform synthesis filter bank operates on these subsequences producing a sequence at the source rate. This sequence is transmitted across the channel using a pulse shape defined by the M-band scaling function. The received signal is match filtered and sampled. The sampled sequence is then operated on by a uniform analysis filter bank producing soft decisions of the transmitted symbols which are forwarded to the appropriate data sink possibly for further processing.

![Figure 3.2. Transceiver Processing For M-band Wavelet Modulation](image-url)
The implementation of the uniform filter banks of Figure 3.2 can be achieved in a number of ways [20][29]. We will adopt in this dissertation the tree structure. The advantages of using trees is in its ability to implement uniform filter banks, discrete wavelet transforms, transforms associated with wavelet packets [9] and time-varying tiling diagrams [28]. The latter two subjects are not addressed in this dissertation but are pertinent research topics. In addition, the regularity of the tree makes for a flexible and modular hardware or software solution. The tree structured uniform synthesis bank is shown in Figure 3.3 while the analysis filter bank is shown in Figure 3.4.

![Diagram of the Transmitter Uniform Synthesis Filter Bank](image)

**Figure 3.3. Detailed Illustration Of The Transmitter Uniform Synthesis Filter Bank**

A particularly useful feature of the M-band scaling function is that we can obtain a full digital implementation while maintaining the theoretical power spectral density.
Namely, using (3.2), the analog pulse shaping filter in Figure 3.2 can be replaced by the processing shown in Figure 3.5.

Figure 3.4. Detailed Illustration Of The Receiver Uniform Analysis Filter Bank

Figure 3.5. Digital Processing To Replace Analog Pulse Shaping Filter
To see this mathematically, rewrite (3.9) analogous to (2.28) as

\[
m(t) = \sqrt{\frac{E}{T}} \sum_{k \in L} x_k \phi \left( \frac{t}{T} - k \right)
\]  

(3.18)

Using (3.2), (3.18) becomes

\[
m(t) = \sqrt{\frac{E}{T}} \sum_{k \in L} x_k \sqrt{M} \sum_{n \in L} h_n \phi \left( \frac{Mt}{T} - Mk - n \right)
\]  

(3.19)

Making the variable substitution \( p = Mk + n \), (3.19) reduces to

\[
m(t) = \sqrt{\frac{E}{T}} \sum_{p \in L} \left[ \sqrt{M} \sum_{k \in L} x_k h_{p-Mk} \right] \phi \left( \frac{Mt}{T} - p \right)
\]  

(3.20)

where the term in brackets corresponds to the digital interpolation of \( x_k \) by \( h_n \) indicated in Figure 3.5. Now, if the analog image rejection filter reasonably approximates the scaling function scaled by \( M \), the analog output will correspond to the modulation \( m(t) \). Further, with \( M \) reasonably large such that the bandwidth of the lowpass filter, \( h_n \), is much less than the Nyquist sampling bandwidth, the requirements on the analog filter are considerably relaxed provided that it causes no distortion over the bandwidth of the lowpass digital filter.

In a similar manner, the analog match filter can be replaced by the processing in Figure 3.6. Again, for the practical case of large \( M \), the requirements on the analog anti-aliasing filter are relaxed significantly. It should also be pointed out that the digital interpolation and decimation filters shown in Figures 3.5 and 3.6, respectively, can be implemented with polyphase networks for additional computational savings [20].
Finally, we note that the processing in Figures 3.5 and 3.6 can be used in Figure 2.4 for MSM. We could have in Chapter 2 made a similar argument based on a dyadic scaling function. But, since M is effectively two in this case, the image rejection and anti-aliasing filters would require modest cutoff characteristics as would the digital lowpass filter, $h_n$. On the other hand, if the dyadic scaling function is also an M-band scaling function, we can use Figures 3.5 and 3.6 directly. As noted we will encounter such a function in Chapter 5.
Chapter 4

Practical Issues With Filter Bank Implementations

4.1 Introduction

In Chapter 2 it was shown that MSM could be implemented with discrete wavelet transforms and that these transforms were a form of nonuniform filter banks. Similarly, in Chapter 3, it was shown that MWM could be implemented with filter banks but in this case they were uniform where uniformity indicates the partitioning of the signal bandwidth. In this chapter, we consider these structures in greater detail emphasizing implementation issues. In the process of this investigation, we identify and solve a number of practical problems arising when finite impulse response (FIR) filters are used in these banks. The essential issue often not addressed in theoretical developments is the nonzero group delay in realizable FIR filters. This combined with decimation operators creates subtle problems in the practical use of these filter banks. In addition, a currently held belief in the signal processing literature is that when the filters in a two channel filter bank are FIR the order of the filters must be odd. In this chapter we show that even order filters can indeed be used in these filter banks, thus admitting linear phase. As a
consequence, a number of these results are of interest to the signal processing community as a whole. Additionally, all the signal processing structures developed in this chapter have been implemented in a set of MATLAB functions [35].

We begin this chapter with some notation. In particular, we will follow the signal processing tradition of using z-transforms for analysis and algorithm derivation. Further, we will adhere to the notation in the filter bank literature, specifically, reference [20]. This notation is more general than the notation of Mallat used in Chapters 2 and 3 and offers additional insight into the practical operation of the filter banks. In section 4.3, we will make consistent the two notations.

4.2 Notation

Definition 4.1 Let $Z_N = \{n \in \mathbb{Z}: 0 \leq n \leq N\}$. An $N$th order finite impulse response (FIR) sequence is a real or complex sequence, $h(n)$, taking the value of zero for all integers $n \notin Z_N$.

The length of an $N$th order FIR sequence is $N + 1$. Let $h(n)$ be an $N$th order FIR sequence then its z-transform is given by

$$H(z) = \sum_{n=0}^{N} h(n)z^{-n}$$  \hspace{1cm} (4.1)

The complex conjugate of (4.1) on the unit circle will be given the special symbol $\tilde{H}(z)$. To obtain $\tilde{H}(z)$ from $H(z)$, we conjugate the sequence coefficients and replace occurrences of $z$ with $z^{-1}$. A useful example of this notation is the following relation

$$z^{-N}\tilde{H}(z) = h^*(N) + h^*(N-1)z^{-1} + \cdots + h^*(0)z^{-N}$$  \hspace{1cm} (4.2)
That is, (4.2) causes a time reversal and conjugation of the sequence. We can also express the power complementary property of Theorem 2.6 in terms of this notation, namely,

\[ H(z)H(z) + H(-z)H(-z) = 2 \]  

(4.3)

Note that \( H(-z) \) plays the part of the high pass component.

In the filter banks encountered in previous chapters and to be considered in this chapter, expanders and decimators play a central role. Given a sequence of samples, an L-fold expander inserts L-1 zeros between each sample of the sequence. The operation is illustrated in Figure 4.1(a) and has the z-transform relation

\[ Y_E(z) = X\left(\frac{1}{z^L}\right) \]  

(4.4)

Note, the rate of the signal out of the expander has been increased by a factor of L. An M-fold decimator takes a sample sequence and only keeps every M\textsuperscript{th} sample. This operation is illustrated in Figure 4.1(b).

Figure 4.1. Multirate Expander And Decimator
The z-transform relation for an M-fold decimator is given by

\[ Y_D(z) = \frac{1}{M} \sum_{k=0}^{M-1} X \left( \frac{1}{z^M} e^{-\frac{j2\pi k}{M}} \right) \]  

(4.5)

In this case, the rate out of the decimator is reduced by a factor of M.

As noted earlier, tree-structured filter banks can be used to treat nonuniform and uniform filter banks in a unified manner. The processing is illustrated in Figure 4.2.

![Figure 4.2. Tree-Structured Analysis-Synthesis Filter Bank](image)

Each processing element in Figure 4.2 is one of the odd order two channel filter banks illustrated in Figure 4.3. A modified version of this structure for even order filters will be
considered in section 4.4. The shaded elements in Figure 4.2 are not present for the DWT and inverse DWT.

![Figure 4.3. Odd Order Analysis-Synthesis Two-Channel FIR Filter Bank](image)

In general with $X(z)$ as the input, the z-transform of the output in Figure 4.3 is given by

$$\hat{X}(z) = T(z)X(z) + A(z)X(-z) \quad (4.6)$$

where $T(z)$ is the distortion transfer function incorporating phase and amplitude distortions and $A(z)X(-z)$ is the aliased component resulting from the decimators.

**Definition 4.2** A two channel filter bank having the input-output relationship in (4.6) is said to have perfect reconstruction if the alias component is zero and $T(z)$ reduces to $z^{-N}$ for some positive integer $N$. Note, the distortion transfer function is allpass.

### 4.3 Smith-Barnwell Odd Order FIR Filter Bank

#### 4.3.1 Two-Channel Filter Bank

We will now consider the two-channel filter bank in Figure 4.3 in greater detail. The following theorem is due to Smith and Barnwell [34].
Theorem 4.3 Let \( N \) be odd. Let \( H_0 \) satisfy the power complementary property of (4.3). Then, the filter bank of Figure 4.3 will have perfect reconstruction provided the remaining filters are related as follows

\[
H_1(z) = -z^{-N}H_0(-z) \tag{4.7}
\]

\[
F_0(z) = H_1(-z) \tag{4.8}
\]

and

\[
F_1(z) = -H_0(-z) \tag{4.9}
\]

Moreover, the distortion transfer function is given by

\[
T(z) = z^{-N} \tag{4.10}
\]

Proof: See [20].

Utilizing (4.2) and standard z-transform properties, the filters can be expressed in the time domain as [20]

\[
h_1(n) = (-1)^n h_0^*(N-n) \tag{4.11}
\]

\[
f_0(n) = h_0^*(N-n) \tag{4.12}
\]

and

\[
f_1(n) = h_1^*(N-n) \tag{4.13}
\]
where \( N \) is the filter order of \( h_0 \) and must be odd. It follows that all filters in the filter bank are FIR with the same order. The relationship of these filters with those in Mallat's wavelet transforms is

\[
h_n = f_0(n) \quad \text{(4.14)}
\]

and

\[
g_n = f_1(n) \quad \text{(4.15)}
\]

where \( h_n \) and \( g_n \) were defined in Chapter 2.

**4.3.2 J-Channel Nonuniform Filter Bank**

With the design dictated by Theorem 4.3, we can use the two-channel filter bank as building blocks in the tree structure of Figure 4.2. As a result of (4.10), there is a nonzero group delay through each of these building blocks. In a uniform tree configuration, this poses no difficulties since every channel consists of the same number of blocks. But, in the nonuniform case where the shaded elements in Figure 4.2 are not utilized the signals in each channel are experiencing different group delays. Thus, we can not simply "short-out" the shaded processing elements to obtain the discrete wavelet transforms when using actual FIR filters. In the literature, the transforms are commonly portrayed in this manner. The following corollary to Theorem 4.3 addresses this issue.

**Corollary 4.4** Let the filters be chosen in accordance with Theorem 4.3. Then for the integer \( J > 1 \), the J-channel nonuniform filter bank can be recursively expressed as in Figure 4.4 and has perfect reconstruction. Moreover, the distortion transfer function is given by

\[
T(z) = z^{-(2^{J-1})N} \quad \text{(4.16)}
\]
where $N$ is the odd filter order.

**Figure 4.4. J-Channel Nonuniform Odd Order FIR Filter Bank**

Proof: The proof is by induction. The starting condition for the induction is provided by Theorem 4.3. Assume the corollary holds for $J-1$ then the $J-1$ channel block in Figure 4.4 can be replaced by the delay $z^{-\left(2^{J-2}-1\right)N}$. Starting from the analysis portion of the bank, we have

\[ X_0(z) = H_0(z)X(z) \]  \hspace{1cm} (4.17)

and

\[ X_1(z) = H_1(z)X(z) \]  \hspace{1cm} (4.18)

Using (4.5) for decimators with $M=2$, we have that

\[ V_0(z) = \frac{1}{2} \left( X_0 \left( z^{-2} \right) + X_0 \left( -z^{-2} \right) \right) \]  \hspace{1cm} (4.19)

and
Using (4.4) for expanders with $L=2$, we have that

$$Y_0(z) = V_0(z^2)z^{-(2^{l-1} - 2)^N} \quad (4.21)$$

and

$$Y_1(z) = V_1(z^2)z^{-(2^{l-1} - 2)^N} \quad (4.22)$$

Using (4.19) and (4.20) in (4.21) and (4.22), respectively, yields

$$Y_0(z) = \frac{1}{2} (X_0(z) + X_0(-z)) z^{-(2^{l-1} - 2)^N} \quad (4.23)$$

and

$$Y_1(z) = \frac{1}{2} (X_1(z) + X_1(-z)) z^{-(2^{l-1} - 2)^N} \quad (4.24)$$

Using (4.17), (4.18), (4.23) and (4.24), we have that the output is

$$\hat{X}(z) = (H_0(z)F_0(z) + H_1(z)F_1(z)) \frac{X(z)}{2} z^{-(2^{l-1} - 2)^N}$$

$$+ \left( H_0(-z)F_0(z) + H_1(-z)F_1(z) \right) \frac{X(-z)}{2} z^{-(2^{l-1} - 2)^N} \quad (4.25)$$

where we can now clearly identify the distortion transfer function and alias component.

Using (4.8) and (4.9) of Theorem 4.3, the second term is zero and we have alias cancellation. Using (4.7)-(4.9) in (4.25), the output becomes

$$\hat{X}(z) = \left( -H_0(z)(-z)^{-N} \tilde{H}_0(z) + z^{-N} H_0(-z) \tilde{H}_0(-z) \right) \frac{X(z)}{2} z^{-(2^{l-1} - 2)^N} \quad (4.26)$$
Since $N$ is odd, (4.26) simplifies to

$$\hat{X}(z) = \left( H_0(z)\hat{H}_0(z) + H_0(-z)\hat{H}_0(-z) \right) \frac{X(z)}{2} z^{-(2^{j-1}-1)N}$$  \hspace{1cm} (4.27)$$

Now since $H_0$ was chosen to be power complementary, using (4.3) we finally get that

$$\hat{X}(z) = z^{-(2^{j-1}-1)N} X(z)$$  \hspace{1cm} (4.28)$$

By Definition 4.2, equation (4.28) establishes perfect reconstruction. Further, the distortion transfer function is readily seen to be (4.16) which establishes the corollary for $J$ channels. This proves the corollary by induction.

Combining the structure in Figure 4.4 with Figure 4.2 produces the realizable filter bank for implementing the wavelet transforms and is illustrated in Figure 4.5.

\textbf{Figure 4.5. Realizable Odd Order Nonuniform Filter Bank}
4.3.3 Even Order Issue

We stress the importance of \( N \) being odd in the simplification of (4.26). If not, the term in parentheses would not be a power complementary relation as required by (4.3). In particular, consider the distortion transfer function for (4.26) evaluated on the unit circle

\[
T(f) = \frac{1}{2} \left( (-1)^{-\frac{(N-1)}{2}} |H_0(f)|^2 + |H_0\left(f - \frac{1}{2}\right)|^2 \right) e^{-j2\pi(2^{-(N-1)}-1)N} \tag{4.29}
\]

Let \( N \) be even and evaluate (4.29) at \( f = \frac{1}{4} \) to produce

\[
T\left(\frac{1}{4}\right) = \frac{1}{2} \left( -|H_0\left(\frac{1}{4}\right)|^2 + |H_0\left(-\frac{1}{4}\right)|^2 \right) e^{-j\frac{\pi}{2}(2^{-(N-1)}-1)N} \tag{4.30}
\]

Now, let \( h_0 \) be real and symmetric such that its phase function is linear. Then its magnitude spectrum is an even function of \( f \). Using this fact, (4.30) goes to zero at \( f = \frac{1}{4} \) resulting in severe amplitude distortion. Hence, linear phase is unattainable with the Smith-Barnwell design. Note that we are excluding half-sample symmetry (i.e., Type 2 filters [20]) due to their inappropriateness in discrete time implementation of communication systems. That is, the signal generation and timing recovery processes would be greatly complicated.

A consequence of this constraint is that linear phase FIR filters can not be used in the efficient filter bank implementations of the modulation formats developed in Chapters 2 and 3. This implies that the scaling function cannot be symmetric. In communication applications, symmetry is often a requirement. Lack of pulse symmetry has been shown to increase loop noise in a widely used timing recovery scheme [25]. Also, symmetry has
been shown to be more robust to nonlinear quantization error in signal compression schemes [8]. One might therefore expect symmetry to be an important issue in nonlinear satellite communication channels. From a practical perspective, symmetry in digital filters reduces the number of multiplies by approximately half enabling higher speed operation. With this motivation, we now derive a new filter bank structure which can utilize even order FIR filters by proving the following theorem.

4.4 Even Order Filter Bank

4.4.1 Two-Channel Filter Bank

Theorem 4.5 Let N be even. Let \( H_0 \) satisfy the power complementary property of (4.3). Then, the filter bank of Figure 4.6 will have perfect reconstruction provided the remaining filters are related as follows

\[ H_1(z) = z^{-N} \tilde{H}_0(-z) \]  

(4.31)

\[ F_0(z) = H_1(-z) \]  

(4.32)

and

\[ F_1(z) = H_0(-z) \]  

(4.33)

Moreover, the distortion transfer function is given by

\[ T(z) = z^{-(N+1)} \]  

(4.34)

Proof: Starting from the analysis portion of the bank, we have

\[ X_0(z) = H_0(z)X(z) \]  

(4.35)

and
Using (4.5) for decimators with \( M=2 \), we have that

\[
V_0(z) = \frac{1}{2} \left( X_0 \left( \frac{1}{z^2} \right) + X_0 \left( -\frac{1}{z^2} \right) \right) \tag{4.37}
\]

and

\[
V_1(z) = \frac{1}{2} \left( X_1 \left( \frac{1}{z^2} \right) + X_1 \left( -\frac{1}{z^2} \right) \right) \tag{4.38}
\]

Using (4.4) for expanders with \( L=2 \), we have that

\[
Y_0(z) = V_0(z^2)z^{-1} \tag{4.39}
\]

and

\[
Y_1(z) = V_1(z^2) \tag{4.40}
\]

Using (4.37) and (4.38) in (4.39) and (4.40), respectively, yields
\[ Y_0(z) = \frac{1}{2}(X_0(z) + X_0(-z))z^{-1} \quad (4.41) \]

and

\[ Y_1(z) = \frac{1}{2}(X_1(z) + X_1(-z)) \quad (4.42) \]

Using (4.35), (4.36), (4.41) and (4.42), we have that the output is

\[
\hat{X}(z) = (H_0(z)F_0(z) + H_1(z)F_1(z)) \frac{X(z)}{2} z^{-1} \\
+ \left( H_0(-z)F_0(z) - H_1(-z)F_1(z) \right) \frac{X(-z)}{2} z^{-1}
\]

\[ (4.43) \]

where we again can clearly identify the distortion transfer function and alias component.

Using (4.32) and (4.33) of the hypothesis, the second term is zero and we have alias cancellation. Using (4.31)-(4.33) in (4.43), the output becomes

\[
\hat{X}(z) = \left( H_0(z)(-z)^{-N} \tilde{H}_0(z) + z^{-N} H_0(-z)\tilde{H}_0(-z) \right) \frac{X(z)}{2} z^{-1}
\]

\[ (4.44) \]

Since \( N \) is even, (4.44) simplifies to

\[
\hat{X}(z) = \left( H_0(z)\tilde{H}_0(z) + H_0(-z)\tilde{H}_0(-z) \right) \frac{X(z)}{2} z^{-(N+1)}
\]

\[ (4.45) \]

Now since \( H_0 \) was chosen to be power complementary, using (4.3) we finally get that

\[
\hat{X}(z) = z^{-(N+1)}X(z)
\]

\[ (4.46) \]
By Definition 4.2, equation (4.46) establishes perfect reconstruction. Further, the distortion transfer function is readily seen to be (4.34). This proves the theorem.

4.4.2 Remarks

We again stress the importance of N being even in this case for the simplification of (4.44). If not, the term in parentheses would not be a power complementary relation as required by (4.3). Also, the solution to overcoming the linear phase problem was quite subtle resulting simply in the judicious insertion of appropriate delays. In fact, the time domain relationship between the filters is identical to the odd order design. Taking the inverse z-transform of (4.31)-(4.33) results in (4.11)-(4.13). Chapter 5 will consider designs for \( h_n \). It should be pointed out also that the filter bank in Figure 4.6 is not unique. From the proof of Theorem 4.5, an apparent symmetry exists. In particular, the delays could have been inserted in the lowpass channel of the analysis bank and the highpass channel of the synthesis bank as an alternative to the choice in Figure 4.6.

4.4.3 J-Channel Nonuniform Filter Bank

Because of the group delay through the two-channel filter bank governed by (4.34), we must again address the nonuniform filter bank as was done for the odd order case. Thus, we have the following corollary.

**Corollary 4.6** Let the filters be chosen in accordance with Theorem 4.5. Then for the integer \( J>1 \), the J-channel nonuniform filter bank can be recursively expressed as in Figure 4.7 and has perfect reconstruction. Moreover, the distortion transfer function is given by

\[
T(z) = z^{-(2^{J-1}-1)(N+1)}
\]  

(4.47)

where \( N \) is the even filter order.
Figure 4.7. J-Channel Nonuniform Even Order Filter Bank

Proof: The proof is by induction and is similar to the proof of Corollary 4.4. The starting condition for the induction is provided by Theorem 4.5. Assume the corollary holds for J-1 then the J-1 channel block in Figure 4.7 can be replaced by the delay $z^{-((2^{J-2}-1)(N+1)))}$.

Starting from the analysis portion of the bank, we have

$$X_0(z) = H_0(z)X(z)$$  \hspace{1cm} (4.48)

and

$$X_1(z) = H_1(z)X(z)z^{-1}$$  \hspace{1cm} (4.49)

Using (4.5) for decimators with $M=2$, we have that

$$V_0(z) = \frac{1}{2} \left( X_0 \left( \frac{1}{z^2} \right) + X_0 \left( -\frac{1}{z^2} \right) \right)$$  \hspace{1cm} (4.50)

and

$$V_1(z) = \frac{1}{2} \left( X_1 \left( \frac{1}{z^2} \right) + X_1 \left( -\frac{1}{z^2} \right) \right)$$  \hspace{1cm} (4.51)

Using (4.4) for expanders with $L=2$, we have that
\[
Y_0(z) = V_0(z^2)z^{-2^{j-1}-2}(N+1)z^{-1} \tag{4.52}
\]

and
\[
Y_1(z) = V_1(z^2)z^{-2^{j-1}-2}(N+1) \tag{4.53}
\]

Using (4.50) and (4.51) in (4.52) and (4.53), respectively, yields
\[
Y_0(z) = \frac{1}{2}(X_0(z) + X_0(-z))z^{-2^{j-1}-2}(N+1)z^{-1} \tag{4.54}
\]

and
\[
Y_1(z) = \frac{1}{2}(X_1(z) + X_1(-z))z^{-2^{j-1}-2}(N+1) \tag{4.55}
\]

Using (4.48), (4.49), (4.54) and (4.55), we have that the output is
\[
\hat{X}(z) = (H_0(z)F_0(z) + H_1(z)F_1(z)) \frac{X(z)}{2} z^{-2^{j-1}-2}(N+1)z^{-1}
+ \frac{H_0(-z)F_0(z) - H_1(-z)F_1(z)}{2} \frac{X(-z)}{2} z^{-2^{j-1}-2}(N+1)z^{-1} \tag{4.56}
\]

where we can again identify the distortion transfer function and alias component. Using (4.32) and (4.33) of Theorem 4.5, the second term is zero and we have alias cancellation. Using (4.31)-(4.33) in (4.56), the output becomes
\[
\hat{X}(z) = (H_0(z)\tilde{H}_0(z) + z^{-N}H_0(-z)\tilde{H}_0(-z)) \frac{X(z)}{2} z^{-2^{j-1}-2}(N+1)z^{-1} \tag{4.57}
\]

Since N is even, (4.57) simplifies to
\[
\hat{X}(z) = (H_0(z)\tilde{H}_0(z) + H_0(-z)\tilde{H}_0(-z)) \frac{X(z)}{2} z^{-2^{j-1}-1}(N+1) \tag{4.58}
\]
Now since $H_0$ was chosen to be power complementary, using (4.3) we finally get that

$$\hat{X}(z) = z^{-(2^{j-1} - 1)(N+1)}X(z) \quad (4.59)$$

By Definition 4.2, equation (4.59) establishes perfect reconstruction. Further, the distortion transfer function is readily seen to be (4.47) which establishes the corollary for $J$ channels. This proves the corollary by induction.

Combining the structure in Figure 4.7 with Figure 4.2 produces the realizable filter bank for implementing the wavelet transforms. The structure is illustrated in Figure 4.8.

![Figure 4.8. Realizable Even Order Nonuniform Filter Bank](image)
4.5 Delay Equalization In Synthesis-Analysis Configurations

As just seen, the practical issue associated with the odd and even order analysis-synthesis filter banks was the nonzero group delay of realizable FIR filters. Another issue attributable to the group delay occurs when using synthesis-analysis banks. Namely, for the communication problem as depicted in Figures 2.4 and 3.2, the synthesis bank is at the transmitter while the analysis bank is at the receiver. So instead of analysis-synthesis, we are performing synthesis-analysis which is sometimes referred to as transmultiplexers [20]. In general, the tree structured synthesis-analysis filter bank is illustrated in Figure 4.9.

![Synthesis-Analysis Configuration For Communication Applications](image)

Figure 4.9. Synthesis-Analysis Configuration For Communication Applications

When converting to a synthesis-analysis configuration, perfect reconstruction remains the essential criteria. But when also using tree structures, a problem surfaces. This problem is a result of the decimators being at the receiver. Specifically, a potential exists for a non integer group delay out of a particular channel.
To see this we begin by noting from (4.5) that the output of a decimator is given by

\[ Y_D(z) = \frac{1}{2} \left( \frac{\Delta}{z^2} + (-z)^{-\frac{\Delta}{2}} \right) \]  \hspace{1cm} (4.60)

when the input \( X(z) = z^{-\Delta} \) and \( M=2 \). We can ignore the aliasing term since we are only interested in perfect reconstruction systems at this point. Then a signal with a given group delay will have this delay divided in half when passed through a decimator. Now considering a J-channel filter bank which has a depth of J-1 in the analysis bank of the receiver, then the number of decimators in the longest path is J-1. So whatever group delay this signal was experiencing, the output group delay would be reduced by \( 2^{J-1} \). Thus, assuming that the filters in a J-channel odd order synthesis-analysis filter bank are chosen for perfect reconstruction, then using (4.16), the signal output of the longest tree path can be expressed as

\[ \hat{X}(z) = X(z)z^{-\Delta} \]  \hspace{1cm} (4.61)

where: \( \Delta = \frac{(2^{J-1}-1)N}{2^{J-1}} \) \hspace{1cm} (4.62)

From (4.62), we see that regardless of J and N the group delay can never be a positive integer as required by Definition 4.2. Further, since J can be arbitrary, this holds for shorter paths in nonuniform trees.

To solve this problem, we borrow from the idea used to solve the even order issue with filter banks. That is, we judiciously insert delays within the analysis tree to insure an integer group delay out of a particular channel. Figure 4.10 illustrates a general
placement of delays, $b_k$, at each node in the tree. The problem is to determine a systematic procedure for selecting the $b_k$ such that these individual delays and the total group delay in a channel are positive integers and thus realizable. We first must determine the effect that these new delays have on the output group delay. This is simply a matter of determining the number of decimators following each inserted delay and then reducing this delay analogous to (4.62). From Figure 4.10, we see that for a J-channel bank the $k^{th}$ delay will be reduced by $2^{J-k}$. Thus, we can rewrite (4.62) as

$$\Delta = \frac{(2^{J-1} - 1) N}{2^{J-1}} + \sum_{k=1}^{J-1} \frac{b_k}{2^{J-k}}$$

(4.63)

Factoring $2^{J-1}$ in (4.63) yields

Figure 4.10. Delay Equalization In Analysis Bank Of The Receiver
\[
\Delta = \frac{(2^{J-1} - 1)N + \sum_{k=1}^{J-1} 2^{k-1} b_k}{2^{J-1}}
\]  
(4.64)

(4.64) can be rewritten as

\[
\Delta = N + \left( \sum_{k=1}^{J-1} 2^{k-1} b_k \right) - N
\]  
\[
\Delta = N + \frac{\left( \sum_{k=1}^{J-1} 2^{k-1} b_k \right) - N}{2^{J-1}}
\]  
(4.65)

The solution readily follows by choosing the \( b_k \) such that

\[
N = \sum_{k=1}^{B} 2^{k-1} b_k
\]  
(4.66)

where \( b_k \in \{0,1\} \) and \( N \) is the odd filter order. Namely, a unit delay is inserted whenever a one appears in the binary expansion of the filter order while no delays are used for the occurrences of zeros in the expansion. For example, the binary expansion for \( N=11 \) is 1011 and \( B=4 \). Unit delays in this case would be inserted at the first, second and fourth nodes in the tree.

Since the filter order and tree depth are unrelated, we have introduced the variable \( B \) in defining the binary expansion. We will next show that the solution obtained is general.

**Case 1**: \( B \leq J-1 \). In this case \( b_k=0 \) for all \( k \geq B+1 \). Using this in (4.65) yields

\[
\Delta = N + \frac{\sum_{k=1}^{B} 2^{k-1} b_k - N}{2^{J-1}}
\]  
(4.67)
Using (4.66), (4.67) reduces to

\[ \Delta = N \]  

which is a valid solution.

**Case 2: \( B > J - 1 \).** In this case, (4.66) in (4.65) yields

\[
\Delta = N + \sum_{k=1}^{B} 2^{k-1} b_k - \sum_{k=1}^{B} 2^{k-1} b_k
\]

(4.69)

Canceling similar terms in (4.69) and bringing the denominator within the sum yields

\[
\Delta = N - \sum_{k=J}^{B} 2^{k-J} b_k
\]

(4.70)

Since \( k \geq J \), the second term is always a positive integer. Further, since \( J \geq 2 \) then from (4.66), \( \Delta \) is a positive integer.

Cases 1 and 2 have shown that a general solution to the delay equalization problem has been obtained for the odd order filter bank. For the even order filter bank, we can obtain a solution in exactly the same manner. In particular, we can start with (4.47) in (4.62) and proceed with precisely the same development. The result is that the binary expansion is now performed on \( N+1 \) where \( N \) is the even filter order. Note that regardless of the order of the filters, a unit delay is always inserted at the first node.

### 4.6 Band Shuffling In Tree-Structured Uniform Analysis Filter Banks

We have just seen how the decimators used in the filter banks create subtle problems in practical FIR filter banks. In tree-structured uniform analysis banks another issue arises which warrants mentioning. In signal analysis, this bank is used to study
different portions of the signal bandwidth. The M outputs of the filter bank correspond to a uniform decomposition of the bandwidth into M bands. One would expect a linear ordering of the signals with respect to the bands. That is, adjacent signals out of the filter bank correspond to adjacent bands in the spectrum. This does not occur for tree structured banks. An apparent shuffling of the bands occurs. We will now show that the shuffling is systematic and that it follows a gray coded ordering [36]. This problem is important to identify since additive noise terms in the received signal will only be subjected to the analysis bank. Understanding the ordering of the bands out of the filter bank is essential to developing techniques to mitigate these interference terms.

We start by recalling the recursive definition of gray coding [36].

**Definition 4.6** Let $G_2 = (0,1)$ and $G_k = \left( g_1, g_2, \cdots, g_{2^k-2}, g_{2^k-1} \right)$ be row vectors then

$$G_{k+1} = \left( 0g_1, 0g_2, \cdots, 0g_{2^k-2}, 0g_{2^k-1}, 1g_{2^k-2}, 1g_{2^k-1}, \cdots, 1g_2, 1g_1 \right) \quad (4.71)$$

$G_k$ is called the index vector for a gray coded sequence. Examples of gray coding are $G_3 = (00,01,11,10)$ and $G_4 = (000,001,011,010,110,111,101,100)$.

**Proposition 4.7** Let the subbands in the signal spectrum follow the linear ordering as indicated in Figure 4.11. That is, the lowpass signal is $X_0$ and the highpass signal is $X_{M-1}$ where $M$ is the number of subbands. Let the subscripts on the subbands be base two then the ordering of the subband signals out of the tree-structured uniform analysis bank is as shown in Figure 4.12 where $G_J$ is an index vector defined in Definition 4.6 with $M = 2^{J-1}$. 
Proof: It suffices to show that the algorithm which generates the subband signals in Figure 4.12 is the same as the recursive algorithm in Definition 4.6. Referring to Figure 4.3, at each stage in the analysis tree, the input signal is lowpass and highpass filtered. These signals are then decimated. For M=2, evaluate (4.5) on the unit circle, namely,

\[ Y_D(f) = \frac{1}{2} \left( X \left( \frac{f}{2} \right) + X \left( \frac{f-1}{2} \right) \right) \]  

(4.72)
Let $X$ be bandlimited to half the Nyquist bandwidth as would occur with the signals out of the filters in Figure 4.3. Then if $X$ is lowpass, the first term in (4.72) dominates and results simply in a rescaling of the frequency axis according to the new decimated rate. Thus, if $X$ is composed of subbands, then these subbands follow the linear ordering in Figure 4.11. But if $X$ is highpass, the second term dominates and (4.72) centers the signal spectrum at $f = \frac{1}{2}$ and then rescales to the decimated rate. In particular, the high frequency components are now close to the new DC point while the lower frequency components are away from the new DC point. Now if this signal is further composed of subbands, these subbands will therefore be reversed with respect to the linear ordering. Further, let $J=2$ in Figure 4.12, the lowpass component is assigned a zero while the highpass component is assigned a one. Hence, from (4.71), we see this is exactly what is occurring in the gray coding. Namely, the lower half of the indices are simply the indices from the previous stage with a zero prepended. The upper half of the indices are the indices from the previous stage reversed with a one prepended. This establishes that the filter bank outputs are generated by the same algorithm as gray coding and therefore proves the proposition.
Chapter 5

Pulse Shape Design

5.1 Introduction

One of the advantages of using wavelets in orthogonally multiplexed signaling is the large number of designs available. Scaling functions with varying temporal and spectral characteristics exist all of which can be implemented with filter banks. This provides considerable flexibility in the system design. The purpose of this chapter then is to investigate these approaches and identify appropriate designs for use in the communication formats being developed in this dissertation. As a result, important connections between MRA bases and familiar communication waveforms will be made. The most important of these is the connection between the Meyer MRA and square root Nyquist pulse since this pulse is so widely used in communications. This also allows us to extend the generalization of the Meyer scaling and wavelet functions. As a consequence of this generalization, we are able to prove the Meyer function is a valid M-band scaling function. A corollary to this proof is a bound relating the roll-off parameter and the multiplicity M. We use the square root raised cosine pulse to design linear phase \( h_n \) filters for use in the new even order filter bank derived in Chapter 4. When linear
phase is not essential, the Smith-Barnwell design will be seen as the practical solution for use in odd order filter banks.

5.2 Extensions Of The Meyer MRA

5.2.1 Meyer Scaling Function

Definition 5.1 The Fourier transform of the scaling function in the Meyer MRA is defined by

\[
\Phi(f) = \begin{cases} 
1 & ,|f| \leq \frac{1-\beta}{2} \\
\cos\left(\frac{\pi}{2} \vartheta\left(\frac{|f|}{\beta} - \frac{1-\beta}{2\beta}\right)\right) & ,\frac{1-\beta}{2} \leq |f| \leq \frac{1+\beta}{2} \\
0 & ,\text{otherwise}
\end{cases}
\] (5.1)

where the function \(\vartheta\) satisfies

\[
\vartheta(x) = \begin{cases} 
0 & ,x \leq 0 \\
1 & ,x \geq 1
\end{cases}
\] (5.2)

and

\[
\vartheta(x) + \vartheta(1-x) = 1
\] (5.3)

In [8], (5.1) is reported for the specific case of \(\beta = \frac{1}{3}\). We generalize this to include all \(\beta \in \left[0, \frac{1}{3}\right]\) as was done in [9] for spline-based wavelet functions. As discussed previously, we need an \(h_n\) filter for efficient implementation which satisfies (2.2). Obtaining the \(h_n\) filter for (5.1) is particularly straightforward since the pulse is strictly bandlimited, that is, compactly supported in frequency. We need only to sample the scaling function according to
Because of its compact support in frequency, the scaling function has infinite support in the time domain. Consequently, (5.4) is an infinite length sequence. We will address this issue further in Section 5.4.

**5.2.2 Square Root Raised Cosine Scaling And Wavelet Functions**

The condition in (5.3) causes odd symmetry about the point \( \left( \frac{1}{2}, \frac{1}{2} \right) \). The simplest form of \( \vartheta \) satisfying (5.2) and (5.3) is linear for \( 0 < x < 1 \), namely

\[
\vartheta(x) = x
\]  

(5.5)

With (5.5) in (5.1), the scaling function spectrum reduces to

\[
 \Phi(f) = \begin{cases} 
  1, & |f| \leq \frac{1-\beta}{2} \\
  \cos \left( \frac{\pi}{4} \left( \frac{2f-1}{\beta} \right) \right), & \frac{1-\beta}{2} \leq |f| \leq \frac{1+\beta}{2} \\
  0, & \text{otherwise}
\end{cases}
\]  

(5.6)

(5.6) corresponds to the pulse spectrum of a classically used communication waveform the square root raised cosine [26]. Now, the parameter \( \beta \) has precise meaning. It is called the rolloff parameter and indicates the excess bandwidth required over Nyquist signaling [12]. The inverse Fourier transform of (5.6) yields the square root raised cosine pulse [26]

\[
\phi(t) = \frac{\sin(\pi(1-\beta)t)+4\beta t\cos(\pi(1+\beta)t)}{\pi\left(1-(4\beta t)^2\right)}
\]  

(5.7)
Figure 5.1 illustrates (5.7) for $\beta = \frac{1}{3}$.

![Graph](image)

**Figure 5.1. Square Root Raised Cosine Scaling Function, $\beta = \frac{1}{3}$**

Having made this connection we can now derive the dyadic wavelet function associated with the square root raised cosine pulse. The spectrum is given in general form by

$$
\Psi(f)e^{j\pi f} = \begin{cases} 
\sin\left(\frac{\pi}{4}\left(\frac{|f| - 1}{\beta} + 1\right)\right), & \frac{1 - \beta}{2} \leq |f| \leq \frac{1 + \beta}{2}, \\
1, & \frac{1 + \beta}{2} \leq |f| \leq (1 - \beta) \\
\cos\left(\frac{\pi}{4}\left(\frac{|f| - 1}{\beta} + 1\right)\right), & (1 - \beta) \leq |f| \leq (1 + \beta) \\
0, & \text{otherwise}
\end{cases}
$$

(5.8)

The inverse Fourier transform of (5.8) yields the time domain square root raised cosine wavelet function, namely,
The details of this derivation are somewhat lengthy and are therefore provided in Appendix A. Figure 5.2 illustrates (5.9) for $\beta = \frac{1}{3}$.

$$
\psi(t) = \frac{4\beta(t-\frac{1}{2}) \cos(\pi(1-\beta)(t-\frac{1}{2})) - \sin(\pi(1+\beta)(t-\frac{1}{2}))}{\pi \left[1 - \left(4\beta(t-\frac{1}{2})\right)^2\right]}(5.9)
$$

$$
+ \frac{\sin(2\pi(1-\beta)(t-\frac{1}{2})) + \sin(2\pi(1+\beta)(t-\frac{1}{2}))}{\pi \left[1 - \left(8\beta(t-\frac{1}{2})\right)^2\right]}(t-\frac{1}{2})
$$

From Figure 5.2, we see the characteristic bandpass behavior, i.e., no DC component, of wavelet functions. We note that the wavelet function is smooth and, in fact, infinitely differentiable as are all Meyer wavelets [8]. The ripple seen in Figure 5.2 is an artifact of the plotting routine.
5.2.3 M-band Extensions

The range on $\beta$ given above is a consequence of (5.4) and the requirements imposed by (2.2) or equivalently (2.3). To further understand this, we turn to M-band wavelets. The generalizations of (2.2) and (2.3) for M-band scaling functions were given in (3.2) and (3.3), respectively. We now prove that the Meyer scaling function constitutes an M-band scaling function.

**Theorem 5.2** Let the integer $M > 1$ and let $\phi$ be the Meyer scaling function defined in Definition 5.1 then there exists a periodic function $H$ with period 1 such that the Fourier transform of $\phi$ satisfies (3.3), namely,

$$\Phi(f) = \frac{1}{\sqrt{M}} H\left(\frac{f}{M}\right) \Phi\left(\frac{f}{M}\right)$$

(5.10)

Hence, $\phi$ is an M-band scaling function.

**Proof:** Let the integer $M > 1$ and let $H$ be defined by

$$H(f) = \sqrt{M} \sum_{k \in I} \Phi(M(f - k))$$

(5.11)

$H$ is periodic with period 1 since

$$H(f + 1) = \sqrt{M} \sum_{k \in I} \Phi(M(f + 1 - k))$$

(5.12)

Making the variable substitution $p = k - 1$ establishes the periodicity. Now, using (5.11) in the right hand side (5.10) we have
To simplify (5.13), consider Figure 5.3 which illustrates the relevant terms in (5.13).

\[
\frac{1}{\sqrt{M}} H\left(\frac{f}{M}\right) \Phi\left(\frac{f}{M}\right) = \sum_{k=1} \Phi(f - Mk) \Phi\left(\frac{f}{M}\right)
\]  

(5.13)

Since \(\beta \leq \frac{1}{3}\) by Definition 5.1, then for \(M > 1\) we have from Figure 5.3

\[
\frac{(1+\beta)M}{2} \leq -\frac{1+\beta}{2} + M
\]

(5.14)

Then for \(k \neq 0\), the support of \(\Phi\left(\frac{f}{M}\right)\) does not intersect with the support of \(\Phi(f - Mk)\).

Thus,

\[
\frac{1}{\sqrt{M}} H\left(\frac{f}{M}\right) \Phi\left(\frac{f}{M}\right) = \Phi(f) \Phi\left(\frac{f}{M}\right)
\]

(5.15)
Since (5.14) implies

$$\frac{1+\beta}{2} \leq \frac{(1-\beta)M}{2}$$

(5.16)

then $\Phi\left(\frac{f}{M}\right) = 1$ on the support of $\Phi(f)$. Using this fact in (5.15) proves the theorem.

A corollary to this theorem extends the range of $\beta$ producing the following bound relating it with the multiplicity.

**Corollary 5.3.** Let $\beta$ be the rolloff parameter for the Meyer scaling function then $\beta$ is bounded above by

$$\beta \leq \frac{M-1}{M+1}$$

(5.17)

where $M>1$ is the multiplicity.

Proof: From (5.14), we have that

$$(1+\beta)M \leq -(1+\beta) + 2M$$

(5.18)

Rearranging, (5.18) becomes

$$\beta(M+1) \leq M - 1$$

(5.19)

(5.17) easily follows from (5.19) proving the corollary.

We observe that (5.17) reduces to one-third as expected for the dyadic case. Additionally, (5.17) approaches unity for large $M$ consistent with the definition of the rolloff parameter in the communication context [12].
A second corollary to Theorem 5.2 provides an explicit means for determining the sequence \( h_n \).

**Corollary 5.4** Let \( \phi \) be a Meyer scaling function with multiplicity \( M \) then the sequence \( h_n \) is given by

\[
h_n = \frac{1}{\sqrt{M}} \phi\left( \frac{n}{M} \right)
\]  

(5.20)

**Proof:** Taking the inverse discrete-time Fourier transform of (5.11) yields

\[
h_n = \sqrt{M} \left\{ \sum_{k=1}^{M-1} \Phi(M(f - k)) e^{j2\pi f_n} df \right\}
\]  

(5.21)

where the integral is over one period [37]. By Definition 5.1 and Corollary 5.3, only the \( k=0 \) term contributes to the integral. Thus, (5.21) becomes

\[
h_n = \sqrt{M} \left\{ \sum_{k=1}^{M-1} \Phi(Mf) e^{j2\pi f_n} df \right\}
\]  

(5.22)

Making the variable substitution \( u=Mf \), (5.22) becomes

\[
h_n = \frac{1}{\sqrt{M}} \left\{ \sum_{k=1}^{M-1} \Phi(Mf) e^{j2\pi f_n} df \right\}
\]  

(5.23)
Using Corollary 5.3 we note that \( \left( -\frac{1+\beta}{2}, \frac{1+\beta}{2} \right) \subset \left( -\frac{M}{2}, \frac{M}{2} \right) \) for all integers \( M > 1 \). By Definition 5.1, the integral in (5.23) is then seen to be the inverse Fourier transform of \( \Phi(f) \) evaluated at \( \frac{n}{M} \), namely \( \phi \left( \frac{n}{M} \right) \), which proves the corollary.

5.3 Compactly Supported Wavelets

5.3.1 Haar MRA

Since a practical communication system necessarily requires finite length filters, we now address the issue of compact support. An example of an MRA with compactly supported scaling and wavelet functions is the Haar MRA whose wavelet function was known to generate an orthonormal basis for \( L^2(\mathbb{R}) \) even before the formal development of wavelets [8]. This MRA also has an interesting connection to communications. In particular, the scaling function corresponds to the Non Return To Zero (NRZ) pulse while the wavelet function corresponds to the Manchester pulse [30]. These waveforms are illustrated in Figure 5.4.

![Figure 5.4. Scaling And Wavelet Function In The Haar MRA](image)
The functions in the Haar MRA have excellent time localization but poor frequency localization [30]. Further, recalling the power spectral density calculation for MSM in Chapter 2, then with J=2 we can combine an NRZ QAM signal with a Manchester QAM signal yielding the effective bandwidth of a twice rate NRZ QAM signal. Conversely, since many systems currently employ the Manchester waveform already, particularly within NASA [33], the throughput of this system can be effectively doubled with no appreciable increase in bandwidth by using MSM with J=2.

5.3.2 Daubechies' Wavelets

Much of the celebrated work by Daubechies has involved scaling and wavelet functions possessing compact support [8]. Unfortunately, the resulting functions also do not have good frequency characteristics desirable in communication problems. In addition, the Daubechies' functions are asymmetric. The following Smith-Barnwell design is a much more systematic and practical approach for asymmetric designs that yield good frequency responses. As such we will not consider Daubechies' functions further in this dissertation.

5.3.3 Smith-Barnwell Design

Recalling the connection made between wavelets and digital filter banks in Chapter 2-4, we alternatively can use certain designs from filter bank theory to define a pulse shape. In this approach the $h_n$ filter is first determined and then the successive approximation algorithm [19] is used to estimate the scaling function. It is important to note that not all filter bank designs are associated with wavelets. Only those designs satisfying the admissibility condition and the power complementary property defined in Chapter 2 will give rise to wavelets [19].

Of the filter bank designs, the Smith-Barnwell technique [34] is one of the more practical solutions achieving good frequency responses. The procedure is quite straightforward. Given transition band and sidelobe level requirements, design an odd
order half-band FIR filter using the Kaiser window design philosophy and then compute its spectral factor [20]. Note that because of the spectral factorization, the sidelobe level used in the design is approximately twice the requirement in dB. Figure 5.5 illustrates a representative frequency response versus digital frequency (1 corresponds to the symbol rate) for an N=19 order $h_n$ filter obtained using this procedure.

![Magnitude Spectra of the $h_n$ Filter In The Smith-Barnwell Design](image_url)

**Figure 5.5. Magnitude Spectra of the $h_n$ Filter In The Smith-Barnwell Design**

An estimate of the scaling function associated with the $h_n$ filter of Figure 5.5 is shown in Figure 5.6. A potential drawback with the Smith-Barnwell design rests in the asymmetric character of its impulse response as indicated in this figure. Indeed, Daubechies has shown that simultaneously achieving compact support and symmetry are incompatible in the real-valued case [8]. It is this connection between odd order filter banks and Daubechies work which is put forth to explain the inability of using linear phase filters. We saw in Chapter 4, however, that an even order filter bank can achieve perfect reconstruction provided the power complementary property is satisfied. In
essence the problem must be decoupled. In Chapter 4, we developed the filter bank structure. We now must determine a design.

![Scaling Function Graph](image)

**Figure 5.6. Estimated Scaling Function For Smith-Barnwell Filter Of Figure 5.5 Using Successive Approximation Algorithm**

5.4 Function Approximation To Meyer Scaling Function

5.4.1 Design Procedure

Since the $h_n$ filter in the Meyer MRA is symmetric and satisfies the power complementary property, we propose as a filter design an approximation to the infinite length $h_n$ filter of (5.4). Since the $h_n$ filter coefficients are the expansion coefficients for the scaling function, truncating this series gives the best $L^2$ approximation to the scaling function. As an example, the square root raised cosine pulse of (5.7) was sampled according to (5.4) and truncated symmetrically about zero to produce a 36th order filter. The rolloff parameter was set to one-third. The frequency response of this filter is shown in Figure 5.7.
Comparing the response of Figure 5.7 to the Smith-Barnwell design in Figure 5.5, we see that the two responses are quite comparable. The only significant difference is that the square root raised cosine design has a slightly sharper cutoff at bandedge but also has a slightly higher first sidelobe. In addition, the computational complexity of the two filters is about the same due to the symmetry of the square root raised cosine filter. Thus, we can expect that for a give set of requirements on the filter design, either design can be used with the only essential issue being a linear phase requirement.

Using the successive approximation algorithm, the estimated scaling function is illustrated in Figure 5.8. The symmetry is apparent and the approximation is quite close to the theoretical square root raised cosine pulse.
5.4.2 Error Analysis

As noted, the estimated scaling function in Figure 5.8 well approximated the theoretical response given in (5.7). From a filter bank perspective, it is expected that some deviation from perfect reconstruction has taken place. To show this qualitatively, a random sequence of ±1 was used as input to the even order two-channel filter bank of Figure 4.6. This input was chosen since its spectrum is essentially flat over the entire Nyquist bandwidth. Figure 5.9 illustrates the input. The corresponding output is shown in Figure 5.10. The simulation used the $h_n$ filter of Figure 5.7. The remaining filters were determined from (4.11)-(4.15). Comparing Figure 5.9 and 5.10 we see that for all practical purposes perfect reconstruction has been achieved. No aliasing has occurred due to the choice of synthesis filters (4.32) and (4.33). The delay in the output is a group delay of 37 samples as predicted by (4.34).
Figure 5.9. Random Input Into Even Order Two-Channel Filter Bank

Figure 5.10. Output Of Even Order Filter Bank Using Figure 5.9 As Input
Thus, the truncated design approach appears to be a viable practical solution for linear phase applications. We will now quantify this deviation and use it to set the filter order, thus, completing the design.

Due to the choice of analysis and synthesis filters in Theorem 4.5, we can express the output of the even order two-channel filter bank as

\[
\hat{X}(z) = \frac{1}{2} \left( H_0(z)\tilde{H}_0(z) + H_0(-z)\tilde{H}_0(-z) \right) X(z)z^{-(N+1)}
\]  

(5.24)

where \( H_0(z) \) can no longer be assumed power complementary as was done in equation (4.45). Since we are considering realizable FIR sequences, the error is defined as

\[
E(z) = X(z)z^{-(N+1)} - \hat{X}(z)
\]

(5.25)

Using (5.24) in (5.25), we have that

\[
E(z) = \left[ 1 - \frac{1}{2} \left( H_0(z)\tilde{H}_0(z) + H_0(-z)\tilde{H}_0(-z) \right) \right] X(z)z^{-(N+1)}
\]

(5.26)

Define the distortion ratio as

\[
D(z) = \frac{E(z)}{X(z)}
\]

(5.27)

Using (5.26) in (5.27), we have that

\[
D(z) = \left[ 1 - \frac{1}{2} \left( H_0(z)\tilde{H}_0(z) + H_0(-z)\tilde{H}_0(-z) \right) \right] z^{-(N+1)}
\]

(5.28)
The distortion energy is given by

\[ \varepsilon_d = \sum_{n=1}^{\infty} |d(n)|^2 \]  

(5.29)

By Parseval's relation [37] and (5.28), the distortion energy becomes

\[ \varepsilon_d = \frac{1}{2} \left[ \frac{1}{2} - \frac{1}{2} \left( \left| H_0(f) \right|^2 + \left| H_0 \left( f - \frac{1}{2} \right) \right|^2 \right) \right]^2 df \] 

(5.30)

Recall the \( h_n \) filter in Mallat's notation is the synthesis low pass filter \( f_0(n) \). By (4.12), the magnitude response of the analysis and synthesis low pass filters are the same. Thus, (5.30) becomes

\[ \varepsilon_d = \frac{1}{2} \left[ \frac{1}{2} - \frac{1}{2} \left( \left| H_N(f) \right|^2 + \left| H_N \left( f - \frac{1}{2} \right) \right|^2 \right) \right]^2 df \] 

(5.31)

where \( H_N(f) \) is the spectrum of a Meyer \( h_n \) filter determined from (5.4) truncated such that an \( N \)th order sequence is produced. Finally, the signal-to-distortion ratio in dB is defined by

\[ SDR = -10 \cdot \log_{10}(\varepsilon_d) \] 

(5.32)

Thus, with (5.31) in (5.32), we can evaluate the filter design and set the order \( N \). For the case of the square root raised cosine filter, Figures 5.11 and 5.12 illustrate plots of (5.32) as a function of the filter order for \( \beta = \frac{1}{5} \) and \( \beta = \frac{1}{3} \), respectively. The integration in
was performed numerically. From these figures, it is seen that SDR increases with increasing filter order as expected. That is, since the truncated series expansion for the scaling function converges with increasing $N$, we would expect that asymptotic perfect reconstruction be achievable. We also see that a 2 to 4 dB increase in SDR is achieved with $\beta = \frac{1}{3}$ as compared to $\beta = \frac{1}{5}$. But, by Definition 5.1, the latter has a narrower bandwidth. In addition, for filter orders in excess of 16, the SDR is greater than 30 dB. For typical $\frac{E}{N_0}$ operating points, these values of SDR imply the distortion noise will be well below the thermal noise levels. For the design in Figure 5.7, the SDR was 41.9 dB.

To summarize the design procedure, determine an acceptable SDR level and use it to determine the filter order. Use (5.4) to obtain the $h_n$ filter which is truncated symmetrically about zero such that an $N^{th}$ order FIR sequence results. Thus, using (4.11)-(4.15), the design of an even order filter bank is remarkably simple. But, as we will now discuss it has some shortcomings.

![Figure 5.11. SDR Versus Filter Order For $\beta = \frac{1}{5}$]
5.5 Remarks

At this point we would like to compare the Smith-Barnwell design with the linear-phase approximation design. The advantage of the Smith-Barnwell design is that the design procedure is systematic with considerable flexibility in controlling the design parameters (e.g., cutoff frequency, transition band and sidelobe levels). In addition, the design is an FIR sequence that can truly achieve perfect reconstruction regardless of the order. In particular, the output of an odd order two-channel filter bank with the random sequence of Figure 5.9 as an input would have no distortion other than the group delay predicted by (4.10). The major disadvantage with this procedure is that the resulting filters are asymmetric and therefore not linear phase. On the other hand, the linear phase approximation design introduced in the preceding section can achieve linear phase. Further, this approach is extremely simple to produce designs. But, the procedure is not very flexible in terms of controlling design parameters. That is, an analytical form for the time domain scaling function must be obtainable and for every value of $\beta$ of interest, a
curve such as Figure 5.11 or 5.12 must be generated. The former issue may be quite difficult depending on the complexity of \( \vartheta(x) \). We have \( \vartheta(x) \) for the square root raised cosine case and the filter designs were quite good. We do emphasize that the procedure is an approximation and only achieves perfect reconstruction asymptotically as the filter order increases.

In summary, if linear phase is not essential then the Smith-Barnwell design is recommended. Otherwise, the linear phase approximation design should be used.
6.1 Introduction

The principal advantage of orthogonally multiplexed communication is a decomposition of the data bandwidth. This allows frequency selective processing for effective mitigation of channel distortions. This concept has been established in bandwidth efficient applications. We now wish to establish this feature for spread spectrum applications by extending the waveforms developed in Chapters 2 and 3 and utilizing the efficient implementations discussed in Chapter 4. The only related work in this area appeared very recently [4]. A direct sequence (DS) spread signal having a square chip pulse was applied to traditional MCM implemented directly in analog form. As will be seen in this section, using a DS spread signal in conjunction with a uniform decomposition of the bandwidth is not warranted, since for a given amount of bandwidth expansion, it reduces the available frequency resolution for mitigating narrowband interference. Additionally, a square pulse suffers heavily from spectral leakage and is
generally ineffective against arbitrary narrowband jamming [21]. Indeed in [21], we showed digital filter banks quite effective in the suppression of narrowband interference.

Utilizing the notions of a super-symbol and tiling diagram, a unified approach to spread spectrum communication will be obtained. In particular, dimensionality in time is achieved with classical DSPN using a scaling function pulse shape, dimensionality in time-frequency is obtained with MSM and dimensionality in frequency is obtained with MWM. This unified viewpoint allows us to generically view the receiver as a projection onto the basis functions defining the particular waveform. We are then able to develop a filter which operates on the coefficients of the projections in an optimum manner where optimality is based on maximum signal-to-noise ratio. Representing the additive interference in the coordinates of the modulation, we will study the behavior of the optimum weights. Specifically it will be seen that the optimum weights tend to a uniform weighting when the interference is spread across most coordinates while the optimum weights tend to an excision rule [38] when the interference is concentrated to a small number of coordinates. With this study we are lead to conclude that the spread spectrum waveform should be designed such that the interference is localized to a minimum number of coordinates in the waveform and then to apply an excision rule as the optimum filter. With this approach the interference is completely eliminated with degradation in performance due only to the energy loss from excision. Further, since excision is traditionally performed in the frequency domain, the unified viewpoint taken in this dissertation generalizes excision to the time and time-frequency domains.

6.2 Spread Spectrum Waveform Development

6.2.1 DSPN With Scaling Function Chip Pulse

In Chapters 2 and 3, the new modulation formats were determined by decomposing a single rate QAM signal into a composition of QAM signals operating at lower rates. For MSM, the lower rate signals were dyadically related while for MWM,
the sub rates were equal. Of importance was the fact that the QAM symbols on each pulse were independent thus producing a bandwidth efficiency on the order of 1 BPS/Hz or greater. In particular, each dimension was used to convey new information. In spread spectrum systems, dimensionality is used to create redundancy which can be exploited at the receiver to combat degradations due to channel distortions and unwanted interference [12]. A frequently used waveform is the direct sequence pseudonoise (DSPN) signal defined by [42]

$$m(t) = \sqrt{\frac{E}{T}} \sum_{j=1}^{M-1} a_j \sum_{k=0}^{J-1} c_k \phi\left(M\left(\frac{t-j}{T}\right) - k\right)$$  \hspace{1cm} (6.1)

where \(a_j \in \{-1,+1\}\) is a random binary data sequence, \(c_k \in \{-1,+1\}\) is the pseudonoise (PN) chip sequence known at the receiver, \(T\) is the data symbol period and \(\phi\) is typically an NRZ square pulse. In this dissertation, we will allow \(\phi\) to be any valid scaling function. In addition, the chips in the PN sequence are assumed to be independent. In practice, they would be generated in a pseudo random manner [42][43]. From (6.1), we see that the data symbol is composed of \(M\) orthogonally translated basis functions. Hence, DSPN achieves dimensionality in time which has the tiling diagram representation for a single data symbol illustrated in Figure 6.1 for the specific case \(M=8\).

The implementation of a DSPN system is illustrated in Figure 6.2. At the transmitter, the PN code is applied to each data symbol in accordance with (6.1). The coded sequence is then transmitted across the channel with a chip pulse shape governed by the scaling function. At the receiver, the signal is match filtered and sampled at the chip rate. At this point, the PN code is removed by multiplying these samples by the known PN code. In a traditional system, the \(M\) resulting despread samples associated with a particular data symbol are simply added together in a coherent manner to produce
a soft decision estimate of the data symbol. This coherent combining affords the system processing gain against channel interference. In this dissertation, we generalize this combining in the form of the filter weighting strategy illustrated in Figure 6.2 where the PN code removal and filter weighting have been combined into a single operation. In Section 6.3, we will consider the optimum setting of these weights. In this development, the despread samples for a specific data symbol are represented as an M-dimensional projection coefficient vector or simply projection vector. The enumeration illustrated in Figure 6.1 defines the elements of the vector with respect to their position in the time-frequency plane of the tiling diagram. Further, we will use the channel model illustrated in Figure 6.3 where $j(t)$ is intentional or unintentional interference assumed statistically independent of the AWGN $n(t)$.
6.2.2 Spread Spectrum Multiscale Modulation

In a manner similar to what was done in Chapter 2, we can start with (6.1) and apply the principles of MSM to obtain a spread spectrum waveform which achieves dimensionality in time and frequency. Specifically, let $\phi$ and $\psi$ be the scaling and wavelet functions in an MRA then the new signal is defined by
We will refer to this waveform as spread spectrum MSM (SS-MSM). Now the $a_j$ symbol is composed of $M = 2^J - 1$ basis functions consisting of both scaled and translated versions of the scaling and wavelet functions. The tiling diagram for this waveform is also illustrated in Figure 6.1. Again one data symbol has been illustrated. In the language of Chapter 2, each data symbol corresponds to a super-symbol. It is clear from the figure that dimensionality for a given data symbol period has been achieved in both time and frequency in contrast to DSPN's time-only dimensionality. Also make careful note of the enumeration of the components in the projection vector for SS-MSM. Specifically, the components in this vector are unraveled with respect to their position in the tiling diagram.

Using results from Chapter 2, the implementation for a SS-MSM system is illustrated in Figure 6.4. At a functional level the implementation is similar to that of

\[
m(t) = \sum_{j \in I} a_j \left( \sqrt{2^{-(J-1)}} \frac{E}{T} c_0 \phi \left( \frac{t}{T} - j \right) + \sum_{n=1}^{J-1} \sqrt{2^{-(J-n)}} \frac{E}{T} \sum_{k=0}^{2^n-1} c_{2^n+k} \psi \left( \frac{t}{T} - j - k \right) \right)
\]

(6.2)
DSPN but with some important differences. Besides the use of wavelet transforms, the process of applying and removing the PN code has been complicated due to the multirate nature of the signals in the subbands. In particular, Figure 6.5 illustrates the required processing for applying the PN code in SS-MSM. To produce the multirate signals required at the input of the IDWT, the coded sequence must be multiplexed together in a systematic manner. It is important to note that these multiplexers are not decimators. Namely, a 2:1 multiplexer takes two signals in and produces a twice rate signal out. As shown in Figure 6.5, groups of signals are multiplexed according to the sub channels in
the tiling diagram of Figure 6.1. Similarly, the PN code removal and filter weight application are shown in greater detail in Figure 6.6. Now, the multirate signals out of the DWT must be demultiplexed into $M$ uniform sub channels. The number of demultiplexers is the same as the number of multiplexers in Figure 6.5. Additionally, the process of demultiplexing the multirate signals out of the DWT follows the sub channel partitioning in the tiling diagram of Figure 6.1.
6.2.3 Spread Spectrum M-band Wavelet Modulation

Finally, we can obtain dimensionality in frequency by applying MWM. Namely, let \( \phi \) be an M-band scaling function and \( \psi_n \) (\( n=1,...,M-1 \)) be wavelet functions in an M-band wavelet system where \( M=2^J-1 \), then

\[
m(t) = \sqrt{\frac{E}{MT}} \sum_{j=1}^{M} a_j \left( c_0 \phi \left( \frac{t}{T} - j \right) + \sum_{n=1}^{M-1} c_n \psi_n \left( \frac{t}{T} - j \right) \right)
\]  

(6.3)

We will refer to this waveform as spread spectrum MWM (SS-MWM). The M basis functions composing a data symbol are now frequency translated as indicated by the tiling diagram in Figure 6.1. It should be noted that the PN sequence is not a necessity in the definition of (6.3). The sequence appears in order to whiten the interference component of the channel model. With dimensionality in frequency, we can rely on the fact that an M-band wavelet system based on a Nyquist pulse constitutes a valid Karhunen-Loeve expansion (see Appendix B). The Nyquist pulse is obtained from the square root raised cosine pulse considered in Chapter 5 with \( \beta=0 \). Furthermore, even though the optimum filter to be developed performs best when the projection coefficients of the interference are uncorrelated, the filter is general and can account for any cross correlation. In practice with \( \beta \neq 0 \), correlation will only exist between adjacent bands. But, to maintain the unified treatment in this dissertation we will keep the PN sequence in the definition. In addition, the general form of (6.3) makes it possible to extend the waveform to code division multiple access [42]. Using results from Chapter 3, the implementation for a SS-MWM system is illustrated in Figure 6.7.

6.2.4 Processing Gain And Dimensionality

From the definitions of the spread spectrum waveforms, the dimensionality in each case is M and we see that the same data symbol occupies each dimension in a
Figure 6.7. Implementation Of SS-MWM System

particular signaling interval. Further, using the results of Chapters 2 and 3, the bandwidth requirement of each waveform is the same. Consequently, for a given data rate, we have increased the bandwidth requirements by a factor of $M$. This apparent redundancy provides the interference suppression capability of the waveform. In particular, mitigation performance improves with increasing dimensionality $M$. In the classical sense, the dimensionality is called the bandwidth expansion factor and is taken to be the processing gain of the spread spectrum system [12]. But since the bandwidth expansion and processing gain coincide only in a very special case, we will delineate the two parameters. As will be seen, employing the optimum filter will result in processing gains much greater than the bandwidth expansion factor. Further, the achievable processing gain will also depend on the specific form of the interference. Additionally, in this dissertation, we will use the more general term of dimensionality as opposed to bandwidth expansion since dimensionality in the particular case of SS-MSM is somewhat obscured between the time and frequency domains.
6.3 Optimum Filter Development

6.3.1 Generic Receiver Processing

We now turn to the development of the optimum filter. From the waveform definitions and Figures 6.2, 6.4 and 6.7, the receiver can be generically viewed as the projection of the received signal upon the basis functions defining the particular waveform over one data symbol. In particular, the despread samples are merely the inner products of the received signal with each basis function followed by multiplication with the appropriate PN chip. Consequently, in the development to follow we need not be concerned with the specific form of the basis functions upon which projections are being obtained. The received signal can thus be written as

\[ r(t) = m(t) + j(t) + n(t) \]  

where \( m(t) \) is one of the three modulations discussed in the previous section, \( j(t) \) is an additive interference signal and \( n(t) \) is the thermal noise component. We can now express the projection coefficients of the received signal for the \( a_j \) symbol after PN code removal as the \( M \) dimensional column vector

\[ r = a_j \sqrt{\frac{E}{M}} \, U + J + N \]  

where \( U \) is the ones vector. The vector \( J \) is given by

\[ J = [c_0 J_0, \ldots, c_{M-1} J_{M-1}]^T \]  

where \( J_k \) is the inner product of \( j(t) \) with the \( k \)th basis function defining the waveform. The vector \( N \) is given by
where $N_k$ is the inner product of $n(t)$ with the $k$th basis function defining the waveform.

6.3.2 Filter Weights

To determine the filter weights, we must first address the criterion for optimality. Since communication performance is determined by symbol error rate which is largely based on signal-to-noise ratio (SNR), a maximum SNR criteria will be employed. The problem is now to determine a weight vector, $\mathbf{W}$, such that the signal-to-noise ratio in

\[ \hat{r} = \mathbf{W}^T \mathbf{r} \]  

is maximized where $T$ denotes transpose. The output SNR is given by

\[ SNR = \frac{E}{M} \frac{|\mathbf{W}^T \mathbf{U}|^2}{\mathbf{W}^T \mathbf{R}_u \mathbf{W}} \]  

where: $\mathbf{R}_u = \mathbf{R}_J + \mathbf{R}_N$

$\mathbf{R}_J$ and $\mathbf{R}_N$ are the correlation matrices of the projection vectors for the interference and noise components, respectively. This problem was solved by Applebaum in the field of sensor arrays [23]. The optimum weights are

\[ \mathbf{W}_{opt} = \mu \mathbf{R}_u^{-1} \mathbf{U} \]
where \( \mu \) is an arbitrary constant. The constant in the solution is a result of the lack of uniqueness in maximizing SNR. Using (6.10) in (6.9), the maximum SNR becomes

\[
SNR_{\text{opt}} = \frac{E}{M} U^T R_u^{-1} U
\]

Note from (6.10) and (6.11) that the generality of the filter can take into account correlation across the dimensions defining the specific waveform. Also, the filter has the maximal ratio combiner [4] as a special case.

6.3.3 Performance Analysis

To evaluate the performance of the filter, we must characterize the statistics of the thermal noise and interference terms. The thermal noise, \( n(t) \), is assumed to be zero mean AWGN and independent of the PN sequence. The binary PN sequence is assumed to be statistically independent with each chip being zero mean and unit variance. Let \( \{ \varphi_k(t) \}_{0 \leq k \leq M-1} \) arbitrarily denote the orthonormal basis set for a specific data symbol. Then, the \( k \)th projection coefficient of \( n(t) \) is given by

\[
N_k = \int_{-\infty}^{\infty} n(t) \varphi_k(t) dt
\]

Since \( n(t) \) is zero mean, by linearity, the coefficient is zero mean. Also by linearity, the coefficient is Gaussian [24]. Since \( n(t) \) is AWGN, its correlation function is \( \frac{N_0}{2} \delta(\tau) \) where \( \delta(\tau) \) is the Dirac delta function. Using this with (6.12) and the orthonormal property of the basis functions, we have that

\[
E[N_j N_k] = \frac{N_0}{2} \delta_{jk}
\]
where $\delta_{jk}$ Kronecker delta function. $E\{\}$ is statistical expectation where it should be apparent from the context whether we are referring to signal energy or expectation. Thus, we have that

$$E\{c_j N_j c_k N_k\} = E\{c_j c_k\} E\{N_j N_k\}$$

(6.14)

Using the statistics of the noise and PN sequence, (6.14) reduces to

$$E\{c_j N_j c_k N_k\} = \frac{N_0}{2} \delta_{jk}$$

(6.15)

The correlation matrix of $N$ now becomes

$$R_N = \frac{N_0}{2} I$$

(6.16)

where $I$ is the MxM identity matrix.

To characterize the interference, we assume the interference and the PN sequence are statistically independent. Now, the $k^{th}$ projection coefficient of $j(t)$ is given by

$$J_k = \int_{-\infty}^{\infty} j(t) \phi_k(t) dt$$

(6.17)

Since we are interested in understanding the behavior of the filter under different interference conditions with respect to the different waveforms being developed, the interference will be modeled in the coordinates of the basis functions defining the particular waveform. Interference energy not in the coordinates of the spread spectrum
waveform is not relevant to data detection [44]. In particular, we are assuming the interference \( j(t) \) is a stationary Gaussian random process such that \( J_k \) is zero mean with

\[
e^\{JJ\}_{kk} = \sqrt{\alpha_j \alpha_k E_j \rho_{jk}}
\]

(6.18)

where \( \alpha_k \) is the percent interference energy in the \( k \)th coordinate, \( E_J = P_J T \) is the total interference energy in a data symbol period, \( P_J \) is the interference power and \( \rho_{jk} \) is the cross correlation coefficient between the \( j \)th and \( k \)th coordinates. A Gaussian assumption for the despread interference is common in analyses [4][12][42]. By linearity of (6.17), \( J_k \) is a Gaussian random variable. Further, we have the following constraint on \( \alpha_k \)

\[
\sum_{k=0}^{M-1} \alpha_k = 1
\]

(6.19)

Similar to (6.14), we have that

\[
e^\{ccJ\}_{jk} = \left\{ \begin{array}{ll}
\alpha_k E_j, & j = k \\
0, & otherwise
\end{array} \right.
\]

(6.20)

Using the statistics of the interference and PN sequence, (6.20) reduces to

\[
e^\{ccJ\}_{jk} = \left\{ \begin{array}{ll}
\alpha_k E_j, & j = k \\
0, & otherwise
\end{array} \right.
\]

(6.21)

From (6.21), the PN sequence is seen to decorrelate the interference. Hence, the \( M \times M \) correlation matrix of \( J \) is diagonal, namely,
Using (6.16) and (6.22), we have that

\[
R_f = P_J T \begin{pmatrix}
\alpha_0 & 0 & 0 & 0 \\
0 & \alpha_1 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \alpha_{M-1}
\end{pmatrix}
\]

(6.22)

Using (6.16) and (6.22), we have that

\[
R_u = \begin{pmatrix}
\alpha_0 P_J T + \frac{N_0}{2} & 0 & 0 & 0 \\
0 & \alpha_1 P_J T + \frac{N_0}{2} & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \alpha_{M-1} P_J T + \frac{N_0}{2}
\end{pmatrix}
\]

(6.23)

Using (6.23) in (6.10), the optimum weight vector becomes

\[
W_{opt} = \mu \begin{bmatrix}
1 \\
\frac{1}{\alpha_0 P_J T + \frac{N_0}{2}} \\
\vdots \\
\frac{1}{\alpha_{M-1} P_J T + \frac{N_0}{2}}
\end{bmatrix}
\]

(6.24)

Using (6.23) in (6.11) yields

\[
SNR_{opt} = E \sum_{k=0}^{M-1} \frac{1}{\alpha_k P_J T + \frac{N_0}{2}}
\]

(6.25)
Since the interference will typically be concentrated in a fraction of the available coordinates, particularly with hostile interferences, we will make the following definitions. Recall from Definition 4.1 that

\[ Z_{M-1} = \{ n \in \mathbb{Z} : n = 0, \ldots, M-1; M < \infty \} \]  

(6.26)

Now define \( N_K \) such that \( N_K \subset Z_{M-1} \) and its cardinality \( |N_K| = K \leq M \). Further, define \( \alpha_k = 0 \quad \forall k \in Z_{M-1} \setminus N_K \). In particular, \( N_K \) represents the set of coordinate indices containing interference energy. With these definitions, (6.25) becomes

\[ SNR_{opt} = \frac{E}{M} \sum_{k \in N_K} \frac{1}{\alpha_k P_j T + \frac{N_0}{2}} + \frac{2E}{N_0} (1 - \rho) \]  

\[ \rho = \frac{K}{M} \]  

(6.27)

where \( \rho = \frac{K}{M} \). (6.27) is the desired general result.

In order to compare the optimum result with some well known suboptimum weighting strategies, let \( \alpha_k = \frac{1}{K} \quad \forall k \in N_K \). That is, the interference is uniformly distributed across a subset of the coordinates. Then, (6.27) becomes

\[ SNR_{opt} = \frac{\rho^2}{P_j} + \frac{\rho}{P_S M + \frac{2E}{N_0}} + \frac{2E}{N_0} (1 - \rho) \]  

\[ \rho = \frac{K}{M} \]  

(6.28)

where \( P_S = \frac{E}{T} \) is the signal power. If in addition \( K=M \), then with \( \mu = \frac{P_j T}{M} + \frac{N_0}{2} \), the weight vector in (6.24) reduces to

\[ W_{opt} = U \]  

(6.29)
(6.29) is called uniform weighting and is the traditional weighting strategy. Based on the assumptions in reaching (6.29), it is an intuitively consistent result. With \( \rho = 1 \) in (6.28), the performance with a uniform weighting is given by

\[
\frac{1}{P_j J} \frac{1}{P_SM^{\frac{1}{2}}} + \frac{2E}{N_0}
\]

which is precisely the performance of traditional DSPN against a single tone interference [43]. From (6.30) with \( P_J = 0 \), we get AWGN performance. Also when the thermal noise is ignored [42], we see that the processing gain is \( M \) where this gain is defined by

\[
PG = \frac{SNR_{\text{spreading}}}{SNR_{\text{non-spreading}}}
\]

The SNR when non-spreading is simply \( SNRU \) evaluated at \( M = 1 \). Thus, the rule of thumb processing gain based on the dimensionality occurs only in a very special case.

Another suboptimum strategy is termed excision. Consider the \( k^{\text{th}} \) component of (6.24)

\[
(W_{\text{opt}})_k = \frac{1}{\alpha_k P_J + \frac{N_0}{2}}
\]

If the interference energy is very large in this coordinate then (6.32) goes to zero and the filter is effectively excising this coordinate. Thus, the excision weighting strategy is given by
With the above definitions, this is equivalent to

$$W_{exc} = \begin{cases} 1 & \kappa = 0 \\ 0 & \kappa \neq 0 \end{cases}$$  \hspace{1cm} (6.33)$$

Then using (6.23) and (6.34) in (6.9), the performance of excision weighting is given by

$$SNR_{exc} = \frac{2E}{N_0(1-\rho)} \hspace{1cm} (6.35)$$

where \( \rho \) is as defined above and can be interpreted as the fractional number of coordinates excised. From, (6.35) and (6.28), we see that excision weighting well approximates the optimal weights when \( \rho \) is small, that is, when the interference is localized to a small number of coordinates in the waveform. Also from (6.35), performance is not a function of the interference power. We can therefore expect significant processing gain since all the interference power will be eliminated. We point out that excision was originally used with DSPN and FFT processors to transform the time domain signal into the frequency domain for easy identification of narrowband interference [39]-[41]. This scheme suffers from both distortion and energy loss. On the other hand, since the waveforms in this chapter have been defined in their particular domain, excision is an inherent part of detection and no additional processors are required. Degradation now is solely due to energy loss. Further, the unified framework taken in this chapter generalizes excision to the time, time-frequency and frequency domains.
For the first comparison of the weighting strategies, we have evaluated (6.28), (6.30) and (6.34) for three different scenarios based on interference-to-signal power ratio \( \left( \frac{P_J}{P_S} \right) \) as a function of the percentage, \( \rho \), of waveform coordinates containing interference energy. In each case, the \( \frac{E_B}{N_0} \) was 10 dB and the dimensionality \( M \) was 1000. The second comparison is based on probability of a bit error (\( P_B \)). Since the interference was assumed Gaussian we can use (2.26) of Chapter 2 to yield

\[
P_B = Q\left(\sqrt{\text{SNR}}\right)
\]

where SNR is the signal-to-noise ratio of the particular weighting strategy. For this second comparison, the dimensionality was again set to 1000 and \( \rho \) was fixed at .4. That is, 40% of the waveform coordinates contain interference.

For the first scenario, \( \frac{P_J}{P_S} \) was 10 dB and represents a low interference scenario where the chosen dimensionality is expected to mitigate the interference. The SNR results are shown in Figure 6.8 while the results for \( P_B \) are shown in Figure 6.9. In Figure 6.8, we have shown the SNR when not spreading so that the actual processing gain can be ascertained from (6.31). Also shown is the SNR in AWGN, which for these cases is \( \frac{2E_B}{\sqrt{N_0}} \). From this figure, the performance of uniform weighting is close to optimum while that of excision weighting is rapidly deteriorating as the number of coordinates excised increases. The deteriorated performance of excision can also be seen in Figure 6.9 for \( P_B \), keeping in mind that \( \rho=.4 \). But in this figure we see that at large values of \( \frac{E_B}{\sqrt{N_0}} \) uniform performance is beginning to degrade with respect to optimum. In Figure 6.10, we have expanded the vertical scale of Figure 6.8 to make clear the behavior in this critical region. In particular, at low values of \( \rho \), excision is close to optimum until approximately .18. At this point, uniform performance is superior to
Figure 6.8. SNR Versus $\rho$ For $\frac{P_J}{P_S} = 10$ dB

Figure 6.9. $P_B$ Versus $\frac{E_B}{N_0}$ For $\frac{P_J}{P_S} = 10$ dB And $\rho=.4$
excision but not as good as optimum. As $\rho$ increases, optimum performance approaches uniform while excision performance degrades rapidly. Finally, from Figures 6.8 and 6.10, we see that the processing gain for uniform weighting is just over 22 dB. The processing gain with excision weighting is approximately 22.5 dB at low $\rho$ but degrades considerably for larger $\rho$. But excision is still better than an unspread system even out to values of $\rho$ extremely close to 1. The processing gain for optimum weighting is maintained between 22 dB and the maximum achievable 23 dB. Recall, the processing gain based on dimensionality is 30 dB in this case.

For the second scenario, $\frac{P_I}{P_S}$ was 20 dB. This represents a moderate interference condition but the available dimensionality should be sufficient to provide reliable communication. The results for SNR are shown in Figure 6.11 while the PB results are shown in Figure 6.12. From Figure 6.11, we see that uniform performance has

Figure 6.10. Expanded View: SNR Versus $\rho$ For $\frac{P_I}{P_S} = 10$ dB
Figure 6.11. SNR Versus $\rho$ For $\frac{P_J}{P_S} = 20$ dB

Figure 6.12. $P_B$ Versus $\frac{E_B}{N_0}$ For $\frac{P_J}{P_S} = 20$ dB And $\rho = 0.4$
degraded relative to the first scenario and that excision is now performing better out to about $\rho=0.65$. At which point, uniform performance is superior with optimum performance approaching it as $\rho$ approaches one. Examining Figure 6.12 with $\rho=0.4$, we also see uniform performance is better at low $\frac{E_b}{N_0}$. Unfortunately, uniform performance is leveling off and appears to be approaching a constant probability for large $\frac{E_b}{N_0}$. This is indeed the case as can be seen from (6.30). Specifically, as $\frac{E_b}{N_0}$ increases, SNR approaches the constant $\frac{P_s M}{P_J}$ which in turn implies $\text{PB}$ is approaching a constant. This is a very undesirable feature with the uniform weighting strategy. From Figure 6.11, we see that uniform weighting achieves a processing gain of about 28 dB while for low values of $\rho$, optimum and excision weighting achieves processing gains of approximately 32 dB. At large $\rho$, optimum weighting approaches about 28 dB while excision weighting falls considerably. We note from Figure 6.12, that even in this moderate interference condition, loss from theoretical using optimum weights is only on the order of 1.7 dB at a typical operating point of $10^{-4}$. For excision weighting, the loss is 2.2 dB which can be computed directly from (6.35) with $\rho=0.4$. These losses are quite small in light of the mitigation performance which has been obtained. With the final scenario, the gains are even more pronounced.

For the last scenario, $\frac{P_J}{P_s}$ was 40 dB. This represents a high interference condition where the available dimensionality is not likely to provide reliable communication. The results for SNR are shown in Figure 6.13 while the $\text{PB}$ results are shown in Figure 6.14. From Figure 6.13, we see that uniform weighting is well below the other two curves but does achieve exactly 30 dB processing gain. On the other hand, over most of the range for $\rho$, excision and optimum performance are nearly identically. We note that very close to $\rho=1$, excision performance will go to zero (linear scale) while the optimum approaches uniform performance. The processing gains for optimum and
Figure 6.13. SNR Versus $\rho$ For $\frac{P_j}{P_s} = 40$ dB

Figure 6.14. $P_B$ Versus $\frac{E_B}{N_0}$ For $\frac{P_j}{P_s} = 40$ dB And $\rho = .4$
excision are well above the 30 dB mark based on dimensionality. In fact at low values of \( p \), nearly 60 dB of gain has been achieved. Even for values of \( p \) out to .8, the processing gain is nearly 50 dB. Thus even in this high interference condition, reliable communication is possible. This is further demonstrated in Figure 6.14 where PB performance is essentially identical for optimum and excision weighting. Since excision performance is not dependent upon the interference power as indicated in (6.35), the loss in Figure 6.14 is 2.2 dB which is a small price to pay for mitigating such a strong interference. Also seen in Figure 6.14 is the fact that for this case uniform weighting is completely ineffective.

### 6.3.4 Remarks

The preceding performance analysis has demonstrated that when the interference is localized to a minimum number of coordinates in the spread spectrum waveform, an excision weighting well approximates the optimum strategy achieving significant levels of processing gain. Further, at moderate to high interference conditions, a large number of components can be excised while maintaining good performance. For a low interference condition which is spread across most coordinates, a uniform filter weighting performed better but the achievable processing gain was modest. We are therefore lead to conclude that the best weighting strategy is an excision rule with a spread spectrum waveform which localizes the interference to a minimum number of coordinates.

We should point out at this time that the weights were assumed to be known at the receiver in this analysis. Clearly, this is not a problem with uniform weighting. Namely, uniform weighting is a nonparametric technique that does not rely on the statistics of the received signal. On the other hand, the optimum weights are highly parametric relying on detailed knowledge of the interference and noise correlation matrices including stationarity across coordinates. With excision, some knowledge of the statistics of the received signal are required but not to the same degree. In fact, typical excision strategies
operate on the magnitude of the received projection coefficients to ascertain the presence of interference. Since this can be done over a small number of data symbols, excision should be effective under time varying conditions. Hence, we are again lead to an excision strategy for not only superior performance but for practical determination of the filter weights.

6.4 Interference Projections On Waveform Basis Functions

In this section, we are interested in seeing the behavior of two common interference sources when projected on the basis functions describing DSPN, SS-MSM and SS-MWM. The two interference sources are the time domain impulse and the tone. These represent extreme signals with regard to time-frequency signal analysis [8]. The dimensionality $M$ was 64 and we used the analysis tree filter bank described in Chapter 4 to determine the projection coefficients. The $h_n$ filter used was the linear phase square root raised cosine FIR filter of Figure 5.7. The plots to be presented are 3-D plots. The vertical axis is the magnitude of the projection coefficient. The axis denoted chips is the indices for the elements of the projection vector which follow the enumeration of the tiling diagram of Figure 6.1 with the zero element being the farthest left. Units on this axis are dependent on the waveform. For DSPN, the units are in time while for SS-MWM, the units are in frequency. For SS-MSM, the units must be interpreted in time-frequency in accordance with the tiling diagram. Since a projection vector is collected each data symbol period, the remaining axis has units of symbol period.

6.4.1 DSPN

Figure 6.15 illustrates the time impulse projected upon the basis functions defining DSPN. As expected since DSPN achieves dimensionality in time, the impulse is well localized in time as indicated by the small number of coefficients in the projection vector exhibiting energy. Figure 6.16, illustrates the tone projected upon the DSPN basis functions. After the initial group delay of the filter bank, we see that the tone is spread
over all the coordinates. The tiling diagram is useful for explaining this operation as indicated in Figure 6.17. From this figure, a time domain impulse will intersect with a minimum number of components. A tone, which is an impulse in frequency, will intersect with all the components as we saw in the earlier figure.

Figure 6.15. Projection Of Time Impulse On DSPN Basis Functions

Figure 6.16. Projection Of Tone On DSPN Basis Functions
6.4.2 SS-MWM

Figure 6.18 illustrates the time impulse projected upon the basis functions defining SS-MWM. This case is the converse of the DSPN case. Namely, since SS-MWM achieves dimensionality in frequency, the impulse is not well localized in time and interference energy is spread across all the components. Figure 6.19 illustrates the tone projected upon the SS-MWM basis functions. Here, the interference is localized to a small number of coordinates as expected. Again, the tiling diagram is useful for explaining this operation as indicated in Figure 6.20. From this figure, a time domain impulse will intersect with all the components. A tone which is an impulse in frequency will intersect with a minimum number of components.
Figure 6.18. Projection Of Time Impulse On SS-MWM Basis Functions

Figure 6.19. Projection Of Tone On SS-MWM Basis Functions
Figure 6.20. SS-MWM Tiling Diagram Representation Of Interferences

6.4.3 SS-MSM

Figure 6.21 illustrates the time impulse projected upon the basis functions defining SS-MSM. From this figure, we see that more components are corrupted than for DSPN but far less than what was seen with SS-MWM. Figure 6.22 illustrates the tone projected upon the SS-MSM basis functions. Similarly, more components are corrupted than with SS-MWM but far less than with DSPN. Again, the tiling diagram is useful for explaining this operation as indicated in Figure 6.23. From this figure, a time domain impulse will intersect with some but not all components of the tiling diagram keeping in mind the enumeration with respect to the projection vector. In fact, we see that the lower frequency components are always corrupted with an impulse while high frequency components are being missed or hit depending upon the position of the impulse within the data symbol period. Referring back to Figure 6.21, this is precisely what is occurring.
Namely, the lower elements of the projection vector have been corrupted while higher elements are intermittently corrupted. Similarly with a tone, only a portion of the
components will be corrupted as indicated in Figure 6.23. In this case, the location of the tone within the band is quite important.

6.4.4 Remarks

Based on the results of the previous and current sections, if one knew a priori that the interference was impulsive then DSPN with time domain excision is the recommended system. On the other hand, if the interference was known to be narrowband then SS-MWM with frequency domain excision is recommended. But, as was seen, SS-MSM appears to be an effective compromise for these two extreme interference sources. As a result, with little a priori knowledge concerning the interference, SS-MSM with time-frequency excision should be considered.
Chapter 7

Conclusions And Future Research

7.1 Summary Of Results

In this dissertation a unified approach to orthogonally multiplexed communication was put forth. In particular, we first used orthonormal dyadic wavelet functions to defined a new waveform called multiscale modulation (MSM). This waveform achieves a nonuniform partitioning of the data bandwidth. All the fundamental characterization of MSM was established including the power spectral density, tiling diagram, bandwidth efficiency and probability of error performance. It was then shown how discrete wavelet transforms provided for an efficient digital implementation. These transforms were seen as special cases of nonuniform filter banks. We next used orthonormal M-band wavelet functions to define a communication waveform which provides for a uniform decomposition of the data bandwidth. The waveform was termed M-band wavelet modulation (MWM). Again all fundamental characterization of the waveform was determined. Additionally, an efficient implementation based on tree structured uniform analysis-synthesis filter banks was introduced.
Treating uniform and nonuniform filter banks in a unified manner, we then considered these structures in greater detail. In particular, it was found that using realizable FIR filters created subtle issues in practical implementations due to their generally nonzero group delay. For the known odd order two-channel filter bank, we derived the realizable J-channel nonuniform filter bank. We then showed that, in contrary to the present literature, two-channel filter banks having even order FIR filters were possible, thus admitting linear phase. With this established, the general realizable J-channel nonuniform filter bank was derived. We then found that when converting an analysis-synthesis bank to a synthesis-analysis bank as required for the communication application, a problem surfaced. This problem was again attributable to the nonzero group delay of the filters in conjunction with decimation operators. In particular, distorted signals occurred at each channel output even if the filters were chosen for perfect reconstruction. We then showed that the solution to this problem was a judicious insertion of variable unit delays at each node in the tree structured analysis filter bank at the receiver. In particular, a unit delay was inserted based on the binary expansion of the filter order for the odd order case. For even order, the expansion was performed on N+1 where N is the even filter order. Thus, a delay was inserted for every one appearing in the binary expansion while no delay was used when a zero occurred. Finally, we showed that the subband signals out of a tree structured analysis filter bank followed a gray coded ordering in contrast to an expected linear frequency ordering.

We next considered pulse shape design, that is, specification of the scaling function and lowpass digital filter which completely determine the efficient implementations of the waveforms. We first made an important connection between the Meyer scaling function and the Nyquist communication pulse. This allowed us to derive the square root raised cosine dyadic wavelet. With this connection we were also able to generalize the domain of definition for the Meyer functions. This further lead to a proof
that the Meyer scaling function is an M-band scaling function. A corollary to this proof was a bound relating the rolloff parameter and the multiplicity. An additional corollary provided a means for determining the M-band low pass digital filter. We then developed a function approximation design for even order linear phase FIR filters for use in the new even order filter bank. An error analysis was conducted to ascertain the deviation from perfect reconstruction and set the filter order. Finally, a comparison between this new design and the Smith-Barnwell design was undertaken. It was concluded that if linear phase is not essential, the Smith-Barnwell design is preferred. Otherwise, the linear phase approximation design is recommended.

As an application of the newly developed waveforms, we introduced a unified framework for spread spectrum communication. In particular, dimensionality in time was obtained with DSPN having a scaling function chip pulse, dimensionality in time-frequency was obtained with spread spectrum MSM and dimensionality in frequency was achieved with spread spectrum MWM. It was found that the receiver could generically be viewed as a projection of the received signal upon the basis functions defining a data symbol. Considering an additive channel consisting of interference statistically independent from the AWGN, we then developed an optimum filter for maximizing SNR which operated on the vector formed from the projection coefficients. Modeling the interference in the coordinates of the spread spectrum waveform a theoretical analysis was conducted. Specifically, it was shown that special cases of the optimum filter were the traditional uniform weighting and the excision weighting which is customarily done in the frequency domain. Thus, the unified framework generalized excision to the time and time-frequency domains. We next performed a comparative study between the optimum, uniform and excision weighting strategies under low, medium and high interference conditions. The comparison was based both on output SNR and probability of bit error. In a low interference condition, the uniform weighting closely approximated
optimum. But for medium and high interference, the excision strategy performed closer to optimum and the processing gains were well above the gain estimated by the dimensionality. In fact, reliable communication was found to be possible even when a relatively large number of coordinates were excised. In all cases, if the interference was localized to a minimum number of waveform coordinates then excision well approximated the optimum strategy. Consequently, we were lead to conclude that the spread spectrum waveform should be design such that the interference is concentrated to a minimum number of waveform coordinates and then to apply an excision strategy for data detection. We finally considered the behavior of two common interference sources projected on the basis functions defining DSPN, SS-MSM and SS-MWM. The two interferences were the time domain impulse and the tone which is theoretically an impulse in frequency. Thus, the two interferences represented extremes which could be encountered in a practical communication system. We found that as expected the DSPN waveform localized the impulse well but the tone was distributed across all the coordinates. Conversely, the SS-MWM waveform localized the tone well but the impulse was distributed across the coordinates. With SS-MSM, a compromise was seen. Namely, both the impulse and the tone were reasonably localized. Thus we were lead to conclude that if an impulsive channel was anticipated, DSPN with time domain excision is recommended. On the other hand, for narrowband interference, SS-MWM with frequency domain excision is recommended. But, if little knowledge about the expected interference condition was known then SS-MSM was recommended with time-frequency excision.
7.2 Recommendations For Further Study

This dissertation has developed the foundations of two new modulation formats and has successfully applied them to a pertinent application. There are numerous directions that can be taken from this point both theoretical and practical. A few of these are now listed.

Wavelet Packets

Wavelet packets provide a general tiling diagram decomposition. Thus, the waveforms in this dissertation can be viewed as special cases of wavelet packets. One could expect with this general decomposition of the data bandwidth that even better effectiveness against channel impairments could be achieved. For example, greater mitigation capability against interfering signals in spread spectrum. But, with this generality comes an increase in the computational load which must be traded for any improvements in performance.

Synchronization: Carrier, Symbol and Automatic Gain Control

Throughout this dissertation synchronization has been assumed. In order to produce a working prototype, approaches to synchronization must be developed and analyzed. This could include both open loop approaches based on parameter estimation techniques and closed loop approaches for highly dynamic conditions. Gain control is a particularly important subject area for the spread spectrum application. That is, the results in this dissertation have shown that very strong interferences can be mitigated. But these strong signals could cause significant problems for RF processing and A/D conversion.
Systematic Design Methodology For Even Order Filter Bank

The design technique based on function approximation of the Meyer scaling function is very simple to use and achieved good results. But the design is not very flexible in terms of controlling design parameters. A systematic design methodology like the Kaiser window design is needed. In addition, more analytical expressions for Meyer wavelet functions are needed which can be evaluated for possible filter designs.

Detailed Performance With Specific Interference Sources

The analysis conducted in this dissertation modeled the interference in the coordinates of the spread spectrum waveform. Now that the general behavior of the filter weighting strategy is understood a more detailed and specific analysis in needed which considers specific interference models.

Joint Fading/Interference Mitigation

The filter developed in Chapter 6 to mitigate additive interference can be generalized to optimally consider both frequency selective fading and additive interference jointly. These distortions are typically treated separately. A thorough analysis is needed to understand the relative impact each distortion has on the other.

Filter Weight Estimation And Adaptation For Multichannel Spread Spectrum Receivers

The analysis presented in this dissertation assumed the filter weights were known by the receiver. Practical approaches to estimating the filter weights are therefore needed. Since the problem formulation resembles the sensor array problem, a starting point would be to consider existing techniques from this field. Adaptability of the estimation procedure should be addressed so that time varying interference can be
effectively mitigated. Also a detailed analysis should be performed to understand the effects that the noisy weights have on theoretical performance.

**Error Control Coding**

For the case of MSM, the multirate nature of the signals invites the possibility of new coding strategies. Using the notion of a supersymbol, one could develop a coding strategy which could compensate for frequency selective distortions in the channel. Also, some multidimensional TCM schemes get their dimensionality in time by treating successive symbols together. As an alternative, again using the notion of a super symbol, one could gain dimensionality in time and frequency to develop TCM approaches which compensate for channels distortions.

**Optimum Receiver Structures In Bandlimited Channels**

In this dissertation, we did not consider bandlimiting effects. The optimum receiver in bandlimited channels for multichannel receivers is known [49]. But, the signals in the subchannels are generally of the same rate. The multirate nature of the signals defining MSM pose a new challenge. The optimum structure is needed as well as practical suboptimum approximations including detailed performance evaluations.

**Spread Spectrum Multiple Access**

The spread spectrum waveforms developed in this dissertation can be extended to multiple access by giving each user an independent PN code. This would produce a multi-user communication system which would be robust to additive interference. Detailed performance analyses are needed to characterize the multiple access interference with and without additive interference.
Bibliography


September 1990.


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Appendix A

Square Root Raised Cosine Dyadic Wavelet

A.1 Derivation

Define $Y(f)$ to be the left hand side of (5.6), namely

$$Y(f) = \Psi(f)e^{j\phi} = \begin{cases} 
\sin\left(\frac{\pi}{4}\left(\frac{|f|-1}{\beta}+1\right)\right), & \frac{1-\beta}{2} \leq |f| \leq \frac{1+\beta}{2}, \\
1, & \frac{1+\beta}{2} \leq |f| \leq (1-\beta) \\
\cos\left(\frac{\pi}{4}\left(\frac{|f|-1}{\beta}+1\right)\right), & (1-\beta) \leq |f| \leq (1+\beta) \\
0, & \text{otherwise}
\end{cases}$$  \hspace{1cm} (A.1)

We will first find $y(t)$ and then use a familiar transform property to obtain $\psi(t)$. Now, since (A.1) is symmetric, the inverse Fourier transform takes the form

$$y(t) = 2 \int_{0}^{\infty} Y(f) \cos(2\pi ft) df$$  \hspace{1cm} (A.2)
Using (A.1) in (A.2), we identify three nontrivial integrals corresponding to each disjoint frequency region in (A.1). By linearity, we will consider each integral individually. Thus, (A.2) becomes

\[ y(t) = I_1 + I_2 + I_3 \]  
(A.3)

where

\[ I_1 = 2 \int_{\frac{1+\beta}{2}}^{1+\beta} \sin \left( \frac{\pi}{4} \left( \frac{2f - 1}{\beta} + 1 \right) \right) \cos(2\pi ft) df \]  
(A.4)

\[ I_2 = 2 \int_{\frac{1+\beta}{2}}^{1-\beta} \cos(2\pi ft) df \]  
(A.5)

\[ I_3 = 2 \int_{\frac{1+\beta}{2}}^{1-\beta} \cos \left( \frac{\pi}{4} \left( \frac{f - 1}{\beta} + 1 \right) \right) \cos(2\pi ft) df \]  
(A.6)

Starting with the first integral and using a familiar trigonometric formula, (A.4) becomes

\[ I_1 = \int_{\frac{1+\beta}{2}}^{\frac{1-\beta}{2}} \sin \left( \frac{\pi}{4} \left( \frac{2f - 1}{\beta} + 1 \right) + 2\pi ft \right) df + \int_{\frac{1+\beta}{2}}^{\frac{1-\beta}{2}} \sin \left( \frac{\pi}{4} \left( \frac{2f - 1}{\beta} + 1 \right) - 2\pi ft \right) df \]  
(A.7)

Performing the integrations, (A.7) becomes
\[ I_1 = \sin(\pi(1+\beta)t) \left( \frac{1}{\pi + 2\pi t} - \frac{1}{\pi - 2\pi t} \right) + \cos(\pi(1-\beta)t) \left( \frac{1}{\pi + 2\pi t} + \frac{1}{\pi - 2\pi t} \right) \]  

(A.8)

Now evaluating the second integral directly yields

\[ I_2 = \frac{1}{\pi t} \left( \sin(2\pi(1-\beta)t) - \sin(\pi(1+\beta)t) \right) \]  

(A.9)

For the third integral, we again use a familiar trigonometric formula to yield

\[ I_3 = \int_{1-\beta}^{1+\beta} \cos \left( \frac{\pi}{4} \left( \frac{f - 1}{\beta} + 1 \right) + 2\pi f \right) \, df + \int_{1-\beta}^{1+\beta} \cos \left( \frac{\pi}{4} \left( \frac{f - 1}{\beta} - 1 \right) - 2\pi f \right) \, df \]  

(A.10)

Performing the integrations, (A.10) becomes

\[ I_3 = \sin(2\pi(1-\beta)t) \left( \frac{1}{\pi - 2\pi t} - \frac{1}{\pi + 2\pi t} \right) + \cos(2\pi(1+\beta)t) \left( \frac{1}{\pi - 2\pi t} + \frac{1}{\pi + 2\pi t} \right) \]  

(A.11)

Combining (A.8), (A.9) and (A.11), we have that \( y(t) \) is given by
\[ y(t) = \cos(\pi(1-\beta)t) \left( \frac{1}{\pi + 2\pi t} + \frac{1}{\pi - 2\pi t} \right) \]

\[ - \sin(\pi(1+\beta)t) \left( \frac{1}{\pi t} - \frac{1}{\pi + 2\pi t} + \frac{1}{\pi - 2\pi t} \right) \]

\[ + \sin(2\pi(1-\beta)t) \left( \frac{1}{\pi t} - \frac{1}{\pi + 2\pi t} + \frac{1}{\pi - 2\pi t} \right) \]

\[ + \cos(2\pi(1+\beta)t) \left( \frac{1}{\pi/4 + 2\pi t} + \frac{1}{\pi/4 - 2\pi t} \right) \]

(A.12)

Equation (A.12) can be simplified to yield

\[ y(t) = \cos(\pi(1-\beta)t) \left( \frac{4\beta}{\pi(1-(4\beta t)^2)} \right) - \sin(\pi(1+\beta)t) \left( \frac{1}{\pi t(1-(4\beta t)^2)} \right) \]

\[ + \sin(2\pi(1-\beta)t) \left( \frac{1}{\pi t(1-(8\beta t)^2)} \right) + \cos(2\pi(1+\beta)t) \left( \frac{8\beta}{\pi(1-(8\beta t)^2)} \right) \]

(A.13)

Combining like terms yields

\[ y(t) = \frac{4\beta t \cos(\pi(1-\beta)t) - \sin(\pi(1+\beta)t)}{\pi(1-(4\beta t)^2)t} + \frac{\sin(2\pi(1-\beta)t) + 8\beta t \cos(2\pi(1+\beta)t)}{\pi(1-(8\beta t)^2)t} \]

(A.14)

Using the delay property of Fourier transforms in accordance with (A.1), we can obtain the wavelet function by
\[ \psi(t) = y\left(t - \frac{1}{2}\right) \]  

(A.15)

Combining (A.14) and (A.15) yields the desired result, namely,

\[
\psi(t) = \frac{4\beta \left(t - \frac{1}{2}\right) \cos \left(\pi (1 - \beta) \left(t - \frac{1}{2}\right)\right) - \sin \left(\pi (1 + \beta) \left(t - \frac{1}{2}\right)\right)}{\pi \left[1 - \left(4\beta \left(t - \frac{1}{2}\right)^2\right)\left(t - \frac{1}{2}\right)\right]}
+ \frac{\sin \left(2\pi (1 - \beta) \left(t - \frac{1}{2}\right)\right) + 8\beta \left(t - \frac{1}{2}\right) \cos \left(2\pi (1 + \beta) \left(t - \frac{1}{2}\right)\right)}{\pi \left[1 - \left(8\beta \left(t - \frac{1}{2}\right)^2\right)\left(t - \frac{1}{2}\right)\right]}
\]  

(A.16)
**Appendix B**

**MWM As A Karhunen-Loeve Expansion**

**B.1 Karhunen-Loeve Expansions**

A Karhunen-Loeve (K-L) expansion is a representation for sample functions from a random process [24][45]. Let \( \{ x_t \}_{t \in \mathbb{R}} \) be a random process on a complete probability measure space \((\Omega, \mathcal{B}, \mathbb{P})\). We are considering sample functions with possibly infinite support. Thus, from [47], we require that the random process be continuous in probability [48] and that almost every sample function have finite energy. Let \( \{ \phi_k(t) \}_{k \in \mathbb{N}} \) be a complete orthonormal set where \( \mathbb{N} \) is the set of natural numbers. A series expansion of a sample function \( x(t) \) is

\[
x(t) = \lim_{n \to \infty} \sum_{k=1}^{n} x_k \phi_k(t)
\]

where:

\[
x_k = \int_{-\infty}^{\infty} x(t) \phi_k^*(t) dt
\]

and where \( l.i.m. \) denotes convergence in the norm of \( L^2(\Omega, \mathcal{B}, \mathbb{P}) \). If for each \( k \), \( \phi_k \) is a solution to the following integral equation
\[ \lambda_k \phi_k(t) = \int_{-\infty}^{\infty} R(t,u) \phi_k(u) \, du \]  
(B.3)

where:  
\[ R(t,u) = E[x(t)x^*(u)] \]  
(B.4)

then

\[ E[x_jx_k^*] = \begin{cases} \lambda_k, & j = k \\ 0, & \text{otherwise} \end{cases} \]  
(B.5)

and (B.1) is called a K-L expansion. In particular, \( \phi_k \) is an eigenfunction with eigenvalue \( \lambda_k \) of the integral equation (B.3). From (B.5), we see the importance of K-L expansions in that the series coefficients are uncorrelated. Further, if the process is Gaussian then the coefficients are Gaussian, hence, independent [24].

**B.2 Nyquist Filter Bank**

In [24], it is shown under certain conditions a DTFT constitutes a K-L expansion, thus, justifying the use of the FFTs in practice. An FFT can be viewed as a uniform analysis filter bank but the subband filters are not particular very good [20]. On the other hand, we have the ideal filter bank with contiguous nonoverlapping subbands. Although unrealizable, this filter bank can be generated from the Nyquist pulse of Chapter 5 with \( \beta = 0 \). It is important to consider this bank since practical realizations try to achieve good subband containment and therefore approximate this ideal case. We now show that this ideal case constitutes a K-L expansion. Let \( \{\phi_k(t)\}_{k \in \mathbb{N}} \) be an orthonormal set such that their Fourier transforms satisfy

\[ \Phi_j(f)\Phi^*_k(f) = 0 \]  
(B.6)
for all \( f \in R \) and \( k \neq j \). Let \( \{x_{t}\}_{t \in R} \) be a stationary random process then with (B.2) in (B.5), we must show that

\[
I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(t-u)\phi_{k}(u)d\phi_{j}^{*}(t)dt = \lambda_{k}\delta_{jk}
\]

(B.7)

The Fourier transform of \( R(\tau) \) is the power spectral density \( S(f) \). Using this, the left side of (B.7) becomes

\[
I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(f)e^{j2\pi(f(t-u))}\phi_{k}(u)d\phi_{j}^{*}(t)dtdf
\]

(B.8)

Rearranging integrals, we have that

\[
I = \int_{-\infty}^{\infty} S(f)\int_{-\infty}^{\infty} \phi_{k}(u)e^{-j2\pi u}du \int_{-\infty}^{\infty} \phi_{j}^{*}(t)e^{j2\pi \eta}d\eta df
\]

(B.9)

which reduces to

\[
I = \int_{-\infty}^{\infty} S(f)\Phi_{k}(f)\Phi_{j}^{*}(f)df
\]

(B.10)

Using (B.6), (B.10) reduces to

\[
I = \begin{cases} 
\int_{-\infty}^{\infty} S(f)|\Phi_{k}(f)|^{2}df, & j = k \\
0, & \text{otherwise}
\end{cases}
\]

(B.11)
thus establishing the claim. From (B.11), we see that the variance of the $k^{th}$ projection corresponding to the $k^{th}$ subband is the integral of the subband output power spectral density over its frequency support. This is an intuitively consistent result.

From (B.10), we can make some additional comments if the subband filters are partially overlapping. First, evaluating (B.10) in this case produces cross correlation between subbands. With well designed filters, the overlap, hence the correlation, will only be between adjacent bands. If we produced a finite number of projections and formed a vector, the resulting correlation matrix would be tri-diagonal. Further, a well designed set of filters will be highly attenuated in the region of mutual support. Thus, the resulting cross correlation will be small.
Orthogonally multiplexed communication is a bandwidth efficient modulation format that places independent QAM symbols on orthogonal pulses. With the traditional approach, this provides a uniform decomposition of the bandwidth allowing for variable compensation techniques within each subband tailored to the channel distortion peculiar to that band. In this dissertation, we propose orthonormal wavelet basis functions for the orthogonal pulses. In particular, dyadic wavelets are used to provide a nonuniform decomposition of the data bandwidth while M-band wavelets are used to generate a uniform decomposition. All the fundamental characterization of these new waveforms is provided and it is shown that digital filter banks enable an efficient discrete time implementation.

These filter banks are studied in detail resulting in the identification and solution of a number of practical problems encountered with FIR filter realizations. Most importantly, an even order FIR filter bank is developed for applications requiring linear phase. We then evaluate candidate pulse designs and make the important connection between the Meyer scaling function and the Nyquist communication pulse. In the dyadic case, we derive an analytical form for the square root raised cosine wavelet. Additionally, this allows us to prove several results in extending the Meyer scaling