RADIATION FROM AN APERTURE
INTO
AN ANISOTROPIC PLASMA HALF-SPACE

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by
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Chapter 1

Introduction

Propagation and reflection of electromagnetic waves from an anisotropic medium have been extensively studied [1,2,3,4]. In a propagation problem, most authors consider the dispersion equation and the polarization of waves. The solutions of the dispersion equation for an anisotropic plasma indicate that there are two waves traveling with different velocities in an arbitrary direction. Each solution of the dispersion equation determines the corresponding direction of electric field intensity and the state of polarization of the wave. In the case of reflection, problems with the incident wave located either in the isotropic or the anisotropic medium have been studied. The boundary conditions on the interface determine the reflection and transmission coefficient matrices. In an anisotropic medium, increasing attention has been given to plasma (an ionized gas). In the presence of a constant magnetic field, the plasma has a dyadic permittivity of the form (2.12) when the static magnetic field is in the c-direction. The electromagnetic waves in plasma have been studied in different aspects in order to understand the physics of the ionosphere.

The radiation problem is that of determining the electromagnetic field generated by a given source distribution. In an anisotropic medium, the important problem in this area is the evaluation of the dyadic Green’s function.
The first paper on radiation of a given current distribution in an anisotropic medium was given by Bunkin [5]. He obtained some basic results by using the dyadic Green’s function. The general solutions were rather complicated, and he restricted explicit discussion to the far field only.

Felsen [8,9] solved a two-dimensional, scalar diffraction problem by Fourier Integral techniques. The solution is examined in the asymptotic range of a large distance from the line source of magnetic currents. This source is located in a uniaxially anisotropic medium. Clemmow [43] studied the radiation from an electric dipole in a uniaxial medium by a procedure of scaling to a vacuum field. Hurd [45] extended the Clemmow technique to include magnetic dipoles in a medium with both electric and magnetic anisotropy. Wu [10] also studied the radiation from an elementary dipole in a magneto-ionic medium. The problems were formulated by reducing the Maxwell Equations to a pair of Equivalent Transmission-Line problems. Asymptotic solutions for the far field are obtained by the Saddle-Point method.

Takaku [11] calculated electric dipole fields in an axially symmetric anisotropic medium by means of the cylindrical coordinates. The field expressions still need to be described in asymptotic form by Saddle-Point method. More than a decade later, not much further development has been made in this area.

Tsalamengas [12,13,14,15] studied the radiation from a
dipole in an anisotropic layer and bianisotropic slabs by combining Fourier Transforms with matrix analysis methods. Transmission phenomena through the medium of interest are investigated by examining the field patterns in the far-zone region.

All of the solutions mentioned above were limited to the propagation of plane waves, the field expression in asymptotic form, evaluation by saddle-point method, or reduction to two-dimensional or transmission-line problems. In an anisotropic plasma, these solutions are in the form of a multiple integral, Bessel or Hankel function.

However, in 1983, Chen [16] obtained an exact expression of dyadic Green’s function of a uniaxially anisotropic medium by means of Fourier and Bessel transforms. The result is expressed in a coordinate-free form. The field due to the given source distribution can be obtained from the dyadic Green’s function.

On the radiation over a perfectly conducting plane, Wu [22] points out that, generally, the Image theory can not be applied in an anisotropic medium. Rao and Wu (T.T.)[21] clarified the applicability of Image theory that was discussed by Wu in Ferrite. The proof indicated that the conventional image theory is applicable only for the case when the static magnetic field is perpendicular to the conducting plane. Kong [19,20] also reached the same conclusion for a bianisotropic medium.
The present study consists of two parts. The first part describes the electromagnetic fields radiated from a given source in an unbounded anisotropic medium. The second part is concerned with the influence of a perfectly conducting plane on the radiation of a given source distribution. Chen's approach was advantageously adopted in formulating the problem to obtain the dyadic Green's function in closed form.

We will begin, in Chapter 2, with the formulation of the problem. The procedure closely follows the work presented by Chen. The dyadic Green's function will be obtained by means of linear analysis [38], and Bessel and Fourier Transform [17,18] techniques. This dyadic Green's function is in closed form with respect to an orthogonal coordinate system $\hat{a}, \hat{b}, \hat{c}$ and can be directly reduced to the case of a uniaxial or an isotropic medium. The radiated field in an unbounded plasma due to a particular orientation of the static magnetic field will then be determined. Some examples of the field patterns will be plotted. These field patterns will be compared with the field patterns in isotropic and uniaxially anisotropic media. Two types of source distribution will be used to plot the field patterns: an infinitesimal oscillating electric dipole and a rectangular aperture.

Since, in an anisotropic medium, the conventional image method does not apply in general; an alternate approach, utilizing the factorization of the dyadic Green's function from the first part, will be developed which will lead to the
reformulation of the image theory. The reformulation of image theory in an isotropic medium and its extension to a uniaxial medium will be considered in Chapter 3. By this reformulation and the dyadic Green's function as specified in Chapter 2, the radiated field in plasma half-space with an arbitrary orientation of the static magnetic field will be obtained in Chapter 4. The field patterns in a uniaxial medium and a plasma half-space will then be determined for several special orientations of the static magnetic field. The field patterns will also be compared to the field pattern in an isotropic medium. All of the notations that will be used can be found in reference [16].

The theories presented in this dissertation may be applied to spacecraft communications for a craft immersed in an anisotropic plasma. When a space shuttle, for example, passes through the ionosphere, there are complications that arise due to changes in signal radiation patterns. Since the ionosphere is an anisotropic plasma, the results presented here can be used to better understand and handle communications problems in space programs. A rectangular aperture is studied, because such an aperture is commonly employed on aircraft and spacecraft.
Chapter 2

Radiation in an Unbounded Anisotropic Plasma

Introduction

In this chapter, the discussion will be focused on the radiation of a given source distribution in an unbounded anisotropic plasma. We will begin with Maxwell’s equations in the source region. After some manipulations, taking advantage of the linearity of the equation, we will be able to obtain the electromagnetic field in terms of the dyadic Green’s function in a closed form and the given source distribution. In determining the dyadic Green’s function, we will follow closely the work presented by Chen [36], who utilized three-dimensional Fourier transforms, linear analysis and Bessel transforms. The dyadic Green’s function will be obtained in the orthogonal coordinate system $\hat{a}, \hat{b}, \hat{c}$. This function will be directly reduced when the medium of interest becomes uniaxial or isotropic. Combining the dyadic Green’s function with a rectangular aperture or a dipole source distribution, the electromagnetic field will be determined. For a given simple source distribution such as an infinitesimal oscillating electric dipole, the total (near, intermediate, and far) field can be directly obtained. The rectangular aperture and dipole field patterns will be plotted and the differences compared between isotropic or uniaxial media, and anisotropic plasma. All of the notations and symbols that will be used can be found in reference [16].
2.1 **Dyadic Green's Function of a Plasma**

In this section we will obtain the dyadic Green's function and the related results. For a given source distribution immersed in an unbounded anisotropic medium characterized by a dielectric tensor \( \varepsilon_c \) and a relative permeability \( \mu \). Maxwell's equations in the frequency domain can be written as:

\[
\nabla \times \overline{E}(\overline{r}) = i\omega \mu_0 \mu \overline{H}(\overline{r}) - \overline{M}(\overline{r}) \quad (2.1)
\]

\[
\nabla \times \overline{H}(\overline{r}) = -i\omega \varepsilon_0 \overline{\varepsilon_c} \cdot \overline{E}(\overline{r}) + \overline{J}(\overline{r}) \quad (2.2)
\]

Eliminating \( \overline{H}(\overline{r}) \) from eqs. (2.1) and (2.2), we obtain

\[
(\nabla \nabla - \nabla^2 \overline{\varepsilon_c} + k_0^2 \mu \overline{\varepsilon_c}) \cdot \overline{E}(\overline{r}) = i\omega \mu_0 \mu \overline{J}(\overline{r}) - \nabla \times \overline{M}(\overline{r}) \quad (2.3)
\]

Since eq. (2.3) is linear, we may write

\[
\overline{E}(\overline{r}) = \iiint \overline{G}(\overline{r}, \overline{f}) \cdot \overline{S}(\overline{f}) \, d^3 f \quad (2.4)
\]

where

\[
\overline{S}(\overline{f}) = i\omega \mu_0 \mu \overline{J}(\overline{f}) - \nabla \times \overline{M}(\overline{f}) \quad (2.5)
\]

Substitution of eq. (2.4) into eq. (2.3), yields

\[
\iiint \{ (\nabla \nabla - \nabla^2 \overline{\varepsilon_c} + k_0^2 \mu \overline{\varepsilon_c}) \cdot \overline{G}(\overline{r}, \overline{f}) \} \cdot \overline{S}(\overline{f}) \, d^3 f = \overline{S}(\overline{r}) \quad (2.6)
\]

Utilizing the three-dimensional delta function

\[
\overline{S}(\overline{r}) = \iiint \overline{S}(\overline{f}) \delta(\overline{r} - \overline{f}) \, d^3 f \quad (2.7)
\]

We can write eq. (2.6) as

\[
\iiint \{ (\nabla \nabla - \nabla^2 \overline{\varepsilon_c} + k_0^2 \mu \overline{\varepsilon_c}) \cdot \overline{G}(\overline{r}, \overline{f}) - \overline{f} \delta(\overline{r} - \overline{f}) \} \cdot \overline{S}(\overline{f}) \, d^3 f = 0 \quad (2.8)
\]

\[
(\nabla \nabla - \nabla^2 \overline{\varepsilon_c} + k_0^2 \mu \overline{\varepsilon_c}) \cdot \overline{G}(\overline{r}, \overline{f}) = \overline{f} \delta(\overline{r} - \overline{f}) \quad (2.9)
\]
Utilizing the three-dimensional Fourier transform, we finally obtain the dyadic Green function as:

\[
\overline{G}(\vec{r}, \vec{r}') = \frac{1}{(2\pi)^3} \iiint \overline{W}^{-1}(\vec{K}) e^{i\vec{K} \cdot (\vec{r} - \vec{r}')} d^3k
\]  

(2.10)

where

\[
\overline{W}(\vec{K}) = k^2 \overline{I} - k_0^2 \mu \overline{\varepsilon}_c - \overline{K}\overline{K}
\]  

(2.11)

In the case of plasma, the dielectric tensor takes the form

\[
\overline{\varepsilon}_c = \varepsilon_1 \overline{I} + i\varepsilon_3 (\hat{c} \times \overline{I}) + (\varepsilon_2 - \varepsilon_1) \hat{c} \hat{c}
\]  

(2.12)

where \(\hat{c}\) is the unit vector in the direction of the static magnetic field. Substitution of eq.(2.12) into eq.(2.11) yields the wave matrix of a plasma as:

\[
\overline{W}(\vec{K}) = (k^2 - k_0^2 \mu \varepsilon_1) \overline{I} - \overline{K}\overline{K} + (\varepsilon_1 - \varepsilon_2) k_0^2 \mu \hat{c} \hat{c} - i\varepsilon_3 k_0^2 \mu (\hat{c} \times \overline{I})
\]  

(2.13)

Using Linear Analysis [23], we find the determinant and adjoint of the wave matrix as:

\[
|\overline{W}(\vec{K})| = -k_0^2 \mu \left[ k^2 \overline{K} \overline{\varepsilon}_c \overline{K} + (k_0^2 \mu)^2 (\varepsilon_1^2 - \varepsilon_3^2) \right.
\]

\[
+ k_0^2 \mu \left\{ (-\varepsilon_3^2 + \varepsilon_2^2) (\hat{K} \times \hat{c})^2 - \varepsilon_1 \varepsilon_2 (\hat{K} \cdot \hat{c})^2 - \varepsilon_1 \varepsilon_2 \right\} k^2 \]  

(2.14)

and

\[
adj\overline{W}(\vec{K}) = \overline{K} k^2 - k_0^2 \mu \varepsilon_2 + (k_0^2 \mu)^2 adj\overline{\varepsilon}
\]

\[
+ k_0^2 \mu \left\{ (\varepsilon_2 - \varepsilon_1) (\hat{c} \times \hat{K}) (\hat{K} \cdot \hat{c} - \varepsilon_3) + i\varepsilon_3 (\overline{K} (\hat{K} \times \hat{c}) - (\hat{K} \times \hat{c}) \overline{K}) - (\overline{K} \cdot \hat{c}) \overline{I} \right\}
\]  

(2.15)

With respect to an orthogonal coordinate system \(\hat{a}, \hat{b}, \hat{c}\),

\(\overline{K}\) may be expressed as:

\[
\overline{K} = k_a \hat{a} + k_b \hat{b} + k_c \hat{c} = k_p \cos \phi_c \hat{a} + k_p \sin \phi_c \hat{b} + k_c \hat{c}
\]  

(2.16)
where \( \hat{c} \) is the same unit vector as defined before and 
\[
\hat{a} = \hat{B} \times \hat{c}, \quad \hat{B} = \hat{c} \times \hat{a} \quad \text{(see Fig. 2.1)}.
\]

![Figure 2.1 The orientation of the coordinate][44]

The determinant eq.(2.14) may also be arranged in the following form

\[
|\overline{\nu}(K)| = k_0^2 \mu \frac{\vec{e}_c \cdot \vec{K}}{2 \vec{e}_c \cdot \vec{K}} \left( k^2 - k_1^2 \right) \left( k^2 - k_2^2 \right) \quad (2.17)
\]

where

\[
k_{1,2}^2 = \left( \frac{k_0^2 \mu}{2 \vec{e}_c \cdot \vec{K}} \right) \left( \varepsilon_3 \varepsilon_2 - \varepsilon_2 \varepsilon_1 \right) + \left( \varepsilon_1 \varepsilon_3 - \varepsilon_3 \varepsilon_1 \right) \varepsilon_1 \varepsilon_3 (1 + (\vec{K} \cdot \vec{c})^2) \]

\[
\pm \sqrt{(\varepsilon_3^2 - \varepsilon_2^2 - \varepsilon_1 \varepsilon_2)^2 (\vec{K} \cdot \vec{c})^4 + 4 \varepsilon_3^2 \varepsilon_2^2 (\vec{K} \cdot \vec{c})^2}
\]

In a propagation problem, eq.(2.17) shows that there exist two types of plane waves with complex propagation constants \( k_1 \) and \( k_2 \) whose values depend on the angle between the direction of propagation and the static magnetic field. For radiation
this implies that there exist two types of radiated fields with complex phase constants $k_1$ and $k_2$ whose values depend on the direction of an observer and the orientation of the static magnetic field. These two fields radiate with different phase velocities which correspond to different complex phase constants. The rate of attenuation of these two fields is due to the imaginary part of their complex phase constants.

To carry out the integration eq. (2.10) with the determinant eq. (2.17) for the dyadic Green’s function, the final solution cannot be obtained in closed form. The expression for the solution is rather complicated. Our purpose here is to rearrange this determinant, allowing us to carry out the integration for the dyadic Green’s function in a closed form. From eq. (2.17), we rearrange the determinant into the following form

$$|W(K)| = k_0^2 \mu \{ (\overline{K} \cdot \overline{e}_c \cdot K - k_3^2) (k^2 - k_4^2) + C \} \quad (2.18)$$

where

$$k_3^2 = k_0^2 \mu a^2 \quad ; \quad k_4^2 = k_0^2 \mu b^2$$

$$a^2 = \varepsilon_2 (\varepsilon_1 + \frac{\varepsilon_3^2}{\varepsilon_1 - \varepsilon_2}) \quad ; \quad b^2 = \varepsilon_1 - \frac{\varepsilon_3^2}{\varepsilon_1 - \varepsilon_2} \quad (2.19)$$

$$C = (k_0^2 \mu)^2 \varepsilon_2 \varepsilon_3^2 \left( \frac{\varepsilon_3^2}{(\varepsilon_1 - \varepsilon_2)^2} - 1 \right)$$

which would allow us to consider the constant term $C$. In order for the constant term $C$ to vanish, one of the following conditions must be satisfied:
A) C equals zero which implies
A.1 $\varepsilon_2$ must equal zero, or
A.2 $\varepsilon_3$ must equal zero, or
A.3 \[ \left( \frac{\varepsilon_1 - \varepsilon_2 - \frac{\varepsilon_3^2}{\varepsilon_1 - \varepsilon_2}}{\varepsilon_1 - \varepsilon_2} \right) \] must equal zero.

or B) C must be far less than $k_j^2k_i^2$, such that when we combine
$k_j^2k_i^2$ and C, C will not affect $k_j^2k_i^2$. Essentially, we can drop
the constant term C.

* Condition A.1 will not be considered, because no known
  substance takes the dielectric tensor of this form.
* Condition A.2 implies uniaxially anisotropic medium.
* Condition A.3 plus A.2 imply an isotropic medium.
* Condition A.3 implies that $\omega_b$ or $\omega_p$ is equal to zero which
  is the isotropic case.

Therefore only condition B will be considered. For this
condition to exist, we must have the term
\[ \frac{\varepsilon_3^2(\varepsilon_3^2 - (\varepsilon_1 - \varepsilon_2)^2)}{\varepsilon_1^2(\varepsilon_1 - \varepsilon_2)^2 - \varepsilon_3^4} \] approach to zero.

Utilizing a computer simulation yields:

R1. $\omega > \frac{\omega_p}{0.4}$ and $\omega > \frac{\omega_b}{0.6}$

or R2. $\omega > \frac{\omega_p}{0.4}$ and $\omega < \omega_b < 2.5\omega$

or R3. $\omega > \omega_p$ and $\omega < \frac{\omega_b}{2.5}$

(The above source frequency restrictions were the rough idea
for a more precise frequency range, one must consult with
condition B)
In reality, the given source distribution can be generated at some specific frequency range. For the following, the above frequency restrictions will apply. After decomposition, the inverse of the wave matrix can be written in the form

\[ \mathbf{W}^{-1}(\mathbf{K}) = \frac{\mathbf{A}_1 \cdot \mathbf{K} \cdot \mathbf{K}}{k_0^2 \mu (\mathbf{K} \cdot \mathbf{c}_c \cdot \mathbf{K} - k_3^2)} + \frac{\mathbf{A}_{2c}}{\mathbf{K} \cdot \mathbf{c}_c \cdot \mathbf{K} - k_3^2} + \frac{\mathbf{A}_{3c}}{k^2 - k_4^2} \] (2.20)

After some manipulations the coefficients of the numerator were obtained as:

\[ \mathbf{A}_1 = -\mathbf{I} \] (2.21)

\[ \mathbf{A}_{2c} = \frac{\text{adj} \mathbf{c}_c}{b^2} - \frac{\mathbf{A}_{3c} a^2}{b^2} \] (2.22)

and

\[ \mathbf{A}_3 = \frac{1}{A_x} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \] (2.23)

\[ A_x = (\varepsilon_1 - \frac{a^2}{b^2}) k_p^2 + (\varepsilon_2 - \frac{a^2}{b^2}) k_c^2 \]

\[ A_{11} = (\varepsilon_1 + \varepsilon_2 - b^2 - \frac{\varepsilon_1 \varepsilon_2}{b^2}) k_a^2 + (\varepsilon_1 - \frac{\varepsilon_1 \varepsilon_2}{b^2}) k_b^2 + (\varepsilon_2 - \frac{\varepsilon_1 \varepsilon_2}{b^2}) k_c^2 \]

\[ A_{12} = i e_3 (1 - \frac{\varepsilon_2}{b^2}) (k_a^2 + k_b^2) - i \frac{\varepsilon_2 e_3}{b^2} k_b^2 + (\varepsilon_2 - b^2) k_a k_b \]

\[ A_{13} = (\varepsilon_1 - b^2) k_a k_c + i e_3 k_b k_c \]

\[ A_{21} = A_{12}^* \]
Substituting eqs. (2.21-2.23) into eq. (2.20), the inverse wave matrix can be obtained as:

\[
\overline{W}^{-1}(\mathbf{K}) = (-\frac{\mathbf{K}\mathbf{K}}{k_0^2\mu} + \frac{\text{adj}\overline{c}}{b^2}) \frac{1}{(\mathbf{K} \cdot \overline{c} \cdot \mathbf{K} - k_0^2\mu a^2)}
- \frac{a^2}{b^2} \frac{\overline{A}_{3c}}{\mathbf{K} \cdot \overline{c} \cdot \mathbf{K} - k_0^2\mu a^2} + \frac{\overline{A}_{3c}}{k^2 - k_0^2\mu b^2}
\]

(2.24)

Now substituting eq. (2.24) back into eq. (2.10), we obtain the dyadic Green’s function in the form of

\[
\overline{G}(\mathbf{K}) = (\frac{\nabla \nabla}{k_0^2\mu} + \frac{\text{adj}\overline{c}}{b^2}) I_{1c}(\mathbf{K}) - \frac{a^2}{b^2} \overline{I}_{3c}(\mathbf{K}) + \overline{I}_{2c}(\mathbf{K})
\]

(2.25)

where

\[
I_{1c}(\mathbf{K}) = \frac{1}{(2\pi)^3} \int \int \int \frac{e^{i\mathbf{R}\mathbf{R}}}{\mathbf{K} \cdot \overline{c} \cdot \mathbf{K} - k_0^2\mu a^2} d^3K
\]

(2.26)

\[
\overline{I}_{3c}(\mathbf{K}) = \frac{1}{(2\pi)^3} \int \int \int \frac{\overline{A}_{3c} e^{i\mathbf{R}\mathbf{R}}}{\mathbf{K} \cdot \overline{c} \cdot \mathbf{K} - k_0^2\mu a^2} d^3K
\]

(2.27)
and
\[ \overline{I}_{2c}(\overline{R}) = \frac{1}{(2\pi)^3} \iiint \frac{\overline{A}_{3c} e^{iR\cdot \overline{r}}}{k^2 - k_0^2 \mu b^2} \, d^3K \] 

(2.28)

Eq. (2.25) is the dyadic Green's function of an anisotropic plasma with the above source frequency restrictions. We put it into this form to enable us to compare previous works, which yielded the same results when we reduced this dyadic Green's function to the case of isotropic or uniaxially anisotropic medium. Next the evaluation of integrals \( I_{1c}(\overline{R}) \), \( \overline{I}_{2c}(\overline{R}) \) and \( \overline{I}_{3c}(\overline{R}) \) will be given.

2.2 Evaluation of Integrals

In this section, we will carry out the integrations in \( I_{1c}(\overline{R}) \), \( \overline{I}_{2c}(\overline{R}) \), and \( \overline{I}_{3c}(\overline{R}) \), by using three-dimensional Fourier and Bessel transforms [17,18,37] in an orthogonal coordinate system \( \hat{a}, \hat{b}, \hat{c} \) (see Fig.1):

\[ \overline{r} = \rho \cos \phi_c \hat{a} + \rho \sin \phi_c \hat{b} + c \hat{c} \] 

(2.29)

\[ k \overline{r} = \rho k_c \cos (\phi_c - \phi_c) + c k_c \] 

(2.30)

\[ d^3k = k_p dk_p dk_c d\phi_c \] 

(2.31)

To cover the whole \( k \)-space, the limits for \( k_c \), \( k \), and \( \phi_c \) are from \(-\infty\) to \(+\infty\), from 0 to \(+\infty\), and from 0 to \(2\pi\), respectively. After carrying out the integration by means of Bessel and Fourier integrals, we obtain
\[ I_1 (\mathbf{R}) = \frac{\sqrt{\mu} a}{4\pi e_1 e_2} \frac{e^{ik_0 R_0}}{R_o}; \quad R_o = \sqrt{\mu} a \sqrt{(R \cdot e_c \cdot R)} \]  

\[ I_2 (\mathbf{R}) = \begin{bmatrix} I_{211} & I_{212} & I_{213} \\ I_{221} & I_{222} & I_{223} \\ I_{231} & I_{232} & I_{233} \end{bmatrix} \]  

\[ I_{211} = \frac{1}{4\pi \left( e_1 - \frac{a^2}{b^2} \right)} \left( \sqrt{\mu} b \frac{e^{ik_0 R_0}}{R_0} \right) \left( e_1 - \frac{e_1 e_2}{b^2} \right) + \left( \frac{e_2 - e_1 e_2}{b^2} \right) \frac{1}{A} 
+ \frac{i (e_2 - b^2) \cos 2 \varphi_c \left( e^{ik_0 R_0} - e^{ik_0 \sqrt{\mu} b \cdot c} \right)}{k_0 \sqrt{\mu} b (R \cdot e_c) \left( 1 + Ak^2 \mu b^2 \right)} \right) \]  

\[ I_{222} = \frac{1}{4\pi \left( e_1 - \frac{a^2}{b^2} \right)} \left( \sqrt{\mu} b \frac{e^{ik_0 R_0}}{R_0} \right) \left( e_1 - \frac{e_1 e_2}{b^2} \right) + \left( \frac{e_2 - e_1 e_2}{b^2} \right) \frac{1}{A} 
+ \frac{i (e_2 - b^2) \cos 2 \varphi_c \left( e^{ik_0 R_0} - e^{ik_0 \sqrt{\mu} b \cdot c} \right)}{k_0 \sqrt{\mu} b (R \cdot e_c) \left( 1 + Ak^2 \mu b^2 \right)} \right) \]  

\[ I_{233} = \frac{\sqrt{\mu} b}{4\pi \left( e_1 - \frac{a^2}{b^2} \right)} \frac{e^{ik_0 R_0}}{R_0} \left( e_1 - \frac{e_1^2 - e_1}{b^2} \right) \frac{1}{A} + \frac{(e_1 - b^2)}{A} \left( \frac{1}{A} \right) \]  

\[ I_{212} = \frac{i \sqrt{\mu} b}{4\pi \left( e_1 - \frac{a^2}{b^2} \right)} \frac{e^{ik_0 R_0}}{R_0} \left( 1 - \frac{e_2 - e_2}{b^2} \right) \]  

\[ I_{221} = \frac{i \sqrt{\mu} b}{4\pi \left( e_1 - \frac{a^2}{b^2} \right)} \frac{e^{ik_0 R_0}}{R_0} \left( 1 - \frac{e_2 - e_2}{b^2} \right) \]  

\[ + \frac{(e_2 - b^2)}{4\pi \left( e_1 - \frac{a^2}{b^2} \right)} \sin 2 \varphi_c \left( \sqrt{\mu} b e^{ik_0 R_0} \right) + \frac{i2 \left( e^{ik_0 R_0} - e^{ik_0 \sqrt{\mu} b \cdot c} \right)}{k_0 \sqrt{\mu} b (R \cdot e_c) \left( 1 + Ak^2 \mu b^2 \right)} \]
\[ I_{313} = I_{233} = I_{231} = I_{332} = 0 \quad ; \quad R_\sigma = \sqrt{\mu b R} \]

and

\[
I_3 (R) = \begin{bmatrix}
I_{311} & I_{312} & I_{313} \\
I_{321} & I_{322} & I_{323} \\
I_{331} & I_{332} & I_{333}
\end{bmatrix}
\]

\[
I_{311} = \frac{1}{4\pi (\varepsilon_1 - \frac{a^2}{b^2})} \left[ \frac{\sqrt{\mu} a}{e_1 \sqrt{\varepsilon_2}} \frac{e^{i k_0 R_\sigma}}{R_\sigma} \left( \varepsilon_1 - \frac{\varepsilon_1\varepsilon_2}{b^2} + (\varepsilon_2 - b^2) \cos^2 \varphi_c + \left( \varepsilon_2 - \frac{\varepsilon_1\varepsilon_2}{b^2} \right) \frac{1}{A} \right) \right] \\
+ i (\varepsilon_2 - b^2) \cos^2 \varphi_c \sqrt{\varepsilon_1} \left( \frac{e^{i k_0 R_\sigma} - e^{i k_0 \sqrt{\varepsilon_2} \cdot \varepsilon_1}}{k_0 \sqrt{\mu} a \varepsilon_2 (R \times \varepsilon)^2 \left( 1 + \frac{A}{\varepsilon_2} k_0^2 \mu a^2 \right)} \right)
\]

\[
I_{322} = \frac{1}{4\pi (\varepsilon_1 - \frac{a^2}{b^2})} \left[ \frac{\sqrt{\mu} a}{e_1 \sqrt{\varepsilon_2}} \frac{e^{i k_0 R_\sigma}}{R_\sigma} \left( \varepsilon_1 - \frac{\varepsilon_1\varepsilon_2}{b^2} + (\varepsilon_2 - b^2) \sin^2 \varphi_c + \left( \varepsilon_2 - \frac{\varepsilon_1\varepsilon_2}{b^2} \right) \frac{1}{A} \right) \right] \\
+ i (\varepsilon_2 - b^2) \cos^2 \varphi_c \sqrt{\varepsilon_1} \left( \frac{e^{i k_0 R_\sigma} - e^{i k_0 \sqrt{\varepsilon_2} \cdot \varepsilon_1}}{k_0 \sqrt{\mu} a \varepsilon_2 (R \times \varepsilon)^2 \left( 1 + \frac{A}{\varepsilon_2} k_0^2 \mu a^2 \right)} \right)
\]

\[
I_{333} = \frac{\sqrt{\mu} a}{4\pi (\varepsilon_1 - \frac{a^2}{b^2}) e_1 \sqrt{\varepsilon_2} R_\sigma} \left[ \varepsilon_1 - \frac{(\varepsilon_1^2 - \varepsilon_2^2)}{b^2} \right] \left( 1 + \frac{1}{A} \right) + (\varepsilon_2 - b^2) \frac{1}{A}
\]

\[
I_{312} = \frac{i \sqrt{\mu} a}{4\pi (\varepsilon_1 - \frac{a^2}{b^2}) e_1 \sqrt{\varepsilon_2} R_\sigma} \varepsilon_3 \left( - \frac{\varepsilon_2}{b^2} - \frac{\varepsilon_2}{Ab^2} \right) \\
+ \frac{(\varepsilon_2 - b^2)^2 \sin^2 \varphi_c}{8 \pi (\varepsilon_1 - \frac{a^2}{b^2})} \left( \frac{\sqrt{\mu} a e^{i k_0 R_\sigma}}{e_1 \sqrt{\varepsilon_2} R_\sigma} + i 2 \sqrt{\varepsilon_1} \frac{\left( \varepsilon^{i k_0 R_\sigma} - e^{i k_0 \sqrt{\varepsilon_2} \cdot \varepsilon_1} \right)}{k_0 \sqrt{\mu} a \varepsilon_2 (R \times \varepsilon)^2 \left( 1 + \frac{A}{\varepsilon_2} k_0^2 \mu a^2 \right)} \right)
\]
In the following, we simplify the final result for the dyadic Green's function in eq.(2.25).

\[ I_{321} = \frac{-i\sqrt{\mu a} e^{ik_R}}{4\pi \left(\varepsilon_1 - \frac{a^2}{b^2}\varepsilon_2 R_0\right)} \varepsilon_3 \left(\frac{\varepsilon_2}{b^2} - \frac{\varepsilon_2}{Ab^2}\right) \]

\[ + \left(\varepsilon_2 - b^2\right) \sin^2 \varphi_c \frac{\sqrt{\mu a} e^{ik_R}}{8\pi \left(\varepsilon_1 - \frac{a^2}{b^2}\varepsilon_2 R_0\right)} \varepsilon_1 + 2\sqrt{\varepsilon_1} \frac{e^{ik_R} - e^{ik_R}\frac{1}{\varepsilon_2 R_0}}{k_0\sqrt{\mu a} \varepsilon_2 (\varepsilon_2 R_0)^2 \left(1 + \frac{\varepsilon_2}{\varepsilon_1} k_0^2 \mu a^2\right)} \]

\[ I_{311} = I_{323} = I_{331} = I_{332} = 0 \]

### 2.3 Dyadic Green's Function in an Orthogonal Coordinate System \( \mathbf{\hat{a}}, \mathbf{\hat{b}}, \mathbf{\hat{c}} \)

In the following, we simplify the final result for the dyadic Green's function in eq.(2.25).

\[ \overrightarrow{\mathbf{G}}(\mathbf{R}) = \left(\nabla^2 + \frac{adj \varepsilon_c}{k_0^2 \mu} \right) I_{1c}(\mathbf{R}) - \frac{a^2}{b^2} I_{3c}(\mathbf{R}) + I_{2c}(\mathbf{R}) \quad (2.35) \]

\[ \frac{\nabla^2}{k_0^2 \mu} I_{1c}(\mathbf{R}) = \{B_1(R) (\overline{\varepsilon}_{xc} \cdot \mathbf{R}) (\overline{\varepsilon}_{xc} \cdot \mathbf{R}) + B_2(R) \overline{\varepsilon}_{xc}\} e^{ik_R} \quad (2.36) \]

\[ B_1(R) = \frac{a}{4\pi k_0^2 \varepsilon_1 \varepsilon_2} \frac{(3 - i3k_0 R_0 - k_0^2 R_0^2)\mu a^4}{R_0^5} \quad (2.37) \]

\[ B_2(R) = \frac{a}{4\pi k_0^2 \varepsilon_1 \varepsilon_2} \frac{(ik_0 R_0 - 1)\mu a^2}{R_0^3} \quad (2.38) \]

\[ \overline{\varepsilon}_{xc} = \frac{1}{\varepsilon_1} \mathbf{\hat{a}} \mathbf{\hat{a}} + \frac{1}{\varepsilon_1} \mathbf{\hat{b}} \mathbf{\hat{b}} + \frac{1}{\varepsilon_2} \mathbf{\hat{c}} \mathbf{\hat{c}} \]

\[ \frac{adj \varepsilon_c}{b^2} I_1(\mathbf{R}) = B_3(R) \overline{\mathbf{A}}_{4c} e^{ik_R} \quad (2.39) \]

\[ B_3(R) = \frac{\sqrt{\mu a}}{4\pi \varepsilon_1 \varepsilon_2 b^2 R_0} \quad ; \quad \overline{\mathbf{A}}_{4c} = adj \varepsilon_c \quad (2.40) \]
\[
\frac{a^2}{b^2} T_3^c (R) = \frac{a^2}{b^2} A_{5c} e^{i \kappa \phi} + \frac{a^2}{b^2} A_{6c} \left( e^{i \kappa \phi} - e^{i \kappa \phi_{R} \sqrt{\frac{a}{b}} c} \right) \quad (2.41)
\]

\[
A_{5aa} = \frac{1}{4\pi (e_1 - \frac{a^2}{b^2})} \frac{\sqrt{\mu a}}{e_1 \sqrt{e_2 R_e}} \left( e_1 \frac{e_2}{b^2} + (e_2 - b^2) \cos^2 \phi_c + \left( e_2 - \frac{e_1 e_2}{b^2} \right) \frac{1}{A} \right)
\]

\[
A_{6aa} = \frac{1}{4\pi (e_1 - \frac{a^2}{b^2})} \frac{i (e_2 - b^2) \cos 2 \phi_c \sqrt{e_1}}{k_0 \sqrt{\mu a e_2} (R \times \hat{c})^2 \left( 1 + \frac{A}{e_2} k_0^2 \mu a^2 \right)}
\]

\[
A_{5bb} = \frac{1}{4\pi (e_1 - \frac{a^2}{b^2})} \frac{\sqrt{\mu a}}{e_1 \sqrt{e_2 R_e}} \left( e_1 \frac{e_2}{b^2} + (e_2 - b^2) \sin^2 \phi_c + \left( e_2 - \frac{e_1 e_2}{b^2} \right) \frac{1}{A} \right)
\]

\[
A_{6bb} = -\frac{1}{4\pi (e_1 - \frac{a^2}{b^2})} \frac{i (e_2 - b^2) \cos 2 \phi_c \sqrt{e_1}}{k_0 \sqrt{\mu a e_2} (R \times \hat{c})^2 \left( 1 + \frac{A}{e_2} k_0^2 \mu a^2 \right)}
\]

\[
A_{5cc} = \frac{\sqrt{\mu a}}{4\pi (e_1 - \frac{a^2}{b^2}) e_1 \sqrt{e_2 R_e}} \left( e_1 \frac{(e_1^2 - e_2^2)}{b^2} \right) \left( 1 + \frac{1}{A} \right) + \left( e_1 - b^2 \right) \frac{1}{A}
\]

\[
A_{5ab} = \frac{i \sqrt{\mu a e_3}}{4\pi \left( e_1 - \frac{a^2}{b^2} \right) e_1 \sqrt{e_2 R_e}} \left( 1 - \frac{e_2}{b^2} - \frac{e_2}{b^2} \right) + \frac{(e_2 - b^2) \sin 2 \phi_c \sqrt{\mu a}}{8\pi \left( e_1 - \frac{a^2}{b^2} \right) e_1 \sqrt{e_2 R_e}}
\]

\[
A_{6ab} = \frac{i 2 \sqrt{e_1} (e_2 - b^2) \sin 2 \phi}{8\pi \left( e_1 - \frac{a^2}{b^2} \kappa_0 \sqrt{\mu a e_2} (R \times \hat{c})^2 \left( 1 + \frac{A}{e_2} k_0^2 \mu a^2 \right)}
\]

\[
A_{5ba} = -\frac{i \sqrt{\mu a e_3}}{4\pi \left( e_1 - \frac{a^2}{b^2} \right) e_1 \sqrt{e_2 R_e}} \left( 1 - \frac{e_2}{b^2} - \frac{e_2}{b^2} \right) + \frac{(e_2 - b^2) \sin 2 \phi_c \sqrt{\mu a}}{8\pi \left( e_1 - \frac{a^2}{b^2} \right) e_1 \sqrt{e_2 R_e}}
\]

\[
A_{6ba} = \frac{i 2 \sqrt{e_1} (e_2 - b^2) \sin 2 \phi}{8\pi \left( e_1 - \frac{a^2}{b^2} \kappa_0 \sqrt{\mu a e_2} (R \times \hat{c})^2 \left( 1 + \frac{A}{e_2} k_0^2 \mu a^2 \right)}
\]
\[ \overline{I}_c(R) = A_{\gamma c} e^{ik_0R} + A_{\delta c} (e^{ik_0R} - e^{ik_0\sqrt{\mu}R}) \quad (2.42) \]

\[ A_{\gamma\alpha\alpha} = \frac{1}{4\pi} \frac{\sqrt{\mu}b}{R_0} \left( \frac{e_1 - e_2}{b^2} + \frac{(e_2 - b^2)}{b^2} \cos^2\varphi_c + \frac{e_2}{b^2} \frac{1}{A} \right) \]

\[ A_{\delta\alpha\alpha} = \frac{1}{4\pi} \frac{i(e_2 - b^2)\cos\varphi_c}{k_0\sqrt{\mu}b(R\delta)^2(1 + Ak_0^2\mu b^2)} \]

\[ A_{\gamma\beta\beta} = \frac{1}{4\pi} \frac{\sqrt{\mu}b}{R_0} \left( \frac{e_1 - e_2}{b^2} + \frac{(e_2 - b^2)}{b^2} \sin^2\varphi_c + \frac{e_2}{b^2} \frac{1}{A} \right) \]

\[ A_{\delta\beta\beta} = \frac{1}{4\pi} \frac{i(e_2 - b^2)\cos\varphi_c}{k_0\sqrt{\mu}b(R\delta)^2(1 + Ak_0^2\mu b^2)} \]

\[ A_{5cc} = \frac{\sqrt{\mu}b}{4\pi} \left[ \frac{(e_1 - (e_1^2 - e_2^2))}{b^2} \left( 1 + \frac{1}{A} \right) + (e_1 - b^2) \frac{1}{A} \right] \]

\[ A_{\gamma ab} = \frac{i\sqrt{\mu}be_3}{4\pi} \frac{1 - e_2 - \frac{e_2}{Ab^2}}{R_0} + \frac{(e_2 - b^2)\sin2\varphi_c\sqrt{\mu}b}{8\pi} \frac{1}{\left( e_1 - \frac{a^2}{b^2} \right) R_0} \]

\[ A_{\delta ab} = \frac{i2(e_2 - b^2)\sin2\varphi}{8\pi} \frac{1}{\left( e_1 - \frac{a^2}{b^2} \right) k_0\sqrt{\mu}b(R\delta)^2(1 + Ak_0\mu b^2)} \]

\[ A_{7ba} = -\frac{i\sqrt{\mu}be_3}{4\pi} \frac{1 - e_2 - \frac{e_2}{Ab^2}}{R_0} + \frac{(e_2 - b^2)\sin2\varphi_c\sqrt{\mu}b}{8\pi} \frac{1}{\left( e_1 - \frac{a^2}{b^2} \right) R_0} \]

\[ A_{8ba} = \frac{i2(e_2 - b^2)\sin2\varphi}{8\pi} \frac{1}{\left( e_1 - \frac{a^2}{b^2} \right) k_0\sqrt{\mu}b(R\delta)^2(1 + Ak_0\mu a^2)} \]
where subscript \( e \) denotes extraordinary field, subscript \( o \) denotes ordinary field, and subscript \( c \) denotes the orientation of the static magnetic field.

With all of the above, we can rearrange eq. (2.35) in the form

\[
\mathbf{G}(\mathbf{R}) = (\overline{A}_9 + \overline{A}_{10}) e^{ik_o \mathbf{R}_o} - \overline{A}_{10} e^{-ik_o \sqrt{\frac{\mathbf{E} \cdot \mathbf{E}}{\varepsilon_2}}} \\
+ (\overline{A}_7 + \overline{A}_8) e^{ik_o \mathbf{R}_o} - \overline{A}_8 e^{ik_o \sqrt{\frac{\mathbf{B} \cdot \mathbf{B}}{\mu_2}}}.
\]

From eq. (2.43) it can be seen clearly that there are fields with different (complex) phase variables radiating in this medium. These fields are not the same as the field in an isotropic medium which has only one phase constant (not depending on direction of radiation) corresponding to only one type of field. Physically, the imaginary part of this complex phase number, which involves \( a \) and \( b \) will effect the attenuation of that field. In other words, the cut-off condition of each particular field will be when \( a^2 \) or \( b^2 \) is less than or equal to zero, respectively.

In the following, we consider the dyadic Green’s function when the medium of interest becomes uniaxial or isotropic respectively.
2.3.1 Dyadic Green's function in a uniaxial medium

If the medium of interest becomes a uniaxial medium (or uniaxial plasma), i.e. \( \varepsilon_3 = 0 \), the resulting solution of the dyadic Green's function in eq.(2.35) will be reduced to

\[
\overline{G_u}(R) = e^{ik_u R_\mu} \left\{ \left( B_{u1}(R) (\overline{e}_{uxc} \cdot \hat{R}) + B_{u2}(R) \overline{e}_{uxc} \right) + B_{u3}(R) \overline{A}_{u4c} \right\} - \varepsilon_2 e^{ik_u R_\mu} \left\{ \overline{A}_{u5c}(R) + \overline{A}_{u6c}(R) \right\} + e^{ik_u R_\omega} \left\{ \overline{A}_{u7c}(R) + \overline{A}_{u8c}(R) \right\}
\]

(2.44)

where subscript \( u \) denotes uniaxial and the detail of the coefficients will be summarized in section 3.2.1. The resulting solution eq.(2.44) agrees with the work by Chen [36].

2.3.2 Dyadic Green's function in an isotropic medium

If the medium is isotropic, i.e. \( \varepsilon_1 = \varepsilon_2 = \varepsilon \) and \( \varepsilon_3 = 0 \), the resulting dyadic Green's function in eq.(2.35) (or 2.44 in uniaxial) reduces to

\[
\overline{G_i}(R) = \frac{e^{ik \sqrt{\mu \varepsilon} R}}{4\pi R} \left\{ C_{i1}(R) \hat{R} + C_{i2}(R) \hat{R} \right\}
\]

(2.45)

\[
C_{i1} = 1 + \frac{i}{k_0 \sqrt{\mu \varepsilon} R} - \frac{1}{k_0^2 \mu \varepsilon R^2}
\]

\[
C_{i2} = -1 - \frac{3i}{k_0 \sqrt{\mu \varepsilon} R} + \frac{3}{k_0^2 \mu \varepsilon R^2}
\]

which agrees with the work by Tai, Chen and Yaghjian [28,29,30].
2.4 Radiation from a Dipole

If the given source distribution is an infinitesimal oscillating electric dipole located at \((x_i, y_i, z_i)\), eq. (2.5) takes the form

\[
\mathcal{E}(\mathbf{r}) = i\omega \mu_0 \mu I I \delta(x-x_i) \delta(y-y_i) \delta(z-z_i) \tag{2.46}
\]

The electric field in an unbounded anisotropic plasma eq. (2.4) becomes

\[
\mathbf{E}(\mathbf{r}) = i\omega \mu_0 I I \\
\left[ e^{ik_0 R} \left( B_1(R) \left( \mathbf{\hat{e}}_{xc} \cdot \mathbf{R} \right) + B_2(R) \mathbf{\hat{e}}_{xc} + B_3(R) \mathbf{\hat{A}}_{5c} \right) \cdot \mathbf{\hat{r}} \right. \\
- \frac{a^2}{b^2} \left\{ e^{ik_0 \mathbf{R} \mathbf{\hat{A}}_{5c}(R)} \cdot \mathbf{\hat{r}} + \left( e^{ik_0 \mathbf{R} \mathbf{\hat{A}}_{5c}(R)} - e^{ik_0 \sqrt{a^2 b^2 R^2} \mathbf{\hat{A}}_{5c}(R)} \right) \right\} \\
+ \left\{ e^{ik_0 \mathbf{R} \mathbf{\hat{A}}_{5c}(R)} \cdot \mathbf{\hat{r}} + \left( e^{ik_0 \mathbf{R} \mathbf{\hat{A}}_{5c}(R)} - e^{ik_0 \sqrt{a^2 b^2 R^2} \mathbf{\hat{A}}_{5c}(R)} \right) \right\} \tag{2.47}
\]

\[
R_\theta = \alpha \sqrt{\mu_R \mathbf{\hat{e}}_{xc} \cdot \mathbf{R}} \\
R_\phi = \sqrt{\mu_R b R} \\
R = \sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}
\]

2.5 Radiation from a rectangular aperture

In this section we will determine the field in an unbounded anisotropic medium with a given rectangular aperture source distribution [24]. Eq. (2.4) can be rewritten as:

\[
\mathbf{E}(\mathbf{r}) = -\int \mathcal{E}(\mathbf{r}, \mathbf{r}') \cdot \nabla \mathcal{H}(\mathbf{r}) \; d^3\mathbf{r}' \\
= -\int \left\{ \mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_3 + \mathcal{X}_4 \right\} d^3\mathbf{r}' \tag{2.48}
\]
where $X_1$, $X_2$, $X_3$ and $X_4$ are the combination of eqs. (2.36), (2.39), (2.41) and (2.42), and

$$\bar{M}_s(r) = E_{oy} \sin\left(\frac{mnX}{a} \right) \cos\left(\frac{mnY}{b} \right) e^{iB_z X - E_{ox} \cos\left(\frac{mnX}{a} \right)} \sin\left(\frac{mnY}{b} \right) e^{iB_z Y}$$

(2.49)

At the aperture $x: -\frac{a}{2} \leq X \leq \frac{a}{2}$; $y: -\frac{b}{2} \leq Y \leq \frac{b}{2}$; $z = 0$, $\bar{M}_s(r)$ becomes

$$\bar{M}_s(r) = E_{oy} \sin\left(\frac{mnX}{a} \right) \cos\left(\frac{mnY}{b} \right) \cos\left(\frac{mnX}{a} \right) \sin\left(\frac{mnY}{b} \right)$$

(2.50)

$$\nabla \times \bar{M}_s(r) = 2\pi \left( \frac{m}{a} E_{ox} + \frac{n}{b} E_{oy} \right) \sin\left(\frac{mnX}{a} \right) \sin\left(\frac{nny}{b} \right)$$

(2.51)

where $E_0 = \frac{m}{a} E_{ox} + \frac{n}{b} E_{oy}$

Next we will consider 3 special cases. The expression for the far field solution will be carried out when the static magnetic field is along the $z$, $x$, or $y$-direction respectively.

2.5.1) The static magnetic field is along $z$-axis

In this case $(\hat{a} = \hat{z}, \hat{b} = \hat{y}, \hat{c} = \hat{x})$, the integrand in eq. (2.48) can be written as:

$$X_1 = \pi E_0 \left\{ \frac{B_1(R)}{\varepsilon_2} \left( \frac{\xi^2 + \eta^2 + \zeta^2}{\varepsilon_1^2} \right) + B_2(R) \varepsilon_2 \right\} \sin\left(\frac{mnX}{a} \right) \sin\left(\frac{nny}{b} \right) e^{ik_0 R_z}$$

$$X_2 = \pi E_0 \left\{ B_3(R) \left( \varepsilon_1^2 - \varepsilon_2^2 \right) \varepsilon_2 \right\} \sin\left(\frac{mnX}{a} \right) \sin\left(\frac{nny}{b} \right) e^{ik_0 R_z}$$

$$X_3 = \pi E_0 \frac{a^2}{b^2} A_{5cc} \varepsilon_2 \sin\left(\frac{mnX}{a} \right) \sin\left(\frac{nny}{b} \right) e^{ik_0 R_z}$$
After substituting all of the above back into eq. (2.48) and carrying out the integration at the aperture we obtain

\[
\overline{E}(\bar{r})_z = \pi E_0 \frac{a_b}{4} \left[
\left\{ \frac{B_1 R}{\varepsilon_2} \left( \frac{x \hat{x}}{\varepsilon_1} + \frac{y \hat{y}}{\varepsilon_1} + \frac{z \hat{z}}{\varepsilon_2} \right) + \frac{B_2 R}{\varepsilon_2} + B_3 (\varepsilon^2_1 - \varepsilon^2_2) \right\} e^{ik_0 R_0} 
\right.
\]

\[
\left. \left( \frac{\sin X_{x11} - \sin X_{x12}}{X_{x11}} \frac{\sin Y_{x11} - \sin Y_{x12}}{Y_{x12}} \right) e^{ik_0 R_0} \right] 
\]

\[
+ \frac{b}{R_0} A_y \left( \frac{\sin X_{x21} - \sin X_{x22}}{X_{x21}} \frac{\sin Y_{x21} - \sin Y_{x22}}{Y_{x22}} \right) e^{ik_0 R_0} \] 

(2.53)

where

\[
A_y = \frac{\sqrt{\mu}}{4\pi \left( \frac{\varepsilon_1 - a^2}{b^2} \right)} \left[ \left( \varepsilon_1 - \frac{(\varepsilon^2_1 - \varepsilon^2_2)}{b^2} \right) \left( 1 + \frac{1}{A} \right) + (\varepsilon_1 - b^2) \frac{1}{A} \right] 
\]

### 2.5.2) The static magnetic field is along x-axis

In this case (\( \delta = \lambda, \varphi = \varphi, \xi = z \)), \( \varphi_c = \varphi_x \) is the angle between the y-axis and the projection of the distance vector on the yz-plane. The angle is in the counterclockwise direction from the y-axis. The integrand in eq. (2.48) becomes

\[
\overline{X}_1 = \pi E_0 B_1(R) \left( \frac{x \hat{x}}{\varepsilon_1} + \frac{y \hat{y}}{\varepsilon_1} + \frac{z \hat{z}}{\varepsilon_2} \right) \sin \left( \frac{m \varepsilon \hat{X}}{a} \right) \sin \left( \frac{n \varepsilon \hat{Y}}{b} \right) e^{ik_0 R_{yy}} 
\]

\[
\overline{X}_2 = \pi E_0 B_3(R) \left( i \varepsilon_2 \varepsilon_3 \hat{Y} + \varepsilon_1 \varepsilon_2 \hat{Z} \right) \sin \left( \frac{m \varepsilon \hat{X}}{a} \right) \sin \left( \frac{n \varepsilon \hat{Y}}{b} \right) e^{ik_0 R_{yy}} 
\]
\[
\chi_3 = A_8 \frac{a^2}{b^2} \left( (A_{5ab} \hat{y} + A_{5bb} \hat{z}) e^{ik_0R_{ex}} + (A_{6ab} \hat{y} + A_{6bb} \hat{z}) (e^{ik_0R_{ex}} - e^{ik_0\sqrt{\frac{b}{a}R_{ey}}}) \right)
\]

\[
\chi_4 = A_8 \left( (A_{7ab} \hat{y} + A_{7bb} \hat{z}) e^{ik_0R_y} + (A_{8ab} \hat{y} + A_{8bb} \hat{z}) (e^{ik_0R_y} - e^{ik_0\sqrt{\frac{b}{a}R_{ey}}}) \right)
\]

(2.54a-d)

After substituting all of the above back into eq. (2.48) and carrying out the integration at the aperture we obtain

\[
\bar{E}(r)_x = \pi E_0 \frac{a_s b_s}{4} \left[ \frac{B_{1f}}{\varepsilon_1} \left( \frac{x \delta + y \hat{y} + z \hat{z}}{\varepsilon_1} \right) + \frac{B_{2f}}{\varepsilon_1} \hat{z} + B_{3f}(i \varepsilon_2 \varepsilon_3 \hat{y} + \varepsilon_1 \varepsilon_2 \hat{z}) 
\right. \\
- \frac{a^2}{b^2} (A_{5ab} + A_{6ab}) \hat{y} + (A_{5bb} + A_{6bb}) \hat{z} \\
(\frac{\sin X_{x11} - \sin X_{x12}}{X_{x11}}) (\frac{\sin Y_{x11} - \sin Y_{x12}}{Y_{x11}}) e^{ik_0R_{ex}} \\
+ \{(A_{7ab} + A_{8ab}) \hat{y} + (A_{7bb} + A_{8bb}) \hat{z}\} \\
(\frac{\sin X_{x21} - \sin X_{x22}}{X_{x21}}) (\frac{\sin Y_{x21} - \sin Y_{x22}}{Y_{x21}}) e^{ik_0R_y} \right]
\]

(2.55)

2.5.3) The static magnetic field is along y-axis

In this case \((\delta = \hat{x}, \varepsilon = \hat{y}, \hat{z} = \hat{z})\), \(\varphi_c = \varphi_y\) is the angle between the z-axis and the projection of the distance vector on the \(xz\)-plane. The angle is in the counterclockwise direction from the z-axis. The integrand in eq. (2.48) becomes

\[
\bar{X}_1 = \pi E_0 \left\{ \frac{B_1(R)}{\varepsilon_1} \left( \frac{x \delta + y \hat{y} + z \hat{z}}{\varepsilon_1} \right) + \frac{B_2(R)}{\varepsilon_1} \hat{z} \right\} \sin \left( \frac{m\pi \xi}{a} \right) \sin \left( \frac{n\pi \hat{y}}{b} \right) e^{ik_0R_{ey}}
\]
After substituting all of the above back into eq. (2.48) and carrying out the integration at the aperture, we obtain

\[
\mathbf{E}(\mathbf{r}) = \pi E_0 \frac{a b}{4} \left[ \frac{B_1 \mathbf{z}}{\varepsilon_1} \left( \frac{x \mathbf{r}}{\varepsilon_1} + \frac{y \mathbf{r}}{\varepsilon_2} + \frac{z \mathbf{r}}{\varepsilon_1} \right) + \frac{B_2 \mathbf{z}}{\varepsilon_1} (\mathbf{r} + i \varepsilon_2 \mathbf{z} + \varepsilon_1 \mathbf{z}) + \frac{B_3 \mathbf{z}}{\varepsilon_1} \mathbf{r} \right]
\]

\[
- \frac{a^2}{b^2} (A_{5ba} + A_{6ba}) \mathbf{X} + (A_{5aa} + A_{6aa}) \mathbf{Z}
\]

\[
\left( \frac{\sin X_{Y11}}{X_{y11}} - \frac{\sin X_{Y12}}{X_{y12}} \right) \left( \frac{\sin Y_{Y11}}{Y_{y11}} - \frac{\sin Y_{Y12}}{Y_{y12}} \right) e^{ik_0 R_y}
\]

\[
+ \pi E_0 \frac{a b}{4} \left( (A_{7ab} + A_{8ab}) \mathbf{X} + (A_{7bb} + A_{8bb}) \mathbf{Z} \right)
\]

\[
\left( \frac{\sin X_{Y21}}{X_{y21}} - \frac{\sin X_{Y22}}{X_{y22}} \right) \left( \frac{\sin Y_{Y21}}{Y_{y21}} - \frac{\sin Y_{Y22}}{Y_{y22}} \right) e^{ik_0 R_y}
\]  

The electric field that we obtained in the above two sections can be directly reduced to the result when the medium becomes uniaxial or isotropic. The dyadic Green's functions which characterize those media will be as specified in section 2.3.1 and 2.3.2.
2.6 **Numerical Solution**

In section 2.3, we obtained the final expression for the dyadic Green's function in a closed form. For a given source distribution as specified by eq. (2.5), the fields will be determined. Some examples of the field patterns in an unbounded anisotropic plasma will be plotted and compared to the field patterns in isotropic and uniaxial media. To plot the field patterns, the procedure and specification will be as follow:

1. Only far-field patterns will be plotted.

2. First, the source will be given as an infinitesimal oscillating electric dipole of length 1 with constant current $I_0$ and frequency $2.85 \, \text{GHz}$. $f = 2.85 \, \text{GHz}$

3. The orientation of the dipole will be 1) along $z$-axis, 2) along $y$-axis, and 3) on $xy$-plane, respectively.

4. The field patterns will be compared with the field patterns in isotropic and uniaxial media when the static magnetic field is along the $z$- and $y$-axis.

5. Second, the given rectangular aperture source distribution will be considered. This aperture is located on $xy$-plane. The configuration of the aperture in the $x$ and $y$-directions are $a_x(=2\lambda)$ and $b_x(=3\lambda)$ respectively.

6. Repeat step 4 considering also a static magnetic field along the $x$-axis.
In the figures of the field patterns, the following notation will be used:

- Solid line (———) represents the magnitude of the electric field of the extraordinary wave in an anisotropic plasma. In Figures 2.2, 2.10-11, 2.20-21 and 2.34-35 the solid line will also be used to represent the magnitude of the electric field in an isotropic medium.

- One dash one dot line (_._._._._._.) represents the magnitude of the electric field of the ordinary wave in an anisotropic plasma.

- Dash line (---) represents the magnitude of the electric field of the extraordinary wave in a uniaxial medium.

- Dot line (••••••••••) represents the magnitude of the electric field of the ordinary wave in a uniaxial medium.

The effect of the orientation of the static magnetic field on the radiation patterns will be discussed in the following section.
Figure 2.2 Dipole (along \( z \)-axis) radiated in an isotropic medium. The field patterns are plotted the same on \( \mathbf{xz} \)-plane and \( \mathbf{yz} \)-plane.

Figure 2.3 Dipole (along \( z \)-axis) radiated in uniaxial and plasma. The static magnetic field is along \( z \)-axis. 
\[ \omega_b = 0.7\omega, \quad \omega_p = 0.3\omega \]
The field patterns are plotted the same on \( \mathbf{xz} \)-plane and \( \mathbf{yz} \)-plane.
Figure 2.4 Dipole (along z-axis) radiated in uniaxial medium and plasma. The static magnetic field is along z-axis. 
\[ \omega_b = 1.2 \omega, \omega_p = 0.3 \omega \] 
The field patterns are plotted the same on xz-plane and yz-plane.

Figure 2.5 Dipole (along z-axis) radiated in uniaxial medium and plasma. The static magnetic field is along z-axis. 
\[ \omega_b = 0.7 \omega, \omega_p = 0.2 \omega \] 
The field patterns are plotted the same on xz-plane and yz-plane.
Figure 2.6 Dipole (along z-axis) radiated in uniaxial medium and plasma. The static magnetic field is along y-axis. 
\[ \omega_b = 0.7\omega, \quad \omega_p = 0.3\omega \]
The field patterns are plotted on xz-plane.

Figure 2.7 Dipole (along z-axis) radiated in uniaxial medium and plasma. The static magnetic field is along y-axis. 
\[ \omega_b = 0.7\omega, \quad \omega_p = 0.3\omega \]
The field patterns are plotted on yz-plane.
Figure 2.8 Dipole (along z-axis) radiated in uniaxial medium and plasma. The static magnetic field is along y-axis. \( \omega_b = 1.2\omega, \ \omega_p = 0.3\omega \) The field patterns are plotted on xz-plane.

Figure 2.9 Dipole (along z-axis) radiated in uniaxial medium and plasma. The static magnetic field is along y-axis. \( \omega_b = 1.2\omega, \ \omega_p = 0.3\omega \) The field patterns are plotted on yz-plane.
Figure 2.10 Dipole (along y-axis) radiated in isotropic medium. The field pattern is plotted on xz-plane.

Figure 2.11 Dipole (along y-axis) radiated in isotropic medium. The field pattern is plotted on yz-plane.
Figure 2.12 Dipole (along y-axis) radiated in uniaxial medium and plasma. The static magnetic field is along z-axis. $\omega_b = 0.7 \omega$, $\omega_p = 0.3 \omega$ The field patterns are plotted on xz-plane.

Figure 2.13 Dipole (along y-axis) radiated in uniaxial medium and plasma. The static magnetic field is along z-axis. $\omega_b = 0.7 \omega$, $\omega_p = 0.3 \omega$ The field patterns are plotted on yz-plane.
Figure 2.14 Dipole (along y-axis) radiated in uniaxial medium and plasma. The static magnetic field is along z-axis. \( \omega_0 = 1.2\omega, \omega_p = 0.3\omega \) The field patterns are plotted on \( \mathbf{xz} \)-plane.

Figure 2.15 Dipole (along y-axis) radiated in uniaxial medium and plasma. The static magnetic field is along z-axis. \( \omega_0 = 1.2\omega, \omega_p = 0.3\omega \) The field patterns are plotted on \( \mathbf{yz} \)-plane.
Figure 2.16 Dipole (along $y$-axis) radiated in uniaxial medium and plasma.
The static magnetic field is along $y$-axis.
\[ \omega_b = 0.7\omega, \quad \omega_p = 0.3\omega \]
The field patterns are plotted on $xz$-plane.

Figure 2.17 Dipole (along $y$-axis) radiated in uniaxial medium and plasma.
The static magnetic field is along $y$-axis.
\[ \omega_b = 0.7\omega, \quad \omega_p = 0.3\omega \]
The field patterns are plotted on $yz$-plane.
Figure 2.18 Dipole (along y-axis) radiated in uniaxial medium and plasma. The static magnetic field is along y-axis. 
\[ \omega_b = 1.2\omega, \omega_p = 0.3\omega \]
the field patterns are plotted on \(xz\)-plane.

Figure 2.19 Dipole (along y-axis) radiated in uniaxial medium and plasma. The static magnetic field is along y-axis. 
\[ \omega_b = 1.2\omega, \omega_p = 0.3\omega \]
The field patterns are plotted on \(yz\)-plane.
Figure 2.20 Dipole (along \(zy\)-axis) radiated in isotropic medium. The field pattern is plotted on \(xz\)-plane.

Figure 2.21 Dipole (along \(zy\)-axis) radiated in isotropic medium. The field pattern is plotted on \(yz\)-plane.
Figure 2.22 Dipole (along $zy$-axis) radiated in uniaxial medium and plasma. The static magnetic field is along $z$-axis. $\omega_b = 0.7\omega$, $\omega_p = 0.3\omega$. The field patterns are plotted on $xz$-plane.

Figure 2.23 Dipole (along $zy$-axis) radiated in uniaxial medium and plasma. The static magnetic field is along $z$-axis. $\omega_b = 0.7\omega$, $\omega_p = 0.3\omega$. The field patterns are plotted on $yz$-plane.
Figure 2.24 Dipole (along zy-axis) radiated in uniaxial medium and plasma. The static magnetic field is along z-axis.
\[ \omega_b = 1.2\omega, \quad \omega_p = 0.3\omega \]
The field patterns are plotted on xz-plane.

Figure 2.25 Dipole (along zy-axis) radiated in uniaxial medium and plasma. The static magnetic field is along z-axis.
\[ \omega_b = 1.2\omega, \quad \omega_p = 0.3\omega \]
The field patterns are plotted on yz-plane.
Figure 2.26 Dipole (along zy-axis) radiated in uniaxial medium and plasma. The static magnetic field is along x-axis. 
\[ \omega_b = 0.7\omega, \quad \omega_p = 0.3\omega \]
The field patterns are plotted on \(xz\)-plane.

Figure 2.27 Dipole (along zy-axis) radiated in uniaxial medium and plasma. The static magnetic field is along x-axis 
\[ \omega_b = 0.7\omega, \quad \omega_p = 0.3\omega \]
The field patterns are plotted on \(yz\)-plane.
Figure 2.28 Dipole (along $zy$-axis) radiated in uniaxial medium and plasma. The static magnetic field is along $x$-axis. $\omega_b = 1.2\omega$, $\omega_p = 0.3\omega$ The field patterns are plotted on $xz$-plane.

Figure 2.29 Dipole (along $zy$-axis) radiated in uniaxial medium and plasma. The static magnetic field is along $x$-axis. $\omega_b = 1.2\omega$, $\omega_p = 0.3\omega$ The field patterns are plotted on $yz$-plane.
Figure 2.30 Dipole (along \(zy\)-axis) radiated in uniaxial medium and plasma. The static magnetic field is along \(y\)-axis. \(\omega_b = 0.7\omega\), \(\omega_p = 0.3\omega\). The field patterns are plotted on \(xz\)-plane.

Figure 2.31 Dipole (along \(zy\)-axis) radiated in uniaxial medium and plasma. The static magnetic field is along \(y\)-axis. \(\omega_b = 0.7\omega\), \(\omega_p = 0.3\omega\). The field patterns are plotted on \(yz\)-plane.
Figure 2.32 Dipole (along zy-axis) radiated in uniaxial medium and plasma. The static magnetic field is along y-axis.

\[ \omega_b = 1.2\omega, \quad \omega_p = 0.3\omega \]

The field patterns are plotted on \(xz\)-plane.

---

Figure 2.33 Dipole (along zy-axis) radiated in uniaxial medium and plasma. The static magnetic field is along y-axis.

\[ \omega_b = 1.2\omega, \quad \omega_p = 0.3\omega \]

The field patterns are plotted on \(yz\)-plane.
Figure 2.34 Rec. Aper. radiated in isotropic medium, reduced from $\omega_0 = 0.7\omega$, $\omega_p = 0.3\omega$ by using the same diagonal elements of dielectric tensor. The field pattern is plotted on $xz$-plane.

Figure 2.35 Rec. Aper. radiated in isotropic medium, reduced from $\omega_0 = 0.7\omega$, $\omega_p = 0.3\omega$ by using the same diagonal elements of dielectric tensor. The field pattern is plotted on $yz$-plane.
Figure 2.36 Rec. Aper. radiated in uniaxial medium and plasma. The static magnetic field is along \( z \)-axis. 
\[ \omega_b = 0.7\omega, \quad \omega_p = 0.3\omega \]
The field patterns are plotted on \( \text{xz-plane} \).

Figure 2.37 Rec. Aper. radiated in uniaxial medium and plasma. The static magnetic field is along \( z \)-axis. 
\[ \omega_b = 0.7\omega, \quad \omega_p = 0.3\omega \]
The field patterns are plotted on \( \text{yz-plane} \).
Figure 2.38 Rec. Aper. radiated in uniaxial medium and plasma.
The static magnetic field is along z-axis.
\( \omega_b = 1.2\omega, \omega_p = 0.3\omega \)
The field patterns are plotted on \( \text{zx-plane} \).

Figure 2.39 Rec. Aper. radiated in uniaxial medium and plasma.
the static magnetic field is along z-axis.
\( \omega_b = 1.2\omega, \omega_p = 0.3\omega \)
The field patterns are plotted on \( \text{yz-plane} \).

Figure 2.40 Rec. Aper. radiated in uniaxial medium and plasma.
The static magnetic field is along y-axis.
\( \omega_p = 0.7\omega, \\omega_p = 0.3\omega \)
The field patterns are plotted on \( xz \)-plane.

Figure 2.41 Rec. Aper. radiated in uniaxial medium and plasma.
The static magnetic field is along y-axis.
\( \omega_p = 0.7\omega, \\omega_p = 0.3\omega \)
The field patterns are plotted on \( yz \)-plane.
Figure 2.42 Rec. Aper. radiated in uniaxial medium and plasma.
The static magnetic field is along y-axis.
\( \omega_b = 1.2\omega, \omega_p = 0.3\omega \)
The field patterns are plotted on \( xz \)-plane.

Figure 2.43 Rec. Aper. radiated in uniaxial medium and plasma.
The static magnetic field is along y-axis.
\( \omega_b = 1.2\omega, \omega_p = 0.3\omega \)
The field patterns are plotted on \( yz \)-plane.
2.7 **Summary**

Utilizing Chen's approach enables us to obtain the general solution of the dyadic Green's function in closed form for an anisotropic plasma. With this final explicit solution, we can

1. characterize plasma by one simple dyadic Green's function in closed form.

2. have the static magnetic field oriented in an arbitrary direction.

3. reduce directly to isotropic and uniaxial media.

4. express the cut-off conditions for each wave when \( a^2 \) or \( b^2 \) is less than or equal to zero. After the computation in the case of plasma, \( b^2 \) always equals unity. This implies that there always exists the field due to \( b^2 \). We will refer to this field as 'ordinary field'. We will refer to the field corresponding to \( a^2 \) as 'extraordinary field'. The cut-off frequency of extraordinary field is given by

\[
2\omega_p^2 + \omega_b^2 < \omega^2
\]

when \( \omega > \omega_b \).

2.7.1 *Comparison of field configurations*

From the numerical solution we can summarize as follows:

A. **Isotropic and uniaxially anisotropic media**

1. The configuration in a uniaxial medium of either the extraordinary or the ordinary field pattern is the same as that of the field pattern in an isotropic medium, except for their amplitudes. These amplitudes depend on the orientation of the optic axis and the plane of observation.

2. The orientation of the dipole is in the direction of
the static magnetic field and coincides with either one of the coordinate axes. The dominant field in uniaxial medium is the extraordinary field. The configuration of the field pattern in a uniaxial medium is the same as the field pattern in an isotropic medium except for its amplitude.

3. The orientation of the dipole is in the direction of the static magnetic field but oblique with the coordinate axes. The configuration of the dominant field pattern in uniaxial medium is slightly different compared with the field pattern in isotropic medium. The amplitudes are different.

4. The orientation of the dipole and the direction of the static magnetic field coincide with the coordinate axes, but not in the same direction. In uniaxial medium, one field dominates while the other vanishes. The dominant field depends on the plane of observation on the coordinate axes.

5. In the case of a rectangular aperture, the field patterns are directional. The resulting field patterns can be summarized as in the case of a dipole oriented along the z-axis.

B. Isotropic, Uniaxial media and anisotropic plasma

1. When $\epsilon_3$ is very small (compared to $\epsilon_1$ and $\epsilon_2$), the configuration of the extraordinary field (ordinary) pattern(s) in plasma and extraordinary field (ordinary) pattern(s) in a uniaxial medium are nearly the same except for their amplitudes.

2. When $\epsilon_3$ is not very small (compared to $\epsilon_1$ and $\epsilon_2$), the
configuration of the field pattern(s) in plasma deviates significantly compared to the field pattern(s) in a uniaxial medium.

3. Regardless of how small $\epsilon_j$ is, neither an extraordinary nor an ordinary field will dominate in plasma. In other words, both fields will always exist in plasma.

4. For a rectangular aperture, the configuration of the field patterns in plasma are totally different from the field patterns in isotropic and uniaxial media.
CHAPTER 3
Reformulation of Image Theory in Isotropic medium
and its extension to Uniaxial Medium

Introduction

The conventional image theory [19,21,39] consists of the following steps: a) placing the image source and removing the conducting plane, b) assuming the same medium over the entire space, and c) solving for the field solution above the conducting plane in the presence of both original and image sources utilizing the Uniqueness theorem [31,32] and the boundary condition on the conducting plane. The field solution thus obtained is valid only above the conducting plane. The conventional image theory is applicable in an isotropic medium. In an anisotropic plasma, it is applicable only when the static magnetic field is perpendicular to the conducting plane. Our objective in this chapter is to reformulate the image theory for an isotropic medium which will enable us to extend it to the uniaxial medium in the second part and to anisotropic plasma in the following chapter. We will use the dyadic Green's function in closed form for this reformulation.

Because of the reflection from the perfectly conducting plane, the total field will be composed of two fields: one radiated from the source directly and the other radiated due to the reflection of the conducting plane. In the conventional image theory the reflected field is considered as the field due to the image source. This image source is for analysis
purposes only. To establish an alternate image theory the exact expression of the field must be given utilizing dyadic Green's function in closed form. The total field due to an infinitesimal oscillating electric dipole will be considered. The unknown source distribution for the reflected field will be derived. Examples will be given when the orientation of the dipole is 1) perpendicular to the conducting plane, and 2) parallel to the conducting plane. First an isotropic medium will be considered, then the result will be extended to the case of a uniaxial medium. The field pattern will be plotted for a uniaxial medium when the optic axis is along the z-, x- or y-axis respectively.

3.1 Reformulation of Image Theory in Isotropic medium

Assume that an isotropic medium is characterized by the dielectric constant $\varepsilon$. Using the dyadic Green's function and a given source distribution, the electric field in an unbounded isotropic medium will be reviewed. The field above the conducting plane will be discussed. The unknown reflected source (for analysis purposes only) in terms of the directed source will be derived. The field must satisfy the boundary condition on the conducting plane. The results will be expressed in both rectangular and spherical coordinates.

3.1.1 Summary of the electric field in an unbounded medium

The electric field in an unbounded region can be written as $[26,27,28]$
$$\mathbf{E}(\mathbf{r}) = \iint \mathcal{G}_d(\mathbf{r}, \mathbf{r}) \cdot \mathbf{S}(\mathbf{r}) \, d^3 r$$  \hspace{1cm} (3.1)$$

where $\mathbf{S}(\mathbf{r})$ is the given source distribution located at $\mathbf{r}$
and $\mathcal{G}_d(\mathbf{r}, \mathbf{r})$ is the dyadic Green's function of an isotropic
medium which is [29,30]

$$\mathcal{G}_d(\mathbf{r}) = \mathcal{G}_d(\mathbf{r}) e^{i k_0 \sqrt{\varepsilon R}} ; \quad \mathbf{R} = \mathbf{r} - \mathbf{r}$$  \hspace{1cm} (3.2)$$

$$\mathcal{G}_d(\mathbf{r}) = \frac{1}{4\pi R} \left\{ C_1(R) \mathbf{r} + C_2(R) \hat{\mathbf{R}} \right\}$$

$$C_1(R) = 1 + \frac{i}{k_0 \sqrt{\varepsilon R}} - \frac{1}{k_0^2 \varepsilon R^2}$$

$$C_2(R) = \frac{3}{k_0^2 \varepsilon R^2} - \frac{3i}{k_0 \sqrt{\varepsilon R}} - 1$$

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\hat{\mathbf{R}} = \frac{1}{R} (\mathbf{r} - \mathbf{r}) = (x-x') \hat{x} + (y-y') \hat{y} + (z-z') \hat{z}$$

3.1.2 Fields above an infinite conducting plane

If the given source distribution is located over an infinite conducting plane, see Fig.3.1, at $\mathbf{r}_1$.

The field due to the direct path from the source would be

$$\mathbf{E}_d(\mathbf{r}) = \iint \mathcal{G}_d(\mathbf{r}_1) e^{i k_0 \sqrt{\varepsilon R_1}} \mathbf{S}_d(\mathbf{r}_1) \, d^3 r_1$$  \hspace{1cm} (3.3)$$

where $R_1 = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}$
and the field due to the reflected path produced from the conducting plane would be

$$\mathbf{E}_r(\mathbf{r}) = \iint \mathcal{G}_d(\mathbf{r}_2) e^{i k_0 \sqrt{\varepsilon R_2}} \mathbf{S}_d(\mathbf{r}_2) \, d^3 r_2$$  \hspace{1cm} (3.4)$$
If the given source distribution is an infinitesimal oscillating electric dipole located at \((x_i, y_i, z_i)\), then the source distribution can be written as

\[
\vec{S}(\vec{r}) = i\omega \mu_0 \vec{J}(\vec{r}) = i\omega \mu_0 \vec{I} \delta(x-x_i) \delta(y-y_i) \delta(z-z_i)
\]  
(3.5)

### 3.1.3 Dipole located at \((0, 0, h)\)

The field due to a direct source

\[
\vec{E}_d(\vec{r}) = i\omega \mu_0 \vec{I} \hat{\vec{r}} \delta(x) \delta(y) \delta(z-h)
\]  
(3.6)

is

\[
\vec{E}_d(\vec{r}) = \frac{\vec{E}_d(\vec{r})}{\sqrt{R_1}} e^{ik_0 \sqrt{x^2+y^2+(z-h)^2}} \cdot i\omega \mu_0 \vec{I} \hat{\vec{r}}
\]  
(3.7)

and the field due to the image source

\[
\vec{E}_d(\vec{r}) = \frac{\vec{E}_d(\vec{r})}{\sqrt{R_2}} e^{ik_0 \sqrt{x^2+y^2+(z-h)^2}} \cdot i\omega \mu_0 \vec{I} \hat{\vec{r}}
\]  
(3.8)

where \(R_2 = \sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}\).
The field above the \( z = 0 \) plane is unique if it satisfies the same boundary condition of the two problems which

a) in the first problem with the conducting plane, states that the tangential component of the electric field (or the normal component of the magnetic field) must vanish on the conducting plane,

and b) in the second problem when the conducting plane is removed, the Uniqueness Theorem \([31,32]\) states that when the tangential components of the electric field (or magnetic field) on the perfect conducting plane is specified, then the field radiated by the given source distribution is unique within that region.

The above conditions imply that the fields of the two problems, one with the conducting plane and the other when the conducting plane is removed, are the same.

By the boundary condition and the Uniqueness Theorem mentioned above, the transformation equations for the fields in the upper and lower half regions can be written as

\[
E_{xr} = -E_{xd} ; E_{yr} = -E_{yd} ; E_{zr} = E_{zd}
\]

or

\[
\mathbf{E}_r(\vec{r}) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{E}_d(\vec{r}) = \mathbf{P} \cdot \mathbf{E}_d(\vec{r}) \bigg|_{z=0(\theta=90^\circ)} \quad (3.10)
\]
\[ \mathbf{F} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -2x - 2y + 2z \]

\[ \mathbf{G}_1(\mathbf{R}_2c) e^{i k_0 \sqrt{x^2 + y^2 + h^2}} \mathbf{i} \omega \mu_0 \mu_0 \mathbf{E}_1 = \mathbf{F} \cdot \mathbf{G}_1(\mathbf{R}_1c) e^{i k_0 \sqrt{x^2 + y^2 + h^2}} \mathbf{i} \omega \mu_0 \mu_0 \mathbf{E}_1 \]

\[ \mathbf{G}_1(\mathbf{R}_2c) \cdot \mathbf{I}_2 = \mathbf{F} \cdot \mathbf{G}_1(\mathbf{R}_1c) \cdot \mathbf{I}_1 \]

\[ R_{2c} = R_{1c} = \sqrt{x^2 + y^2 + h^2} \]

\[ \mathbf{I}_2 = [\mathbf{G}_1^{-1}(\mathbf{R}_2c) \cdot \mathbf{F} \cdot \mathbf{G}_1(\mathbf{R}_1c)] \cdot \mathbf{I}_1 = [\mathbf{S}_p] \cdot \mathbf{I}_1 \]  \hspace{1cm} (3.11)

\[ \mathbf{S}_p = \mathbf{G}_1^{-1}(\mathbf{R}_2c) \cdot \mathbf{F} \cdot \mathbf{G}_1(\mathbf{R}_1c) \]  \hspace{1cm} (3.12)

In an isotropic medium: \( \mathbf{S}_p = \mathbf{F} \); hence from eq. (3.8), we have

\[ \mathbf{S}_x(\mathbf{I}) = i \omega \mu_0 \mu_0 \mathbf{E}_1 (\mathbf{F} \cdot \mathbf{I}_1) \delta(\mathbf{x}) \delta(\mathbf{y}) \delta(\mathbf{z} + h) \]  \hspace{1cm} (3.13)

The electric field above the \( z = 0 \) plane will be

\[ \mathbf{E}_T(\mathbf{r}) = \mathbf{E}_d(\mathbf{r}) + \mathbf{E}_x(\mathbf{r}) \]

\[ = i \omega \mu_0 \mu_0 \mathbf{E}_1 e^{i k_0 \sqrt{\mathbf{R}_1^2}} \mathbf{G}_1(\mathbf{R}_1c) \cdot \mathbf{I}_1 + i \omega \mu_0 \mu_0 \mathbf{E}_1 e^{i k_0 \sqrt{\mathbf{R}_2^2}} \mathbf{G}_1(\mathbf{R}_2c) \cdot \mathbf{I}_2 \]

\[ = \frac{i \omega \mu_0 \mu_0 \mathbf{E}_1 e^{i k_0 \sqrt{\mathbf{R}_1^2}}}{4\pi \mathbf{R}_1} \left\{ C_1(\mathbf{R}_1) \mathbf{I} + C_2(\mathbf{R}_1) \mathbf{R}_1 \cdot \mathbf{I}_1 \right\} \]

\[ + \frac{i \omega \mu_0 \mu_0 \mathbf{E}_1 e^{i k_0 \sqrt{\mathbf{R}_2^2}}}{4\pi \mathbf{R}_2} \left\{ C_1(\mathbf{R}_2) \mathbf{I} + C_2(\mathbf{R}_2) \mathbf{R}_2 \cdot \mathbf{I}_2 \right\} (\mathbf{F} \cdot \mathbf{I}_1) \]

\[ R_1 = \sqrt{x^2 + y^2 + (z - h)^2} \]
\[ R_2 = \sqrt{x^2 + y^2 + (z + h)^2} \]
\[ \hat{R}_1 = \frac{1}{R_1} (x\hat{x} + y\hat{y} + (z - h) \hat{z}) \]
\[ \hat{R}_2 = \frac{1}{R_2} (x\hat{x} + y\hat{y} + (z + h) \hat{z}) \]
\[ \hat{\mathbf{t}}_1 = x\hat{x} + y\hat{y} + z\hat{z} ; \quad \sqrt{x_1^2 + y_1^2 + z_1^2} = 1 \]
\[ \overline{P} \cdot \hat{\mathbf{t}}_1 = -x\hat{x} - y\hat{y} + z\hat{z} \]
\[ (\hat{R}_1 \cdot \hat{\mathbf{t}}_1) \hat{R}_1 = \frac{1}{R_1^2} (xx_1 + yy_1 + (z - h) z_1)(x\hat{x} + y\hat{y} + (z - h) \hat{z}) \]
\[ (\hat{R}_2 \cdot \overline{P} \cdot \hat{\mathbf{t}}_1) \hat{R}_2 = \frac{1}{R_2^2} (-xx_1 - yy_1 + (z + h) z_1)(x\hat{x} + y\hat{y} + (z + h) \hat{z}) \]

Check on the \( z = 0 \) plane

\[ z = 0 , \quad R_{1c} = R_{2c} = \sqrt{x^2 + h^2} , \quad C_1(R_{1c}) = C_1(R_{2c}) , \quad C_2(R_{1c}) = C_2(R_{2c}) \]
\[ (\hat{R}_1 \cdot \hat{\mathbf{t}}_1) \hat{R}_1 = \frac{1}{R_{1c}^2} (xx_1 + yy_1 - h z_1)(x\hat{x} + y\hat{y} - h \hat{z}) \]
\[ (\hat{R}_2 \cdot \overline{P} \cdot \hat{\mathbf{t}}_1) \hat{R}_2 = \frac{1}{R_{2c}^2} (-xx_1 - yy_1 + h z_1)(x\hat{x} + y\hat{y} + h \hat{z}) \]

The electric field on the \( z = 0 \) plane will be

\[ \overline{E}_{tc}(\overline{r}_c) = 2 \frac{i\omega \mu_0 I e^{ik_{tc} R_{1c}}}{4\pi R_{1c}} \left\{ C_1(R_{1c}) z_1 - C_2(R_{1c}) \frac{h}{R_{1c}^2} (xx_1 + yy_1 - h z_1) \right\} \hat{z} \]

(3.15)

The tangential components of the electric field satisfy the same boundary condition at the \( z = 0 \) plane. Hence according to uniqueness theorem the fields in the region \( z > 0 \) for the two problems are the same.
3.1.3.1 Dipole perpendicular to the conducting plane

In this case, \( x_1 = y_1 = 0, z_1 = 1 ; \vec{I}_1 = \hat{z} ; \vec{I}_2 = \vec{r} \cdot \vec{I}_1 = 2 \)

\[
\bar{E}_d(\vec{r}) = \frac{i \omega \mu_0 I e^{ik_0 \sqrt{R_1}}}{4 \pi R_1} \left\{ C_1(R_1) \vec{I}_1 + C_2(R_1) (\vec{R}_1 \cdot \vec{I}_1) \vec{R}_1 \right\}
\]

\[
\vec{R}_1 \cdot \vec{I}_1 = \frac{1}{R_1} (x \hat{x} + y \hat{y}) \cdot \hat{z} = \frac{z-h}{R_1}
\]

\[
\bar{E}_x(\vec{r}) = \frac{i \omega \mu_0 I e^{ik_0 \sqrt{R_2}}}{4 \pi R_2} \left\{ C_1(R_2) \vec{I}_2 + C_2(R_2) (\vec{R}_2 \cdot \vec{I}_2) \vec{R}_2 \right\}
\]

\[
\vec{R}_2 \cdot \vec{I}_2 = \frac{1}{R_2} (x \hat{x} + y \hat{y}) \cdot (z+h) \hat{z} = \frac{z+h}{R_2}
\]

The rectangular components of the electric field above the \( z = 0 \) plane will be

\[
\bar{E}_T(\vec{r}) = \bar{E}_d(\vec{r}) + \bar{E}_x(\vec{r})
\]

\[
= \frac{i \omega \mu_0 I e^{ik_0 \sqrt{R_1}}}{4 \pi R_1} \left\{ C_1(R_1) \hat{z} + C_2(R_1) \left( \frac{z-h}{R_1} \hat{z} \right) \frac{(x \hat{x} + y \hat{y}) \cdot \hat{z}}{R_1} \right\}
\]

\[
+ \frac{i \omega \mu_0 I e^{ik_0 \sqrt{R_2}}}{4 \pi R_2} \left\{ C_1(R_2) \hat{z} + C_2(R_2) \left( \frac{z+h}{R_2} \hat{z} \right) \frac{(x \hat{x} + y \hat{y}) \cdot \hat{z}}{R_2} \right\}
\]

or

\[
E_{Tx} = \frac{i \omega \mu_0 I e^{ik_0 \sqrt{R_1}}}{4 \pi R_1} C_2(R_1) \left( \frac{r \cos \theta - h}{R_1} \right) \frac{r \sin \theta \cos \phi}{R_1}
\]

\[
+ \frac{i \omega \mu_0 I e^{ik_0 \sqrt{R_2}}}{4 \pi R_2} C_2(R_2) \left( \frac{r \cos \theta + h}{R_2} \right) \frac{r \sin \theta \cos \phi}{R_2}
\]

\[
E_{Ty} = \frac{i \omega \mu_0 I e^{ik_0 \sqrt{R_1}}}{4 \pi R_1} C_2(R_1) \left( \frac{r \cos \theta - h}{R_1} \right) \frac{r \sin \theta \sin \phi}{R_1}
\]
or in spherical components

\[
E_{Tr} = \frac{i \omega \mu_0 I e^{ik_0 \sqrt{\epsilon}}}{4\pi R_1} \left\{ C_1(R_1) \cos \theta + C_2(R_1) \left( \frac{R}{R_1} \cos \theta - \frac{h}{R_1} \right) \left( \frac{R}{R_1} - \frac{h}{R_1} \cos \theta \right) \right\}
\]

\[
+ \frac{i \omega \mu_0 I e^{ik_0 \sqrt{\epsilon}}}{4\pi R_2} \left\{ C_1(R_2) \cos \theta + C_2(R_2) \left( \frac{R}{R_2} \cos \theta + \frac{h}{R_2} \right) \left( \frac{R}{R_2} + \frac{h}{R_2} \cos \theta \right) \right\}
\]

\[
E_{T\theta} = \frac{i \omega \mu_0 I e^{ik_0 \sqrt{\epsilon}}}{4\pi R_1} \left\{ -C_1(R_1) \sin \theta + C_2(R_1) \left( \frac{R}{R_1} \cos \theta - \frac{h}{R_1} \right) \frac{h}{R_1} \sin \theta \right\}
\]

\[
+ \frac{i \omega \mu_0 I e^{ik_0 \sqrt{\epsilon}}}{4\pi R_2} \left\{ -C_1(R_2) \sin \theta - C_2(R_2) \left( \frac{R}{R_2} \cos \theta + \frac{h}{R_2} \right) \frac{h}{R_2} \sin \theta \right\}
\]

\[
E_{T\phi} = 0
\]

(3.18a–c)

which agree with the results of the previous works [33,34,35].

3.1.3.2 Dipole parallel to the conducting plane

In this case, \( x_1 = z_1 = 0, y_1 = 1 \); \( \hat{T}_1 = \hat{y}; \hat{T}_2 = \hat{F} \cdot \hat{T}_1 = -\hat{y} \)

\[
\overline{E}_d(\vec{r}) = \frac{i \omega \mu_0 I e^{ik_0 \sqrt{\epsilon}}}{4\pi R_1} \left\{ C_1(R_1) \hat{T}_1 + C_2(R_1) (\hat{R}_1 \cdot \hat{T}_1) \hat{R}_1 \right\}
\]

\[
\hat{R}_1 \cdot \hat{T}_1 = \frac{1}{R_1} (x \hat{\epsilon} + y \hat{\epsilon} + (z - h) \hat{z}) \cdot \hat{y} = \frac{V}{R_1}
\]

\[
\overline{E}_z(\vec{r}) = \frac{i \omega \mu_0 I e^{ik_0 \sqrt{\epsilon}}}{4\pi R_2} \left\{ C_1(R_2) \hat{T}_2 + C_2(R_2) (\hat{R}_2 \cdot \hat{T}_2) \hat{R}_2 \right\}
\]

\[
\hat{R}_2 \cdot \hat{T}_2 = \frac{1}{R_2} (x \hat{\epsilon} + y \hat{\epsilon} + (z + h) \hat{z}) (-\hat{y}) = -\frac{V}{R_2}
\]
The electric field in rectangular coordinates above the $z = 0$ plane will be

$$E_T = E_d + E_r$$

$$= \frac{i\mu_0 I e^{ik_0\sqrt{R_1}}}{4\pi R_1} \left\{ C_1(R_1) \hat{y} + C_2(R_1) \left( \frac{y}{R_1} \frac{(x^2+y^2+(z-h)^2)}{R_1} \right) \right\}$$

$$+ \frac{i\mu_0 I e^{ik_0\sqrt{R_2}}}{4\pi R_2} \left\{ -C_1(R_2) \hat{y} + C_2(R_2) \left( -\frac{y}{R_2} \frac{(x^2+y^2+(z+h)^2)}{R_2} \right) \right\}$$

(3.19)

Check for the far field above the $z = 0$ plane

In the far-zone [41], we have

$R_1 = r-h\cos\theta$, $R_2 = r+h\cos\theta$; for phase term

$R_1 = R_2 = r$; for amplitude variation

$$\frac{h}{R_1} = \frac{h}{R_2} \to 0$$

Hence, the rectangular components of the electric field in eq.(3.19) reduce to

$$E_{Txf} = \frac{i\mu_0 I e^{ik_0\sqrt{r}}}{4\pi r} \left[ 2i\sin(k_0\sqrt{\epsilon h}\cos\theta) \sin^2\theta \sin\phi \cos\phi \right]$$

$$E_{Tyf} = \frac{i\mu_0 I e^{ik_0\sqrt{r}}}{4\pi r} \left[ 2i\sin(k_0\sqrt{\epsilon h}\cos\theta) \right] (\sin^2\theta \sin^2\phi - 1)$$

$$E_{Tzf} = \frac{i\mu_0 I e^{ik_0\sqrt{r}}}{4\pi r} \left[ 2i\sin(k_0\sqrt{\epsilon h}\cos\theta) \right] \sin\theta \cos\theta \sin\phi \sin\phi$$

(3.20a-c)

or in spherical components, eq.(3.19) becomes

$$E_{Txf} = 0$$
\[ E_{\phi \phi} = \frac{i \omega \mu_0 I l e^{i k_0 \sqrt{\varepsilon} r}}{4 \pi r} (-\cos \theta \sin \varphi)[2i \sin(k_0 \sqrt{\varepsilon} h \cos \theta)] \]

\[ E_{\phi \psi} = \frac{i \omega \mu_0 I l e^{i k_0 \sqrt{\varepsilon} r}}{4 \pi r} (-\cos \varphi)[2i \sin(k_0 \sqrt{\varepsilon} h \cos \theta)] \] (3.20d-f)

which agree with the known results [33,34,35].

3.1.4 Dipole located at \((x_i, y_i, h)\)

When the given direct source distribution is located at \((x_i, y_i, z_i) = (x_i, y_i, h)\) which can be written as

\[ \mathcal{F}_d(\mathbf{r}) = i \omega \mu_0 I l \hat{F}_i \delta(x-x_i) \delta(y-y_i) \delta(z-h) \] (3.21)

The field becomes

\[ \mathcal{E}_{d}\mathbf{l}(\mathbf{r}) = \mathcal{G}_1(\mathcal{R}_{11}) e^{i k_0 \sqrt{\varepsilon}(x-x_i)^2 + (y-y_i)^2 + (z-h)^2} i \omega \mu_0 I l \hat{F}_i \] (3.22)

When the unknown source distribution from the reflection of the direct source is located at \((x_i, y_i, -h)\) which can be written as

\[ \mathcal{F}_x(\mathbf{r}) = i \omega \mu_0 I l \hat{F}_2 \delta(x-x_i) \delta(y-y_i) \delta(z+h) \] (3.23)

The reflected field becomes

\[ \mathcal{E}_{x}\mathbf{l}(\mathbf{r}) = \mathcal{G}_1(\mathcal{R}_{21}) e^{i k_0 \sqrt{\varepsilon}(x-x_i)^2 + (y-y_i)^2 + (z+h)^2} i \omega \mu_0 I l \hat{F}_2 \] (3.24)

By the boundary condition and the Uniqueness theorem mentioned in section 3.1.3, the transformation equations for the fields in the upper and lower half \((z > 0 \text{ and } z < 0)\) regions can be written as

\[ \mathcal{E}_x(\mathbf{r}) = \mathcal{P} \cdot \mathcal{E}_d(\mathbf{r}) \mid_{z=0(\theta=90^\circ)} \]
\[ \overline{C}_1(\overline{R}_{21c}) e^{ik_0\sqrt{E_{R_{11c}}}}i\omega \mu_0 I I_1 = \overline{\mathbf{F}} \cdot \overline{C}_1(\overline{R}_{1c}) e^{ik_0\sqrt{E_{R_{11c}}}}i\omega \mu_0 I I_1 \]

\[ R_{11c} = R_{21c} = \sqrt{(x-x_1)^2 + (y-y_1)^2 + h^2} \]

\[ \overline{C}_1(\overline{R}_{21c}) \cdot I_2 = \overline{\mathbf{F}} \cdot \overline{C}_1(\overline{R}_{11c}) \cdot I_1 \]

\[ \hat{I}_2 = \left[ \overline{C}_1^{-1}(\overline{R}_{21c}) \cdot \overline{\mathbf{F}} \cdot \overline{C}_1(\overline{R}_{11c}) \right] \cdot I_1 = \left[ \overline{\mathbf{F}}_{\mathbf{F}_1} \right] \cdot I_1 \quad \text{(3.25)} \]

\[ \overline{\mathbf{F}}_{\mathbf{F}_1} = \overline{C}_1^{-1}(\overline{R}_{21c}) \cdot \overline{\mathbf{F}} \cdot \overline{C}_1(\overline{R}_{11c}) \quad \text{(3.26)} \]

In an isotropic medium, \( \overline{\mathbf{F}}_{\mathbf{F}_1} = \overline{\mathbf{F}} \); hence from eq. (3.23) we have

\[ \overline{\mathbf{F}}_r(\overline{\mathbf{F}}) = i\omega \mu_0 I I (\overline{\mathbf{F}} \cdot \hat{I}_1) \delta(x-x_1) \delta(y-y_1) \delta(z+h) \quad \text{(3.27)} \]

The electric field above the \( z = 0 \) plane will be

\[ \overline{E}_r(\overline{\mathbf{F}}) = \overline{E}_d(\overline{\mathbf{F}}) + \overline{E}_r(\overline{\mathbf{F}}) \]

\[ = i\omega \mu_0 I I e^{ik_0\sqrt{E_{R_{11c}}}} \overline{C}_1(\overline{R}_{11c}) \cdot \hat{I}_1 + i\omega \mu_0 I I e^{ik_0\sqrt{E_{R_{21c}}}} \overline{C}_1(\overline{R}_{21c}) \cdot \hat{I}_2 \]

\[ = \frac{i\omega \mu_0 I I e^{ik_0\sqrt{E_{R_{11}}}}}{4\pi R_{11}} \left\{ C_1(R_{11}) \overline{I} + C_2(R_{11}) \hat{k}_{11} \hat{k}_{11} \right\} \cdot \hat{I}_1 \]

\[ + \frac{i\omega \mu_0 I I e^{ik_0\sqrt{E_{R_{21}}}}}{4\pi R_{21}} \left\{ C_1(R_{21}) \overline{I} + C_2(R_{21}) \hat{k}_{21} \hat{k}_{21} \right\} \cdot (\overline{\mathbf{F}} \cdot \hat{I}_1) \quad \text{(3.28)} \]

\[ R_{11} = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-h)^2} \]

\[ R_{21} = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z+h)^2} \]

\[ \hat{k}_{11} = \frac{1}{R_{11}} \left\{ (x-x_1) \hat{x} + (y-y_1) \hat{y} + (z-h) \hat{z} \right\} \]

\[ \hat{k}_{21} = \frac{1}{R_{21}} \left\{ (x-x_1) \hat{x} + (y-y_1) \hat{y} + (z+h) \hat{z} \right\} \]
\( \mathbf{\vec{E}}_{11} \cdot \mathbf{\hat{r}}_1 \mathbf{\hat{r}}_{11} = \frac{1}{R_{11}^2} \left\{ (x-x_1)x_1 + (y-y_1)y_1 + (z-h)z_1 \right\} \)

\( \{ (x-x_1)\hat{x} + (y-y_1)\hat{y} + (z-h)\hat{z} \} \)

\( \mathbf{\vec{E}}_{21} \cdot \mathbf{\hat{r}}_1 \mathbf{\hat{r}}_{21} = \frac{1}{R_{21}^2} \left\{ -(x-x_1)x_1 - (y-y_1)y_1 + (z+h)z_1 \right\} \)

\( \{ (x-x_1)\hat{x} + (y-y_1)\hat{y} + (z+h)\hat{z} \} \)

**Check on the z = 0 plane**

\[ z = 0 \Rightarrow R_{11c} = R_{21c} = \sqrt{(x-x_1)^2 + (y-y_1)^2 + h^2} \]

\[ C_1(R_{11c}) = C_1(R_{21c}) \quad ; \quad C_2(R_{11c}) = C_2(R_{21c}) \]

\( \mathbf{\vec{E}}_{11} \cdot \mathbf{\hat{r}}_1 \mathbf{\hat{r}}_{11} = \frac{1}{R_{11c}^2} \left\{ (x-x_1)x_1 + (y-y_1)y_1 - hz_1 \right\} \)

\( \{ (x-x_1)\hat{x} + (y-y_1)\hat{y} - \hat{z}h \} \)

\( \mathbf{\vec{E}}_{21} \cdot \mathbf{\hat{r}}_1 \mathbf{\hat{r}}_{21} = \frac{1}{R_{21c}^2} \left\{ -(x-x_1)x_1 - (y-y_1)y_1 + hz_1 \right\} \)

\( \{ (x-x_1)\hat{x} + (y-y_1)\hat{y} + \hat{z}h \} \)

The electric field on the z = 0 plane will be

\[ \overline{E}_{Tc}(\mathbf{\vec{r}}_c) = \frac{2i\omega \mu_0 I e^{ik_0 R_{11c}}}{4\pi R_{11c}} \]

\[ \left[ C_1(R_{11c})z_1 - C_2(R_{11c}) \frac{h}{R_{11c}^2} \left\{ (x-x_1)x_1 + (y-y_1)y_1 - hz_1 \right\} \right]^2 \quad (3.29) \]

The tangential components of the electric field satisfy the same boundary condition at the z = 0 plane. Hence according to the Uniqueness theorem the fields in the region z > 0 for the two problems are the same.

This derivation will enable us to superimpose complex
sources, such as a rectangular aperture which is on the plane parallel to the perfect conducting plane in the following chapter.

3.2 Reformulation of Image Theory for Uniaxial medium

We will extend the same procedure from the previous section to the case of a uniaxially anisotropic medium which is characterized by the dielectric tensor

\[ \boldsymbol{\varepsilon}_{uc} = \varepsilon_1 \mathbb{I} + (\varepsilon_2 - \varepsilon_1) \hat{c}\hat{c} \]  

(3.30)

where \( \hat{c} \) is the direction of the optic axis.

The electric field radiated from a given source distribution in an unbounded uniaxial medium utilizing the dyadic Green's function will be reviewed. The field over an infinite conducting plane will be discussed. The unknown source distribution from the reflected field (for analysis purposes only) in term of the given directed source distribution will be derived. The field of the two problems that are specified in the previous part must satisfy the same boundary condition on the \( z = 0 \) plane.

3.2.1 Summary of the electric field in an unbounded uniaxial medium

The electric field in an unbounded region can be written as

\[ \overline{E}(\overline{r}) = \iint \overline{G}_u(\overline{r}, \overline{r}') \cdot \overline{S}(\overline{r}') d^3r' \]  

(3.31)

where \( \overline{S}(\overline{r}') \) is the given source distribution located at \( \overline{r}' \).
and $\overline{G}_u(\overline{R})$ is the dyadic Green's function of a uniaxial medium ($\overline{R} = \overline{r} - \overline{R}$) which takes the form [36]

$$\overline{G}_u(\overline{R}) = e^{ik_{o}R_{u}}\left\{ \left[ B_{u1}(R) (\overline{e}_{xc} \cdot \overline{R}) + B_{u2}(R) \overline{e}_{xc} \right] + B_{u3}(R) A_{u4c}(R) \right\}$$

$$- e^2 e^{ik_{o}R_{u}} \left\{ A_{u5c}(R) + A_{u6c}(R) \right\} + e^{ik_{o}R_{u}} \left\{ A_{u7c}(R) + A_{u8c}(R) \right\}$$

(3.32)

$$R_{ue} = \sqrt{\mu \varepsilon_{1} \varepsilon_{2} \overline{R} \cdot \overline{e}_{xc} \cdot \overline{R}}$$

$$R_{uo} = \sqrt{\mu \varepsilon_{1} R}$$

$$\overline{e}_{xc} = \frac{\overline{T}}{\varepsilon_{1}} + \left( \frac{1}{\varepsilon_{2}} - \frac{1}{\varepsilon_{1}} \right) \overline{e}_{c}$$

$$B_{u1}(R) = \frac{\sqrt{\varepsilon_{2}}}{4\pi R(\overline{R} \cdot \overline{e}_{xc} \cdot \overline{R})^{3}} \left( \frac{3}{k_{o}^{2}R_{o}^{2}} - \frac{3i}{k_{o}R_{o}} - 1 \right)$$

$$B_{u2}(R) = \frac{\sqrt{\varepsilon_{2}}}{4\pi R \overline{R} \cdot \overline{e}_{xc} \cdot \overline{R}} \left( \frac{i}{k_{o}R_{o}} - \frac{1}{k_{o}^{2}R_{o}^{2}} \right)$$

$$B_{u3}(R) = \frac{1}{4\pi R e^{2} \overline{R} \cdot \overline{e}_{xc} \cdot \overline{R}}$$

$$A_{u4c} = \varepsilon_{1} \varepsilon_{2} \overline{T} + (\varepsilon_{2} - \varepsilon_{1}) \overline{e}_{c}$$

$$A_{u5c}(R) = \begin{bmatrix} A_{u5aa}(R) & A_{u5ab}(R) & 0 \\ A_{u5ba}(R) & A_{u5bb}(R) & 0 \\ 0 & 0 & A_{u5cc}(R) \end{bmatrix}$$

$$A_{u5aa}(R) = \frac{\sin^{2}\varphi_{c}}{4\pi R e_{1} \overline{R} \cdot \overline{e}_{xc} \cdot \overline{R}}$$

$$A_{u5bb}(R) = \frac{\cos^{2}\varphi_{c}}{4\pi R e_{1} \overline{R} \cdot \overline{e}_{xc} \cdot \overline{R}}$$

$$A_{u5cc}(R) = 0$$

$$A_{u5ab}(R) = -\frac{\sin(2\varphi_{c})}{8\pi R e_{1} \overline{R} \cdot \overline{e}_{xc} \cdot \overline{R}}$$

$$A_{u5ba}(R) = -\frac{\sin(2\varphi_{c})}{8\pi R e_{1} \overline{R} \cdot \overline{e}_{xc} \cdot \overline{R}}$$
\[ \overline{A}_{u7c}(R) = \begin{bmatrix} A_{u7aa}(R) & A_{u7ab}(R) & 0 \\ A_{u7ba}(R) & A_{u7bb}(R) & 0 \\ 0 & 0 & A_{u7cc}(R) \end{bmatrix} \]

\[
A_{u7aa}(R) = \frac{\sin^2 \varphi_c}{4\pi R} \quad ; \quad A_{u7bb}(R) = \frac{\cos^2 \varphi_c}{4\pi R} \quad ; \quad A_{u7cc}(R) = 0
\]

\[
A_{u7ab}(R) = -\frac{\sin(2\varphi_c)}{8\pi R} \quad ; \quad A_{u7ba}(R) = -\frac{\sin(2\varphi_c)}{8\pi R}
\]

\[
\overline{A}_{u6c}(R) = \begin{bmatrix} A_{u6aa}(R) & A_{u6ab}(R) & 0 \\ A_{u6ba}(R) & A_{u6bb}(R) & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
A_{u6aa}(R) = -\frac{ic\cos(2\varphi_c)}{4\pi k_0 \varepsilon_\perp \mu_\perp (R\times\hat{c})^2} \quad ; \quad A_{u6bb}(R) = \frac{ic\cos(2\varphi_c)}{4\pi k_0 \varepsilon_\perp \mu_\perp (R\times\hat{c})^2}
\]

\[
A_{u6ab}(R) = -\frac{isin(2\varphi_c)}{4\pi k_0 \varepsilon_\perp \mu_\perp (R\times\hat{c})^2} \quad ; \quad A_{u6ba}(R) = -\frac{isin(2\varphi_c)}{4\pi k_0 \varepsilon_\perp \mu_\perp (R\times\hat{c})^2}
\]

\[
\overline{A}_{u8c}(R) = \begin{bmatrix} A_{u8aa}(R) & A_{u8ab}(R) & 0 \\ A_{u8ba}(R) & A_{u8bb}(R) & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
A_{u8aa}(R) = -\frac{ic\cos(2\varphi_c)}{4\pi k_0 \varepsilon_\perp \mu_\perp (R\times\hat{c})^2} \quad ; \quad A_{u8bb}(R) = \frac{ic\cos(2\varphi_c)}{4\pi k_0 \varepsilon_\perp \mu_\perp (R\times\hat{c})^2}
\]

\[
A_{u8ab}(R) = -\frac{isin(2\varphi_c)}{4\pi k_0 \varepsilon_\perp \mu_\perp (R\times\hat{c})^2} \quad ; \quad A_{u8ba}(R) = -\frac{isin(2\varphi_c)}{4\pi k_0 \varepsilon_\perp \mu_\perp (R\times\hat{c})^2}
\]

Tensors \( \overline{A}_{u5c}(R) \), \( \overline{A}_{u6c}(R) \), \( \overline{A}_{u7c}(R) \) and \( \overline{A}_{u8c}(R) \) are in \( \hat{a}, \hat{b}, \hat{c} \) - coordinate-free form as specified in chapter 2.

3.2.2 Fields above an infinite conducting plane; Optic axis along the z-axis

We convert the \( \hat{a}, \hat{b}, \hat{c} \) -coordinates to \( \hat{x}, \hat{y}, \hat{z} \) -coordinates by letting \( \hat{c} = 2, \hat{a} = \hat{x}, \) and \( \hat{b} = \hat{y} \). \( \varphi_c \) is the same angle \( \varphi \) in the
rectangular coordinate system. If the given source distribution is located at $\vec{r}$ over the conducting plane, see Fig. 3.1, the fields due to the direct source and the unknown source distribution for the reflected field would be

$$\vec{E}_{du}(\vec{r}) = \iint \left[ e^{ik_0R_{ue1s}} \left( (B_{u1}(R_1)(\vec{e}_{xz} \cdot \vec{R}_1) + B_{u2}(R_1)\vec{e}_{xz}) + B_{u3}(R_1)\vec{A}_{u45} \right) 
- \varepsilon_2 e^{ik_0R_{ue1s}} \left( \vec{A}_{u57}(R_1) + \vec{A}_{u6y}(R_1) \right) + e^{ik_0R_{uo1}} \left( \vec{A}_{u7x}(R_1) + \vec{A}_{u8y}(R_1) \right) \right] \vec{B}_d(\vec{r}) d^3\vec{r}_1 \tag{3.33}$$

$$\varepsilon_{xz} = \frac{\varepsilon_1}{\varepsilon_2} + \left( \frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1} \right)$$

$$R_{ue1s} = \sqrt{\mu_1 \varepsilon_1 \varepsilon_2 R_1 \cdot \vec{e}_{xz} \cdot \vec{R}_1}$$

$$R_{uo1} = \sqrt{\mu_1 \varepsilon_1 R_1}$$

$$R_1 = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}$$

$$\vec{E}_{uuu}(\vec{r}) = \iint \left[ e^{ik_0R_{ue2s}} \left( (B_{u1}(R_2)(\vec{e}_{xz} \cdot \vec{R}_2) + B_{u2}(R_2)\vec{e}_{xz}) + B_{u3}(R_2)\vec{A}_{u45} \right) 
- \varepsilon_2 e^{ik_0R_{ue2s}} \left( \vec{A}_{u57}(R_2) + \vec{A}_{u6y}(R_2) \right) + e^{ik_0R_{uo2}} \left( \vec{A}_{u7x}(R_2) + \vec{A}_{u8y}(R_2) \right) \right] \vec{B}_d(\vec{r}) d^3\vec{r}_2 \tag{3.34}$$

$$R_{ue2s} = \sqrt{\mu_1 \varepsilon_1 \varepsilon_2 R_2 \cdot \vec{e}_{xz} \cdot \vec{R}_2}$$

$$R_{uo2} = \sqrt{\mu_1 \varepsilon_1 R_2}$$

$$R_2 = \sqrt{(x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2}$$

If the given source distribution is as in eq. (3.5) and located at $(0, 0, h)$, we must reevaluate the unknown reflected source in terms of the real source utilizing the procedure in section 3.1. For simplicity, this may be done separately for each tensor. The source factors will be derived
as $\overline{S}_{fu1}$, $\overline{S}_{fu2}$, $\overline{S}_{fu3}$, $\overline{S}_{fu4}$, $\overline{S}_{fu5}$ and $\overline{S}_{fu6}$

which depend on $(B_{u1}(R) (\varepsilon_{xz} \cdot \hat{R}) (\varepsilon_{xz} \cdot \hat{R}) + B_{u2}(R) \varepsilon_{xz})$, $B_{u3}(R) \overline{A}_{u4}$, $\overline{A}_{u5}(R)$, $\overline{A}_{u6}(R)$, $\overline{A}_{u7}(R)$ and $\overline{A}_{u8}(R)$ respectively. In this way the source factors will not be functions of $R$. The source factors can be simplified as

$\overline{S}_{f1z} = \overline{D}_{u1}(\overline{x}_c) \cdot \overline{P} \cdot \overline{D}_{u1}(\overline{x}_c) = \overline{P}$

$\overline{D}_{u1}(\overline{x}_c) = B_{u1}(R) (\varepsilon_{xz} \cdot \overline{R}) (\varepsilon_{xz} \cdot \overline{R}) + B_{u2}(R) \varepsilon_{xz}$

$\overline{S}_{f2z} = \frac{\varepsilon_c \cdot \overline{P} \cdot \text{adj} \overline{e}_z}{|\varepsilon_c|} = \overline{P}$

$\overline{S}_{f3z} = \overline{S}_{f5z} = \overline{A}_{u5c} \cdot \overline{P} \cdot \overline{A}_{u5c} = \overline{P}$

$\overline{S}_{f4z} = \overline{S}_{f7z} = \overline{A}_{u6c} \cdot \overline{P} \cdot \overline{A}_{u6c} = \overline{P}$

$\overline{A}_{u5c} = \begin{bmatrix} (\varepsilon_1 - \varepsilon_2) \sin^2 \varphi_c & \frac{(\varepsilon_2 - \varepsilon_1) \sin(2\varphi_c)}{2} & 0 \\ \frac{(\varepsilon_2 - \varepsilon_1) \sin(2\varphi_c)}{2} & (\varepsilon_1 - \varepsilon_2) \cos^2 \varphi_c & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\overline{A}_{u6c} = \begin{bmatrix} \cos(2\varphi_c) & \sin(2\varphi_c) & 0 \\ \sin(2\varphi_c) & -\cos(2\varphi_c) & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Substitution of the above into eq.(3.34) gives rise to the unknown source distribution for the reflected field in terms of the given direct source distribution. The electric field above the $z = 0$ plane can be summarized as
\[ \mathbf{E}_{\text{tot}} = \mathbf{E}_{du}(\mathbf{r}) + \mathbf{E}_{\text{ru}}(\mathbf{r}) = i\omega \mu_0 I \]

\[
\left[ e^{ik_0 r_1} \left\{ \left( B_1(R_1) \begin{pmatrix} \epsilon_{xx} \cdot \hat{R}_1 \\ \epsilon_{xx} \cdot \hat{R}_1 \end{pmatrix} + B_2(R_1) \epsilon_{xs} \hat{R}_1 \right) + B_3(R_1) \hat{A}_{4x} \right\} \cdot \hat{I}_1 \\
+ e^{ik_0 r_2} \left\{ \left( B_1(R_2) \begin{pmatrix} \epsilon_{xx} \cdot \hat{R}_2 \\ \epsilon_{xx} \cdot \hat{R}_2 \end{pmatrix} + B_2(R_2) \epsilon_{xs} \hat{R}_2 \right) + B_3(R_2) \hat{A}_{4x} \right\} \cdot (\hat{P} \cdot \hat{I}_1) \\
+ \left\{ -\epsilon_2 e^{ik_0 r_3} \hat{A}_{5x}(R_1) + \hat{A}_{6x}(R_1) \right\} + e^{ik_0 r_3} \hat{A}_{7x}(R_1) + \hat{A}_{8x}(R_1) \right\} \cdot \hat{I}_1 \\
+ \left\{ -\epsilon_2 e^{ik_0 r_3} \hat{A}_{5x}(R_2) + \hat{A}_{6x}(R_2) \right\} + e^{ik_0 r_3} \hat{A}_{7x}(R_2) + \hat{A}_{8x}(R_2) \right\} \cdot (\hat{P} \cdot \hat{I}_1) \right] (3.35)
\]

which agrees with eq. [3.14] when the medium becomes isotropic.

Check on the \( z = 0 \) plane

\[ \hat{r}_{1c} \cdot \epsilon_{xx} \cdot \hat{r}_{1c} = \hat{r}_{2c} \cdot \epsilon_{xx} \cdot \hat{r}_{2c} = \frac{1}{R_1^2} \left( \frac{x^2}{\epsilon_1} + \frac{y^2}{\epsilon_1} + \frac{h^2}{\epsilon_2} \right) ; \quad R_{1c} = R_{2c} = \sqrt{x^2 + h^2} \]

\[ R_{01c} = R_{02c} ; \quad R_{01c} = R_{02c} ; \quad B_1(R_{1c}) = B_1(R_{2c}) ; \quad B_2(R_{1c}) = B_2(R_{2c}) ; \]

\[ B_3(R_{1c}) = B_3(R_{2c}) \]

\[ \left( \left( \epsilon_{xx} \cdot \hat{r}_{1c} \right) \cdot \hat{I}_1 \right) \left( \epsilon_{xx} \cdot \hat{r}_{1c} \right) = \frac{1}{R_1^2} \left( \frac{x X_1}{\epsilon_1} + \frac{y Y_1}{\epsilon_1} - \frac{h Z_1}{\epsilon_2} \right) \left( \frac{x}{\epsilon_1} + \frac{y}{\epsilon_1} + \frac{h}{\epsilon_2} \right) \]

\[ \left( \left( \epsilon_{xx} \cdot \hat{r}_{2c} \right) \cdot (\hat{P} \cdot \hat{I}_1) \right) \left( \epsilon_{xx} \cdot \hat{r}_{2c} \right) = \frac{1}{R_2^2} \left( -\frac{x X_1}{\epsilon_1} - \frac{y Y_1}{\epsilon_1} + \frac{h Z_1}{\epsilon_2} \right) \left( \frac{x}{\epsilon_1} + \frac{y}{\epsilon_1} + \frac{h}{\epsilon_2} \right) \]

\[ \hat{A}_{4x} \cdot \hat{I}_1 = \epsilon_1 e_2 x_1 \hat{x} + \epsilon_1 e_2 y_1 \hat{y} + \epsilon_1 e_2 z_1 \hat{z} \]

\[ \hat{A}_{4x} \cdot (\hat{P} \cdot \hat{I}_1) = -\epsilon_1 e_2 x_1 \hat{x} - \epsilon_1 e_2 y_1 \hat{y} + \epsilon_1 e_2 z_1 \hat{z} \]

\[ \hat{A}_{4x}(R_{1c}) \cdot \hat{I}_1 = \{ A_{iab}(R_{1c}) x_1 + A_{iab}(R_{1c}) y_1 \} \hat{x} + \{ A_{iab}(R_{1c}) x_1 + A_{iab}(R_{1c}) y_1 \} \hat{y} \]

\[ \hat{A}_{4x}(R_{2c}) \cdot (\hat{P} \cdot \hat{I}_1) = -\{ A_{iab}(R_{2c}) x_1 + A_{iab}(R_{2c}) y_1 \} \hat{x} - \{ A_{iab}(R_{2c}) x_1 + A_{iab}(R_{2c}) y_1 \} \hat{y} \]

\[ (i = 5 - 8) \]

The electric field on the \( z = 0 \) plane will be
From eq. (3.36) it can be seen clearly that no tangential components exist which satisfy the same boundary condition at the \( z = 0 \) plane. Hence according to the Uniqueness theorem the fields in the region \( z > 0 \) for the two problems are the same.

### 3.2.2.1 Dipole perpendicular to the conducting plane

In this case,

\[
x_1 = y_1 = 0, z_1 = 1 ; \hat{1}_1 = \hat{2} ; S_{\text{Full}} \cdot \hat{1}_1 = \vec{E} \cdot \hat{2} = 2, i = 1-6
\]

The rectangular components of the electric field above the \( z = 0 \) plane can be summarized as

\[
\begin{align*}
\vec{E}_{\text{ruz}}(\vec{r}) &= \vec{E}_{\text{dus}}(\vec{r}) + \vec{E}_{\text{ruz}}(\vec{r}) = i\omega \mu_0 I l \\
&= \left[ e^{ikr \hat{r}_{\text{rul}}} \left\{ B_1(R_1) \frac{(z-h)}{R_1^2 \epsilon_2} \left( \frac{x}{\epsilon_1} + \frac{y}{\epsilon_1} + \frac{(z-h)}{\epsilon_2} \right) + \frac{B_2(R_1)}{\epsilon_2} \hat{2} + B_3(R_1) \epsilon_1^2 \hat{2} \right\} \\
&+ e^{ikr \hat{r}_{\text{rul}}} \left\{ B_1(R_2) \frac{(z+h)}{R_2^2 \epsilon_2} \left( \frac{x}{\epsilon_1} + \frac{y}{\epsilon_1} + \frac{(z+h)}{\epsilon_2} \right) + \frac{B_2(R_2)}{\epsilon_2} \hat{2} + B_3(R_2) \epsilon_1^2 \hat{2} \right\} \right]
\end{align*}
\]

or

\[
E_{xx} = i\omega \mu_0 I l e^{ikr \hat{r}_{\text{rul}}} \frac{B_1(R_1)}{\epsilon_1 \epsilon_2} \left( \frac{r \cos \theta}{R_1} - \frac{h}{R_1} \right) \sin \theta \cos \phi \\
+ i\omega \mu_0 I l e^{ikr \hat{r}_{\text{rul}}} \frac{B_1(R_2)}{\epsilon_1 \epsilon_2} \left( \frac{r \cos \theta}{R_2} + \frac{h}{R_2} \right) \sin \theta \cos \phi
\]
\[ E_{xy} = i\omega \mu_0 I e^{ik_0 r_{xx}} \frac{B_1(R_1)}{\varepsilon_1 \varepsilon_2} \left( \frac{x}{R_1} \cos \theta - \frac{h}{R_1} \right) \sin \theta \sin \phi \]

\[ + i\omega \mu_0 I e^{ik_0 r_{xx}} \frac{B_1(R_2)}{\varepsilon_1 \varepsilon_2} \left( \frac{x}{R_2} \cos \theta + \frac{h}{R_2} \right) \sin \theta \sin \phi \]

\[ E_{xx} = i\omega \mu_0 I e^{ik_0 r_{xx}} \left\{ \frac{B_1(R_1)}{\varepsilon_2} \left( \frac{x}{R_1} \cos \theta - \frac{h}{R_1} \right)^2 + \frac{B_2(R_1)}{\varepsilon_2} + \varepsilon_1^2 B_3(R_1) \right\} \]

\[ + i\omega \mu_0 I e^{ik_0 r_{xx}} \left\{ \frac{B_1(R_2)}{\varepsilon_2} \left( \frac{x}{R_2} \cos \theta + \frac{h}{R_2} \right)^2 + \frac{B_2(R_2)}{\varepsilon_2} + \varepsilon_1^2 B_3(R_2) \right\} \]

(3.38a–c)

which 1) agrees with the work by Chen and Clemmow [42,43] and
2) agrees with eq.(3.16) when the medium becomes isotropic.

3.2.2.2 Dipole parallel to the conducting plane

In this case,
\[ x_1 = z_1 = 0, y_1 = 1 ; \mathbf{i}_1 = \hat{y} ; \mathbf{S}_{\text{Full}} \cdot \mathbf{i}_1 = \mathbf{E} \cdot \hat{y} = -\hat{y}, i = 1-6 \]

The rectangular components of the electric field above the \( z = 0 \) plane can be summarized as

\[ \mathbf{E}_{\text{ruz}}(\mathbf{r}) = \mathbf{E}_{\text{dusz}}(\mathbf{r}) + \mathbf{E}_{\text{ruz}}(\mathbf{r}) = i\omega \mu_0 I \]

\[ \left[ e^{ik_0 r_{xx}} \left\{ \frac{B_1(R_1)}{\varepsilon_1} \left( \frac{x}{R_1} \hat{x} + \frac{y}{\varepsilon_1} \hat{y} + \frac{(z-h)}{\varepsilon_2} \hat{z} \right) + \frac{B_2(R_1)}{\varepsilon_1} \hat{y} + B_3(R_1) \varepsilon_1 \varepsilon_2 \hat{y} \right\} \right. \]

\[ + e^{ik_0 r_{xx}} \left\{ \frac{B_1(R_2)}{\varepsilon_1} \left( \frac{x}{R_2} \hat{x} + \frac{y}{\varepsilon_1} \hat{y} + \frac{(z+h)}{\varepsilon_2} \hat{z} \right) - \frac{B_2(R_2)}{\varepsilon_1} \hat{y} - B_3(R_2) \varepsilon_1 \varepsilon_2 \hat{y} \right\} \]

\[ - \varepsilon_2 e^{ik_0 r_{xx}} \left\{ (\mathbf{A}_{5ab}(R_1) + \mathbf{A}_{6ab}(R_1)) \hat{x} + (\mathbf{A}_{5bb}(R_1) + \mathbf{A}_{6bb}(R_1)) \hat{y} \right\} \]

\[ - \varepsilon_2 e^{ik_0 r_{xx}} \left\{ - (\mathbf{A}_{5ab}(R_2) + \mathbf{A}_{6ab}(R_2)) \hat{x} - (\mathbf{A}_{5bb}(R_2) + \mathbf{A}_{6bb}(R_2)) \hat{y} \right\} \]

\[ + e^{ik_0 r_{01}} \left\{ (\mathbf{A}_{7ab}(R_1) + \mathbf{A}_{8ab}(R_1)) \hat{x} + (\mathbf{A}_{7bb}(R_1) + \mathbf{A}_{8bb}(R_1)) \hat{y} \right\} \]
which 1) agrees with the work by Chen and Clemmow [42,43] and
2) agrees with eq.(3.19) when the medium becomes isotropic.

For the far field above the \( z = 0 \) plane

In the far-zone, we have

\[
R_{\text{e1x}} = \sqrt{\frac{\mu_1 \varepsilon_2 \sin^2 \theta + \cos^2 \theta}{\varepsilon_1 \varepsilon_2}} (r-h \cos \theta); \text{ for phase term}
\]

\[
R_{\text{e2x}} = \sqrt{\frac{\mu_1 \varepsilon_2 \sin^2 \theta + \cos^2 \theta}{\varepsilon_1 \varepsilon_2}} (r+h \cos \theta); \text{ for phase term}
\]

\( R_1 = R_2 = r; \) for amplitude variation

The rectangular components of the far-zone electric field in eq.(3.39) can be written as

\[
E_{zx} = \frac{C_{zfe}(-f_{uze}(h) + f_{uz}(h))}{4\pi r} \sqrt{\frac{\sin^2 \theta + \cos^2 \theta}{\varepsilon_1 \varepsilon_2}} \left( \frac{\sin^2 \theta}{\varepsilon_1 + \cos^2 \varepsilon_2} \right)^{-1} \frac{\sin \theta \cos \theta \sin \phi}{4\pi}
\]

\[
E_{zy} = \frac{C_{zfe}(-f_{uze}(h) + f_{uz}(h))}{4\pi r} \sqrt{\frac{\sin^2 \theta + \cos^2 \theta}{\varepsilon_1 \varepsilon_2}} \left( \frac{\sin^2 \theta}{\varepsilon_1 + \cos^2 \varepsilon_2} \right)^{-1} \frac{\sin \theta \cos \theta \sin \phi}{4\pi}
\]

\[
E_{zz} = \frac{C_{zz}(-f_{uz}(h) + f_{uz}(h))}{4\pi r} \sqrt{\frac{\sin^2 \theta + \cos^2 \theta}{\varepsilon_1 \varepsilon_2}} \sin \theta \cos \theta \sin \phi \left( \frac{\sin^2 \theta}{\varepsilon_1 + \cos^2 \varepsilon_2} \right)^{\frac{3}{2}} \quad (3.40a-c)
\]
\[ C_{zfe} = i\omega \mu_0 I e^{ik_0 \sqrt{\mu_1 \varepsilon_2} \left( \frac{\sin^2 \theta}{\varepsilon_1} + \frac{\cos^2 \theta}{\varepsilon_2} \right) x} \]

\[ f_{uze}(h) = e^{ik_0 \sqrt{\mu_1 \varepsilon_2} h \cos \theta} \]

\[ C_{zfo} = i\omega \mu_0 I e^{ik_0 \sqrt{\mu_1 \varepsilon_2} x} \]

\[ f_{uzo}(h) = e^{ik_0 \sqrt{\mu_1 \varepsilon_2} h \cos \theta} \]

3.2.3 Field over an infinite conducting ground plane;

Optic axis along the \( x \)-axis

The \( \delta, \hat{\delta}, \hat{c} \) -coordinates will be converted to \( \xi, \gamma, \zeta \) -coordinates by letting \( \hat{c} = \xi, \hat{\delta} = \gamma, \) and \( \hat{\delta} = \zeta \). \( \varphi_c \) is the angle between the \( y \)-axis and the projection of the distance vector on the \( yz \)-plane. The angle is in the counterclockwise direction from the \( y \)-axis.

The tensors of the dyadic Green's function in eq. (3.32) become

\[ \bar{\varepsilon}_{xx} = \frac{\bar{\varepsilon}}{\varepsilon_1} + \left( \frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1} \right) \bar{\varepsilon}_\xi \]

\[ \bar{A}_{u4x} = \varepsilon_1 \varepsilon_2 \bar{\varepsilon} + (\varepsilon_1^2 - \varepsilon_1 \varepsilon_2) \bar{\varepsilon}_\xi \]

\[ \bar{A}_{uix}(R) = \begin{bmatrix} A_{uixx}(R) & 0 & 0 \\ 0 & A_{uiya}(R) & A_{uiab}(R) \\ 0 & A_{uiba}(R) & A_{uibb}(R) \end{bmatrix}, \quad i = 5-8 \]

The fields due to the directed source and the unknown source distribution for the reflected field can be written as

\[ \bar{E}_{dux}(\bar{r}) = \iint \left\{ e^{ik_0 \bar{R}_{ux}} \left[ B_1(R_1) \left( \bar{\varepsilon}_{xx} \cdot \bar{R}_1 \right) \left( \bar{\varepsilon}_{xx} \cdot \bar{R}_1 \right) + B_2(R_1) \bar{\varepsilon}_{xx} + B_3(R_1) \bar{A}_{4x} \right] \right\} \]
The source factors for the given source distribution as given in the previous section will be obtained as

\[ \overline{S}_{Fu1} = \overline{S}_{Fu2} = \overline{S}_{Fu3} = \overline{S}_{Fu4} = \overline{S}_{Fu5} = \overline{S}_{Fu6} = \overline{P} \]

Substituting the source factors above back into eq. (3.42), we obtain the unknown source distribution for the reflected field in terms of the direct source. The rectangular components of the electric field above the \( z = 0 \) plane become

\[
\overline{E}_{Tux} = \overline{E}_{dux}(\overline{r}) + \overline{E}_{rux}(\overline{r}) = i \omega \mu_0 Il
\]

\[
\left[ e^{ik_0R_{ux}} \left( \left( B_1(R_1) \left( \overline{e}_{xx} \cdot \overline{R}_1 \right) + B_2(R_1) \overline{e}_{xx} \right) + B_3(R_1) \overline{A}_{4x} \right) \cdot \hat{I}_1 
+ e^{ik_0R_{ux}} \left( \left( B_1(R_2) \left( \overline{e}_{xx} \cdot \overline{R}_2 \right) + B_2(R_2) \overline{e}_{xx} \right) + B_3(R_2) \overline{A}_{4x} \right) \cdot (\overline{P} \cdot \hat{I}_1) 
+ \left\{ -\varepsilon_2 e^{ik_0R_{ux}} \left( \overline{A}_{5x}(R_1) + \overline{A}_{6x}(R_1) \right) + e^{ik_0R_{ux}} \left( \overline{A}_{7x}(R_1) + \overline{A}_{8x}(R_1) \right) \right\} \cdot \hat{I}_1 
+ \left\{ -\varepsilon_2 e^{ik_0R_{ux}} \left( \overline{A}_{5x}(R_2) + \overline{A}_{6x}(R_2) \right) + e^{ik_0R_{ux}} \left( \overline{A}_{7x}(R_2) + \overline{A}_{8x}(R_2) \right) \right\} \cdot (\overline{P} \cdot \hat{I}_1) \right]
\]

Check on the \( z = 0 \) plane

\[
\hat{R}_{1c} \overline{e}_{xx} \cdot \hat{R}_{1c} = \hat{R}_{2c} \overline{e}_{xx} \cdot \hat{R}_{2c} = \frac{1}{R_1^2} \left( \frac{x^2}{\varepsilon_2} + \frac{y^2}{\varepsilon_1} + \frac{z^2}{\varepsilon_1} \right); \quad R_{1c} = R_{2c} = \sqrt{x^2 + y^2 + z^2}
\]
The electric field on the z = 0 plane will be

\[ \vec{E}_{Tuc}(\vec{r}_c) = 2i\omega\mu_0 I \]

\[
\left[ e^{ik_R z_{1c}} \left( -B_1(R_{1c}) \frac{h}{R_{1c}e_1} \left( \frac{XX_1}{e_2} + \frac{YY_1}{e_2} \frac{hZ_1}{e_1} \right) + \frac{h}{R_{1c}e_1} \left( B_2(R_{1c}) \frac{z_1}{e_1} + B_3(R_{1c}) e_1 e_2 z_1 \right) - e_2(A_{6bb}(R_{1c}) + A_{6bb}(R_{1c}) z_1) + e^{ik_R z_{1c}}(A_{6bb}(R_{1c}) + A_{6bb}(R_{1c}) z_1) \right] \right] \]

From eq.(3.44) it can be seen clearly that no tangential components exist which satisfy the same boundary condition at the z = 0 plane. Hence according to the Uniqueness theorem the fields in the region z > 0 for the two problems are the same.

3.2.3.1 Dipole perpendicular to the conducting plane

In this case,

\[ x_1 = y_1 = 0, z_1 = 1 ; \hat{I}_1 = \hat{z} ; S_{Pui} \hat{I}_1 = \vec{P} \cdot \hat{z} = 2, i = 1 - 6 \]

The rectangular components of the electric field above the z = 0 plane become
\[ \vec{E}_{ux}(\vec{r}) = \vec{E}_{dx}(\vec{r}) + \vec{E}_{ux}(\vec{r}) = i \omega \mu_0 I \]

\[
\left[ e^{ik_0 r_{ex}} \left\{ B_1(R_1) \frac{(z-h)}{R^2 \varepsilon_1} \left( \frac{X}{\varepsilon_2} \hat{x} + \frac{Y}{\varepsilon_1} \hat{y} + \frac{(z-h)}{\varepsilon_1} \hat{z} \right) + \frac{B_2(R_1)}{\varepsilon_1} \hat{z} + B_3(R_1) \varepsilon_1 \varepsilon_2 \hat{\theta} \right\} + e^{ik_0 r_{ex}} \left\{ B_1(R_2) \frac{(z+h)}{R^2 \varepsilon_1} \left( \frac{X}{\varepsilon_2} \hat{x} + \frac{Y}{\varepsilon_1} \hat{y} + \frac{(z+h)}{\varepsilon_1} \hat{z} \right) + \frac{B_2(R_2)}{\varepsilon_1} \hat{z} + B_3(R_2) \varepsilon_1 \varepsilon_2 \hat{\theta} \right\} \\
- \varepsilon_2 e^{ik_0 r_{ex}} \left\{ (A_{5ab}(R_1) + A_{6ab}(R_1)) \hat{y} + (A_{5bb}(R_1) + A_{6bb}(R_1)) \hat{z} \right\} \\
- \varepsilon_2 e^{ik_0 r_{ex}} \left\{ (A_{5ab}(R_2) + A_{6ab}(R_2)) \hat{y} + (A_{5bb}(R_2) + A_{6bb}(R_2)) \hat{z} \right\} \\
+ e^{ik_0 r_{ex}} \left\{ (A_{7ab}(R_1) + A_{8ab}(R_1)) \hat{y} + (A_{7bb}(R_1) + A_{8bb}(R_1)) \hat{z} \right\} \\
+ e^{ik_0 r_{ex}} \left\{ (A_{7ab}(R_2) + A_{8ab}(R_2)) \hat{y} + (A_{7bb}(R_2) + A_{8bb}(R_2)) \hat{z} \right\} \right] \tag{3.45}

which agree with eq. (3.16) when the medium becomes isotropic.

For the far field above the \( z = 0 \) plane

The rectangular components of the far-zone electric field can be written as

\[
E_{xx} = -C_{xfo} \left\{ f_{ux}(-h) - f_{ux}(h) \right\} \frac{\sin \theta \cos \theta \cos \phi}{4\pi \varepsilon_1 \sqrt{\varepsilon_2} \left( \hat{R} \cdot \varepsilon_{xx} \cdot \hat{R} \right)^{3/2}}
\]

\[
E_{xy} = -C_{xfo} \left\{ f_{ux}(-h) - f_{ux}(h) \right\} \frac{\sqrt{\varepsilon_2}}{4\pi \varepsilon_1 \sqrt{\hat{R} \cdot \varepsilon_{xx} \cdot \hat{R}}} \left( \frac{\sin \theta \cos \theta \sin \phi}{\varepsilon_1 \hat{R} \cdot \varepsilon_{xx} \cdot \hat{R}} - \sin \phi_c \cos \phi_c \right)
\]

\[
- C_{xfo} \left\{ f_{ux}(-h) - f_{ux}(h) \right\} \frac{\sin \phi_c \cos \phi_c}{4\pi \varepsilon_1 \sqrt{\hat{R} \cdot \varepsilon_{xx} \cdot \hat{R}}}
\]

\[
E_{xz} = -C_{xfo} \left\{ f_{ux}(-h) - f_{ux}(h) \right\} \frac{\sqrt{\varepsilon_2}}{4\pi \varepsilon_1 \sqrt{\hat{R} \cdot \varepsilon_{xx} \cdot \hat{R}}} \left( \frac{\cos^2 \theta}{\varepsilon_1 \hat{R} \cdot \varepsilon_{xx} \cdot \hat{R}} - 1 + \cos^2 \phi_c \right)
\]

\[
+ C_{xfo} \left\{ f_{ux}(-h) - f_{ux}(h) \right\} \frac{\cos \phi_c}{4\pi \varepsilon_1 \sqrt{\hat{R} \cdot \varepsilon_{xx} \cdot \hat{R}}} \tag{3.46a-c}
\]

\[
\hat{R} \cdot \varepsilon_{xx} \cdot \hat{R} = \frac{\sin^2 \theta \cos^2 \phi}{\varepsilon_2} + \frac{\sin^2 \phi \cos^2 \phi}{\varepsilon_1} + \frac{\cos^2 \phi}{\varepsilon_1}
\]
\[ C_{xfo} = i\omega\mu_0 Il e^{ik_0\sqrt{\mu_1}\frac{R}{R_{xx}}R} \]

3.2.3.2 Dipole parallel to the conducting plane

\[ x_1 = z_1 = 0, y_1 = 1; \; I_1 = \hat{y}; \; S_{1ul} I_1 = \vec{P} \cdot \hat{y} = -\hat{y}, \; i = 1-6 \]

The electric field above the z = 0 plane will be

\[ E_{mux}(\vec{r}) = E_{dux}(\vec{r}) + E_{txu}(\vec{r}) = i\omega\mu_0 Il \]

\[ \left[ e^{i\alpha_{R1}} \left\{ \frac{B_1(R_1)}{R_1^2 \varepsilon_1} \left( \frac{X}{\varepsilon_2} + \frac{Y}{\varepsilon_1} + \frac{(z-h)}{\varepsilon_1} \right) + \frac{B_2(R_1)}{\varepsilon_1} \hat{y} + B_3(R_1) \varepsilon_1 \varepsilon_2 \right\} \right. \]

\[ + e^{i\alpha_{R2}} \left\{ \frac{B_1(R_2)}{R_2^2 \varepsilon_1} \left( \frac{X}{\varepsilon_2} + \frac{Y}{\varepsilon_1} + \frac{(z+h)}{\varepsilon_1} \right) + \frac{B_2(R_2)}{\varepsilon_1} \hat{y} - B_3(R_2) \varepsilon_1 \varepsilon_2 \hat{y} \right\} \]

\[ - \varepsilon_2 e^{i\alpha_{R1}} \left\{ (A_{5aa}(R_1) + A_{5ba}(R_1)) \hat{y} + (A_{5ba}(R_1) + A_{6ba}(R_1)) \hat{y} \right\} \]

\[ - \varepsilon_2 e^{i\alpha_{R2}} \left\{ -(A_{5aa}(R_2) + A_{5ba}(R_2)) \hat{y} - (A_{5ba}(R_2) + A_{6ba}(R_2)) \hat{y} \right\} \]

\[ + e^{i\alpha_{R1}} \left\{ (A_{7aa}(R_1) + A_{7ba}(R_1)) \hat{y} + (A_{7ba}(R_1) + A_{8ba}(R_1)) \hat{y} \right\} \]

\[ + e^{i\alpha_{R2}} \left\{ -(A_{7aa}(R_2) + A_{7ba}(R_2)) \hat{y} - (A_{7ba}(R_2) + A_{8ba}(R_2)) \hat{y} \right\} \]

(3.47)

which agrees with eq. (3.19) when the medium becomes isotropic.

For the far field above the z = 0 plane

In rectangular coordinates, eq. (3.47) becomes

\[ E_{xx} = -C_{xfe}(f_{ux}(-h) - f_{ux}(h)) \frac{\sin^2 \theta \sin \phi \cos \phi}{4\pi \varepsilon_1 \sqrt{\varepsilon_2} \left( \hat{R} \cdot \hat{e}_{xx} \cdot \hat{R} \right)^{3/2}} \]

\[ E_{xy} = -C_{xfe}(f_{ux}(-h) - f_{ux}(h)) \frac{\sqrt{\varepsilon_2}}{4\pi \varepsilon_1 \sqrt{\hat{R} \cdot \hat{e}_{xx} \cdot \hat{R}}} \left( \frac{\sin^2 \theta \sin^2 \phi}{\varepsilon_1 \hat{R} \cdot \hat{e}_{xx} \cdot \hat{R}} - 1 + \sin^2 \phi_c \right) \]

\[ + C_{xfo}(f_{ux}(-h) - f_{ux}(h)) \frac{\sin^2 \phi_c}{4\pi \hat{R}} \]
3.3 Numerical Solution

From the results in a uniaxial medium as derived above, the field patterns above the conducting plane will be plotted for the orientation of the optic axis along the z-, x- and y-directions, respectively. The configuration of the given source distribution is as previously specified in section 2.6 with the height \( h = 0 \) above the conducting plane. The lines that represent the field patterns are also the same as specified in section 2.6. The resulting far-zone field patterns will be compared with the known field patterns in isotropic medium (see figs. 2.34 and 2.35 in chapter 2). Also the effect of the orientation of the optic axis on the radiation patterns will be discussed in the next section.

\[
E_{xz} = C_{xze}(f_{ux}(-h) - f_{ux}(h)) \frac{\sqrt{\varepsilon_2}}{4\pi \varepsilon_1 \sqrt{\varepsilon - \varepsilon_{xx} \cdot \varepsilon}} \left( \frac{\sin \theta \cos \theta \sin \varphi - \sin \varphi \cos \varepsilon}{\varepsilon_1^{\frac{1}{2}} \cdot \varepsilon_{xx}^{\frac{1}{2}} \cdot \varepsilon} \right) - C_{xzo}(f_{ux}(-h) - f_{ux}(h)) \frac{\sin \varphi \cos \varphi}{4\pi r} \tag{3.48a-c}
\]
Figure 3.2 Rec. Aper. on the conducting plane radiated in uniaxial medium. The optic axis is along $z$-axis. 
\[ \omega_b = 0.7\omega, \; \omega_p = 0.3\omega \]
The field pattern is plotted on $xz$-plane.

Figure 3.3 Rec. Aper. on the conducting plane radiated in uniaxial medium. The optic axis is along $z$-axis. 
\[ \omega_b = 0.7\omega, \; \omega_p = 0.3\omega \]
The field pattern is plotted on $yz$-plane.
Figure 3.4 Rec. Aper. on the conducting plane radiated in uniaxial medium.
The optic axis is along z-axis.
\( \omega_b = 1.2\omega, \omega_p = 0.3\omega \)
The field pattern is plotted on \( \mathbf{zz} \)-plane.

Figure 3.5 Rec. Aper. on the conducting plane radiated in uniaxial medium.
The optic axis is along z-axis.
\( \omega_b = 1.2\omega, \omega_p = 0.3\omega \)
The field pattern is plotted on \( \mathbf{yz} \)-plane.
Figure 3.6 Rec. Aper. on the conducting plane radiated in uniaxial medium. The optic axis is along z-axis. 
\( \omega _b = 0.7 \omega , \, \omega _p = 0.2 \omega \) 
The field pattern is plotted on \( xx \)-plane.

Figure 3.7 Rec. Aper. on the conducting plane radiated in uniaxial medium. The optic axis is along z-axis. 
\( \omega _b = 0.7 \omega , \, \omega _p = 0.2 \omega \) 
The field pattern is plotted on \( yz \)-plane.
Figure 3.8 Rec. Aper. on the conducting plane radiated in uniaxial medium.
The optic axis is along x-axis.
ω₀ = 0.7ω, ωₚ = 0.3ω
The field patterns are plotted on zz-plane.

Figure 3.9 Rec. Aper. on the conducting plane radiated in uniaxial medium.
The optic axis is along x-axis.
ω₀ = 0.7ω, ωₚ = 0.3ω
The field patterns are plotted on yz-plane.
Figure 3.10 Rec. Aper. on the conducting plane radiated in uniaxial medium. The optic axis is along $x$-axis.
\( \omega_h = 1.2\omega, \omega_p = 0.3\omega \)
The field patterns are plotted on $xz$-plane.

Figure 3.11 Rec. Aper. on the conducting plane radiated in uniaxial medium. The optic axis is along $x$-axis.
\( \omega_h = 1.2\omega, \omega_p = 0.3\omega \)
The field patterns are plotted on $yz$-plane.
Figure 3.12 Rec. Aper. on the conducting plane radiated in uniaxial medium. The optic axis is along x-axis. 
\[ \omega_b = 0.7\omega, \quad \omega_p = 0.2\omega \]
The field patterns are plotted on xz-plane.

Figure 3.13 Rec. Aper. on the conducting plane radiated in uniaxial medium. The optic axis is along x-axis. 
\[ \omega_b = 0.7\omega, \quad \omega_p = 0.2\omega \]
The field patterns are plotted on yz-plane.
Figure 3.14 Rec. Aper. on the conducting plane radiated in uniaxial medium. The optic axis is along $y$-axis.

$\omega_b = 0.7\omega$, $\omega_p = 0.3\omega$

The field patterns are plotted on $xz$-plane.

Figure 3.15 Rec. Aper. on the conducting plane radiated in uniaxial medium. The optic axis is along $y$-axis.

$\omega_b = 0.7\omega$, $\omega_p = 0.3\omega$

The field patterns are plotted on $yz$-plane.
Figure 3.16 Rec. Aper. on the conducting plane radiated in uniaxial medium. The optic axis is along y-axis. 
\[ \omega_h = 1.2\omega, \quad \omega_p = 0.3\omega \]
The field patterns are plotted on xz-plane.

Figure 3.17 Rec. Aper. on the conducting plane radiated in uniaxial medium. The optic axis is along y-axis. 
\[ \omega_h = 1.2\omega, \quad \omega_p = 0.3\omega \]
The field patterns are plotted on yz-plane.
Figure 3.18 Rec. Aper. on the conducting plane radiated in uniaxial medium.
The optic axis is along $y$-axis.
$$\omega_h = 0.7\omega, \ \omega_p = 0.2\omega$$
The field patterns are plotted on $xz$-plane.

Figure 3.19 Rec. Aper. on the conducting plane radiated in uniaxial medium.
The optic axis is along $y$-axis.
$$\omega_h = 0.7\omega, \ \omega_p = 0.2\omega$$
The field patterns are plotted on $yz$-plane.
3.4 Summary

The above derivation has been accomplished utilizing the dyadic Green's function in closed form. This allows us to factorize the dyadic Green's function in isotropic and uniaxial media. In the isotropic case, it factorizes into a tensor and an exponential, both functions of \( R \). In the uniaxial case, it factorizes into multiple terms of the same form as the isotropic case. By this term (or these terms) with boundary condition on the conducting plane and the Uniqueness Theorem, the unknown source distribution for the reflected field can be obtained in terms of the direct source distribution. So, the field above the \( z = 0 \) plane is determined.

Comparing the numerical solutions of sections 3.1 and 3.2, we can summarize as follows:

a) In an isotropic medium, the resulting solutions (eq.(3.14),(3.16) and (3.19)) agree with the results by Balanis, Maclean and Collin [33,34,35]. The far field above the \( z = 0 \) plane is equal to the product of a factor which is a function of \( r \) (distance from the origin to the observation point) and a factor which is a function of the height \( h \) (the distance from the given source distribution to the conducting plane).

b) In uniaxial medium, the dielectric tensor of eq.(3.30) has only the diagonal elements so the conventional image theory is applicable. The resulting solutions (eqs. (3.35) and
(3.43)) agree with the conventional image theory.

c) When the optic axis is along the $z$-direction, only the extraordinary field exists in a uniaxial medium. The configuration of the field pattern is the same as the field pattern in an isotropic medium.

d) When the optic axis is not along the $z$-axis such as in Figs. 3.8-3.19, both ordinary and extraordinary field are significant. The field patterns are totally different from the field pattern in isotropic medium.

e) In a uniaxial medium, the ordinary and extraordinary far field solutions can each be written as the product of a factor which is a function of $r$ (distance from the origin to the observation point) and a factor which is a function of the height $h$ (the distance from the given source distribution to the conducting plane).

The resulting field solution from the reformulation of the image theory agrees with the resulting field solution from the conventional image method in both isotropic and uniaxial media. The same procedure will be applied to the case of plasma in the following chapter.
CHAPTER 4

Radiation in Plasma Half-Space

Introduction

According to the recent research in anisotropic medium [19,20,21,22], the conventional image theory is applicable only when the static magnetic field is perpendicular to the conducting plane. Otherwise the conventional image method is not applicable since the solution yields two set of sources situated in two different media. Taking advantage of the reformulation of image theory that was previously developed in chapter 3, the field in plasma half-space (above the conducting plane) can be determined.

Utilizing the same procedure as in section 3.2, the field which radiates from a given source distribution above an infinite conducting plane will be determined. This field must satisfy the boundary condition on the conducting plane. The unknown reflected source (for analysis purposes only) in terms of the directed source will be derived. We will begin with the given source distribution as an infinitesimal oscillating electric dipole. Then the radiated field from a rectangular aperture will be considered. The field will be determined when the static magnetic field is along the $z$-, $x$-, and $y$-axes, respectively.
4.1 *Radiation from a Dipole*

In the case of plasma where the dyadic Green's function is specified by eq. (2.35), if the source distribution is as given in eq. (3.5) and located at \((0, 0, h)\), the electric field due to the directed source and the unknown source distribution from the reflected field can be written as

\[
\vec{E}_{dp}(\vec{r}) = i\omega \mu_0 Il
\]

\[
\left[ e^{i k_0 R_{eld}} \left( B_1(R_1) (\vec{e}_{xc} \cdot \vec{R}_1) + B_2(R_1) \vec{e}_{xc} + B_3(R_1) \vec{A}_4 \right) \cdot \hat{f}_1 
- \frac{a^2}{b^2} \left( e^{i k_0 R_{old}} \vec{A}_5(R_1) \cdot \hat{f}_1 + e^{i k_0 \sqrt{\mu b R_1}} \vec{A}_6 \cdot \hat{f}_1 \right) \right]
+ \left\{ e^{i k_0 R_{old}} \vec{A}_7(R_1) \cdot \hat{f}_1 + e^{i k_0 \sqrt{\mu b R_1}} \vec{A}_8 \cdot \hat{f}_1 \right\}
\]

\[(4.1)\]

\[
R_{eld} = a\sqrt{\mu R_1 \cdot \vec{e}_{xc} \cdot \vec{R}_1}
\]

\[
R_{old} = \sqrt{\mu b R_1}
\]

\[
R_1 = \sqrt{x^2 + y^2 + (z-h)^2}
\]

\[
\vec{E}_{ip}(\vec{r}) = i\omega \mu_0 Il
\]

\[
\left[ e^{i k_0 R_{elr}} \left( B_1(R_2) (\vec{e}_{xc} \cdot \vec{R}_2) + B_2(R_2) \vec{e}_{xc} + B_3(R_2) \vec{A}_4 \right) \cdot \hat{f}_2 
- \frac{a^2}{b^2} \left( e^{i k_0 R_{olr}} \vec{A}_5(R_2) \cdot \hat{f}_2 + e^{i k_0 \sqrt{\mu b R_2}} \vec{A}_6 \cdot \hat{f}_2 \right) \right]
+ \left\{ e^{i k_0 R_{olr}} \vec{A}_7(R_2) \cdot \hat{f}_2 + e^{i k_0 \sqrt{\mu b R_2}} \vec{A}_8 \cdot \hat{f}_2 \right\}
\]

\[(4.2)\]

\[
R_{elr} = a\sqrt{\mu R_2 \cdot \vec{e}_{xc} \cdot \vec{R}_2}
\]

\[
R_{olr} = \sqrt{\mu b R_2}
\]

\[
R_2 = \sqrt{x^2 + y^2 + (z+h)^2}
\]
We must reevaluate the unknown source distribution from the reflection at the conducting plane in terms of the given directed source distribution due to each tensor separately. The source factors will be obtained as:

$$\overline{S}_{plc} = \overline{D}^{-1}_{1c}(\overline{r}_c) \cdot \overline{P} \cdot \overline{D}_{1c}(\overline{r}_c)$$

$$\overline{D}_{1c}(\overline{r}_c) = B_1(R_c)(\overline{e}_{xc} \cdot \overline{R}_c)(\overline{e}_{xc} \cdot \overline{R}_c) + B_2(R_c)\overline{e}_{xc}$$

$$\overline{S}_{p2c} = \frac{\overline{e}_c \cdot \overline{P} \cdot \text{adj} \overline{e}_c}{|\overline{e}_c|}$$

$$\overline{S}_{p1c} = \overline{A}_{(i+2)}(\overline{r}_c) \cdot \overline{P} \cdot \overline{A}_{(i+2)}(\overline{r}_c), \; i = 3 - 6$$

(4.3a-c)

The electric field above the conducting plane can be written as

$$\overline{E}_{tc}(\overline{r}) = \overline{E}_{dp}(\overline{r}) + \overline{E}_{dr}(\overline{r}) = i\omega \mu_0 Il$$

$$[e^{ik_0R_{sid}} \left\{ B_1(R_1)(\overline{e}_{xc} \cdot \overline{R}_1)(\overline{e}_{xc} \cdot \overline{R}_1) + B_2(R_1)\overline{e}_{xc} \cdot \overline{R}_1 \right\} \cdot \overline{f}_1 + B_3(R_1)\overline{A}_{4c} \cdot \overline{f}_1]$$

$$+ e^{ik_0R_{sid}} \left\{ B_1(R_2)(\overline{e}_{xc} \cdot \overline{R}_2)(\overline{e}_{xc} \cdot \overline{R}_2) + B_2(R_2)\overline{e}_{xc} \cdot \overline{R}_2 \right\} \cdot \left\{ \overline{S}_{p1c} \cdot \overline{f}_1 + B_3(R_2)\overline{A}_{4c} \cdot (\overline{S}_{p2c} \cdot \overline{f}_1) \right\}$$

$$- \frac{a_2}{b^2} \left\{ e^{ik_0R_{sid}} \overline{A}_{5c}(R_1) \cdot \overline{f}_1 + \left\{ e^{ik_0R_{sid}} - e^{ik_0\sqrt{\frac{x_1}{t_2}} a_{2c} \cdot \overline{R}_2} \right\} \overline{A}_{6c}(R_1) \cdot \overline{f}_1 \right\}$$

$$- \frac{a_2}{b^2} \left\{ e^{ik_0R_{sid}} \overline{A}_{5c}(R_2) \cdot (\overline{S}_{p3c} \cdot \overline{f}_1) + \left\{ e^{ik_0R_{sid}} - e^{ik_0\sqrt{\frac{x_1}{t_2}} a_{2c} \cdot \overline{R}_2} \right\} \overline{A}_{6c}(R_2) \cdot (\overline{S}_{p4c} \cdot \overline{f}_1) \right\}$$

$$+ \left\{ e^{ik_0R_{sid}} \overline{A}_{7c}(R_1) \cdot \overline{f}_1 + \left\{ e^{ik_0R_{sid}} - e^{ik_0\sqrt{\frac{y_1}{t_2}} b_{2c} \cdot \overline{R}_2} \right\} \overline{A}_{8c}(R_1) \cdot \overline{f}_1 \right\}$$

$$+ \left\{ e^{ik_0R_{sid}} \overline{A}_{7c}(R_2) \cdot (\overline{S}_{p3c} \cdot \overline{f}_1) + \left\{ e^{ik_0R_{sid}} - e^{ik_0\sqrt{\frac{y_1}{t_2}} b_{2c} \cdot \overline{R}_2} \right\} \overline{A}_{8c}(R_2) \cdot (\overline{S}_{p4c} \cdot \overline{f}_1) \right\}$$

(4.4)

Due to the complexity of the field solution, three special cases will be derived when the static magnetic field is along the z-, x-, and y-axes, respectively.
4.1.1 The static magnetic field is along the z-axis

In eq.(4.3a-c), the source factors become

\[ \overline{S}_{P1z} = \overline{D}_{1}^{-1}(\overline{r}_c) \cdot \overline{B}_1(\overline{r}_c) = \overline{P} \]

\[ \overline{B}_1(\overline{r}_c) = B_1(\overline{e}_{xz} \cdot \overline{R}_c) (\overline{e}_{xz} \cdot \overline{R}_c) + B_2 \overline{e}_{xz} \]

\[ \overline{S}_{P2z} = \frac{\overline{e}_z \cdot \overline{P} \cdot \text{adj} \overline{e}_z}{|\overline{e}_z|} = \overline{P} \]

\[ \overline{S}_{P3z} = \overline{S}_{P5z} = \overline{A}_{5c}^{-1} \cdot \overline{P} \cdot \overline{A}_{5c} = \overline{P} \]

\[ \overline{S}_{P4z} = \overline{S}_{P7z} = \overline{A}_{6c}^{-1} \cdot \overline{P} \cdot \overline{A}_{6c} = \overline{P} \]

\[ \overline{A}_{5c} = \begin{bmatrix} T_1 + (\varepsilon_2 - b^2) \cos^2 \phi_c & i \varepsilon_3 T_3 + \frac{(\varepsilon_2 - b^2) \sin(2\phi_c)}{2} & 0 \\ -i \varepsilon_3 T_2 + \frac{(\varepsilon_2 - b^2) \sin(2\phi_c)}{2} & T_1 + (\varepsilon_2 - b^2) \sin^2 \phi_c & 0 \\ 0 & 0 & T_3 \end{bmatrix} \]

\[ \overline{A}_{6c} = \begin{bmatrix} \cos(2\phi_c) & \sin(2\phi_c) & 0 \\ \sin(2\phi_c) & -\cos(2\phi_c) & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ T_1 = \varepsilon_1 (1 - \frac{\varepsilon_2}{b^2}) + \frac{\varepsilon_2}{A}(1 - \frac{\varepsilon_1}{b^2}) \]

\[ T_2 = 1 - \frac{\varepsilon_2}{b^2} - \frac{\varepsilon_2}{Ab^2} \]

\[ T_3 = \left( \varepsilon_1 - \frac{(\varepsilon_1^2 - \varepsilon_3^2)}{b^2} \right) (1 + \frac{1}{A}) + (\varepsilon_1 - b^2) \frac{1}{A} \]

Substituting the above source factors into eq.(4.4), we obtain the electric field above the z = 0 plane as:

\[ \overline{E}_{tz}(\overline{r}) = \overline{E}_{dz}(\overline{r}) + \overline{E}_{tz}(\overline{r}) = i\omega \mu_0 I \]

\[ \left[ e^{ik_0 r_{dx}} \left( (B_1(\overline{R}_1) (\overline{e}_{xz} \cdot \overline{R}_1) (\overline{e}_{xz} \cdot \overline{R}_1) + B_2(\overline{R}_1) \overline{e}_{xz}) + B_3(\overline{R}_1) \overline{A}_4 \right) \right] \cdot \hat{I}_1 \]
which agrees with eq. (3.35) when the medium becomes uniaxial.

We next check the boundary condition at the \( z = 0 \) plane

In this case,

\[
\vec{R}_1 \cdot \vec{e}_{xx} \cdot \vec{R}_1 = \vec{R}_2 \cdot \vec{e}_{xx} \cdot \vec{R}_2 = \frac{1}{R_1} \left( \frac{x^2}{\varepsilon_1} + \frac{y^2}{\varepsilon_1} + \frac{h^2}{\varepsilon_2} \right) ; \quad R_1 = R_2 = \sqrt{r^2 + h^2}
\]

\[
R_{01} = R_{e2} ; \quad R_{01} = R_{e2} ; \quad B_1 (R_{1c}) = B_1 (R_{2c}) ; \quad B_2 (R_{1c}) = B_2 (R_{2c})
\]

\[
B_3 (R_{1c}) = B_3 (R_{2c})
\]

\[
\left( \vec{e}_{xx} \cdot \vec{R}_1 \right) \cdot \vec{I}_1 = \frac{1}{R_1} \left\{ \frac{XX_1}{\varepsilon_1} + \frac{YY_1}{\varepsilon_1} - \frac{hZ_1}{\varepsilon_2} \right\} \left\{ \frac{x}{\varepsilon_1} \hat{\varkappa} + \frac{y}{\varepsilon_1} \hat{\gamma} + \frac{h}{\varepsilon_2} \hat{\zeta} \right\}
\]

\[
\left( \vec{e}_{xx} \cdot \vec{R}_2 \right) \cdot \left( \vec{I}_1 \right) = \frac{1}{R_2} \left\{ - \frac{XX_1}{\varepsilon_1} + \frac{YY_1}{\varepsilon_1} + \frac{hZ_1}{\varepsilon_2} \right\} \left\{ \frac{x}{\varepsilon_1} \hat{\varkappa} + \frac{y}{\varepsilon_1} \hat{\gamma} + \frac{h}{\varepsilon_2} \hat{\zeta} \right\}
\]

\[
\overline{A}_4 \cdot \vec{I}_1 = \left( \varepsilon_1 \varepsilon_2 x_1 + i e_2 e_3 y_1 \right) \hat{\varkappa} + \left( -i e_2 e_3 x_1 + \varepsilon_1 \varepsilon_2 y_1 \right) \hat{\gamma} + (e_1^2 - e_3^2) z_1 \hat{\zeta}
\]

\[
\overline{A}_4 \cdot \left( \vec{P} \cdot \vec{I}_1 \right) = -\left( \varepsilon_1 e_2 x_1 + i e_2 e_3 y_1 \right) \hat{\varkappa} - \left( -i e_2 e_3 x_1 + \varepsilon_1 \varepsilon_2 y_1 \right) \hat{\gamma} + (e_1^2 - e_3^2) z_1 \hat{\zeta}
\]

\[
\overline{A}_4 (R_{1c}) \cdot \vec{I}_1 = \left\{ A_{iab} (R_{1c}) x_1 + A_{iab} (R_{1c}) y_1 \right\} \hat{\varkappa} + \left\{ A_{iab} (R_{1c}) x_1 + A_{iab} (R_{1c}) y_1 \right\} \hat{\gamma}
\]
\[ \mathbf{A}_i(R_{2c}) \cdot (\mathbf{B} \cdot \hat{r}_1) = -A_{iaa}(R_{2c})x_1 + A_{iab}(R_{2c})y_1 A_{iba}(R_{2c})x_1 + A_{iab}(R_{2c})y_1 \] 

\[ (i = 5 - 8) \]

The electric field on the z = 0 plane becomes

\[ \mathbf{E}_{zc}(r_c) = i \omega \mu_0 I \ell e^{ikr} \]

\[ \left[ -B_1(R_{1c}) \frac{h}{R_1^2 \varepsilon_2} \left( \frac{xx_1}{\varepsilon_1} + \frac{yy_1}{\varepsilon_1} - \frac{hz_1}{\varepsilon_2} \right) + B_2(R_{1c}) \frac{z_1}{\varepsilon_2} + B_3(R_{1c}) (\varepsilon_1^2 - \varepsilon_3^2) z_1 \right] \hat{z} \quad (4.6) \]

That is, the tangential components of the electric field satisfy the same boundary condition (see in section 3.1.3 p.57) at the z = 0 plane. Hence according to the Uniqueness theorem the fields in the region z > 0 for the two problems are the same. So, the electric field solution in eq.(4.5) is the only field solution above the z > 0 plane.

4.1.2 The static magnetic field is along the x-axis

In eq.(4.3a-c), the source factors become

\[ \mathbf{S}_{p1x} = \mathbf{S}_{p4x} = \mathbf{S}_{p6x} = \mathbf{p} \]

\[ \mathbf{S}_{p2x} = \left[ \begin{array} {ccc} -1 & 0 & 0 \\ 0 & -d_{2a} & -d_{2b} \\ 0 & -d_{2b} & d_{2a} \end{array} \right] \]

\[ d_{2a} = \frac{\varepsilon_1^2 + \varepsilon_2^2}{\varepsilon_1^2 - \varepsilon_3^2} ; \quad d_{2b} = \frac{2j \varepsilon_1^2 \varepsilon_3}{\varepsilon_1^2 - \varepsilon_3^2} \]

\[ \mathbf{S}_{p3x} = \mathbf{S}_{p5x} = \left[ \begin{array} {ccc} -1 & 0 & 0 \\ 0 & -d_{3a} & -2d_{3b} \\ 0 & 2d_{3c} & d_{3a} \end{array} \right] \]
\[ d_{3a} = \frac{T_1(T_1 + e_2 - b_2) + \varepsilon_3^2 T_2}{T_1(T_1 + e_2 - b_2) - \varepsilon_3^2 T_2} \]
\[ d_{3b} = \frac{i e_3 T_2 T_1}{T_1(T_1 + e_2 - b_2) - \varepsilon_3^2 T_2} \]
\[ d_{3c} = \frac{-i e_3 T_2(T_1 + e_2 - b_2)}{T_1(T_1 + e_2 - b_2) - \varepsilon_3^2 T_2} \]

and the electric field above the \( z = 0 \) plane can be summarized as

\[ \overline{E}_{Tx}(\vec{r}) = \overline{E}_{dx}(\vec{r}) + \overline{E}_{ix}(\vec{r}) = i \omega \mu_0 I \]

\[ \left[ e^{i k_0 R_{a1x}} \left\{ \left( B_1(R_1) \left( \overline{e}_{xx} \cdot \overline{R}_1 \right) \left( \overline{e}_{xx} \cdot \overline{R}_1 \right) + B_2(R_1) \left( \overline{e}_{xx} \cdot \overline{R}_1 \right) \right) \overline{T}_1 + B_3(R_1) \overline{A}_{4x} \cdot \overline{F}_1 \right\} + e^{i k_0 R_{a2x}} \left\{ \left( B_1(R_2) \left( \overline{e}_{xx} \cdot \overline{R}_2 \right) \left( \overline{e}_{xx} \cdot \overline{R}_2 \right) + B_2(R_2) \left( \overline{e}_{xx} \cdot \overline{R}_2 \right) \right) \overline{F}_1 \right\} + B_3(R_2) \overline{A}_{4x} \cdot \overline{F}_1 \right] \]

\[ - \frac{a^2}{b^2} \left\{ e^{i k_0 R_{a1x}} \overline{A}_{5x}(R_1) \cdot \overline{T}_1 + \left( e^{i k_0 R_{a1x}} - e^{i k_0 \sqrt{\varepsilon_2} \varepsilon_1 R_1} \right) \overline{A}_{6x}(R_1) \cdot \overline{T}_1 \right\} \]

\[ - \frac{a^2}{b^2} \left\{ e^{i k_0 R_{a2x}} \overline{A}_{5x}(R_2) \cdot \overline{T}_1 + \left( e^{i k_0 R_{a2x}} - e^{i k_0 \sqrt{\varepsilon_2} \varepsilon_1 R_2} \right) \overline{A}_{6x}(R_2) \cdot \overline{T}_1 \right\} \]

\[ + \left\{ e^{i k_0 R_{o1x}} \overline{A}_{7x}(R_1) \cdot \overline{T}_1 + \left( e^{i k_0 R_{o1x}} - e^{i k_0 \sqrt{\varepsilon_2} \varepsilon_1 R_1} \right) \overline{A}_{8x}(R_1) \cdot \overline{T}_1 \right\} \]

\[ + \left\{ e^{i k_0 R_{o2x}} \overline{A}_{7x}(R_2) \cdot \overline{T}_1 + \left( e^{i k_0 R_{o2x}} - e^{i k_0 \sqrt{\varepsilon_2} \varepsilon_1 R_2} \right) \overline{A}_{8x}(R_2) \cdot \overline{T}_1 \right\} \right] \]

(4.7)

which agrees with eq.(3.43) when the medium becomes uniaxial.

Checking the boundary condition on the \( z = 0 \) plane

In this case,

\[ \hat{R}_{1c} \cdot \overline{e}_{xx} \cdot \overline{R}_{1c} = \hat{R}_{2c} \cdot \overline{e}_{xx} \cdot \overline{R}_{2c} = \frac{1}{R_1} \left( \frac{x^2}{\varepsilon_2} + \frac{y^2}{\varepsilon_1} + \frac{h^2}{\varepsilon_1} \right) ; \quad R_{1c} = R_{2c} = \sqrt{r^2 + h^2} \]

\[ R_{e1xc} = R_{e2xc} ; \quad R_{o1xc} = R_{o2xc} ; \quad B_1(R_{1c}) = B_1(R_{2c}) ; \quad B_2(R_{1c}) = B_2(R_{2c}) \]
The electric field on the \( z = 0 \) plane can be summarized as

\[
\vec{E}_{exc}(\vec{r}_c) = 2i\omega\mu_0 I I
\]

\[
\left( e^{i\kappa R_{exc}}\right) \left( -B_1(R_{1c}) \frac{h}{R_1^2 R_1} \left( \frac{XX_1}{\epsilon_2} + \frac{YY_1}{\epsilon_1} - \frac{hz_1}{\epsilon_1} \right) + B_2(R_{1c}) \frac{Z_1}{\epsilon_1} + (-i \epsilon_2 \epsilon_3 y_1 + \epsilon_1 \epsilon_2 z_1) \right)
\]

\[
- \frac{a^2}{b^2} \left( e^{i\kappa R_{exc}}(A_{ba}(R_{1c}) y_1 + A_{bb}(R_{1c}) Z_1) \right)
\]

\[
- A_{bb}(R_{1c}) Z_1 \left( e^{i\kappa R_{exc}} - e^{i\kappa \sqrt{b^2 + bR_{1c}^2}} \right)
\]

\[
+ \left( e^{i\kappa R_{exc}}(A_{ba}(R_{1c}) y_1 + A_{bb}(R_{1c}) Z_1) - A_{bb}(R_{1c}) Z_1 \left( e^{i\kappa R_{exc}} - e^{i\kappa \sqrt{b^2 + bR_{1c}^2}} \right) \right) \]
Hence the tangential components of the electric field vanish on the \( z = 0 \) plane. According to the boundary condition on the \( z = 0 \) plane and the Uniqueness theorem, the field in the region \( z > 0 \) for the two problems are the same. So the field solution in eq.(4.7) is the solution to the problem.

4.1.3 The static magnetic field is along the \( y \)-axis

In eq.(4.3a-c), the source factors become

\[
\overline{S}_{Fy} = \overline{S}_{FY} = \overline{S}_{F5y} = \overline{p}
\]

\[
\overline{S}_{F2y} = \begin{bmatrix}
-d_{2a} & 0 & d_{2b} \\
0 & -1 & 0 \\
d_{2b} & 0 & d_{2a}
\end{bmatrix}
\]

\[
\overline{S}_{F3y} = \overline{S}_{F5y} = \begin{bmatrix}
-d_{3a} & 0 & -2d_{3b} \\
0 & -1 & 0 \\
2d_{3b} & 0 & d_{3a}
\end{bmatrix}
\]

and the electric field above the \( z = 0 \) plane can be summarized as

\[
\overline{E}_{TY}(\overline{r}) = \overline{E}_{DY}(\overline{r}) + \overline{E}_{XY}(\overline{r}) = i\omega \mu_0 I I
\]

\[
\left[ e^{ik_0 R_{1y}} \left\{ \left( B_1(R_1) \left( \overline{e}_{xy} \cdot \overline{R}_1 \right) + B_2(R_1) \overline{e}_{xy} \right) \cdot \overline{F}_1 + B_3(R_1) \overline{A}_{4y} \cdot \overline{F}_1 \right\} ight]
\]

\[
+ e^{ik_0 R_{2y}} \left\{ \left( B_1(R_2) \left( \overline{e}_{xy} \cdot \overline{R}_2 \right) + B_2(R_2) \overline{e}_{xy} \right) \left( \overline{F}_1 + B_3(R_2) \overline{A}_{4y} \cdot \overline{F}_1 \right) \right\}
\]

\[
- \frac{a^2}{b^2} \left\{ e^{ik_0 R_{1y}} \overline{A}_{5y}(R_1) \cdot \overline{F}_1 + \left( e^{ik_0 R_{1y}} - e^{ik_0 \frac{\sqrt{\mu_1 \rho}}{\epsilon_2} R_1 \cdot \overline{F}_1} \right) \overline{A}_{6y}(R_1) \cdot \overline{F}_1 \right\}
\]

\[
- \frac{a^2}{b^2} \left\{ e^{ik_0 R_{2y}} \overline{A}_{5y}(R_2) \cdot \left( \overline{S}_{F3y} \cdot \overline{F}_1 \right) + \left( e^{ik_0 R_{2y}} - e^{ik_0 \frac{\sqrt{\mu_2 \rho}}{\epsilon_2} R_2 \cdot \overline{F}_1} \right) \overline{A}_{6y}(R_2) \cdot \left( \overline{F}_1 \right) \right\}
\]

\[
+ \left\{ e^{ik_0 R_{3y}} \overline{A}_{7y}(R_1) \cdot \overline{F}_1 + \left( e^{ik_0 R_{3y}} - e^{ik_0 \frac{\sqrt{\mu_2 \rho}}{\epsilon_2} R_1 \cdot \overline{F}_1} \right) \overline{A}_{8y}(R_1) \cdot \overline{F}_1 \right\}
\]
Checking the boundary condition on the $z = 0$ plane

In this case,

$$\vec{R}_{1c}, \vec{e}_{xy} \cdot \vec{R}_{1c} = \vec{R}_{2c}, \vec{e}_{xy} \cdot \vec{R}_{2c} = \frac{1}{R_1^2}\left( \frac{x^2}{e_1} + \frac{y^2}{e_2} + \frac{h^2}{e_1} \right) \; ; \; R_{1c} = R_{2c} = \sqrt{x^2 + h^2}
$$

$$R_{xyc} = R_{o2yc} ; \; R_{y1yc} = R_{o2yc} ; \; B_1 (R_{1c}) = B_1 (R_{2c}) ; \; B_2 (R_{1c}) = B_2 (R_{2c})$$

$$B_3 (R_{1c}) = B_3 (R_{2c})$$

$$\left( (\vec{e}_{xy} \cdot \vec{R}_{1c}) \cdot \vec{t}_1 \right)(\vec{e}_{xy} \cdot \vec{R}_{1c}) = \frac{1}{R_1^2}\left\{ \frac{xx_1}{e_1} + \frac{yy_1}{e_2} - \frac{hz_1}{e_1} \right\}\left\{ \frac{x}{e_1} \hat{\mathcal{R}} + \frac{y}{e_2} \hat{\mathcal{P}} - \frac{h}{e_1} \hat{\mathcal{Q}} \right\}
$$

$$\left( (\vec{e}_{xy} \cdot \vec{R}_{2c}) \cdot (\vec{P} \cdot \vec{t}_1) \right)(\vec{e}_{xy} \cdot \vec{R}_{2c}) = \frac{1}{R_2^2}\left\{ \frac{xx_1}{e_1} - \frac{yy_1}{e_2} + \frac{hz_1}{e_1} \right\}\left\{ \frac{x}{e_1} \hat{\mathcal{R}} + \frac{y}{e_2} \hat{\mathcal{P}} + \frac{h}{e_1} \hat{\mathcal{Q}} \right\}
$$

$$\vec{A}_{4y} \cdot \vec{t}_1 = (e_1 e_2 x_1 - ie_2 e_3 z_1) \hat{\mathcal{R}} + (e_1^2 - e_3^2) y_j \hat{\mathcal{P}} + (ie_2 e_3 x_1 + e_1 e_2 z_1) \hat{\mathcal{Q}}
$$

$$\vec{A}_{4y} \cdot (\vec{e}_{xy} \cdot \vec{t}_1) = -(e_1 e_2 x_1 - ie_2 e_3 z_1) \hat{\mathcal{R}} - (e_1^2 - e_3^2) y_j \hat{\mathcal{P}} + (ie_2 e_3 x_1 + e_1 e_2 z_1) \hat{\mathcal{Q}}
$$

$$\vec{A}_{my} (R_{1c}) \cdot \vec{t}_1 = (A_{mbb} (R_{1c}) x_1 + A_{mba} (R_{1c}) z_1) \hat{\mathcal{R}} + A_{mcc} (R_{1c}) y_j \hat{\mathcal{P}} + (A_{mba} (R_{1c}) y_1 + A_{maa} (R_{1c}) z_1) \hat{\mathcal{Q}}
$$

$$\vec{A}_{my} (R_{2c}) \cdot (\vec{e}_{xy} \cdot \vec{t}_1) = -(A_{mbb} (R_{2c}) x_1 + A_{mba} (R_{2c}) z_1) \hat{\mathcal{R}} - A_{mcc} (R_{2c}) y_j \hat{\mathcal{P}} + (A_{mba} (R_{2c}) y_1 + A_{maa} (R_{2c}) z_1) \hat{\mathcal{Q}}
$$

$$\vec{A}_{nx} (R_{1c}) \cdot \vec{t}_1 = A_{nbb} (R_{1c}) x_1 \hat{\mathcal{R}} - A_{naa} (R_{1c}) z_1 \hat{\mathcal{Q}}
$$

$$\vec{A}_{nx} (R_{2c}) \cdot (\vec{P} \cdot \vec{t}_1) = - A_{nbb} (R_{2c}) x_1 \hat{\mathcal{R}} - A_{naa} (R_{2c}) z_1 \hat{\mathcal{Q}} \; ; \; m = 5, 7 \; ; \; n = 6, 8
$$

The electric field on the $z = 0$ plane can be summarized as

$$\vec{E}_{zyc} (\vec{t}_c) = 2i \omega \mu_0 \vec{I}I$$
\[
\left[ e^{ik_0R_1e_{1y}} \left\{-B_1(R_{1c}) \frac{h}{R_1^2 \varepsilon_1} \left( \frac{XX_1}{\varepsilon_1} + \frac{YY_1}{\varepsilon_2} - \frac{hZ_1}{\varepsilon_1} \right) + B_2(R_{1c}) \frac{Z_1}{\varepsilon_1} + \left( i \varepsilon_2 \varepsilon_3 X_1 + \varepsilon_1 \varepsilon_2 Z_1 \right) \right\} \right]
\]

\[- \frac{a^2}{b^2} \left( e^{ik_0R_1e_{1y}} (A_{5ab}(R_{1c})X_1 + A_{5aa}(R_{1c})Z_1) \right) \]

\[-A_{6aa}(R_{1c}) Z_1 \left( e^{ik_0R_1e_{1c}} - e^{ik_0 \sqrt{\mu_1 \varepsilon_1} R_{1c}'} \right) \]

\[+ \left( e^{ik_0R_1e_{1y}} (A_{7ab}(R_{1c})X_1 + A_{7aa}(R_{1c})Z_1) - A_{8aa}(R_{1c}) Z_1 \left( e^{ik_0R_1e_{1c}} - e^{ik_0 \sqrt{\mu_1 \varepsilon_1} R_{1c}'} \right) \right) \] \quad (4.10)

Hence, the tangential components of the electric field vanish on the \( z = 0 \) plane. According to the boundary condition and the Uniqueness theorem, the electric field in eq. (4.9) is the solution to the problem.

### 4.2 Radiation from a Rectangular Aperture

If the given source distribution is as specified in section 2.4, which is

\[ \mathcal{S}(\overline{r}) = -\nabla \times \overline{M}_s(\overline{r}) = -\pi E_0 \sin \left( \frac{mnX}{a_a} \right) \cos \left( \frac{mnY}{b_a} \right) \] \quad (4.11)

where \( \overline{M}_s(\overline{r}) \) represents magnetic surface current density, then the volume integral in eq. (2.4) reduces to the surface integral. The exact field solution can be obtained ideally by superposing the given source distribution into some small segments. Each of these segments radiates a separate field as in the derivation in section 3.1.4. Combination of these fields yields the exact field solution. Since this exact
solution is very difficult to obtain, approximate methods must be used. For most practical antennas with overall lengths greater than a wavelength, a maximum total phase error of $\pi/8$ rad does not cause much effect in the analytical formulation [46]. Using this as a criterion and the source factor from the reformulation of the image theory as derived in section 4.1, the far-field solution can be determined. The field patterns will be plotted for different orientation of the static magnetic field and for the source distribution as specified by eq. (4.11).

4.2.1 The static field is along the z-axis

In this case, the electric far-field above the $z = 0$ plane can be summarized as

$$
\bar{E}_{zf} = \pi E_0 \frac{a_a b_a}{2} \left[ \begin{array}{c} 
\frac{B_{1f}(r) z}{\epsilon_2} \left( \frac{x \phi}{\epsilon_1} + \frac{y \phi}{\epsilon_1} + \frac{z \phi}{\epsilon_2} \right) + B_3(r) (\epsilon_1^2 - \epsilon_3^2) \frac{a_a}{b^2} A_{5cc}(r) \phi \right] \\
\left( \frac{\sin x_{s11}}{x_{s11}} - \frac{\sin x_{s12}}{x_{s12}} \right) \left( \frac{\sin y_{s11}}{y_{s11}} - \frac{\sin y_{s12}}{y_{s12}} \right) e^{ik_0 r_{es}} \\
+ A_{7cc}(r) \left( \frac{\sin x_{s21}}{x_{s21}} - \frac{\sin x_{s22}}{x_{s22}} \right) \left( \frac{\sin y_{s21}}{y_{s21}} - \frac{\sin y_{s22}}{y_{s22}} \right) e^{ik_0 r} \right] (4.12)
$$

4.2.2 The static field is along the x-axis

In this case, the electric far-field above the $z = 0$ plane can be summarized as

$$
\bar{E}_{xf} = \pi E_0 \frac{a_a b_a}{4} \left[ \begin{array}{c} 
\end{array} \right]
$$
\[
\left\{ \frac{2 B_{1f}(r) z}{\varepsilon_1} \left( \frac{x \phi}{\varepsilon_2} + \frac{y \phi}{\varepsilon_1} + \frac{z \phi}{\varepsilon_1} \right) + 2 B_{3}(r) \varepsilon_1 \varepsilon_2 \right\} \varepsilon_2^2 \\
- \frac{a^2}{b^2} \left\{ (A_{5ab}(r) (1 + d_3) - 2 d_{3b} A_{5aa}(r)) \phi + (A_{5bb}(r) (1 + d_3) - 2 A_{5ba}(r) d_{3b}) \phi \right\}
\]

\[
\left( \frac{\sin x_{Y11} - \sin x_{Y12}}{x_{Y11} - x_{Y12}} \right) \left( \frac{\sin y_{Y11} - \sin y_{Y12}}{y_{Y11} - y_{Y12}} \right) e^{i k_{r_X}} \\
+ \left( \frac{\sin x_{Y21} - \sin x_{Y22}}{x_{Y21} - x_{Y22}} \right) \left( \frac{\sin y_{Y21} - \sin y_{Y22}}{y_{Y21} - y_{Y22}} \right) e^{i k_{r_Y}} \right] \tag{4.13}
\]

\[\text{4.2.3 The static field is along the y-axis}\]

In this case, the electric far-field above the \(z = 0\) plane can be summarized as

\[
\tilde{E}_{yf} = \pi E_0 \frac{a_s b_s}{4} \left[ \\
\left\{ \frac{2 B_{1f}(r) z}{\varepsilon_1} \left( \frac{x \phi}{\varepsilon_2} + \frac{y \phi}{\varepsilon_1} + \frac{z \phi}{\varepsilon_1} \right) + 2 B_{3}(r) \varepsilon_1 \varepsilon_2 \right\} \varepsilon_2^2 \\
- \frac{a^2}{b^2} \left\{ (A_{5ba}(r) (1 + d_3) - 2 d_{3b} A_{5bb}(r)) \phi + (A_{5aa}(r) (1 + d_3) - 2 A_{5ab}(r) d_{3b}) \phi \right\}
\]

\[
\left( \frac{\sin x_{Y11} - \sin x_{Y12}}{x_{Y11} - x_{Y12}} \right) \left( \frac{\sin y_{Y11} - \sin y_{Y12}}{y_{Y11} - y_{Y12}} \right) e^{i k_{r_X}} \\
+ \left( \frac{\sin x_{Y21} - \sin x_{Y22}}{x_{Y21} - x_{Y22}} \right) \left( \frac{\sin y_{Y21} - \sin y_{Y22}}{y_{Y21} - y_{Y22}} \right) e^{i k_{r_Y}} \right] \tag{4.14}
\]
4.3 **Numerical Solution**

From the results of the previous section, the field patterns for a rectangular aperture will be plotted with the static magnetic field pointing along the z-, x-, and y-axes, respectively. The resulting far field patterns will be compared to those of an isotropic and uniaxial medium as discussed in chapter 3. The effect of the orientation of the static magnetic field on the radiation patterns will also be discussed in the next section.
Figure 4.1 Rec. Aper. on the conducting plane radiated in plasma. The static magnetic field is along \( z \)-axis. \( \omega_b = 0.7\omega, \omega_p = 0.3\omega \) The field patterns are plotted on \( \text{xz} \)-plane.

Figure 4.2 Rec. Aper. on the conducting plane radiated in plasma. The static magnetic field is along \( z \)-axis. \( \omega_b = 0.7\omega, \omega_p = 0.3\omega \) The field patterns are plotted on \( \text{yz} \)-plane.
Figure 4.3 Rec. Aper. on the conducting plane radiated in plasma.
The static magnetic field is along $z$-axis.
\[ \omega_b = 1.2\omega, \omega_p = 0.3\omega \]
The field patterns are plotted on $xz$-plane.

Figure 4.4 Rec. Aper. on the conducting plane radiated in plasma.
The static magnetic field is along $z$-axis.
\[ \omega_b = 1.2\omega, \omega_p = 0.3\omega \]
The field patterns are plotted on $yz$-plane.
Figure 4.5 Rec. Aper. on the conducting plane radiated in plasma. The static magnetic field is along z-axis.
\[ \omega_b = 0.7\omega, \omega_p = 0.2\omega \]
The field patterns are plotted on xz-plane.

Figure 4.6 Rec. Aper. on the conducting plane radiated in plasma. The static magnetic field is along z-axis.
\[ \omega_b = 0.7\omega, \omega_p = 0.2\omega \]
The field patterns are plotted on yz-plane.
Figure 4.7 Rec. Aper. on the conducting plane radiated in plasma.
The static magnetic field is along $x$-axis.
$\omega_b = 0.7\omega, \omega_p = 0.3\omega$
The field patterns are plotted on $xz$-plane.

Figure 4.8 Rec. Aper. on the conducting plane radiated in plasma.
The static magnetic field is along $x$-axis.
$\omega_b = 0.7\omega, \omega_p = 0.3\omega$
The field patterns are plotted on $yz$-plane.
Figure 4.9 Rec. Aper. on the conducting plane radiated in plasma. 
The static magnetic field is along $x$-axis. 
$\omega_b = 1.2\omega$, $\omega_p = 0.3\omega$ 
The field patterns are plotted on $xz$-plane.

Figure 4.10 Rec. Aper. on the conducting plane radiated in plasma. 
The static magnetic field is along $x$-axis. 
$\omega_b = 1.2\omega$, $\omega_p = 0.3\omega$ 
The field patterns are plotted on $yz$-plane.
Figure 4.11 Rec. Aper. on the conducting plane radiated in plasma. The static magnetic field is along x-axis.
\[ \omega_b = 0.7\omega, \omega_p = 0.2\omega \]
The field patterns are plotted on xz-plane.

Figure 4.12 Rec. Aper. on the conducting plane radiated in plasma. The static magnetic field is along x-axis.
\[ \omega_b = 0.7\omega, \omega_p = 0.2\omega \]
The field patterns are plotted on yz-plane.

\[ c \text{ in } x, yz\text{-plane}, \omega_b = 0.7\omega, \omega_p = 0.2\omega \]
Figure 4.13 Rec. Aper. on the conducting plane radiated in plasma.
The static magnetic field is along y-axis.
\( \omega_b = 0.7\omega, \omega_p = 0.3\omega \)
The field patterns are plotted on \( \text{xz-plane} \).

Figure 4.14 Rec. Aper. on the conducting plane radiated in plasma.
The static magnetic field is along y-axis.
\( \omega_b = 0.7\omega, \omega_p = 0.3\omega \)
The field patterns are plotted on \( \text{yz-plane} \).
Figure 4.15 Rec. Aper. on the conducting plane radiated in plasma. The static magnetic field is along y-axis. 
\[ \omega_b = 1.2\omega, \omega_p = 0.3\omega \]
The field patterns are plotted on \( \text{xz} \)-plane.

Figure 4.16 Rec. Aper. on the conducting plane radiated in plasma. The static magnetic field is along y-axis. 
\[ \omega_b = 1.2\omega, \omega_p = 0.3\omega \]
The field patterns are plotted on \( \text{yz} \)-plane.
Figure 4.17 Rec. Aper. on the conducting plane radiated in plasma. The static magnetic field is along y-axis. 
\[ \omega_h = 0.7\omega, \quad \omega_p = 0.2\omega \] 
The field patterns are plotted on \( zz \)-plane.

Figure 4.18 Rec. Aper. on the conducting plane radiated in plasma. The static magnetic field is along y-axis. 
\[ \omega_h = 0.7\omega, \quad \omega_p = 0.2\omega \] 
The field patterns are plotted on \( yz \)-plane.
4.4 **Summary**

The procedure for the reformulation of the image theory utilizing the factorization of the dyadic Green's function in the closed form can be summarized as follows:

1) Make use of the boundary condition on the conducting plane, solve for the unknown source distribution for the reflected field in terms of the given direct source distribution.

2) Use the resulting source distribution as another source and remove the conducting plane.

3) The radiating field from the combination of the direct and resulting source distribution is the only field solution that is valid above the conducting plane (based on the boundary condition and the uniqueness theorem).

From **numerical results**, we conclude that

1) In a plasma, there always exist both ordinary and extraordinary fields. Either of these fields is dominant. This is unlike a uniaxial medium, where there exists only the extraordinary field when the static magnetic field is perpendicular to the conducting plane.

2) The configuration and amplitude of the field patterns in plasma are totally different from the field pattern in an isotropic medium.

3) When the static magnetic field is along the z-axis, the configurations of the field patterns in plasma are totally
different from that in uniaxial medium.

4) When the static magnetic field is along the x- or y-axis, the field patterns in uniaxial media and plasmas are more directional in the normal direction to the conducting plane.

From the resulting solutions in eqs. (4.5, 4.7, 4.9 and 4.12-14), we can

1) Reduce the closed-form solution to the isotropic case which produced the same results as utilizing the conventional image method.

This theory is verified, see section 3.1 for details.

2) Check in the case of an anisotropic medium; when the static magnetic field is in the direction perpendicular to the conducting screen, the final field solution should be the same as in the conventional image method.

The results are verified, see section 4.1 for details.

3) Check the case of a uniaxial medium; since the dielectric tensor has only the diagonal elements, the solution should be the same as utilizing the conventional image method.

The results are verified, see section 3.2 for details.

Consequently, we can conclude that the reformulation of image theory is applicable in isotropic and uniaxial media, and anisotropic plasma.
Chapter 5

Overall Summary

In this dissertation, Chen's approach has been applied to treat problems previously not considered. The first problem was to describe the radiation from a given source distribution in an unbounded anisotropic medium. Using the three-dimensional delta function and three-dimensional Fourier transform, we obtain the dyadic Green's function in the form of eq. (2.10). Using linear analysis, and Bessel and Fourier transforms, we evaluated the dyadic Green's function with respect to an orthogonal coordinate system \( \hat{\alpha}, \hat{\beta}, \hat{\gamma} \). We found the dyadic Green's function for an anisotropic plasma in closed form, with a source frequency restriction. This function is also expressed in terms of coordinate system \( \hat{\alpha}, \hat{\beta}, \hat{\gamma} \) and can directly be reduced to the case of an isotropic or a uniaxial medium. This allows us to consider the characteristics of an anisotropic plasma by one simple dyadic Green's function in closed form. The final solution eq. (2.35) (or eq. (2.43)) shows that there exists two types of fields: one depends on the orientation of the static magnetic field and the other does not. The cut-off condition is determined by the determinant of the wave matrix. The radiated field from a rectangular aperture or an electric dipole was determined for specific orientations of the static magnetic field. The field patterns were plotted and compared with the field patterns in isotropic and uniaxial media.
On the problem of the presence of a perfectly conducting plane, we reformulated the image theory which leads to suitable and more convenient formulas for calculating the radiated field in terms of a given source distribution in an anisotropic plasma. This reformulation includes the factorization of the dyadic Green's function and application of the boundary condition on the conducting plane. The final solution is valid for an arbitrary orientation of the static magnetic field.
REFERENCES:


31. Maclean, T.S.M., "Principles of Antennas, wire and


Appendix 1

Some Useful Integrals

The following are some integrals that have been used in this dissertation. The detailed derivation can be found in references [17,18,37].

\[ \int_0^{2\pi} e^{i\cos(\phi - \varphi)} d\phi = 2\pi J_0(x) \]

\[ \int_0^{2\pi} \cos\phi e^{i\cos(\phi - \varphi)} d\phi = i2\pi \cos\varphi J_1(x) \]

\[ \int_0^{2\pi} \sin\phi e^{i\cos(\phi - \varphi)} d\phi = i2\pi \sin\varphi J_1(x) \]

\[ \int_0^{2\pi} \sin\phi \cos\phi e^{i\cos(\phi - \varphi)} d\phi = \pi \sin(2\varphi) \left\{ \frac{J_0(x) - \frac{2J_1(x)}{x}}{x} \right\} \]

\[ \int_0^{2\pi} \cos^2\phi e^{i\cos(\phi - \varphi)} d\phi = 2\pi \left\{ \cos^2\varphi J_0(x) - \frac{\cos(2\varphi)J_1(x)}{x} \right\} \]

\[ \int_0^{2\pi} \sin^2\phi e^{i\sin(\phi - \varphi)} d\phi = 2\pi \left\{ \cos^2\varphi J_0(x) + \frac{\cos(2\varphi)J_1(x)}{x} \right\} \]

\[ \int_0^{\frac{\pi}{2}} J_0(\rho k_p) \frac{k_p}{\sqrt{k_p^2 + a^2}} dk_p = K_0(\rho a); \: \rho > 0, \: \text{Re\,}a > 0 \]

\[ \int_{-a}^{a} \frac{e^{i|k_c|}}{\sqrt{k_c^2 + a^2}} dk_c = \frac{\pi e^{-|a|}}{a}; \: \text{Re\,}a > 0 \]

\[ \int_{-a}^{a} K_0(\rho \sqrt{k_c^2 - a^2}) e^{i|k_c|} dk_c = \frac{\pi}{\rho} e^{ia\sqrt{\rho^2 + a^2}} \]

\[ \int_{-a}^{a} J_1(\rho k_p) \frac{1}{\sqrt{k_p^2 - c^2}} dk_p = \frac{1}{i\rho b} \left( e^{ib\sqrt{\rho^2 + c^2}} - e^{ibc} \right) \]
\[ \int_a^\infty \frac{k e^{i k x}}{k^2 + a^2} dk = i \pi e^{-ca} \quad ; \quad c > 0 \]

\[ \int_0^a e^{-c \sqrt{k^2 - b^2}} J_1(\rho k) dk = e^{-lbc} - \frac{c e^{-lb \sqrt{c^2 + \rho^2}}}{\sqrt{c^2 + \rho^2}} \]

\[ \int_{-a/2}^{a/2} \sin \left( \frac{m \pi x}{a} \right) e^{iAx} dx = \frac{ia}{2} \left\{ \frac{\sin \left( \frac{Aa + m\pi}{2} \right)}{\frac{Aa + m\pi}{2}} - \frac{\sin \left( \frac{Aa - m\pi}{2} \right)}{\frac{Aa - m\pi}{2}} \right\} \]

\[ \int_{-a/2}^{a/2} \cos \left( \frac{m \pi x}{a} \right) e^{iAx} dx = \frac{a}{2} \left\{ \frac{\sin \left( \frac{Aa + m\pi}{2} \right)}{\frac{Aa + m\pi}{2}} + \frac{\sin \left( \frac{Aa - m\pi}{2} \right)}{\frac{Aa - m\pi}{2}} \right\} \]
Appendix 2

Table of Coefficients

The following are some of the coefficients that have been used in this dissertation:

\[ X_{i1,2} = \frac{k_{0}A_{i1}a_{1}}{2} \pm \frac{m\pi}{2} \; ; \; i = x, y, z, f \]

\[ Y_{i1,2} = \frac{k_{0}A_{i2}a_{2}}{2} \pm \frac{n\pi}{2} \; ; \; i = x, y, z, f \]

\[ X_{i21,2} = \frac{k_{0}A_{i3}b_{1}}{2} \; ; \; i = x, y, z \]

\[ Y_{i21,2} = \frac{k_{0}A_{i4}b_{1}}{2} \; ; \; i = x, y, z \]

\[ A_{1f} = \sin\theta\cos\phi \]

\[ A_{2f} = \sin\theta\sin\phi \]

\[ A_{1x} = \frac{\sqrt{\mu r_{1}}}{\varepsilon_{2}r_{x}} \sin\theta\cos\phi \]

\[ A_{2x} = \frac{\sqrt{\mu r_{2}}}{\varepsilon_{1}r_{x}} \sin\theta\sin\phi \]

\[ A_{3x} = \sqrt{\mu} \sin\theta\cos\phi \]

\[ A_{4x} = \sqrt{\mu} \sin\theta\sin\phi \]

\[ A_{1y} = \frac{\sqrt{\mu r_{1}}}{\varepsilon_{1}r_{y}} \sin\theta\cos\phi \]

\[ A_{2y} = \frac{\sqrt{\mu r_{2}}}{\varepsilon_{2}r_{y}} \sin\theta\sin\phi \]

\[ A_{3y} = \sqrt{\mu} \sin\theta\cos\phi \]

\[ A_{4y} = \sqrt{\mu} \sin\theta\sin\phi \]
\[ A_{1z} = \frac{\sqrt{\mu a r}}{\varepsilon_1 r_z} \sin \theta \cos \phi \]
\[ A_{2z} = \frac{\sqrt{\mu a r}}{\varepsilon_1 r_z} \sin \theta \sin \phi \]
\[ A_{3z} = \sqrt{\mu b} \sin \theta \cos \phi \]
\[ A_{4z} = \sqrt{\mu b} \sin \theta \sin \phi \]
\[ r_x = \sqrt{\frac{x^2}{\varepsilon_2} + \frac{y^2}{\varepsilon_1} + \frac{z^2}{\varepsilon_1}} \]
\[ r_y = \sqrt{\frac{x^2}{\varepsilon_1} + \frac{y^2}{\varepsilon_2} + \frac{z^2}{\varepsilon_1}} \]
\[ r_z = \sqrt{\frac{x^2}{\varepsilon_1} + \frac{y^2}{\varepsilon_1} + \frac{z^2}{\varepsilon_2}} \]
\[ R_{1x} = \sqrt{\mu a r_x} \]
\[ R_{1y} = \sqrt{\mu a r_y} \]
\[ R_{1z} = \sqrt{\mu a r_z} \]
\[ A_8 = \pi e_0 \sin \left( \frac{mn x}{a_x} \right) \sin \left( \frac{n \pi y}{b_y} \right) \]
\[ \overline{A}_9(R) = B_1(R) \left( \overline{e_{xc} \cdot R} \right) \left( \overline{e_{xc} \cdot R} \right) + B_2(R) \overline{e_{xc}} + B_3(R) \overline{A_{4c}} - \frac{a^2}{b^2} \overline{A_{5c}}(R) \]
\[ \overline{A}_{10c} = \frac{a^2}{b^2} \overline{A}_{6c}(R) \]
\[ B_{1f}(R) = \frac{-a^2}{4 \pi \varepsilon_1 \sqrt{\varepsilon_2 R} (\overline{R \cdot e_{xc} \cdot R})^{5/2}} \]
\[ B_{2f} = 0 \]
Radiation from an Aperture into an Anisotropic Plasma Half-Space. (126pp.)

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The present studies are concerned with the radiation from a given source distribution immersed in an anisotropic plasma. These studies will be separated into two parts: 1) radiation in an unbounded plasma and 2) radiation in plasma half-space over a conducting plane.

In the first part, the problem is formulated by means of linear analysis, Bessel and Fourier-transform techniques. The integrals of the dyadic Green's function for an anisotropic plasma are evaluated with respect to an orthogonal coordinate system $\hat{a}, \hat{b}, \hat{c}$. The final solution is expressed in a closed form. This dyadic Green's function can be directly reduced to the case of a uniaxial or an isotropic medium. The rectangular aperture and dipole fields are determined for some specific orientation of the static magnetic field.

Conventional image theory is applicable for the cases of isotropic, uniaxial and anisotropic plasmas. However, in an anisotropic plasma, this theory is only applicable when the static magnetic field is perpendicular to the conducting plane. In the second part, the image theory will be reformulated by using the dyadic Green's function obtained in the first part, the boundary condition on the conducting plane, and the Uniqueness theorem. This reformulation technique can be applied to an arbitrary orientation of the