Optimal control of a high speed overhead crane including hoisting

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ABSTRACT

In this thesis we derive a dynamical nonlinear model of the two dimensional crane including hoisting. We study the behavior of the load displacement during the hoisting operation. When the load is disturbed from its equilibrium position during hoisting it swings about the origin. This condition is considered hazardous in the industry. In this thesis a control is derived for a crane such that it can dampen out this oscillation as quickly as possible. The optimal control theory is applied to formulate the control law. This control is simulated on the crane model. The results show that a crane can be transferred to a desired place in such a way that at the end of transfer the swing of the load decays in the minimum time.
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CHAPTER ONE

INTRODUCTION

1.1 Background

The development of the modern high-speed crane dates back to 1880 A.D. In this year, Dr. Gottlirb Daimler developed an internal combustion engine in Germany [6]. His effort to load this engine was, in fact, the primitive hoisting operation. His effort did not intend to hoist the load, however, it was the first machine operated hoisting. The early 1920's saw a period of change and refinement in the engine design. This design improvement awakened the engineers to the fact that it was quite practical to use this machine for hoisting a load. In next several years, production of hoisting machines and crane became a business. Since that time, major changes have been made in the crane design. We observe that through all these changes, however, runs the theme of refinement rather then reconstruction; of evolution rather then revolution. In this process of evolution, the mechanical engineering has remained an important engineering discipline. It has constantly responded to the need for industrial renewal since the dawn of modern science. But this need of renewal has become urgent today then ever before. Because we are facing profound issues of productivity and competitiveness in all the phases of engineering activities. For example, the energy was considered one of the most expensive parameters of an industrial practice. But with the advancement of high-speed machineries the operational time has become more important than the energy. This is because the total operational time determines the cost of energy. So if we can reduce or control the operational time we can control the energy cost. These issues are not simple. They can not be solved by routine engineering solutions. They require competitive and innovative engineering solutions. Such type of
solutions can only be obtained through engineering exercises which require fresh and updated approaches to the existing engineering materials.

In earlier times, the engineers had adopted an ad hoc approach to improve the design of the system under consideration. This approach requires mature experience and the results of experimentation of years. This approach is referred to as the "state-of-the-art" approach. This approach does not develop a mathematical model for the system. It also does not carry out simulations for the system. Instead, it sets up the experiment for the specific system. The cost of such experiment is usually very high. The results of such experiments are valid only for the system on which the experiment is performed. For these reasons, this approach is not considered economical or efficient for the most of the modern systems. In view of these deficiencies, this approach is not popular in the present times.

The alternate method to address the design problems is to involve a mathematical model. The advantage of this approach is that it yields broad and general design procedures. These procedures can be applied to other engineering systems, cases and problems. Once the designer has formulated a model for the given system, he can experiment with it using available computer facilities. Such an experiment on the model with the help of computers is known as a simulation approach. However, formulation of a mathematical model is not an easy task. But then there comes the applied mathematics to help it. One of the features of applied mathematics is its ability to frame the mere facts of the system into a powerful mathematical expressions. This process of modeling can give a better handle for complicated engineering problems then would be otherwise possible.

In a hoist, the energy required for lifting a load is derived from a number of sources, such as compressed air, internal combustion engines, hydraulic power, steam and
electric power. The pneumatic drives may be either a direct lift supplied by air acting on a piston connected to the load or a drive employed by a compressed air engine. The crankshaft of this engine is geared to the piston or some other type of hoisting apparatus. There are two types of the internal combustion engines, namely, the gasoline engine and the diesel engine. The gasoline engine is generally used for the light capacity hoist. The diesel engine is used for heavier hoists. The diesel engine has the advantage over other drives for portable services such as locomotives and power shovels.

The essential parts of a hoist are a rope or chain which is wrapped around a drum or drive sheave. A hook, grapnel magnet or other device for handling the load is attached to the free end. The rotation of the drum winds up the rope, thus shortening the distance between the drum and the load. If the drum is fixed in position over the load, the process is called a pure hoisting. To drive the drum, one of the power supplies just mentioned is connected with the drum through a suitable speed reducing and torque increasing mechanism. A gear train is often used. A brake is provided to control the speed of hoisting during lowering of loads.

Figure 1-1. The overhead crane including hoisting
In this thesis, we wish to consider the hoisting problem. We have adopted the simulation approach. The mathematical models are developed for the various cases of hoisting. They all are in terms of nonlinear or linear ordinary or partial differential equations with at most three dependent variables. In order to discuss this problem in the context of control theory, we describe the hoisting system as follow.

An average contemporary overhead crane is shown in the figure 1-1. In this system, the load consists of a mass \( m \), while the crane consists of a mass \( M \). The load is attached to a flexible and massless rope. The length of the rope is \( L \). The rope is wound around a pulley \( R \). The load can swing in a circular arc about the pivot. This pivot is located at the center of the pulley \( R \). The straight down position finds the load at the rest. If it is disturbed by an angle from the vertical and then let go, it is expected to swing in the plane back and forth about its vertical position. This vertical position is designated as the rest position. It is also known as the equilibrium position.

When the mass is displaced from the rest position, the only restoring force is that of gravity, which tends to pull it back downward. We wish to study this case when the rope is wound around a pulley. In other words, we carry out hoisting when the load is disturbed from its rest position. In the absence of hoisting, we know from experiments that the load oscillates and comes to the rest position after some time \( t \). But is it the case when the load is also wound around the pulley?

The Structural Mechanics Division, Dayton, Ohio [3] investigated this case at Wright-Patterson Air Force Base of Dayton. They had adopted the ad hoc approach. They mounted a high-speed rescue hoist on an H-53 helicopter. They ran this experiment with various initial load disturbances and hoisting speeds. They observed that when a rope is wound around a pulley, the load begins to oscillate. This oscillation gradually increases as the rope length decreases. This condition is
hazardous especially when human beings are involved with a high-speed rescue operation.

To identify this problem as a control problem we should first introduce the nomenclature, definitions and control terminology as well the concepts. This brief introductory discussion will enable us to state the objective of this thesis in the formal terms of control engineering.

1.2 Nomenclature

\[ A = \text{Plant matrix} \]
\[ B = \text{Control vector} \]
\[ B_L = \text{Boom length} \]

CPU = Central Processing Unit

d = Distance of the crane measured from the vertical axes

E = Modules of elasticity

\[ F_d = \text{Disturbing force} \]

\[ F_n = \text{Normal force acting on a load} \]

\[ F_t = \text{Tangential force acting on a load} \]

\[ F_x = \text{Force acting on a crane in x direction} \]

\[ g = \text{Gravity acceleration} \]

H = Hamilton function
\[ J(u) = \text{The performance index.} \]

\[ k = \text{Spring stiffness of the rope} \]

\[ L = \text{Rope length} \]

\[ L_i = \text{Initial rope length} \]

\[ L_f = \text{Final rope length} \]

\[ L_R = \text{Load radius} \]

\[ L = \text{Laplace function} \]

\[ m = \text{Mass of a load} \]

\[ M = \text{Mass of the crane} \]

\[ P = \text{Costate vector} \]

\[ p_i = \text{Costate variables (i = 1, 2...)} \]

\[ P_T = \text{Pressure Transducer} \]

\[ q_i = \text{Generalized coordinates (i = 1, 2...)} \]

\[ R = \text{Pulley} \]

\[ s = \text{Distance of the load measured from the axes on the crane.} \]

\[ T = \text{Tension in the rope} \]

\[ T_H = \text{Tip height} \]

\[ t = \text{Time} \]
\( u = \text{control (input)} \)

\( V_a = \text{Velocity vector in x direction} \)

\( V_b = \text{Velocity vector in y direction} \)

\( V = \text{Resultant crane velocity} \)

\( v_h = \text{Hoisting speed} \)

\( V_T = \text{Crane velocity} \)

\( W_o = \text{Normalized frequency} \)

\( y = \text{Distance of the load from horizontal axes} \)

\( \Lambda = \text{Lagrangian function} \)

\( \alpha = \text{Boom angle} \)

\( \phi = \text{Deflection angle of the load} \)

\[ \frac{d\phi}{dt} = \text{Time derivative of } \phi. \phi \text{ is a deflection angle of the load} \]

\[ \frac{dx}{dt} = \text{Time derivative of } x. \ x \text{ is a vector function of time.} \]

\[ \frac{\partial x}{\partial t} = \text{Partial time derivative of } x. \ x \text{ is a vector function of time.} \]

\[ \frac{dP}{dx} = \text{First derivative of } P \text{ with respect to } x. \ P \text{ is a scalar or vector function.} \]

\[ \frac{d^2 P}{dx^2} = \text{Second derivative of } P \text{ with respect to } x. \]

\( \text{KE} = \text{Kinetic energy} \)
PE = Potential energy

1.3 Definitions

**Set:** A collection of objects which have common properties is called a set.

**System:** A system is a combination of components that act together and perform certain task.

**Dynamic system:** The following aggregate of mathematical components is called a dynamical system.

a. A set \( \Omega \) of real \( m \)-vectors \( u \) such that \( \Omega \) is the subset of \( \mathbb{R}^m \).

b. The Euclidean \( n \)-space of real \( n \)-vectors \( x \) such that \( x \) belongs to \( \mathbb{R}^n \).

c. An interval \( (t_0, T] \) in \( \mathbb{R} \).

d. A function \( g(x,u,t) \) from \( \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \) into \( \mathbb{R}^l \).

e. A vector differential equation

\[
\frac{dx}{dt} = f(x,u,t)
\]

If there exists a unique and continuous function \( x(t,T) \) which satisfy \( (e) \) almost everywhere in \( (t_0,T) \) with given initial condition \( x(t_0) = x_0 \) then \( x(t) \) is known as the state or state trajectory or the solution of \( (e) \).

**Plant:** A plant is a set of machine parts functioning together, the purpose of which is to perform a particular function. In view of a dynamical system, we identify the vector differential equation \( (e) \) as a plant.

**Process:** A natural, progressively continuing operation that leads to a particular result is known as a process.
Input: If certain signals are applied to the system such that it forces the system to change its state from one condition to the another then this signal is known as an input. A function \( u(t,T) \) is an input function.

Output: Output is the resultant state signal observed after an input is applied to a system. The function \( y(t) = g(x,u,t) \) is called the output.

Disturbance: It is a signal that tends to adversely affect the value of the output of a system.

Feedback control: Feedback control refers to an operation that, in the presence of disturbances, tends to reduce the difference between the output of a system and some reference input.

Time-invariant system: A time-invariant control system is one whose parameters do not vary with time. The response of such a system is independent of the time at which an input is applied.

Time-variant system: A time-variant control system is a system in which one or more parameters vary with time. The response depends on the time at which an input is applied.

state: The state of the dynamic system is the smallest set of variable such that the knowledge of these variables at \( t = t_0 \), together with the knowledge of the input at \( t > t_0 \), completely determines the behavior of the system for any time \( t > t_0 \).

State variables: The variables mentioned in the definition of the state are called the state variables.

Equilibrium state: In the system of equation (e), a state \( x_e \) where
Lyapunov stability: If we denote a spherical region of radius $\rho$ about an equilibrium state $x_e$ as

$$\| x - x_e \| \leq \rho$$

where $\| x - x_e \|$ is the Euclidean norm. Let $S(\delta)$ consist of all points such that the condition $\| x_0 - x_e \| \leq \delta$ is satisfied and let $S(\varepsilon)$ consist of all points such that the condition $\| \phi(t; x_0, t_0) - x_e \| \leq \varepsilon$ for all $t > t_0$ then an equilibrium state $x_e$ of the system of equation (e) is said to be stable.

Uniform stability: If corresponding to each $S(\varepsilon)$, there is an $s(\delta)$ such that trajectories starting in $S(\delta)$ do not leave $s(\varepsilon)$ as time $t$ increases indefinitely and real number $\delta$ depends on $\varepsilon$ and also depend on $t_0$. If $\delta$ does not depend on $t_0$, the equilibrium state is said to be uniformly stable.

Asymptotic stability: An equilibrium state $x_e$ of the system of equation (e) is said to be asymptotically stable if it is stable and if every solution starting within $S(\delta)$ converges, without leaving $S(\varepsilon)$, to $x_e$ as the time $t$ increases indefinitely.

Asymptotic stability in the large: If asymptotic stability holds for all states from which trajectories originate, the equilibrium state is said to be asymptotically stable in the large.

Instability: An equilibrium state $x_e$ is said to be unstable if for some number $\varepsilon > 0$ and any real number $\delta > 0$, no matter how small, there is always a state $x_0$ in $S(\delta)$ such that the trajectory starting at this state leaves $S(\varepsilon)$. 

$$f(x_e, t) = 0 \text{ for all } t$$

is called an equilibrium state of the system.
Performance Index: A performance index is a scalar function of the system parameters. It indicates the behavior of the states of the system and is expressed as an integral. For example, with the initial condition $\chi_0$ for $(e)$, the integral

$$J(u) = \int_0^t F(x,u,t) \, dt$$

is the performance index. Where $F$ is the function of $x$, $u$ and $t$ for the system $(e)$.

1.4 Control concepts

(a) Lagrange's equation of motion: In order that we may develop a model for the high-speed overhead crane with hoisting, we use Lagrangian approach. This approach develops the equations of motion from scalar quantities like kinetic energy, potential energy and virtual work. The use of Lagrangian's equation is convenient in developing the dynamic relationships of multi-degree of freedom systems. Important concepts and definitions, however, have to be first introduced. This is done in section 1.3. We introduced the concepts of the system generalized coordinates and the virtual work in chapter two. The virtual work is used to develop the generalized forces associated with the system generalized coordinates. These concepts are then used to develop Lagrange's equations of motion for a high-speed overhead crane with hoisting. A more complete account of Lagrange's equation is presented in Appendix A. Using these concepts, a nonlinear mathematical model for the high speed overhead crane including hoisting is developed in chapter two.

(b) Laplace transform: The Laplace transform method is an operational method. This method can be used for solving differential equations. By use of Laplace transforms, we convert many common functions into algebraic functions of a complex variable $s$. The operations such as differentiation and integration can be replaced by algebraic operations in the complex plane. Thus, a differential equation can be transformed into an algebraic equation in a complex variable $s$. If the algebraic equation in $s$ is solved
for the dependent variable, then the solution of the differential equation may be found by use of a Laplace transform table. An advantage of the Laplace transform method is that it allows the use of phase plane graphical technique. This technique predicts the system performance without actually solving system differential equations. A. Martinnen et al [5] used this approach to solve the velocity controlled crane problem. They derived a transfer function for the linearized mathematical model of a high-speed crane. The details are given in chapter three.

(c) Lyapunov's second method: In the beginning of this chapter we mentioned the experiments carried out by Wright-Patterson Air Force Base of Dayton on the helicopter based rescue hoist. At that time, we talked about the increasing load oscillation during actual hoisting. This phenomenon is known as the instability of a hoisting system in the Lyapunov sense. The most important problem in control system is concerned with the problem of the stability of the system. Lyapunov's second method gives an efficient method to arrive at the conclusion for stability of the system without actually solving the differential equation. The principal idea of his method is explained in the following reasoning.

If the rate of change \( \frac{dE(x)}{dt} \) of the energy \( E(x) \) of an isolated physical system is negative for every possible state \( x \), except for a single equilibrium state \( x_e \), then the energy continually decreases until it finally assumes its minimum value \( E(x_0) \). A dissipating system perturbed from its equilibrium state will always return to it. This idea, developed in a precise mathematical tool, is discussed by Kalman et al [1].

In 1892, A. M. Lyapunov presented two methods for determining the stability of dynamic systems by ordinary differential equations. The first method consists of procedures in which the explicit form of the solution of the differential equations are used for the analysis. The second method does not require the solutions of the
differential equations. Thus, it is a useful technique especially when we are dealing with nonlinear or time-variant control system.

(d) Phase plane trajectories: We used phase plane trajectories technique to interpret the qualitative behavior of the nonlinear system of a crane with hoisting. In order to understand how the phase plane technique works, we discuss the oversimplified mathematical model of our system. We, for a while, assume that our system is no more than a simple harmonic oscillator.

\[
d\frac{d^2\phi}{dt^2} + \frac{g}{L} \phi = 0
\]

Where \( \phi, \ g \) and \( L \) are same as defined before. We normalize the angular frequency \( \frac{g}{L} \) to one. The kinetic energy and potential energy of the system are given by,

\[
KE = \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2
\]

\[
PE = \frac{1}{2} (\phi)^2
\]

The total energy is the summation of the kinetic and the potential energies given by,

\[
E(\phi) = \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 + \frac{1}{2} (\phi)^2
\]

From the principle of conservation of energy, we know that \( E(\phi) \) is constant for this ideal system. Since \( E(\phi) \) is constant, equation 1-3 is the equation of an ellipse in the \( \phi, \ \frac{d\phi}{dt} \) plane. As \( E(\phi) \) varies, we get a family of concentric ellipse. If we let \( x_1 = \phi \) and \( x_2 = \frac{d\phi}{dt} \) then \( \phi, \ \frac{d\phi}{dt} \) plane is identical to \( x_1 \) and \( x_2 \) phase plane. Thus these phase plane trajectories are concentric ellipse. One ellipse for each value of \( E(\phi) \). This
value of $E(\phi)$ is the each energy level of $E(\phi)$. For general linear and nonlinear models, these energy levels are a family of concentric closed surfaces surrounding the origin. Thus $E(\phi)$ is a measure of the distance of the state $x$ from the origin in the phase plane. The surface $E(x) = h$ lies inside the surface $h(x) = f$ whenever $h < f$. Thus we can say that $y$ is further away from origin than $x$ if $E(y) > E(x)$. If the trajectories of equation 1-4 cross the boundary of every energy level than we can say that the system is asymptotically stable. In other words, if the distance between the origin and the state $x(t)$ is continually decreasing as $t$ goes to infinity, then $x(t)$ approaches the origin.

(e) Pontryagin's minimum principle: The function of any given control algorithm is to determine the nature of the control. This control should be such that it may optimize the performance of the system. This system may be subject to certain constraints. The performance of the system is expressed as some measure of a deviation from ideal or desired behavior. Such a measured entity can be expressed by a criterion of optimization. This criterion is known as the performance index. The performance index is a difference function. Its value indicates how close the actual performance is to the ideal performance. Choosing the performance index for a given system is not easy. Because performance index depends on the requirements of the system as well on the restrictions imposed on the system.

The dynamic models are used to predict the behavior of physical processes. Once it is determined that how the system behaves in its natural mode, it becomes necessary to change the course of its behavior. Because the behavior pattern of the process in its natural mode may or may not be the the best pattern from performance point of view. In other words it becomes necessary to minimize, maximize or optimize the behavior of a system. The feedback control is the one way to modify the performance. We here seek a control such that it should not only modify but also improve the performance of
a system. This entails an introduction to the Pontryagin's minimum principle. Because this principle achieves the goal just mentioned in an effective manner.

If the dynamic system is given as in equation (e) we ask for a control $u$ that transfers the state variable $x(t_0)$ to $x(T)$ in some minimum time $T$. We can apply the Pontryagin's minimum principle to our system and derive the control $u$ in such a way that the given state variable arrives at the desired state in the minimum time. The rigorous proof of this theorem is discussed in [2]. The necessary conditions for the minimum principle is discussed in chapter four.

(f) Pole placement design and optimal control: The regulator systems are, in fact, the feedback control systems. Because the regulator systems attempt to bring the given states of the system to the origin. The regulator systems are designed by constructing an asymptotically stable closed-loop system. Such a closed-loop system can be set up by specifying the desired locations for the loop poles. In this method we assume the control vector to be $u = -kx$, where $k$ is the gain matrix. We determine the feedback gain $k$ such that the system will have a desired characteristic equation. This design method is known as a pole placement technique.

The other approach to the design of regulator system is to determine the gain $k$ such that a quadratic performance index is optimized. This approach is known as the quadratic optimal control problem. In chapter five, we solve the problem of a crane including hoisting with these two approaches.

1.5 Objective

About 100,000 overhead cranes with an average age of 26 years exist in industry. Some of this equipment was installed before the turn of the century and is still putting in full-production work shifts. In recent years, the overhead crane has improved a lot.
The modern crane models are equipped with digital controller with central processing unit. But very few attempts are made to improve the working of the cranes of an old era. Many of these cranes do not function in the safe operational environment. An urgent need exists for renewal of the old cranes. Some commercial control packages are available that may impose safe guidelines on the existing crane machines. But the most of them require additional accessories if installed on the old model. And many of them can not be installed on the old equipment. For example, PAT Equipment Incorporation of Pennsylvania offers a control package. This control package takes care of boom length $B_L$, boom angle $\alpha$, tip height $T_H$, load radius $L_R$, actual load $m$, permissible load and relative load moment. These parameters are shown in the figure 1-1. Among these parameters, boom length, boom angle etc. are not difficult to monitor parameters. But relative load moment is near to impossible parameter to control. The objective of this thesis is to find means of controlling these undesirable load moment or the load oscillation. The PAT control package works well on relatively modern crane systems but does not fit on old machines. We intend to develop control algorithm such that it depends on the available machinery of the crane like crane motor or the hoist motors. And also this control algorithm should be such that it can be employed to the heavy duty high speed industrial cranes. In this thesis we develop the nonlinear hoist model when crane is travelling. We introduce the state-space representation of the dynamic system to identify it as a control problem. We also study the behavior of state variables of the system when the crane is in motion. While deriving the basic conditions for optimality we restructure the general properties of a crane model for Pontryagin's maximum principle. We also present specific results for an important class of the optimization problem that includes linear plants and quadratic performance criteria. Also this control minimizes the given cost functional. Thus this
scheme saves control energy while operating the dynamic system in the desired fashion.

The Structural Mechanics Division, Dayton, Ohio [3] investigated the case of a rescue hoist mounted on an H-53 helicopter at Wright-Patterson Air Force Base of Dayton. They carried out the investigation with various initial load disturbances and hoisting speeds. They concluded that the hoisting direction should be reversed to decrease the oscillation as quickly as possible. But they considered only one case of hoisting. They considered the load hoisting when the helicopter is in the hovering position. Thus the base of the hoist is stationary. Here, we intend to expand the scope of this case by considering various other possibilities as discussed in the chapter two.

Also, we intend to show that the model developed here has the potential to consider other important factors for the crane design like girder span, wheel loads, structural requirements, general layout of the plant, system vibration and strength etc. In other words, we want to show that the nonlinear models developed here of various hoists and cranes are capable of accessing not only the load oscillation but also the factors like productivity, duty cycle, performance, capacity lift, structural integrity, maintenance expense, downtime and replacement parts cost. Thus this work, if modified, can substantially reduce operating and maintenance costs and improve manufacturing productivity and quality. It also can modernize an old overhead handling system with the minimum changes in the existing hardware of the system.

1.6 Outline of the thesis

The contents of the rest of this thesis will now be discussed. The second chapter derives a two dimensional model of the crane. It is a nonlinear mathematical model. In this chapter, we consider various special cases of a crane and hoisting. Each case is numerically simulated on computer.
The third chapter reduces this nonlinear and coupled differential equation to the form which is widely used in the industry practice. This equation is decoupled by noticing the practice of safe crane operation in industry. For safety reasons, the crane is transferred from one point to the another only when the hoisting is off. Keeping this condition in sight, the model is reduced to the form that describes the actual condition. Also, this chapter is discussing the special practical cases currently in use in various type of industries.

The fourth chapter opens with the discussion of a necessity to treat this problem as a time optimal control problem. It presents the analytical solution of the Pontryagin’s minimum time problem. Considering the fact that crane transfer time is more important then the hoisting time, the solution is derived for the system developed in third chapter. Results are numerically computed and are shown on the graphs. The conclusions are derived from the graphical results.

The fifth chapter develops the safe operating envelope for a hoist crane when it travels from one point to another with a heavy load. This objective is achieved by deriving the feedback control of the system developed in the third chapter. The Riccati equation is solved on MATLAB to develop the optimal control. The simulated results are discussed with due remarks.
CHAPTER TWO

A MODEL OF A HIGH SPEED CRANE INCLUDING HOISTING

2.1 Background

We intend to use Lagrangian approach to develop a model for the high-speed overhead crane including hoisting. The Lagrange's technique was developed by the French mathematician J. L. Lagrange in 1788. This approach gives one of the simplest means of establishing the equations of motion of a fairly complicated system. This approach involves terms like kinetic energy, potential energy and virtual work to develop the system of differential equations. The complex dynamic relationships of the systems with multi-degree of freedom can be easily handled by this method. Here we introduce the concept of the system generalized coordinates and the virtual work. The virtual work is used to develop the generalized forces associated with the system generalized coordinates. These concepts are then used to develop Lagrange’s equation of motion for a high-speed overhead crane with hoisting.

2.2 Generalized coordinates and generalized forces

The position and orientation of a body may be given in the rectangular coordinates. In general these are dependent and are interrelated by some equations of constraints. To avoid the complications of these interrelated equations, it is an advantage if some independent quantities can be found to describe the system. These quantities are called generalized coordinates. They are just enough of them to specify the position of the system. The number of generalized coordinates is equal to the number of degrees of freedom of the system. It is customary to use the following notation for the generalized coordinates of a system with n degrees of freedom.
generalized coordinates = \( (q_1, q_2, q_3, \ldots, q_n) \)

In our case, we denote the crane position by the symbol \( x \) and the load position by \( \phi \) and \( s \).
We choose the generalized coordinates as follow,
\[ q_1 = \phi, \text{ the load position variable} \]
\[ q_2 = x, \text{ the crane position variable} \]
\[ q_3 = L, \text{ the rope length} \]

A generalized force is defined as the force that produces the work done \( W \) when only one generalized coordinate \( q \) is changed to the extent \( +\delta q \). The generalized forces employed in the crane model are hoist-motor torque \( F_m \) and crane-motor torque \( F_x \). In our case, we have assumed that the crane models use the armature controlled d-c motors. The control circuit of each motor can be designed in such a way that the torques are adjusted by adjusting the armature currents.

### 2.3 Virtual work of the system

Since the generalized coordinates are sufficient to completely identify the configuration of a system of particles, the position vector of an arbitrary particle in the system can be expressed in terms of the generalized coordinates. The virtual displacement is defined as a change in the configuration of the system as the result of an arbitrary infinitesimal change in the vector of system generalized coordinates. These system generalized coordinates are consistent with the forces and constraints imposed on the system at the given instant of time \( t \). The displacement is called virtual to distinguish it from an actual displacement of the system occurring in a time interval \( \delta t \).

Let \( F \) be the vector of forces that act at the point whose position vector is defined by the vector \( r \). The virtual work due to this force vector is defined by the dot product of the force and the virtual displacement as \( \delta W = F \delta r \).
2.4 Equations of constraint and degrees of freedom

Before we determine the degree of freedom of our system, we show the state variables in a schematic diagram. Once we visualize the state space arrangement, we will be able to derive the equations of constraint of the system from the diagram of a crane that includes hoisting. We denote the time varying rope length by \( L \) instead of the notation \( L(t) \) unless there is a special need to clear the point. The schematic diagram is presented below.

Figure 2-1. The schematic diagram of a crane shows the state variables of the system.
The concepts of degrees of freedom and equations of constraints are of fundamental importance in dynamics. The number of degrees of freedom of a particle is defined as the number of independent coordinates required to determine the position of the particle. A fixed particle has a zero degree of freedom. Our system has three degrees of freedom. If the two coordinates $x$ and $y$ specify the position of a particle and if they are related by some equation $f(x) = y$ then they are not independent coordinates. In this case function $f(x)$ is called the constraint equation. In our case the constraint equations are given as

\[ s = x + L \sin \phi \]  
\[ y = L \cos \phi \]

These equations give the position of the load with respect to the position of the crane. They are also called as the position vectors. If we differentiate equation 2-1 and 2-2 with respect to time we get velocity vectors as follow

\[ \frac{ds}{dt} = \frac{dx}{dt} + L \cos \phi \frac{d\phi}{dt} + \sin \phi \frac{dL}{dt} \]  
\[ \frac{dy}{dt} = -L \sin \phi \frac{d\phi}{dt} + \cos \phi \frac{dL}{dt} \]

We denote above velocity vectors by the following notations

\[ v_a = \frac{ds}{dt} \]  
\[ v_b = \frac{dy}{dt} \]

### 2.5 The Kinetic and Potential energies of the system

It is necessary to obtain an expression for the kinetic energy of the system. It formulates the dynamic equations of motion using Lagrange's approach. In our application it was possible to express the position vector in terms of generalized coordinates. It facilitated to
write the velocity vector in equation 2-4 and 2-5. If we multiply velocity vectors by themselves, we get

\[ v_a^2 = (\frac{dx}{dt})^2 + 2\frac{dx}{dt}L\cos\phi \frac{d\phi}{dt} + L^2 \left( \frac{d\phi}{dt} \right)^2 \cos^2 \phi + 2L \frac{dL}{dt} \frac{d\phi}{dt} \cos\phi \sin\phi + 2\frac{dx}{dt} \frac{dL}{dt} \sin\phi \]  

\[ v_b^2 = (-L \sin \phi \frac{d\phi}{dt})^2 + \left( \frac{dL}{dt} \right)^2 \cos^2 \phi - 2L \frac{dL}{dt} \frac{d\phi}{dt} \sin \phi \cos \phi \]  

The resultant velocity is

\[ V = \sqrt{v_a^2 + v_b^2} \]

\[ V^2 = \left( \frac{dL}{dt} \right)^2 + L^2 \left( \frac{d\phi}{dt} \right)^2 + \left( \frac{dx}{dt} \right)^2 + 2\frac{dx}{dt} \left( \frac{dL}{dt} \cos \phi + \frac{dL}{dt} \sin \phi \right) \]

The kinetic energy of the system is

\[ KE = \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} M V^2 \]

\[ KE = \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} M \left\{ \left( \frac{dL}{dt} \right)^2 + L^2 \left( \frac{d\phi}{dt} \right)^2 + \left( \frac{dx}{dt} \right)^2 + 2L \frac{dL}{dt} \frac{d\phi}{dt} \cos \phi + \frac{dL}{dt} \sin \phi \right\} \]

For our system the potential energy is expressed by the following equation.

\[ PE = M g L (1-\cos \phi) \]

Using these two terms of potential and kinetic energies, we can write the Lagrangian function as
\[ \Lambda = KE - PE \]

\[ \Lambda = \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + \]

\[ \frac{1}{2} M \left\{ \left( \frac{dL}{dt} \right)^2 + L^2 \left( \frac{d\theta}{dt} \right)^2 + \left( \frac{dx}{dt} \right)^2 + 2 \frac{dx}{dt} \left( L \frac{d\theta}{dt} \cos \theta + \frac{dL}{dt} \sin \theta \right) \right\} \]

\[- M g L (1 - \cos \theta) \]

Where \( \Lambda \) denotes the Lagrangian function.

### 2.6 Derivation of Lagrange's equations of motion

To derive the Lagrangian equations of motion, we write the generalized coordinates

\[ q = [x \ \theta \ L]^T \]

For \( q_1 = x \) as our generalized coordinate, we take partial derivative of the Lagrangian function \( \Lambda \) with respect to \( x \).

\[ \frac{\partial \Lambda}{\partial x} = 0 \]

We also take partial derivative of Lagrangian function \( \Lambda \) with respect to the derivative of \( x \).

\[ \frac{\partial \Lambda}{\partial \frac{dx}{dt}} = (m+M) \frac{dx}{dt} + M L \cos \theta \frac{d\theta}{dt} + M \frac{dL}{dt} \sin \theta \]

We take the time derivative of equation 2-20 to obtain

\[ \frac{d}{dt} \left( \frac{\partial \Lambda}{\partial \frac{dx}{dt}} \right) = (m + M) \frac{d^2 x}{dt^2} + M L \cos \theta \frac{d^2 \theta}{dt^2} - M L \sin \theta \left( \frac{d \theta}{dt} \right)^2 + 2 M \cos \theta \frac{d \theta}{dt} \frac{dL}{dt} \]

\[ + M \frac{d^2 L}{dt^2} \sin \theta \]

2-21
Therefore

\[ \frac{d}{dt} \left( \frac{\partial \Lambda}{\partial \dot{x}} \right) - \frac{\partial \Lambda}{\partial x} = F_x \]  

2-22

gives

\[ \frac{d^2 x}{dt^2} + \frac{M \cos \phi}{L} \frac{d^2 \phi}{dt^2} - \frac{ML \sin \phi}{dt^2} \left( \frac{d \phi}{dt} \right)^2 + 2M \cos \phi \frac{d \phi}{dt} \frac{dL}{dt} + \frac{M L}{dt^2} \sin \phi = F_x \]  

2-23

Where \( F_x \) is the generalized force.

For generalized coordinate \( q_2 = \phi \), we take partial derivative of the Lagrangian function \( \Lambda \) with respect to \( \phi \).

\[ \frac{\partial \Lambda}{\partial \phi} = -M g L \sin \phi - M \sin \phi L \frac{dx}{dt} \frac{d \phi}{dt} + M \frac{dx}{dt} \frac{dL}{dt} \cos \phi \]  

2-24

We take partial derivative of Lagrangian function \( \Lambda \) with respect to the time derivative of \( \phi \).

\[ \frac{\partial \Lambda}{\partial \dot{\phi}} = M \frac{dx}{dt} L \cos \phi + M L^2 \frac{d \phi}{dt} \]  

2-25

We take the time derivative of equation 2-25 to obtain

\[ \frac{d}{dt} \left( \frac{\partial \Lambda}{\partial \dot{x}} \right) = M (L \cos \phi \frac{d^2 x}{dt^2} + L^2 \frac{d^2 \phi}{dt^2} + 2L \frac{d \phi}{dt} \frac{dL}{dt} + \cos \phi \frac{dL}{dt} \frac{dx}{dt} \frac{d \phi}{dt} + L \sin \phi \frac{d \phi}{dt}) \]  

2-26

Therefore

\[ \frac{d}{dt} \left( \frac{\partial \Lambda}{\partial \dot{\phi}} \right) - \frac{\partial \Lambda}{\partial \phi} = T_\phi \]  

2-27

gives
\[ ML \cos \phi \frac{d^2 x}{dt^2} + M L^2 \frac{d^2 \phi}{dt^2} + 2 M \frac{d\phi}{dt} L \frac{dL}{dt} + M g L \sin \phi = T_\phi \]  

where \( T_\phi \) is the generalized torque. As in this case \( T_\phi \) is zero.

For generalized coordinate \( q_3 = L \), we take partial derivative of the Lagrangian function \( \Lambda \) with respect to \( L \). We obtain

\[ \frac{\partial \Lambda}{\partial L} = \frac{1}{2} M \left( 2 \frac{dx}{dt} \frac{d\phi}{dt} \cos \phi + 2 L \left( \frac{d\phi}{dt} \right)^2 \right) - M g (1 - \cos \phi) \]

We take partial derivative of Lagrangian function \( \Lambda \) with respect to the time derivative of \( L \).

\[ \frac{\partial \Lambda}{\partial \frac{dL}{dt}} = M \frac{dL}{dt} + M \frac{dx}{dt} \sin \phi \]

The Lagrangian equation of motion is given by

\[ M \frac{d^2 L}{dt^2} + M \frac{d^2 x}{dt^2} \sin \phi - M L \frac{d\phi}{dt} \frac{d\phi}{dt} + M g (1 - \cos \phi) = F_m \]

where \( F_m \) is hoisting motor torque.

Equations 2-23, 2-28 and 2-32 are the set of equations of motion for the travelling crane that includes hoisting. As we see, there are 2nd order non-linear coupled differential equations. Given appropriate initial conditions these equations can be solved numerically.
2.7 State space realization

As we see in equations 2-23 and 2-28, the crane model with hoisting is a complex system. Its parameters like rope length, crane position and load displacement are interrelated in a complicated manner. To analyze such a system on computer, it is essential to reduce the complexity of the mathematical expressions. In state space approach, we describe the system equations in terms of \( n \) first-order differential equations. These equations may be combined into a first-order vector matrix.

To set up a vector matrix, we choose the state variables as

\[
\begin{align*}
  x_1 &= \phi \\
  x_2 &= \frac{d\phi}{dt} \\
  x_3 &= x \\
  x_4 &= \frac{dx}{dt}
\end{align*}
\]

These relations will give

\[
\begin{align*}
  \frac{dx_1}{dt} &= x_2 \\
  \frac{dx_2}{dt} &= (-x_2^2 \cos x_1 \sin x_1) + \left(\frac{2Mv \cos x_1^2 x_2}{mL}\right) + \left(\frac{x_2^2 \sin 2x_1}{2}\right) \\
  &\quad - (2m+M+M \cos 2x_1) \left(\frac{g \sin x_1}{2Lm} + \frac{v x_2}{LM}\right) \\
  &\quad + \left(\frac{g \sin 2x_1 (1-\cos x_1)}{2L}\right) + \left(\frac{F_x \cos x_1}{mL}\right) + \left(\frac{F_m \sin 2x_1}{2ML}\right)
\end{align*}
\]
2.8 Linearization of the equations of motion

It is a common usage in engineering to consider only small deviation from the operating point. In practice, a normal operation of the hoisting system is always around an equilibrium point. The load disturbance is small around the equilibrium. Therefore, it is possible to approximate the nonlinear system by a linear system. This is done by expanding the system in a Taylor series about the operating point and neglecting all higher order terms. A rigorous proof of the legitimacy of this procedure is given in [1] by means of the Lyapunov's second method. In this section, we intend to show that a linear system is equivalent to the nonlinear system provided the variables are within a limited operating range. Since higher order terms of Taylor series expansion are small enough, we retain only the linear terms. In this case, variables deviate only slightly from the operating condition. If $\phi$ is small we can take $\cos\phi \approx 1, \left(\frac{d\phi}{dt}\right)^2 \approx 0$ and $\sin\phi \approx \phi$. We rewrite equations as

\[
\frac{dx_3}{dt} = x_4
\]

\[
\frac{dx_4}{dt} = \left( L x_2^2 \sin x_1 \right) + \left( \frac{M g \cos x_1 \sin x_1}{J} \right) - \left( \frac{x_2^2 M L \sin x_1}{J} \right) \\
+ \left( \frac{M g (1-\cos x_1) \sin x_1}{J} \right) - \left( \frac{F_m \sin x_1}{J} \right) + \left( \frac{F_x}{J} \right)
\]
\[
\frac{d^2 x}{dt^2} - \frac{d\phi}{dt} \left( M L \frac{d\phi}{dt} \right) = F_m
\]

This is the linear mathematical description of the model of a crane with hoisting. The last equation determines the motor torque \( F_m \). If the above linearized system near equilibrium is uniformly asymptotically stable and is a uniformly good approximation to the original system near \( \chi \), then equilibrium state \( \chi \) is locally uniformly asymptotically stable.

### 2.9 Special cases of hoisting

The Cranes are used in power houses, public utilities, turbine and motor rooms, repair shops, assembly operations, service buildings, warehousing, machine shops, paper mill, foundries, fabricating plants, steel warehouses, container yards, lumber mills, mines, scrap yards, cement mills, fertilizer plants and at many other facilities. The Crane Manufacturers Association of America (CMAA) has published a complete list of the industrial areas where crane is used. This shows that the crane can not have just one type of design. It should vary in the type and nature and the capacity. Here, we intend to discuss various special cases of the crane type. We develop a model for each type, simulate the model and compare the results to show similarity between types of the crane.

**Case A:** The first case that we want to examine is the general form of the crane model. It is the nonlinear set of equations. It describes the system in its true perspective. In this case the crane velocity as well the hoisting speed are time varying terms. For the safety reasons, this case is not practiced in the industry except when in the rescue operations. In truck type cranes, the hoisting is carried out when the crane is moving. But in such cases the tip height is small, the truck velocity is low and varying and the load is moderate to light. The care is taken to keep the work place clear of human beings.
\[
\frac{dx_1}{dt} = x_2
\]

\[
\frac{dx_2}{dt} = \left(-x_2^2 \cos{x_1} \sin{x_1}\right) + \left(\frac{2Mv \cos{x_1}^2}{mL}\right) + \left(\frac{x_2^2 \sin{2x_1}}{2}\right) - \left(2m + M + M \cos{2x_1}\right) \left(\frac{gsin{x_1}}{2Lm} + \frac{v \ x_2}{LM}\right)
\]

\[
+ \left(\frac{g \sin{2x_1}(1 - \cos{x_1})}{2L}\right) + \left(\frac{F_x \cos{x_1}}{mL}\right) + \left(\frac{F_m \sin{2x_1}}{2ML}\right)
\]

\[
\frac{dx_3}{dt} = x_4
\]

\[
\frac{dx_4}{dt} = \left(L x_2^2 \sin{x_1}\right) + \left(\frac{Mg \cos{x_1} \sin{x_1}}{m}\right) - \left(\frac{x_2^2 M L \sin{x_1}}{m}\right)
\]

\[
+ \left(\frac{Mg (1 - \cos{x_1}) \sin{x_1}}{m}\right) - \left(\frac{F_m \sin{x_1}}{m}\right) + \left(\frac{F_x}{m}\right)
\]

**Case B:** This is the linearized form of equations 2-36. If the load disturbance is small around the equilibrium state, this form of the equations successfully approximate the nonlinear system. Since initial load disturbance is not more than the fraction of a radian, linearization is justified for our purpose. In this case the rope length varies on the time axes. The crane velocity is nonzero and nonstationary term. This case is used in fertilizer plant where the load lift is large due to the heap of the fertilizer. The load is transferred over a moderate distance and crane does not move on a smooth surface. Thus inducing
nonstationary crane velocity. The care is taken that the work place is clear of human beings.

\[
\frac{dx_1}{dt} = x_2
\]

\[
\frac{dx_2}{dt} = \left( \frac{x_1 (m+M) g}{m L} \right) - \left( \frac{2 v x_2}{L} \right) + \left( \frac{v (m+M) x_4}{m L^2} \right) + \left( \frac{F_x}{LM} \right)
\]

\[
\frac{dx_3}{dt} = x_3
\]

\[
\frac{dx_4}{dt} = \left( \frac{g m x_1}{M} \right) - \left( 2 v x_2 \right) + \left( \frac{v M x_4}{LM} \right) + \left( \frac{F_x}{M} \right)
\]

**Case C:** This is a modified form of equation 2-37. In this case the rope length remains constant. The crane velocity is nonzero and nonstationary term. This case is used in earth excavating projects where the load lift is very small. The load is transfered over a long distance and crane does not move on a smooth surface. Thus inducing nonstationary crane velocity.

\[
\frac{dx_1}{dt} = x_2
\]

\[
\frac{dx_2}{dt} = \left( \frac{x_1 (m+M) g}{m L} \right) + \left( \frac{F_x}{LM} \right)
\]

\[
\frac{dx_3}{dt} = x_3
\]
\[
\frac{dx_4}{dt} = \left( \frac{g \ m \ x_1}{M} \right) + \left( \frac{F_x}{M} \right)
\]

**Case D:** In this case, we consider the hoisting during constant crane velocity. The top running double girder, top running single girder, double girder underhung and single girder underhung are the cases where constant crane velocities are maintained. This is because crane wheels are set into the monorails which provide smooth surfaces to the crane wheels. The hoisting is rarely carried out when the crane is moving during actual working. This case is run only during safety inspection or equipment testing.

\[
\frac{dx_1}{dt} = x_2
\]

\[
\frac{dx_2}{dt} = \left( \frac{x_1 (m+M) g}{m \ L} \right) - \left( \frac{2 \ v \ x_2}{L} \right) + \left( \frac{v (m+M) x_4}{m \ L^2} \right) + \left( \frac{F_x}{L \ M} \right)
\]

\[
\frac{dx_3}{dt} = \text{constant}
\]

**Case E:** This is the modified form of the equation 2-39. In this case the rope length remains constant when the crane travels. The crane velocity is constant due to smooth monorail surface. This case of hoisting is considered safe. It is recommended and required by OSHA. This case is used almost everywhere such as in machine shops, foundries, warehouses, construction, scrap yards etc.

\[
\frac{dx_1}{dt} = x_2
\]

\[
\frac{dx_2}{dt} = \left( \frac{x_1 (m+M) g}{m \ L} \right) + \left( \frac{F_x}{L \ M} \right)
\]
\[ \frac{dx_3}{dt} = \text{constant} \]

**Case F:** This is a case of hoisting when crane remains stationary. This case is investigated by Wright-Patterson Air Force Base of Dayton Ohio. They used it for a rescue hoist mounted on a helicopter. The helicopter remains in the hovering position when the rescued is lifted and taken into the first aid cabin of the helicopter.

\[ \frac{dx_1}{dt} = x_2 \]

\[ \frac{dx_2}{dt} = \frac{g}{L} x_1 - \frac{2v}{L} x_2 \]

**Case G:** This is a modified case of equation 2-41. In this case the rope does not change any more. It, in fact, is a case of simple harmonic motion of a pendulum. It is an oversimplified case of a crane. It is included here for the comparison purpose. Also, this oversimplify system of a hoist help us explain some of the special features of the actual hoist system. By noticing the behavior of simple systems like this, we can get clues to improve the real systems like equations 2-36 to 2-41.

\[ \frac{dx_1}{dt} = x_2 \]

\[ \frac{dx_2}{dt} = \frac{g}{L} x_1 \]
2.10 Simulation and results

In this section we shall obtain computer solution of state equations 2-36 to 2-42. One of the most important computations involved in the control system analysis and design is computations of the transient responses to a given input. Several methods such as Runge-Kutta method, Euler's method and modified Euler's method are available for such purposes. We wish to use a general method for computing the response to a given control input. We choose the fourth-order Runge-Kutta method for our purpose. It is used in obtaining the numerical solution of the differential equations derived in the previous section.

We simulate the general and nonlinear model of the crane and compare its results with the linearized model. We interpreted the nonlinear terms of the general model in view of the simplified form of the crane model. We identified the load oscillation as the wave traveling in the medium of the crane rope of the length L. Also, we showed that the linearized model provides sufficient information for the hoisting that is nonlinear in its nature.
The figure 2-2 shows the general nonlinear case of hoisting. The irregular oscillation of the load implies that two or more displacement waves of different wavelengths are acting simultaneously on a load. The resultant displacement of the load is equal to the sum of displacement that would have produced by each wave acting independently. These displacement waves, as can be seen in the crane model, originate from the nonlinear crane velocity and the derivative of the load displacement.

Figure 2-2. The load displacement is simulated for the general nonlinear case A
The derivative of the load displacement explicitly shows the nonlinear nature in its simulation. It shows that the load velocity is not uniform. Here at this place the linear model clearly differs from the nonlinear one. The care should be taken such that this nonlinear term does not outweigh the control magnitude $|u|$ to be determined in the next chapter.

Figure 2-3. The derivative of load displacement is simulated on time axes for case A
The figure 2-4 shows the crane displacement on the time axes. The crane velocity is greatly influenced by the load oscillation especially when the crane mass is marginally small. The force required to move the crane depends on the mass of the load and initial disturbance of the load. It also depends upon the mass of the crane. The velocity of the crane clearly influence the load oscillation. This can be shown from the first part of the equation 2-36. It has the crane velocity term $x_3$ on its right side. The crane velocity, if utilized, can help dampen out the load oscillation.

Figure 2-4. The crane displacement on the time axes for the case A
Figure 2-5 shows the crane velocity on the time axes. The crane velocity is not uniform. The rate of change of the crane velocity is changing with respect to time $t$. This simulation reveals the dangerous behavior of this type of crane on the production floor. This machine should be designed with due consideration. Also, this term determines the wear and tear of the crane parts as well the energy cost of the system.

Figure 2-5. The crane velocity on the time axes for the case A
The figure 2-6 shows the load displacement of the general linearized model of the crane B. The initial condition chosen for this simulation is $y(t) = 0.15$ radian. This is a common industrial condition in the crane operations. This occurs when the gripper of the hook grips the load in such a way that the hook remains inclined on the load. This condition will induce the initial load displacement. When the load is hoisted, it swings around the operating point.

Figure 2-6. The load displacement is simulated for the general nonlinear case B.
The figure 2-7 shows the derivative of the load displacement of the general linearized model of the crane. The initial condition chosen for this simulation is \( \frac{d}{dt} = 0.15 \) radian per second. This is also a common industrial condition in the crane operations. This occurs when the gripper of the hook grips the load in such a way that the hook remains inclined on the load. And the load collides with some other object during hoisting. This condition will induce the initial load displacement with an acceleration in the load velocity.

Figure 2-7. The derivative of load displacement is simulated on time axes for case B
The figure 2-8 shows the crane displacement on the time axes. The crane velocity is greatly influenced by the load oscillation especially when the crane mass is marginally small. The force required to move the crane depends on the mass of the load and initial disturbance of the load. It also depends upon the mass of the crane. The velocity of the crane clearly influence the load oscillation. The initial condition chosen for the crane displacement is $s_0 = 0.0$ meter. The crane begins from the origin.

Figure 2-8. The crane displacement on the time axes for the case B
Figure 2-9 shows the crane velocity on the time axes for the case B. The crane velocity is not uniform. The rate of change of the crane velocity is changing with respect to time $t$. This term determines the wear and tear of the crane parts and the energy cost of the system. The initial condition chosen for this simulation is $v_0 = 0.0$ meter per second. The crane velocity depends on the horizontal force exerted on the crane. This force can be monitored by controlling the motor torque that drives the crane.

![Graph showing crane velocity over time for case B](image)

**Figure 2-9.** The crane velocity on the time axes for the case B.
The figure 2-10 shows the load displacement of the general linearized model of the crane C. The initial condition chosen for this simulation is $t_1(t) = 0.15$ radian. In this case, the rope length is not changing. The gripper of the hook grips the load in such a way that the hook remains inclined on the load. This condition will induce the initial load displacement. When the load is hoisted, it swings around the operating point.

Figure 2-10. The load displacement is simulated for the general nonlinear case C.
The figure 2-11 shows the derivative of the load displacement of the generalized case of crane C. The derivative of the load displacement explicitly shows the nonlinear nature in its simulation. It shows that the load velocity is not uniform. The initial condition chosen for this simulation is $\dot{\phi}(t) = 0.15$ radian per second. In this case the rope length does not change. This case is used when heavy load is transferred over long distances.

Figure 2-11. The derivative of load displacement is simulated on time axes for case C
The figure 2-12 shows the load displacement of the general linearized model of the crane D. The initial condition chosen for this simulation is \( \phi(t) = 0.15 \) radian. This is a common industrial condition in the crane operations. This occurs when the gripper of the hook grips the load in such a way that the hook remains inclined on the load. This condition will induce the initial load displacement. When the load is hoisted, it swings around the operating point.

Figure 2-12. The load displacement is simulated for the general nonlinear case D.
The figure 2-13 shows the derivative of the load displacement of the general linearized model of the crane. The initial condition chosen for this simulation is $\xi(t) = 0.15$ radian per second. The rope length is changing at the hoisting speed $v$. The crane travels at the moderate speed. It stops between the lifts. The crane velocity greatly influences the load displacement.

Figure 2-13. The derivative of load displacement is simulated on time axes for case D.
The figure 2-14 shows the crane displacement for the case of a crane D. When we maintain a constant crane velocity, the crane displacement steadily increases on the time axes. Such a state of a crane motion can be achieved by using smooth surfaced rails along with grooved crane wheels. The initial condition chosen for this simulation is zero meter. The crane begins from the origin. The crane distance is determined by the crane velocity and the total operating time $T$.

![Graph showing crane displacement over time for case D](image)

Figure 2-14. The crane displacement on the time axes for the case D
The figure 2-15 shows the general linear case of crane E. The simulation result of the load displacement implies that the system is unstable. The crane travels at the constant speed. The rope length does not change. The initial condition chosen for this simulation is $\theta_0 = 0.15$ radians.

Figure 2-15. The load displacement is simulated for the general nonlinear case E.
The derivative of the load displacement explicitly shows the nonlinear nature in its simulation. It shows that the load velocity is not uniform. Here at this place the linear model clearly differs from the nonlinear one. The care should be taken such that this nonlinear term does not outweigh the control magnitude $|u|$ to be determined in the next chapter.

Figure 2-16. The derivative of load displacement is simulated on time axes for case E
Since the crane velocity is constant, the crane steadily goes away from the origin. The crane velocity and total operating time $T$ determines the distance that crane travels during the lifts. The initial condition chosen for this simulation was zero meter.

Figure 2-17. The crane displacement on the time axes for the case E
The figure 2-18 simulates the case of the crane F. In this case, the crane does not move. This case is studied by the Wright-Patterson Air Force Base in detail. Their experiments with various initial conditions of the crane proved that this system is not good for the delicate purposes like human rescue operation or fire extinguishing operations.

Figure 2-18. The load displacement is simulated for the general nonlinear case F
The derivative of the load displacement explicitly shows the nonlinear nature in its simulation. It shows that the load velocity is not uniform. The derivative of the load displacement affects the behavior of the load displacement.

Figure 2-19. The derivative of load displacement is simulated on time axes for case F.
The figure 2-20 shows the load displacement of the over simplified model of the crane. The initial condition chosen for this simulation is $\theta(t) = 0.15$ radian. The system is harmonic. The displacement is uniform and equal in amplitudes. This is an ideal system. It is never found in the common practice.

Figure 2-20. The load displacement is simulated for the general nonlinear case G.
2.11 Summary of the interesting behavior observed

In the nonlinear case, the motion begins in the proximity of the equilibrium state $\xi$. In other words, $\xi_0$ is sufficiently near to $\xi$. In a simulation, the motion does not converge to $\xi$ as the time assumes a large value. In order to express this idea into a precise mathematical tool, we define a real number $r(t_0) > 0$ such that $\| x_0 - x_e \| < r(t_0)$.

What we will notice in this simulation is an absence of a real number $\mu$ such that a difference function $\| \phi(t; x_0, t_0) - x_e \| < \mu$. Where $\phi(t; x_0, t_0)$ is a transition function of the system equation 2-36. It is a solution of equation 2-36, with fixed $u(t)$, going through state $\xi_0$ at time $t(0)$, observed at time $t$. Therefore, it is also known as a solution function of the equation 2-36. This function shows how fast or in what manner $x_0$ is transformed into $x_t$. It specifies the infinitesimal state transition $x \rightarrow x + dx$ corresponding to the infinitesimal change in time $t \rightarrow t + dt$. Since we can not find $m$ corresponding to $r(t_0)$ such that $\| \phi(t; x_0, t_0) - x_e \| < \mu$, the energy of this system is increasing as time approaches infinity. In our case we define initial conditions as follow.

$r_1(t_0) = \phi$ at $t_0$ is an initial angular displacement of load mass $m$.

$r_2(t_0) = \frac{d\phi}{dt}$ at $t_0$ is an initial angular velocity of load mass $m$.

$r_3(t_0) = s$ at $t_0$ is an initial crane position.

$r_4(t_0) = v$ at $t_0$ is an initial crane velocity.

The stability is the property of a system which causes it to develop the forces opposing any motion that may disturb its equilibrium position. In our case, the simulation results show, the load assumes entirely new positions when it is disturbed from its rest position during the hoisting. The load, on the time axis, does not show any sign of returning back to its
rest position. This phenomenon can be explained in the view of Lyapunov's second theorem. As a rule there is no specific way of defining energy when the equations of motion are given in a purely mathematical form. However, we noticed in the second chapter that the Lagrangian function $\Lambda$ is a function of kinetic and potential energies. The second method of Lyapunov says that, if the energy of an isolated physical system is negative for every possible state $x$, except for a single equilibrium state $x_e$, then the energy will continually decrease until it finally assumes its minimum value. If we look at the damping term of equation (e), we notice that this term increases with the decreasing rope length $L - \int_0^t \nu \, dt$. This is because the term $L - \int_0^t \nu \, dt$ is the function of the hoisting speed $\nu$. This term will tend towards infinity as the load approaches the pivot point. This is because at point $P$, $\int_0^t \nu \, dt$ equals $L$. We also notice that this term is directly proportional to the hoisting speed. That means higher velocity will hasten the rate of divergence of the system. However, the low hoisting speed will not eliminate the problem. In other words, the excessive load oscillation will always be experienced regardless of the hoist speed. This fact is confirmed by the experiments carried out by [3]. In order that we may mend this situation, we have to adopt entirely new solution approach. This solution should be such that it does not involve the hoisting speed $\nu$ in its control input function $u$.

**Case A:** The first case that we examined was the general form of the crane model. The data for computer simulation were taken from a real crane made by Lenton Cranes Inc., PA. They are as follow.

\[
\begin{align*}
\text{m} &= 4,000 \text{ kg} \\
L_i &= 20.00 \text{ meter} \\
\text{M} &= 2,240 \text{ kg} \\
L_f &= 0.9 \text{ meter} \\
\dot{\theta}_1 (t_0) &= 0.16 \text{ radian} \\
\dot{\theta}_2 (t_0) &= 0.0 \text{ radian / minute} \\
\dot{\theta}_3 (t_0) &= 0.0 \text{ meter} \\
\dot{\theta}_4 (t_0) &= 0.0 \text{ meter / minute}
\end{align*}
\]
\[ F_m = 4.31 \times 10^4 \text{ N m} \quad F_x = 2.00 \times 10^4 \text{ N m} \]

\[ T_\phi = 0.00 \times 10^4 \text{ N m} \quad v_h = 2.00 \text{ meter / minute} \]

In this case the crane velocity as well the hoisting speed are time varying terms. For the safety reasons, this case is strictly prohibited by OSHA under the regulation of 29AFR 1949 (b) 2. This case is limited to the rescue operations. In truck type cranes, the hoisting is carried out when the crane is moving. But in such cases the tip height is small, the truck velocity is low and varying and the load is moderate to light. The figure 2-2 shows the general nonlinear case of hoisting. The irregular load oscillation implies that two or more displacement waves of different wavelengths are acting simultaneously on a load. These displacement waves originate from the highly nonlinear crane velocity and the derivative of the load displacement. The resultant displacement of the load is equal to the sum of displacement that would have produced by each wave acting independently. The figure 2-3 shows the derivative of the load displacement. We notice that nonlinear characteristics of load displacement is carried further on to this term too. The angular velocity is changing at an unpredictable rate. The x axis is scaled as 1 minute = 10 units. The y axis is scaled as 1 radian or radian / minute = 1 unit. In the duration of first eight minutes, the behavior in both the cases are moderate. But as the rope length approaches the pivot R, oscillations become very large. This makes the rescue operation extremely difficult. The figure 2-4 shows the crane displacement. The crane displacement is affected by the strong and large load oscillations. The crane does not travel in the expected manner. It wavers on its way. This behavior can be explained from the figure 2-5. The figure 2-5 shows the crane velocity. The crane velocity assumes large values as time proceeds. The acceleration exists. This acceleration is ever increasing. It produces erratic crane displacement.
**Case B:** The second case that we examined was the linearized form of the general crane model. The data for computer simulation were taken from a real crane made by Lenton Cranes Inc., PA. They are as follow.

\[
\begin{align*}
    m &= 4,000 \text{ kg} \\
    L_1 &= 20.00 \text{ meter} \\
    \tau_1 (t_0) &= 0.16 \text{ radian} \\
    \tau_3 (t_0) &= 0.0 \text{ meter} \\
    F_m &= 4.31 \times 10^4 \text{ N} \\
    T_\phi &= 0.00 \times 10^4 \text{ N m}
\end{align*}
\]

\[
\begin{align*}
    M &= 2,240 \text{ kg} \\
    L_f &= 0.9 \text{ meter} \\
    \tau_2 (t_0) &= 0.0 \text{ radian / minute} \\
    \tau_4 (t_0) &= 0.0 \text{ meter / minute} \\
    F_x &= 2.00 \times 10^4 \text{ N m} \\
    v_h &= 2.00 \text{ meter / minute}
\end{align*}
\]

If the load disturbance is small around the rest position, this form of the equations successfully approximate the general crane model. In this case the crane velocity as well the hoisting speed are time varying terms. The figure 2-6 shows the general nonlinear case of hoisting. The initial condition is induced due to the inclination of hook while gripping the load. This is a common industrial situation in routine crane operations. This figure shows that the load oscillation increases as the rope length decreases. It shows the unstable behavior of the system. The figure 2-7 shows the derivative of the load displacement on the time axis. We notice that the angular velocity also shows unstable characteristics. The angular velocity is increasing on the time axis. The x axis is scaled as 1 minute = 10 units. The y axis is scaled as 1 radian or radian / minute = 1 unit. In the duration of first eight minutes, the behavior in both the cases are moderate. But as the rope length approaches the pivot R, oscillations become very large. The figure 2-8 shows the crane displacement. The crane displacement is affected by the strong and large load oscillations. The crane does not travel in the expected manner. It wavers on its way. This behavior can be explained
from the figure 2-9. The figure 2-9 shows the crane velocity. The crane velocity assumes large values as time proceeds. The acceleration exists. This acceleration is ever increasing. It produces erratic crane displacement. Such a behavior of the equipment can cause a very heavy wear and tear especially on the long run. Therefore frequent inspection of the machine and the maintenance are essential features of this system.

**Case C:** The third case that we examined was the modified form of the case B. In this case the rope length remains constant. The data for computer simulation were taken from a real crane made by Lenton Cranes Inc., PA. They are as follow.

\[
\begin{align*}
    m &= 4,000 \text{ kg} \\
    L_i &= 20.00 \text{ meter} \\
    \xi_i (t_0) &= 0.16 \text{ radian} \\
    \xi_f (t_0) &= 0.0 \text{ radian} \\
    F_m &= 4.31 \times 10^4 \text{ N m} \\
    \phi &= 0.00 \times 10^4 \text{ N m}
\end{align*}
\]

\[
\begin{align*}
    M &= 2,240 \text{ kg} \\
    L_f &= 20.0 \text{ meter} \\
    \xi_i (t_0) &= 0.0 \text{ radian / minute} \\
    \xi_f (t_0) &= 0.0 \text{ radian / minute} \\
    F_x &= 2.00 \times 10^4 \text{ N m} \\
    \nu_h &= 0.00 \text{ meter / minute}
\end{align*}
\]

In this case the crane velocity is time varying term. The hoisting velocity is zero when the crane is travelling. For the safety reasons, this case is widely used for the earth excavating projects. In this type of work the load lift required is very small. The figure 2-10 shows the nonlinear case of hoisting. The initial condition is induced due to the inclination of hook while gripping the load. This is a common industrial situation in routine crane operations. This figure shows that the load oscillates in a simple harmonic motion. This behavior is considered safe if this oscillation is small. The magnitude of the load oscillation is determined by the initial load inclinations. The figure 2-11 shows the derivative of the load displacement on the time axis. We notice that the angular velocity also shows stable
characteristics. The angular velocity oscillates at a constant rate on the time axis. The x
axis is scaled as 1 minute = 10 units. The y axis is scaled as 1 radian or radian / minute =
1.0E-3 unit. The figure 2-12 shows the crane displacement. The crane displacement is
not affected by the strong and large load oscillations. The crane travels in the expected
manner. It does not waver on its way. This behavior can be explained from the figure 2-
13. The figure 2-13 shows the crane velocity. The crane velocity does not assume large
values as time proceeds. Such a behavior of the equipment will keep the wear and tear
low. Therefore frequent inspection of the machine and the maintenance are not essential for
this type of systems.

Case D: The fourth case that we examined was the modified form of the case C. In this
case crane velocity remains constant but the the rope length does not. The data for
computer simulation were taken from a real crane made by Lenton Cranes Inc., PA. They
are as follow.

\[
\begin{align*}
m &= 4,000 \text{ kg} \\
L_1 &= 20.00 \text{ meter} \\
\phi_1 (t_0) &= 0.16 \text{ radian} \\
F_m &= 4.31 \times 10^4 \text{ N m} \\
T_\phi &= 0.00 \times 10^4 \text{ N m} \\
M &= 2,240 \text{ kg} \\
L_f &= 0.9 \text{ meter} \\
\phi_2 (t_0) &= 0.0 \text{ radian / minute} \\
\phi_3 (t_0) &= 0.0 \text{ meter / minute} \\
F_x &= 2.00 \times 10^4 \text{ N m} \\
\nu_h &= 2.00 \text{ meter / minute}
\end{align*}
\]

In this case the crane velocity is time non-varying term. The hoisting velocity is non zero
when the crane is travelling. For the safety reasons, this case is used only for the safety
inspection or equipment testing. The figure 2-14 shows the nonlinear case of hoisting.
The initial condition is induced due to the inclination of hook while gripping the load. This
is a common industrial situation in routine crane operations. This figure shows that the load oscillation increases as the rope length decreases. This behavior is considered safe if the load oscillation is small when it approaches the pivot R. The magnitude of the load oscillation is determined by the initial load inclinations. The figure 2-15 shows the derivative of the load displacement on the time axis. We notice that the angular velocity also shows unstable characteristics. The angular velocity does not remain constant on the time axis. The x axis is scaled as 1 minute = 10 units. The y axis is scaled as 1 radian or radian / minute = 1.0E-3 unit. The figure 2-16 shows the crane displacement. The crane displacement is not affected by the strong and large load oscillations. The crane travels in the expected manner. It does not waver on its way. This behavior can be explained from the crane velocity of this system. The crane velocity is constant throughout the operation. Such a behavior of the equipment will keep the wear and tear low. Therefore frequent inspection of the machine and the maintenance are not essential for this type of systems.

Case E: The fifth case that we examined was the modified form of the case D. In this case the rope length does not change. The crane velocity remains constant. The data for computer simulation were taken from a real crane made by Lenton Cranes Inc., PA. They are as follow.

\begin{align*}
m &= 4,000 \text{ kg} \\
L_1 &= 20.00 \text{ meter} \\
\zeta_1 (t_0) &= 0.16 \text{ radian} \\
\zeta_2 (t_0) &= 0.0 \text{ radian / minute} \\
\zeta_3 (t_0) &= 0.0 \text{ meter} \\
\zeta_4 (t_0) &= 0.0 \text{ meter / minute} \\
F_m &= 4.31 \times 10^4 \text{ N m} \\
T_\phi &= 0.00 \times 10^4 \text{ N m} \\
M &= 2,240 \text{ kg} \\
L_f &= 20.0 \text{ meter} \\
F_x &= 2.00 \times 10^4 \text{ N m} \\
\chi &= 0.00 \text{ meter / minute}
\end{align*}
In this case the crane velocity is time non varying term. The hoisting velocity is zero when the crane is travelling. The constant crane velocity is maintained by installing the trolley wheels on the smooth monorail surface. The figure 2-17 shows the linear case of hoisting. The initial condition is induced due to the inclination of hook while gripping the load. This is a common industrial situation in routine crane operations. This simulation shows that the load oscillates in a simple harmonic motion. This behavior is considered safe if this oscillation is small. The magnitude of the load oscillation is determined by the initial load inclinations. The figure 2-18 shows the derivative of the load displacement on the time axis. We notice that the angular velocity also shows stable characteristics. The angular velocity changes at a constant rate on the time axis. The x axis is scaled as 1 minute = 10 units. The y axis is scaled as 1 radian or radian / minute = 1.0E-3 unit. The figure 2-19 shows the crane displacement. The crane displacement is not affected by the strong and large load oscillations. The crane travels in the expected manner. It does not waver on its way. Such a behavior of the equipment will keep the wear and tear low. Therefore frequent inspection of the machine and the maintenance are not essential for this type of systems.

**Case F:** The sixth case that we examined was the modified form of the case E. In this case crane remains stationary at one position during the hoisting. The data for computer simulation were taken from a real crane made by Lenton Cranes Inc., PA. They are as follow.

\[
\begin{align*}
   & m = 4,000 \text{ kg} & & M = 2,240 \text{ kg} \\
   & L_1 = 20.00 \text{ meter} & & L_f = 0.9 \text{ meter} \\
   & \xi_1 (t_0) = 0.16 \text{ radian} & & \xi_2 (t_0) = 0.0 \text{ radian / minute} \\
   & \xi_3 (t_0) = 0.0 \text{ meter} & & \xi_4 (t_0) = 0.0 \text{ meter / minute}
\end{align*}
\]
In this case the crane velocity is zero. The hoisting velocity is non zero. This case is investigated by Wright-Patterson Air Force Base of Dayton, Ohio. They used this type of crane for a rescue-hoist mounted on an H-53 helicopter. The figure 2-18 shows the nonlinear case of hoisting. The initial condition is induced due to the inclination of hook while gripping the load. This figure shows that the load oscillation increases as the rope length decreases. The magnitude of the load oscillation is determined by the initial load inclinations. The figure 2-21 shows the derivative of the load displacement on the time axis. The angular velocity oscillates on the time axis and this oscillation assumes large values as the time proceeds. The time axis is scaled as 1 minute = 10 units. The y axis is scaled as 1 radian or radian / minute = 1.0E-3 unit. The unstable behavior is explained in view of Lyapunov's second theorem in this section.

Case G: The seventh case that we examined was the modified form of the case F. This is a over simplified mathematical model of the case F. In fact, this is a case of simple harmonic motion of a pendulum.

\[
\begin{align*}
F_m &= 4.31 \times 10^4 \text{ N} \cdot \text{m} \\
F_x &= 2.00 \times 10^4 \text{ N} \cdot \text{m} \\
T_\phi &= 0.00 \times 10^4 \text{ N} \cdot \text{m} \\
v_h &= 2.00 \text{ meter} / \text{minute} \\
m &= 4,000 \text{ kg} \\
M &= 0.0 \text{ kg} \\
L_i &= 20.00 \text{ meter} \\
L_f &= 20.00 \text{ meter} \\
\dot{\theta}_i (t_0) &= 0.16 \text{ radian} \\
\dot{\theta}_x (t_0) &= 0.0 \text{ radian} / \text{minute} \\
F_m &= 0.00 \times 10^4 \text{ N} \cdot \text{m} \\
F_x &= 0.00 \times 10^4 \text{ N} \cdot \text{m}
\end{align*}
\]
CHAPTER THREE

A MODEL OF A HIGH SPEED CRANE EXCLUDING HOISTING

3.1 Background

The Crane Manufacturers Association of America (CMAA) has published the complete classification of the industrial cranes. Since we are concerned with overhead cranes only, we discuss selected types of the crane in this section. The popular types of overhead cranes are mentioned below.

1. Top running double girder
2. Top running single girder
3. Double girder underhung
4. Single girder underhung
5. Double leg gantry
6. Single leg gantry
7. Floor mounted column jib crane
8. Wall mounted jib crane
9. Pit mounted jib crane
10. Crowler crane

The top running double girder cranes offer the greatest lifting heights, highest tonnages, widest spans and heaviest duties. While Top running, single girder cranes have a one beam bridge that rides on a rail atop the runway. The capacity is generally limited to 30 tons. The double girder underhung cranes have the hoist mounted above the bridge to attain a bit more headroom than the single girder version. The capacities range to 50 tons. The single girder underhung cranes have the bridge end trucks running on the lower flanges of the runway beams. The capacities range to 10 tons. The double leg gantry cranes move along floor rails or guide paths. The capacities range to several hundred tons. The single leg gantry cranes substitute a wall mounted runway for the second leg. They are usually designed to handle loads of less than 20 tons for a specific operation. The jib cranes are used where precise handling of equipment at slow speeds, with long idle
periods between lifts, is required. The capacities depend on the structure of the cranes, usually they range to 10 tons or less.

Figure 3-1 The top running double girder crane
Figure 3-2 The top running single girder crane

Figure 3-3 The double girder underhung crane
Figure 3-4 The single girder underhung crane

Figure 3-5 The schematic diagram of a crane shows the state variables of the system.
In all these special types of crane, the load swings in a vertical plane under the action of gravity. In the Lagrangian theory, the motion of a load is a vector function. The position of a load is a function of time. We determined the velocity and acceleration of the load by differentiating this function once and twice respectively. The acceleration of the load yielded the force acting on the load. The term of normal acceleration derived was \( L \left( \frac{d\phi}{dt} \right)^2 \) and the tangential was \( L \frac{d^2\phi}{dt^2} \). The corresponding forces were \( m L \left( \frac{d\phi}{dt} \right)^2 \) and \( m L \frac{d^2\phi}{dt^2} \).

The other forces acting on load \( m \) were the rope tension \( T \), the gravity force \( mg \) and hoist motor torque \( T_m \). The motor torque is applied only when the rope length is varied. We found that the normal force \( F_n \) is \( T - mg \cos \phi \). And the tangential force \( F_t \) is \(-mgsin\phi\).

The minus sign for \( F_t \) is due to the fact that the force is in the opposite direction to \( \phi \). If we project these forces on a radial and the tangent line to \( m \), we get

\[
T - mg\cos\phi = m L \left( \frac{d\phi}{dt} \right)^2 \tag{3-1}
\]

\[
-mgsin\phi = m L \frac{d^2\phi}{dt^2} \tag{3-2}
\]

This is the case when we consider a pure hoisting. Here in this model, we want to include the crane movement into the equation derived above. If the mass of the crane is \( M \), we select \( x \) and \( \phi \) as the system generalized coordinates.

\[
q = [x \ \phi]^T \tag{3-3}
\]

\[
\left( \frac{m+M}{L} \right) \frac{d^2x}{dt^2} + m \frac{d^2\phi}{dt^2} = \frac{F_x}{L} \tag{3-4}
\]

In this case the generalizes force \( F_x \) is zero.
For safety reasons a crane hoists the load only when it is not moving. Similarly, a crane is moved from one point to another only in the absence of the pure hoisting. The crane model developed in equation 3-2 describes the conditions just mentioned. Therefore, this model is of more interest than others from practical point of view.

3.2 Laplace transformation of the equations of motion

We wish to select a crane velocity as a control. This control intends to dampen out the load oscillation as quickly as it is possible by controlling the velocity of the crane. We control the crane velocity when the crane has just arrived at its destination. The crane velocity should be controlled in such a way that the load oscillation diminishes as quickly as possible. We use the Laplace transfer techniques to arrive at the desired conclusion. The Laplace transform function is used to characterize the input-output relationships of components of the system. It is necessary that this system is time-invariant and a linear system. The transfer function is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input under the assumption that all initial conditions are zero. Thus, the transfer function of a system is, in fact, a mathematical model as that we derived in chapter two. But this function is an operational method of expressing the same model that relates the output variables to the input variables. It does not depend on the magnitude of the input, though it includes the units of the input to relate the input to output.
If $x(t)$ is a real-valued function of time, then we denote the $X(s)$ as the Laplace transform of $x(t)$ and can write $X(s) = \mathcal{L}\{x(t)\}$. And if we are given $X(s)$ then we write the inverse Laplace transform as $x(t) = \mathcal{L}^{-1}\{X(s)\}$.

We take the Laplace transform of equation 3-2

$$\left(s^2 + \frac{g}{L}\right) \varnothing(s) = \frac{1}{L} s^2 X(s) \quad 3-6$$

If we replace velocity $V$ in Equation 3-8 in place of $s X(s)$, we get

$$\left(s^2 + \frac{g}{L}\right) \varnothing(s) = \frac{s V(s)}{L} \quad 3-7$$

and we can write the transform function as follow,

$$\frac{\varnothing(s)}{V(s)} = \frac{s}{L \left(s^2 + \frac{g}{L}\right)} \quad 3-8$$

If we assume $W_o = \frac{g}{L}$ and normalize the frequency to unity, we obtain the transform function as

$$\frac{\varnothing(s)}{V(s)} = \frac{s}{L \left(s^2 + 1\right)} \quad 3-9$$

Now, if we take the Laplace inverse of equation 3-10, we get

$$\frac{d^2 \varnothing}{dt^2} + \varnothing = \frac{1}{L} \frac{d^2 x}{dt^2} \quad 3-10$$

We rewrite this equation as
If we choose $x_L = L \phi$ as our state variable, we obtain

$$x_L = \frac{d^2x}{dt^2} - \frac{d^2\phi}{dt^2}$$  

3-12

Here, we define $v_T = \frac{dx}{dt}$. Thus, the input $v_T$ is a crane velocity. If we replace these terms in equation 3-10, we get

$$\frac{d^2x_L}{dt^2} + x_L = \frac{dv_T}{dt}$$  

3-13

This is the model of the crane for the velocity controlled hoisting.

### 3.3 State space approach

To simulate this model on a computer we define the state variables as

$$x_1 = x_L = \int x_2 + v_T \, dt$$  

3-14

$$x_2 = \frac{dx_L}{dt} = \int x_1 \, dt$$  

3-15

$$x_3 = \int v_T \, dt$$  

3-16
And differentiate equations 3-14 to 3-16

\[
\frac{dx_1}{dt} = x_2 + v_T
\]  \hspace{1cm} 3-17

\[
\frac{dx_2}{dt} = -x_1
\]  \hspace{1cm} 3-18

\[
\frac{dx_3}{dt} = v_T
\]  \hspace{1cm} 3-19

In the matrix form, this model becomes

\[
\begin{bmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt} \\
\frac{dx_3}{dt}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} +
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}[u]
\]  \hspace{1cm} 3-20

This crane model is the reduced form of the equation 2-32, but simulation in next chapter will prove that it preserves all the characteristics of the equation 2-32. Thus it can be said it is a reliable model of the crane.

3.4 Simulation and results

We used the fourth order Runge-Kutta method with a step size \( h = \frac{T}{200} \). First we considered the load oscillation when crane was not in the motion. This is shown in Fig. 3.2. We see that it behaves like a simple pendulum and has a simple harmonic motion. But as the crane is forced to move, we notice the difference in the pattern of load oscillation. Now, the load displacement does not show similar features anymore. Result is shown in Figure 3.5. Thus we conclude that this seemingly simple and reduced model
of the crane preserves all the important properties of the actual crane. Also the results of simulation of this system show the need to set up a solution procedure to dampen out the oscillation as the crane moves using the available control variable. We answer this question in the next chapter.

**Figure 3-6** The load displacement is simulated for the equation 3-20
Here, we explain the two different crane behaviors in two different circumstances by the following reasoning. First we consider the case when the crane is not in motion and load is disturbed from its rest position. It is convenient to define the potential energy to be zero when the load is at its lowest point, so its energy is entirely kinetic at this position.

![Graph showing the state variable $x_2$ simulated for the equation 3-20](x.x92)

Figure 3-7 The state variable $x_2$ is simulated for the equation 3-20
When the load is at its highest point, the velocity is zero, and the energy is entirely potential. Since no forces other than gravity do work, the total mechanical energy is conserved, and the maximum potential energy is equal to the maximum kinetic energy. As the load swings back and forth, there is a continual conversion of kinetic to potential energy and then back to kinetic energy. The presence of dissipative forces such as friction or air resistance will bring the load to a rest position after some time $t$. But when a crane is put into motion, the energy is supplied back to replace the losses. This energy typically depends on the crane velocity $v$. However the exact dependence varies and can be quite complicated for this system. To obtain the first approximation, it is assumed that the disturbing force $F_d$ is linearly proportional to the crane velocity $v$, or $F_d = \gamma v$. Here, $\gamma$ is the proportional constant and $v$ is the crane velocity. The positive sign indicates that the force $F_d$ assist the system behavior. If $\gamma$ is zero, oscillations continue with the same amplitude indefinitely. With a small value of $\gamma$, the oscillations steadily increase in amplitude until they are very large. This explanation leads us to the conclusion that the crane velocity can be a strong candidate for the hoist control. This is shown in the next chapter. We use the celebrated principle of Pontryagin to obtain a time optimal velocity control such that the load oscillation is dampen out in the minimum time for this system.
CHAPTER FOUR

THE TIME OPTIMAL CONTROL OF A HOIST

4.1 Background

The dynamic models are useful in predicting the behavior of physical processes especially when it is not possible to build the actual system itself. There is however another context for dynamic modeling in terms of being able to optimize the behavior of a system. We have seen how feedback controls can modify performance. In this chapter, we wish to show that with a proper choice of control one can not only modify but also improve performance of a system. This entails an introduction to certain aspects of optimal control theory.

In particular, we want to examine the problem of hoisting a load, which is unstable in its time-varying mode. We were able to stabilize it by a suitable linear feedback control in chapter three. Later, it will be seen that this control is the consequence of a natural optimality condition.

We define the crane transfer time as the time that crane takes to arrive at point B from point A. This transfer time plays a vital role in production scheduling. Because the crane transfer time determines the down time on the production floor. Longer the crane transfer time higher the production cost. The shorter crane transfer time can lower the production cost. Therefore, it is important to minimize the crane transfer time for any given operation. Here in this chapter we derive the time optimal control of the dynamic system developed in chapter three. We use Pontryagin’s minimum principle.
If the dynamic system is given as in equation 3-12, we ask for a control \( u \) that will transfer the state variable \( x(t_0) \) to \( x(T) \) in some minimum time \( T \). In fact, such problems were considered by Pontryagin in the year 1962. We apply his theory to our system and derive the control \( u \) in such a way that a given state variable arrives at the desired state in the minimum time. The rigorous proof of this theorem is discussed in [2]. In the following section necessary conditions for the maximum principle is briefly discussed.

4.2 Pontryagin's minimum principle

Given the dynamic system

\[
\frac{dx}{dt} = A \cdot x + B \cdot u
\]

Where \( u \) is a control input function, \( x \) a vector in \( \mathbb{R}^k \), \( u \) a vector in \( \mathbb{R}^m \), and \( \frac{dx}{dt} \) is a smooth vector-valued function of \( x \) and \( u \) with components in \( \mathbb{R}^m \). We assume that there is a unique smooth trajectory of 4-1 for each choice of initial condition \( x(0) = \chi_0 \) and input function \( u \) in a certain admissible class \( F \). Each choice of an admissible \( u \) alters the right side of 4-1. This different choice of \( u \) induces a different solution to emerge from \( \chi_0 \). For this reason, \( u \) is called a control function.

Our aim is to minimize the integral

\[
J = \int_{t_0}^{t_f} dt
\]

over some interval \([t_0, t_f]\) by a suitable choice of admissible control \( u \).

Each \( u \) in a certain admissible class \( F \) determines a solution of 4-1 for each given \( \chi_0 \) and therefore a corresponding value of the integral 4-2. Let \( u^* \) be a control in a certain admissible class \( F \) that minimizes the integral and denote the resulting trajectory by \( \chi^* \). In our case, we have
\[
\begin{bmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt} \\
\frac{dx_3}{dt}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix} [u]
\]

\begin{equation}
\frac{dx}{dt}
\end{equation}

Obviously, we have

\[
\frac{dx}{dt} = \begin{bmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt} \\
\frac{dx_3}{dt}
\end{bmatrix};
\quad A = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix};
\quad B = \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}
\text{ in equation 4-1.}
\]

A function \( H \) is defined by

\[
H = 1 + P^T (Ax + Bu)
\]

where \( P \) is the costate vector. \( H \) is the Hamilton function. The costate vector is the solution of the equation 4-4b given below. The necessary conditions for optimality are given as,

\[
\frac{dx}{dt} = -\frac{\partial H}{\partial P}
\]

\[
\frac{dP}{dt} = -\frac{\partial H}{\partial x}
\]

with the boundary conditions,

\[
\begin{align*}
x_1(t_0) &= 0.16 \text{ radian} \\
x(t_0) &= x_b \Rightarrow x_2(t_0) &= 0.0 \text{ radian / second} \\
x_3(t_0) &= 0.0 \text{ meter}
\end{align*}
\]
The control that may minimize the Hamilton function is given by,

\[ \frac{\partial H}{\partial u} = 0 \]

When the optimal control is given by

\[ u^* = - \text{sgn} \left\{ \frac{\partial H}{\partial u} \right\} \]

A control that may minimize the transition time from an initial state to a final state is known as the minimum-time control. The Pontryagin's maximum principle gives the necessary and sufficient conditions that should be satisfied by so called optimal control. We will see that our search for the control is easily accomplished. However, the real problem is to distinguish the optimal control from the admissible controls and obtain it. Such a control can be obtained by fulfilling the conditions laid down by Pontryagin. We show that this task is successfully accomplished.

### 4.3 Application of Pontryagin's principle

We define the Hamilton function for the equation 3-24 as follow,

\[ H = 1 + x_2 p_1 - x_1 p_2 + u (p_1 + p_3) \]

where \( p_1, \ p_2 \) and \( p_3 \) are costate variables. The control that may minimize the Hamilton function can be given by,
\[ u = - \text{sgn} (p_1 + p_3) \]

where \( p_1, p_2 \) and \( p_3 \) can be found out by solving the following differential equations simultaneously with Equation 4.7.

\[
\frac{dp_1}{dt} = - \frac{\partial H}{\partial x_1} = p_2
\]

\[
\frac{dp_2}{dt} = - \frac{\partial H}{\partial x_2} = - p_1
\]

\[
\frac{dp_3}{dt} = - \frac{\partial H}{\partial x_3} = 0
\]

Now if we assume that initial conditions for \( p_1, p_2 \) and \( p_3 \) are given as

\[
p_1(0) = a
\]
\[
p_2(0) = b
\]
\[
p_3(0) = c
\]

where \( a, b, c \) are nonzero constants.

We can write solution of equations 4-11 to 4-13 as

\[
p_1 = a \cos t
\]
\[
p_2 = b \sin t
\]
\[
p_3 = c
\]

We assume that our control is constrained in magnitude. We can draw the trajectories of this system using different initial conditions. One such trajectory is shown in Fig. 4.2. This trajectory shows how the two given state variables assume new positions with respect
to each other. The time variable is no more present in this illustration. Therefore, we can determine the path for the state variables that does not depend on time.

Figure 4-1. The bang-bang control $u$ is shown on the time axis for the system 3-4.
Figure 4-2. The state variable $x_1$ is projected on Y axis against the state variable $x_2$ on X axis. Only one such trajectory is shown in this figure.
4.4 Simulation and results

Now, we look at the simulation results. We notice that the load should approach the axis of origin from the initial state $x_0$ in some minimum time. This can be achieved only if we can force the state at time $t > t_0$ on to the optimal trajectory that passes through the origin.

![Figure 4-3](image.png)

Figure 4-3. The variable $x_3$ is shown on the time axis for the system 3-4.
Figure 4.2 shows only one trajectory with respect to chosen initial conditions a and b. It is a circle around the center (u,0). Similar circle can be obtained using control of the same amplitude but with negative value. We obtain a circle around the center (-u,0). If we prepare the chart of such trajectories using several values of a and b, then we get a map that can help us choose the control such that system can be brought to the desired final states in minimum time.

In Figure 4.1, we notice that the control u does not remain constant over the entire time. But we have employed a constant control. This means that it has to be piecewise constant and has to switch between two values +u and -u. This type of control is known as the bang-bang control. And is, as seen in the result, a very effective control strategy to drive the system to the desired state in the minimum time. We also notice that the minimum time is a function of initial states, the target set and the control input function.
5.1 Background

The difficulties involved in determining the optimal control solving set of the canonical equations lead us to the reduced form of crane model as shown in chapter three. This procedure is not general. The general analytical solutions are difficult to derive even if the system is linear [2]. In this chapter we turn to the general method of Riccati's equation to solve our system. We use the LQR algorithm from the software called MATLAB. The name MATLAB stands for matrix laboratory. MATLAB is a high-performance interactive software package for scientific and engineering numeric computation. MATLAB can carry out numerical analysis, matrix computation, and signal processing.

The system designed by modern control theory via state-space method enables the engineer to design such systems having optimal controls. These optimal controls are obtained with respect to given performance indexes. Also, modern control theory enables the designer to include initial conditions in the design. The state-space methods are suited for digital-computer computations. This relieves the designer of a burden of difficult computations otherwise necessary. Also, it is not necessary that the state variables represent physical quantities of the system. The variables that do not represent physical quantities may be chosen as state variables.

In this section, we show how the state-space method aids in obtaining a solution of the Riccati's equation. A solution of the Riccati's equation eventually leads us to an optimal control. The derived control $u$ forces the state variable $x(t_0)$ to $x(T)$ without spending too much of the control energy. In fact, given any initial state $x(t_0)$, the optimal control
problem is to find an allowable control vector \( u \) that transfers the state to the desired region of the state space for which the performance index is minimized. The system that optimizes the given performance index, by definition, is optimal. The important point is that the design prepared by using the quadratic performance index yields a stable control system. An optimal control based a quadratic performance index is a linear function of the state variables. This implies that we need to feedback all state variables. This requires that all such variables be available for feedback. If not all state variables are available for feedback, then we need to employ a state observer to estimate unmeasurable state variables and use estimated values to generate optimal control signals. This state variables are the combination of the measurements of the output and control variables.

As we show, Hamilton function \( H \) becomes a linear function of costate \( \lambda \). The canonical equations reduce to a system of homogeneous linear differential equations. The transversality condition on the costates at the terminal time \( T \) specifies that the costate \( \lambda \) at \( T \) be a linear function of the terminal state \( x(T) \). This conclusion leads us to the result that the optimal control \( u \) is the solution of the Riccati Equation.

5.2 Derivation of the Riccati Equation

If we are given the dynamic system,

\[
\frac{dx}{dt} = Ax + Bu
\]

Where \( x = \) state vector (n-vector)

\( u = \) control vector (r-vector)

\( A = n \times n \) constant matrix

\( B = n \times r \) constant matrix

In our case, we have
\[
\begin{bmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt} \\
\frac{dx_3}{dt}
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix} [u]
\]

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Obviously, we have

\[
\frac{dx}{dt} = \begin{bmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt} \\
\frac{dx_3}{dt}
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

in equation 5-1.

We want to find the control vector \( u \) such that a given performance index \( J \) is minimized. Where \( J \) is given as follow,

\[
J = \frac{1}{2} \int_{0}^{\infty} (x^T Q x + u^T R u) \, dt
\]

Where \( Q \) is a positive-definite or positive-semidefinite real symmetric matrix, \( R \) is a positive definite real symmetric matrix and \( u \) is unconstrained control vector. The optimal control has to minimize the performance index such that a system is stable.

Using Pontryagin's minimum principle we write Hamiltonian function as follow,
\[ H = \frac{1}{2} (x^T Q x + u^T R u) + \Lambda^T (A x + B u) \]

where \( \Lambda \) are the costate variables. Then necessary conditions for optimality are,

\[ \frac{\partial H}{\partial u} = 0 \]

\[ B^T \Lambda + R u = 0 \]

or we can say,

\[ u = -R^{-1} B^T \Lambda \]

The canonical equations are given as,

\[ \frac{\partial H}{\partial x} = -\frac{d\Lambda}{dt} \]

\[ = -(A^T \Lambda + Q x) \]

\[ \frac{\partial H}{\partial \Lambda} = \frac{dx}{dt} \]

\[ = A x + B u \]

If we assume \( \Lambda = P x \) where \( P \) is \( R^{n \times n} \) symmetric positive definite matrix then we can say,

\[ \frac{dx}{dt} = A x + B (-R^{-1} B^T \Lambda) \]

\[ = A x + B (-R^{-1} B^T P x) \]

\[ = A x - B R^{-1} B^T P x \]

While,
\[
\frac{d}{dt}(Px) = \frac{dP}{dt} x + P \frac{dx}{dt} \\
= -(A^T(Px) + Qx) \\
\frac{dP}{dt} x + P(A - BR^{-1}B^TP)x = -(A^TPx + Qx) \\
(\frac{dP}{dt} + PA + A^TP - PBR^{-1}B^TP + Q)x = 0
\]

This must hold for all \(x\),

\[
( PA + A^TP - PBR^{-1}B^TP + Q ) = \frac{dP}{dt}
\]

as \(t \to \infty\), \(-\frac{dP}{dt} \to 0\)

\[
( PA + A^TP - PBR^{-1}B^TP + Q ) = 0
\]

is known as the Riccati's equation. The control \(u\) is given by

\[
u = -R^{-1}B^TPx
\]

5.3 Simulation and results

The design steps may be stated as follow:
1. Solve equation 5-16, the reduced matrix Riccati equation for the matrix \(P\).
2. Substitute matrix \(P\) into equation 5-5. The resulting control \(u\) is the optimal one.

The LQR algorithm from MATLAB is a sophisticated algorithm that solves the Riccati equation. We provide matrices \(R\) and \(Q\) as the input data. The LQR algorithm solves the continuous time-linear-quadratic regulator problem and the associated Riccati equation.

\(K = \text{lqr}(A,B,Q,R)\) calculates the optimal feedback gain matrix \(K\) such that the feedback law \(u = -Kx\) minimizes the cost function \(J\) subject to the constraint equation 5-1. This algorithm returns the unique positive definite solution matrix to the Riccati equation.

We choose the matrices \(R\) and \(Q\) as follow
\[
R = \begin{bmatrix} 0.5 \end{bmatrix}
\]

\[
Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

This algorithm returned the positive definite matrix \( P \) as follow.

\[
P = \begin{bmatrix} 1.7287 & 0.1072 & -1.1020 \\ 0.1072 & 2.7055 & -0.8864 \\ -1.1020 & -0.8864 & 1.8091 \end{bmatrix}
\]

\[
u = -0.31 x_1 + 0.38 x_2 - 0.35 x_3
\]

The result obtained by employing this control on the system is shown in the figure 5-1.

Figure 5-1. The optimal control \( u \) is employed on an unstable system 3-4. The resultant system is stable.
The matrix P was substituted in equation 5-5. The resulting control $u$ was employed on the system given in equation 3-4. We simulated this system numerically using the Runge-Kutta method. The result shown in Figure 5-1 is for a time-invariant method. The same technique can be modified for the time variant system. The Figure 5-2 shows the result for the system given in the equation 2-36. This is a general and non-linear system of a hoist. In both the cases the optimal controls are able to dampen out the load oscillation as quickly as possible.

Figure 5-2. The control $u$ is employed on an unstable time variant system. The resultant system becomes a stable system.
In this appendix, we show how Lagrange's equation of motion is developed. In the analysis presented in this section, we use a system of particles. Nonetheless, the obtained Lagrange's equation can be applied to the dynamics of rigid bodies as well, since a rigid body can be considered as a collection of a large number of particles.

In the following discussion, we assume that the system consists of \( n \) particles. The displacement \( r \) of the \( i \)th particle is assumed to depend on a set of generalized coordinates \( q \).

\[
r^i = r^i(q_1, q_2, \ldots, q_n, t)
\]  \hspace{1cm} a-1

Differentiating equation (a-1) with respect to time using the chain rule of differentiation yields

\[
\frac{dr^i}{dt} = \frac{\partial r^i}{\partial q_1} \frac{dq_1}{dt} + \frac{\partial r^i}{\partial q_2} \frac{dq_2}{dt} + \cdots + \frac{\partial r^i}{\partial q_n} \frac{dq_n}{dt} + \frac{dr}{dt}
\]  \hspace{1cm} a-2

The virtual displacement \( \delta r \) can be expressed in terms of the virtual change of the generalized coordinates as

\[
\delta r = \frac{\partial r^i}{\partial q_1} \delta q_1 + \frac{\partial r^i}{\partial q_2} \delta q_2 + \cdots + \frac{\partial r^i}{\partial q_j} \delta q_j \quad (j = 1, 2, \ldots)
\]  \hspace{1cm} a-3

The dynamic equilibrium of the particle can be defined using Newton's second law as

\[
\frac{dp^i}{dt} = F^i
\]  \hspace{1cm} a-4
where $\frac{dp^i}{dt}$ is the vector of linear momentum and $F^i$ is the vector of the total forces acting on the particle $i$. The linear momentum vector $p^i$ is defined as

$$p^i = m^i \frac{dr^i}{dt}$$

where $m^i$ is the mass of the particle $i$ which is assumed to be constant. Consequently,

$$\frac{dp^i}{dt} = m^i \frac{d^2 r^i}{dt^2}$$

Substituting equation a-5 into equation a-4 yields

$$m^i \frac{d^2 r^i}{dt^2} - F^i = 0$$

Therefore

$$\left( m^i \frac{d^2 r^i}{dt^2} - F^i \right) \delta r^i = 0$$

By summing up these expressions for the particles and using a-3, one obtains

$$\sum_{i=1}^{n_p} \left( m^i \frac{d^2 r^i}{dt^2} - F^i \right)^T \sum_{j=1}^{n} \left( \frac{\partial r^i}{\partial q_j} \right) = 0$$
which can also be written as

\[
\sum_{i=1}^{n_p} \sum_{j=1}^{n_p} \left( m_i \frac{d^2 r_i}{dt^2} - F_i \right)^T \left( \frac{\partial r_i}{\partial q_j} \right) = 0
\]  

Define the generalized force \( \mathbf{Q} \) associated with the generalized coordinate \( q_j \) as

\[
\mathbf{Q} = \sum_{i=1}^{n_p} \mathbf{F}_i^T \frac{\partial r_i}{\partial q_j}
\]

One can show that

\[
\sum_{i=1}^{n_p} (m_i \frac{d^2 r_i}{dt^2} \frac{\partial r_i}{\partial q_j}) = \sum_{i=1}^{n_p} \left[ \frac{d}{dt} \left( m_i \frac{dr_i}{dt} \frac{\partial r_i}{\partial q_j} \right) - m_i \frac{dr_i}{dt} \frac{d}{dt} \left( \frac{\partial r_i}{\partial q_j} \right) \right]
\]

It is however, clear from equation a-2 that

\[
\frac{d}{dt} \left( \frac{\partial r_i}{\partial q_j} \right) = \sum_{k=1}^{n} \frac{\partial^2 r_t}{\partial q_j \partial q_k} \frac{\partial q_k}{\partial t} + \frac{\partial^2 r_t}{\partial q_j \partial t} - \frac{\partial^2 r_t}{\partial q_j \partial t}
\]

Substituting the results of equation a-13 into equation a-12 yields the Lagrangian equation of motion for the \( i \)th particle.

\[
\sum_{i=1}^{n_p} (m_i \frac{d^2 r_i}{dt^2} \frac{\partial r_i}{\partial q_j}) = \sum_{i=1}^{n_p} \left( \frac{\partial}{\partial q_j} \left( \frac{1}{2} m_i r_i^T r_i \right) - \frac{\partial}{\partial q_j} \left( \frac{1}{2} m_i r_i^T r_i \right) \right)
\]
\[ \sum_{i=1}^{n_p} \left( m_i \frac{d^2 r_i}{dt^2} \right) = \sum_{i=1}^{n_p} \frac{d}{dt} \left( \frac{\partial T_i}{\partial q} - \frac{\partial T_i}{\partial q} \right) \]

Where \( T^i \) is the kinetic energy of the particle \( i \) defined as

\[ T^i = \left( \frac{1}{2} m_i v_i^T v_i \right) \]

Therefore equation a-15 can be written in terms of kinetic energy of the system of particles as

\[ \sum_{i=1}^{n_p} \left( m_i \frac{d^2 r_i}{dt^2} \right) = \sum_{i=1}^{n_p} \frac{d}{dt} \left( \frac{\partial T_i}{\partial q} \right) - \frac{\partial T_i}{\partial q} \]

If we substitute equation a-8 and a-17 into a-7 the above equation will yield \( n \) equations defined as

\[ \sum_{i=1}^{n_p} \frac{d}{dt} \left( \frac{\partial T_i}{\partial q} \right) - \frac{\partial T_i}{\partial q} - Q = 0 \quad j=1,2,3,...,n \]

This equation is called Lagrange's equation of motion. Clearly, there are as many equations as the number of generalized coordinates.
References


2. Michael Athans, Peter L. Falb, Optimal control, McGRAW-HILL BOOK COMPANY, 1966


