Dynamic Analysis of Circular Plate on Elastic Foundation Using Modified Vlasov Model

A Thesis Submitted to
The Faculty of the College of Engineering and Technology
Ohio University

In Partial Fulfillment of the Requirements for the Degree of
Master of Science in Civil Engineering

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March 1992
This thesis has been approved
for the Department of Civil Engineering
and the College of
Engineering and Technology

Russ Professor of Civil Engineering

Dean of the College of
Engineering and Technology
Dedicated

to My Loving Daughter Harshi, Spouse Mangala

and

to the Memory of My Late Father Alwis Peiris
Acknowledgement

A very special tribute goes to my advisor Dr. Shad M. Sargand for his guidance, inspiration, encouragement and support without which this work would have never been completed.

Also, my special regards and appreciation go to Dr Glenn A. Hazen for his interest and suggestions in the course of this work.

I further express my sincere thanks to the other committee member Dr. Larry Snyder for helpful discussions and comments.

I express my deepest thank and utmost appreciation to my loving daughter Harshi and husband Mangala for their patience, spiritual support and self-sacrifice during the course of my graduate studies and research.

The financial support provided by the Ohio department of Transportation during this study is gratefully acknowledged.
Abstract

An analytical method to estimate the elastic properties and modal characteristics of a two layer plate-foundation system is presented. A modified two parameter Vlasov foundation model is developed for the dynamic analysis of a finite circular plate resting on an elastic foundation. The load is considered as a triangular impact which simulates the Falling Weight Deflectometer (FWD) blows. The decay rate-a parameter that governs the displacement profile in the soil foundation-is treated as a function of the characteristics of natural vibration of the system. The foundation layer is treated as a semi-infinite elastic continuum and hence the reflected waves in the horizontal directions are assumed as negligible. A rigorous theoretical basis is outlined in the formulation using a variational principle.

The solution to the resulting transcendental equations is found using the Bessel functions for various cases that arise due to the change of material properties and depths of layers. A fast, nondestructive and less costly solution, to the difficulties that stem in the selection of effective material properties due to inhomogeneity and variable thicknesses of soil media may be obtained by the back calculation of soil properties using this model. Therefore this versatile and relatively more accurate method can be very useful in the design of engineering structures such as highways and airport runways.

The application of the developed model in the impact loading analysis is illustrated with the numerical results obtained and the results were compared with the records of the FWD field test results. A parametric study is conducted for different elastic properties of the elastic foundation and thicknesses of the plate. The results are produced for the variation of properties and behavior during the course of impact loading and after the impact is diminished.
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\( A \)  
An integration constant

\( A_n & B_n \)  
Coefficients for the homogeneous solution

\( B_1, B_2, B_3 & B_4 \)  
Integration constants

\( c_1 & c_2 \)  
Material and geometric constants defined in eq. (2.8)

\( C \)  
A parameter of the modified Vlasov plate-foundation model

\( C_n \)  
Modal coefficients of the eigen functions, used to define the load distribution

\( D \)  
Flexural rigidity of the circular plate

\( E_0 \)  
Modulus for the state of Plane strain

\( E_2 \)  
Elastic modulus of middle, unbound, granular base layer

\( E_3 \)  
Subgrade modulus

\( E_s \)  
Young's modulus of the foundation

\( f \)  
Subscript to denote properties of the foundation

\( G \)  
Shear modulus of the foundation

\( h \)  
Thickness of the plate or pavement of the road

\( h_2 \)  
Thickness of the subgrade

\( H \)  
Thickness of the foundation

\( I_0 \)  
Zeroth-order, first kind modified Bessel function

\( J_0 \)  
Zeroth-order, first kind Bessel function

\( k \)  
Modulus of the foundation

\( k_1 \)  
A parameter related to the interaction mechanism of a foundation model as defined by eq. (2.2)

\( K \)  
A parameter of the modified Vlasov plate-foundation model

\( K_0 \)  
Zeroth-order second kind modified Bessel function
\( K \) Kinetic energy of the system
\( m_0 \) Mass density of the pavement as a homogeneous elastic plate
\( \bar{m}_0 \) Mass density of elastic foundation
\( M \) A model parameter related to the mass densities of the plate and foundation as defined in eq. (3.11)
\( M_1 \) Bending moment at the edge of the circular plate
\( p \) Foundation reaction, subscript to denote properties of the plate
\( P \) Compressive wave
\( P(r,t) \) Impulse loading function
\( q \) Intensity of applied distributed load
\( Q_1 \) Shear force acting at the edge of the circular plate inside the cylindrical section considered
\( Q_2 \) Shear force acting at the edge of the circular plate outside the cylindrical section considered
\( r \) radial coordinate
\( R \) Radius of the finite circular plate, Rayleigh wave
\( R_L \) Radius of the loading pad
\( s \) Subscript to denote soil properties
\( S \) Shear wave
\( t \) Time coordinate in space
\( T \) Membrane tension
\( \bar{U} \) Strain energy of the system
\( V_1, V_2, V_3 \) Elements of the first row of characteristic matrix
\( V_4 \)
\( w \) Static surface deflection of the plate
\( w_{0n} \) Initial displacement at the start of an momentary impulse
\( \dot{w}_{0n} \) Initial velocity at the start of an momentary impulse
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<td>$W_1$</td>
<td>Surface deflection as a function of time and radial distance</td>
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<td>$W_n$</td>
<td>$n$th modal shape</td>
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<tr>
<td>$\bar{W}$</td>
<td>Work potential of the externally applied loads</td>
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<td>$x$</td>
<td>Horizontal coordinate</td>
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<td>$X_1, X_2, X_3 &amp; X_4$</td>
<td>Elements of the second row of characteristic matrix</td>
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<td>$y$</td>
<td>Transverse coordinate</td>
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<td>$Y_0$</td>
<td>Zeroth-order second kind Bessel function</td>
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<td>$Y_1 &amp; Y_2$</td>
<td>Elements of the third row of characteristic matrix</td>
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<td>Depth measured from the surface of the foundation</td>
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<td>$\delta$</td>
<td>Variational operator</td>
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<td>Strains</td>
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<td>$\phi$</td>
<td>Displacement profile function</td>
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<td>$\gamma$</td>
<td>Dimensionless decay rate of the foundation</td>
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<td>$\lambda$</td>
<td>Eigen values of the system</td>
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<td>$\nu_0$</td>
<td>Poisson's ratio for the state of Plane strain</td>
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<tr>
<td>$\nu_s$</td>
<td>Poisson's ratio of the foundation</td>
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<tr>
<td>$\Pi$</td>
<td>Functional</td>
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<td>$\theta$</td>
<td>Arc angle</td>
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<td>$\sigma$</td>
<td>Stresses</td>
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<td>$\tau$</td>
<td>A spontaneous time during the impact</td>
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<td>$\Omega$</td>
<td>Volume of the elastic medium</td>
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<tr>
<td>$\omega$</td>
<td>Natural frequencies of the system</td>
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<tr>
<td>$\bar{\omega}$</td>
<td>Nondimensionalized natural frequency of the system</td>
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<td>$\nabla^2$</td>
<td>Laplacian operator</td>
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1 Introduction

1.1 Research Problem

The study of homogeneous foundation layers can be encountered in the problems such as characterization of existing or design of new highway and airport runway pavements and machine foundations etc. For the simplicity of analysis, the number of layers in these applications can be reduced to two elastic layers and hence can be analyzed as a composite plate on elastic foundation. In the actual practice, the plate represents the first layer made of finite, thin cement or asphalt concrete surface while the elastic foundation becomes the homogeneous second layer. In general, the analysis and the design of structures in foundation engineering applications are based on the assumed material properties of the elastic foundation.

The research has been systematically conducted in the search of accurate models to represent practical engineering problems of plates on elastic foundation. These investigations have been primarily focused in the development of various shear interaction mechanisms at the plate-foundation interface. In the case of elastodynamic analyses of plates on foundations, the deflection profile and hence the state of strain along the depth have been treated as invariable and determined priori irrespective of the characteristics of the system. These assumptions considerably reduce the complexity of the real problem but cannot be used accurately in the vibration analyses. Therefore the development of a new model without these assumptions is the basis for the present study.

Another aspect of the development of theoretical models of plate on elastic foundation is that they can be used to estimate the elastic properties of the foundations
based on nondestructive field test results. Previously, it has been a common practice, to conduct a series of field and laboratory tests to determine the soil parameters required for the analysis and design of foundations. These laborious, costly and destructive testing procedures produce the properties of individual constituents of the foundation at different depths below the ground surface. As commonly can be found, in the case of nonuniform test results and irregular thicknesses of different soil layers, it becomes very difficult to simulate an effective foundation model. Therefore it is more appropriate to investigate alternative methods to determine effective material properties of a proposed or existing site where the concept of plate on elastic foundation can be applied in the analysis. The records of existing nondestructive field tests such as Falling Weight Deflectometer (FWD), the Dynaflect and the Road Rator in the pavement characterization may be used in the application of such new methods.

1.2 Scope and Objectives

As explored in the preceding section, there exists a need for a more accurate model for elastodynamic analysis of pavements as plates on elastic foundations. Furthermore a more realistic method to estimate the effective in-situ material properties of an integrated plate-foundation system is demanding. As such the scope of this study lies in the development of an improved two-layer foundation model and a procedure to determine effective elastic constants of the foundation using the results of impact load tests on a paved highway or similar infrastructure. The paved top layer may be treated as a circular plate and the soil foundation is considered as an elastically compounded single layer which rests on a hard rock type media. The loading is considered as a pulse load (a short loading time) equivalent to a FWD blow acting on a small circular steel pad at the center of the circular
plate. The reflecting waves associated with the pulse load are neglected in the analytical formulations, assuming a semi infinite soil foundation domain.

Within the scope of a comprehensive study, to alleviate destructive and costly field testings followed by an estimation of soil properties empirically, the objectives of the present study are two folds,

(i) The development of an analytical procedure with a rigorous theoretical basis using the variational principle. In this process the decay rate of the soil is treated as a function of the dynamic characteristics which depend on parameters of the integrated system of the plate on foundation.

(ii) The development and implementation of an efficient iterative scheme for solving the governing transcendental equations emanating from various possible cases of solutions written in terms of Bessel functions.

The present analytical study is limited for a linear elastic, isotropic foundation media. The dissipation of energy due to material damping (attenuation) of the excited foundation is neglected since these effects are secondary compared to the geometric (radiation) damping. An explanation to this will be given in the literature review.

1.3 Thesis Overview

In Chapter 2, a detailed background of the problem is discussed by critically reviewing some of previous work in the modeling of the soil foundation including their merits as well as shortfalls. The available nondestructive techniques are also discussed in
this chapter. The theoretical development of the two parameter modified Vlasov model is presented in Chapter 3. The variational principle is used in this rigorous formulation process to derive the governing equations of motion. Then a close form solution is obtained using the Bessel and Trigonometric functions for all cases pertaining to possible material properties of the system.

Chapter 4 illustrates the implementation of the developed solution scheme by iterative means. The iterative procedure is coded in Fortran language and used to obtain numerical results. In Chapter 5, the results obtained using the modified Vlasov model for various foundation properties and plate thicknesses are presented. Also in this chapter, the effects of these parameters on the decay rate, nondimensionalized natural frequency, and the deflection of the surface before and after impulse are discussed. The Chapter 6 contains the conclusions and the future improvements to the developed analytical model and to its applications.
2 Literature Review

2.1 Overview

In this chapter a detailed survey of some of the work conducted by previous researchers related to the present study will be discussed. Firstly, various foundation models developed for the study of response of the surface due to applied loads is presented. In this case, schematic representations of previously proposed mechanisms, mathematical models and their analytical equations are reviewed. Next, the application of these models as well as others for the study of plates on elastic foundation are discussed. In this case structure is represented by two layers with distinct material properties. Finally, the techniques that are available for the nondestructive testing of pavements are surveyed. In this part of the study, major instrumental and analytical techniques are presented with a greater emphasis on the impact loading techniques.

2.2 Foundation Models

The response of a foundation at the contact area of a load line has been the prime interest in many foundation engineering applications. In the analysis of such problems since the behavior inside the foundation material is not important relatively simple mathematical models have been used in the past. Among them, one of the most widely used simple models has been based on the Winkler [1867] hypothesis. This model represents a continuous elastic foundation by a set of closely spaced independent linear springs, see Fig. 2.1. Since the foundation is characterized only by spring constant the Winkler type
model is also called one-parameter model. The vertical displacement at the foundation surface, \( w \) is related to the intensity of applied load (\( q \)) or to the foundation reaction (\( p \)) by

\[
q(x,y) = p(x,y) = k \cdot w(x,y)
\]  

(2.1)

where \( k \) is the modulus of the foundation and is defined as the reaction of the foundation per unit surface area per unit deflection.
It can be seen from eq. 2.1 that when the load is uniform, the deflection also becomes uniform. But this is not true for most of the foundation materials, except for a rigid stamp. In addition there is no deflection outside the loaded area which is also physically incorrect. Therefore this simple mathematical model does not represent actual behavior of many foundation materials which come across in engineering practice. In the search of a model that is physically more representative and mathematically simple, a continuous investigation has been done in the last five decades, mainly by introducing some kind of interaction between the independent springs in the Winkler model. This class of mathematical models have another constant parameter which characterizes the interaction implied between springs and hence are called two-parameter models. The condition of equilibrium in transverse (z) direction of these 3-Dimensional models yields the following load-displacement relation,

\[ q(x,y) = k w(x,y) - k_1 \nabla^2 w(x,y) \]  

(2.2)

where \( \nabla^2 \) is the Laplacian operator and \( k_1 \) is defined by the introduced interaction mechanism of the improved model.

Filonenko-Borodich [1940] developed a new foundation model by connecting the top ends of springs in Winkler model with an elastic membrane subjected to a constant tension, \( T \) as shown in Fig. 2.2 and eq. (2.2) becomes,

\[ q(x,y) = k w(x,y) - T \nabla^2 w(x,y) \]  

(2.3)
Hetényi [1950] presented his modified model in which each spring was connected to a plate that can deform only in *bending*, see Fig. 2.3. With this model he showed that eq. (2.2) has the following form,

\[ q(x,y) = k w(x,y) + D \nabla^4 w(x,y) \]  \hspace{1cm} (2.4)

where \( D \) is the flexural rigidity of the plate.
Pasternak [1954] improved the one parameter model by assuming the existence of shear interactions between spring elements. The shear effects were mathematical represented by a layer of incompressible vertical elements which can deform only by transverse shearing. This model is illustrated by Fig. 2.4 and the \( k_1 \) in eq. (2.2) is equal to the shear modulus \( (G) \) of the assumed isotropic, homogeneous foundation. In this case, considering the equilibrium of deformed shear elements in the \( zx- \) or \( zy- \) plane, he derived that

\[
q(x,y) = p(x,y) - G \nabla^2 w(x,y)
\]  

(2.5)

where \( p \) is the foundation pressure and is given by

\[
p(x,y) = k w(x,y)
\]  

(2.6)

Another approach of analyzing foundations subjected to surface loads was based on the theory of elasticity. In this case the foundation was considered as a semi-infinite elastic
continuum and the solution became mathematically more complex than a simple model. A number of solutions derived by using the theory of isotropic elastic solids were presented by Gorbunov-Pasadov [1949]. But it became obvious, as pointed out by Kerr [1964], that the elasticity approach was not suitable for materials such as soil since they do not characterize the behavior of a homogeneous isotropic medium. To overcome this problem, the continuum approach was simplified by imposing some constraints with respect to expected displacements and/or stresses.

Reissner [1958] used a foundation layer in which all in-plane stresses were negligible, i.e. $\sigma_x = \sigma_y = \tau_{xy} = 0$, and the horizontal displacements at the top and bottom surfaces of the foundation were zero. The analytical solution to the elastic foundation problem obtained from this method is given by,

$$q(x,y) - \frac{c_2}{4c_1} \nabla^2 q(x,y) = c_1 w(x,y) - c_2 \nabla^2 w(x,y)$$

(2.7)

in which the two constants relate to the material and geometry of the foundation by,

$$c_1 = \frac{E}{H} \quad \text{and} \quad c_2 = \frac{HG}{3}$$

(2.8)

where $E$, $G$ and $H$ are Young's modulus, shear modulus and depth of the foundation respectively. It can be seen that for a constant or linearly varying load, eq. (2.7) becomes identical to eq. (2.5) when the two constants are redefined, namely $c_1 = k$ and $c_2 = G$. But this model gave physically unrealistic results due to the fact that its predicted shear stresses in the $zx$- and $zy$-planes were independent of the $z$-coordinate. Therefore similar to Winkler type simple models, this model may be applied only to study the response near loading contact area and not to study stresses inside the foundation.
Vlasov [1960] adopted the isotropic elastic continuum approach and derived a two parameter model based on the variational principle. In his method the foundation was treated as an elastic layer and the constraints were imposed by restricting the deflection within the foundation to an appropriate shape. The two parameter Vlasov model accounts for complete shear interaction.

\[ p(x,y) = kw(x,y) - c\nabla^2w(x,y) \quad (2.9) \]

\[ w(r,z) = w(r) \phi(z) \quad (2.10) \]

\[ \phi(z) = \frac{\text{Sinh } \gamma(1-z/H)}{\text{Sinh } \gamma} \quad (2.11) \]

### 2.3 Modeling of a Plate on Elastic Foundation

The application of the Winkler model to solve the problems of plate on elastic foundation has been the most popular method in the past. But it can be seen even in the recent analytical studies, see Celep [1988] and Zheng and Zhou [1988]. A detail study of the theory of plates on elastic foundation based on the Winkler model can be seen in Timoshenko [1959].

The governing equation of bending of a thin plate under a distributed lateral pressure is well established and given by,

\[ D\nabla^4w(x,y) = q(x,y) - p(x,y) \quad (2.12) \]
If the plate is resting on a *Winkler-type foundation* this equation can be rewritten after substituting for \( p(x,y) \) from eq. (2.1). The governing equation of a plate on *Winkler-type foundation* becomes,

\[
D \nabla^4 w(x,y) = q(x,y) - kw(x,y)
\]  

(2.13)

![Plate Resting on Pasternak Foundation Model](image)

**Fig. 2.5 Plate Resting on Pasternak Foundation Model**

For a plate resting on the *Pasternak-type foundation* as shown in Fig. 2.5, the eq. (2.12) can be written using eq. (2.5) as

\[
D \nabla^4 w(x,y) - G \nabla^2 w(x,y) + kw(x,y) = q(x,y)
\]

(2.14)

The differential equation of a plate on an elastic foundation, identical to eq. (2.14), has also been derived by Yu [1957] using a generalized foundation model. The generalized foundation model is developed by assuming, in addition to the foundation reaction \( p(x,y) \) proportional to the lateral deflection (the Winkler hypothesis), that there is also a moment \( m \)
(x,y) exerted by the foundation which is proportional to the slope at the surface (see Pasternak [1937], Galletly [1959] and Kerr [1964]).

A two parameter Vlasov model has been developed using the variational principle for the static and dynamic analysis of a plate on elastic foundation, Vlasov [1960]. This model has been later used by Sargand et. al. [1987] to analyze dynamics of an axisymmetric plate on elastic foundation. In this application the decay rate, $\gamma$ (see eq. 2.11) has been kept constant. But the decay rate is a function of system parameters especially when the plate-foundation system is subjected to dynamic excitations.

2.4 Nondestructive Techniques for Testing and Analysis of Foundations

The measurement of recoverable resilient deflection of a flexible pavement structure can provide valuable information about the structural capacity of the pavement. Routine testing of highways and airfields type structures is necessary to study the performance of these pavements specially because of the seasonal effects the strength of the pavement structure is changed as shown in Fig 2.6. A significant progress has been made in structural testing and evaluation methodology in the last few decades both in analysis techniques and in field equipments.
A description of nondestructive testing techniques which are currently being used is given by Moore et. al. [1978]. These methods have been identified and divided into four categories, i.e., response measured with static loading, steady state dynamic loading, Impact loading and wave propagation techniques.

The static deflection techniques were the first to be used in the pavement performance evaluations. In this testing method, a loaded truck (for highway pavements) or an aircraft (for airfield pavement) is moved away from the test point at a creep (slow) speed and the maximum rebound deflection was measured. The response was measured using an instrument called Benkelmen beam (BB) which operates on the simple lever arm principle. When the load moves an eight feet long probe beam pivots around a point on a reference beam that rests on the pavement beyond the area of influence and other end of the probe
beam connects to a dial. Sometimes the load is moved past the test point at a creep speed (quasi-dynamic measurements) where BB is located. *Travelling Deflectometer* and *LaCroix Deflectograph* which also operate on the BB principle are two other instruments which can be used with a slow steadily moving load. With this type of equipment, probes are also moved or shifted with or ahead of truck and hence they can be used for multiple individual measurements in one loading cycle.

The *steady state dynamic* (SSD) *loading* equipments induce a steady state sinusoidal vibration in the pavement and the deflections are usually measured with inertial motion sensors such as velocity transducers (geophones) or accelerometers. For a pure sinusoidal motion at any fixed frequency above the sensor's resonance, the output of the sensor is directly proportional to deflection. If the motion characterizes a range of frequencies an electronic integrator is employed with the motion sensor to measure the deflections over the frequency range. The dynamic deflections at any specific driving frequency are approximately proportional to the amplitude of the applied load and this is similar to the static loading case discussed before. But this proportionality factor (or dynamic stiffness, i.e. amplitude of dynamic force required to produce unit amplitude in deflection on the foundation surface) is frequency dependent. Thus when the resonances in force generator cause forces to be present at frequencies different to the driving frequency, errors in interpretation can result. A comprehensive analysis of the SSD testing methods has been presented by Moore et. al. [1978].

The most commonly used testing equipments in steady state dynamic loading techniques are the *Dynaflect* and *Road Rater* (RR). The force generator in the *Dynaflect* consists of a counter rotating masses to apply a peak to peak dynamic load of 1000 lbs at a fixed frequency of 8 Hz. The force generator of *RR* uses a hydraulic actuated vibrator to produce various magnitudes of dynamic forces at a range of driving frequencies (5-100hz).
Before applying dynamic forces with these equipments an initial static load of magnitude less than half the peak-to-peak dynamic force, is exerted on the pavement to insure continuous contact of the vibrator with the pavement. But these periodic loading devices can produce erroneous results due to the spurious resonances in the pavement strata, specially when the operating frequency is close to the natural frequency of the structure, Sebaaly et. al [1985] and Mamlouk and Davies [1984].

In the impact loading response technique, a transient load is delivered on the pavement by dropping a weight (1 kN) from a preselected height onto an impact plate. By varying the drop height the impulse load can be varied. The impact plate consists of a spring-damping system and is designed to impart a suitable force impulse which is closely approximates a half-sine wave. The 30 cm diameter loading platen is normally rubber coated to distribute the impulse load uniformly over it. The response is measured with inertial motion sensors similar to those used in steady state deflection techniques.

The impact testing techniques require only few seconds to measure an impulse response. The data obtained in such a small duration contains the same information that contained in a number of SSD tests. The impact loading devices associated with surface deflection measurements can simulate the traffic load intensities and durations with greater fidelity and less labor than static and SSD loading devices, Hoffmann and Thompson [1982 a]. The impact loading techniques have become more popular during the last decade in the USA, and most widely used impact loading equipment has been the falling weight deflectometer (FWD), see Fig. 2.7. Based on experimental studies by Claessen et. al. [1976] in Holland and Hoffman and Thompson [1982 a] concluded that the FWD is the most realistic and suitable nondestructive device compared to static and SSD devices.
The interpretation of FWD response data for back calculation of foundation modulus is performed by different ways. Claessen et. al. [1976] and Koole et. al. [1979] started with three-layer system and assumed values for Poisson's ratios of all layers and the thickness of the base layer since they have little effect on the response. The elastic modulus ($E_2$) of middle (unbound, granular base) layer is given an effective value which is related to the subgrade modulus ($E_3$) and its thickness ($h_2$) as given by eq. (2.15). Based on a graphical method finally two unknowns ($E_3$ and thickness or modulus of asphalt layer) are found using the measured deflection bowl.

\[
E_2 = 0.2h_2^{4.5}E_3 \tag{2.15}
\]

where the units of $h_2$ is in millimeters.
Using a stress-dependent finite element model for axisymmetric three-layer system, Hoffman and Thompson [1982 b] estimated the subgrade nonlinear resilient modulus (i.e. the ratio of the deviator stress to the recoverable strain). For this finite element method (FEM), only the deflection bowl measured from FWD data were used and no inertia effects were considered. A dynamic analysis has been performed on a layered continuum by Sebaaly et. al [1985] based on Fourier analysis for a periodic FWD impulses. They assumed thicknesses and material properties of all layers and calculated the deflection response at the center of the impact plate. Based on their findings they concluded that inertia of the pavement is instrumental in the interpretation of response data of the pavement. But their predicted deflection pattern shows a constant phase lag upto the peak deflection as shown in Fig. 2.8. Hence contrary to experimental data, their analysis showed a little difference in the peak acceleration between the falling mass itself and pavement.

In the steady-state dynamic and impulse analyses, usually, the damping of the structure should also be considered. The granular materials (sand, etc.) exhibit hysteretic damping behavior and the typical values of damping ratio approximately 5%, Richart et. al.
The clay type materials inherit even smaller values. The dissipation of energy due to material damping of the excited foundation is neglected. This is due to the fact that the major component of energy dissipation in a foundation structure results from radiation (geometric) damping, i.e., the energy dissipation from the source of excitation to the far field. Hence as explained by Mamlouk and Davies [1984] and Sebaaly et. al. [1985] the energy dissipation due to material damping is secondary importance and may be neglected in practice.

<table>
<thead>
<tr>
<th>Wave Type</th>
<th>Percentage of Energy Dissipated</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>7</td>
</tr>
<tr>
<td>$S$</td>
<td>26</td>
</tr>
<tr>
<td>$R$</td>
<td>67</td>
</tr>
</tbody>
</table>

Table 2.1 Dissipation of Energy in an Elastic Half-Space

The response measurement techniques discussed above characterize the entire pavement structure and do not provide direct evaluation of individual pavement layers. On the other hand the Wave propagation technique offers promise of providing elastic moduli of individual layers. The wave propagation method involves the measurement of the velocity and wave length of the surface propagating away from the vibratory source. When a pavement surface is subjected to vibration there are three types of waves transmitted, viz., compression ($P$), Shear ($S$), and Rayleigh ($R$) waves. The $R$ wave is a surface wave and the $P$ and $S$ waves are body waves.
The dissipation of energy input from a vibrator on a semi-infinite half-space has been found to be as shown in the Table 2.1 as given by Miller and Pursey [1955] and Moore et. al. [1978]. In the wave propagation techniques, $R$ waves are the type of waves measured since they become dominant with less rate of attenuation compared to $P$ and $S$ waves. The interpretation of wave velocity measurements to obtain material elastic constants remains unresolved. A suitable interpretation scheme for multi-layered pavement is yet to be devised and hence this method is not very popular.
3 Theoretical Development

3.1 Overview

In this chapter, a modified two-parameter Vlasov model is derived for the analysis of a pavement structure subjected to impact loading. The formulation of the problem is based on a finite axisymmetric plate resting on an elastic foundation. The Hamilton's principle is applied in the development of motion equations for the assumed deformable body subjected to dynamic forces. Then the free vibration equations derived in the form of forth- and second-order differential equations are solved using Bessel and trigonometric functions for all possible cases that result in after substituting various suitable material properties. Finally the pulse load is introduced into the analysis and the deflections are determined using a modal superposition scheme. The impact load is assumed to be distributed evenly over the impact plate and is taken as a series of eigen functions of the modal values of the foundation surface deflection.

3.2 Derivation of Governing Equations

Consider a flexible pavement structure of a highway or an aircraft landing ground as an axisymmetric compound elastic plate resting on an elastic foundation which is underlain by a rigid bedrock at a finite depth. A schematic view of the assumed finite circular plate and the elastic foundation is shown in Fig. 3.1. The thicknesses of the plate and foundation are denoted by $h$ and $H$ respectively. The radius of the finite circular plate is taken as $R$. The usual assumptions of material linearity and isotropy are invoked to represent each layer as a homogeneous composite medium. Each layer is characterized by
two elastic constants \((E\) and \(v)\) and the mass density \((m)\). A perfect bonding (no slip) is assumed at the interface between the plate and foundation. The internal energy dissipation due to material damping (attenuation) of the foundation is neglected. The incoming surface waves are neglected by assuming the elastic foundation to be infinite in the horizontal direction.

![Diagram of the Circular Plate-Soil Foundation Model](image)

**Fig. 3.1** The Circular Plate-Soil Foundation Model

The deflections in the horizontal directions are considered to be negligible, Winkler [1867] and hence in this axisymmetric analysis the displacement in \(r\)-direction is taken as equal to zero. It is also assumed that the vertical displacement in the foundation at time \(t\), inside a point at \((r,z)\) can be written as

\[
w(r,z,t) = w_1(r,t) \phi(z)
\]  

(3.1)
where $w_1(r,t)$ is the surface deflection at time $t$ at an $r$ distance away from the center of the plate and $\phi(z)$ is a function which describes the pattern of decrease of $w$ along the depth of the foundation.

The function, $\phi(z)$, is chosen to satisfy the top and bottom surface boundary conditions, i.e. $\phi(0) = 1$ and $\phi(H) = 0$, see Fig. 3.1. Therefore the deflection at the bottom of the foundation layer is zero and as a result of this assumption, the reflection of waves in the vertical direction is diminished.

### 3.2.1 Variational Principle

Let the flexural rigidity of the circular plate be $D$ and the applied load on the plate be $q(r,t)$. Now for the axisymmetric elastic body shown in Fig. 3.1 considering individual elements of arc angle $d\theta$ at a radius $r$ and depth $z$ in the foundation and plate separately, the total strain energy can be written with respect to polar coordinates $(r,\theta,z)$ as

$$
\overline{U} = \frac{1}{2} \int_{\Omega_f} \left( \sigma_r e_r + \sigma_\theta e_\theta + \sigma_z e_z + \tau_{r\theta} \gamma_{r\theta} + \tau_{r\theta} \gamma_{r\theta} + \tau_{z\theta} \gamma_{z\theta} \right) r dr d\theta dz 
+ \frac{D}{2} \left[ \left( \frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r} \frac{\partial w_1}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w_1}{\partial \theta^2} \right)^2 \right]_{\Omega_p} 
- 2(1-\nu_p) \left[ \frac{\partial^2 w_1}{\partial r^2} \left( \frac{1}{r} \frac{\partial w_1}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w_1}{\partial \theta^2} \right) \right]_{\Omega_p} 
\left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial w_1}{\partial \theta} \right) \right]_{\Omega_p}^2 r dr d\theta
$$

where $\Omega$ and $S$ are respectively, the volume and the surface area of the layer considered. The subscripts $p$ and $f$ denote the plate and elastic foundation, respectively. Since there is
no \theta dependency the stress components \sigma_\theta, \tau_{\theta r} and \tau_{\theta z} become zero. Furthermore since
the displacement in the r-direction is assumed to be zero, \varepsilon_r = 0.

Since the present analysis considers the dynamic behavior of the pavement
structure, the kinetic energy of the system also need to be included. The total kinetic energy
of the system is given by

\[ K = \frac{1}{2} \int_{\Omega_f} \rho \phi^2 \left( \frac{\partial w_1}{\partial t} \right)^2 r \, dr \, d\theta \, dz + \frac{1}{2} \int_{\Omega_p} \rho_0 \phi^2 \left( \frac{\partial w_1}{\partial t} \right)^2 r \, dr \, d\theta \, dz \]

(3.3)

where \( m_0 \) and \( \bar{m}_0 \) the mass density of the composite material of circular plate and the elastic
foundation, respectively.

The work potential of externally applied distributed axisymmetric surface loads can
be expressed by

\[ W = - \int_{\Omega_p} q(r,t) w_1(r,t) \, r \, dr \, d\theta \]

(3.4)

The total potential energy of the system is defined by a function of deflection
\( w_1(r,t) \) and \( \phi(z) \) and

\[ \Pi = \bar{U} + \bar{W} - \bar{K} \]

(3.5)

and therefore is a function of surface deflection \( w_1(r,t) \) and \( \phi(z) \).
Now substituting for non zero stress ($\sigma_z$ and $\tau_{xz}$) and strain ($e_z$ and $\gamma_{xz}$) components in eq. (3.2) and substituting into eq. (3.5) from eqs. (3.2) to (3.4), the total potential energy can be expressed in the expanded form,

$$\Pi(w_1, \phi) = \frac{1}{2} \int_{\Omega_f} \left[ \frac{E_0}{(1-\nu_0)^2} w_1^2 \left( \frac{d\phi}{dz} \right)^2 + \frac{E_0}{2(1+\nu_0)} \phi^2 \left( \frac{\partial w_1}{\partial r} \right)^2 \right] r dr d\theta dz$$

$$+ \frac{D}{2} \int_{S_p} \left[ (\nabla^2 w_1)^2 - 2(1-\nu_p) \frac{\partial^2 w_1}{\partial r^2} \frac{1}{r} \frac{\partial w_1}{\partial r} \right] r dr d\theta$$

$$- \int_{S_p} q(r,t)w_1(r,t) r dr d\theta$$

$$- \frac{1}{2} \int_{\Omega_f} m_0 \phi^2 \left( \frac{\partial w_1}{\partial t} \right)^2 r dr d\theta dz - \frac{1}{2} \int_{\Omega_p} m_0 \left( \frac{\partial w_1}{\partial t} \right)^2 r dr d\theta dz$$

where $E_0 = \frac{E_s}{(1-\nu_s)^2}$ and $\nu_0 = \frac{\nu_s}{1-\nu_s}$. The constants $E_s$ and $\nu_s$ are the Young's modulus and the Poisson's ratio of the foundation, respectively. $\nabla^2$ is the Laplacian operator written in terms of the coordinate $r$.

The Hamilton's principle for a continuous system of elastic body can be expressed as

$$\delta \int_{t_1}^{t_2} \Pi dt = 0$$

(3.7)

where $\delta$ denotes the variation with respect to variables of the function $\Pi$ and when the motion of the elastic system satisfies the boundary conditions at time $t_1$ and $t_2$. The possible
path of the motion is given by an extremum of $\Pi$ (a minimum or a maximum or an inflection) which is determined from the eq. (3.7).

Now the variation of eq. (3.7) with respect to $w_1$ and $\phi$ leads to the following two partial differential equations, respectively:

$$
\frac{E_0}{(1-\nu_0)^2} \int_0^H (\frac{d\phi}{dz})^2 dz \cdot w_1 - \frac{E_0}{2(1+\nu_0)} \int_0^H \phi^2 dz \nabla^2 w_1 + D \nabla^4 w_1
$$

$$
+ \left( \frac{\bar{m}}{m_0} \int_0^H \phi^2 dz + m_0 h \right) \frac{\partial^2 w_1}{\partial t^2} = q
$$

Eq. (3.8) represents the governing equation of motion of the circular plate on elastic foundation. Therefore when the term containing the flexural rigidity of the plate is removed it gives the equation of the surface of the foundation, i.e. the foundation outside the plate. The displacement of the entire foundation is governed by the eq. (3.9)
3.2.2 Free Vibration of the Plate

The resonance frequencies and the mode shapes can be found by solving the eq. (3.8) without the applied load, \( q(r,t) \). Then eq. (3.8) can be reduced to

\[
\nabla^4 w_1 - 2C \nabla^2 w_1 + Kw_1 = -M \frac{\partial^2 w_1}{\partial t^2}
\]

(3.10)

in which the two parameters of the plate-foundation system \( C \) and \( K \) can be written in the elastic material constants and the displacement profile function \( \phi(z) \). The constant \( M \) is related to the plate and foundation mass densities, thicknesses and the function \( \phi(z) \). These parameters are defined by

\[
C = \frac{E_0}{4D(1+\nu)} \int_0^H \phi^2 \, dz
\]

\[
K = \frac{E_0}{D(1-\nu^2)} \int_0^H (\frac{d\phi}{dz})^2 \, dz
\]

\[
M = \frac{1}{D} \left[ m_0 h + \bar{m}_0 \int_0^H \phi^2 \, dz \right]
\]

(3.11)

Now for a given system of plate and foundation, if \( \phi(z) \) is known the parameters \( C \), \( K \) and \( M \) can be calculated. Hence a solution to the partial differential equation can be found. This was the case when the function \( \phi(z) \) is independent of modal parameters (frequency and surface deflection) as assumed in the Vlasov model. But in the present model the function \( \phi(z) \) is derived as a function of modal parameters in addition to the geometric and elastic properties.
Now to find a solution to the eq. (3.9), it can be rewritten as a second-order ordinary differential equation as shown below,

\[ \phi'' - \left( \frac{\alpha}{H} \right)^2 \phi = 0 \]  

(3.12)

where prime (') indicates the derivative with respect to \( z \)-coordinate. The coefficient \( \gamma \) is defined in the following form

\[
\left( \frac{\alpha}{H} \right)^2 = \frac{\int_{0}^{\infty} E_0 \left( \frac{\partial w}{\partial r} \right)^2 - \bar{m} \left( \frac{\partial w}{\partial r} \right)^2 r \, dr}{\int_{0}^{\infty} \frac{E_0 \omega_1^2}{(1+v_0)} r \, dr}
\]  

(3.13)

A solution to the second-order differential equation (3.12) can be found in several ways. In this case, \( \phi(z) \) is assumed in terms of hyperbolic functions of nondimensionalized variables and is written in the following form:

\[
\phi(z) = \frac{\text{Sinh} \, \gamma (1-z/H)}{\text{Sinh} \, \gamma}
\]  

(3.14)

It can be seen from eq. (3.14) that when the coefficient \( \gamma \) varies, the function \( \phi \) has different shapes within the limits of 1 (at \( z = 0 \)) and 0 (at \( z = H \)). As a consequence of that the displacement profile along the depth of the foundation also varies within the limiting values of \( w_1 \) and 0. Therefore \( \gamma \) is known as the decay rate of the foundation and in this case it has been taken as a nondimensionalized quantity.
3.2.3 Free Vibration of the Foundation Outside the Plate

In order to analyze the foundation away from the plate \((r > R)\), a solution to eq (3.8) is found dropping the term with plate flexural rigidity. Hence the governing equation of the foundation beyond the circular plate can be written in terms of the deflection of the surface outside the plate, \(w_2(r,t)\) such that

\[-c \nabla^2 w_2 + k w_2 = -\bar{m} \frac{\partial^2 w_2}{\partial r^2}\]  

(3.15)

where \(c = 2D.C\), \(k = D.K\), and \(\bar{m} = \bar{m}_0 \int_0^H 2 \, dz\)  

(3.16)

3.3 Solutions to the Governing Equations of the Plate and Foundation

3.3.1 Deflection of the Plate and Foundation Under the Plate

To find a solution to the eq. (3.10), the plate deflection is written after the separation of variables in the following form

\[w_1(r,t) = W(r).T(t)\]  

(3.17)

Then the eq. (3.10) can be rewritten as two independent equations such that

\[\frac{\nabla^4 W(r) - 2C \nabla^2 W(r)}{W(r)} = -\frac{KT + MT^\prime}{T} = \lambda^4\]  

(3.18)
in which \( \lambda \) is a parameter which will be determined later in terms of the natural frequency of vibration and foundation parameters.

Eqs. (3.18) can be separated into the following form of two equations:

\[
T'' + \omega^2 T = 0 \tag{3.19}
\]

and

\[
\nabla^4 W - 2C \nabla^2 W + (K - M \omega^2) W = 0 \tag{3.20}
\]

where \( \omega^2 = \frac{(K + \lambda^4)}{M} \)

Now solution to the time function \( T(t) \) in eq. (3.19) can be found after considering the limiting condition when \( t \to \infty \), \( w_1(r,t) = 0 \) as

\[
T = Ae^{-i\omega t} \tag{3.21}
\]

where \( \omega \) is the natural frequency of the system and \( A \) is an integration constant.

The solution to eq. (3.20) gives mode shapes of the free vibration of the plate on elastic foundation. By putting \( \nabla^2 W = nW \) this forth-order differential equation can be reduced to two second-order differential equations as follows:

\[
\nabla^2 W + aW = 0 \tag{3.22}
\]

\[
\nabla^2 W - bW = 0
\]
in which \( a = \sqrt{C^2 + M\omega^2 - K} - C \) 
\( b = \sqrt{C^2 + M\omega^2 - K} + C \)

Now a solution to Eqs. (3.22) can be written in terms of Bessel functions. The solution depends on whether \( a \) and \( b \) are real, imaginary, positive or negative values. There are four different cases which are based on the values of \( C,K,M,\overline{m} \) and \( \omega \) and solutions to these cases are given below.

**Case 1:** \( M\omega^2 - K > 0 \)

\[
w_1(r,t) = \left[ B_1 J_0(\sqrt{a}r) + B_2 I_0(\sqrt{b}r) \right] e^{i\omega t}
\]

where \( J_0 \) and \( I_0 \) are the zeroth-order first kind and zeroth-order first kind of modified Bessel functions respectively. \( B_1 \) and \( B_2 \) are integration constants yet to be determined based on the boundary conditions at the edge of the finite circular plate.

**Case 2:** \( M\omega^2 - K = 0 \)

\[
w_1(r,t) = \left[ B_1 + B_2 I_0(\sqrt{b}r) \right] e^{i\omega t}
\]

**Case 3:** \( M\omega^2 - K < 0 \) and \( C^2 + M\omega^2 - K \geq 0 \)

\[
w_1(r,t) = \left[ B_1 J_0(\sqrt{|a|}r) + B_2 I_0(\sqrt{b}r) \right] e^{i\omega t}
\]
Case 4: $M\omega^2 - K < 0$ and $C^2 + M\omega^2 - K < 0$

For this case since both $a$ and $b$ becomes imaginary, the solution can be expressed in terms of trigonometric functions,

$$w_1(r,t) = \left[ B_1 \sum_{n=0}^{\infty} \left( \frac{r\beta}{2} \right)^{2n} \cos 2n\phi \left( \frac{r\beta}{2} \right)^{2n} \sin 2n\phi \right] e^{-i\omega t} \quad (3.27)$$

where $\beta = \sqrt{K - M\omega^2}$, $\phi = \frac{1}{2} \tan^{-1} \left( \sqrt{K - C^2 - M\omega^2} / C \right)$

3.3.2 Solution to the Foundation Outside the Plate

The deflection of the foundation outside the plate can be found by solving eq. (3.15) in terms of Bessel functions. In this case let $y = \frac{\bar{m}\omega^2 - k}{c}$ and depending on the sign of $y$ there can be three different solutions as given below:

Case A: $\bar{m}\omega^2 - k > 0$

$$w_2(r,t) = \left[ B_3 J_0 \left( \sqrt{y} \ r \right) + B_4 Y_0 \left( \sqrt{y} \ r \right) \right] e^{-i\omega t} \quad (3.28)$$

where $Y_0$ is the zeroth-order second kind Bessel function, $B_3$ and $B_4$ are the integration constants.

Case B: $\bar{m}\omega^2 - k = 0$

$$w_2(r,t) = B_3 e^{-i\omega t} \quad (3.29)$$
Case C: \( \overline{m} \omega^2 - k < 0 \)

\[
    w_2(r,t) = B_3 K_0 \left( \sqrt{|y|} r \right) e^{-i\omega t} \tag{3.30}
\]

where \( K_0 \) is the zeroth-order second kind of modified Bessel function. In these three cases, the time function is assumed to be the same as that for region \( r < R \). The solution \( w_2 \) satisfies the limiting boundary condition when \( r \) reaches infinity, deflection becomes zero. It can be seen from the above three equations that for Case A, two integration constants are required to determine a solution for \( w_2 \).

### 3.4 Boundary Conditions and Determination of Integration Constants

There are twelve possible cases depending upon the magnitudes of parameters \( C, K, M \) and \( \omega \). The number of integration constants to be determined is either three or four. The actual case of motion of the existing system is determined by these constants. Now to find a solution to the governing equations of the developed model it is required to calculate the unknown constants based on the boundary conditions. The assumed displacement functions as a solution to the problem automatically satisfy the conditions at the center of the plate and initial conditions. In this case the following four boundary conditions are considered at a cylindrical section along the edge of the circular plate. The last boundary condition is used only when the Case A is applied in the solution equation.
At any time $t$, when the radius $r = R$,

$$w_1(R,t) = w_2(R,t)$$

$$Q_1(R,t) = Q_2(R,t)$$

$$M_1(R,t) = 0$$

$$\frac{\partial w_1(R,t)}{\partial r} = \frac{\partial w_2(R,t)}{\partial r} \quad \text{(additional b.c. for Case A)}$$

where $Q_1$ is the shear force acting at the edge of the circular plate inside the cylindrical section considered ($r = R^-$) and $Q_2$ is the shear force at the same section when the foundation outside the plate is considered ($r = R^+$). $M_1$ is the bending moment at the edge.
of the circular plate, see Fig. 3.2. The equations of bending moment and shear force for the plate and foundation system are given in Appendix A.

The displacement solutions from any possible case can be substituted into eqs. (3.31). Then the resulting determinant yields the characteristic equation for any possible case:

\[
\begin{vmatrix}
V_1 & V_2 & V_3 & V_4 \\
X_1 & X_2 & X_3 & X_4 \\
Y_1 & Y_2 & 0 & 0 \\
Z_1 & Z_2 & Z_3 & Z_4
\end{vmatrix} = 0
\] (3.32)

The elements of this determinant can be evaluated for all cases using their displacement solutions. All these elements can be expressed as follows:

**Case 1:**

\[
V_1 = J_0 \left( \sqrt{aR} \right)
\]

\[
X_1 = J_0'' \left( \sqrt{aR} \right) + \frac{1}{R} J_0'' \left( \sqrt{aR} \right) - \left( \frac{1}{R^2} + \frac{c}{D} \right) J_0' \left( \sqrt{aR} \right)
\]

\[
Y_1 = J_0'' \left( \sqrt{aR} \right) + \frac{\nu}{R} J_0' \left( \sqrt{aR} \right)
\]

\[
Z_1 = J_0' \left( \sqrt{aR} \right)
\]

\[
V_2 = I_0 \left( \sqrt{bR} \right)
\]

\[
X_2 = I_0'' \left( \sqrt{bR} \right) + \frac{1}{R} I_0'' \left( \sqrt{bR} \right) - \left( \frac{1}{R^2} + \frac{c}{D} \right) I_0' \left( \sqrt{bR} \right)
\]

\[
Y_2 = I_0'' \left( \sqrt{bR} \right) + \frac{\nu}{R} I_0' \left( \sqrt{bR} \right)
\]

\[
Z_2 = I_0' \left( \sqrt{bR} \right)
\]
Case 2:

\[ V_1 = 1 \]
\[ X_1 = 0 \]
\[ Y_1 = 0 \]
\[ Z_1 = 0 \]
\[ V_2 = I_0 \left( \sqrt{bR} \right) \]
\[ X_2 = I_0'' \left( \sqrt{bR} \right) + \frac{1}{R} I_0'' \left( \sqrt{bR} \right) - \left( \frac{1}{R^2} + \frac{c}{D} \right) I_0' \left( \sqrt{bR} \right) \]
\[ Y_2 = I_0'' \left( \sqrt{bR} \right) + \frac{v}{R} I_0' \left( \sqrt{bR} \right) \]
\[ Z_2 = I_0' \left( \sqrt{bR} \right) \]

Case 3:

\[ V_1 = I_0 \left( \sqrt{aR} \right) \]
\[ X_1 = I_0'' \left( \sqrt{aR} \right) + \frac{1}{R} I_0'' \left( \sqrt{aR} \right) - \left( \frac{1}{R^2} + \frac{c}{D} \right) I_0' \left( \sqrt{aR} \right) \]
\[ Y_1 = I_0'' \left( \sqrt{aR} \right) + \frac{v}{R} I_0' \left( \sqrt{aR} \right) \]
\[ Z_1 = I_0' \left( \sqrt{aR} \right) \]
\[ V_2 = I_0 \left( \sqrt{bR} \right) \]
\[ X_2 = I_0'' \left( \sqrt{bR} \right) + \frac{1}{R} I_0'' \left( \sqrt{bR} \right) - \left( \frac{1}{R^2} + \frac{c}{D} \right) I_0' \left( \sqrt{bR} \right) \]
\[ Y_2 = I_0'' \left( \sqrt{bR} \right) + \frac{v}{R} I_0' \left( \sqrt{bR} \right) \]
\[ Z_2 = I_0' \left( \sqrt{bR} \right) \]
Case 4:

\[
V_1 = \sum_{n=0}^{\infty} \left( \frac{R\beta}{2} \right)^{2n} \frac{\cos 2n\phi}{(n!)^2}
\]

\[
X_1 = \sum_{n=0}^{\infty} \left[ \frac{n(2n-1)(n-1)}{2} \beta^3 \left( \frac{R\beta}{2} \right)^{2n-3} + \frac{n(2n-1)}{2R} \beta^2 \left( \frac{R\beta}{2} \right)^{2n-2} \right. \\
- \left( \frac{1}{R^2} + \frac{c}{D} \right) n\beta \left( \frac{R\beta}{2} \right)^{2n-1} \right] \frac{\cos 2n\phi}{(n!)^2}
\]

\[
Y_1 = \sum_{n=0}^{\infty} \left[ \frac{n(2n-1)}{2} \beta^2 \left( \frac{R\beta}{2} \right)^{2n-2} + \frac{n\nu}{R} \beta \left( \frac{R\beta}{2} \right)^{2n-1} \right] \frac{\cos 2n\phi}{(n!)^2}
\]

\[
Z_1 = \sum_{n=0}^{\infty} n\beta \left( \frac{R\beta}{2} \right)^{2n-1} \frac{\cos 2n\phi}{(n!)^2}
\]

\[
V_2 = \sum_{n=0}^{\infty} \left( \frac{R\beta}{2} \right)^{2n} \frac{\sin 2n\phi}{(n!)^2}
\]

\[
X_2 = \sum_{n=0}^{\infty} \left[ \frac{n(2n-1)(n-1)}{2} \beta^3 \left( \frac{R\beta}{2} \right)^{2n-3} + \frac{n(2n-1)}{2R} \beta^2 \left( \frac{R\beta}{2} \right)^{2n-2} \right. \\
- \left( \frac{1}{R^2} + \frac{c}{D} \right) n\beta \left( \frac{R\beta}{2} \right)^{2n-1} \right] \frac{\sin 2n\phi}{(n!)^2}
\]

\[
Y_2 = \sum_{n=0}^{\infty} \left[ \frac{n(2n-1)}{2} \beta^2 \left( \frac{R\beta}{2} \right)^{2n-2} + \frac{n\nu}{R} \beta \left( \frac{R\beta}{2} \right)^{2n-1} \right] \frac{\sin 2n\phi}{(n!)^2}
\]

\[
Z_2 = \sum_{n=0}^{\infty} n\beta \left( \frac{R\beta}{2} \right)^{2n-1} \frac{\sin 2n\phi}{(n!)^2}
\]
Case A:

\[ V_3 = -J_0 \left( \sqrt{y R} \right) \]
\[ X_3 = \frac{\xi}{D} J' \left( \sqrt{y R} \right) \]
\[ Z_3 = -J'_0 \left( \sqrt{y R} \right) \]
\[ V_4 = -Y_0 \left( \sqrt{y R} \right) \]
\[ X_4 = \frac{\xi}{D} Y'_0 \left( \sqrt{y R} \right) \]
\[ Z_4 = -Y'_0 \left( \sqrt{y R} \right) \]  \hspace{1cm} (3.37)

Case B:

\[ V_3 = -1 \]
\[ X_3 = 0 \]  \hspace{1cm} (3.38)

Case C:

\[ V_3 = -K_0 \left( \sqrt{y R} \right) \]
\[ X_3 = \frac{\xi}{D} K'_0 \left( \sqrt{y R} \right) \]  \hspace{1cm} (3.39)

For Case B or Case C the elements \( V_4, X_4, Z_3 \) and \( Z_4 \) are not required.

Substitution for \( w_1 \) into eq. (3.13) yields that,

\[
\gamma'^2 = \frac{\left( \frac{H}{R} \right)^2 \int \left[ \left( \frac{\partial W(r)}{\partial r} \right)^2 + \varphi^2 \left( \frac{W(r)}{R} \right)^2 \right] r dr}{\frac{2}{(1-v_0)} \int \left( \frac{W(r)}{R} \right)^2 r dr} \]  \hspace{1cm} (3.40)
in which $\bar{\omega} = \sqrt{\frac{m_0\omega^2 R^2}{G}}$ is the nondimensional natural frequency. $G$ and $\gamma$ are the shear modulus and dimensionless decay rate of the foundation, respectively.

### 3.5 Impact Loading Analysis

The equation of motion given in the eq. (3.8) is used to find the response of the plate due to an impact loading. Now after substituting from eq. (3.11), the eq. (3.8) can be written in the following form

$$\nabla^4 w - 2C\nabla^2 w + Kw + M\frac{\partial^2 w}{\partial t^2} = q(r,t)$$

where $w$ is the surface deflection of the plate (in this case the subscript 1 is omitted for simplicity).

A general solution to the Eq. (3.41) can be obtained after calculating the homogeneous and particular solutions. The homogeneous solution can be written in terms of modal parameters as follows

$$w(r,t) = \sum_{n=1}^{\infty} w_n(r,t) = \sum_{n=1}^{\infty} W_n(r) T_n(t)$$

This can be written for the $n$th mode in the following form, assuming $T_n(t)$ in terms of trigonometric functions.

$$w_n(r,t) = W_n(r) \left( A_n \sin \omega_n t + B_n \cos \omega_n t \right)$$
where \( W_n \) is the \( n \)th modal shape and the coefficients \( A_n \) and \( B_n \) for the homogeneous solution can be found using the initial conditions at \( t = 0 \). The particular solution can be found by considering instantaneous impulses after dividing the total impact loading function into small intervals. The deflection for time \( t > \tau \) is determined by the initial conditions at time \( t = \tau \), i.e.,

\[
\begin{align*}
{w_n}(r, \tau) &= w_{0n} \\
{\dot{w}_n}(r, \tau) &= \dot{w}_{0n}
\end{align*}
\]  

(3.44)

After the separation of variables, the impulse load \( P(r,t) \) can be written in terms of \( P(r) \) and \( T(t) \) as,

\[
P(r,t) = P(r).T(t)
\]  

(3.45)

Fig. 3.3 An Idealized FWD Impact Load
and if the function \( P(r) \) is taken as a uniformly distributed unit load (UDUL, \( P(r) = 1 \)) over the loading pad then \( T(t) \) shows the distribution of the total impact over the time which is shown in Fig. 3.3.

The distribution of load, at a particular time \( t \), can be expressed in terms of all modal shapes of vibration at that time. Hence by superposition, \( P(r) \) can be written in a series of eigen functions of \( W_n(r) \) such that,

\[
P(r) = \sum_{n=1}^{\infty} C_n W_n(r)
\]  

(3.46)

where the constants \( C_n \) can be found multiplying both sides by \( W_n(r) \) and integrating over the surface of the foundation, i.e.,

\[
C_n = \frac{\int_{0}^{R_L} W_n(r) rdr}{\int_{0}^{\infty} W_n^2(r) rdr}
\]  

(3.47)

where \( R_L \) is the radius of the loading pad on the pavement.

First consider how to determine the homogeneous solution. It can be seen from the Fig. 3.3 that at \( \tau = 0 \), unlike a step impulse, the applied impact load is zero and before the impact load is applied, the system is assumed to be at rest. Hence the initial displacement \( (w_{0n}) \) and velocity \( (\dot{w}_{0n}) \) both are zero at \( t = 0 \). Consequently both the constants, \( A_n \) and \( B_n \) in the homogeneous eq. (3.43) are zero and hence only the particular part of solution exists.
Fig. 3.4 Division of FWD Impact Load into Momentary Impulses

Now to determine the particular solution at \( t (\tau \geq t > \tau) \), it is required to find the initial conditions at time \( t = \tau \) after the instantaneous impulses applied during \( t = 0 \) to \( t = \tau \). This has been done using the assumed instantaneous impulses as shown in Fig. 3.4.

Substituting eq. (3.44) into eq. (3.43) and solving for \( A_n \) and \( B_n \), the particular solution of \( n \)th modal shape at \( t = \tau \) can be found as

\[
\dot{w}_n(t) = w_{0n} \cos \omega_n (t - \tau) + \frac{\dot{w}_{0n}}{\omega_n} \sin \omega_n (t - \tau)
\]  

(3.48)

At \( t = 0 \), \( w_{0n} = 0 \)

(3.49a)

and hence using eq. (3.48) the shape of the \( n \)th mode of vibration becomes,

\[
\dot{w}_n(t) = \frac{\dot{w}_{0n}}{\omega_n} \sin \omega_n (t - \tau)
\]  

(3.49b)
and the first momentary impulse can be used to determine the $\dot{w}_{0n}$ term. By equating the momentum of the plate-foundation system and the applied pulse at $t=0$,

$$m\dot{w}_{0n} = P(r,t) = C_n W_n(r).T(\tau).\Delta \tau$$

(3.50)

where $m = D.M = m_0 h + \bar{m}_0 \int_0^2 dz$

(3.51)

Now substituting for $\dot{w}_{0n}$ in eq. (3.49b) from eq. (3.50) the deflection of $n$th mode due to a momentary impulse over the infinitesimal time $\Delta \tau$ at $t=0$, is given by

$$w_n(r,t) = \frac{C_n}{m \omega_n} W_n(r) T(\tau) \sin \omega_n (t-\tau) \Delta \tau$$

(3.52)

The deflection during the application of total impact can be found by integrating the eq. (3.52) over the time $t=0$ to $\bar{t}$, i.e.,

$$w(r,t) = \sum_{n=1}^{\infty} \frac{C_n}{m \omega_n} \int_0^{\tau} T(\tau) \sin \omega_n (t-\tau) \, d\tau$$

(3.53)

Free Vibration $t > \bar{t}$ is given by the following homogeneous solution,

$$w(r,t-\bar{t}) = \sum_{n=1}^{\infty} W_n(r) \left( A_n \sin \omega_n (t-\bar{t}) + B_n \cos \omega_n (t-\bar{t}) \right)$$

(3.54)

The initial conditions at $t=\bar{t}$ are found from the conditions at the end of the impact loading. Let the deflection and the velocity at $t=\bar{t}$ be $w(r,\bar{t})$ and $\dot{w}(r,\bar{t})$ hence
\[ w(r, \bar{r}) = \sum_{n=1}^{\infty} W_n(r) B_n \]
\[ \dot{w}(r, \bar{r}) = \sum_{n=1}^{\infty} \omega_n W_n(r) A_n \]  
(3.55)

Now multiplying both sides of eqs. (3.55) by \( W_n(r) \) and integrating over the area of the foundation surface results in

\[ A_n = \frac{\int_{0}^{\infty} \dot{w}(r, \bar{r}) W_n(r) \, r \, dr}{\int_{0}^{\infty} \omega_n \int_{0}^{\infty} W_n^2(r) \, r \, dr} \]  
(3.56a)

\[ B_n = \frac{\int_{0}^{\infty} w(r, \bar{r}) W_n(r) \, r \, dr}{\int_{0}^{\infty} \int_{0}^{\infty} W_n^2(r) \, r \, dr} \]  
(3.56b)
4 Numerical Implementation

4.1 Overview

The close form solution found in the chapter 3 cannot be used directly to find a direct solution to the problem. This is due to the fact that the final transcendental equations need a certain initial values to begin with. Hence in this chapter a numerical method is developed using an iterative procedure to solve the resulting transcendental equations. The response of the plate on elastic foundation during the impact loading and free vibration that followed is calculated using the principle of modal superposition. A computer code is written based on this iterative method and a set of pavement problems are solved with a simulated impulse loading equivalent to the falling weight deflectometer records.

4.2 Geometric and Material Parameters of the System

The derived equations in the previous chapter cannot be solved by a simple mathematical calculation. The unknown parameters of the system—the natural frequency and the decay rate are coupled each other in the solution equations in a nonlinear form. First, the geometric parameters of the circular plate \((R, h)\) and the foundation \((H)\) as well as the material properties of the plate \((E, \nu)\) are required to be known. Then by assuming the Poisson's ratio and the Young's modulus of the foundation, the system parameters \(C, K\) and \(M\) can be determined for each mode of vibration using a predetermined decay rate \(\gamma\).

The eigen characteristics of the system are evaluated for the first three modes of the system. By considering the modal superposition, for the each eigen value \((\omega)\), the
normalized modal shape of the foundation surface is calculated. In this case, the modal shape of the circular plate is checked against the estimated eigen value to verify the mode. This is done by considering the slope variation of the resulting normalized axisymmetric modal shape. This procedure is necessary since the assumed decay rate can produce spurious modes away from the resonance frequencies. Until the decay rate and the estimated eigen value of the system are simultaneously satisfied with the governing equations of motion, this difficulty exists in the computation. The calculation of decay rates and the corresponding set of eigen characteristics is accomplished by using the iterative procedure described in the next section.

4.3 Iterative Procedure

The computation starts for the given properties of the plate-foundation system by assuming a decay rate for a particular mode. The respective eigen value (natural frequency) of the system is calculated using the determinant given in eq. (3.32). The size of the determinant is either 3x3 or 4x4 depending upon the resulting case among cases 1 to 4 as given in section 3.4 for the assumed decay rate. The determinant is set to zero, first using a course frequency increment and then a fine increment of a 2% of the coarse value. The normalized modal shape is calculated for this eigen value. If the eigen value and the modal shape do not represent the correct modal characteristics of the system, i.e. the smallest eigen value gives the first modal shape, and so on, then a new value for the decay rate is assumed for the expected mode and recalculate the eigen value until the mode and the natural frequency match correctly.

Next the decay rate of the experiencing mode can be determined from the governing equations of the system. If the assumed and the calculated \( \gamma \) values lie within a relative
error of 0.5% then the natural frequency, the decay rate and the mode shape are considered as correct. If the error is large then a new value for $\gamma$ is set which is taken as the mean value of the calculated and assumed $\gamma$ values and proceed again from the beginning of the iterative method. When $\gamma$ is determined within the tolerance then go to the next mode. The main steps of this iterative computational procedure is illustrated by the algorithm shown in Fig. 4.1. The response during and after the impact load is calculated as described in the next section.

During the numerical integration along the radial coordinate of the plate, the plate is divided into sixty small divisions. The integration outside the plate is carried out up to a distance of four times the radius of the plate and the same incremental distance is used. This radial coordinate increment is also used in the integration over the loading pad.

### 4.4 Forced Vibration Response and Free Vibration After the Impact

The forced vibration response and the surface deflection during this excitation is the key to the backcalculation procedure. The impact load on the circular plate is assumed as a triangular load as given in Fig. 3.3 and this is the simulated applied force of a falling weight deflectometer loading record. The impact load is also divided into sixty momentary impulses and the response is calculated using the initial conditions at the starting instant time of each momentary impulse. Since the complete impact load diagram starts from zero the displacement as well as the velocity at the initiation of impact is considered as zero. The deflection response at the surface of the foundation is calculated during the impact loading using the first three modes. The conditions at the end of the impact is used to obtain the homogeneous solution of the motion equations which gives the free vibration response. A
computer code is written based on the derived formulation and the algorithm in Fig. 4.1 for the complete analysis of the finite circular plate on elastic foundation problem. The computer code and a typical data file are listed in Appendix B.
Assume a Value for \( \gamma = \gamma_0 \) (for Mode I)

Calculate \( C, K \) and \( M \)

Calculate \( \omega \) and Mode Shape

Is Mode Shape \( \equiv \) Mode, I

no

Assume Different \( \gamma_0 \)

yes

Calculate \( \gamma \) Using Eq. (3.40)

If abs \( (\gamma - \gamma_0) \) \( \Rightarrow \) TOL

no
\( \gamma \Rightarrow (\gamma_0 + \gamma)/2 \)

yes

Divide Impact Load into \( N \) intervals

Mode, \( I \Rightarrow 1,2,3 \)

Time index \( J=1,2,\ldots,N \)

Find \( w_n(r,t) \)

During the pulse \( w(r,t) = \sum w_n \)

Fig. 4.1 Algorithm for Iterative Procedure
5  Presentation of Results and Discussion

5.1 Overview

The results of the elastodynamic analysis of a plate on elastic foundation and the simulation of the falling weight deflectometer load on a pavement are presented using the developed two layer modified foundation model. The effect of the thickness of the plate and the modulus of the foundation are studied in this chapter. Numerical results are obtained during the course of impact as well as during free vibration just after the impact. The surface response and the maximum deflection at the point of contact of the impact are also discussed.

5.2 Physical Properties of the System

The material of the circular plate can be chosen to represent either asphalt or cement concrete pavements. This is determined by the elastic constants of the plate. The Young’s modulus of the foundation is varied with a fixed plate thickness to study the effect of it on the response of the foundation during free and transient vibration. The radius of the plate, thickness of the foundation, Poison's ratios of both the plate and foundation, and the Young's modulus of the plate are kept invariant for all the cases studied. The values used for these properties are given below.

- Radius of the plate \( (R) \) = 12.5 ft
- Mass density of the plate \( (m) \) = 125 lbs/ft\(^3\)
- Young's modulus of the plate \( (E_p) \) = 0.4x10\(^6\) psi
Poisson's ratio of the plate \( (v_p) \) = 0.41

Thickness of the foundation \( (H) \) = 25 ft

Mass density of the foundation \( (\rho) \) = 110 lbs/ft\(^3\)

Poisson's ratio of the foundation \( (v_s) \) = 0.4

The eigen characteristics and dynamic responses are calculated for plate thickness, \( h \) equals to 6", 8", 9" and 10". The Young's modulus of the foundation is also varied from 4000 psi to 18,000 psi. For an 8" thick plate, the numerical analysis is also conducted using foundations with Young's modulus equal to 3000 and 3500 psi.

5.3 Natural Frequency of the System

The modified two parameter model can be used for free vibration and transient load analysis. The natural frequencies \( (\omega) \) of plates with thicknesses equal to 6", 8", 9" and 10" are shown in Figs. C.1 to C.4 respectively in Appendix C. The results are presented for the first three symmetric modes of vibration. It can be seen from these curves that the resonance frequency of the system increases when the stiffness of the foundation increases for a particular mode. The variation of nondimensionalized natural frequency \( (\tilde{\omega}) \) which is defined in eq. (3.40) are calculated from the results shown in Appendix C. Figs. 5.1 to 5.4 show these values for the above four thicknesses of plates. For the first mode of vibration, there is no significant change in nondimensionalized frequency of a given plate resting on different elastic foundations. The symmetric second mode of vibration shows slight decrease in the nondimensionalized values for stiffer foundations. For the third symmetric mode, the variation of nondimensionalized resonance frequency becomes significant.
Furthermore, these results show that with the variable thickness of plates resting on a given elastic foundation, the increase or decrease of natural frequency does not have a monotonically varying pattern. This can be due to the nonlinear relation between the plate thickness and other system parameters. Therefore in the case of pavement testing and performance analysis, the thickness of the pavement has a very significant effect on the response.

Fig. 5.1 Natural Frequency of 6" Plate-Foundation System
Fig. 5.2 Natural Frequency of 8" Plate-Foundation System

Fig. 5.3 Natural Frequency of 9" Plate-Foundation System
5.4 Decay Rate of the System

The decay rate of the present modified model is considered as a system parameter which varies with the change of any other geometric or material property. The nondimensionalized decay rate ($\gamma$) which consistently satisfies the governing equations of the system, is plotted in parallel to the natural frequency graphs. The Figs. 5.5 to 5.8 show the decay rate of 6", 8", 9" and 10" plates resting on various foundations. The changes in the decay rate follow a similar pattern to changes in the nondimensionalized frequency. For higher modes the decay rate decreases with increasing Young’s modulus of the foundation. The influence of plate thickness on the decay rate also does not have a regular pattern. Therefore this study reveals that due to the nonlinearity of the system, the
pavement performance predictions may not be correct unless the data is correlated from test results from equal thicknesses of pavements.

The assumed solution to the governing equation (3.12) is given in eq. (3.14). The influence of \( \gamma \) on the transverse deflection is shown in the Fig. 5.9. For smaller values of \( \gamma \) the surface disturbance have a greater influence along the depth than a larger decay rate. In other words since the first mode vibration results in a smaller \( \gamma \) the response of the first mode of vibration attenuates slowly along the depth compared to other modes. On the other hand the response of higher modes is more restricted to the region close to the surface. Therefore these results show that when a pavement is subjected to vibration the dissipation of energy from higher modes of vibration is only in the vicinity of the surface. Hence these modes may intend to cause more damage to the pavement.

It can also be seen that the validity of assumed function \( \phi(z) \) can be judged by the attenuation of the response along the depth. The error incurred due to neglecting reflecting waves in the transverse direction is not significant for higher modes since the deflection diminishes at an increasing rate compared to the first mode.
Fig. 5.5 Decay Rate of 6" Plate-Foundation System

Fig. 5.6 Decay Rate of 8" Plate-Foundation System
Fig. 5.7 Decay Rate of 9" Plate-Foundation System

Fig. 5.8 Decay Rate of 10" Plate-Foundation System
Fig. 5.9 Effect of Decay Rate on Transverse Deflection

Fig. 5.10 Central Deflection of 6" Plate due to Impact
5.5 Central Deflection During and After the Impact Loading

The maximum response due to an impact load acting at the center of 6”, 8”, 9” and 10” thick plates are shown in Figs. 5.10 to 5.13. These results as expected become consistent with an static analysis based on different plate thicknesses and foundation moduli. When the foundation becomes stiffer, each plate produces lesser deflections. Also when a plate resting on a given foundation becomes thicker, due to the high rigidity of the plate the deflections become less. These peak values of deflections are triggered after the highest intensity of the impact on the plate. The variation of deflection with time from the beginning of the application of impact is plotted in Fig. 5.14 for four different plates. It can be observed from this plot that when the plate becomes thicker the peak deflection shifts further increasing the time lag between the peak load and peak deflection. This may be due to the rigidity of the plate and when the rigidity increases the response to a transient load delays. Therefore in the case of a moving vehicle on a highway, the response becomes quicker if the thickness of the pavement is smaller.

Next, the deflection variations with the impact load of 30 milliseconds duration and just after the impact are analyzed. The Figs. 5.15 to 5.18 show the changes in deflection with various Young’s modulus of foundations for 6”, 8”, 9” and 10” thick plates respectively. The applied simulated load has a peak at 5 milliseconds and therefore the extremum of deflection occur with a time lag for each foundation. These response curves also show that there are two peak values during the course of the impact. Therefore this impact is long enough to produce forced vibration response within the 30 millisecond period of time. When the applied load diminishes the plate bounce back to its original position and continues to produce deflection opposite to the direction of the impact. This is the case for the assumed elastic foundation with continuing contact at the two material
interface. But in the case of soil foundations the tension in the soil is negligible and this upward motion may result in loss of contact between the pavement and the subgrade.

A falling weight deflectometer test conducted on a 8” thick asphalt pavement is plotted on the graph shown in Fig. 5.16. The actual impact load of this test can be seen in Fig. 5.19. The simulated load is also marked in this figure. This simulation is based on the roughly equal areas under the curves and maintaining the peak load unchanged. Under these conditions the peak of the simulated load became close to 5 milliseconds on the time scale and hence the load triangle is drawn with peak at 5 millisecond point. It can be seen from these results that the Young’s modulus of the foundation can be estimated as equal to 4500 psi. The difference in shape of the experimental results and theoretical graphs may be due to the simulated shape of the actual impact load. The response after the impact is recorded flat in the test record. Therefore a comparison between theoretical and experimental results cannot be made in the free vibration response. As the theoretical results show the response of the free vibration decreases with the increase of foundation stiffness. It can also be seen that the free vibration reduces when the plate thickness increases for a given foundation.
Fig. 5.11 Central Deflection of 8" Plate due to Impact

Fig. 5.12 Central Deflection of 9" Plate due to Impact
Fig. 5.13 Central Deflection of 10" Plate due to Impact

Fig. 5.14 Maximum Transverse Deflection of Plates of Various Thickness
(Elastic Modulus of Foundation is 4000 psi)
Fig. 5.15 Central Deflection of 6" Plate with Different Foundations
Fig. 5.16 Central Deflection of 8" Plate with Different Foundations
Fig. 5.17 Central Deflection of 9" Plate with Different Foundations
Fig. 5.18 Central Deflection of 10\(^\circ\) Plate with Different Foundations

<table>
<thead>
<tr>
<th>Pressure (psi)</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>4000 psi</td>
<td>⌂</td>
</tr>
<tr>
<td>4500 psi</td>
<td>■</td>
</tr>
<tr>
<td>5000 psi</td>
<td>▲</td>
</tr>
<tr>
<td>8000 psi</td>
<td>●</td>
</tr>
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<tr>
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<td>▲</td>
</tr>
<tr>
<td>14000 psi</td>
<td>●</td>
</tr>
<tr>
<td>16000 psi</td>
<td>■</td>
</tr>
</tbody>
</table>

Central Deflection of Plate (mm)
5.6 Deflection Along the Radius During and After the Impact Loading

To study the deflections of the foundation surface along the radius of the plate, the transverse deflections were calculated at distances equal to 0", 8", 12", 24", 36", 50" and 70" from the center. These plots were shown in Figs. 5.20 to 5.23 for the 8" thick plate with foundation modulus equal to 3000 psi, 4000 psi, 4500 psi and 5000 psi respectively. It can be seen from these plots that the deflection at the center of the plate reaches its maximum first and then at the rest of the points follow thereafter according to their distances from the center. The region close to the center also shows two peaks both occur after the peak of the impact load which occurs at 5 milliseconds. It is also clear from these graphs that after the impact load starts to impart on at the plate center, the center region
starts to move in the direction of the load while points at 70" radius begins to move in the opposite direction. This produces sagging and hogging regions in the plate and hence the plate top surface undergoes tension and compression at different sections at certain times.

The Fig. 5.24 shows the deflected shapes of the plate at time intervals of 5 milliseconds from the beginning of the impact for Young's modulus of 4500 psi and plate thickness of 8". The experimental values also plotted on this graph from the available falling weight deflectometer test results. The experimental measurements were taken at 0", 7.9", 11.8", 23.6", 35.6", 49", and 70.9" from the point of loading. The theoretical and experimental center deflections were close for the foundation of 4500 psi modulus. When the distance increases from the center the actual measurements decrease faster than the theoretical results. The reasons for this can be due to the following differences in assumed and actual conditions:

The plate is considered in the numerical analysis as a homogeneous, elastic and isotropic plate. But the actual asphalt pavement may not behave as a uniform and homogeneous media since it constitutes different phases of materials such as gravel and asphalt. The pavement may also have different forms of cracks which depends upon the age of the road. When the plate moves upward as shown by theoretical results, due to elastic conditions of foundation the plate and the foundation remains in contact. But since the soil is weak in tension the actual pavement and the foundation may lose the contact if the actual pavement starts to move causing more cracks.

Since the weather conditions may affect the test results it is required to note the temperature and moisture conditions. Then the backcalculated modulus values based on these experimental results also should be marked with the field conditions of the tests used for the analysis.
Fig. 5.20 Transverse Deflection Along the Radius of 8" Plate and Elastic Foundation of 3000 psi
Fig. 5.21 Transverse Deflection Along the Radius of 8" Plate and Elastic Foundation of 4000 psi
Fig. 5.22 Transverse Deflection Along the Radius of 8" Plate and Elastic Foundation of 4500 psi
Fig. 5.23 Transverse Deflection Along the Radius of 8" Plate and Elastic Foundation of 5000 psi
Fig. 5.24 Deflected Shape of the Plate During the Impact
6 Conclusion and Proposed Future Work

6.1 Summary

A modified two parameter model is developed for the analysis of plate on elastic foundations. The decay rate of the foundation is treated as a system parameter and the analysis is conducted for a finite circular plate. This theoretical approach is used for the backcalculation of the foundation elastic properties based on the falling weight deflectometer test records. The natural frequency, decay rate, response during and after the impact are presented in with the results. The application of this method for estimation of material properties is more efficient and accurate for the nondestructive impact load analysis of pavements.

6.2 Theoretical Analysis

The dynamic analysis of plate on elastic foundation is studied with the free vibration response and under transient loads. A variational principle is used in the derivation of rigorous theoretical formulations. The presented analytical procedure becomes more accurate compared to the existing models of two parameter foundations. The following contributions are made in the theoretical approach of this study:

- A finite circular plate resting on an elastic foundation subjected to vibration is analyzed, while satisfying the deflection and slope continuity conditions along the plate edge.
• The transverse deflection distribution along the depth of the foundation is treated as dependent to the dynamic parameters of the system.

6.3 Application of the Developed Model

A numerical analysis is conducted using a simulated highway pavement test series. This can be used to study the effects of foundation elastic modulus and the thickness of the pavement on system parameters and dynamic response. The usefulness and effectiveness of this model on the backcalculation of material properties of foundations can be outlined as follows:

• The presented procedure can be used to determine the resonance frequency, modal shapes of a given system.

• This model gives the rate of change of transverse deflection for an assumed solution in the form of eq. (3.14).

• Transient as well as free vibration response analysis can be performed for a actual problem which can be modelled as a two parameter foundation.

• The iterative procedure coded in the solution scheme to converge to a solution with first three symmetric modes of vibration can be used to backcalculate the foundation modulus based on nondestructive impact load tests such as the falling weight deflectometer.
6.4 Recommendations and Future Work

The modified two parameter foundation model shows that the decay rate which has been considered as a constant so far, is a function of system properties as well as the characteristics of natural vibration. This model predicts the frequencies of a circular plate resting on a compound single soil layer due to axisymmetric vibration. It also gives modal shapes, the decay rate and the deflection profile within the soil for three predominant modes. This model shows an accurate numerical procedure. The presented analysis is capable to estimate soil properties without any destructive field tests and hence may become a very useful tool for the designer. The efficiency of the convergence to a solution based on the initialized values for the decay rate and the expected range of frequency for a particular mode. The accuracy of the convergence can be set upto 0.1% of the final answer. The following areas are suggested as extensions of the present study for future research:

- The present model can be further extended for a plate on two layer soil foundations. This may become useful in the case of two very distinctive nonhomogeneous layers that exist in the proposed field.

- A development of a set of charts for all possible soil properties as well as plate thicknesses such as asphalt thicknesses in the case of a highway performance analysis may be useful to read the effective properties directly with respect to the impact test records.
References


Appendix A

The shear force at the inner side of a cylindrical section at a radius \( r \) through the plate and foundation is given by,

\[
Q_r = -D \frac{d}{dr} \left[ \frac{d^2 W_1(r,t)}{dr^2} + \frac{1}{r} \frac{dW_1(r,t)}{dr} \right] + c \frac{dW_1(r,t)}{dr} \tag{A.1}
\]

where \( D \) is the flexural rigidity of the plate, \( W_1 \) is the transverse deflection of the plate \((r \leq R)\) and \( c \) is a foundation parameter which is defined in eq. (3.16)

The shear force at the outer side of a cylindrical section at a radius \( r \) through the foundation is given by,

\[
Q_f = c \frac{dW_2(r,t)}{dr} \tag{A.2}
\]

where \( W_2 \) is the transverse deflection of the foundation surface, outside the plate \((r \geq R)\)

The bending moment of the circular plate at a radius \( r \) is given by

\[
M_r = -D \left[ \frac{d^2 W_1(r,t)}{dr^2} + \frac{v}{r} \frac{dW_1(r,t)}{dr} \right] \tag{A.3}
\]
Appendix B

C234567

PROGRAM PULSE
IMPLICIT REAL*4(A-H,P-Z,K,M)

****************************************************************
** DYNAMIC ANALYSIS OF A CIRCULAR FOOTING ON ELASTIC **
** FOUNDATION MODIFIED TO USE 4X4 DETERMINANT **
****************************************************************

PARAMETER (NMAXX=250,NMAXT=65)

COMMON/COEF/K,C,R,VP,MF,K1,C2,E0,V0,W,HF
DIMENSION W1(NMAXX),DW1(NMAXX),XX(NMAXX)
& ,Z1(NMAXX),Z2(NMAXX)

DIMENSION A(4,4),BJ(3),BI(3), WW(3,NMAXX) ,PT1(65),PT2(65)
& ,TT1(65),AWN(NMAXX),BWN(NMAXX),TWP(80,65),TFW(80,65)
& ,TWDOTP(80,65),TTW(3,80,65),POINT(10)
& ,OMEGA(5),TT2(50),WSQR(3),WSQ(NMAXX),WRT(3,NMAXX)

DIMENSION TTWDOT(3,80,65),PTT1(65),PTT2(65),TTT1(65)

DIMENSION IP(10)
CHARACTER*1 CASE
CHARACTER*72 SUBTIT
DATA PI/3.14159265/

C *** PLATE PARAMETERS :
C EP:  Modulus of plate (lbs/sq.in)i.e.(psi)
C HP:  Thickness of plate (ft)
C L:  # of divisions of the plate used for integration
C MP:  Mass of plate (lb mass/c.ft)
C R:  Radius of plate (ft)
C VP:  Poisson's ratio of plate
C *** FOUNDATION PARAMETERS :
C       C2: Small c (fdn.parameter)
C       E0: EF/(1.- VF**2)
C       EF: Modulus of foundation (psi)
C       G: Decay rate (non dimensional) = Gamma*H
C       HF: Thickness of foundation (ft)
C       K1: Small k (fdn.parameter)
C       MF: Mass of foundation (lb mass/c.ft)
C       V0: VF/(1.- VF)
C       VF: Poission's ratio of foundation

C *** SYSTEM PARAMETERS :
C       DG: Increment of G
C       G3: Upper limit of G in matching
C       W: Natural freq.(rad/sec)
C       WI: Freq. increment (corse)
C       WI1: Freq. increment start with WI1 later make it fine to get
C            exact freq. s.t.determinant=0
C       WLIMIT: Upper limit of freq. used in iteration
C       WT: Initial freq. assumed in matching (lower bound)

C *** IMPULSE PARAMETERS :
C       DT: TIME2/NN
C       NN: # of intervals of the load triangle
C       P1: 1st vertex of the load triangle
C       P2: 2nd vertex of the load triangle
C       P3: 3rd vertex of the load triangle
C       RL: Radius of the loading pad
C       TIME1: Time at 2nd vertex
C       TIME2: Time at the end of impulse

C *** GENERAL PARAMETERS FOR FREE VIBRATION :
C       D: Flexural rigidity of the plate
C       EFF: EF
C       EPP: EP
C       ERR: Tolerance of G value
C ISTEP: # of modes (currently only 3)
C LMODE: Mode # (only 1,2,&3 are considered)

C ***VARIABLES FOR FREE VIBRATION:
C A: The characteristic matrix used to find for eigen values
C ALP: SQRT ((MBAR*W2-K1)/C2)
C AN: A constant in the solution of the homogeneous equation
C ANN: Integrated value of AWN over the plate radius
C AWN: An array of the product of modal deflection,velocity at the
C end of impulse and horizontal coord.(used for free vibration)
C BI(I): Ith order 1st kind modified Bessel functions
C BI(N): The array of 1,2,3th order 1st kind modified Bessel fn.
C BIP: The 1st deriv.of 0th order 1st kind modified Bessel fn.=J0'
C BIPP: The 2nd deriv.of 0th order 1st kind modified Bessel fn.=J0"
C BIPPP: The 3rd deriv.of 0th order 1st kind modified Bessel fn.=J0'"
C BJ(I): Ith order 1st kind Bessel functions
C BJ(N): Array of 1,2,3 th order 1st kind Bessel function
C BJP: The 1st derivative of 0th order 1st kind Bessel function= J0'
C BJPP: The 2nd derivative of 0th order 1st kind Bessel function= J0"'
C BJPPP: The 3rd derivative of 0th order 1st kind Bessel function=J0'"
C BN: A constant in the solution of the homogeneous equation
C BNN: Integrated value of BWN over the plate radius
C BWN: An array of the product of modal deflection,displace. at the
C end of impulse and horizontal coord.(used for free vibration)
C DW1(I): Array of the derivative of deflection of foundation surface
C for a particular mode at selected L points
C GG: The G value calculated from using system parameters
C GGPREV: GG of previous iteration
C GPREV: Ave. value of G & GG in previous iteration (GV)
C GSTAR1: Current starting G value (use when G>G3 in iteration)
C GSTAR2: Current starting G value (use when G>lower limit of G in
C iteration)
C GV: (G+GG)/2
C IGG: 1000*current GG
C IGGPRE: 1000*GG of prev. iteration (1000*GGPREV)
C IW5: 100*WPREV
IW: 100*current freq. (100*W)

OMEGA(I): Array representing natural freq.s of modes 1,2 & 3

OMET: Frequency*time

PHI: 0.5*ATAN [SQRT(K-C**2-M*Omega**2/C)]

RR: Horiz. coord. along the plate

TWDOTP: The array of velocity of the plate at an integration point

of the plate at integration times of the impulse

W1(I): Array of deflection of the foundation surface for a particular

mode at selected L points

W2: Square of freq.

WF1: The term AN*Sin(OMET)+ BN*Cos(OMET)

WPREV: Freq. from previous iteration

WSQR(IS): Array in integrated value of XX*W(I)**2 along plate radius,

along the radius of the pad (one value for each mode)

WSQ(I): Array XX*(W1)**2

WW: Array to store deflection for modes 1,2 & 3

X: The product of ALPHA and R or BETA and R

XI: The product of ALPHA and RR or BETA and RR at an integra. pt.

XX(I): Array of radial distance to L points

XX: Horizontal coord. vector used for integration (L points)

YYD: A term used for calculation of Bessel functions

Z1: The array of product of X coord. and square of the slope of

the mode shape

Z2: The array of product of X coord. and square of the modal values

*** VARIABLES FOR IMPULSE LOADING:

DT: Interval of pulse duration used for integration =delta(time)

CN: Coefficient of a mode shape to represent load distribution

JSTEP: # of divisions covered by the loading base

OT: Product of frequency of a mode and a time TT within the

impulse duration

PT1(I): Array of the product of ordinates of the 1st part of pulse
times Sin(OT)

PT2(I): Array of the product of ordinates of the 1st part of pulse
times Cos(OT)

PTT1(I): Array of the product of ordinates of the 2nd part of pulse
times Sin(OT)

C PTT2(I): Array of the product of ordinates of the 2nd part of pulse
C times Cos(OT)

C TT1(I): Array of time coordinate at integration points (for TIME<TIME1)
C TT2(I): Array of time coordinate for the 2nd part of impulse
C TTT1(I): Time vector used for numerical integration over the time
C TTWP: Due to pulse, the deflection at a given integration radius
C point, at a given integration time (i.e. Wn)
C TTWDOT: Array of velocity for a mode at integration point at the
C radius at integration times of the pulse
C TWP: 2-d array for total deflection when pulse acts for given
C integration points, for given time
C YDOT: YT2*Cos(OT)+YT1*Sin(OT)
C Y1: Integrated value of PT1 over t=0 to a time TT within the
C impulse duration
C YT2: Integrated value of PT2 over t=0 to a time TT within the
C impulse duration
C YT: YT2*Cos(OT)-YT1*Cos(OT)

*Begin
   ALP = 0.
   PHI = 0.

C: NOTE:: Initialized plate parameters R=150 in, MP=125 lbmass/cu. feet
C: VP=.41 and EP=400000 psi
C: R = 150./12.
   MP=125.
   VP=.41
   EP=4.E5

C: Read HP
   READ(5,*)HP
C: Read MF, HF, VF, EF
   READ(5,'(A)')SUBTIT
   READ(5,*)MF, HF, VF, EF
C *** PULSE DATA
C:   P1-P3 in (lb) and PMASS in (lb mass)
     RL= 5.9/12.
     P1= 0.
     P2= 6600.*32.2
     P3= 0.
     TIME1= .005
     TIME2= .03
     PMASS= 2500.
     NN = 60

CC: Points where deflection to be printed as output during pulse
     POINT(1) = 0.
     POINT(2) = 8./12.
     POINT(3) = 12./12.
     POINT(4) = 24./12.
     POINT(5) = 36./12.
     POINT(6) = 50./12.
     POINT(7) = 70./12.
     POINT(8) = 100./12.
     POINT(9) = 120./12.
     POINT(10) = 140./12.

     EPP= EP
     EFF= EF

C: Read G,DG,G3,WT, WI1, WLIMIT, ERR for mode1
     READ(5,*)LMO
     READ(5,'(A)')SUBTIT
     READ(5,*)G,DG,G3,WT, WI1, WLIMIT, ERR
     WI= WI1
     WT1= WT
     L= 60
     GSTAR1= G
     GSTAR2= G
     GG= 0.
GPREV = 0.
GGPREV = 0.
WPREV = 0.

** INITIAL VALUES FOR THE ITERATION ***

*MODE* = 'LMODE

** To convert E values from psi to lbf & ft

EP = EP * 144. * 32.2
EF = EF * 144. * 32.2
E0 = EF / (1.-VF**2)
V0 = VF / (1.-VF)

** Initialize the number of modes ISTEP = 3

ISTEP = 3
IFLAG = 0
DO 150 I = 1, ISTEP
  OMEGA(I) = 0.
  WSQR(I) = 0.
150 CONTINUE

DO 5 IS = 1, ISTEP
  GOTO 1

** Now for the given G value DET. = 0, but the difference of GG & G

is not in the given tolerance. Now G(updated) = (G + GG) / 2

IF (G.GT.G3) THEN

** Now G > G3, then G is updated as follows

G = GSTAR1 + DG / 4.
GSTAR1 = G
IF (GSTAR1.GT.G3) THEN
  WT = WT + 1.
IF (WT.GT.WLIMIT) THEN
WRITE(6,'(WT IS GR. THAN ',WLIMIT,
' REDUCE WT & LIMIT OF WT')

GOTO 4000
END IF
GSTAR1 = GSTAR2
END IF
END IF
END IF
IF (G.LT.GSTAR2) THEN
  G = GSTAR2 + DG
  GSTAR2 = G
END IF

CONTINUE
C: Calculate K, M, C ****************
IF (G.GT.80.) THEN
  K1 = EO * G / (2. * (1 - V0**2)) / HF
  C1 = HF / 2 / G
  GOTO 3
END IF
KK1 = (COSH(G) * SINH(G) + G) / (SINH(G))**2
K1 = E0 * G / (2. * (1 - V0**2)) * KK1 / HF
C1 = (SINH(G) * COSH(G) - G) * HF / (2 * G * (SINH(G))**2)

M1 = (MF * C1 + MP * HP)
MBAR = MP * C1

C * C2 = small c (fdn. parameter), small c = 2*D*C
  C2 = E0 / (2. * (1 + V0)) * C1
  K = K1 / D
  C = C2 / D / 2.
  M = M1 / D

C: Initialize 4*4 matrix of characteristic eq. ******
  DO 15 I = 1, 4
    DO 15 J = 1, 4
      A(I, J) = 0.
  15 CONTINUE
C: Initialize Bessel functions******
C: BJ(I)= Ith order 1st kind Bessel functions
C: BI(I)= Ith order 1st kind modified Bessel functions
    DO 20 I=1,3
        BJ(I) = 0.
        BI(I) = 0.
    20    CONTINUE

C: LF= A flag
C: DD= The initialized value of the determinant
    LF= 0
    DD= 100.
    W= WT
    WI= WI1

1100    CONTINUE

C: ALPHA= Square root of 'a' in the eqs
C: BETA= Square root of 'b' in the eqs
C: ABAR= SQRT(K-C**2-M*W2)
C: RR= Horizontal coordinate along the plate
    ALPHA= 0.
    BETA= 0.
    ABAR= 0.
    RR= 0.
    W2=W**2.

C  * ICASE= The # of the case (1,2,3,4) i.e.within plate fdn.surface
C  * Initialize the case #
    ICASE= 0
    CASE = '0'

C  * Cases 1,2,3,4 start ***********
    IF(W2*M-K)600,400,200

C  * Case 1 starts here ******
200    ALPHA=SQRT(SQRT(C**2+M*W2)-K-C)
ICASE = 1
BETA = SQRT(SQRT(C**2 + M*W**2 - K) + C)
X = ALPHA*R

CC**
N = 0
CALL BESJ(X, N, Y)
A(1, 1) = Y
DO 210 N = 1, 3
   CALL BESJ(X, N, Y)
   BJ(N) = Y
210 CONTINUE
X = BETA*R
N = 0
CALL BESI(X, N, Y)
A(1, 2) = Y
DO 220 N = 1, 3
   CALL BESI(X, N, Y)
   BI(N) = Y
220 CONTINUE
BJP = -1.0*ALPHA*BJ(1)
BJPP = -ALPHA**2*(A(1, 1) - BJ(2))/2.
BJPPP = ALPHA**3*(3.0*BJ(1) - BJ(3))/4.
BIP = BETA*BI(1)
BIPP = BETA**2*(A(1, 2) + BI(2))/2.
BIPPP = BETA**3*(3.0*BI(1) + BI(3))/4.
A(2, 1) = BJPP + BJP/R - BJP*(1.0/R**2 + 2.*C)
A(2, 2) = BIPP + BIP/R - BIP*(1.0/R**2 + 2.*C)
A(3, 1) = BJPP + VP/R*BJP
A(3, 2) = BIPP + VP/R*BIP
A(4, 1) = BJP
A(4, 2) = BIP
GOTO 800

201 CONTINUE
DO 202 I = 1, L + 1
   XX(I) = 0.
XX(I) = RR
XI = ALPHA * RR
CALL BESJ(XJ,0,YJ)
CALL BESJ(XJ,1,YJ1)
XI = BETA * RR
CALL BESI(XI,0,YI)
CALL BESI(XI,1,YI1)
W1(I) = 0.
DW1(I) = 0.
W1(I) = A1 * YJ + A2 * YI
DW1(I) = -A1 * ALPHA * YJ1 + A2 * BETA * YI1
RR = RR + R/L

202 CONTINUE
GOTO 2000

C * Case 2 starts here **************

400 BETA = SQRT(2. * C)
X = BETA * R
ICASE = 2
N = 0
CALL BESI(X,N,Y)
A(1,2) = Y
DO 420 N = 1,3
   CALL BESI(X,N,Y)
   BI(N) = Y
420 CONTINUE
A(1,1) = 1.
A(2,1) = .0
A(3,1) = .0
BIP = BETA * BI(1)
BIPPP = BETA ** 2 * (A(1,2) + BI(2)) / 2.
BIPPP = BETA ** 3 * (3. * BI(1) + BI(3)) / 4.
A(2,2) = BIPPP + BIPPP / R - BIP*(1./R**2 + 2.*C)
A(3,2) = BIPPP + VP/R * BIP
A(4,1) = 0.
A(4,2) = BIP
GO TO 800

401    CONTINUE
   DO 402 I=1,L+1
      XX(I) = 0.
      XX(I) = RR
      XI = BETA*RR
      CALL BESI(XI,0,YI)
      CALL BESI(XI,1,YII)
      W1(I) = 0.
      DW1(I) = 0.
      W1(I) = A1 + A2*YI
      DW1(I) = A2*BETA*YII
      RR = RR + R/L
402    CONTINUE
   GOTO 2000

600    CONTINUE
C   * Case 3 starts here *******************
   IF(C**2+M*W2-K)700,610,610
610    ALPHA=SQRT(C-SQRT(C**2+M*W2-K))
    BETA=SQRT(C+SQRT(C**2+M*W2-K))
    X=ALPHA*R
    ICASE = 3
    N=0
    CALL BESI(X,N,Y)
    A(1,1)=Y
   DO 620 N=1,3
      CALL BESI(X,N,Y)
      BI(N)=Y
620    CONTINUE
   BIP = ALPHA*BI(1)
   BIPP = ALPHA**2*(A(1,1)+BI(2))/2.
   BIPPP= ALPHA**3*(3.*BI(1)+BI(3))/4.
   A(2,1)= BIPPP+BIPP/R-BIP*(1./R**2+2.*C)
   A(3,1)= BIPP+VP/R*BIP
A(4,1) = BIP
X = BETA * R
N = 0
CALL BESI(X, N, Y)
A(1, 2) = Y
DO 630 N = 1, 3
   CALL BESI(X, N, Y)
   BI(N) = Y
630 CONTINUE
BIP = BETA * BI(1)
BI(1) = BI(1)
BIPP = BETA**3 * (BI(1) + BI(2))/2.
BI(2) = BI(2)
A(2, 2) = BIPP + BIPP*R - BIP*(1./R**2 + 2.*C)
A(3, 2) = BIPP + VPI*R*BIP
A(4, 2) = BIP
GO TO 800
601 CONTINUE
DO 602 I = 1, L + 1
   XX(I) = 0.
   XX(I) = RR
   XI = ALPHA * RR
   CALL BESI(XI, 0, YIA)
   CALL BESI(XI, 1, YIA1)
   XI = BETA * RR
   CALL BESI(XI, 0, YIB)
   CALL BESI(XI, 1, YIB1)
   W1(I) = 0.
   DW1(I) = 0.
   W1(I) = A1*YIA + A2*YIB
   DW1(I) = A1*ALPHA*YIA1 + A2*BETA*YIB1
   RR = RR + R/L
602 CONTINUE
GOTO 2000
C * Case 4 starts here **************

700    ABAR=SQRT(K-C**2-M*W2)
      ICASE = 4
      BETA = (K-M*W2)**.25
      X=BETA*R
      IF(C.EQ.0.) GO TO 750
      PHI=.5*ATAN(ABAR/C)
      GOTO 760

    750    PHI = PI/4.
    760    CONTINUE

      N=0
      CALL SUMCOS(X,BETA,PHI,YC,YC1,YC2,YC3)
      A(1,1)=YC
      A(2,1)= YC3+YC2/R-YC1*(1./R**2+2.*C)
      A(3,1)=YC2+YCl*VP/R
      A(4,1)= YCl
      CALL SUMSIN(X,BETA,PHI,YS,YS1,YS2,YS3)
      A(1,2) = YS
      A(2,2) = YS3+YS2/R-YS1*(1./R**2+2.*C)
      A(3,2) = YS2+YS1*VP/R
      A(4,2) = YS1
      GOTO 800

701    CONTINUE

    DO 702 I=1,L+1
      XX(I) = 0.
      XX(I) = RR
      X = BETA*RR
      IF(C.EQ.0.) THEN
        PHI = PI/4.
        GOTO 703
      END IF
      PHI = .5*ATAN(ABAR/C)
    703    CONTINUE

      N=0
CALL SUMCOS(X,BETA,PHI,YC,YC1,YC2,YC3)
CALL SUMSIN(X,BETA,PHI,YS,YS1,YS2,YS3)
W1(I) = 0.
DW1(I) = 0.
W1(I) = A1*YC + A2*YS
DW1(I) = A1*YC1 + A2*YS1
RR = RR + R/L
702 CONTINUE
GOTO 2000

800 CONTINUE
Y2=(MBAR*W2-K1)/C2
IF(Y2)820,860,880

880 ALP=SQRT(Y2)
N = 0.
X = ALP*R
CASE = 'A'
CALL BESJ(X,N,Y)
CALL BESY0(X,Y,YY)
A(1,3)= -1.*Y
A(1,4)= -1.*YY
N = 1
CALL BESJ(X,N,Y)
CALL BESY1(X,Y,YY)
A(2,3)= -2.*C*ALP*Y
A(2,4)= -2.*C*ALP*YY
A(3,3)= 0.
A(3,4)= 0.
A(4,3)= ALP*Y
A(4,4)= ALP*YY
GO TO 900

860 A(1,3)= -1.
A(2,3)=.0
A(3,3)=.0
CASE = 'B'
GO TO 940

820   ALP=SQRT(-1.*Y2)
     X=ALP*R
CASE = 'C'
     N=0
     CALL BESI(X,N,Y)
     CALL BESK0(X,Y,YY)
     A(1,3)=-1.*YY
     N=1
     CALL BESI(X,N,Y)
     CALL BESK1(X,Y,YY)
     A(2,3)=-2.*C*ALP*YY
     A(3,3)=.0

900   CONTINUE
     DET= A(1,3)*(A(2,1)*A(3,2)-A(2,2)*A(3,1))-
         1          A(2,3)*(A(1,1)*A(3,2)-A(1,2)*A(3,1))
     IF(A(1,4).NE.0.OR.A(2,4).NE.0.) GOTO 945
     GO TO 908

940   DET=A(2,1)*A(3,2)-A(3,1)*A(2,2)
     GOTO 908

945   DET= A(4,4)*DET + A(2,4)*(A(1,3)*(A(3,1)*A(4,2)
         1          -A(3,2)*(A(4,1)) + A(4,3)*(A(1,1)*A(3,2) - A(3,1)*A(1,2))
         2          - A(1,4)*(A(4,3)*(A(2,1)*A(3,2) - A(3,1)*A(2,2))
         3          + A(2,3)*(A(3,1)*A(4,2) - A(4,1)*A(3,2)))

908   CONTINUE
     IF(LF.EQ.0) THEN
         IF(DET.NE.0.) GOTO 1005
     END IF

CC ** To check determinant crosses zero ******************************************
     IF(DD*DET.LE.0.) GOTO 1000
     P= 0.
CC ** To check whether determinant is diverging ********************
IF(ABS(DD).LT.ABS(DET).AND.D1.LT.ABS(DD)) THEN
  IF(GG.NE.0.AND.G.LT.GPREV) THEN
    G=GPREV
  ELSE
    G=G+DG
  END IF
CC ** GPREV = Ave. value of G * GG of previous iteration *******
END IF
CC ** P (a flag)=1, when determinant diverges **************
P= 1.
END IF
CC ** When DET. diverges & G>G3, G & WT update as follows. *******
IF(G.GT.G3) THEN
  WT= WT+ WI
C: If MODEFL=1 then increase WT by another WI to avoid short ccting
  IF(MODEFL.EQ.1) WT= WT + WI
  G= GSTAR1
  WRITE(6,*)':::::: G>G3,G=',G,'NEW WT=',WT
CC ** IF G>W LIMIT then stop ********************
IF(WT.GT.WLIMIT) THEN
  WRITE(6,*)'WT IS OR. THAN ',WLIMIT
  GOTO 4000
END IF
GOTO 1
END IF
IF(P.EQ.1.) GOTO 1
1005 CONTINUE
CC ** When DET. converges,till it crosses zero,increase W to W+WI
CC and go back to 1100 ************
D1= ABS(DD)
DD=DET
W=W+WI
LF = 1
GOTO 1100
1000 CONTINUE
IF(IFLAG.EQ.1) GOTO 1010
W = W - WI
WI = WI/50.
IFLAG = 1
GOTO 1100

1010 CONTINUE
CC ** Make WI = initial WI & IFLAG = 0 (initialize)*****************
WI = WI1
IFLAG = 0
C: To calculate the coefficients A1, A2, A3, A4
IF(A(1,4).NE.0.OR.A(2,4).NE.0.) THEN
   CALL COEFF4(A, A1, A2, A3, A4)
ELSE
   CALL COEFF(A, A1, A2, A4)
   A3 = 0.
END IF

3000 CONTINUE
CC ** By now DET. = 0, G is fixed, W known & coefficients known ********
IF((W2*M-K).GT.0.) GO TO 201
IF((W2*M-K).EQ.0.) GO TO 401
IF((W2*M-K).LT.0.).AND.((C**2+M*W2-K).GE.0.).GO TO 601
IF((W2*M-K).LT.0.).AND.((C**2+M*W2-K).LT.0.).GO TO 701

2000 CONTINUE
CC ** Check to avoid rigid body motions of the plate **************
CHECK = 0.
DO 2005 I = 1, L + 1
CC ** Find Max. abs. plate deflection value ***************
   IF(CHECK.LE.ABS(W1(I))) THEN
      CHECK = ABS(W1(I))
   END IF

2005 CONTINUE
DO 2010 I=1,L+1
CC ** W1 values are divided by check to normalize.(also DW1)**********
   W1(I)= W1(I)/CHECK
   DW1(I) = DW1(I)/CHECK
2010 CONTINUE
IF(W1(1).EQ.W1(2).AND.W1(2).EQ.W1(3)) THEN
   IF(W1.LT.1.) W=W+1.
   IF(W1.GE.1.) W=W+WI*2.
   WRITE(6,*)("DEFLECTION IS SAME OR ZERO,W=",W
   GOTO 1100
ENDIF
LK= 1
DO 2015 I=2,L
   IF(DW1(I+1).GE.0.AND.DW1(I).LT.0.) GOTO 2016
   IF(DW1(I+1).LE.0.AND.DW1(I).GT.0.) GOTO 2016
   GOTO 2015
2016 LK= LK+1
2015 CONTINUE
IF(LK.EQ.LMODE) GOTO 2020
CC ** If we get a modal shape higher than the reqd. mode #, do as follows
C: MODEFL uses to avoid short circuiting when calculated mode is
C: greater than required mode and when G value exceeds G3
   MODEFL=0
   IF(LK.GT.(LMODE)) THEN
      G= G-DG
      WT= WT - WI
      IF(G.GT.G3) G=G3
      WRITE(6,*)'MODE,W,G=',LK,W,G
      MODEFL= 1
      GOTO 1
   END IF
CC ** If we get a modal shape less than the reqd. mode #, as follows
IF(WT.GT.WLIMIT) THEN
   WRITE(6,*)'WT IS GR. THAN ',WLIMIT
   'REDUCE WI & LIMIT OF WT'
199
GOTO 4000
END IF
G = G + DG
GOTO 1

2020    CONTINUE
CC  ** Fill XX, W1, DW1 arrays beyond the plate ******************
CC  ** For cases C,B,A ******************
    IF(Y2)2500,2600,2700

CC  * CASE A ***
2700    CONTINUE
    RR = R + R/L
    DO 2710 I = L + 2, 4*L + 1
         W1(I) = 0.
         DW1(I) = 0.
         XX(I) = RR
         X = ALP*RR
         CALL BESJ(X,0,Y)
         CALL BESY0(X,Y,YY)
         W1(I) = (A3*Y + A4*YY)/CHECK
         CALL BESJ(X,1,Y)
         CALL BESY1(X,Y,YY)
         DW1(I) = -(A3*ALP*Y + A4*ALP*YY)/CHECK
         RR = RR + R/L
    2710 CONTINUE
    GOTO 2900

2600    CONTINUE
    RR = R + R/L
    DO 2610 I = L + 2, 4*L + 1
         XX(I) = RR
         W1(I) = A4/CHECK
         DW1(I) = 0.
         RR = RR + R/L
    2610 CONTINUE
GOTO 2900

2500 CONTINUE
RR = R + R/L
DO 2510 I = L + 2, 4*L + 1
   W1(I) = 0.
   DW1(I) = 0.
   XX(I) = RR
   X = ALP * RR
   CALL BESI(X, 0, Y)
   CALL BESK0(X, Y, YY)
   W1(I) = A4 * YY / CHECK
   CALL BESI(X, 1, Y)
   CALL BESK1(X, Y, YY)
   DW1(I) = -A4 * ALP * YY / CHECK
   RR = RR + R/L
2510 CONTINUE

2900 CONTINUE

CC ** To check eq. 3.40 for G (if G is correct eq. 3.40 is satisfied)
CC ** GG = THE G VALUE CALCULATED FROM EQ.31
CALL GAM(XX, W1, DW1, GG, L)
WRITE(6, 2901) LK, W, WT, G, GG
2901 FORMAT(1X, '******* MODE, W, WT, G, GG, ', I4, 4E12.4)
WRITE(6, '*') 'CASE, CASE, ', ERR

CC ** Check the tolerance of G (GG & assumed G are within err. limit)
CC (By now DET. = 0 & found a G value i.e. GG to satisfy DET. = 0)
GV = (G + GG) / 2.
IF(ABS((G - GV)/GV).LE.ERR) GOTO 2100

CC * IGG = 1000*CURRENT GG
CC * IG5 = 1000*GG OF PREV. ITERATION i.e. 1000*IGGPRE
CC * IW = 100*CURRENT FREQ. i.e. 100*W
CC * IW5 = 100*W FROM PREV. ITERATION i.e. 100*WPREV
CC * G5 = GG OF PREV. ITERATION i.e. GGPREV
IGG= GG*1000
IGGPRE= GGPREV*1000
IW= W*100
IW5= WPREV*100
IF(IGG.EQ.IGGPRE.AND.IW.EQ.IW5) THEN
   G= (G+GG)/2.
   WT= WT+WI
   WRITE(6,*)'GG=G & W=W (SHORT CCT):
   G,W,WT=',G,W,WT
1
   GOTO 2
END IF

CC ** Update G4,G5 with current G,GG,W resp.(use with next iteration
CC values to check short circuit) ******************
CC
GPREV=G
G= (GG + G)/2.
GGPREV=GG
WPREV=W
GOTO 2

CC ** Calculation of G,W (i.e.freq.)& modal shape(IS) over for a
CC particular mode **********************************************
2100 OV = (0+00)/2.
   WRITE(6,*)'G IS ',GV,' WITHIN ',100.*(G-GV)/GV,' % ERROR'
CC ** Model1, Mode2 and Mode 3 deflection values are stored separately in a
CC three dimentional array & print *****************************
DO 2110 I=I,L+1
   WW(IS,I) = W1(I)
2110 WRITE(6,*)'X,DISP=',XX(I),W1(I)
WRITE(6,30)MP,MF
WRITE(6,31)HP,HF
WRITE(6,32)EPP,EFF
WRITE(6,33)VP,VF,R
WRITE(6,34)ICASE,CASE
WRITE(6,*)'FREQ. =',W,'DECAY RATE =',OV
WRITE(6,*)'G IS ',GV,'WITHIN',100.*(G-GV)/GV,'% ERROR'
WRITE(6,*)'INIT. FREQ. & INCR.=',WT,WT
WRITE(6,*)'WLIMT,G3,ERR: ',WLIMIT, G3,ERR
WRITE(6,*)'K= , C= ',K,C
WRITE(6,*)'M= ,MBAR=',M,MBAR
WRITE(6,*)'L=',L,'MODE=',LK
WRITE(6,*)'ALPHA,BETA,ABAR,PHI',ALPHA,BETA,ABAR,PHI
WRITE(6,'(///)')

CC **** IMPULSE ANALYSIS ****
OMEGA(IS)= W
WRITE(6,('" OMEGA, IS",D12.3,I4)')OMEGA(IS),IS
DO 7 I =1,L+1
7 WSQ(I)= XX(I)*Wl(I)**2
WSQR(IS)= AVINT(XX,WSQ,NMAXX,L+1,0.,R,IND)
JSTEP= RL*L/R +1

CC ** JSTEP= # of Integration steps over the loading pad *****
IF(JSTEP.LT.3) GOTO 36
CC ** CN= Coefficient of a mode shape to represent load distribution **
CN= AVINT(XX,W1,NMAXX,JSTEP,0.,RL,IND)/WSQR(IS)
GOTO 37
36 CN= W1(I)/WSQR(IS)
37 CONTINUE

WRITE(6,('"CN,W1,OMEGA",3(2X,EI2.3))')CN,W1(1),OMEGA(IS)

CC
DT=TIME2/NN
TT=0.
DO 141 J=1,L+1
   WSQ(J)= 0.
   DO 142 I=1,NN+1
      TTWP(IS,J,I)= 0.
142 CONTINUE

CC* TTWP= Due to pulse ,the deflection at a given integration radius
TTWDOT(IS,J,I)= 0.
CONTINUE
DO 41 I=1,NN+1
   PT1(I)= 0.
   PT2(I)= 0.
   PTT1(I)= 0.
   PTT2(I)= 0.
   TT1(I)= 0.
   TTT1(I)= 0.
   DO 42 J=I,L+1
      TWP(J,I) = 0.
      TWDOTP(J,I) = 0.
   41 CONTINUE
DO 46 I=1,NN+1
   YT1= 0.
   YT2= 0.
   OT= OMEGA(IS)*TT
   TT1(I) = TT
   IF(TT.GT.TIME1+DT/4.) GOTO 57
   PT1(I)= ((P2-P1)*TT/TIME1+P1)*SIN(OT)
   PT2(I)= ((P2-P1)*TT/TIME1+P1)*COS(OT)
   IF(I.LT.3) GOTO 49
   YT1= AVINT(TT1,PT1,NMAXT,I,0.,TT,IND)
   YT2= AVINT(TT1,PT2,NMAXT,I,0.,TT,IND)
   IF (TT.LE.TIME1+DT/4.) THEN
      YYT1= YT1
      YYT2= YT2
      PPT1= PT1(I)
      PPT2= PT2(I)
      TTT1(I)= TIME1
      PTT1(I)= PPT1
      PTT2(I)= PPT2
      I1=I
   END IF
GOTO 58
** Second part of the pulse triangle starts ****************************

57 CONTINUE

PT = P2 * (TIME2 - TT)/(TIME2 - TIME1)
II = I - I + 1
TTT1(II) = TT
PTT1(II) = PT * SIN(OT)
PTT2(II) = PT * COS(OT)
IF (II, LT, 3) GOTO 49

YT1 = AVINT (TTT1, PTT1, NMAXT, II, TIME1, TT, IND)
YT2 = AVINT (TTT1, PTT2, NMAXT, II, TIME1, TT, IND)

58 YT = YT2 * SIN(OT) - YT1 * COS(OT)
YDOTT = YT2 * COS(OT) + YT1 * SIN(OT)
IF (TT, GT, TIME1 + DT/4.) THEN
  YT = YT + YYT2 * SIN(OT) - YYT1 * COS(OT)
  YDOTT = YDOTT + YYT2 * COS(OT) + YYT1 * SIN(OT)
END IF

DO 6 J = 1, L + 1

TTWP(IS, J, I) = W1(J) * CN * YT / M1 / OMEGA(IS)
TTWDOT(IS, J, I) = W1(J) * CN * YDOTT / M1

6 CONTINUE

49 TT = TT + DT

46 CONTINUE

** Initialization for mode 2 ****************************

READ(5, *) LMODE
IF (LMODE, EQ, 2) THEN

C: Read G, DG, G3, WT, W1, WLIMIT, ERR for MODE2

READ(5, '(A)') SUBTIT
READ(5, *) G, DG, G3, WT, W1, WLIMIT, ERR
WRITE(6, *) ' * INITIAL VALUES FOR THE ITERATION* '
WRITE(6, '(*') MODE = ' ', LMODE
WRITE(6, '(*')'G', , DG = , W = , W1 = ', G, DG, WT, W1
ELSEIF (LMODE, EQ, 3) THEN

C: Read G, DG, G3, WT, W1, WLIMIT, ERR for MODE3

READ(5, '(A)') SUBTIT
READ(5, *) G, DG, G3, WT, W1, WLIMIT, ERR
WRITE(6,*)' * INITIAL VALUES FOR THE ITERATION* '
WRITE(6,*)'MODE= ',LMODE
WRITE(6,*)'G=,DG= ,W=, ,WI= ',G,DG,WT,WI1
ENDIF
WI=WI1
WT1=WT
GSTAR1=G
GSTAR2=G
GG=0.
GPREV=0.
5 CONTINUE
DO 12 I=1,NN+1
DO 12 J=I,L+l
   IF(I.EQ.(II+1)) GOTO 12
CC ** First three modes are considered in the calculation of deflection
   TWP(J,I) = TTWP(1,J,I)+TTWP(2,J,I)+TTWP(3,J,I)
   TWDOTP(J,I) = TTWDOT(1,J,I)+TTWDOT(2,J,I)+TTWDOT(3,J,I)
12 CONTINUE
DR = R/FLOAT(L)
DO 51 JP=I,10
   IP(JP) = POINT(JP)/DR
51 CONTINUE
WRITE(6,'(4X,"TIME",TI5,"D 1 S T A N C E F R O M 
& C E N T E R")')
WRITE(6,'(2X,TI5,F4.1, "IN",T25,F5.1,"IN",T38,F5.1,"IN", 
& T51,F5.1,"IN",T64,F5.1,"IN")')POINT(1)*12.,POINT(2)*12.,
& POINT(3)*12.,POINT(4)*12.,POINT(5)*12.
DO 13 I=1,NN+1
   IF(I.EQ.(II+1)) GOTO 13
   IF(IP(1).EQ.0.AND.POINT(1).EQ.0.) THEN
CC ** WPT1,2,3,4,5,...are total def. at reqd. pts. during the pulse***
   WPT1 = TWP(1,I)
   ELSEIF(IP(1).EQ.0.AND.POINT(1).NE.0.) THEN
      WPT1 = TWP(1,I) + ( TWP(2,I)-TWP(1,I) )/DR*POINT(1)
   ELSE
WPT1 = TWP(IP(1),I) + ( TWP(IP(1)+1,I)
& -TWP(IP(1),I) )/DR*( POINT(1)-IP(1)*DR)
ENDIF

WPT2 = TWP(IP(2),I) + ( TWP(IP(2)+1,I)-TWP(IP(2),I) )/DR*
& ( POINT(2)-IP(2)*DR)

WPT3 = TWP(IP(3),I) + ( TWP(IP(3)+1,I)-TWP(IP(3),I) )/DR*
& ( POINT(3)-IP(3)*DR)

WPT4 = TWP(IP(4),I) + ( TWP(IP(4)+1,I)-TWP(IP(4),I) )/DR*
& ( POINT(4)-IP(4)*DR)

WPT5 = TWP(IP(5),I) + ( TWP(IP(5)+1,I)-TWP(IP(5),I) )/DR*
& ( POINT(5)-IP(5)*DR)

WRITE(6,25)TT1(I),WPT1,WPT2,WPT3,WPT4,WPT5

CONTINUE

WRITE(6,'(//4X,"TIME",TI5,"D I S T A N C E FROM")'
& C E N T E R")')
WRITE(6,'(2X,TI5,F4.1, "IN",T27,F5.1, "IN",T38,F5.1, "IN",T51,F5.1,"IN",T64,F5.1,"IN")')POINT(6)*12.,POINT(7)*12.,
& POINT(8)*12.,POINT(9)*12.,POINT(10)*12.

DO 17 I=I,NN+1
IF(LEQ,(I1+1)) GOTO 17
WPT6 = TWP(IP(6),I) + ( TWP(IP(6)+1,I)-TWP(IP(6),I) )/DR*
& ( POINT(6)-IP(6)*DR)
WPT7 = TWP(IP(7),I) + ( TWP(IP(7)+1,I)-TWP(IP(7),I) )/DR*
& ( POINT(7)-IP(7)*DR)
WPT8 = TWP(IP(8),I) + ( TWP(IP(8)+1,I)-TWP(IP(8),I) )/DR*
& ( POINT(8)-IP(8)*DR)
WPT9 = TWP(IP(9),I) + ( TWP(IP(9)+1,I)-TWP(IP(9),I) )/DR*
& ( POINT(9)-IP(9)*DR)
WPT10= TWP(IP(10),I) + ( TWP(IP(10)+1,I)
& -TWP(IP(10),I) )/DR*( POINT(10)-IP(10)*DR)

WRITE(6,25)TT1(I),WPT6,WPT7,WPT8,WPT9,WPT10

CONTINUE
CC ***** FREE VIBRATION *****
CC ** To plot center deflection after pulse18 time intervals selected ********
   N= 18
   TT= TT-DT
CC ** TT-DT=TIME 2
   TTT= TT
   DO 19 I=1,N
       TWF(I,I)= 0.
CC ** Time coordinate value after TIME 2 **********************
   IT2(I)= 0.
   19 CONTINUE
   DO 14 IS=1,ISTEP
       TT= TTT
       DO 9,J=I,L+1
           AWN(J)= WW(IS,J)*TTWDOT(IS,J,NN+1)*XX(J)
           BWN(J)= WW(IS,J)*TTWP(IS,J,NN+1)*XX(J)
       9 CONTINUE
   ANN = AVINT(XX,AWN,NMAXX,L+1,0.,R,IND)
   BNN = AVINT(XX,BWN,NMAXX,L+1,0.,R,IND)
   AN= ANN/WSQR(IS)/OMEGA(IS)
   BN= BNN/WSQR(IS)
   TT= TT-TIME2
   WRITE(6,23)TT,AN,BN,OMEGA(IS),IS
   DO 8 I= 1,N
       OMET = OMEGA(IS)*TT
       WF1 = AN*SIN(OMET) + BN*COS(OMET)
CC ** TWF =Total center deflection for free vibration **
   TWF(I,I) = TWF(I,I) + WW(IS,1)*WF1
   IT2(I)= IT + TIME2
   IT= IT + .0005
   8 CONTINUE
14 CONTINUE
   DO 16 I=1,N
   16 WRITE(6,22)IT2(I),TWF(I,I)
4000 STOP
SUBROUTINE BESJ(X,N,Y)
IMPLICIT REAL*4(A-H,P-Z,K,M)

** Aim : To determine 1st kind BESSEL FUNCTIONS

C: REF: Handbook of Math. fns. with formulas, graphs & math. tables

C: N= Ith order of Bessel function (N= 0,1 or 2) (input)
C: X= Argument (input)
C: Y= Required Bessel function value at X (output)

EXTERNAL FACT
DATA PI/3.14159265/

C: Initialize the value of BESSEL FUNCTION i.e. Y
Y=0.
IF(X.EQ.0.) THEN
   IF(N.EQ.0) Y= 1.
   IF(N.EQ.1) Y= 0.
   GOTO 15
END IF
CCC ** For X>10. Use EXP SERIES **
C: Y=JN(X)=(2/(PI*X)**0.5* COS(X-PI/4-N*PI/2) ********

IF(X.GT.10.) THEN
  Y= (2./PI/X)**.5*COS(X-PI/4.-N*PI/2.)
  GOTO 15
END IF

DO 10 I=1,70
  J=I-1
  JJ=J+N
  IF(JJ.GE.57) GOTO 15
  IF(FACT(J).LE.0..OR.FACT(JJ).LE.0.) GOTO 15

C: Y=JN(X)=(0.5*X)**N*SUM((-1)**J* (X/2)**(J+N)* (X/2)**J)/J!*(J+N)
C: YD=DELTA Y

YD= (-1.)**J*((X/2.)**((N+J)/FACT(JJ))*(X/2.)**J)/FACT(J)
Y = Y + YD

10 CONTINUE

15 CONTINUE
C WRITE(6,*)'YD=',YD,'Y=',Y,'X=',X,'N=',N,'J=',J
RETURN
END

*--------------------------------------------------------------------------------------------------------
C234567

SUBROUTINE BESI(X,N,Y)
IMPLICIT REAL *4(A-H,P-Z,K,M)

** Aim: To determine 1st kind modified BESSEL FUNCTIONS

C: REF: Handbook of Math. fns. with formulas, graphs & math. tables

C: N= Ith order of Bessel function (N= 0,1 or 2) (input)
C: X= Argument (input)
C: Y= Required Bessel function value at X (output)
EXTERNAL FACT
DATA PI/3.14159265/

C: Initialize the value of BESSEL FUNCTION i.e. Y
Y = 0
IF (X.EQ.0.) THEN
   IF (N.EQ.0) Y = 1.
   IF (N.EQ.1) Y = 0.
   GOTO 15
END IF
IF (X.GT.49.) THEN
   IF (X.GT.170.) THEN
      Y = 1.E36
      GOTO 15
   END IF
   Y = EXP(X)/(2.*PI*X)**.5
   GOTO 15
END IF
DO 10 I = 1, 56
   J = I-1
   JJ = J+N
   IF (FACT(J).LE.0. OR. FACT(JJ).LE.0.) GOTO 15
   YD = ((X/2.)**(N+J)/FACT(JJ))*((X/2.)**(N)/FACT(J))
   Y = Y+YD
   YYD = (X/2.)**(N+J)
   IF (ABS(YYD).LT.1.E-36) GOTO 15
10 CONTINUE

15 CONTINUE
C WRITE(6,*)'YD=',YD,'Y=',Y,'X=',X,'N=',N,'J=',J
RETURN
END
SUBROUTINE BESY0(X,Y,YY)
IMPLICIT REAL*4(A-H,P-Z,K,M)

**
Aim: To determine 2nd kind BESSEL FUNCTIONS of order 0

C: REF: Handbook of Math. fns. with formulas, graphs & math. tables
C: X= Argument (input)
C: Y= BESJ function value used to calculate BESY0 (input)
C: YY= Required Bessel function value (output)

EXTERNAL FACT
DATA PI/3.14159265/
BG= .5772156

IF(X.GT.10.) THEN
  YY= (2./PI/X)**.5*SIN(X-PI/4.)
  GOTO 15
END IF
FR= 0.
YY= 2*(ALOG(X/2.)+BG)*Y/PI
DO 10 I=1,56
  FR= FR+1./FLOAT(I)
  YYD= -2.*(-1)**I*((X/2.)**I/FACT(I))**2*FR/PI
  YY = YY + YYD
10 CONTINUE
15 CONTINUE
C WRITE(6,*)'BESY0 YY=',YY,'I=',I,'X=',X
RETURN
END
SUBROUTINE BESY1(X,Y,YY)
IMPLICIT REAL*4(A-H,P-Z,K,M)

** Aim : To determine 2nd kind BESSEL FUNCTIONS of order 1

C: REF: Handbook of Math. fns. with formulas, graphs & math. tables

C: X= Argument (input)
C: Y= BESJ function value used to calculate BESY0 (input)
C: YY= Required Bessel function value (output)

EXTERNAL FACT
DATA PI/3.14159265/
BG= .5772156

IF(X.GT.10.) THEN
  YY= (2./PI*X)**.5*SIN(X-3.*PI/4.)
  GOTO 15
END IF
FR= 1.
YY= 2*(ALOG(X/2.)+BG)*Y/PI-(2./X + X/2.)/PI
DO 10 I=1,56
  IF(I.EQ.1) THEN
    YYD= -(1)**I*(X/2.)**I/FACT(I)**2*FR*X/2./(I+1)/PI
    YY= YY + YYD
  ELSE
    YYD= YYD - XYD
    YY= YY + YYD
  END IF
  IF(ABS(YYD).LT.1.E-12.) GOTO 15
10 CONTINUE
15 CONTINUE

C WRITE(6,*)'BESY1 YY=',YY,'I=',I,'X=',X
RETURN
END
SUBROUTINE BESK0(X,Y,YY)
IMPLICIT REAL*4(A-H,P-Z,K,M)

** Aim : To determine 2nd kind modified BESSEL FUNCTIONS of order 0

C: REF: Handbook of Math. fns. with formulas, graphs & math. tables

C: X= Argument (input)
C: Y= BESJ function value used to calculate BESY0 (input)
C: YY= Required Bessel function value (output)

EXTERNAL FACT
DATA PI/3.14159265/
BG= .5772156

IF(X.GT.2.5) THEN
  IF(X.GT.165) THEN
    YY= 1.E-36
    GOTO 15
  END IF
  YY= EXP(-X)/(2.*PI*X)**.5
  GOTO 15
END IF
FR= 0.
YY= -(ALOG(X/2.)+BG)*Y
DO 10 I=1,56
  FR= FR+1./I
  YYD= ((X/2.)**I/FACT(I))**2*FR
  YY= YY +YYD
  IF(ABS(YYD).LT.1.E-12.0 .AND. ABS(YY).GT.1.E-12) GOTO 15
  IF(ABS(YY).LT.1.E-12) GOTO 15
10 CONTINUE
15 CONTINUE
SUBROUTINE BESK1(X,Y,YY)
IMPLICIT REAL*4(A-H,P-Z,K,M)

** Aim: To determine 2nd kind modified BESSEL FUNCTIONS of order 1

C: REF: Handbook of Math. fns. with formulas, graphs & math. tables

C: X = Argument (input)
C: Y = BESJ function value used to calculate BESY0 (input)
C: YY = Required Bessel function value (output)

EXTERNAL FACT
DATA PI/3.14159265/
BG= .5772156

IF(X.GT.2.5) THEN
   IF(X.GT.165) THEN
      YY= 1.E-36
      GOTO 15
   END IF
   YY= EXP(-X)/(2.*PI*X)**.5
   GOTO 15
END IF
FR= 1.
YY= (ALOG(X/2.)+BG)*Y +1./X -X/4.
DO 10 I=1,56
   FR= FR +1./I +1./((I+1)
   YYD= -(X/2.)**I/FACT(I)**2*FR*X/4./((I+1)
   YY= YY +YYD
IF(ABS(YY).LT.1.E-12) GOTO 15
10 CONTINUE

15 CONTINUE
C WRITE(6,*')BESK1 YY=',YY,'X=',X,'I=',I
RETURN
END

*--------------------------------------------------------------------------------------------------------
C234567

SUBROUTINE SUMCOS(X,BETA,PHI,YC,YC1,YC2,YC3)
IMPLICIT REAL*4(A-H,P-Z,K,M)
**
Aim: Calculate a summation of Cosine functions for CASE 4
C: BETA= (K-M*Omega**2)**.25 (input)
C: PHI=.5*ATAN [SQRT(K-C**2-M*Omega**2/C)] (input)
C: X= Coordinate distance in the radial direction (input)
C: YC= V1 value in the 4X4 matrix for CASE 4 (output)
C: YC1= X1 value in the 4X4 matrix for CASE 4 (output)
C: YC2= Y1 value in the 4X4 matrix for CASE 4 (output)
C: YC3= Z1 value in the 4X4 matrix for CASE 4 (output)
EXTERNAL FACT
DATA PI/3.14159265/

YC = 0.  
YC1 = 0.  
YC2 = 0.  
YC3 = 0.  
DO 10 I=1,55
   NPI=0
   J=I-1
   CS = 0.
   IF(FACT(J).LE.0.) GOTO 15
   THETA= PHI*2.*(FLOAT(J))
   ...
AT = ABS(THETA)
IF (AT.GT.(2.*PI)) THEN
  NPI = (AT/2./PI)
ELSE
  GOTO 11
END IF
IF (THETA.GT.0.) THEN
  THETA = THETA - FLOAT(NPI)*2.*PI
ELSE
  THETA = THETA + (FLOAT(NPI)+1.)*2.*PI
END IF
11
CS = COS(THETA)/(FACT(J)**2.)
IF (X.EQ.0.) THEN
  YC = CS
  YC1 = 0.
  GOTO 15
END IF
DCC = (X/2)**(2*J)
DYC = (X/2.)**(2*J)*CS
DYC1 = J*BETA*(X/2.)**(2*J-1)*CS
DYC2 = J*(2*J-1)*BETA**2.*(X/2.)**(2*J-2)*CS/2.
YC = YC + DYC
YC1 = YC1 + DYC1
YC2 = YC2 + DYC2
YC3 = YC3 + DYC3
IF (DYC.EQ.0.) GOTO 10
IF (ABS(YC).GT.1.E12) GOTO 15
10 CONTINUE
15 CONTINUE
C WRITE(6,*)'SUBROUT SUMCOS YC-YC3 =',YC,YC1,YC2,YC3,'J=',J
RETURN
END
SUBROUTINE SUMSIN(X,BETA,PHI,YS,YS1,YS2,YS3)
IMPLICIT REAL*4(A-H,P-Z,K,M)

** Aim: Calculate a summation of Sine functions for CASE 4

C: BETA= (K-M*Omega**2)**.25 (input)
C: PHI= .5*ATAN [SQRT(K-C**2-M*Omega**2)] (input)
C: X= Coordinate distance in the radial direction (input)
C: YS= V2 value in the 4X4 matrix for CASE 4 (output)
C: YS1= X2 value in the 4X4 matrix for CASE 4 (output)
C: YS2= Y2 value in the 4X4 matrix for CASE 4 (output)
C: YS3= Z2 value in the 4X4 matrix for CASE 4 (output)

EXTERNAL FACT
DATA PI/3.14159265/
YS = 0.
YS1 = 0.
YS2 = 0.
YS3 = 0.
NPI= 0
DO 10 I=1,57
   J=I-1
   SN = 0.
   IF(FACT(J).LE.0.) GOTO 15
   THETA= PHI*2.*FLOAT(J)
   AT= ABS(THETA)
   IF(AT.GT.(2.*PI)) THEN
      NPI= (AT/2./PI)
   ELSE
      GOTO 11
   END IF
   IF(THETA.GT.0.) THEN
      THETA= THETA- FLOAT(NPI)*2.*PI
   ELSE
      GOTO 15
   END IF
10 CONTINUE
11 END
ELSE
  THETA= THETA+ FLOAT(NPI+1)*2.*PI
END IF

11
SN = SIN(THETA)/(FACT(J)**2.)
IF(X.EQ.0.) THEN
  YS= SN
  YS1=0.
  GOTO 15
END IF
DSS= (X/2.)**(2*J)
IF(ABS(DSS).GT.1.E36) GOTO 15
DYS= (X/2.)**(2*J)*SN
DYS1 = J*BETA*(X/2.)**(2*J-1)*SN
DYS2 = J*(2*J-1)*BETA**2.*(X/2.)**(2*J-2)*SN/2.
DYS3 = J*(2*J-1)*(2*J-2)*BETA**3.*(X/2.)**(2*J-3)*SN/4.
YS  = YS + DYS
YS1 = YS1+DYS1
YS2 = YS2+DYS2
YS3 = YS3+DYS3
IF(DYS.EQ.0.) GOTO 10
IF(ABS(YS).GT.1.E12) GOTO 10
10 CONTINUE

15 RETURN
END

*--------------------------------------------------------------------------------------------------------
C234567

REAL FUNCTION FACT(NF)
IMPLICIT REAL*4(A-H,P-Z,K,M)

**  Aim: Calculates factorial of a given number

INTEGER NF,LM
IF(NF.EQ.0) THEN
   FACT = 1
ELSE
   FACT = NF
   LM = NF
5   IF(LM.LE.1) GO TO 10
C   WRITE(6,*),'LM=',LM
   LM = LM - 1
   FACT = FACT * LM
   GO TO 5
10  CONTINUE
END IF

RETURN
END

*-----------------------------------------------------------------------------------------------
C234567

SUBROUTINE COEFF(A,A1,A2,A4)
IMPLICIT REAL*4(A-H,P-Z,K,M)

**   Aim: To solve 3 singular simultaneous equations
**   This is used for cases B & C with cases 1, 2, 3 and 4

DIMENSION A(4,4)

S1 = A(1,1)
S2 = A(2,1)
S3 = A(3,1)
T1 = A(1,2)
T2 = A(2,2)
T3 = A(3,2)
U1 = A(1,3)
U2 = A(2,3)
U3 = A(3,3)
IF(S1/S2.EQ.T1/T2.AND.S1/S2.EQ.U1/U2) GO TO 10
IF(S1/T1.EQ.S2/T2.AND.S1/T1.EQ.S3/T3) GOTO 20
IF(S1/U1.EQ.S2/U2.AND.S1.EQ.0.) GOTO 30
A4 = 1.
IF(S1.NE.0..AND.S2.NE.0.) THEN
  A2 = -(U1/S1-U2/S2)/(T1/S1-T2/S2)*A4
  A1 = -(T1*A2 + U1*A4)/S1
ELSEIF(T1.NE.0..AND.T2.NE.0.) THEN
  A1 = -(U1/T1-U2/T2)/(S1/T1-S2/T2)*A4
  A2 = -(S1*A1 + U1*A4)/T1
ENDIF
GOTO 50

10 A4 = 1.
IF(S1.NE.0..AND.S2.NE.0.) THEN
  A2 = -(U1/S1-U3/S3)/(T1/S1-T3/S3)*A4
  A1 = -(T1*A2 + U1*A4)/S1
ELSEIF(T1.NE.0..AND.T2.NE.0.) THEN
  A1 = -(U1/T1-U3/T3)/(S1/T1-S3/T3)*A4
  A2 = -(S1*A1 + U1*A4)/T1
ENDIF

20 A4 = 0.
A2 = 1.
A1 = -T1/S1*A2
GOTO 50

30 A2 = 0.
A4 = 1.
A1 = -U1/S1*A4

50 CONTINUE
C WRITE(6,*)'SUBR. COEFF : A1,A2,A4= ',A1,A2,A4
RETURN
END
SUBROUTINE GAM(X,W1,DW1,GG,L)
IMPLICIT REAL*4(A-H,P-Z,K,M)

** Aim: To calculate Gamma (G) using system parameters

PARAMETER (NMAXX=250,NMAXT=65)

COMMON/COEF/K,C,R,VP, MF,K1,C2,E0,V0,W,HF
DIMENSION X(NMAXX),W1(NMAXX),DW1(NMAXX)
& .Z1(NMAXX),Z2(NMAXX)
EXTERNAL AVINT

DO 10 I=1,4*L+1
   Z1(I) = 0.
   Z2(I) = 0.
   Z1(I) = X(I)*(DW1(I))**2
   Z2(I) = X(I)*(W1(I))**2
   IF(Z1(I).NE.0.AND.Z1(I).LE.1.E-30) GOTO 20
   L1 = I
10 CONTINUE
20 CONTINUE
   LL = L1
   FI = AVINT(X,Z1,NMAXX,LL,.0,4*R,IND)
   SI = AVINT(X,Z2,NMAXX,LL,.0,4*R,IND)
   TOPP= E0/(2.*(1+V0))
   TOP = TOPP*FI + MF*SI*W**2
   SGG = TOP*(1-V0**2)/(E0*SI)
   GG = SQRT(ABS(SGG))*HF
30 FORMAT(2X,'TOPP,EO,VO,MF,W',5E12.3)

RETURN
END
REAL FUNCTION AVINT(X,Y,NMAX,N,XLO,XUP,IND)
IMPLICIT REAL*4(A-H,P-Z,K,M)

** Aim: Integrate Y w.r.t. X over N points in the range XLO & XUP
C Reference: Methods of Numerical Integration by P.J.Davis & P.Rabinowitz

C N: Number of integration points used (input)
C NMAX: Maximum dimensions of the arrays X and Y
C (as defined in the calling program)
C X: Array of integration coordinate points
C Y: Array of function values at integration points
C XLO: Lower limit of integration
C XUP: Upper limit of integration

DIMENSION X(NMAX), Y(NMAX)

IND=0
IF(N.LT.3) RETURN
DO 10 I=2,N
     IF(X(I).LE.X(I-1)) RETURN
10 CONTINUE
SUM=0.
IF(XLO.LE.XUP)GO TO 5
SYL=XUP
XUP=XLO
XLO=SYL
IND=-1
GO TO 6
5 IND=1
SYL=XLO

6 IB=1
J=N
DO 1 I=1,N
     IF(X(I).GE.XLO)GO TO 7
     IB=IB+1
1
CONTINUE

IB=MAX0(2,IB)
IB=MIN0(IB,N-1)
DO 2 I=1,N
   IF(XUP.GE.X(J))GO TO 8
   J=J-1
2 CONTINUE

J=MIN0(J,N-1)
J=MAX0(IB,J-1)
DO 3 JM=IB,J
   X1=X(JM-1)
   X2=X(JM)
   X3=X(JM+1)
   TERM1=Y(JM-1)/((X1-X2)*(X1-X3))
   TERM2=Y(JM)/((X2-X1)*(X2-X3))
   TERM3=Y(JM+1)/((X3-X1)*(X3-X2))
   A=TERM1+TERM2+TERM3
   B=-(X2+X3)*TERM1-(X1+X3)*TERM2-(X1+X2)*TERM3
   C=X2*X3*TERM1+X1*X3*TERM2+X1*X2*TERM3
   IF(JM.GT.IB)GO TO 14
   CA=A
   CB=B
   CC=C
   GO TO 15
3 CONTINUE
AVINT=SUM+CA*(XUP**3-SYL**3)/3.+CB*.5*(XUP**2-SYL**2)+CC*(XUP-SYL)

IF(IND.EQ.1)RETURN
IND=1
SYL=XUP
XUP=XLO
XLO=SYL
AVINT=-AVINT

RETURN
END

--------------------------------------------------------------------------------------------------------

C234567

SUBROUTINE COEFF4(A,A1,A2,A3,A4)
IMPLICIT REAL*4(A-H,P-Z,K,M)

** Aim: To solve 4 singular simultaneous equations
** This is used for case A with cases 1,2,3 and 4

DIMENSION A(4,4),B(4,4),C(4,4)

N=4
A1= 0.
A2= 0.
A3= 0.
A4= 0.
DO 10 I=1,4
   IF(A(I,1).EQ.0.) GOTO 10
   I1= I
   GOTO 11
10 CONTINUE
11 DO 7 I=1,N
    DO 7 J=1,N
       B(I,J)= A(I,J)
7 CONTINUE
DO 15 I=1,N
   IF(I1.GT.N) I1=I1-N
   DO 16 J=1,N
      A(I,J) =B(I1,J)
   16 CONTINUE
   I1=I1+1
15 CONTINUE
AA1= A(1,1)
DO 18 J=1,N
   A(1,J) = A(1,J)/AA1
18 CONTINUE
DO 19 I=2,N
   B1= A(I,1)
   DO 20 J=1,N
      A(I,J)= A(I,J)-A(1,J)*B1
20 CONTINUE
19 CONTINUE
DO 30 I=2,N
   I2= 0
   IF(ABS(A(I,2)).LE.1.E-14) GOTO 30
   I2= 1
   GOTO 33
30 CONTINUE
IF(I2.EQ.0) THEN
   IF(A(2,3)/A(3,3).NE.A(2,4)/A(3,4).OR.A(2,3)/A(4,3).NE.A(2,4)/
      1 A(4,4))THEN
      A4= 0.
      A3= 0.
      A2= 1.
      A1= -A(1,2)/A(1,1)
      RETURN
   ELSE
      WRITE(6,*),'ONLY 2 INDEPENDENT EQNS.'
      STOP
   END IF
END IF
DO 35 I=1,N
   DO 35 J=1,N
   C(I,J) = A(I,J)
   DO 40 I=2,4
      IF (I2.GT.N) I2=I2-N+1
   DO 41 J=1,N
   A(I,J) = C(I2,J)
   I2= I2+1
   40 CONTINUE
   AA2= A(2,2)
   DO 42 J=1,N
      A(2,J)= A(2,J)/AA2
   DO 45 I=3,N
      B2= A(I,2)
   DO 46 J=2,N
      A(I,J)= A(I,J)- A(2,J)*B2
   45 CONTINUE
   A33= A(3,3)/A(4,3)
   A34= A(3,4)/A(4,4)
   AAA= (A33-A34)/(A33+A34)*2.
   IF (ABS(AAA).GT.0.01) THEN
      A4= 0.
      A3= 0.
      A2= 1.
      A1= -A(1,2)/A(1,1)
   ELSE
      A4= 1.
      A3= -A(3,4)/A(3,3)
      A2= -(A3*A(2,3) + A4*A(2,4))/A(1,1)
      A1= -(A2*A(1,2) + A3*A(1,3) + A4*A(1,4))/A(1,1)
   END IF

RETURN

*----------------------------------------------------------------------------------------------

END
Appendix C

Fig. C.1 Natural Frequency of 6” Thick Plate on Elastic Foundation

Fig. C.2 Natural Frequency of 8” Thick Plate on Elastic Foundation
Fig. C.3 Natural Frequency of 9" Thick Plate on Elastic Foundation

Fig. C.4 Natural Frequency of 10" Thick Plate on Elastic Foundation