INTEGRATED SUPPLY CHAIN OPTIMIZATION MODEL USING
MATHEMATICAL PROGRAMMING AND CONTINUOUS APPROXIMATION

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This dissertation entitled

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This research presents an integrated approach to optimize the different functions in a supply chain on strategic and operational levels. This overall optimization is achieved using mathematical programming for modeling the supply chain functions such as capacitated location, production, and distribution (CLPD) functions. Continuous approximation is then used to optimize the inventory costs, penalty costs, and transportation costs. These two techniques have been used in a phased manner to achieve combined optimization results for the different decisional levels in a multi-product multi-echelon production-distribution system. The integrated supply chain model has been formulated as a profit maximization problem and solved using computer software in the first phase. Results from this mathematical formulation are used in the second phase along with continuous approximation of inventory distribution patterns. It is then followed by application of classical optimization techniques for determination of closed form expressions for optimal number of shipments.

Approved:

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Chapter 1 Introduction

Supply chain management (SCM) is the systematic analysis and educated decision-making within the different business functions of an organization resulting in smooth and cost-effective flows of resources – material, information, and money. In other words, it is the coordination and synchronization of the flow of resources in the network of suppliers, manufacturing facilities, distribution centers and customers. These network elements form the different echelons of the supply chain. The supply chain is the network that sources raw material from suppliers, transforms it into finished products at the manufacturing facilities, and distributes the finished products to the final customers through the distribution centers. These activities constitute the individual business functions of the company’s supply chain network.

Decisions are made across the supply chain on three levels: strategic, tactical and operational. Strategic decisions are long term decisions where the time horizon may be anything from one year to several years i.e. it involves multiple planning horizons. These decisions may be made on an organizational level or the supply chain level with the aim for global optimization. Tactical decisions are taken over a shorter period of time, maybe a few months. These are more localized decisions taken to keep the organization on the track set at the strategic level. Operational decisions are similar to day-to-day decisions for planning a few days worth of operations. These take into consideration the most profitable way to carry out daily activities for satisfying immediate requirements.
The field of supply chain management uses an increasing set of tools and techniques to effectively coordinate the decision-making process. Quantitative models have been developed to analyze the various dimensions of a company’s supply chain. These modeling attempts have been targeted to effectively portray the governing factors and their interactions in deciding the overall performance. Attempts have also been made to model the qualitative aspects taking into consideration the ensuing subjectivity in the analysis. These efforts have been aimed at gaining a better understanding of the overall nature of the problem and to model it in the most accurate manner. Tools from various areas such as operations research, statistical analysis, quality control, location science, inventory control, theory of constraints, and so on have been used for modeling purposes.

1.1 Supply chain functions

The supply chain of an organization consists of different functions at each planning stage. According to Ganeshan and Harrison [22], these functions can be broadly classified in the following four categories – location, production, inventory, and transportation. Each of these functions plays a major role in the overall performance of the supply chain. So it becomes essential to execute each one of them in an optimal manner to ensure an efficient supply chain performance.

The location function deals with the decision about where to locate a facility keeping in mind the different constraints such as fixed costs, capacities, demands, labor costs, space constraints, distance from target markets, repulsion factors if any, local taxes, government regulations and so on. This is one of the strategic decisions that the
organization has to make because it will affect the future performance of the organization for a long period of time. One of the factors that categorize the location decisions is capacity. Depending on whether capacity constraint is present or not, the problem is referred to as the capacitated or uncapacitated location problem. It is relatively difficult to obtain optimal solutions to capacitated location problems (CLP). Optimization algorithms as well as heuristic procedures have been proposed to solve these problems.

In a manufacturing facility, one of the most important functions is to prepare a production plan for a given time horizon. According to Berry et al., a master production schedule is developed based on the sales forecasts taking into account the resource constraints of the plant. It drives the detailed capacity and material planning process and presents a strategy that states the company’s production goals. The master production schedule for a longer time period justifies purchase of equipment thus influencing the budgeting decisions as well. It thus forms a vital link between various supply chain functions making it important for inclusion in an integrated framework.

Managing the inventory process is an important task due to its costly nature. Stored inventory consumes space, incurs holding costs and handling costs, and most importantly locks up a huge amount of capital. Inventory may include raw materials, spare parts, factory supplies, work-in-process or finished products. Most often the amount of inventory that needs to be stored can be controlled by internal decisions processes such as production planning, materials requirement planning, distribution planning, etc. Moreover, this leads to an inter-relation and inter-dependence between the inventory
function, production function, and distribution function. The internal decisions can also help in optimizing factors such as economic order quantity, safety stock, reorder level, active and reserve storage inventory, and so on. As a result, inventory management assumes an important role in supply chain optimization.

Transportation costs form one of the largest parts of the overall logistics costs in a company’s functional structure. It represents the physical linking of the different phases and entities of the supply chain system. The various factors governing transportation costs include travel distance, number of shipments, hourly salaries, fixed costs, fuel charges, etc. Companies have to decide between different strategies, for example, maintaining an own fleet of trucks versus opting for a third party logistics provider, making TL or LTL shipments and so on. Routing and scheduling plans are also an important part of the overall transportation function. These decisions affect not just the costs but also customer satisfaction based on timely deliveries, reliability, and safety.

These functions have been mostly studied and optimized individually. However, in the recent years, there have been attempts to consider and analyze these functions collectively to study their interactions and provide a better solution for overall optimization. Integration refers to use of tools and techniques for a combined analysis of supply chain functions or entities to model their combinatorial behavior for determining the best policies for implementation. There are numerous ways to measure the supply chain performance. According to National Research Council (NRC), some of the most commonly used metrics are profitability, total revenues, costs, return on investments,
response times, market share, quality, customer satisfaction, risk minimization, waste reduction, and so on. It also lists some detail metrics used by companies for internal operations such as inventory levels and capacity utilization, customer service, lead-times, accuracy of forecasts, logistics costs, obsolescence, turnovers, etc. Although these metrics are for internal measurement, they definitely measure the overall performance of the company with respect to the complete supply chain. In cases involving integration, it becomes important that the performance metrics for all participants are aligned to ensure a common focal point for all the efforts.

1.2 The need to integrate

Supply chains have been more or less integrated to some extent – albeit on a low scale – as a whole, or in parts. Integration, if done at all, has been mostly done in patches throughout the supply chain. In many cases, this has been driven more by the need to survive and improvise, than by the willingness to improve and advance further. For example, suppliers have been coordinating with manufacturers to implement quality assurance programs in order to meet the ever increasing stringent quality requirements. These quality standards are mostly driven by market conditions which would not allow the manufacturer to accept material from the supplier if it does not meet those standards, thus risking the supplier’s business. As a more recent example, the RFID (radio frequency identification) compliance mandates imposed by Wal-Mart and Department of Defense (DOD), have compelled their suppliers to implement RFID technology in order to continue doing business with these giants. Although it is a relatively new technology and initially expensive to implement, there are long term gains like effective supply chain
management, real-time inventory tracking, cost-effective warehouse management, and so on. Moreover, implementation also requires a high degree of supply chain integration. These long term gains are far more attractive and outweigh the short-term difficulties. Still, most of the suppliers are initially reluctant to implement, and agree to comply with those mandates only to avoid losing business from a stable client.

Traditionally the different echelons of the supply chain and also the different decisions have been considered separately for planning purposes. They have been dealt with on an individual basis rather than collectively. In other words, problems in these domains have been solved separately and/or sequentially which, although makes them a lot easier to solve, may not necessarily consider their interrelationships. However, the manufacturing environment has changed a lot recently with respect to the ideological changes and technological advancements. As a result, this isolation of decisional components and the individualistic handling of the different business functions, may often lead to local optimization of the supply chain instead of global optimization. There are a lot of disadvantages to the traditional way of handling un-integrated supply chain operations. Some of them are listed below:

i. Cost-overruns: Due to the lack of coordination between suppliers and manufacturers, or between manufacturers and distributors, there are frequent cases of lost sales, for example, due to stock-outs, or excess inventory holding costs due to overstocks. This happens when the demand and production are not
synchronized and manufacturing is done solely on either poor estimates, or on an ad-hoc basis.

ii. Lack of data sharing: Although the necessary data is present across the supply chain at various points, it is not used at the right places and at the right times thus creating an illusion of lack of data. This leads to unreasonable assumptions resulting in poor strategies. For example, small changes in the demands does not reach the production department from the marketing department in real-time, there may be some un-satisfied customers at the end of the supply chain. This leads to an overall decrease in customer satisfaction levels which may result in some lost business.

iii. Internal competition: Different functions within the supply chain of an organization are supposed to go hand-in-hand to increase the overall value of the organization. However, these functions may end up competing for some resources at the same time thus creating un-necessary competition internally. This could be reversed by integrated planning thus resulting in optimum usage of available internal resources.

iv. Duplication of efforts: Absence of coordination may also lead to execution of similar tasks being performed at multiple points along the supply chain. This incurs some avoidable expenses that could be used for better purposes. For example, maintaining and updating databases of finished products inventory independently at the manufacturing location and distribution location will lead to duplication. Instead, a central location of the database being maintained and
updated in real-time, and accessed from anywhere throughout the supply chain, will avoid the duplication expenses.

v. Lack of vision: Short-term focus on individual functions of the supply chain may provide a quick solution to a temporary problem, but it may not provide a strategic edge which comes with a long-term vision. Its consequences will be evident in the longer time horizon when initial strategic decisions play a major role.

Management philosophy needs to take into consideration the dynamism of today’s industrial practices. This necessitates a comprehensive view of the activities that constitute the supply chain. An important foundational basis for a sound SCM system is active cooperation between its members resulting in collaborative planning at each stage. This collaborative planning should encompass the functionalities of supply chain entities, their interactions, and their integrations. This not only improves the overall performance but also captures the true essence of a SCM system. According to a book published by the NRC, there are some worldwide trends that are necessitating the integration of supply chain activities. The following factors can be counted as the driving forces for an integrated approach to supply chain management:

- Increased cost competitiveness
- Shorter product life cycles
- Faster product development cycles
- Globalization and customization of product offerings
• Faster response times for higher customer satisfaction
• Higher overall quality

Integration can be carried out for different aspects of a company’s supply chain. According to Shapiro, there are four dimensions to integration -

i. Functional integration: It relates to the different functions performed by the organization, for example, purchasing, location, manufacturing, warehousing, and transportation.

ii. Spatial integration: This type of integration is done over a target group of supply chain entities – vendors, manufacturing facilities, warehouses, and markets.

iii. Hierarchical integration: It refers to integration of the overlapping decisions in the strategic, tactical, and operational planning horizons. For example, it is important to strategically locate the manufacturing facilities with respect to the vendors to help in optimizing the supply policies.

iv. Enterprise integration: It emphasizes the importance of strategic and tactical planning decisions like, integration of supply chain management with demand management and financial management to maximize the revenues and also to increase long-term returns on investments.

There are numerous benefits of integration that can help the company to improve its performance by way of collaborative planning and execution. According to NRC, one of the most visible and obvious benefits for a manufacturing company is in the area of inventory control. Due to combined optimization of the various functions in a
manufacturing environment, such as, location, production, and distribution, inventory can be controlled at optimum levels leading to lower holding costs, reduction of required warehouse space, reduced material handling activities, and timely deliveries. Another benefit mentioned is the reduction in transaction costs due to high level of information sharing in an integrated environment. Real-time updates, reduced paperwork due to e-transactions, and reduced execution times, leads to lower transaction costs for each entity in the supply chain. Additional benefits include the following:

- Function and procedural synergy between participants
- Lower response times to changing market conditions
- Reduced bottlenecks
- Decrease in level of redundant activities
- Cheaper cost of manufacturing operations
- Reduction is investment levels due to elimination of redundant manufacturing capacity
- Increased competitiveness

1.3 Configuration parameters

Due to an increasing popularity of trends like globalization and mass customization, supply chain configurations involving multiple facility locations for different target markets, and multi-product systems to cater to a vastly diverse customer base, are becoming prevalent in multinational companies. This, although, makes the supply chain more complex to handle, holds a lot of promise as a potentially huge cost saving tool.
1.3.1 Multi-facility location

Organizations are increasingly adopting a strategy to establish multiple facilities to service varied markets in different geographical locations. This strategy provides a lot of advantages – first, it results in proximity to the market and customers thus providing a better understanding of both. It also results in quicker reading of changing market conditions and thus lower response times for adjusting future plans accordingly. Also, the social aspect of this decision is providing higher employment opportunities to the local community so it feels a sense of belonging to the products they are buying. Secondly, due to a relatively higher number of plants, each of which is specially suited for the target market, there is a huge opportunity for consolidation of the company’s product portfolio. Every target market has its own set of favorite products which the local plant can concentrate upon. This helps in increasing the production efficiencies of those plants with respect to products due to mass production or near-mass production opportunities. Thirdly, multiple plant locations can increase system reliability (i.e. in case of failure or shut down of one facility, there would still be a number of others facilities to service the on-going demand thus decreasing the risk of losing customers and market share). The demands could be diverted or spread out to other facilities with excess or idle capacities. The fourth advantage is the reduction of transportation times due to proximity to the markets. Thus demands can be satisfied in quicker times leading to higher customer satisfaction. Finally, it can also provide more options for future expansion and diversification plans. The plants servicing markets with high growth rates could be expanded to service specific demands without adding any redundancy in those plants.
On the other hand, there are some disadvantages to this strategy. First, setting up of each additional facility entails some fixed costs which would be incurred every time a new facility is set up. These costs include the land costs, building costs, etc. Then, it also requires taking measures to adhere to some local laws that are specific for each location. Third, advantages of mass production may be lost due to spreading out of production quantities. Also, although the transportation times decrease, some costs related to transportation may be added due to maintenance of additional fleets for each facility and additional drivers for each plant’s individual transportation network. In case of fewer facilities there could be lesser idle times for trucks as well as drivers thus increasing their productivities.

### 1.3.2 Multi-product systems

Most big companies now-a-days tend to keep a diverse product portfolio to cater to a large and ever-changing market. New trends like mass-customization of products has presented the customers with almost limitless opportunities thus making it important for companies to have diverse production capabilities at each plant. This helps to satisfy a huge customer base, although rendering the operations management function more difficult. Multi-product systems increases the overall complexity in managing the operations due to increased number of inputs, additional production lines, setup and changeover costs involved, storage and transportation requirements, and so on. On the other hand, failure to analyze these issues would result in inefficient handling of these
systems, which may result in reduction of market share, decreased popularity, or overall growth stagnation.

The advantages of the above-mentioned strategies of multi-facility and multi-product systems make them a better option to adopt. The disadvantages, although, have to be considered in the decision-making process while finalizing a particular organization’s policies. This, however, can be done not just by evaluating each of these strategies individually but by integrated decision-making involving all the governing factors and supply chain functions together, in a single model. This collective approach analyzes quantitatively all the constraints – individual as well as conflicting – to prescribe the best performing solution from a large solution space.

Collaborative functioning and collective decision making is thus turning out to be the key for effective planning. A holistic study of the interactions of supply chain activities is becoming the need of the hour. In other words, an analytical treatment of this collective problem domain will lead to global optimization of the different business functions and in turn, of the whole supply chain. In light of these observations, this research attempts to address the need for an integrated approach to supply chain management.

The organization of this document is as follows. Chapter 1 provides an introduction to supply chain management – its basics, functions, configurations. It also provides a brief account of the need for integration in supply chain modeling. Chapter 2 presents a literature review of the previous work done in the field of supply chain optimization in
general, and integrated analysis, in particular. Chapter 3 presents the proposed integrated supply chain optimization model in detail. It is then followed by an application of the model for a sample problem in Chapter 4.
Chapter 2 Literature review

There is a vast amount of literature available on supply chain management research dealing with the different aspects of the subject. Numerous models in the literature, conceptual as well as quantitative, refer to the planning and quantitative aspects of the different business functions – location, production, inventory and transportation. Research has also been done in considering these areas for combined optimization. Proposed models include a combination of two or more of these areas for integration. This section presents a review of some of the work done till date which deals with these areas on an individual as well as on a collective basis.

2.1 Integrated models

One of the first steps to run an effective supply chain is the strategic positioning of the manufacturing facilities, warehouses and distribution centers. Canel and Khumawala [8] provide a review of the literature for the uncapacitated multi-period international facilities location problem (IFLP). They formulate a mixed-integer programming (MIP) model and solve it using the branch and bound method. The decision variables include the countries to locate manufacturing facilities, and their production and shipping levels. They also provide a case study with an application of the solution procedure. The performance measurement criteria used here is the number of nodes needed to reach the optimal solution, and to verify the optimality. Computational time was also considered as another performance indicator. The research also identifies the scope for multi-stage problems wherein the location of manufacturing facilities would be accompanied by the
optimal location of distribution centers (DC). Canel et al. [9] later develop heuristic procedures for solving the MIP model of a similar IFLP. The heuristic procedures are tested for their computational efficiency. The profit maximization problem also considers the other important factors for international location problems, such as, exchange rates, export incentives, tariffs, taxes, and so on. The multi-period model also takes into account the time-dependent variations in prices, costs, and demands.

A number of quantitative models use mixed-integer programming (MIP) to solve the supply chain optimization problems. One of the first attempts was done by Geoffrion and Graves [24], where a MIP model was formulated for the multicommodity location problem. This seminal research involved the determination of distribution center (DC) locations, their capacities, customer zones and transportation flow patterns for all commodities. A solution to the location portion of the problem was presented, based on Bender’s Decomposition (BD). The transportation portion of the problem is decoupled into a separate classical transportation problem for each commodity. Their approach shows a high degree of effectiveness and advantage of using BD over branch-and-bound. The technique has been applied on a real problem to test its performance. However, the computational requirements and technical resources required for its implementation make it a difficult choice.

Cohen and Lee 1988 [11], develop an analytical model to establish a materials requirements policy based on stochastic demand. They develop four different sub-models with a minimum-cost objective. A mathematical algorithm at the end decides the optimal
ordering policies to minimize the costs. Cohen and Lee 1989 [12] also develop a deterministic analytical MIP model to maximize the global after-tax profits through optimal policies for facility network design and material flows. The decision variables for the network design issues include location and capacities of all production facilities whereas those for material management issues include sourcing decisions, production and distribution planning. The model decides the optimal resource deployment for a particular policy option. It thus shows the robustness of the global manufacturing network which provides the company with increased flexibility in responding to changing scenarios by adjusting the sourcing, production, and distribution plans.

A MIP model for a production, transportation, and distribution problem has been developed by Pirkul and Jayaraman [35] to represent a multi-product tri-echelon capacitated plant and warehouse location problem. The model minimizes the sum of fixed costs of operating the plants and warehouses, and the variable costs of transporting multiple products from the plants to the warehouses and finally to the customers. A solution procedure is provided based on Lagrangian relaxation (LR) to find the lower bound, followed by a heuristic to solve the problem. The research shows computationally stable results for this combined approach. The heuristic, in particular, performs well with respect to approximations of optimality and solution times. Fumero and Vercellis [20] also use LR to solve a MIP model for integrated multi-period optimization of production and logistics operations. Two approaches are presented – integrated and decoupled – and results are compared. The objective in the integrated approach is to minimize the inventory, setup and logistics costs by employing LR which enables the separation of
production and logistics functions although it facilitates global optimization of the objective function in the integrated model. In other words, the main model is decomposed into four sub-models: production, inventory, distribution, and routing. The approach separates the capacitated lot-sizing and vehicle routing decisions. In the decoupled approach, the production and logistics problems are analyzed and solved separately in two different models. The results from these two approaches are then compared to analyze their performance. Over a domain of problems of varying sizes, the overall performance ratio varied by around 10%. The final analysis shows that for problems of greater sizes (number of products, customers, time periods), the integrated approach gains relevance.

Sabri and Beamon [37] present a multi-objective multi-product multi-echelon stochastic model that simultaneously address strategic and operational planning while taking into consideration the uncertainty in demand, and production and supply lead-times. The authors point to some gaps in the SC literature – most stochastic models presented so far consider only up to two echelons. Conversely, other larger models are mostly deterministic in nature. Moreover, these deterministic models consider only profitability and ignore other performance measures. Another observation is that strategic and operational levels are not considered simultaneously. The main model presented here consists of a mixed integer linear programming (MILP) sub-model for the strategic level, to determine the optimal number and location of manufacturing facilities and DCs, and assignment of service regions to DCs. The stochastic operational level sub-model is an extension of Cohen et al. 1988 [11], in that, it considers simultaneous optimization of
non-linear production, distribution and transportation costs. A solution algorithm is then presented to integrate these two sub-models to achieve an overall supply chain performance vector. The approach presented here mainly emphasizes on integration of strategic and operational level decisions while considering demand uncertainty.

Ganeshan [21] presents an integrated model that synchronizes the inventory at retailers and warehouse, and the demand at the warehouse. One of the important contributions of this paper is the development of an inventory-logistics framework that includes inventory and transportation components in the total cost function. It also identifies the need for a model that caters to more than two echelons in the supply chain and a better way to model random delay at the warehouse. Barbarosoglu et al. [2] use Lagrangean Relaxation (LR) to solve a MIP model for an integrated production-distribution system. A heuristic is also designed to aid in the problem solving thus further increasing the overall efficiency of the procedure. Due to the increased complexity of large scale integrated models, the paper highlights the importance of developing alternative solution techniques which are able to provide near optimal solutions.

Dhaenens-Flipo and Finke [16] consider an integrated production-distribution problem in a multi-facility, multi-product and multi-period environment. The problem is modeled as a network flow problem with an objective to match products with production lines to minimize the related costs. The network with three elements – productions lines, warehouses, and customers – has been initially modeled for a single period and then advanced to obtain a multi-period model. Due to the large number of variables and
constraints involved in the multi-period model, it is significantly reduced in size by reducing the density of the network. In other words, of all the possible warehouse links going to a customer, only the cheapest 20% links are considered thus reducing the number of variables significantly. The problems are generated randomly and solved using CPLEX software.

Chandra and Fisher [10] compare the computational aspects of solving the production and distribution problems separately and in a combined model. A number of cases with different values for the various parameters for these models are analyzed to compare the performances. These parameters include duration of planning period, number of products, number of customer, fixed costs, inventory costs etc. A cost minimization model is developed along with constraints to present the problem in its integrated form. In the segregated approach, the production scheduling problem is solved using previously presented solutions by Barany et al. 1984 and then a heuristic solution to the vehicle routing problem is presented. In the combined approach, the results from the individual problems are taken and an improvement heuristic is applied to obtain modifications to the solutions that reduce costs. Final analysis shows an improvement of up to 20% in some cases for the integrated approach. The authors propose that a segregated approach is suitable if there is a sufficient reserve of finished products between the production and shipping phases. On the other hand, an integrated approach is suitable for cases where the distribution costs are significantly higher and the related networks are dense.
Lodree, Klien, and Jang [31] consider the integration of customer waiting time with the production-distribution functions in a supply chain. Optimization policies for models with combinations of production, inventory and transportation costs, along with customer waiting times have been proposed to determine the production rate and the sequence of vehicle shipments.

Previous work in the integration of supply chains has generally concentrated on two domains – production/distribution problems explained above and production-location problems. Hurter and Martinich [28] present a detailed study of integrated production-location (PL) problems. Their book presents two categorizes of problems – deterministic and stochastic in nature – with respect to parametric certainty. Quantitative models have been presented for these categories of problems on a line, plane and network respectively. These include not just single-facility but also multi-facility models to highlight their computational complexity. It also presents capacitated as well as uncapacitated formulations of production-location problems and the algorithms to obtain solutions to these problems. Since this research considers only capacitated location problems, it refers to these type of formulations presented in the book. The models presented in the book provide a better illustration of how to integrate the production and location functions and the modeling issues involved. This book underlines the interdependence of these functions and emphasizes the need to consider them simultaneously in strategic decision-making. Its also states that in general PL problems, when the potential plant locations are finite, then the cost functions consist of a fixed cost and a constant cost times the output. The PL problem then reduces to a fixed charge (FC) problem wherein the optimal
location of the plants, the output levels, and the shipping plan for supplying to the customers, are determined. The model considered in this research tilts more towards a FC type of problem because the plants are to be located from a finite set of potential locations.

Bhutta et al. [5] presents a mixed integer linear programming model for international facility location decisions considering exchange and tariff rates. Along with location, production and distribution functions, investment level was also considered as one of the decision variables. This profit maximization model represents the integration of all of the above mentioned factors thus providing an insight into how they are affected due to global factors such as exchange and tariff rates. Determination of international facility location decisions in this model is thus based on a collective analysis of the various supply chain factors such as production capacity, distribution patterns and also investment levels. Encouraging results have been obtained in terms of the model performance and results thus emphasizing the need for integrated supply chain analysis.

Benjamin [4] analyzes the inventory and transportation cost minimization problem addressing the decisions such as production lot size at each facility, total amount shipped from each facility and the amount shipped every few weeks after receiving the order. This problem is treated as a combination of the economic production lot size problem, the transportation problem and the economic order quantity problem. Three basic costs have been identified – production cost, transportation cost and inventory cost. The author concludes that integrated formulation can be applied to a wide variety of problems such
as multi-stage trans-shipment problems with setup costs. Also, setup and fixed transportation costs can be analyzed simultaneously.

### 2.2 Continuous approximation models

Masel and Pujari [32] consider the integration of customer waiting costs in integrated supply chain analysis using a continuous approximation approach. Expressions for customer waiting cost, inventory cost, and the transportation cost are obtained in three separate sub-models. These sub-models are then analyzed collectively in a single comprehensive minimization model for simultaneous optimization. Classical optimization is then used to solve this model to obtain a closed form solution to determine the optimal number of shipments that minimizes the total costs. This is one of the few attempts to include qualitative aspects of the supply chain, such as customer satisfaction in terms of waiting time reduction, in the overall analysis. Another important aspect of this paper is the use of continuous approximation to model the inventory distribution patterns, which has been adopted in this research as well. This approach helps in obtaining a clear understanding of and accurate modeling of the inventory patterns given their tendency to follow periodic cycles. The optimal value for number of shipments obtained in the end results in a total minimum cost that includes customer waiting cost, inventory cost, and transportation cost.

Dasci and Verter [14] consider the integration of production and distribution functions by proposing an alternative approach based on the use of continuous approximation of costs and demands as opposed to discrete MIP models. Simultaneous, instead of sequential
optimization of configuration decisions in production-distribution networks is proposed to avoid sub-optimality. Decision variables are the number and locations of facilities and their service regions. Closed form solutions are obtained to minimize the fixed costs of facility location and operation, and the transportation costs. Burns et. al. [6] focus on minimizing the transportation and inventory costs by comparing two strategies – direct shipping and peddling. They provide an analytic approach while using spatial density of customers instead of discrete locations and obtain expressions for shipment size.

Langevin et al. [30] present an overview of continuous approximation models developed for freight distribution problems. The authors categorize these models into six classes depending on one or more origins, one or more destinations, and with or without transshipments. The review shows that use of continuous approximation models in conjunction with optimization methods proves to be a powerful tool for problem solving. Also, these models provide significant analysis capabilities for decisions on strategic as well as operational levels due to their use of concise summaries of data rather than detailed data points. The authors point to the sparse interaction in the past between mathematical programming and continuous approximation but also predict an increased development of combined models due to their complementary nature.

Other research carried out in this area, for example, Hall [27], Geoffrion [23], Daganzo [13], Newell [34], Daskin [15], advocate the use of continuous approximation models to supplement mathematical programming instead of replacing it. This is due to the advantages of continuous approximation such as ease of interpretation, use of minimal
information, and reliance on distribution functions rather than exact data points. Robuste et al. [36] and Klincewicz et al. [29] also suggest a combinatorial approach using continuous approximation and mathematical programming to solve a variety of problems.

2.3 Model implementation: Case-studies

Arntzen et al. [1] develop a Global Supply Chain Model (GSCM) at Digital Equipment Corporation, wherein a MILP takes into account multiple products, facilities, echelons, time periods and transportation modes. It minimizes cost and/or weighted cumulative production and distribution times. It is solved using branch-and-mound enumeration. Several nontraditional methods are used such as elastic constraints, row factorization, cascaded problem solution, and constraint-branching enumeration. The global bill of materials has been used for multi-stage and multi-location model implementation. The GSCM provides a general approach for quick-response deterministic modeling of global supply chain.

Glover et al. [25] present a case study at Agrico Chemical Company where an integrated production, distribution and inventory (PDI) system was implemented to assist in long range and short range decision making such as location and size of DCs long term inventory investments, product allocation etc. The PDI planning system consisting of a network model helped in making these decisions by minimizing production costs, inventory costs, transportation costs etc. This case study shows the actual monetary benefits of implementing an integrated PDI system along with non-quantifiable benefits such as thoroughly understanding the complete supply chain of the company.
Camm et al. [7] decomposed the supply chain problem at Proctor & Gamble into two sub-problems: a distribution-location problem and a product-sourcing problem. A combination of integer programming, network optimization models, and geographical information system (GIS) was used to solve this problem. It shows the integration of the sub-models and highlights the importance of a hybrid approach containing human judgment and optimization. These papers provide a good account of the commercial implementation of developed models thus detailing the issues arising from real world applications.

2.4 Model reviews

Many researchers have also presented a review of the work done in integrated SCM models. These analyses provide a chronological progress of the evolution of SCM models, a description of the solution techniques, specific issues addressed in the models and also the directions for future research. Vidal and Goetschalckx [41] provide a critical and comprehensive review of the production-distribution models in global supply chain systems. An explanation of the different optimization models is presented with emphasis on MIP models, followed by a brief review of analytic approach. Further comments on some modeling issues include review of work done to explain the additional aspects of formulating the models such as aggregation of suppliers, qualitative aspects of inventory management, common errors in aggregation of customers, and so on. Areas for future research are suggested, which include considering stochastic aspects of global supply chains, bill of materials (BOM) constraints, qualitative factors, exchange rates, taxes,
duties, etc. The authors point to the lack of research in integrating the different components of a production-distribution system.

Sarmiento and Nagi [38] review the research done in combined optimization of the production-distribution functions. Only models with two echelons are considered and they are classified according to the production, distribution and the related inventory decisions taken at each echelon. The classification is also based on finite/infinite time horizons. Research in inventory/routing problems is also analyzed followed by a comprehensive account of suggestions for future research. The areas that need to be considered in more details for inventory/distribution problems include constrained transportation systems, complex networks, multiple product systems, location of multiple customers and depots, emergency shipments, vehicle routing, and inventory level setting. Apart from searching for optimal solutions, more research could be done in the area of heuristic procedures and validation methods. Stochastic demands is another area that needs further exploration. For production/distribution problems, further research is proposed in minimum safety stock requirement for guaranteed reliability, analytical formulations, and study of interrelations between production and distribution. The authors also express the need for simultaneous integration of multiple functions along with considering non-linear transportation costs, variable travel times in the analysis of distribution systems.

Erenguc, Simpson, and Vakharia [17] provide a review of the important decision elements in each of the three stages of the supply chain network – supplier, plant, and
distribution. The supplier stage analysis deals with issues like supplier selection, number of suppliers, and the volumes of shipments from each of them. A multi-product multi-stage inventory model is formulated for the plant stage considering linear inventory holding costs, fixed costs for replenishment, and no capacity constraints. The distribution stage reviews in detail the important factors for issues such as distribution network, location/allocation decisions, and inventory decisions. General multi-product cost minimization models are presented for the last two issues. The directions for future research emphasize on an integrative approach to inventory management, importance of information sharing, and development of analytical and simulation models to integrate the three stages of the supply chain.

Goetschalckx, Vidal, and Dogan [26] review some mathematical programming models and their solution algorithms in global logistics systems. Two models are then presented to focus on the development of an integration methodology for strategic and tactical decisions. The first model deals with the determination of transfer prices for global supply chains. The second one is a MILP model that deals with multi-period production and distribution allocation in a domestic environment, which is solved by using Bender’s decomposition method. It is concluded that such models require a significant amount of technical expertise to handle. Areas that require further research include consideration of transportation mode selection in global logistics systems, allocations of transportation costs among subsidiaries, considering inventory costs simultaneously, and nonlinear effects of international taxation.
Beamon [3] presents a comprehensive review of the research in multi-stage supply chain modeling along with a direction for future research in this area. Multi-stage supply chain models have been categorized into deterministic, stochastic, economic and simulation models. The paper categorizes supply chain performance measures into two categories - qualitative and quantitative. It also lists the decision variables usually used in supply chain modeling. Selecting the correct set of performance measures for the chosen decision variables is suggested to be an important area for future research. Various other modeling issues such as product postponement, demand distortion and supply chain classification schemes are also considered important for future research.

Thomas and Griffin [40] present a review of models addressing coordinated multi-stage supply chain planning. Their research observes a dearth of efforts in this area due to the poor quality of available data as well as increased complexity of search models. The paper presents a review of models in two major categories – strategic and operational. Decision variables in strategic models mostly include plant or DC locations, resource allocation, etc. whereas those for operational models include selection of batch size, transportation mode, production quantity. The paper concludes that independent management of supply chain entities results in poor overall behavior. Moreover, the recent trend of globalization necessitates coordinated supply chain planning although it leads to increased complexity. An important observation is that transportation cost is the largest component of logistics cost thus emphasizing its importance. Also, there is insufficient research on supply chain coordination at an operational level. These
observations thus justify the necessity of an integrated multi-stage supply chain optimization model.

Schmidt and Wilhelm [39] present a review of the work done on different decisional levels in the supply chain with respect to time frames – strategic, tactical and operational. Modeling issues are discussed at each level and a prototype formulation is provided as an extension of the discussion. The objective of the deterministic multi-product prototype model is to maximize profit by considering fixed costs of locating facilities and sourcing materials, and variable costs of manufacturing and transportation operations. This research considers mainly high-technology industries for analysis of logistics networks. One of the conclusions states that decisions at higher levels affect those at lower levels by way of influencing the constraints. Also, the different time frames at each level must be integrated to achieve the desired performance levels.

The problem of simultaneous analysis of different supply chain functions has attracted many researchers and has produced a considerable amount of literature in this area. Fredendall and Hill [19] and Mentzer [33] are good examples in this context. However, there are some unexplored areas that need to be researched to fill the voids in the field of integrated supply chain modeling. This research represents an attempt in this direction along with suggesting a path for extrapolation of research efforts needed to develop an integrated approach. As noted earlier, majority of the models presented so far have been concentrated mostly on the integration of location-production or production-distribution functions. So the aim here is to develop an integrated model which, along with optimal
capacitated facility location, also represents the cost optimization of the production and distribution functions, and inventory flow in the supply chain of an organization. A capacitated facility location function in a multi-product multi-echelon production-distribution system is considered here, with deterministic demand and linear transportation costs. The specific aim is to develop a quantitative model to represent a capacitated-location/production/distribution (CLPD) problem. Analysis of each of these functions in an integrated framework would help in better coordination of the supply chain.
Chapter 3 Supply chain integration model

In this research we consider a multi-product, multi-facility, and multi-customer location-production-distribution system. The system contains a set of manufacturing facilities with limited production capacities situated within a geographical area. Each of these facilities can produce one or all of the products in the company’s portfolio. The customer demands for multiple products are to be satisfied from this set of manufacturing facilities. There are fixed costs associated with each facility location which may include land costs, construction and fabrication costs etc. Although we assume that the customer allocation has to be done within the existing set of manufacturing facilities, sometimes it may be necessary to make changes or expansions in the current facilities to accommodate the production quantities which ultimately will prove to be beneficial. Costs for these changes would be included in the fixed costs. So the production capacities of each of these facilities effectively represent its current and potential capacities.

Although a product requires multiple inputs, the input costs provided for each facility represents the total cost of procuring all the inputs required for producing each unit of output. This includes the purchasing costs as well as transportation costs for the inputs. The unit manufacturing costs for producing the products at each facility includes the setup costs, machining costs, assembly costs and similar processing costs. The unit input costs and manufacturing costs for the same product may be different for each facility due to various factors such as location, proximity to input sources, regional labor costs, transportation costs, taxes, government regulations etc. These factors will therefore
understandably play a role in the selection of that facility and the extent to which it will satisfy the customer demand. The customer locations and their distances from each of the facilities are also known. So the transportation costs required to ship the products from the manufacturing facilities to the customers would also be important for plant location and customer allocation. Moreover, sometimes urgent orders need that the products be shipped to the customers at the earliest which means transportation times are also important for customer satisfaction.

The integrated model developed here is solved using a two-phased approach. Phase I involves the formulation of a mixed integer problem which helps in determining the strategic level location-allocation decisions. Phase II uses the results from first phase as an input and fine-tunes them further to provide a detailed distribution plan that optimizes the transportation and inventory costs. The main model is an integration of the following functions:

- **Capacitated location function**: This function deals with the capacitated location of the manufacturing facilities and the allocation of customer demand to these facilities. Taking into consideration the different constraints such as capacities, demands, and fixed and variable costs at each facility, this function helps in the optimal location of manufacturing facilities to satisfy customer requirements within the constraint framework.
• Production function: The capacities of the manufacturing facilities and the customer demands act as some of the inputs to the production function which determines the production quantities of every product at each of the facilities.

• Distribution function: The demand for each product from the customers and other parameters such as holding costs, fixed costs and transportation costs determine the inventory control policies and the average inventory levels. This function receives inputs such as travel times, transportation costs, etc. to optimize the decisions such as number of shipments, shipment schedules, and shipment sizes.

This research considers two approaches for solving the capacitated-location/production/distribution (CLPD) problem:

• Mixed integer programming – A MIP model is developed to represent the integrated problem and solved in the first phase using OPL software which deploys the Branch and Bound (BB) algorithm. The output from this model is the optimal location of manufacturing facilities and the customer demand allocation to these facilities.

• Continuous approximation – Phase II uses continuous approximation to model the material flow in the system and to determine the optimal distribution plan based on the different shipping patterns considered. It determines the number of
shipments, and the shipment sizes for each shipping pattern thus providing a customized solution for each company-customer relationship.

A combinatorial strategy is thus developed wherein the two approaches are combined to first obtain the upper level solutions in the decision hierarchy, followed by lower level optimization. This phased approach helps in reducing the complexity of the model as well as to achieve the overall system integration.

3.1 Model development

The objective of this research is to develop an integrated model to represent a capacitated location/production/distribution (CLPD) problem. The problem, therefore, is modeled as a mixed integer linear programming formulation that seeks to optimize capacitated facility location and customer allocation decisions, and production quantities at these locations to satisfy customer demands. Additionally, the model also aims at optimizing the number of shipments from these facilities to minimize the inventory and transportation costs. A broad outline of the integrated model is shown below in Figure 3.1.
As described earlier, the integrated model is implemented in two phases. The first phase uses the given data to optimize the decisions with respect to location, production, and customer allocation. These results are then used in the second phase along with additional data to obtain the number of shipments, shipment sizes, and optimal inventory and transportation costs. Along with including data such as inventory holding costs, customer waiting costs, and so on, this additional data will also include the “inventory distribution patterns” that may be unique not only for each customer but also for each of its products. These patterns are explained later in the next section. This two-phased approach lets us use the initial solution to the CLPD problem from the first phase and further improve its quality in the next phase. In other words, it meets the broader objectives by fixing the plant locations, and allocating customer demands of each product to each of these plants thus setting their production levels while taking into consideration the plant capacities,
input and manufacturing costs, and shipping logistics. This solution is then fine-tuned and taken one step further to obtain a shipment plan such that inventory and transportation costs are minimized. Separating the two phases is essential because after obtaining the plant-to-customer shipping pairs for each product, this approach lets us customize the shipment plans for each of these pairs according to the customer’s needs specified by their inventory distribution patterns. A detailed framework of the two-phased integrated model is shown below in Figure 3.2.
Figure 3.2 Detailed framework of integrated model
3.2 Assumptions

1. Each manufacturing facility is able to produce all of the products.

2. The selling price for a product may vary from customer to customer depending on the negotiations, order sizes, discounts, historical relationships, etc.

3. The company and customer have agreed beforehand on the inventory distribution pattern so the shipping plans would be formulated accordingly.

4. Shipments of each product from each facility are going in truckload quantities and no truck carries mixed shipments.

5. The number of shipments is always even.

3.3 Phase I - Model formulation

The parameters and the decision variables used to formulate the model are shown in Table 3.1 below.
### Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{kp}$</td>
<td>Selling price of product $p$ for customer $k$</td>
</tr>
<tr>
<td>$F_j$</td>
<td>Fixed cost for plant location at site $j$</td>
</tr>
<tr>
<td>$u_{jp}$</td>
<td>Total cost of inputs required for product $p$ being produced for customer $k$ at plant $j$</td>
</tr>
<tr>
<td>$m_{jp}$</td>
<td>Unit manufacturing cost for product $p$ being produced for customer $k$ at plant $j$</td>
</tr>
<tr>
<td>$r_{jk}$</td>
<td>Fixed cost for a shipment from plant $j$ to customer $k$</td>
</tr>
<tr>
<td>$t_{jk}$</td>
<td>Travel time per shipment from plant $j$ to customer $k$</td>
</tr>
<tr>
<td>$g_{jk}$</td>
<td>Transportation cost per unit time per shipment from plant $j$ to customer $k$</td>
</tr>
<tr>
<td>$D_{kp}$</td>
<td>Total demand for product $p$ from customer $k$</td>
</tr>
<tr>
<td>$W_{jp}$</td>
<td>Total plant capacity for product $p$ at plant $j$</td>
</tr>
</tbody>
</table>

### Decision variables

<table>
<thead>
<tr>
<th>Decision variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_j$</td>
<td>Binary variable to select a plant location at site $j$</td>
</tr>
<tr>
<td>$Q_{jkp}$</td>
<td>Production quantity of product $p$ for customer $k$ at plant $j$</td>
</tr>
</tbody>
</table>
The formulation of the objective function and constraints is shown below. J, K, P represent the total number of locations, customers, and products respectively.

Maximize

\[
\sum_{j=1}^{J} \sum_{k=1}^{K} \sum_{p=1}^{P} S_{kp} Q_{jkp} - \left( \sum_{j=1}^{J} \sum_{k=1}^{K} \sum_{p=1}^{P} \left( F_{j} x_{j} + \sum_{k=1}^{K} \sum_{p=1}^{P} \left( u_{jp} + m_{jp} \right) Q_{jkp} + \left( r_{jk} + t_{jk} g_{jk} \right) \frac{Q_{jkp}}{e_{p}} \right) \right)
\]

subject to

\[
\sum_{j=1}^{J} \sum_{p=1}^{P} Q_{jkp} = \sum_{p=1}^{P} D_{kp} \quad \forall k \quad (3.1)
\]

\[
\sum_{j=1}^{J} \sum_{k=1}^{K} Q_{jkp} = \sum_{k=1}^{K} D_{kp} \quad \forall p \quad (3.2)
\]

\[
\sum_{j=1}^{J} Q_{jkp} = D_{kp} \quad \forall k, p \quad (3.3)
\]

\[
\sum_{k=1}^{K} Q_{jkp} \leq W_{jp} \quad \forall j, p \quad (3.4)
\]

\[
\sum_{k=1}^{K} \sum_{p=1}^{P} Q_{jkp} \leq M x_{j} \quad \forall j \quad (3.5)
\]

\[
x_{j} \in \{0, 1\} \quad (3.6)
\]

\[
S_{kp}, Q_{jk}, W_{jp}, D_{kp} \geq 0 \quad \forall j, k, p \quad (3.7)
\]

\[
u_{jp}, m_{jp} \geq 0 \quad \forall j, p \quad (3.8)
\]

\[
r_{jk}, t_{jk}, g_{jk} \geq 0 \quad \forall j, k \quad (3.9)
\]

\[
e_{p} > 0 \quad \forall p \quad (3.10)
\]
The model is formulated as a profit maximization problem. The objective function seeks to maximize the profit by subtracting total costs from the total revenues generated.

Constraint (3.1) ensures that the total amount of products being manufactured at all plants for a particular customer is equal to the total demand of all products from that customer. Similarly constraint (3.2) ensures that the total amount of a particular product being manufactured at all plants for all customers is equal to the total demand of that product from all customers. It is important to note here that the first two constraints are stated separately to show better accountability of the total demands from all customers and for all products respectively. Whereas the first two constraints consider the demands from all customers and for all products in terms of total quantities, constraint (3.3) considers the demand of each product from each customer individually. It thus ensures that the total amount of a specific product being manufactured for a particular customer at all plants is equal to the demand of the product from that customer. Constraint (3.4) presents the capacity constraint i.e. the total amount of a product being manufactured at a particular plant for all customers is less than or equal to the plant capacity for that product. Constraint (3.5) makes sure that a plant is located when and only if there is a demand for any product. In other words, when the demand is zero, no plant is located, whereas, in the case of a non-zero demand, a plant is definitely located to satisfy that demand. Here, \( M \) is a very large positive integer. Constraint (3.6) shows that \( x_j \) is a binary variable, and constraints (3.7)-(3.10) avoid any negative quantities.
3.3.1 Total revenues

The total revenues are obtained by multiplying the selling price of each product for each customer by its production quantity. The summation is carried out for all products (P) for all customers (K) at all plants (J). As mentioned earlier the sales price for a product may vary from customer to customer depending on the negotiations done with each one beforehand.

\[ \sum_{j=1}^{J} \sum_{k=1}^{K} \sum_{p=1}^{P} S_{kp} Q_{jkp} \]

3.3.2 Total costs

The total costs required to satisfy demand for all products from all the customers is a combination of fixed costs, inputs costs, manufacturing costs and transportation costs as is evident from the objective function. Each of these costs is explained in detail below.

3.3.2.1 Fixed costs

Locating a facility at a site usually requires fixed costs such as land acquisition costs, facility construction costs etc. For this model we assume that the manufacturing facilities are to be located on an existing set of potential locations. These potential locations may or may not have a fully functional manufacturing facility. Moreover, the existing facilities may not have sufficient capacities to meet the demand requirements. In these cases, the facility may need to be fully constructed from the beginning or may undergo some expansions to meet the projected demands. In this model, the costs related to these construction/expansion activities are considered to be fixed costs. It is assumed here that
these fixed costs (F\textsubscript{j}) for each potential location j are known beforehand based on the current capacity and projected demand. So the fixed costs for a potential location will only incur if the facility is ultimately located there. The binary variable x\textsubscript{j} takes a value of one or zero, depending on whether the facility is located at site j or not.

\[ \text{Fixed costs} = \sum_{j=1}^{J} F_j x_j \]

### 3.3.2.2 Input costs

In a mass production industry environment, usually multiple inputs are required to produce a single product. These may be obtained from multiple sources with varying prices. The purchasing function looks into the aspect of purchasing all the inputs at the best available price in the market. The model presented here, considers that the input sources have already been selected and the prices are already known. Also, there may be transportation charge for procuring these inputs from the suppliers. So the input costs (u\textsubscript{jp}) for product p considered here include the purchasing costs (p\textsubscript{c,jp}) of all the inputs required for product p, and also the transportation costs (s\textsubscript{c,jp}) to procure them, as shown below.

\[ \text{Input costs, } u_{jp} = p_{c,jp} + s_{c,jp} \]
Although the same inputs are required to produce a product at any plant, the costs required to obtain those inputs may vary for different plants depending on the location of the plant, its distance from the input sources, market rates in that area, etc.

3.3.2.3 *Manufacturing costs*

After the inputs have been purchased and transported to the manufacturing facility, different manufacturing operations are carried out to transform the inputs into the finished products. These processes range from pre-treating, for example, in a chemical industry to casting and heat treatment in steel industry. The final stages may also include some assembly operations to finally obtain the required output. The costs required to carry out all these operations are considered as the manufacturing costs in the model.

\[ \text{Manufacturing costs} = m_{ip} \]

As in the case of input costs, the manufacturing costs for the same product also may vary for different plants. This is because these costs depend on factors such as labor rates, overheads, etc. which may be vary significantly for each plant.

3.3.2.4 *Transportation costs*

In this model we assume that the finished products are transported from the manufacturing facility to the customer’s warehouse or the customer’s manufacturing facility for further use. Further, these products are transported in truckloads at regular intervals. There is a fixed cost \( r_{jk} \) for every shipment which is incurred every time a
shipment takes place. Depending on the plant-customer allocation pairs, the travel times \( t_{jk} \) may vary according to the distances. It is assumed that the travel time is linearly proportional to the distance between the plant and the customer. However, the transportation costs may or may not be exactly proportional to the travel times because the transportation costs per unit time per shipment \( g_{jk} \) may vary for each plant-customer pair depending on the route conditions, climate conditions, geographical factors, etc.

\[
\frac{\text{Transportation cost per unit time}}{\text{shipment}} = g_{jk}
\]

For every product, \( e_p \) represents the maximum number of units of product \( p \) that can be shipped on every shipment. This quantity may vary for each product depending on the size, weight, or other special handling requirements for that product. For example, a computer-box will occupy less amount of space as compare to a washing machine. The total transportation cost is thus given by the addition of fixed costs and variable costs for every shipment. The quantity \( (Q_{jkp}/e_p) \) represents a uniform benchmark to compare the transportation costs for each plant-customer pair for every product.

3.3.3 Capacities and demands

Out of the existing set of potential locations for setting up a manufacturing facility, majority is assumed to be locations having an existing facility with a current capacity to produce a set of products, although some of them may be greenfield locations with no capacity at all. The capacity of an existing facility, in turn, may not be sufficient to satisfy the customer demands which would entail expanding the capacities as required. The
capacities for a plant refer to those for multiple products as each plant may contain several production lines for each of the products. So expansion, in some cases, would mean adding extra productions lines for products with high demands. Depending on various constraints, demand for a particular product from a single customer may be satisfied from different manufacturing facilities.

3.3.4 Output

As stated earlier, the output of the model formulated above in the first phase is the location of the manufacturing facilities and the allocation of customer demands to each of these facilities. In other words, the output is obtained in the form of values for the following decision variables:

1. \( x_j \): Plant locations
2. \( Q_{jkp} \): Production quantity of product \( p \) for customer \( k \) at plant \( j \)

The facilities are chosen from the given set of existing locations for which the capacities, fixed costs, input and manufacturing costs, and transportation costs are known beforehand. The customer demands are also known which are satisfied from the capacities of the located facilities. Phase I provides a general optimization of the CLPD problem on a broader scale thus representing higher level decisions in the model hierarchy. It takes into consideration an overall account of the various constraints to suggest a solution that is optimal from a long-term perspective. The decision to locate a
facility at a particular site has a strategic impact on the system thus influencing its performance over a greater time horizon.

3.4 Phase II – Model enhancement through optimized distribution

After obtaining the plant locations and customer demand allocations there is a need to fine-tune the distribution plan. As the inventory distribution patterns have been pre-decided, the next step is to determine the optimal number of shipments that would minimize the total inventory holding costs, penalty costs, and transportation costs. The inventory distribution patterns are governed by the following factors:

- **Unit production time (y):** This is the time taken to produce one unit of a product at any manufacturing facility. It is assumed that for a given product, the unit production time does not vary for each facility.

- **Penalty cost (w):** It is described as the cost associated with the time the customer has to wait to receive the product after placing the order. It is assumed that the customer expects the delivery of the products as soon as possible. However, since the products are manufactured after an order being placed, it takes some time for the products to be delivered to the customer. The penalty cost is incurred for each product in the order for each time unit until the product is delivered. The penalty cost is negotiated with the customer beforehand at the time of placing the order. It may also be formulated internally in the organization as a tool to measure the losses due to tardy customer deliveries. These losses could be explained as short-
term losses such as order cancellations or long-term losses such as losing business from customers due to reduced satisfaction levels. Moreover, this may also result in a deterioration of the company’s image in the market. Although some of these may not be purely tangible losses, they still need to be included in the analysis in order to increase the accuracy of the model with respect to the costs incurred.

- **Inventory holding cost (h):** It is the cost associated with storing the product inventory until it is shipped to the customer. This cost is incurred between when the product is manufactured at the facility, and its delivery to the customer.

- **Lead time between shipments (L):** The products may not be shipped to the customer as soon as they are manufactured due to factors such as non-availability of trucks or insufficient workers. There is also some time required for actual loading of the products in the trucks. Sometimes the products may first travel to the DC from where they may be finally shipped to the customer. The time taken due to these factors is collectively considered here as the lead time between shipments.

There are other factors also that indirectly affect the inventory distribution pattern such as number of machines available for a given time period, number of trucks available at any instant, shipping rate, travel time, unit travel costs, and shipment size.
A list of parameters and decision variables used in phase II is given below in Table 3.2.

Table 3.2 Parameters and decision variables for phase II

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_p$</td>
<td>Unit production time for product $p$</td>
</tr>
<tr>
<td>$w_p$</td>
<td>Penalty cost for product $p$ per unit time</td>
</tr>
<tr>
<td>$h_p$</td>
<td>Inventory holding cost of product $p$</td>
</tr>
<tr>
<td>$L_p$</td>
<td>Lead time between shipments for product $p$</td>
</tr>
<tr>
<td>$Q_{jkp}$</td>
<td>Quantity of product $p$ produced for customer $k$ at plant $j$</td>
</tr>
<tr>
<td>$r_{jk}$</td>
<td>Fixed cost of shipment from plant $j$ to customer $k$</td>
</tr>
<tr>
<td>$t_{jk}$</td>
<td>Travel time per shipment from plant $j$ to customer $k$</td>
</tr>
<tr>
<td>$g_{jk}$</td>
<td>Transportation cost per unit time per shipment from plant $j$ to customer $k$</td>
</tr>
</tbody>
</table>
The second phase represents a production/distribution supply chain in which the products are manufactured at the plants and shipped to customers in multiple shipments at regular intervals, until the demand is satisfied. After the commencement of production, products equivalent to a shipment size (V) are produced. The products would be stored in the inventory if the shipment is not possible immediately. If it is possible, then the products are loaded on the trucks to be shipped to the customer. Products continue to be produced during the time the first batch is being shipped. As explained earlier, this time is considered as the lead time between shipments. The plant continues manufacturing the
product until customer demand is met. It is possible to store the whole order and ship it at the end of production. However, this option would incur higher inventory cost for storing a large number of products for a long time. It will also incur penalty costs because the customer would have to wait till the end of production to receive the products. It is assumed here that the customer is ready to accept the products as and when the shipment takes place. Another assumption for ease of exposition is that the number of shipments (N) is even for all cases considered.

There can be different cases of inventory distribution patterns based on the difference between production rate (PR) and shipping rate (SR) (i.e. \( i = 1, 2, 3 \)), continuous or intermittent production and/or shipping, and instantaneous or gradual shipping. These patterns will in turn influence the inventory costs, penalty costs and transportation costs. Six different cases are described below along with the expressions for each of the costs. These six cases have been identified to model as many scenarios as possible in a generalized framework. Although there can be almost infinite number of cases depending on variations in each of the governing conditions, the cases described here represent the most generic situations that may occur due to these variations. In order to categorize the problem domain it was necessary to draw the line somewhere, albeit, in such a way that any of the possible scenarios can be categorized in at least one of these cases.
3.4.1 Case 1: Continuous production, instantaneous shipping; PR = SR (i.e. i = 1)

In this case the production and shipping functions are synchronized and occur at the same rate (i.e. i = 1). As soon as products equivalent to the first shipment size are produced, the shipping activity is commenced resulting in products being shipped to the customer at the end of time (Vy+L). This represents the time taken to produce V products and the lead time for the first shipment. The subsequent shipments occur at regular intervals of time (Vy) until the full order is shipped. This case has been considered in Masel and Pujari [32], where closed form expressions have been obtained for inventory costs, customer waiting costs, and transportation costs. This can be considered as the simplest and one of the most commonly occurring cases. Figure 3.3 below shows the inventory distribution pattern for case 1.

Figure 3.3 Case 1: Inventory distribution pattern
Total inventory cost, $C_I$:

Total inventory,

$$\frac{L^2}{2y} + \frac{N'y^2 - V^2y}{2} + (NVL) + \left(\frac{V^2}{2} - VL + \frac{L^2}{2y}\right) + \left(VL - \frac{L^2}{y}\right)$$  \hspace{1cm} (3.4.1.1)

$$= \frac{N'y^2 + NVL}{2} = \frac{Q^2}{2N} + QL \quad \text{[because...]} V = \frac{Q}{N}$$  \hspace{1cm} (3.4.1.2)

The shipment size ($V$) is obtained by dividing the total production quantity ($Q$) by the number of shipments ($N$). This relation is used to determine the expression for inventory cost in the subsequent cases.

Total inventory holding cost,

$$C_i = \left(\frac{Q^2}{2N} + QL\right)(h) = \frac{Q^2y^h}{2N} + QLh$$  \hspace{1cm} (3.4.1.3)

Total customer waiting cost (penalty cost), $C_P$:

Waiting cost incurred for the first shipment,

$$= (V_y + L + t)wV$$  \hspace{1cm} (3.4.1.4)

Total waiting cost for $N$ shipments,

$$C_P = \sum_{i=1}^{N} (iV_y + L + t)wV = \left(\frac{Qy(N+1)}{2N} + L + t\right)wQ$$  \hspace{1cm} (3.4.1.5)

Total transportation cost, $C_T$:

$$C_T = (r + tg)N$$  \hspace{1cm} (3.4.1.6)
These costs are then added, and differentiated with respect to N and equated to zero, to obtain the expression for optimal number of shipments which minimizes total distribution costs.

\[
\text{Minimize} \quad C_{\text{ip}} = \left( \frac{Qy(N + 1)}{2N} + L + t \right) wQ + \frac{Q^2yh}{2N} + QLh + (r + tg)N
\]  \hspace{1cm} (3.4.1.7)

\[
\frac{\partial}{\partial N} C_{\text{ip}} = \frac{\partial}{\partial N} \left( \left( \frac{Qy(N + 1)}{2N} + L + t \right) wQ + \frac{Q^2yh}{2N} + QLh + (r + tg)N \right) = 0
\]  \hspace{1cm} (3.4.1.8)

Optimal number of shipments,

\[
N = \sqrt{\frac{Q^2y(h + w)}{2(r + tg)}}
\]  \hspace{1cm} (3.4.1.9)

By substituting the value of N, in equations (3.4.1.2), (3.4.1.3), (3.4.1.5), and (3.4.1.6), the optimal value for shipment size and minimum values for inventory, penalty, and transportation costs are obtained.
3.4.2 Case 2: Two-tiered production rate

In this case production occurs at two different rates while the shipping rate remains constant. As soon as products equivalent to the first shipment size are produced, the shipping activity is commenced resulting in products being shipped to the customer at the end of time (Vy). It represents the time taken to produce V products and the lead time for the first shipment. The subsequent shipments occur at regular intervals until the full order is shipped. However, due to the two-tiered production rate the duration of these intervals changes as the production rate changes.

The change in production rate may occur due to various constraints such as availability of machines, workers, machine sharing due to multi-product lines, and so on. Alternately, there could be constraints due to inventory storage limitations on the customer side due to which it may not be possible for the customer to take delivery of the products. So the production rate could be decreased to avoid inventory build-up at the manufacturing facility.

However, there is a choice in deciding the difference between the initial (PR1) and latter (PR2) production rates. So we consider that their relation is given by,

\[ PR1 = i \times (PR2) \]  

(3.4.2.1)

where,

\[ i \in R, i > 1 \]

For the case where \( i < 1 \), the overall inventory level will be almost similar to that for \( i > 1 \), the magnitude of \( i \) on one side being inverse of that on the other side. The inventory
costs will depend on the actual value of i. However, the penalty costs will be higher for i < 1 because higher number of products will be shipped late. The case with i > 1 is considered here for analysis as shown in Figure 3.4.

The shipments occur at regular intervals of time (V_y) during the first half and time (iV_y) during the second half until the full order is shipped. For ease of computation, it is assumed that the number of shipments is even.

Also, for ease of computation and exposition, it is assumed that the production rate is faster for the first half and slower for the next half of the production cycle. It is also assumed that the change does not occur while a shipment is being produced, but at the end of production. There can be potentially infinite number of cases depending on when
and how many times the production rate changes throughout the cycle. So to model this system behavior to some extent, it becomes necessary to consider some degree of specification in the scenario.

**Inventory cost**

The total inventory held during the full shipping cycle is equal to the total area covered by the graph. It is divided into two sub-areas which are calculated as follows:

\[
\text{Area I: } \left[ \frac{1}{2} (iV_y)(3V) \right] \frac{N}{2} = \frac{NV^2 y}{4} \tag{3.4.2.2}
\]

\[
\text{Area II: } \left[ \frac{1}{2} (iV_y)(V) \right] \frac{N}{2} = \frac{iNV^2 y}{4} \tag{3.4.2.3}
\]

Total inventory = \(\frac{(i + 1)NV^2 y}{4}\) \(\tag{3.4.2.4}\)

This expression actually represents the total inventory held over the entire time horizon. It can thus be expressed as the total inventory-timeunits which, when multiplied by the inventory holding cost, determines the total inventory cost.

So the inventory cost is given by,

\[
C_I = \left( \frac{(i + 1)NV^2 y}{4} \right) h = \frac{(i + 1)Q^2 yh}{4N} \tag{3.4.2.5}
\]

**Penalty cost**

Time required for first shipment = \((iV_y + t)\) \(\tag{3.4.2.6}\)

Penalty cost for first shipment = \((iV_y + t)wV\) \(\tag{3.4.2.7}\)
Similarly, penalty cost for second shipment = \((2V_y + t)wV\)  \hspace{1cm} (3.4.2.8)

So, penalty cost for \([N/2]\)\(^{th}\) shipment,

\[
= \left\{ \frac{N}{2} \sum_{i=1}^{N} (nV_y) + t \right\} wV = \frac{N^2 V_y^2 yw}{8} + \frac{NV_y^2 yw}{4} + NV_tw
\hspace{1cm} (3.4.2.9)
\]

Similarly,

Penalty cost for \([N/2 + 1]\)\(^{th}\) shipment = \(\left(\frac{N}{2} + 1\right)V_y + t\)\)wV  \hspace{1cm} (3.4.2.10)

Penalty cost for \([(N-1)/3 + 2]\)\(^{th}\) shipment = \(\left(\frac{N}{2} + 2\right)V_y + t\)\)wV  \hspace{1cm} (3.4.2.11)

So, penalty cost for \(N\)\(^{th}\) shipment,

\[
= \left\{ \sum_{i=1}^{N} \left(\frac{N}{2} + n\right) iV_y + t \right\} wV = \frac{3iN^2 V_y^2 yw}{8} \frac{iNV_y^2 yw}{4} + NV_tw
\hspace{1cm} (3.4.2.12)
\]

So total penalty cost for 1 to \(N\) shipments is given by,

\[
C_p = \frac{(3i+1)N^2 V_y^2 yw}{8} + \frac{(i+1)NV_y^2 yw}{4} + NV_tw
\hspace{1cm} (3.4.2.13)
\]

\[= \frac{(3i+1)Q^2 yw}{8} + \frac{(i+1)Q^2 yw}{4N} + Qtw\]

\[\text{Transportation cost}\]

Transportation cost is given by,

\[
C_T = (r + tg)N
\hspace{1cm} (3.4.2.14)
\]
The costs are then added, and differentiated with respect to $N$ and equated to zero, to obtain the expression for optimal number of shipments which minimizes total distribution costs.

\[
\text{Minimize } C_{ipt} = \frac{(i+1)Q^2 yh}{4N} + \frac{(3i+1)Q^2 yw}{8} + \frac{(i+1)Q^2 yw}{4N} + Qtw + (r + tg)N \quad (3.4.2.15)
\]

\[
\frac{\partial}{\partial N} C_{ipt} = -\frac{(i+1)Q^2 yh}{4N^2} - \frac{(i+1)Q^2 yw}{4N^2} + (r + tg) = 0 \quad (3.4.2.16)
\]

Optimal number of shipments,

\[
N = \sqrt[4]{\frac{(i+1)Q^2 y(h+w)}{4(r + tg)}} \quad (3.4.2.17)
\]
3.4.3 Case 3: Continuous production, instantaneous shipping; PR > SR

Here the production function occurs at a faster rate as compared to the shipping function. Time \((Vy)\) represents the time taken to produce \(V\) products. The first shipment of products is shipped to the customer after time \(L\), which is the lead time between two shipments. The lead time between shipments is high enough to permit production of more than \(V\) units between each shipment. The higher lead time between shipments may occur due to factors such as limited availability of transport services, limited availability of workers, or other constraints. The expression for lead time is determined by the difference between production rate and shipping rate. Let us consider that their relation is given by,

\[
PR = i \times (SR) \tag{3.4.3.1}
\]

where,

\[
i \in R, i > 1
\]

Then lead time is given by,

\[
L = i \times V \times y \tag{3.4.3.2}
\]

For the case where \(i \leq 1\), Case 3 becomes Case 1.

The subsequent shipments occur at regular intervals of time \((L=iVy)\) until the full order is shipped. For ease of computation, it is assumed that the number of shipments is even. In this type of scenario, optimization of number of shipments, and in turn, shipment size would be needed mostly when transportation costs are sufficiently higher than inventory costs. If they are not, it would still be cost effective to ship products as and when shipping is possible, without optimization of shipment size.
Depending on the value of \( i \), the total number of shipments (\( N \)), and the inventory distribution pattern, the number of shipments sent during and after production follow a fixed ratio. After analysis of a number of cases and their distribution patterns, the following Table 3.3 is prepared which accurately determines the ratio for a particular case.

Table 3.3 Ratio of shipments during and after production

<table>
<thead>
<tr>
<th>Value of ( i ) for ( PR=i(SR) )</th>
<th>Number of shipments</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N=2 )</td>
<td>( N=4 )</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>( i=2 )</td>
<td>1:1</td>
</tr>
<tr>
<td>( i=3 )</td>
<td>1:3*</td>
</tr>
<tr>
<td>( i=4 )</td>
<td>1:3</td>
</tr>
<tr>
<td>( i=5 )</td>
<td>1:5*</td>
</tr>
<tr>
<td>( i=6 )</td>
<td>1:5</td>
</tr>
<tr>
<td>( i=7 )</td>
<td>1:7*</td>
</tr>
</tbody>
</table>
The table shows ratios \( x:y \) where, \( x \) and \( y \) are the number of shipments during and after production respectively. In most of the cases, the production and shipment activities are carried out simultaneously until the production stops. After that, shipments are continued from the existing inventory levels. However, in special situations marked with an asterisk (*) when values of \( i \) are odd numbers, for some cases the production may need to be extended to produce an additional shipment quantity to obtain an even number of total shipments. This slight modification in the inventory pattern is carried out to satisfy our underlying assumption of considering only even number of total shipments so that all cases could be treated on an equal scale. These modifications are evident from the inventory patterns in cases with \( i = 3 \). The table shows the ratios for limited values of \( N \) and \( i \), the basic purpose being to show the underlying patterns of periodicities in the ratios. This table could be further extended for more values of \( N \) and \( i \) for future research purposes.

Also, based on the value of \( i \), there can be two sub-cases as described below. For this research, it is assumed that \( i \) will take values ranging from 1 to 3 in integers. This assumption takes into consideration real-life situations where it would be feasible to expect that production rate will not exceed the shipping rate by a factor of more than 3. Also, for the ease of exposition, it is assumed that \( i \) will take only integer values. This assumption helps in discretization of the problem domain, making it easier to analyze. Since the case for \( i = 1 \) has already been considered, we now consider two sub-cases for \( i = 2 \) and \( i = 3 \) respectively.
Case 3a: Continuous production, instantaneous shipping; \( PR = i \left( SR \right); \quad i = 2 \)

In this sub-case, the production rate is twice the shipping rate as shown in Figure 3.5. This implies that products equivalent to two shipment sizes are produced during the lead time between shipments. Further, the production of entire order is completed when only half the number of shipments is completed. The remaining products are stored in the inventory while shipments are being carried out at regular intervals until the entire order is shipped and inventory level reaches zero.

Figure 3.5 Case 3a: Inventory distribution pattern
Inventory cost

The total inventory held during the full shipping cycle is equal to the total area covered by the graph. It is divided into two sub-areas which are calculated as follows:

\[ \text{Area I: } \left[ \left( \frac{1}{2} \right) (2Vy)(2y) \right] \frac{N}{2} = N^2 y \cdot 2 \quad (3.4.a.3) \]

\[ \text{Area II: } 2 \sum_{n=1}^{N-1} nN (2Vy) + \left( \frac{N}{2} \right) V(2Vy) = \frac{N^2 y^2 y}{2} - N^2 y^2 y + N^2 y^2 y \quad (3.4.a.4) \]

\[ \text{Total inventory} = N^2 y + \frac{N^2 y^2 y}{2} = \frac{N^2 y^2 y}{N^2} + \frac{N^2 y^2 y}{2N^2} \quad (3.4.a.5) \]

So the inventory cost is given by,

\[ C_i = \left( \frac{N^2 y^2 y}{N^2} + \frac{N^2 y^2 y}{2N^2} \right) h = \frac{Q^2 y h}{N} + \frac{Q^2 y h}{2} \quad (3.4.a.6) \]

Penalty cost

Time required for first shipment = \((2Vy + t)\) \quad (3.4.a.7)

Penalty cost for first shipment = \((2Vy + t)wV\) \quad (3.4.a.8)

Penalty cost for second shipment = \((4Vy + t)wV\) \quad (3.4.a.9)

Total penalty cost from 1 to \(N/2\) shipments,

\[ = \sum_{n=1}^{N} (n \cdot 2Vy + t)wV = \frac{N^2 y^2 w}{4} + \frac{NV^2 y w}{2} + \frac{NVtw}{2} \quad (3.4.a.10) \]

Similarly,
Penalty cost for \((N/2+1)^{th}\) shipment = \(\left(\left(\frac{N}{2} + 1\right)2Vt + t\right)wV\) \hspace{1cm} (3.4.3a.11)

Penalty cost for \((N/2+2)^{th}\) shipment = \(\left(\left(\frac{N}{2} + 2\right)2Vt + t\right)wV\) \hspace{1cm} (3.4.3a.12)

Total penalty cost from \(N/2+1\) to \(N\) shipments,
\[
= \sum_{n=1}^{N} \left(\left(\frac{N}{2} + n\right)2Vt + t\right)wV = \frac{3N^2V^2yw}{4} + \frac{NV^2yw}{2} + \frac{NVtw}{2}
\] \hspace{1cm} (3.4.3a.13)

So total penalty cost for 1 to \(N\) shipments is given by,
\[
C_p = N^2V^2yw + NV^2yw + NVtw = Q^2yw + \frac{Q^2yw}{N} + Qtw
\] \hspace{1cm} (3.4.3a.14)

**Transportation cost**

Transportation cost is given by,
\[
C_T = (r + tg)N
\] \hspace{1cm} (3.4.3a.15)

The costs are then added, and differentiated with respect to \(N\) and equated to zero, to obtain the expression for optimal number of shipments which minimizes total distribution costs.

\[\text{Minimize} \; C_{ipr} = \frac{Q^2yh}{2} + \frac{Q^2yh}{N} + Q^2yw + \frac{Q^2yw}{N} + Qtw + (r + tg)N\] \hspace{1cm} (3.4.3a.16)

\[
\frac{\partial}{\partial N} C_{ipr} = -\frac{Q^2yh}{N^2} - \frac{Q^2yw}{N^2} + (r + tg) = 0
\] \hspace{1cm} (3.4.3a.17)

Optimal number of shipments,
\[
N = \sqrt{\frac{Q^2y(h+w)}{2(r+tg)}}
\] \hspace{1cm} (3.4.3a.18)
Case 3b(i): Continuous production, instantaneous shipping; \( PR = i \times (SR); \quad i = 3; \)

\( N \) not a multiple of 3

In this sub-case, the production rate is thrice the shipping rate. This implies that products equivalent to three shipment sizes are produced during the lead time between shipments. Further, the production of entire order is completed when only one-third number of shipments are completed. This sub-case has been further classified into two cases depending on whether the total number of shipments is a multiple of three or not. This classification helps in accurately modeling the inventory and penalty costs which are different due to the slight change in the distribution patterns. Due to our underlying assumption that \( N \) is an even number, when \( N \) is not a multiple of three (case 3b(i)), a batch of \( V \) products has to be manufactured at the end of production. When \( N \) is a multiple of three (case 3b(ii)), there is no need to produce this small batch, and shipment can commence right after the last batch of \( 3V \) products has been produced. This is evident in Figures 3.6 and 3.7, at the top of the distribution curves.
Inventory cost

The total inventory held during the full shipping cycle is equal to the total area covered by the graph. It is divided into three sub-areas which are calculated as follows:

Area I: \[
\left[ \frac{1}{2} (3V_y)(3V) \right] \frac{N-1}{3} = \frac{3NV^2y}{2} - \frac{3V^2y}{2}
\] (3.4.3bi.1)

Area II: \[
\left( \frac{1}{2} \right) (V_y)(V) + (2V_y)(V) = \frac{5V^2y}{2}
\] (3.4.3bi.2)

Area III: \[
2 \left[ \sum_{n=1}^{N-1} n(2V_y)(3V_y) \right] + \left[ \sum_{n=1}^{N-1} (2n-1)(V_y)(3V_y) \right] = N^2V^2y - V^2y
\] (3.4.3bi.3)

Total inventory = \[
\frac{3NV^2y}{2} + N^2V^2y
\] (3.4.3bi.4)
So the inventory cost is given by,

\[ C_t = \left( \frac{3N^2y^2}{2} + N^2V^2y \right)h + \frac{3Q^2yh}{2N} + Q^2yh \] (3.4.3bi.5)

**Penalty cost**

Time required for first shipment = \((3Vy + t)\) (3.4.3bi.6)

Penalty cost for first shipment = \((3Vy + t)wV\) (3.4.3bi.7)

Similarly, penalty cost for second shipment = \((6Vy + t)wV\) (3.4.3bi.8)

So, penalty cost for \([N-1]/3\)th shipment,

\[
= \sum_{n=1}^{N-1} \left( n(3Vy) + \frac{N-1}{3} t \right) wV = \frac{N^2V^2yw}{6} + \frac{NV^2yw}{6} - \frac{V^2yw}{3} + \frac{NVtw}{3} - \frac{Vtw}{3} \] (3.4.3bi.9)

Similarly,

Penalty cost for \([N-1]/3 + 1\)th shipment = \(\left( \frac{N-1}{3} + 1 \right)3Vy + t \)wV (3.4.3bi.10)

Penalty cost for \([N-1]/3 + 2\)th shipment = \(\left( \frac{N-1}{3} + 2 \right)3Vy + t \)wV (3.4.3bi.11)

So, penalty cost for Nth shipment,

\[
= \sum_{n=1}^{2N+1} \left( \frac{N-1}{3} + n \right)3Vy + t \)wV = \frac{8N^2V^2yw}{6} + \frac{8NV^2yw}{6} + \frac{V^2yw}{3} + \frac{2NVtw}{3} + \frac{Vtw}{3} \] (3.4.3bi.12)

So total penalty cost for 1 to N shipments is given by,

\[
C_p = \frac{3N^2V^2yw}{2} + \frac{3NV^2yw}{2} + NVtw = \frac{3Q^2yw}{2} + \frac{3Q^2yw}{2N} + Qtw \] (3.4.3bi.13)
Transportation cost

Transportation cost is given by,

\[ C_T = (r + tg)N \]  (3.4.3bi.14)

The costs are then added, and differentiated with respect to \( N \) and equated to zero, to obtain the expression for optimal number of shipments which minimizes total distribution costs.

\[
\text{Minimize } C_{iPT} = \frac{3Q^2y_h}{2N} + \frac{3Q^2y_w}{2} + \frac{3Q^2y_w}{2N} + Qtw + (r + tg)N
\]  (3.4.3bi.15)

\[
\frac{\partial}{\partial N} C_{iPT} = -\frac{3Q^2y_h}{2N^2} - \frac{3Q^2y_w}{2N^2} + (r + tg) = 0
\]  (3.4.3bi.16)

Optimal number of shipments,

\[
N = \sqrt{\frac{3Q^2y(h+w)}{2(r+tg)}}
\]  (3.4.3bi.17)
**Case 3b(ii): Continuous production, instantaneous shipping;**  
\[ PR = i \ (SR); \quad i = 3; \]

*N is a multiple of 3*

Figure 3.7 shows the inventory distribution pattern for this case.

![Inventory distribution pattern](image)

Figure 3.7 Case 3b(ii): Inventory distribution pattern

**Inventory cost**

The total inventory held during the full shipping cycle is equal to the total area covered by the graph. It is divided into two sub-areas which are calculated as follows:

\[
\text{Area I:} \quad \left[ \frac{1}{2} (3V) (3V) \right] \frac{N}{3} = \frac{3NV^2}{2} y \quad (3.4.3\text{bii.1})
\]
Area II: 
\[2 \left( \sum_{n=1}^{\frac{N-1}{3}} n(3V_y + t) \right) + \left( \sum_{n=1}^{N} (2n-1)(3V_y + t) \right) + \left( \frac{2N}{3} \right)(3V_y + t) = N^2V^2y \]  
(3.4.3bii.2)

Total inventory = \[\frac{3NV^2y}{2} + N^2V^2y\]  
(3.4.3bii.3)

So the inventory cost is given by,
\[C_I = \left( \frac{3NV^2y}{2} + N^2V^2y \right)h = \frac{3Q^2y}{2N} + Q^2yh\]  
(3.4.3bii.4)

**Penalty cost**

Time required for first shipment = \(3V_y + t\)  
(3.4.3bii.5)

Penalty cost for first shipment = \((3V_y + t)wV\)  
(3.4.3bii.6)

Similarly, penalty cost for second shipment = \((6V_y + t)wV\)  
(3.4.3bii.7)

So, penalty cost for \([\frac{N}{3}]^{th}\) shipment,
\[= \left( \sum_{n=1}^{\frac{N}{3}} (n(3V_y)) + t \right)wV = \frac{N^2V^2y}{6} + \frac{NV^2yw}{2} + \frac{NVtw}{3}\]  
(3.4.3bii.8)

Similarly,

Penalty cost for \([\frac{N}{3} + 1]^{th}\) shipment = \(\left( \frac{N}{3} + 1 \right)3V_y + t \)\(wV\)  
(3.4.3bii.9)

Penalty cost for \([\frac{N-1}{3} + 2]^{th}\) shipment = \(\left( \frac{N}{3} + 2 \right)3V_y + t \)\(wV\)  
(3.4.3bii.10)

So, penalty cost for \(N^{th}\) shipment,
\[= \left( \sum_{n=1}^{\frac{2N}{3}} \left( \left( \frac{N}{3} + n \right)3V_y + t \right) \right)wV = \frac{4N^2V^2yw}{3} + NV^2yw + \frac{2NVtw}{3}\]  
(3.4.3bii.11)

So total penalty cost for 1 to \(N\) shipments is given by,
The optimal number of shipments, $N$, is given by:

$$N = \sqrt{\frac{3Q^2y(h+w)}{2(r+tg)}}$$  \hspace{1cm} (3.4.3bii.16)
3.4.4 Case 4: Gradual shipping

In this case production and shipping functions occur simultaneously and every shipment is dispatched gradually in lots instead of one instantaneous dispatch. So while the lots are being shipped gradually, the inventory decreases proportionally. On the other hand, simultaneous production increases the inventory. Thus the resultant inventory levels depend on the relative difference between the production and shipping rates.

When products equivalent to the first shipment size are produced, the shipping activity is commenced at the end of time (Vy). The products are shipped in lots, where the lot size is smaller than the shipment size. Each shipment consists of ‘a’ lots. The lead time for shipping each lot is assumed to be negligible due to smaller size. It is also assumed that the shipment of lots for each shipment can be started only after the whole shipment has been produced. Each shipment occurs at regular intervals until the full order is shipped. There are three sub-cases depending on the difference between PR and SR.

Case 4a: Gradual shipping: \( PR = SR \)

Here the production and shipping occur at same rates as shown in Figure 3.8. When V products are produced, the first shipment commences. However, each shipment is shipped gradually in ‘a’ lots. Thus the shipping lot size \( (V/a) \) is shipped while production is in progress. Subsequent lots are shipped at regular intervals until the entire shipment is shipped. The cycle is repeated for all shipments until the entire order is shipped. It is assumed here that all lot sizes are equal for each shipment.
Inventory cost

The total inventory held during the full shipping cycle is equal to the total area covered by the graph. It is divided into three sub-areas which are calculated as follows:

Area I: \( \left( \frac{1}{2} \right)(V_y)(V) = \frac{V^2y}{2} \) \hspace{1cm} (3.4.4a.1)

Area II: \( (V_y)(V)(N-1) = NV^2y - V^2y \) \hspace{1cm} (3.4.4a.2)

Area III: \( \left( \frac{1}{2} \right)(V_y)(V) = \frac{V^2y}{2} \) \hspace{1cm} (3.4.4a.3)

Total inventory = \( NV^2y \) \hspace{1cm} (3.4.4a.4)
So the inventory cost is given by,

\[ C_I = (NV^2y)_h = \frac{Q^2y}{N} \]  \hspace{1cm} (3.4.4a.5)

**Penalty cost**

Time required for first lot of first shipment

\[ [V_y + \left( \frac{V}{a} \right)_y + t] \]  \hspace{1cm} (3.4.4a.6)

Penalty cost for first lot of first shipment

\[ [V_y + \left( \frac{V}{a} \right)_y + t]w\left( \frac{V}{a} \right) \]  \hspace{1cm} (3.4.4a.7)

Similarly,

Penalty cost for second lot of first shipment

\[ [V_y + \sum_{n=1}^{a} \left( \frac{V}{a} \right)_y + t]w\left( \frac{V}{a} \right) \]  \hspace{1cm} (3.4.4a.8)

So, penalty cost for a\textsuperscript{th} lot of first shipment

\[ [V_y + \sum_{n=1}^{a} \left( \frac{V}{a} \right)_y + t]w\left( \frac{V}{a} \right) \]  \hspace{1cm} (3.4.4a.9)

Similarly,

Penalty cost for a\textsuperscript{th} lot of second shipment

\[ [2V_y + \sum_{n=1}^{a} \left( \frac{V}{a} \right)_y + t]w\left( \frac{V}{a} \right) \]  \hspace{1cm} (3.4.4a.10)

Penalty cost for a\textsuperscript{th} lot of N\textsuperscript{th} shipment

\[ [N V_y + \sum_{n=1}^{a} \left( \frac{V}{a} \right)_y + t]w\left( \frac{V}{a} \right) \]  \hspace{1cm} (3.4.4a.11)

So total penalty cost for 1 to N shipments is given by,

\[ C_P = \sum_{b=1}^{N} b V_y + \sum_{n=1}^{a} \left( \frac{V}{a} \right)_y + t \] \quad w\left( \frac{V}{a} \right) = \frac{N^2 V^2 w}{2a} + \frac{a + 2}{2a} NV^2 y_w + \frac{NVtw}{a} \]  \hspace{1cm} (3.4.4a.12)

\[ C_P = \frac{Q^2 yw}{2a} + \frac{(a + 2)Q^2 yw}{2aN} + \frac{Qtw}{a} \] \hspace{1cm} (3.4.4a.13)
Transportation cost

Transportation cost is given by,

\[ C_T = (r + tga)N \]  

(3.4.4a.14)

The costs are then added, and differentiated with respect to \( N \) and equated to zero, to obtain the expression for optimal number of shipments which minimizes total distribution costs.

\[
\text{Minimize } C_{Ipt} = \frac{Q^2 yh}{N} + \frac{Q^2 yw}{2a} + \frac{(a + 2)Q^2 yw}{2aN} + \frac{Qtw}{a} + (r + tga)N
\]

(3.4.4a.15)

\[
\frac{\partial}{\partial N} C_{Ipt} = -\frac{Q^2 yh}{N^2} - \frac{(a + 2)Q^2 yw}{2aN^2} + (r + tga) = 0
\]

(3.4.4a.16)

Optimal number of shipments,

\[
N = \sqrt{\frac{Q^2 y \left[ 2h + \left( 1 + \frac{2}{a} \right) w \right]}{2(r + tga)}}
\]

(3.4.4a.17)
Case 4b: Gradual shipping: \( PR = i \ (SR); \quad i = 2 \)

Figure 3.9 shows the inventory distribution pattern for this case.

Inventory cost

The total inventory held during the full shipping cycle is equal to the total area covered by the graph. It is divided into two sub-areas which are calculated as follows:

Area I: 
\[
\left[ \frac{1}{2} \right] \left( 2V_y \right) (V) \right] N = NV^2y
\]

Area II: 
\[
2 \sum_{n=1}^{N-1} n(V) (2V_y) = \frac{N^2V^2y}{2} - NV^2y
\]
Total inventory = \( \frac{N^2y^2}{2} \) \hspace{1cm} (3.4.4b.3)

So the inventory cost is given by,

\[
C_i = \left( \frac{N^2y^2}{2} \right)h = \frac{Q^2yh}{2} \hspace{1cm} (3.4.4b.4)
\]

**Penalty cost**

Time required for first lot of first shipment = \( \left[ \left( \frac{V}{a} \right)y + t \right] \) \hspace{1cm} (3.4.4b.5)

Penalty cost for first lot of first shipment = \( \left[ \left( \frac{V}{a} \right)y + t \right]w\left( \frac{V}{a} \right) \) \hspace{1cm} (3.4.4b.6)

Similarly,

Penalty cost for second lot of first shipment = \( \left[ 2\left( \frac{V}{a} \right)y + t \right]w\left( \frac{V}{a} \right) \) \hspace{1cm} (3.4.4b.7)

So, penalty cost for a\(^{th}\) lot of first shipment = \( \left[ \sum_{n=1}^{a} \left( \frac{V}{a} \right)y + t \right]w\left( \frac{V}{a} \right) \) \hspace{1cm} (3.4.4b.8)

Similarly,

Penalty cost for a\(^{th}\) lot of second shipment = \( \left[ 2Vy + \sum_{n=1}^{a} \left( \frac{V}{a} \right)y + t \right]w\left( \frac{V}{a} \right) \) \hspace{1cm} (3.4.4b.9)

Penalty cost for a\(^{th}\) lot of N\(^{th}\) shipment = \( \left[ NVy + \sum_{n=1}^{a} \left( \frac{V}{a} \right)y + t \right]w\left( \frac{V}{a} \right) \) \hspace{1cm} (3.4.4b.10)

So total penalty cost for 1 to N shipments is given by,

\[
C_p = \sum_{b=1}^{b-1} \left[ (b-1)2Vy + \sum_{n=1}^{a} \left( \frac{V}{a} \right)y + t \right]w\left( \frac{V}{a} \right) \hspace{1cm} (3.4.4b.11)
\]

\[
C_p = \frac{N^2V^2}{a} + \frac{(a-1)NVy}{2a} + \frac{NVtw}{a} = \frac{Q^2y}{a} + \frac{(a-1)Q^2y}{2aN} + \frac{Qtw}{a} \hspace{1cm} (3.4.4b.12)
\]
Transportation cost

Transportation cost is given by,

\[ C_T = (r + tga)N \]  \hspace{1cm} (3.4.4b.13)

The costs are then added, and differentiated with respect to \( N \) and equated to zero, to obtain the expression for optimal number of shipments which minimizes total distribution costs.

\[
\text{Minimize } C_{ipt} = \frac{Q^2y_h}{2} + \frac{Q^2yw}{a} + \frac{(a-1)Q^2yw}{2aN} + \frac{Qtw}{a} + (r + tga)N
\]  \hspace{1cm} (3.4.4b.14)

\[
\frac{\partial}{\partial N} C_{ipt} = -\frac{(a-1)Q^2yw}{2aN^2} + (r + tga) = 0
\]  \hspace{1cm} (3.4.4b.15)

Optimal number of shipments,

\[
N = \sqrt[2a(r + tga)]{(a-1)Q^2yw}
\]  \hspace{1cm} (3.4.4b.16)
Case 4c(i): Gradual shipping; \( PR = i \text{ (SR)}; \ i = 3; \ N \text{ not a multiple of } 3 \)

Figure 3.10 shows the inventory distribution pattern for this case.

![Inventory distribution pattern](image.png)

**Inventory cost**

The total inventory held during the full shipping cycle is equal to the total area covered by the graph. It is divided into four sub-areas which are calculated as follows:

\[
\text{Area I: } \left[ \left( \frac{1}{2} \right) (3V) (2V) \right] \left( \frac{N-1}{3} \right) = NV^2 y - V^2 y \quad (3.4.4ci.1)
\]

\[
\text{Area II: } \left[ \left( \frac{1}{2} \right) (3V) (3V) \right] \left( \frac{2(N-1)}{3} \right) = NV^2 y - V^2 y \quad (3.4.4ci.2)
\]
Area III: 
\[
\left[ \sum_{n=1}^{N-1} n(2V)(3V_y) \right] + \left[ \sum_{n=1}^{2N-5} n(V)(3V_y) \right] = N^2V^2y - 2NV^2y + V^2y \]  \quad (3.4.4ci.3)

Area IV: 
\[
\left( \frac{1}{2} \right)(3V_y)\left( \frac{2V}{3} \right) = V^2y \]  \quad (3.4.4ci.4)

Total inventory = \(N^2V^2y\)  \quad (3.4.4ci.5)

So the inventory cost is given by,
\[
C_i = (N^2V^2y)h = Q^2yh \]  \quad (3.4.4ci.6)

**Penalty Cost**

Time required for first lot of first shipment = \(\left[ \left( \frac{V}{a} \right) + t \right] \)  \quad (3.4.4ci.7)

Penalty cost for first lot of first shipment = \(\left[ \left( \frac{V}{a} \right) + t \right] w\left( \frac{V}{a} \right) \)  \quad (3.4.4ci.8)

Similarly,

Penalty cost for second lot of first shipment = \(2\left[ \left( \frac{V}{a} \right) + t \right] w\left( \frac{V}{a} \right) \)  \quad (3.4.4ci.9)

So, penalty cost for \(a\)th lot of first shipment = \(\sum_{n=1}^{a} n\left[ \left( \frac{V}{a} \right) + t \right] w\left( \frac{V}{a} \right) \)  \quad (3.4.4ci.10)

Similarly,

Penalty cost for \(a\)th lot of second shipment = \(3V_y + \sum_{n=1}^{a} n\left[ \left( \frac{V}{a} \right) + t \right] w\left( \frac{V}{a} \right) \)  \quad (3.4.4ci.11)

Penalty cost for \(a\)th lot of \(N\)th shipment = \(NV_y + \sum_{n=1}^{a} n\left[ \left( \frac{V}{a} \right) + t \right] w\left( \frac{V}{a} \right) \)  \quad (3.4.4ci.12)

So total penalty cost for 1 to \(N\) shipments is given by,
\[
C_p = \sum_{b=1}^{N} \left( (b-1)3V_y + \sum_{n=1}^{a} n\left[ \left( \frac{V}{a} \right) + t \right] w\left( \frac{V}{a} \right) \right) \]  \quad (3.4.4ci.13)
\[ C_p = \frac{3N^2V^2yw}{2a} + \frac{(a-2)NV^2yw}{2a} + \frac{NVtw}{a} = \frac{3Q^3yw}{2a} + \frac{(a-2)Q^2yw}{2aN} + \frac{Qtw}{a} \]  (3.4.4ci.14)

**Transportation cost**

Transportation cost is given by,

\[ C_r = (r + tga)N \]  (3.4.4ci.15)

The costs are then added, and differentiated with respect to \( N \) and equated to zero, to obtain the expression for optimal number of shipments which minimizes total distribution costs.

\[
\text{Minimize } C_{ipr} = Q^2yh + \frac{3Q^2yw}{2a} + \frac{(a-2)Q^2yw}{2aN} + \frac{Qtw}{a} + (r + tga)N
\]  (3.4.4ci.16)

\[
\frac{\partial}{\partial N} C_{ipr} = -\frac{(a-2)Q^2yw}{2aN^2} + (r + tga) = 0
\]  (3.4.4ci.17)

Optimal number of shipments,

\[ N = \sqrt[2a(r + tga)]{\frac{(a-2)Q^2yw}{2a(r + tga)}} \]  (3.4.4ci.18)
Case 4c(ii): Gradual shipping; \( PR = i \ (SR); \ i = 3; \ N \) is a multiple of 3

Figure 3.11 shows the inventory distribution pattern for this case.

Inventory cost

The total inventory held during the full shipping cycle is equal to the total area covered by the graph. It is divided into four sub-areas which are calculated as follows:

\[
\text{Area I: } \left[ \frac{1}{2} \left(3V_y\right) \left(2V\right) \right] \left( \frac{N}{3} \right) = NV^2y
\]

(3.4.4cii.1)
Area II: \[ \left( \frac{1}{2} \right) (3V^2y) \left( \frac{2N}{3} \right) = NV^2y \] (3.4.4cii.2)

Area III: \[ N \left( \sum_{n=1}^{N} n(2V)3V^2y \right) + \left( \sum_{n=1}^{N} (2n-1)(V)3V^2y \right) = N^2V^2y - 2NV^2y \] (3.4.4cii.3)

Total inventory = \( NV^2y \) (3.4.4cii.4)

So the inventory cost is given by,

\[ C_i = (NV^2y)h = Q^2yh \] (3.4.4cii.5)

**Penalty cost**

Time required for first lot of first shipment = \( \left( \frac{V}{a} \right)^y + t \) (3.4.4cii.6)

Penalty cost for first lot of first shipment = \( \left( \frac{V}{a} \right)^y + t w \left( \frac{V}{a} \right) \) (3.4.4cii.7)

Similarly,

Penalty cost for second lot of first shipment = \( 2 \left( \frac{V}{a} \right)^y + t w \left( \frac{V}{a} \right) \) (3.4.4cii.8)

So, penalty cost for a \( ^{th} \) lot of first shipment = \( \sum_{n=1}^{a} n \left( \frac{V}{a} \right)^y + t w \left( \frac{V}{a} \right) \) (3.4.4cii.9)

Similarly,

Penalty cost for a \( ^{th} \) lot of second shipment = \( 3V^y + \sum_{n=1}^{a} n \left( \frac{V}{a} \right)^y + t w \left( \frac{V}{a} \right) \) (3.4.4cii.10)

Penalty cost for a \( ^{th} \) lot of N\( ^{th} \) shipment = \( NV^y + \sum_{n=1}^{a} n \left( \frac{V}{a} \right)^y + t w \left( \frac{V}{a} \right) \) (3.4.4cii.11)

So total penalty cost for 1 to N shipments is given by,

\[ C_p = \sum_{b=1}^{N} \left[ (b-1)3V^y + \sum_{n=1}^{a} n \left( \frac{V}{a} \right)^y + t w \left( \frac{V}{a} \right) \right] \] (3.4.4cii.12)
Transportation cost

Transportation cost is given by,

\[ C_T = (r + tga)N \]  

(3.4.cii.14)

The costs are then added, and differentiated with respect to \( N \) and equated to zero, to obtain the expression for optimal number of shipments which minimizes total distribution costs.

Minimize \( C_{ipt} = Q^2yw + \frac{3Q^2yw}{2a} + \frac{(a - 2)Q^2yw}{2aN} + \frac{Qtw}{a} + (r + tga)N \)  

(3.4.cii.15)

\[
\frac{\partial}{\partial N} C_{ipt} = -\frac{(a - 2)Q^2yw}{2aN^2} + (r + tga) = 0
\]

(3.4.cii.16)

Optimal number of shipments,

\[
N = \sqrt{\frac{(a - 2)Q^2yw}{2a(r + tga)}}
\]

(3.4.cii.17)
Case 4d: Gradual shipping: \( PR < SR \) (i.e. \( i < 1 \))

Figure 3.12 shows the inventory distribution pattern for this case.

When production rate is less than shipping rate, optimization for \( N \) is not required because it would be economical to ship the products as soon as products equivalent to one shipment size, \( V \), are produced. Also, the availability of trucks is not an issue due to lower production rate.
3.4.5 Case 5: Discontinuous and gradual shipping

Here the production occurs continuously at a constant rate. However, shipping takes place intermittently and the shipments too are gradually dispatched in lots. As in the previous case, the inventory decreases at a constant rate due to the lots being shipped gradually. On the other hand, it also continues to increase due to simultaneous production. Thus the resultant inventory levels depend on the relative difference between the production and shipping frequencies and rates. The shipping activity starts as soon as products equivalent to the first shipment size are produced. The subsequent shipments occur at regular intervals, including interruptions, until the full order is shipped.

The periodic interruptions in shipping may arise due to due to various constraints such as availability of transportation services, availability of employees, vehicle sharing due to multi-customer deliveries, and so on. Alternately, there could be constraints due to inventory storage limitations on the customer side due to which it may not be possible for the customer to take delivery of the shipments instantaneously. These factors may enforce a discontinuous and gradual approach for shipping.

As noted earlier, there can be situations when PR and SR may not be equal. However, there is a choice in deciding the difference between PR and SR. Let us consider that their relation is given by,

\[ PR = i \times (SR) \]  \hspace{1cm} (3.4.5.1)

where,

\[ i = 1,2,3 \]
Shipments occur at intervals until the full order is shipped. For ease of computation, it is assumed that the number of shipments is even. Based on the value of $i$, there are two sub-cases as described below.

**Case 5a: Discontinuous and gradual shipping;** \( PR = SR \)

Figure 3.13 shows the inventory distribution pattern for this case.

*Figure 3.13 Case 5a: Inventory distribution pattern*

**Inventory cost:**

The total inventory held during the full shipping cycle is equal to the total area covered by the graph. It is divided into two sub-areas which are calculated as follows:
Area I: \[ \left[ \frac{1}{2} (V_y)(V) + (V_y)(V) \right] N = \frac{3NV^2y}{2} \] (3.4.5a.2)

Area II: \[ 2 \sum_{n=1}^{N-1} n(V)(2V_y) = \frac{N^2V^2y}{2} - NV^2y \] (3.4.5a.3)

Total inventory = \[ \frac{N^2V^2y}{2} + \frac{NV^2y}{2} \] (3.4.5a.4)

So the inventory cost is given by,

\[ C_i = \left( \frac{N^2V^2y}{2} + \frac{NV^2y}{2} \right) + \frac{Q^2y}{2N} \] (3.4.5a.5)

**Penalty cost**

Time required for first lot of first shipment = \[ V_y + \left( \frac{V}{a} \right) y + t \] (3.4.5a.6)

Penalty cost for first lot of first shipment = \[ V_y + \left( \frac{V}{a} \right) y + t \] w\( \left( \frac{V}{a} \right) \) (3.4.5a.7)

Similarly,

Penalty cost for second lot of first shipment = \[ V_y + 2\left( \frac{V}{a} \right) y + t \] w\( \left( \frac{V}{a} \right) \) (3.4.5a.8)

So, penalty cost for \( a \)th lot of first shipment = \[ V_y + \sum_{n=1}^{a} n\left( \frac{V}{a} \right) y + t \] w\( \left( \frac{V}{a} \right) \) (3.4.5a.9)

Similarly,

Penalty cost for \( a \)th lot of second shipment = \[ 3V_y + \sum_{n=1}^{a} n\left( \frac{V}{a} \right) y + t \] w\( \left( \frac{V}{a} \right) \) (3.4.5a.10)

Penalty cost for \( a \)th lot of \( N \)th shipment = \[ NV_y + \sum_{n=1}^{a} n\left( \frac{V}{a} \right) y + t \] w\( \left( \frac{V}{a} \right) \) (3.4.5a.11)
So total penalty cost for 1 to N shipments is given by,

\[ C_p = \sum_{b=1}^{N} \left[ (2b-1)\beta y + \sum_{n=1}^{d} n\left(\frac{V}{a}\right)y + t \right] w\left(\frac{V}{a}\right) \]  \hspace{1cm} (3.4.5a.12)

\[ C_p = \frac{N^2 V^2 yw}{a} + \frac{(a+1)NV^2 yw}{2a} + NVtw = \frac{Q^2 yw}{a} \frac{(a+1)Q^2 yw}{2aN} + \frac{Qtw}{a} \]  \hspace{1cm} (3.4.5a.13)

**Transportation cost**

Transportation cost is given by,

\[ C_T = (r + tga)N \]  \hspace{1cm} (3.4.5a.14)

The costs are then added, and differentiated with respect to N and equated to zero, to obtain the expression for optimal number of shipments which minimizes total distribution costs.

\[ \text{Minimize } C_{ipt} = \frac{Q^2 yh}{2} + \frac{Q^2 yh}{2N} + \frac{Q^2 yw}{a} + \frac{(a+1)Q^2 yw}{2aN} + \frac{Qtw}{a} + (r + tga)N \]  \hspace{1cm} (3.4.5a.15)

\[ \frac{\partial}{\partial N} C_{ipt} = -\frac{Q^2 yh}{2N^2} - \frac{(a+1)Q^2 yw}{2aN^2} + (r + tga) = 0 \]  \hspace{1cm} (3.4.5a.16)

Optimal number of shipments,

\[ N = \sqrt{\frac{Q^2 y\left[h + \left(1 + \frac{1}{a}\right)w\right]}{2(r + tga)}} \]  \hspace{1cm} (3.4.5a.17)
**Case 5b(i): Discontinuous and gradual shipping;** \( PR = i (SR); \ i = 2; \)

**N not a multiple of 3**

Figure 3.14 shows the inventory distribution pattern for this case.

![Figure 3.14 Case 5b(i): Inventory distribution pattern](image)

**Inventory cost:**

The total inventory held during the full shipping cycle is equal to the total area covered by the graph. It is divided into four sub-areas which are calculated as follows:

Area I: \[
\frac{1}{2} (V_y)(V) + 2(V_y)(V) + \left( \frac{1}{2} \right) (2V_y)(V) \left( \frac{N-1}{3} \right) = \frac{7N V^2 y}{6} - \frac{7V^2 y}{6} \quad (3.4bi.1)
\]

Area II: \[
\left( \frac{1}{2} \right) (V_y)(V) + \left( \frac{1}{2} \right) (2V_y)(V) = \frac{3V^2 y}{2} \quad (3.4bi.2)
\]
Area III: \[
\left[ (V_y)(V) + \left( \frac{1}{2} \right) (2V_y)(V) \right] \frac{2N - 2}{3} = \frac{4NV^2y}{3} - \frac{4V^2y}{3}
\] (3.4.5bi.3)

Area IV: \[
2 \left[ \sum_{n=1}^{N-4} n(2V)(3V_y) \right] + \left[ \sum_{n=1}^{N-1} (2n-1)(V)(3V_y) \right] + \left[ \left( \frac{2N-2}{3} \right) (V)(3V_y) \right]
\]

\[= N^2V^2y - 2NV^2y + V^2y \] (3.4.5bi.4)

Total inventory = \[N^2V^2y + \frac{NV^2y}{2} \] (3.4.5bi.5)

So the inventory cost is given by,
\[C_i = \left( N^2V^2y + \frac{NV^2y}{4} \right) h = Q^2yh + \frac{Q^2y^2h}{2N} \] (3.4.5bi.6)

**Penalty cost**

Time required for first lot of first shipment = \[V_y + \left( \frac{V}{a} \right) y + t \] (3.4.5bi.7)

Penalty cost for first lot of first shipment = \[V_y + \left( \frac{V}{a} \right) y + t \] \[w\left( \frac{V}{a} \right) \] (3.4.5bi.8)

Similarly,

Penalty cost for second lot of first shipment = \[V_y + \left( \frac{V}{a} \right) y + t \] \[w\left( \frac{V}{a} \right) \] (3.4.5bi.9)

So, penalty cost for a\(^{th}\) lot of first shipment = \[V_y + \sum_{n=1}^{a} n \left( \frac{V}{a} \right) y + t \] \[w\left( \frac{V}{a} \right) \] (3.4.5bi.10)

Similarly,

Penalty cost for a\(^{th}\) lot of second shipment = \[4V_y + \sum_{n=1}^{a} n \left( \frac{V}{a} \right) y + t \] \[w\left( \frac{V}{a} \right) \] (3.4.5bi.11)

Penalty cost for a\(^{th}\) lot of N\(^{th}\) shipment = \[NVy + \sum_{n=1}^{a} n \left( \frac{V}{a} \right) y + t \] \[w\left( \frac{V}{a} \right) \] (3.4.5bi.12)
So total penalty cost for 1 to N shipments is given by,

\[ C_p = \sum_{b=1}^{N} \left( 3b - 2Nw + \sum_{n=1}^{a} n \left( \frac{V}{a} \right)^n \right) w \left( \frac{V}{a} \right) = \frac{3N^2V^2yw}{2a} + \frac{NV^2yw}{2} + \frac{NVtw}{a} \]  

(3.4.5bi.13)

\[ C_p = \frac{3Q^2yw}{2a} + \frac{Q^2yw}{2N} + \frac{Qyw}{a} \]  

(3.4.5bi.14)

Transportation cost

Transportation cost is given by,

\[ C_T = (r + tga)N \]  

(3.4.5bi.15)

The costs are then added, and differentiated with respect to N and equated to zero, to obtain the expression for optimal number of shipments which minimizes total distribution costs.

\[ \text{Minimize } C_{opt} = Q^2yh + \frac{Q^2yh}{2N} + \frac{3Q^2yw}{2a} + \frac{Q^2yw}{2N} + \frac{Qyw}{a} + (r + tga)N \]  

(3.4.5bi.16)

\[ \frac{\partial}{\partial N} C_{opt} = -\frac{Q^2yh}{2N^2} - \frac{Q^2yw}{2N^2} + (r + tga) = 0 \]  

(3.4.5bi.17)

Optimal number of shipments,

\[ N = \sqrt{\frac{Q^2y(h+w)}{2(r+tga)}} \]  

(3.4.5bi.18)
**Case 5b(ii): Discontinuous and gradual shipping;**  \( PR = i (SR); \ i = 2; \)

**N is a multiple of 3**

Figure 3.15 shows the inventory distribution pattern for this case.

![Image of inventory distribution pattern](image)

**Inventory cost:**

The total inventory held during the full shipping cycle is equal to the total area covered by the graph. It is divided into three sub-areas which are calculated as follows:

\[
\text{Area I: } \left[ \frac{1}{2} \left( V_y \right) + 2 \left( V_y \right) + \frac{1}{2} \left( 2V_y \right) \right] \left( \frac{N}{3} \right) = \frac{7N V_y^2}{6} \quad (3.45bii.1)
\]
Area II: \[
\left( \frac{1}{2} \right) (V_y)(V) + \left( \frac{1}{2} \right) (2V_y)(V) \] \[\frac{2N}{3} = \frac{4NV^2y}{3}\] (3.4.bii.2)

Area III: \[
2 \left[ \sum_{n=1}^{N} n(2V)(3V_y) \right] + \left[ \sum_{n=1}^{N} (2n-1)(V)(3V_y) \right] = N^2V^2y - 2NV^2y \] (3.4.bii.3)

Total inventory = \[N^2V^2y + \frac{NV^2y}{2}\] (3.4.bii.4)

So the inventory cost is given by,

\[C_i = \left( N^2V^2y + \frac{NV^2y}{2} \right) h = Q^2yh + \frac{Q^2yh}{2N}\] (3.4.bii.5)

**Penalty cost**

Time required for first lot of first shipment = \[V_y + \left( \frac{V}{a} \right) y + t\] (3.4.bii.6)

Penalty cost for first lot of first shipment = \[V_y + \left( \frac{V}{a} \right) y + t \] \[w \left( \frac{V}{a} \right)\] (3.4.bii.7)

Similarly,

Penalty cost for second lot of first shipment = \[V_y + 2\left( \frac{V}{a} \right) y + t \] \[w \left( \frac{V}{a} \right)\] (3.4.bii.8)

So, penalty cost for \(a\)th lot of first shipment = \[V_y + \sum_{n=1}^{a} n \left( \frac{V}{a} \right) y + t \] \[w \left( \frac{V}{a} \right)\] (3.4.bii.9)

Similarly,

Penalty cost for \(a\)th lot of second shipment = \[4V_y + \sum_{n=1}^{a} n \left( \frac{V}{a} \right) y + t \] \[w \left( \frac{V}{a} \right)\] (3.4.bii.10)

Penalty cost for \(a\)th lot of \(N\)th shipment = \[NV_y + \sum_{n=1}^{a} n \left( \frac{V}{a} \right) y + t \] \[w \left( \frac{V}{a} \right)\] (3.4.bii.11)

So total penalty cost for 1 to \(N\) shipments is given by,
\[
C_p = \sum_{b=1}^{N} \left[ (3b-2)V_y + \sum_{n=1}^{a} n \left( \frac{V}{a} \right)^y + t \right] \left( \frac{V}{a} \right)
\]  
(3.4.5bii.12)

\[
C_p = \frac{3N^2V^2_yw}{2a} + \frac{NV^2yw}{2} + \frac{NVtw}{a} = \frac{3Q^2yw}{2a} + \frac{Q^2yw}{2N} + \frac{Qtw}{a}
\]  
(3.4.5bii.13)

**Transportation cost**

Transportation cost is given by,

\[
C_T = (r + tga)N
\]  
(3.4.5bii.14)

The costs are then added, and differentiated with respect to \( N \) and equated to zero, to obtain the expression for optimal number of shipments which minimizes total distribution costs.

\[
\text{Minimize } C_{\text{ipT}} = Q^2yh + \frac{Q^2yh}{2N} + \frac{3Q^2yw}{2a} + \frac{Q^2yw}{2N} + \frac{Qtw}{a} + (r + tga)N
\]  
(3.4.5bii.15)

\[
\frac{\partial}{\partial N} C_{\text{ipT}} = -\frac{Q^2yh}{2N^2} - \frac{Q^2yw}{2N^2} + (r + tga) = 0
\]  
(3.4.5bii.16)

Optimal number of shipments,

\[
N = \sqrt{\frac{Q^2y(h+w)}{2(r+tga)}}
\]  
(3.4.5bii.17)
Case 5c: Discontinuous and gradual shipping; \( PR = i(SR); \ i = 3 \)

Figure 3.16 shows the inventory distribution pattern for this case..

Inventory cost:

The total inventory held during the full shipping cycle is equal to the total area covered by the graph. It is divided into three sub-areas which are calculated as follows:

Area I:
\[
\left[ \frac{1}{2} (V_y)(V) + 3(V_y)(V) + \left( \frac{1}{2} \right) 3(V_y)(V) \right] \left( \frac{N}{4} \right) = \frac{13NY^2y}{8}
\]
(3.4.5c.1)

Area II:
\[
\left[ (V_y)(V) + \left( \frac{1}{2} \right) 3(V_y)(V) \right] \left( \frac{N}{4} \right) = \frac{15NY^2y}{8}
\]
(3.4.5c.2)
Area III: 
\[
\sum_{n=1}^{N-1} n(3V y) + \sum_{n=1}^{N} (n+1)(V y) = \frac{3N^2V^2y}{2} - 3NV^2y \quad (3.4.5c.3)
\]

Total inventory = \( \frac{3N^2V^2y}{2} + \frac{NV^2y}{2} \) \quad (3.4.5c.4)

So the inventory cost is given by,
\[
C_i = \left( \frac{3N^2V^2y}{2} + \frac{NV^2y}{2} \right) h = \frac{3Q^2yh + Q^2yh}{2N} \quad (3.4.5c.5)
\]

**Penalty cost**

Time required for first lot of first shipment = \( V_y + \left( \frac{V}{a} \right)_y + t \) \quad (3.4.5c.6)

Penalty cost for first lot of first shipment = \( V_y + \left( \frac{V}{a} \right)_y + t \) \[w\left( \frac{V}{a} \right)\] \quad (3.4.5c.7)

Similarly,

Penalty cost for second lot of first shipment = \( V_y + 2\left( \frac{V}{a} \right)_y + t \) \[w\left( \frac{V}{a} \right)\] \quad (3.4.5c.8)

So, penalty cost for \( a \)th lot of first shipment = \( V_y + \sum_{n=1}^{a} n\left( \frac{V}{a} \right)_y + t \) \[w\left( \frac{V}{a} \right)\] \quad (3.4.5c.9)

Similarly,

Penalty cost for \( a \)th lot of second shipment = \( 5V_y + \sum_{n=1}^{a} n\left( \frac{V}{a} \right)_y + t \) \[w\left( \frac{V}{a} \right)\] \quad (3.4.5c.10)

Penalty cost for \( a \)th lot of \( N \)th shipment = \( NV_y + \sum_{n=1}^{a} n\left( \frac{V}{a} \right)_y + t \) \[w\left( \frac{V}{a} \right)\] \quad (3.4.5c.11)

So total penalty cost for 1 to \( N \) shipments is given by,
\[
C_p = \sum_{h=1}^{N} \left[ (4b-3)V_y + \sum_{n=1}^{a} n\left( \frac{V}{a} \right)_y + t \right] \[w\left( \frac{V}{a} \right)\] \quad (3.4.5c.12)
\[ C_p = \frac{2N^2V^2yw}{a} + \frac{(a-1)NV^2yw}{2a} + \frac{NVtw}{a} = \frac{2Q^2yw}{a} + \frac{(a-1)Q^2yw}{2aN} + \frac{Qtw}{a} \] (3.4.5c.13)

**Transportation cost**

Transportation cost is given by,

\[ C_r = (r + tga)N \] (3.4.5c.14)

The costs are then added, and differentiated with respect to N and equated to zero, to obtain the expression for optimal number of shipments which minimizes total distribution costs.

\[ \text{Minimize } C_{ipt} = \frac{3Q^2yh}{2} + \frac{Q^2yh}{2N} + \frac{2Q^2yw}{a} + \frac{(a-1)Q^2yw}{2aN} + \frac{Qtw}{a} + (r + tga)N \] (3.4.5c.15)

\[ \frac{\partial}{\partial N} C_{ipt} = -\frac{Q^2yh}{2N^2} - \frac{(a-1)Q^2yw}{2aN^2} + (r + tga) = 0 \] (3.4.5c.16)

Optimal number of shipments,

\[ N = \sqrt{\frac{Q^2y\left[h + \left(1 - \frac{1}{a}\right)w\right]}{2(r + tga)}} \] (3.4.5c.17)
3.4.6 Case 6: Discontinuous production, discontinuous and gradual shipping

Both the production and shipping, in this case, occur intermittently and the shipments too are gradually dispatched in lots. As in the previous case, the inventory decreases at a constant rate due to the lots being shipped gradually. On the other hand, it also continues to increase due to simultaneous production. Thus the resultant inventory levels depend on the relative difference between the production and shipping frequencies and rates. Shipments occur at regular intervals, including interruptions, until the full order is shipped.

The periodic interruptions in production may occur due to various constraints such as availability of machines, availability of employees, machine sharing due to multi-product deliveries, and so on. The periodic interruptions in shipping may arise due to various constraints such as availability of transportation services, availability of employees, vehicle sharing due to multi-customer deliveries, and so on. Alternately, there could be constraints due to inventory storage limitations on the customer side due to which it may not be possible for the customer to take delivery of the shipments instantaneously. These factors may enforce a discontinuous and gradual approach for production as well as shipping.

As in previous cases, let us consider that the relation between PR and SR is given by,

\[ PR = i \times (SR) \]  \hspace{1cm} (3.4.6.1)

where,

\[ i = 1, 2, 3 \]
The subsequent shipments occur at regular intervals of time (iVy) until the full order is shipped. For ease of computation, it is assumed that the number of shipments is even. Based on the value of i, there are two sub-cases as described below.

**Case 6a: Discontinuous production, discontinuous and gradual shipping: PR = SR**

Figure 3.17 shows the inventory distribution pattern for this case.

![Inventory distribution pattern](image)

**Inventory cost:**

The total inventory held during the full shipping cycle is equal to the total area covered by the graph. It is divided into four sub-areas which are calculated as follows:

\[
\text{Area I: } \left( \frac{1}{2} \right)(i)\sqrt{V} = \frac{V^2y}{2}
\]  

(3.4.6a.1)
Area II: \[
\left( \frac{1}{2} \right) (V_y)(V) = \frac{V^2 y}{2}
\] (3.4.6a.2)

Area III: \[
\left[ \left( \frac{1}{2} \right) (2V_y)(V) \right] \frac{N}{2} = \frac{NV^2 y}{2}
\] (3.4.6a.3)

Area IV: \[
\left[ (2V_y)(V) \right] \frac{N}{2} - (V_y)(V) = 2NV^2 y - V^2 y
\] (3.4.6a.4)

Total inventory = \[
\frac{5NV^2 y}{2}
\] (3.4.6a.5)

So the inventory cost is given by,

\[
C_i = \left( \frac{5NV^2 y}{2} \right) h = \frac{5Q^2 yh}{2N}
\] (3.4.6a.6)

**Penalty cost**

Time required for first lot of first shipment = \[
2V_y + \left( \frac{V}{a} \right) y + t
\] (3.4.6a.7)

Penalty cost for first lot of first shipment = \[
2V_y + \left( \frac{V}{a} \right) y + t w \left( \frac{V}{a} \right)
\] (3.4.6a.8)

Similarly,

Penalty cost for second lot of first shipment = \[
2V_y + 2 \left( \frac{V}{a} \right) y + t w \left( \frac{V}{a} \right)
\] (3.4.6a.9)

So, penalty cost for a\(^{th}\) lot of first shipment = \[
2V_y + \sum_{n=1}^{a} n \left( \frac{V}{a} \right) y + t w \left( \frac{V}{a} \right)
\] (3.4.6a.10)

Similarly,

Penalty cost for a\(^{th}\) lot of second shipment = \[
4V_y + \sum_{n=1}^{a} n \left( \frac{V}{a} \right) y + t w \left( \frac{V}{a} \right)
\] (3.4.6a.11)

Penalty cost for a\(^{th}\) lot of N\(^{th}\) shipment = \[
NV_y + \sum_{n=1}^{a} n \left( \frac{V}{a} \right) y + t w \left( \frac{V}{a} \right)
\] (3.4.6a.12)
So total penalty cost for 1 to N shipments is given by,

\[
C_p = \sum_{b=1}^{N} \left[ (2b)V_y + \sum_{n=1}^{a} n \left( \frac{V}{a} \right)^n t \right] \left( \frac{V}{a} \right)
\]  

(3.4.6a.13)

\[
C_p = \frac{N^2V^2yw}{a} + \frac{(a+3)NV^2yw}{2a} + \frac{NVtw}{a} = \frac{Q^2yw}{a} + \frac{(a+3)Q^2yw}{2aN} + \frac{Qtw}{a}
\]  

(3.4.6a.14)

**Transportation cost**

Transportation cost is given by,

\[
C_T = (r + tga)N
\]  

(3.4.6a.15)

The costs are then added, and differentiated with respect to N and equated to zero, to obtain the expression for optimal number of shipments which minimizes total distribution costs.

\[
\text{Minimize } C_{ipt} = \frac{5Q^2yh}{2N} + \frac{Q^2yw}{a} + \frac{(a+3)Q^2yw}{2aN} + \frac{Qtw}{a} + (r + tga)N
\]  

(3.4.6a.16)

\[
\frac{\partial}{\partial N} C_{ipt} = -\frac{5Q^2yh}{2N^2} - \frac{(a+3)Q^2yw}{2aN^2} + (r + tga) = 0
\]  

(3.4.6a.17)

**Optimal number of shipments,**

\[
N = \sqrt[2]{\frac{Q^2y \left[ 5h + \left( 1 + \frac{3}{a} \right) w \right]}{2(r + tga)}}
\]  

(3.4.6a.18)
Case 6b: Discontinuous production, discontinuous and gradual shipping; \( PR = i \) (SR);

\( i = 2 \)

Figure 3.18 shows the inventory distribution pattern for this case.

![Figure 3.18 Case 6b: Inventory distribution pattern](image)

**Inventory cost:**

The total inventory held during the full shipping cycle is equal to the total area covered by the graph. It is divided into four sub-areas which are calculated as follows:

\[
\text{Area I: } \left[ \left( \frac{1}{2} \right) (V_y)(V') + 2(V_y)(V') + \left( \frac{1}{2} \right) (2V_y)(V') \right] \frac{N}{4} = \frac{7NV^2}{8} \tag{3.4.6b.1}
\]

\[
\text{Area II: } \left[ \left( \frac{1}{2} \right) (V_y)(V') + \left( \frac{1}{2} \right) (2V_y)(V') \right] \frac{N}{4} = \frac{3NV^2}{8} \tag{3.4.6b.2}
\]

\[
\text{Area III: } \left[ (V_y)(V') + \left( \frac{1}{2} \right) (2V_y)(V') \right] \frac{N}{2} = NV^2 \tag{3.4.6b.2}
\]
Area IV: \[
\left[ \sum_{n=1}^{N} n(2V)(6V) \right] + \left[ \sum_{n=1}^{N} n(V)(3V) \right] = \frac{3N^2V^2y}{4} - \frac{3NV^2y}{4} \quad (3.4.6b.3)
\]

Total inventory = \[
\frac{3N^2V^2y}{4} + \frac{3NV^2y}{2} \quad (3.4.6b.3)
\]

So the inventory cost is given by,

\[
C_i = \left( \frac{3N^2V^2y}{4} + \frac{3NV^2y}{2} \right) h = \frac{3Q^2yh}{4} + \frac{3Q^2y}{2N} \quad (3.4.6b.4)
\]

**Penalty cost**

Time required for first lot of first shipment = \[
V_y + \left( \frac{V}{a} \right)_y + t \quad (3.4.6b.5)
\]

Penalty cost for first lot of first shipment = \[
V_y + \left( \frac{V}{a} \right)_y + t \left[ \left( \frac{V}{a} \right) \right] w\left( \frac{V}{a} \right) \quad (3.4.6b.6)
\]

Similarly,

Penalty cost for second lot of first shipment = \[
V_y + 2\left( \frac{V}{a} \right)_y + t \left[ \left( \frac{V}{a} \right) \right] w\left( \frac{V}{a} \right) \quad (3.4.6b.7)
\]

So, penalty cost for \( a \)th lot of first shipment = \[
V_y + \sum_{n=1}^{a} n\left( \frac{V}{a} \right)_y + t \left[ \left( \frac{V}{a} \right) \right] w\left( \frac{V}{a} \right) \quad (3.4.6b.8)
\]

Similarly,

Penalty cost for \( a \)th lot of second shipment = \[
4V_y + \sum_{n=1}^{a} n\left( \frac{V}{a} \right)_y + t \left[ \left( \frac{V}{a} \right) \right] w\left( \frac{V}{a} \right) \quad (3.4.6b.9)
\]

Penalty cost for \( a \)th lot of \( N \)th shipment = \[
NV_y + \sum_{n=1}^{a} n\left( \frac{V}{a} \right)_y + t \left[ \left( \frac{V}{a} \right) \right] w\left( \frac{V}{a} \right) \quad (3.4.6b.10)
\]

So total penalty cost for 1 to \( N \) shipments is given by,

\[
C_p = \sum_{b=1}^{N} \left[ (3b-2)V_y + \sum_{n=1}^{a} n\left( \frac{V}{a} \right)_y + t \right] w\left( \frac{V}{a} \right) \]
Transportation cost

Transportation cost is given by,

\[ C_r = (r + tga)N \]  \hspace{1cm} (3.4.6b.13)

The costs are then added, and differentiated with respect to \( N \) and equated to zero, to obtain the expression for optimal number of shipments which minimizes total distribution costs.

\[ \text{Minimize } C_{tot} = \frac{3Q^2 yh}{4} + \frac{3Q^2 yh}{2N} + \frac{3Q^2 yw}{2a} + \frac{Q^2 yw}{2N} + \frac{Qtw}{a} + (r + tga)N \]  \hspace{1cm} (3.4.6b.14)

\[ \frac{\partial}{\partial N} C_{tot} = -\frac{3Q^2 yh}{2N^2} - \frac{Q^2 yw}{2N^2} + (r + tga) = 0 \]  \hspace{1cm} (3.4.6b.15)

Optimal number of shipments,

\[ N = \sqrt{\frac{Q^2 y(3h + w)}{2(r + tga)}} \]  \hspace{1cm} (3.4.6b.16)
Case 6c(i): Discontinuous production, discontinuous and gradual shipping; \( PR = i \ (SR) \);
\( i = 3; \ N \) not a multiple of 4

Figure 3.19 shows the inventory distribution pattern for this case.

![Inventory distribution pattern](image)

Figure 3.19 Case 6c(i): Inventory distribution pattern

**Inventory cost:**

The total inventory held during the full shipping cycle is equal to the total area covered by the graph. It is divided into five sub-areas which are calculated as follows:

Area I:
\[
\left[ \frac{1}{2} \right] (V_y)(V') + (3V_y)(V') = \frac{7V^2y}{2}
\]

Area II:
\[
\left[ \frac{1}{2} \right] (3V_y)(2V') \left[ \frac{N-2}{4} \right] = \frac{3NV^2y}{4} - \frac{3V^2y}{2}
\]
Area III: \[
\left[ \frac{1}{2} (V_y)(V) + \left( \frac{1}{2} \right) (3V_y)(V) \right] \left( \frac{N - 2}{4} + 1 \right) = \frac{NV^2 y}{2} + V^2 y
\] (3.4.6ci.3)

Area IV: \[
\left[ (V_y)(V) + \left( \frac{1}{2} \right) (3V_y)(V) \right] \left( \frac{N}{2} \right) = \frac{5NV^2 y}{4}
\] (3.4.6ci.4)

Area V: \[
\left[ \sum_{n=1}^{N-1} (2n - 1)(V)(8V_y) - 8V^2 y \right] + \left[ \sum_{n=1}^{N-2} n(V)(4V_y) \right] = N^2 V^2 y + NV^2 y - 6V^2 y
\] (3.4.6ci.5)

Total inventory = \[
N^2 V^2 y + \frac{7NV^2 y}{2} - 3V^2 y
\] (3.4.6ci.6)

So the inventory cost is given by,

\[
C_i = \left( N^2 V^2 y + \frac{7NV^2 y}{2} - 3V^2 y \right) h = Q^2 yh + \frac{7Q^2 yh}{2N} - \frac{3Q^2 yh}{N^2}
\] (3.4.6ci.7)

**Penalty cost**

Time required for first lot of first shipment = \[
V_y + \left( \frac{V}{a} \right) y + t
\] (3.4.6ci.8)

Penalty cost for first lot of first shipment = \[
\left[ V_y + \left( \frac{V}{a} \right) y + t \right] \omega \left( \frac{V}{a} \right)
\] (3.4.6ci.9)

Similarly,

Penalty cost for second lot of first shipment = \[
\left[ V_y + 2 \left( \frac{V}{a} \right) y + t \right] \omega \left( \frac{V}{a} \right)
\] (3.4.6ci.10)

So, penalty cost for a\(^th\) lot of first shipment = \[
\left[ V_y + \sum_{n=1}^{a} n \left( \frac{V}{a} \right) y + t \right] \omega \left( \frac{V}{a} \right)
\] (3.4.6ci.11)

Similarly,

Penalty cost for a\(^th\) lot of second shipment = \[
5V_y + \sum_{n=1}^{a} n \left( \frac{V}{a} \right) y + t \right] \omega \left( \frac{V}{a} \right)
\] (3.4.6ci.12)

Penalty cost for a\(^th\) lot of N\(^th\) shipment = \[
NVy + \sum_{n=1}^{a} n \left( \frac{V}{a} \right) y + t \right] \omega \left( \frac{V}{a} \right)
\] (3.4.6ci.13)
So total penalty cost for 1 to N shipments is given by,

\[ C_P = \sum_{b=1}^{N} \left[ (4b - 3)w^y + \sum_{n=1}^{a} n \left( \frac{V'}{a} \right)^y + t \right] w \left( \frac{V'}{a} \right) \]  
(3.4.6ci.14)

\[ C_P = \frac{2N^2V'^2yw}{a} + \frac{(a-1)NV'^2yw}{2a} + \frac{NVtw}{a} = \frac{2Q'^2yw}{a} + \frac{(a-1)Q'^2yw}{2aN} + \frac{Qtw}{a} \]  
(3.4.6ci.15)

**Transportation cost**

Transportation cost is given by,

\[ C_T = (r + tga)N \]  
(3.4.6ci.16)

The costs are then added, and differentiated with respect to N and equated to zero, to obtain the expression for optimal number of shipments which minimizes total distribution costs.

\[ \text{Minimize } C_{IP} = Q'^2yh + \frac{7Q'^2yh}{2N} - \frac{3Q'^2yh}{N^2} + \frac{2Q'^2yw}{a} + \frac{(a-1)Q'^2yw}{2aN} + \frac{Qtw}{a} + (r + tga)N \]  
(3.4.6ci.17)

\[ \frac{\partial}{\partial N} C_{IP} = -\frac{7Q'^2yh}{2N^2} + \frac{6Q'^2yh}{N^3} - \frac{(a-1)Q'^2yw}{2aN^2} + (r + tga) = 0 \]  
(3.4.6ci.18)

Please refer to the explanation about expression for N after case 6c(ii).
Case 6c(ii): Discontinuous production, discontinuous and gradual shipping; \( PR = i \ (SR) \);

\[ i = 3; \quad N \text{ is a multiple of } 4 \]

Figure 3.20 shows the inventory distribution pattern for this case.

Figure 3.20 Case 6c(ii): Inventory distribution pattern

Inventory cost:

The total inventory held during the full shipping cycle is equal to the total area covered by the graph. It is divided into five sub-areas which are calculated as follows:

\[
\text{Area I: } \left[ \frac{1}{2} (V_y)(V) + (3V_y)(V) \right] = \frac{7V^2y}{2} \quad \text{(3.4.6cii.1)}
\]

\[
\text{Area II: } \left[ \frac{1}{2} (3V_y)(2V) \right] = \frac{3NV^2y}{4} \quad \text{(3.4.6cii.2)}
\]
Area III: \[ \frac{1}{2}(V_y)(V) + \frac{1}{2}(3V_y)(V) \left( \frac{N}{4} - 1 \right) = \frac{NV^2}{2} - 2V^2y \] (3.4.6cii.3)

Area IV: \[ (V_y)(V) + \frac{1}{2}(3V_y)(V) \left( \frac{N}{2} + 1 \right) = \frac{5NV^2 y}{4} + \frac{5V^2 y}{2} \] (3.4.6cii.4)

Area V: \[ \sum_{n=1}^{N} (2n - 1)(V)(8V_y) - 8V^2 y \] \[ + \sum_{n=1}^{N} n(V)(4V_y) \right] = N^2V^2y + NV^2y - 8V^2y \] (3.4.6cii.5)

Total inventory = \[ N^2V^2y + \frac{9NV^2y}{4} - 4V^2y \] (3.4.6cii.6)

So the inventory cost is given by,

\[ C_i = \left( N^2V^2y + \frac{9NV^2y}{4} - 4V^2y \right) \begin{pmatrix} y \\ v \end{pmatrix} + \begin{pmatrix} Q^2 \end{pmatrix} yh + \frac{9Q^2y}{4N} \begin{pmatrix} 4Q^2yh \end{pmatrix} - \frac{4Q^2y}{N^2} \] (3.4.6cii.7)

**Penalty cost**

Time required for first lot of first shipment = \[ V_y + \left( \frac{V}{a} \right) y + t \] (3.4.6cii.8)

Penalty cost for first lot of first shipment = \[ V_y + \left( \frac{V}{a} \right) y + t \] \[ w \left( \frac{V}{a} \right) \] (3.4.6cii.9)

Similarly,

Penalty cost for second lot of first shipment = \[ V_y + 2 \left( \frac{V}{a} \right) y + t \] \[ w \left( \frac{V}{a} \right) \] (3.4.6cii.10)

So, penalty cost for a\(^{th}\) lot of first shipment = \[ V_y + \sum_{n=1}^{a} n \left( \frac{V}{a} \right) y + t \] \[ w \left( \frac{V}{a} \right) \] (3.4.6cii.11)

Similarly,

Penalty cost for a\(^{th}\) lot of second shipment = \[ 5V_y + \sum_{n=1}^{a} n \left( \frac{V}{a} \right) y + t \] \[ w \left( \frac{V}{a} \right) \] (3.4.6cii.12)

Penalty cost for a\(^{th}\) lot of N\(^{th}\) shipment = \[ NVy + \sum_{n=1}^{a} n \left( \frac{V}{a} \right) y + t \] \[ w \left( \frac{V}{a} \right) \] (3.4.6cii.13)
So total penalty cost for 1 to N shipments is given by,

\[
C_P = \sum_{h=1}^{N} \left[ (4b-3)W_y + \sum_{n=1}^{a} \left( \frac{V}{a} \right)^y + t \right] w \left( \frac{V}{a} \right)
\]  \hspace{1cm} (3.4.6cii.14)

\[
C_P = \frac{2N^2 V^2 yw}{a} + \frac{(a-1)N^2 V^2 yw}{2a} + \frac{NVtw}{a} = \frac{2Q^2 yw}{a} + \frac{(a-1)Q^2 yw}{2aN} + \frac{Qtw}{a}
\]  \hspace{1cm} (3.4.6cii.15)

**Transportation cost**

Transportation cost is given by,

\[
C_T = (r + tga)N
\]  \hspace{1cm} (3.4.6cii.16)

The costs are then added, and differentiated with respect to N and equated to zero, to obtain the expression for optimal number of shipments which minimizes total distribution costs.

\[
Minimize \quad C_{ipt} = Q^2 yh + \frac{9Q^2 yh}{4N} - \frac{4Q^2 yh}{N^2} + \frac{2Q^2 yw}{a} + \frac{(a-1)Q^2 yw}{2aN} + \frac{Qtw}{a} + (r + tga)N
\]  \hspace{1cm} (3.4.6cii.17)

\[
\frac{\partial}{\partial N} C_{ipt} = -\frac{9Q^2 yh}{4N^3} + \frac{8Q^2 yh}{N^3} - \frac{(a-1)Q^2 yw}{2aN^2} + (r + tga) = 0
\]  \hspace{1cm} (3.4.6cii.18)

In cases 6(c)i and 6(c)ii, i.e. when \( i = 3 \), the total cost expression contains a \( N^2 \) term in the denominator. It results in a \( N^3 \) term in the denominator after taking the derivative with respect to N in the next step. This cubic term thus results in a cube-root expression for the final value of N. It can lead to a speculation that for cases with discontinuous production where PR is i times SR, and i takes up values in multiples of 3, the final expression for
optimal value of N results in a cube-root expression. This observation is significant due to the possibility that the cube-root expression may result in some imaginary roots.
Chapter 4 Implementation

The model has been solved using OPL Studio 3.7. A program has been written to represent the objective function and the constraints. The program consisting of two parts – the main module containing the actual program and the data file containing data for the various parameters – for an example problem is shown in Appendix A. The data file changes according to the input parameters, and it is called by the main module which is executed to produce the final output. The program was executed on a Pentium IV machine with a 3.0 GHz processor and 1.0 GB RAM.

4.1 Example

An example problem consisting of 5 potential plant locations, 2 customers, and 3 products (5L-2C-3P) is solved using the OPL program. The problem data and results are shown below. The results from phase I indicate that plants 1, 2, 3, and 5 should be located to satisfy customer demands. The quantity of each product to be produced at each plant for each customer is also determined as shown in the output below. The final profit is $110280.00.
Phase I data for example problem 5L-2C-3P:

\begin{verbatim}
NofPlants = 5;
NofCustomers = 2;
NofProducts = 3;
Demand = [
    [1700, 3500, 2200],
    [2300, 1500, 2800]
];
Capacity = [
    [1300, 1200, 1300],
    [1200, 1800, 2000],
    [1500, 2000, 1700],
    [1000, 1000, 1000],
    [1000, 1000, 1000]
];
UnitsPerTruck = [1, 1, 1];
SellPrice = [
    [40, 56, 82],
    [42, 58, 75]
];
FixedCost = [50000, 30000, 40000, 60000, 20000];
InputCost = [
    [11, 23, 36],
    [9, 27, 32],
    [13, 26, 35],
    [40, 50, 60],
    [10, 15, 20]
];
ManufCost = [
    [10, 17, 15],
    [12, 12, 18],
    [14, 15, 16],
    [20, 25, 30],
    [5, 10, 15]
];
TranspFixedCost = [
    [1, 2],
    [3, 2],
    [3, 4],
    [5, 4],
    [2, 3]
];
TranspTime = [
    [5, 7],
    [9, 10],
\end{verbatim}
TranspUnitCost = [
    [0.5, 0.7],
    [0.6, 0.4],
    [0.6, 0.5],
    [0.5, 0.5]
];
Phase I results for example problem 5L-2C-3P:

Optimal Solution with Objective Value: 110280.0000

x[1] = 1
x[2] = 1
x[3] = 1
x[4] = 0
x[5] = 1

Qty[1,1,1] = 1300
Qty[1,1,2] = 1200
Qty[1,1,3] = 1300
Qty[1,2,1] = 0
Qty[1,2,2] = 0
Qty[1,2,3] = 0
Qty[2,1,1] = 0
Qty[2,1,2] = 300
Qty[2,1,3] = 0
Qty[2,2,1] = 1200
Qty[2,2,2] = 1500
Qty[2,2,3] = 2000
Qty[3,1,1] = 0
Qty[3,1,2] = 1000
Qty[3,1,3] = 0
Qty[3,2,1] = 500
Qty[3,2,2] = 0
Qty[3,2,3] = 700
Qty[4,1,1] = 0
Qty[4,1,2] = 0
Qty[4,1,3] = 0
Qty[4,2,1] = 0
Qty[4,2,2] = 0
Qty[4,2,3] = 0
Qty[5,1,1] = 400
Qty[5,1,2] = 1000
Qty[5,1,3] = 900
Qty[5,2,1] = 600
Qty[5,2,2] = 0
Qty[5,2,3] = 100
The solution for example problem, 5L-2C-3P is shown below in Figure 4.1. The results for this problem indicate that plants 1, 2, 3, and 5 should be located to satisfy customer demands. The quantity of each product to be produced at each plant for each customer along with the final profit amount is also determined.

![Figure 4.1 Plant location and customer allocation for all products for example 5L-2C-3P](image_url)

Phase II considers each plant-customer-product relationship individually and minimizes the inventory and distribution costs by determining the optimal number of shipments. For phase II, let us consider the following data for the example problem:
Table 4.1 Phase II data for example problem

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Notation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit production time</td>
<td>y</td>
<td>0.5 hours/unit</td>
</tr>
<tr>
<td>Unit penalty cost</td>
<td>w</td>
<td>0.001 $/unit/hour</td>
</tr>
<tr>
<td>Unit inventory holding cost</td>
<td>h</td>
<td>0.05 $/unit/hour</td>
</tr>
<tr>
<td>Quantity produced</td>
<td>Q</td>
<td>2000 units</td>
</tr>
<tr>
<td>Fixed cost of shipment</td>
<td>r</td>
<td>500 $</td>
</tr>
<tr>
<td>Travel cost per unit time per shipment</td>
<td>g</td>
<td>50 $/hour/ship</td>
</tr>
<tr>
<td>Travel time for a shipment</td>
<td>t</td>
<td>5 hours</td>
</tr>
</tbody>
</table>

If we consider the relation 2-2-3, product 3 is produced for customer 2 at plant 2. The optimal value obtained from phase I is \( Q_{223} = 2000 \) units. If we consider the relation being modeled by case 3a, the optimal number of shipments is obtained by,

\[
N = \sqrt{\frac{Q^2 y(h+w)}{(r+tg)}}
\]
Substituting the values we obtain the following results:

Number of shipments = 12

Shipment size = 167 units

Inventory cost = $29166.67

Penalty cost = $2176.67

Transportation cost = $9000.00

Total cost = $40343.33

We can continue the same calculations for each value of Q and determine the optimal number of shipments depending on the case being modeled for the relation represented by that Q.

Another example problem 15L-10C-10P was tested with 285 constraints and 1515 variables. The solution was obtained in 0.22 seconds. A list of example problems and results is shown below in Table 4.2.
Figure 4.2 Plant location, customer allocation for all products for example 15L-10C-10P
### Table 4.2 Phase I results for example problems

<table>
<thead>
<tr>
<th>Problem size:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Potential Locations (L) – Customers (C) – Products (P)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>No. of constraints</th>
<th>No. of variables</th>
<th>No. of iterations</th>
<th>Solving time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3L-2C-3P</td>
<td>23</td>
<td>21</td>
<td>30</td>
</tr>
<tr>
<td>5L-2C-3P</td>
<td>31</td>
<td>35</td>
<td>91</td>
</tr>
<tr>
<td>15L-10C-10P</td>
<td>285</td>
<td>1515</td>
<td>494</td>
</tr>
<tr>
<td>30L-20C-15P</td>
<td>815</td>
<td>9030</td>
<td>1581</td>
</tr>
<tr>
<td>50L-25C-30P</td>
<td>2355</td>
<td>37550</td>
<td>1937</td>
</tr>
<tr>
<td>75L-40C-50P</td>
<td>5915</td>
<td>150075</td>
<td>6036</td>
</tr>
<tr>
<td>100L-80C-60P</td>
<td>11040</td>
<td>480100</td>
<td>17522</td>
</tr>
</tbody>
</table>

### 4.2 Analysis

A study of results from both phases presents the following analysis:

- Results from phase I shown in Table 4.2 indicate that the performance of the model is significant in terms of the solving time required for large sized problems. For example, the fourth problem with the configuration 30L-20C-15P containing 9,030 variables and 815 constraints took 494 iterations and 1.86 seconds to
produce an optimal solution. Going further, the seventh problem with the configuration 100L-80C-60P containing 480,100 variables and 11,040 constraints took 17,522 iterations and 284.56 seconds to produce an optimal solution. This sufficiently large-sized problem with close to half a million variables was solved in less than 5 minutes producing an optimal solution, thus showing a high degree of solvability.

- Analysis of expressions from Table 5.1 shows that the final expression for optimal value of N does not change depending on the condition that N is or is not a multiple of 3 or 4. This condition though, definitely affects the inventory distribution pattern. Specifically, when N is not a multiple of 3, it necessitates the slight adjustment in production (discussed previously). However, it does not affect the inventory, penalty, and transportation costs thus maintaining the same expression for optimal value of N.

- These results show a reasonably good performance for large-sized problems in terms of solving time required to arrive at an optimal solution. The largest problem tested here, 100L-80C-60P having more than eleven thousand constraints and almost half million variables was solved in less than 5 minutes. This shows the suitability of the model for large-scale problems which could be faced in real-life applications requiring a quick but reliable solution.
Chapter 5 Conclusions

Following conclusions can be drawn from the results obtained from both phases of the integrated model:

1. This research proposes a mixed integer programming model that achieves strategic level optimization of capacitated location, production and distribution functions. It thus achieves the goal of total cost minimization of these supply chain functions.

2. Further, in conjunction to the MIP model, a continuous approximation procedure is also proposed in a phased approach for tactical level optimization of inventory and distribution functions.

3. Six different cases with a total of 18 different sub-cases have been identified to model the various scenarios arising out of varying production and shipping practices.

4. Closed-form expressions have been obtained for the optimal number of shipments in each case. Table 5.1 below presents a comprehensive list of the closed form expressions for all cases analyzed here.
5. The closed form expressions for optimal number of shipments obtained in phase II and the continuous approximation procedure applied show a high degree of solvability and understandability. This is one of the main reasons for originally opting for this approach in order to solve the integrated model for minimizing total costs.

6. All the cases presented in phase II cover most of the real-life situations that may arise from varying production and shipping rates, continuous and discontinuous production, and continuous and gradual shipping. This provides a comprehensive coverage of inventory distribution patterns to minimize inventory, penalty, and transportation costs.

7. The results from phase I and phase II are combined to achieve overall cost optimization for the integrated supply chain model thus minimizing the total location, production, inventory, penalty, and transportation costs.
Table 5.1 List of cases and corresponding closed-form expressions for N

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Case description</th>
<th>Case No.</th>
<th>Case No.</th>
<th>Constraint on the value of N</th>
<th>Expression for optimal value of N</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Continuous production, instantaneous shipping</td>
<td>1</td>
<td>1</td>
<td>-</td>
<td>$N = \frac{Q^2y(h + w)}{2(r + tg)}$</td>
</tr>
<tr>
<td>2</td>
<td>Two-tiered production rate</td>
<td>2</td>
<td>i</td>
<td>-</td>
<td>$N = \frac{(i + 1)Q^2y(h + w)}{4(r + tg)}$</td>
</tr>
<tr>
<td>3</td>
<td>Continuous production, instantaneous shipping</td>
<td>3a</td>
<td>2</td>
<td>-</td>
<td>$N = \frac{Q^2y(h + w)}{r + tg}$</td>
</tr>
<tr>
<td>4</td>
<td>Continuous production, instantaneous shipping</td>
<td>3b(i)</td>
<td>3</td>
<td>N not a multiple of 3</td>
<td>$N = \frac{3Q^2y(h + w)}{2(r + tg)}$</td>
</tr>
<tr>
<td>5</td>
<td>Continuous production, instantaneous shipping</td>
<td>3b(ii)</td>
<td>3</td>
<td>N is a multiple of 3</td>
<td>$N = \frac{3Q^2y(h + w)}{2(r + tg)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>4a</td>
<td>1</td>
<td>-</td>
<td>[N = \sqrt{\frac{Q^2y\left[2h + \left(1 + \frac{2}{a}\right)w\right]}{2(r + tga)}}]</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>4b</td>
<td>2</td>
<td>-</td>
<td>[N = \sqrt{\frac{(a-1)Q^2yw}{2a(r + tga)}}]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gradual shipping</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>4c(i)</td>
<td>3</td>
<td>N not a multiple of 3</td>
<td>[N = \sqrt{\frac{(a-2)Q^2yw}{2a(r + tga)}}]</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>4c(ii)</td>
<td></td>
<td>N is a multiple of 3</td>
<td>[N = \sqrt{\frac{(a-2)Q^2yw}{2a(r + tga)}}]</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>4d</td>
<td>&lt;1</td>
<td>-</td>
<td>Not required</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>5a</td>
<td>1</td>
<td>-</td>
<td>[N = \sqrt{\frac{Q^2y\left[h + \left(1 + \frac{1}{a}\right)w\right]}{2(r + tga)}}]</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>Discontinuous and gradual shipping</td>
<td>5b(i)</td>
<td>2</td>
<td>N not a multiple of 3</td>
<td>[N = \sqrt{\frac{Q^2y(h + w)}{2(r + tga)}}]</td>
</tr>
<tr>
<td>13</td>
<td></td>
<td>5b(ii)</td>
<td>N is a multiple of 3</td>
<td>[N = \sqrt{\frac{Q^2y(h + w)}{2(r + tga)}}]</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>5c</td>
<td>3</td>
<td>-</td>
<td>[N = \sqrt{\frac{Q^2y\left[h + \left(1 - \frac{1}{a}\right)w\right]}{2(r + tga)}}]</td>
<td></td>
</tr>
</tbody>
</table>
Table 5.1 continued

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th>N = \sqrt{\frac{Q^2y\left[5h + \left(1 + \frac{3}{a}\right)w\right]}{2(r + tga)}}</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>6a</td>
<td>1</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>N = \sqrt{\frac{Q^2y(3h + w)}{2(r + tga)}}</td>
</tr>
<tr>
<td></td>
<td>6b</td>
<td>2</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td></td>
<td></td>
<td></td>
<td>N not a multiple of 4</td>
</tr>
<tr>
<td></td>
<td>6c(i)</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td></td>
<td></td>
<td></td>
<td>N is a multiple of 4</td>
</tr>
</tbody>
</table>

5.1 Future research

Future research can be directed towards exploring the following areas:

1. Considering stochastic demand instead of deterministic demand will help in considering a more realistic picture in situations where there is a lot of uncertainty in the market. Often, an assumption of a deterministic behavior of demand may seem unrealistic due to various factors causing uncertainty such as, seasonal variations, new product launch, entering a new market and so on. These factors would be better represented by a stochastic model providing it more credibility and accuracy.
2. In the second phase, the sub-cases introduced here consider the production rate up to three times the shipping rate. Although this presents a reasonably realistic situation to analyze, it still poses some limits in terms of a generic analysis. Future research could be directed towards finding a generic expression for the optimal number of shipments for a given value of \( i \), where \( i \) is the number of times PR is higher than SR.

3. Another area of future research is to evaluate routing decisions within this quantitative model. This could be done along with consideration of TL as well as LTL quantities as compared to just TL quantities considered here. Although it increases the model complexity, it would also provide more options to the user to reduce costs.

4. The table of ratios presented in this research presents a limited set of ratios for \( N \) and \( i \) values. This table could be further expanded to study the ratio patterns and propose expressions to predict ratios for a given combination of \( N \) and \( i \) values. Although it may not be possible to obtain these ratios for any combination of \( N \) and \( i \), it may still be possible to develop expressions for certain combinations. There may also be governing factors such as upper boundaries to these values which could be considered in determining the ratios.

5. Inventory management is also an important SCM function which could be introduced in the phase I formulation to present a more comprehensive solution
for the integrated model. Inventory decisions could be taken at a strategic level through phase I results which could then be fine-tuned according to tactical conditions using the cases presented in phase II.
References


Appendix A

OPL program for phase I

-----------------------------
int+ NofPlants = ...;
int+ NofCustomers = ...;
int+ NofProducts = ...;

range
  Plants 1..NofPlants,
  Customers 1..NofCustomers,
  Products 1..NofProducts;

int+ Demand[Customers,Products] = ...;
int+ Capacity[Plants,Products] = ...;
int+ UnitsPerTruck[Products] = ...;
int+ M = 10000;

float+ FixedCost[Plants] = ...;
float+ SellPrice[Customers,Products] = ...;
float+ InputCost[Plants,Products] = ...;
float+ ManufCost[Plants,Products] = ...;
float+ TranspTime[Plants,Customers] = ...;
float+ TranspCost[Plants,Customers] = ...;

var
  int x[Plants] in 0..1,
  int Qty[Plants,Customers,Products] in 0..maxint;

maximize
  sum(j in Plants, k in Customers, p in Products) (SellPrice[k,p]*Qty[j,k,p]) -
  ( sum(j in Plants)
    (FixedCost[j]*x[j]) +
    sum(k in Customers, p in Products)
      ((InputCost[j,p]+ManufCost[j,p])*Qty[j,k,p] +
      (TranspCost[j,k]*TranspTime[j,k])*Qty[j,k,p]/UnitsPerTruck[p])
  )

subject to {
  forall(k in Customers)
\[
\text{forall}(p \text{ in Products}) \quad \text{sum}(j \text{ in Plants}, k \text{ in Customers}) \, \text{Qty}[j,k,p] = \text{sum}(k \text{ in Customers}) \, \text{Demand}[k,p];
\]

\[
\text{forall}(k \text{ in Customers}, p \text{ in Products}) \quad \text{sum}(j \text{ in Plants}) \, \text{Qty}[j,k,p] = \text{Demand}[k,p];
\]

\[
\text{forall}(j \text{ in Plants}, p \text{ in Products}) \quad \text{sum}(k \text{ in Customers}) \, \text{Qty}[j,k,p] \leq \text{Capacity}[j,p];
\]

\[
\text{forall}(j \text{ in Plants}) \quad \text{sum}(k \text{ in Customers}, p \text{ in Products}) \, \text{Qty}[j,k,p] \leq M \ast x[j];
\]

-----------------------------
Appendix B

OPL data file for phase I

--------------------------
NofPlants = 5;
NofCustomers = 2;
NofProducts = 3;
Demand = 
    [[1700, 3500, 2200],
     [2300, 1500, 2800]];
Capacity = 
    [[1300, 1200, 1300],
     [1200, 1800, 2000],
     [1500, 2000, 1700],
     [1000, 1000, 1000],
     [1000, 1000, 1000]];
UnitsPerTruck = [1, 1, 1];
SellPrice = 
    [[40, 56, 82],
     [42, 58, 75]];
FixedCost = [50000, 30000, 40000, 60000, 20000];
InputCost = 
    [[11, 23, 36],
     [9, 27, 32],
     [13, 26, 35],
     [40, 50, 60],
     [10, 15, 20]];
ManufCost = 
    [[10, 17, 15],
     [12, 12, 18],
     [14, 15, 16],
     [20, 25, 30],
     [5, 10, 15]];
TranspFixedCost = [1, 2, 3, 2, 3, 4, 5, 4,
TranspTime = [
  [5, 7],
  [9, 10],
  [12, 8],
  [15, 20],
  [10, 10]
];
TranspUnitCost = [
  [0.5, 0.7],
  [0.6, 0.4],
  [0.6, 0.5],
  [1, 2],
  [0.5, 0.5]
];

--------------------------