NONLINEAR EVOLUTION EQUATIONS
AND OPTIMIZATION PROBLEMS IN BANACH SPACES

A dissertation presented to
the faculty of
the College of Arts and Sciences of Ohio University

In partial fulfillment
of the requirements for the degree
Doctor of Philosophy

Haewon Lee
August 2005
This dissertation entitled

NONLINEAR EVOLUTION EQUATIONS AND OPTIMIZATION
PROBLEMS IN BANACH SPACES

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The aim of this study is two-fold, namely to develop an existence theory for a class of nonlinear evolution equations and to find higher order optimality conditions for nonlinear programming problems in a Banach space.

The first objective has been achieved by using the theory of differential equations governed by $m-$accretive operators, compactness methods and fixed-point techniques. The investigation related to the second objective has been carried out by using higher order tangential cones.

The main contributions of our work are the following. First, we improve earlier results on the existence of solutions for nonlinear nonlocal Cauchy problems in Banach spaces, both in the time-independent case and in the time-dependent case. Next, for a nonlocal problem containing a multivalued perturbation, we show the applicability of our theory to the control of a distributed parameter system. Finally, for a class of abstract programming problems, we extend previous results on second order optimality conditions to higher order ones and give several significant examples.

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This dissertation is dedicated to the Lord who gives me life and victory.
Acknowledgments

Throughout this study there have been many people who have supported me. First of all, I would like to express my sincere gratitude to my advisors Professor Sergiu Aizicovici and Professor Nicolae H. Pavel for their kind guidance and patience during the course of my Ph.D. study. I also would like to thank Professor Alexander Arhangelskii, Professor Archil Gulisashvili and Professor M.S.K. Sastriy for their invaluable teaching. Moreover, I acknowledge the financial support that I have received from the Department of Mathematics. Next, I would like to express my gratitude toward Pastor KeunSang Lee and brothers and sisters in Athens Korean Church for their warm love and constant prayer for my family. Also, I would like to thank my parents, brothers and a sister for their concern and support in hard times. In addition, during this study I would like to acknowledge the financial support and earnest prayer for me from my parents-in-law. Finally, special thanks go to my wife SuMi, without whom I could never have completed this study. I would sincerely like to express my love and gratitude to her for being the wise mother of KyungJoon, Caleb and HeeJoo, as well as for being a wife of noble character for me.
Contents

Abstract 4

Dedication 5

Acknowledgments 6

Introduction 8

1 Preliminaries 12
  1.1 Basic Results ........................................ 12
  1.2 Vector-Valued Functions and Sobolev Spaces .............. 17
  1.3 Accretive Operators and Abstract Evolution Equations .... 21

2 Nonlocal Cauchy Problems 31
  2.1 The Autonomous Case ................................... 31
  2.2 The Nonautonomous Case ................................ 41
  2.3 The Multivalued Case .................................. 51

3 Optimization Problems 63
  3.1 Introduction ........................................... 63
  3.2 Auxiliary Results ...................................... 63
  3.3 Necessary Conditions for Optimization .................... 70
  3.4 Sufficient Conditions for Optimization in Finite Dimension . 76

References 79
**Introduction**

This dissertation is based on the author’s original research on abstract nonlinear evolution equations and optimization problems. More precisely, this work is concerned with classes of first-order abstract nonlinear nonlocal Cauchy problems associated with nonlinear $m-$accretive operators in Banach spaces, as well as with finding conditions of optimality for nonlinear programming problems by using higher order tangential cones. The plan of the dissertation is as follows.

Chapter 1 provides some definitions and general results in nonlinear analysis and differential equations, including a brief discussion of $m$-accretive operators and semigroup theory.

Chapter 2 is devoted to the study of the existence of solutions for abstract first-order nonlinear nonlocal Cauchy problems. In particular, Section 2.1 focuses on the autonomous case, that is, on problems of the form

$$
\begin{align*}
\begin{cases}
    u'(t) + Au(t) &\ni f(t, u(t)), \quad t \in [0, T] \\
u(0) &= g(u),
\end{cases}
\end{align*}
$$

(P_1)

in a real Banach space $X$. Here, $A : D(A) \subset X \to 2^X$ is a nonlinear $m-$accretive operator such that $-A$ generates a compact semigroup $S(t)$ ($t > 0$), $f : [0, T] \times X \to X$ and $g : C([0, T]; X) \to \overline{D(A)}$. The study of abstract nonlocal semilinear initial-value problems was initiated by Byszewski [27, 29] and subsequently has been investigated in many papers [4, 7, 79, 80, 88]. The motivation for these studies is that nonlocal Cauchy problems have better effects in applications than the traditional Cauchy problem with an initial value of the type $u(0) = u_0$. The
The main result of this section is a nonlinear version of a result by Liang et al. [80]. There, \(-A\) is assumed to be an unbounded linear operator which generates a compact operator semigroup for \(t > 0\), \(f\) satisfies a Lipschitz condition in \(u\), while \(g\) is completely determined on \([\delta, T]\) for some small \(\delta > 0\) (cf. condition \((H_3)\) Sec 2.1) Here we extend the result of [80] to the fully nonlinear case, by using the theory of differential equations governed by \(m\)-accretive operators in Banach spaces, compactness methods and fixed-point techniques.

The second section of Chapter 2 extends the theory in the autonomous case to the time-dependent problem:

\[
\begin{cases}
  \quad u'(t) + A(t)u(t) \ni f(t, u(t)), \quad t \in [0, T] \\
  \quad u(0) = g(u),
\end{cases}
\]

in a real Banach space \(X\). Here, \(A(t) : D(A(t)) \subset X \to 2^X\) is a nonlinear (possibly multivalued) \(m\)-accretive operator for each \(t \in [0, T]\) such that \(-A(t)\) generates an evolution operator \(U(t, s)\), and \(f : [0, T] \times X \to X\) and \(g : C([0, T]; X) \to D(A(t)), \quad t \in [0, T]\). The study of abstract nonlocal time-dependent Cauchy problems was earlier done by a few authors. Aizicovici and Gao in [4] obtained existence and uniqueness results under Lipschitz conditions on both \(f\) and \(g\). In [7] Aizicovici and McKibben established the existence of integral solutions under Carathéodory and some boundedness conditions on \(f\), and continuity and compactness conditions on \(g\). Byszewski in [28] showed the existence and uniqueness of weak solutions to some nonautonomous functional differential equations with nonlocal initial conditions under Lipschitz type assumptions on
both $f$ and $g$. Here we use weaker conditions on $g$ and consider a general class of time-dependent multivalued operators in a Banach space.

In Section 2.3 we study the existence of integral solutions for the nonlinear evolution equation with a nonconvex multivalued perturbation:

\[
\begin{cases}
    u'(t) \in -Au(t) + F(t, u(t)), & t \in [0, T] \\
    u(0) = g(u),
\end{cases}
\]

in a real Banach space $X$. Here, $A : D(A) \subset X \to 2^X$ is a nonlinear $m$–accretive operator such that $-A$ generates a compact semigroup $S(t)$ ($t > 0$), $F : [0, T] \times X \to 2^X \setminus \{\emptyset\}$ and $g : C([0, T]; X) \to D(A)$. The study of semilinear evolution inclusions with nonlocal conditions was earlier done in [18]-[21]. It has a significant application in controllability testing (see, e.g., [19]). Aizicovici and McKibben [7] obtained the existence of integral solutions of problem $(P_3)$ with $A$ nonlinear, $F$ lower semicontinuous, and $g$ of Lipschitz type. In [9] Aizicovici and Staicu have considered the case when $F$ is upper semicontinuous and convex valued. Here we obtain the existence of integral solutions to the problem $(P_3)$ in the case when $F$ is a closed valued, lower semicontinuous, multifunction, under weaker conditions than those of Theorem 3.8 [7].

In Chapter 3 we discuss the higher order optimality conditions and their applications. Here we obtain some higher order necessary conditions of optimality for nonlinear programming problems in a Banach space by using the higher order tangential cones, which were introduced in [100]. We improve and extend the result of [101]. We first present the generalization of the result in [102] and then give higher order necessary conditions for a local minimum. We also give signifi-
cant examples to show the applicability of these results. Finally we present some higher order sufficient conditions for a local minimum in a finite dimensional space and then give counterexamples to show that these results are not applicable in an infinite dimensional space. These counterexamples point out that our sufficient conditions for optimality may not remain valid in infinite dimensional spaces.

Parts of this work have been included in the papers [6, 78]. Some of the results of Chapter 2 were announced in [5].
1 Preliminaries

In this chapter, we present certain definitions and general results in nonlinear analysis and differential equations, including a brief discussion of $m$-accretive operators, semigroup theory and other well-known theorems to be used in the sequel.

1.1 Basic Results

First of all, we present several basic inequalities that we use in the sequel.

**Theorem 1.1 (Gronwall’s Inequality)** Let $u$, $h$ and $k$ be given functions from $[t_0, T)$ into $\mathbb{R}$, where $T \leq +\infty$. If $u$ is continuous, $h \in L^\infty_{loc}([t_0, T))$, $k \in L^1_{loc}([t_0, T), \mathbb{R}^+)$ and

$$u(t) \leq h(t) + \int_{t_0}^{t} k(s)u(s)ds,$$

for each $t \in [t_0, T)$, then

$$u(t) \leq h(t) + \int_{t_0}^{t} h(s)k(s)\exp(\int_{s}^{t} k(\tau)d\tau)ds,$$

for each $t \in [t_0, T)$.

If, in addition, $h(t) = u_0$ for each $t \in [t_0, T)$, from the inequality above we deduce:

**Theorem 1.2 (Bellman’s Inequality)** Let $u \in C([t_0, T), \mathbb{R})$, $u_0 \in \mathbb{R}$, $k \in L^1_{loc}([t_0, T), \mathbb{R}^+)$ and

$$u(t) \leq u_0 + \int_{t_0}^{t} k(s)u(s)ds,$$
for each $t \in [t_0, T)$, then
\[ u(t) \leq u_0 \exp(\int_{t_0}^{t} k(s)ds), \]
for each $t \in [t_0, T)$.

**Theorem 1.3** *(Quadratic Form of Gronwall’s Inequality)* Let $u \in C([t_0, T], \mathbb{R})$ satisfy
\[ \frac{1}{2} \|u(t)\|^2 \leq \frac{1}{2} \|u_0\|^2 + \int_{t_0}^{t} k(s)\|u(s)\|ds, \]
for each $t \in [t_0, T)$, where $k \in L^1_{loc}([t_0, T), \mathbb{R}_+)$, then
\[ \|u(t)\| \leq \|u_0\| + \int_{t_0}^{t} k(s)ds, \]
for each $t \in [t_0, T)$.

**Definition 1.4** A subset $K$ in a metric space $X$ is called:

(i) compact, if every sequence in $K$ has at least one subsequence which converges to some element of $K$,

(ii) relatively compact, if its closure is compact.

**Definition 1.5** Let $X$ be a topological space. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called lower semicontinuous (l.s.c.) if for each $k \in \mathbb{R}$, the set \( \{x \in X : f(x) > k\} \) is open in $X$. A function $g : X \to \mathbb{R} \cup \{-\infty\}$ is called upper semicontinuous (u.s.c.) if $-g$ is l.s.c.

**Proposition 1.6** Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function. Then the following conditions are equivalent:
(i) $f$ is l.s.c.,

(ii) for each $k \in \mathbb{R}$, the set $\{y \in X : f(y) \leq k\}$ is closed in $X$,

(iii) for each $y \in X$, we have $\lim_{x \to y} \inf f(x) = f(y)$.

**Proposition 1.7** If $Y$ is a compact subset of $X$ and $f : Y \to \mathbb{R} \cup \{+\infty\}$ is l.s.c. and $f \not\equiv +\infty$, then $f$ is bounded from below and there exists at least one element $y \in Y$ such that $f(y) = \inf \{f(x) : x \in Y\}$.

Let $X$ be a real Banach space with norm $\| \cdot \|$ and dual $X^*$.

**Definition 1.8** The mapping $J : X \to 2^{X^*}$, defined by $J(x) := \{x^* \in X^* : x^*(x) = \|x\|^2 = \|x^*\|^2\}$, is called the duality mapping of $X$.

By the Hahn-Banach Theorem, it follows that $J(x)$ is nonempty for each $x \in X$. Moreover, $J(x)$ is closed, convex and bounded in $X^*$, for each $x \in X$.

**Definition 1.9** Let $X$ be a normed space. Then

(i) $X$ is said to be strictly convex if, for each $x, y \in X$ with $\|x\| = \|y\| = r$, and $0 < \lambda < 1, r > 0$, one has $\|\lambda x + (1 - \lambda)y\| < r$.

(ii) $X$ is said to be uniformly convex if, for each $\epsilon \in (0, 2]$, there is $\delta = \delta(\epsilon) > 0$ such that, whenever $x, y \in X$ with $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, it follows that $\|x + y\| \leq 2(1 - \delta(\epsilon))$, or equivalently if whenever $x_n, y_n \in X$ with $\|x_n\| \leq 1, \|y_n\| \leq 1$ and $\lim_{n \to \infty} \|x_n + y_n\| = 2$, it follows that $\lim_{n \to \infty} \|x_n - y_n\| = 0$. 
Obviously, a uniformly convex space is strictly convex. It is well-known that any uniformly convex space is a reflexive Banach space and also each real Hilbert space is uniformly convex.

**Proposition 1.10** Let $X$ be a uniformly convex space and $\{x_n\} \subset X$. If $x_n \to x$ as $n \to \infty$ and $\lim_{n \to \infty} \sup \|x_n\| \leq \|x\|$ (i.e. $\|x_n\| \to \|x\|$), then $x_n \to x$.

**Theorem 1.11** (Clarkson) If $M$ is a bounded measurable subset in $\mathbb{R}^n$, $n \geq 1$ and $1 < p < \infty$, then $L^p(M)$ endowed with the usual norm is uniformly convex.

**Theorem 1.12** (Kato)

(i) If $X^*$ is strictly convex, then the duality mapping $J$ is single-valued.

(ii) If $X^*$ is uniformly convex, then the duality mapping $J : X \to X^*$ is uniformly continuous on bounded subsets of $X$.

Let $X$ be a real Banach space of norm $\| \cdot \|$, with dual $X^*$ and duality mapping $J$. Let us define the following functions from $X \times X$ into $\mathbb{R}$.

\[
<y, x>_s = \lim_{h \downarrow 0} \frac{1}{2h}(||x + hy||^2 - \|x\|^2), \quad <y, x>_+ = \lim_{h \downarrow 0} \frac{1}{h}(|x + hy| - \|x\|)
\]

\[
<y, x>_i = \lim_{h \uparrow 0} \frac{1}{2h}(||x + hy||^2 - \|x\|^2), \quad <y, x>_- = \lim_{h \downarrow 0} \frac{1}{h}(|x + hy| - \|x\|),
\]

where $x, y \in X$ and $h \in \mathbb{R}, h \neq 0$. Since $h \mapsto \|x + hy\|^2$ and $h \mapsto \|x + hy\|$ are convex, the above functions are well-defined. Set

\[
<y, x>_h = \frac{1}{h}(|x + hy| - \|x\|), \quad x, y \in X, h \neq 0.
\]

Obviously it follows that
\[(i) \ < y, x >_s = \| x \| \ < y, x >_+, \ < y, x >_i = \| x \| \ < y, x >_-, \]

\[(ii) \ < y, x >_+ \leq \ < y, x >_h \leq \| y \|, h > 0, \]

\[(iii) \ < y, x >_h \leq \ < y, x >_-, h < 0. \]

These functions are closely related with the duality mapping \( J \) as follows:

**Proposition 1.13**  For every \( x, y \in X \) we have

\[(i) \ < y, x >_s = \sup \{ x^*(y) ; x^* \in J(x) \} \]

\[(ii) \ < y, x >_i = \inf \{ x^*(y) ; x^* \in J(x) \} \]

\[(iii) \ The \ following \ properties \ are \ equivalent: \]

\[(a) \ there \ exists \ x^* \in J(x) \ such \ that \ x^*(y) \geq 0, \]

\[(b) \ < y, x >_s \geq 0, \ (or \ < y, x >_+ \geq 0), \]

\[(c) \ \| x \| \leq \| x + ty \|, t > 0. \]

For further properties of \(<, >_s, <, >_i, <, >_+ \text{ and } <, >_-\) we refer the reader to Pavel [95, 96].

Let \( I \) be an open interval in \( \mathbb{R} \). A function \( u : I \to X \) is said to be *weakly differentiable* at \( t_0 \in I \) if, for each \( x^* \in X^* \), the real valued function \( t \mapsto < x^*, u(t) > \) is differentiable at \( t = t_0 \). Denote by \( u'(t_0) \) the weak derivative of \( u \) at \( t_0 \), that is

\[
\lim_{h \to 0} < x^*, \frac{u(t_0 + h) - u(t_0)}{h} > = < x^*, u'(t_0) >, \forall x^* \in X^*. 
\]
Theorem 1.14 (Kato) Suppose that \( u : I \to X \) is weakly differentiable at \( t_0 \in I \) and \( t \mapsto \|u(t)\| \) is differentiable at \( t = t_0 \). Then
\[
\frac{d}{dt} \|u(t)\| \bigg|_{t=t_0} = \|u(t_0)\| \cdot \frac{d}{dt} \|u(t_0)\| = \langle u'(t_0), u(t_0) \rangle_s = \langle u'(t_0), u(t_0) \rangle_i .
\]

1.2 Vector-Valued Functions and Sobolev Spaces

In this section we present basic notations, definitions and general results on vector-valued functions and Sobolev spaces.

Let \([0, T]\) be a fixed real interval \((0 < T < +\infty)\) and let \(X\) be a real Banach space. An \(X\)-valued function \( x(t) \) defined on \([0, T]\) is said to be absolutely continuous on \([0, T]\) if for each \( \varepsilon > 0 \), there exists \( \delta(\varepsilon) > 0 \) such that
\[
\sum_n \|x(t_n) - x(s_n)\| \leq \varepsilon \text{ whenever } \sum_n |t_n - s_n| \leq \delta(\varepsilon) \text{ and } (t_n, s_n) \cap (t_m, s_m) \neq \emptyset \text{ for } m \neq n.
\]
Here \((t_n, s_n)\) are arbitrary intervals contained in \([0, T]\). It is well-known that every real-valued absolutely continuous function \( x(t) \) on a real interval \([0, T]\) is almost everywhere differentiable on \((0, T)\) and it is expressed as the indefinite integral of the derivative. But this fails when \( x(t) \) is absolutely continuous from \([0, T]\) to a general Banach space \(X\), while this is true in a reflexive Banach space.

Theorem 1.15 (Kōmura) Let \(X\) be a reflexive Banach space. Then every \(X\)-valued absolutely continuous function \( x(t) \) on \([0, T]\), is a.e. differentiable on \((0, T)\) and
\[
x(t) = x(0) + \int_0^t \frac{dx}{ds}(s)ds, \quad 0 \leq t \leq T.
\]

Let \(D(0, T)\) denote the space of all real valued functions defined on \([0, T]\) which are infinitely differentiable on \([0, T]\) and have compact support in \((0, T)\). If \(X\)
is a Banach space, we denote by $\mathcal{D}'([0,T],X)$ the space of all linear continuous operators from $\mathcal{D}(0,T)$ to $X$. An element $u$ of $\mathcal{D}'([0,T],X)$ is called an $X$-valued distribution on $(0,T)$. If $u \in \mathcal{D}'([0,T],X)$, then

$$D^k u(\varphi) = (-1)^k u(D^k \varphi), \quad \forall \varphi \in \mathcal{D}(0,T)$$

defines another vector-valued distribution $D^k u \in \mathcal{D}(0,T), \forall k \in \mathbb{Z}^+$. The distribution $D^k u$ is called the derivative of order $k$ of $u$. If $1 \leq p \leq \infty$, we denote by $L^p([0,T],X)$ the space of all $X$-valued strongly measurable functions $x(t)$ defined a.e. on $(0,T)$ such that $\|x(t)\|^p$ is Lebesgue integrable over $(0,T)$. It is well-known that $L^p([0,T],X)$ is a Banach space with the norm defined by

$$\|x\|_p = \left( \int_0^T \|x(t)\|^p dt \right)^{\frac{1}{p}}, \quad 1 \leq p \leq \infty$$

with the usual modification in the case $p = \infty$. If $X$ is reflexive, then $L^p([0,T],X)$ ($0 < p < \infty$) is reflexive too, and its dual space is $L^q([0,T],X^*)$ ($\frac{1}{p} + \frac{1}{q} = 1$). We note that corresponding to every $u \in L^1([0,T],X)$ there is a distribution uniquely defined by

$$u(\varphi) = \int u(t)\varphi(t)dt, \quad \forall \varphi \in \mathcal{D}(0,T).$$

Thus $L^1([0,T],X)$ can be regarded as a linear subspace of $\mathcal{D}'([0,T],X)$. In the sequel a function $u \in L^1([0,T],X)$ will be identified with the corresponding vector-valued distribution. Let $k \in \mathbb{Z}^+$ and $1 \leq p \leq \infty$. The symbol $W^{k,p}([0,T],X)$ denotes the space of all vector-valued distributions $u \in \mathcal{D}'([0,T],X)$ with the property that

$$D^j u \in L^p([0,T],X) \text{ for } j = 0, 1, 2, \ldots, k$$

where $D^j$ is the derivative in the sense of distributions. We denote by $A^{k,p}([0,T],X)$ the space of all absolutely continuous functions $u$ from $[0,T]$ to $X$. 

whose derivatives $\frac{d^j u}{dt^j}$ are a.e. absolutely continuous for $j = 1, 2, \cdots, k - 1$ and belong to $L^p([0, T], X)$ for $j = 0, 1, 2, \cdots, k$. In particular, $A^{1,p}([0, T], X)$ consists of all absolutely continuous functions $u : [0, T] \to X$ with the property that function $t \mapsto \frac{du}{dt}(t)$ exists a.e. on $(0, T)$ and belongs to $L^p([0, T], X)$. If $X$ is reflexive, then Theorem 1.15 implies that function $u : [0, T] \to X$ belongs to $A^{1,p}([0, T], X)$ if and only if there is $g \in L^p([0, T], X)$ such that $u(t) = u(0) + \int_0^t g(s)ds$ for every $t \in [0, T]$.

It turns out that the space $W^{k,p}([0, T], X)$ can be identified with $A^{k,p}([0, T], X)$ as follows:

**Theorem 1.16** Let $X$ be a Banach space and let $u \in L^p([0, T], X)$, $1 \leq p \leq \infty$. Then the following two conditions are equivalent:

(i) $u \in W^{k,p}([0, T], X)$.

(ii) There is $u_1 \in A^{k,p}([0, T], X)$ such that $u(t) = u_1(t)$ a.e. on $(0, T)$.

**Proposition 1.17** Let $X$ be reflexive and let $f \in L^p([0, T], X)$ with $1 \leq p \leq \infty$. Then the following two conditions are equivalent:

(i) There exists $f_1 \in W^{1,p}([0, T], X)$ such that $f(t) = f_1(t)$ a.e. on $(0, T)$.

(ii) $\int_0^{T-h} \|f(t + h) - f(t)\|^p dt \leq Ch^p$, for every $h \in (0, T)$.

Now we introduce some basic notations and definitions in the theory of scalar distributions defined in $\mathbb{R}^n$. Let $\Omega$ be an open subset of $\mathbb{R}^n$. Denote by $\mathcal{D}(\Omega)$ the space of all real-valued infinitely differentiable functions defined in $\Omega$ and with
compact support in $\Omega$. By $C^k(\Omega)$, $0 \leq k \leq \infty$, we denote the set of all real-valued functions defined in $\Omega$ which have continuous partial derivatives of order up to and including $k$. Define on $\mathcal{D}(\Omega)$ the usual topology. Denote by $\mathcal{D}'(\Omega)$ the space of all scalar distributions defined on $\Omega$, i.e. the dual space of $\mathcal{D}(\Omega)$. We use the multi-index notation

$$D^\alpha u(x) = D_1^{\alpha_1}D_2^{\alpha_2} \cdots D_n^{\alpha_n}u(x), \ x \in \Omega, \ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

where $D_i = \frac{\partial}{\partial x_i}$, $i = 1, 2, \ldots, n$. The distribution $D^\alpha u$ defined by $D^\alpha u(\varphi) = (-1)^{|\alpha|}u(D^\alpha \varphi)$, $\forall \varphi \in \mathcal{D}(\Omega)$ is called the derivative of order $\alpha$ of $u \in \mathcal{D}'(\Omega)$. Here $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Let $L^p(\Omega), 1 \leq p \leq \infty$, denote the usual Banach space of Lebesgue measurable functions from $\Omega$ to $\mathbb{R}$ under the norm

$$\|u\|_p = \left(\int_\Omega |u(x)|^pdx\right)^{\frac{1}{p}}, \text{ if } 1 \leq p < +\infty,$$

$$\|u\|_\infty = \text{ess sup}_{x \in \Omega} |u(x)|.$$

For $1 \leq p \leq \infty$ and $k \geq 1$, we denote by $W^{k,p}(\Omega)$ the set of all functions $u(x)$ defined in $\Omega$ such that $u$ and its distributional derivatives $D^\alpha u$ of order $|\alpha| \leq k$ all belong to $L^p(\Omega)$. $W^{k,p}(\Omega)$ is a Banach space under the norm

$$\|u\|_{k,p} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_p. \quad (1.1)$$

In particular, $W^{k,2}(\Omega)$ is a Hilbert space by the scalar product

$$(u, v)_k = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u(x)D^\alpha v(x)dx.$$

Let $C^k_0(\Omega)$ denote the space of all functions $u \in C^k(\Omega)$ with compact support in $\Omega$. The completion of the space $C^k_0(\Omega)$ with respect to the norm (1.1) will be denoted by $W^{k,p}_0(\Omega)$. For simplicity we denote $W^{k,2}(\Omega) = H^k(\Omega)$ and similarly
\( W^{k,2}_0(\Omega) = H^k_0(\Omega) \). Finally, denote by \( W^{-k,q}\)(\( \Omega \)), \( 1 \leq q \leq \infty \), the set of all distributions \( u \in \mathcal{D}'(\Omega) \) which can be represented as

\[
  u = \sum_{|\alpha| \leq k} D^\alpha f_\alpha, \quad f_\alpha \in L^q(\Omega).
\]

\( W^{k,p}_0(\Omega) \) and \( W^{k,p}(\Omega) \) are called Sobolev spaces.

**Theorem 1.18** The dual space of \( W^{k,p}_0(\Omega) \) coincides with the space \( W^{-k,q}(\Omega) \), \( 1 \leq p \leq \infty \), \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Theorem 1.19** Assume that \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \) with a sufficiently smooth boundary. Then

(i) If \( \frac{1}{r} > \frac{1}{p} - \frac{k}{n} \) where \( 1 \leq p \leq \infty \), \( 1 \leq r \leq \infty \) and \( k \geq 1 \), then \( W^{k,p}(\Omega) \subset L^r(\Omega) \). Moreover, if \( p < \infty \), \( r < \infty \), then the injection mapping of \( W^{k,p}(\Omega) \) into \( L^r(\Omega) \) is completely continuous.

(ii) If \( m < k - \frac{n}{p} \), then \( W^{k,p}(\Omega) \subset C^m(\overline{\Omega}) \).

### 1.3 Accretive Operators and Abstract Evolution Equations

Let \( X \) and \( Y \) be linear spaces. By a *multivalued operator* we mean a mapping \( A : X \to Y \), defined by

\[
  Ax := \{ y \in Y : [x, y] \in A \}.
\]
Thus \([x, y] \in A\) indicates that \(x \in D(A)\) and \(y \in Ax\), and \(A\) is regarded as a subset of \(X \times Y\). The domain of \(A\) is defined by

\[
D(A) := \{x \in X : Ax \neq \phi\}
\]

and the range of \(A\) is defined by

\[
R(A) := \bigcup_{x \in D(A)} Ax.
\]

If \(A, B \subset X \times Y\) and \(\lambda\) is a real number, we define

(i) \(A + B = \{[x, y + z] : x \in D(A) \cap D(B), y \in Ax \text{ and } z \in Bx\}\)

(ii) \(\lambda A = \{[x, \lambda y] : y \in Ax\}\)

(iii) \(A^{-1} = \{[y, x] : [x, y] \in A\}\)

**Definition 1.20** An operator \(A : D(A) \subset X \rightarrow 2^X\) is called accretive if \(<y_1 - y_2, x_1 - x_2 > \geq 0\), for every \([x_i, y_i] \in A, i = 1, 2\), or equivalently if for every \(x_1, x_2 \in D(A)\), there is \(w \in J(x_1 - x_2)\) such that \(<w, y_1 - y_2 > \geq 0\) for all \(y_i \in Ax_i, i = 1, 2\).

It readily follows that \(A\) is accretive if and only if \(<y_1 - y_2, x_1 - x_2 > \geq 0\), for every \([x_i, y_i] \in A, i = 1, 2\).

**Definition 1.21** Let \(A : D(A) \subset X \rightarrow 2^X\) be an operator and \(\lambda > 0\).

(i) The operator \(J_\lambda : D(J_\lambda) \subset X \rightarrow 2^X\), defined by \(J_\lambda := (I + \lambda A)^{-1}\), is called the Resolvent of \(A\), where \(D(J_\lambda) = R(I + \lambda A)\).

(ii) The operator \(A_\lambda : D(A_\lambda) \subset X \rightarrow 2^X\), defined by \(A_\lambda := \frac{1}{\lambda}(I - J_\lambda)\), is called the Yosida Approximation of \(A\), where \(D(A_\lambda) = R(I + \lambda A)\).
Proposition 1.22 An operator $A : D(A) \subset X \to 2^X$ is accretive if and only if for each $\lambda > 0$, the resolvent $J_\lambda$ is single-valued and nonexpansive, i.e. $\|J_\lambda x - J_\lambda y\| \leq \|x - y\|$ for each $x, y \in R(I + \lambda A)$.

Theorem 1.23 If $A : D(A) \subset X \to 2^X$ is accretive and $\lambda > 0$, then

(i) $A_\lambda$ is single-valued, accretive and Lipschitz continuous on $R(I + \lambda A)$ having the Lipschitz constant $\frac{1}{\lambda}$,

(ii) $A_\lambda x \in AJ_\lambda x$, for each $x \in R(I + \lambda A)$,

(iii) $\|A_\lambda x\| \leq |Ax|$ for each $x \in R(I + \lambda A) \cap D(A)$, where $|Ax| := \inf\{\|y\| : y \in Ax\}$, for each $x \in D(A),$

(iv) $\lim_{\lambda \downarrow 0} J_\lambda x = x$ strongly in $X$ for each $x \in \bigcap_{\lambda > 0} R(I + \lambda A) \cap D(A),$

(v) if $x \in R(I + \lambda A)$ and $\mu > 0$, then $\frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} J_\lambda x \in R(I + \mu A)$, and $J_\lambda x = J_\mu(\frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} J_\lambda x)$ (the resolvent identity),

(vi) if $X^*$ is uniformly convex, then for each $u \in \bigcap_{\lambda > 0} R(I + \lambda A)$, the function $\lambda \mapsto \|A_\lambda u\|$ is nonincreasing on $(0, +\infty)$.

Definition 1.24 An operator $A : D(A) \subset X \to 2^X$ is called $m-$accretive if it is accretive and $R(I + \lambda A) = X$, for each $\lambda > 0$.

Now we give our attention on a special class of $m-$accretive operators that are significant in the theory of nonlinear partial differential equations.

Definition 1.25 Let $X$ be a real Banach space. Then
(i) A function \( \varphi : X \rightarrow \mathbb{R} \cup \{+\infty\} \) is called convex if \( \varphi(\lambda x + (1 - \lambda)y) \leq \lambda \varphi(x) + (1 - \lambda)\varphi(y) \), for each \( x, y \in X \) and \( \lambda \in (0, 1) \).

(ii) A function \( \varphi : X \rightarrow \mathbb{R} \cup \{+\infty\} \) is called proper if \( \varphi \not\equiv +\infty \).

(iii) Let \( \varphi : X \rightarrow \mathbb{R} \cup \{+\infty\} \) be a proper convex function. Then the set \( D(\varphi) := \{ x \in X : \varphi(x) < +\infty \} \) is called the effective domain of \( \varphi \).

**Proposition 1.26** Let \( X \) be reflexive and let \( \varphi \) be a proper, l.s.c. convex function defined on \( X \). Suppose that \( \lim_{\|x\| \to \infty} \varphi(x) = +\infty \). Then there exists \( x_0 \in X \) such that \( \varphi(x_0) = \inf \{ \varphi(x) : x \in X \} \).

**Definition 1.27** Let \( X \) be a real Banach space with dual \( X^* \), and \( \varphi : X \rightarrow \mathbb{R} \cup \{+\infty\} \) a proper, l.s.c. convex function. Then the set \( \partial \varphi(x) := \{ z \in X^* : \varphi(x) \leq \varphi(y) + \langle x - y, z \rangle, \text{ for each } y \in X \} \) is called the subdifferential of \( \varphi \) calculated at \( x \). The operator \( \partial \varphi : D(\partial \varphi) \subset X \rightarrow 2^{X^*} \) which assigns to each \( x \in X \) the subset \( \partial \varphi(x) \) in \( X^* \) is called the subdifferential of \( \varphi \). We recall that \( D(\partial \varphi) = \{ x \in X : \partial \varphi(x) \neq \emptyset \} \).

From the definition of \( \partial \varphi \), it is obvious that \( \partial \varphi(x) \) is always a (possibly empty) closed convex set. We also note that \( \varphi(x) = \min \{ \varphi(y) : y \in X \} \) if and only if \( 0 \in \partial \varphi(x) \).

**Theorem 1.28** Let \( \varphi : X \rightarrow \mathbb{R} \cup \{+\infty\} \) be a proper, l.s.c. convex function. Then \( D(\partial \varphi) = D(\varphi) \).

**Proposition 1.29** Let \( X \) be reflexive and \( A = \partial \varphi \) where \( \varphi : X \rightarrow \mathbb{R} \cup \{+\infty\} \) is a proper, l.s.c. convex function on \( X \). Then the following conditions are equivalents:
\( (i) \quad \lim_{\|x\| \to \infty} \frac{\varphi(x)}{\|x\|} = +\infty. \)

\( (ii) \quad R(A) = X^* \) and \( A^{-1} \) is bounded.

For every \( \lambda > 0 \), define
\[
\varphi_\lambda(x) := \inf_{u \in X} \left\{ \frac{1}{2\lambda} \|x - u\|^2 + \varphi(u) \right\}, \quad x \in X.
\]

where \( \varphi : X \to \mathbb{R} \cup \{+\infty\} \) is a proper, l.s.c. convex function on \( X \). Obviously, the function \( \varphi_\lambda \) is convex and every finite on \( X \).

\textbf{Theorem 1.30} Let \( X \) and \( X^* \) be reflexive and strictly convex and let \( \varphi \) be a proper, l.s.c. convex function on \( X \). Let \( A = \partial \varphi \). Then the function \( \varphi_\lambda \) is Gâteaux differentiable on \( X \) and \( A_\lambda = \partial \varphi_\lambda \). In addition, we have

\( (i) \quad \varphi_\lambda(x) = \frac{1}{2\lambda} \|x - J_\lambda x\|^2 + \varphi(J_\lambda), \) for every \( \lambda > 0 \) and \( x \in X \).

\( (ii) \quad \lim_{\lambda \to 0} \varphi_\lambda(x) = \varphi(x), \) for every \( x \in X \).

\( (ii) \quad \varphi(J_\lambda x) \leq \varphi_\lambda(x) \leq \varphi(x), \) for every \( \lambda > 0 \) and \( x \in X \).

\textbf{Corollary 1.31} In Theorem 1.30 assume that \( X = H \) is a real Hilbert space.

Then the function \( \varphi_\lambda \) is Fréchet differentiable on \( H \) and \( A_\lambda = \partial \varphi_\lambda \) is Lipschitz on \( H \).

\textbf{Theorem 1.32} (Minty, Moreau) Let \( H \) be a real Hilbert space with the inner product \( \langle \cdot, \cdot \rangle \). If \( \varphi : H \to \mathbb{R} \cup \{+\infty\} \) is a proper, l.s.c. convex function, then its subdifferential \( \partial \varphi : D(\partial \varphi) \subset H \to 2^H \) is an \( m \)-accretive operator.

In general, an \( m \)-accretive operator \( A : D(A) \subset H \to 2^H \) is not the subdifferential of a certain proper, l.s.c. convex function except when \( H = \mathbb{R} \).
**Theorem 1.33** Each $m$–accretive operator $\beta : D(\beta) \subset \mathbb{R} \to 2^\mathbb{R}$ is the subdifferential of a proper, l.s.c. convex function.

Now we consider the evolution equations governed by $m$–accretive operators. Let $A : D(A) \subset X \to 2^X$ be an $m$–accretive operator. Let $f : [0, T] \to X$ be a given function and let us consider the quasi-autonomous equation:

$$\frac{du}{dt}(t) + Au(t) \ni f(t), \quad t \in [0, T].$$

**Definition 1.34** Let $f \in L^1([0, T], X)$. Then a function $u : [0, T] \to X$ is called a strong solution of (1.2) on $[0, T]$ if

(i) $u(t) \in D(A)$ a.e. for $t \in (0, T),$

(ii) $u \in W^{1,1}((0, T], X)$ and there exists $g \in L^1((0, T), X), g(t) \in Au(t)$ a.e. for $t \in (0, T)$ such that

$$\frac{du}{dt}(t) + g(t) \ni f(t) \text{ a.e. for } t \in (0, T).$$

**Theorem 1.35** Let $X$ be reflexive and let $A : D(A) \subset X \to 2^X$ be an $m$–accretive operator. Then for each $u_0 \in D(A)$ and $f \in W^{1,1}([0, T], X)$ there exists a unique strong solution $u$ of (1.2) on $[0, T]$ which satisfies: $u(0) = u_0, u \in W^{1,\infty}([0, T], X)$ and

$$\|\frac{du}{dt}(t)\| = |Au(t) + f(t)| \leq |Au_0 + f(0)| + \int_0^T \|\frac{df}{ds}(s)\|ds \text{ a.e. for } t \in (0, T),$$

where $|Ax + z| = \inf\{\|y + z\| : y \in Ax\}$ for each $x \in D(A).$
**Definition 1.36** Let $f \in L^1([0, T], X)$. Then a function $u : [0, T] \to X$ is called an integral solution of (1.2) on $[0, T]$ if $u \in C([0, T], X)$, $u(t) \in \overline{D(A)}$ for each $t \in [0, T]$, and $u$ satisfies:

$$\|u(t) - x\| \leq \|u(s) - x\| + \int_s^t < u(\tau) - x, f(\tau) - y >_+ d\tau, \quad \forall 0 \leq s \leq t \leq T,$$

for each $x \in D(A), y \in Ax$.

Next we define next another type of solution which is equivalent to that of the integral solution.

**Definition 1.37** Let $f : \in L^1([0, T], X)$. A mild solution of the problem (1.2) is a function $u \in C([0, T], X)$ satisfying: for each $0 < c < T$ and $\varepsilon > 0$ there exist

(i) $0 < t_0 < t_1 < \cdots < c \leq t_n < T, t_k - t_{k-1} \leq \varepsilon$, for $k = 1, 2, \cdots, n$,

(ii) $f_1, f_2, \cdots, f_n \in X$ with $\sum_{i=1}^{n} \int_{t_{k-1}}^{t_k} \|f(t) - f_k\|dt \leq \varepsilon$,

(iii) $v_0, v_1, \cdots, v_n \in X$ satisfying:

$$\frac{v_k - v_{k-1}}{t_k - t_{k-1}} + Av_k \ni f_k, \text{ for } k = 1, 2, \cdots, n, \text{ and such that }$$

$$\|u(t) - v_k\| \leq \varepsilon, \text{ for } t \in [t_{k-1}, t_k), k = 1, 2, \cdots, n.$$

**Theorem 1.38** Let $A : D(A) \subset X \to 2^X$ be an $m-$accretive operator, let $f \in L^1([0, T], X)$ and $u_0 \in \overline{D(A)}$. If $u : [0, T] \to X$ is a mild solution of (1.2) on $[0, T]$ satisfying $u(0) = u_0$, then $u$ is the unique integral solution of (1.2) on $[0, T]$ satisfying $u(0) = u_0$.

**Theorem 1.39** Let $A : D(A) \subset X \to 2^X$ be an $m-$accretive operator, let $f, g \in L^1([0, T], X)$ and let $u, v$ be two mild solutions of (1.2) corresponding to $f$ and to
g, respectively. Then
\[ \|u(t) - v(t)\|^2 \leq \|u(s) - v(s)\|^2 + 2 \int_s^t < u(\tau) - v(\tau), f(\tau) - g(\tau) > d\tau, \]
and
\[ \|u(t) - v(t)\| \leq \|u(s) - v(s)\| + \int_s^t < u(\tau) - v(\tau), f(\tau) - g(\tau) > d\tau, \forall 0 \leq s \leq t \leq T. \]

This theorem implies the Lipschitz continuous dependence of mild solutions of (1.2) on the data.

**Corollary 1.40** Let \( A : D(A) \subset X \rightarrow 2^X \) be an \( m \)-accretive operator, let \( f, g \in L^1([0,T], X) \) and let \( u, v \) be two mild solutions of (1.2) corresponding to \( f \) and to \( g \), respectively. Then
\[ \|u(t) - v(t)\| \leq \|u(s) - v(s)\| + \int_s^t \| f(\tau) - g(\tau) \| d\tau, \forall 0 \leq s \leq t \leq T. \]

Now we introduce the semigroups generated by \( m \)-accretive operators. Let \( X \) be a real Banach space and let \( C \) be a nonempty subset in \( X \).

**Definition 1.41** A family of functions \( \{S(t) : S(t) : C \rightarrow C, t \geq 0\} \) is called a semigroup of nonexpansive mappings on \( C \) if

(i) \( S(t + s) = S(t)S(s) \), for each \( t, s \geq 0 \),
(ii) \( S(0) = I \), where \( I \) is the identity on \( C \),
(iii) \( \lim_{t \downarrow 0} S(t)x = x \), for each \( x \in C \),
(iv) \( \|S(t)x - S(t)y\| \leq \|x - y\| \), for each \( x, y \in C \) and \( t \geq 0 \).
Theorem 1.42 (Crandall and Liggett) Let $A : D(A) \subset X \to 2^X$ be an $m-$accretive operator. Then

$$S_A(t)x = \lim_{n \to \infty} (I + \frac{t}{n}A)^{-n}x,$$

exists for each $x \in \overline{D(A)}$ and uniformly for $t$ in every compact in $\mathbb{R}_+$. In addition, \{${S_A(t) : S_A(t) : \overline{D(A)} \to \overline{D(A)}, t \geq 0}$\} is a semigroup of nonexpansive mappings on $\overline{D(A)}$, and for each $x \in D(A)$ and $t > 0$, we have $\|S_A(t)x - x\| \leq t|Ax|$, where $|Ax| := \inf\{\|y\| : y \in Ax\}$. $S_A$ is called the semigroup generated by $-A$.

The relationship between the behaviour of $S(t)$ and that of $J_\lambda$ where $t$ and $\lambda$ approach 0 independently is given due to Brezis.

Theorem 1.43 (Brezis) If $x \in \overline{D(A)}$, $t > 0$ and $\lambda > 0$, then

$$\|S(t)x - x\| \leq (2 + \frac{t}{\lambda})\|J_\lambda x - x\|$$

and

$$\|J_\lambda x - x\| \leq \frac{2}{\lambda}(1 + \frac{\lambda}{t})\int_0^t \|S(s)x - x\|ds.$$ 

Recall that an operator $T : C \subset X \to X$ is called compact if it is continuous and it maps bounded subsets in $C$ into relatively compact subsets in $X$.

Definition 1.44 A semigroup \{${S(t) : S(t) : C \to C, t \geq 0}$\} of nonexpansive mappings on $C \subset X$ is called compact if for each $t > 0$, $S(t)$ is a compact operator.

Generally speaking, compact semigroups are generated, either by $m-$accretive operators acting in finite dimensional Banach spaces, or by $m-$accretive operators arising in the study of parabolic problems.
Theorem 1.45 (Brezis) Let $A : D(A) \subset X \to 2^X$ be an $m$–accretive operator and \{S(t) : S(t) : \overline{D(A)} \to \overline{D(A)}, t \geq 0\} be the semigroup generated by $-A$ on $\overline{D(A)}$. Then the following conditions are equivalent:

(i) The semigroup \{S(t) : S(t) : \overline{D(A)} \to \overline{D(A)}, t \geq 0\} on $\overline{D(A)}$ is compact,

(ii) (a) $J_\lambda$ is a compact operator, for all $\lambda > 0$,

(b) The semigroup \{S(t) : S(t) : \overline{D(A)} \to \overline{D(A)}, t \geq 0\} on $\overline{D(A)}$ is equicontinuous.

Theorem 1.46 (Pazy) Let $A : D(A) \subset X \to 2^X$ be a densely defined linear, $m$–accretive operator and \{S(t) : S(t) : X \to X, t \geq 0\} be the semigroup of linear operators generated by $-A$ on $X$. Then the following conditions are equivalent:

(i) The semigroup \{S(t) : S(t) : X \to X, t \geq 0\} on $X$ is compact,

(ii) (a) $J_\lambda$ is a compact operator, for all $\lambda > 0$,

(b) The mapping $t \mapsto S(t)$ is continuous in the uniform operator topology from $(0, +\infty)$ into $L(X)$. 


2 Nonlocal Cauchy Problems

2.1 The Autonomous Case

In this section we are concerned with the existence of an integral solution for a nonlinear Cauchy problem with nonlocal initial conditions of the general form:

\[
\begin{aligned}
  &u'(t) + Au(t) \ni f(t, u(t)), \quad t \in [0, T] \\
  &u(0) = g(u),
\end{aligned}
\] (2.1)

in a real Banach space \(X\). Here, \(A : D(A) \subset X \to 2^X\) is a nonlinear \(m\)-accretive operator such that \(-A\) generates a compact semigroup \(S(t)\) \((t > 0)\),\(\) \(f : [0, T] \times X \to X\) and \(g : C([0, T]; X) \to D(A)\). The study of abstract nonlocal semilinear initial-value problems was initiated by Byszewski [27, 29] and subsequently has been investigated in many papers [4, 7, 79, 80, 88]. The motivation for these studies is that nonlocal Cauchy problems have better effects in applications than the traditional Cauchy problem with an initial value of the type \(u(0) = u_0\). The present work is a nonlinear version of a result by Liang et al. [80]. There, \(-A\) is assumed to be an unbounded linear operator which generates a compact operator semigroup for \(t > 0\), \(f\) satisfies a Lipschitz condition in \(u\), while \(g\) is completely determined on \([\delta, T]\) for some small \(\delta > 0\) (cf. condition \((H_3)\) below.) Here we extend the result of [80] to the fully nonlinear case, by using the theory of differential equations governed by \(m\)-accretive operators in Banach spaces, compactness methods and fixed-point techniques.

For further background and details of this section we refer the reader to V. Barbu [16]. Let \(X\) be a real Banach space of norm \(\| \cdot \|\). A set-valued operator \(A\) in \(X\) with domain \(D(A)\) and range \(R(A)\) is said to be accretive if \(\|x_1 - x_2\| \leq \]
\[ \|x_1 - x_2 + \lambda(y_1 - y_2)\|, \text{ for all } \lambda > 0 \text{ and } y_i \in Ax_i, i = 1, 2. \]

A is called \(m\)-accretive if it is accretive and \(R(I + \lambda A) = X\), for all \(\lambda > 0\). (Here, \(I\) stands for the identity on \(X\).) In the case when \(X\) is a Hilbert space, \(m\)-accretivity is equivalent to maximal monotonicity.

Next consider the Cauchy problem:

\[
\begin{cases}
  u'(t) + Au(t) \ni f(t), & t \in [0, T] \\
  u(0) = u_0,
\end{cases}
\tag{2.2}
\]

where \(A\) is \(m\)-accretive in \(X\), \(f \in L^1(0, T; X)\) and \(u_0 \in \overline{D(A)}\). It is well-known that (2.2) has a unique integral solution \(u \in C([0, T]; \overline{D(A)})\) (see, e.g., [16], p.124). If \(u\) and \(v\) are integral solutions of \(u' + Au \ni f\) and \(v' + Av \ni g\) respectively, with \(f, g \in L^1(0, T; X)\) then they satisfy the Benilan’s inequality, i.e.

\[
\|u(t) - v(t)\| \leq \|u(0) - v(0)\| + \int_0^t \|f(\tau) - g(\tau)\|d\tau, \forall t \in [0, T].
\tag{2.3}
\]

The following two theorems play a key role in the proof of our main result.

**Theorem 2.1 (Ascoli’s theorem).** Let \(\mathcal{F} \subset C([a, b]; X)\) satisfy:

(i) For any \(t \in [a, b]\), \(\{f(t) : f \in \mathcal{F}\}\) is relatively compact in \(X\);

(ii) \(\mathcal{F}\) is equicontinuous on \([a, b]\), that is, for any \(\varepsilon > 0\) and any \(t \in [a, b]\), there exists \(\delta > 0\) such that \(\|f(t) - f(s)\| < \varepsilon\), for any \(s \in [a, b]\) satisfying \(|t - s| < \delta\), and all \(f \in \mathcal{F}\).

Then \(\mathcal{F}\) is relatively compact.
Theorem 2.2 (Schauder’s fixed-point theorem). Let $C$ be a nonempty bounded convex closed subset in $X$. If $F : C \rightarrow C$ is continuous and $F(C)$ is relatively compact, then $F$ has at least one fixed point.

Let $(X, \| \cdot \|)$ be a real Banach space, and let $A$ be an $m$–accretive operator in $X$ with domain $D(A)$. It is well-known ([16]) that $-A$ generates a nonlinear contraction semigroup $S(t)$ on $\overline{D(A)}$, which is given by the formula

$$S(t)x = \lim_{n \to \infty} (I + \frac{t}{n}A)^{-n}x,$$

for all $x \in \overline{D(A)}$ and all $0 \leq t \leq T$. The semigroup $S(t)$ is said to be compact if $S(t)$ maps bounded subsets of $\overline{D(A)}$ into precompact subsets of $\overline{D(A)}$, for all $0 \leq t \leq T$. We can now make precise what we mean by an integral solution of (2.1).

Definition 2.3 A function $u \in C([0,T];\overline{D(A)})$ is said to be an integral solution of (2.1) if $u$ is an integral solution of (2.2) in the sense of Benilan with $f(t, u(t))$ in place of $f(t)$ and $g(u)$ in place of $u_0$.

Let $r$, $T$ be finite positive constants, and set $B_r := \{x \in X : \|x\| \leq r\}$, and $K_r := \{\phi \in C([0,T]; X) : \phi(t) \in B_r, \forall t \in [0,T]\}$.

Now, we are able to formulate our main result:

Theorem 2.4 Assume that:

(H$_1$) $S(t)$ is compact for all $t > 0$;

(H$_2$) $f : [0,T] \times X \rightarrow X$ is continuous in $t$ on $[0,T]$ and there exists a constant $L(r) > 0$ such that $\|f(t,u) - f(t,v)\| \leq L(r)\|u - v\|, \forall t \in [0,T], \forall u, v \in B_r;$
\((H_3)\) \(g : C([0, T]; X) \rightarrow \overline{D(A)}\) is a continuous mapping which maps \(K_r\) into a bounded set, and there is a \(\delta = \delta(r) \in (0, T)\) such that \(g(\phi) = g(\psi)\) for any \(\phi, \psi \in K_r\) with \(\phi(s) = \psi(s), s \in [\delta, T]\);

\((H_4)\) \(T \sup_{t \in [0, T], \phi \in K_r} \|f(t, \phi(t))\| + \sup_{\phi \in K_r} \|S(t)g(\phi)\| \leq r.\)

Then (2.1) has at least one integral solution.

**Remark 2.1** Condition \((H_4)\) makes sense, since \(\|f(t, \phi(t))\|\) is bounded for \(t \in [0, T]\) and \(\phi \in K_r\) (cf. \((H_2)\)), and

\[
\|S(t)g(\phi)\| \leq \|S(t)g(\phi) - S(t)x_0\| + \|S(t)x_0\| \\
\leq \|g(\phi) - x_0\| + \|S(t)x_0\| \\
\leq \sup_{\phi \in K_r} \|g(\phi)\| + \|x_0\| + \max_{t \in [0, T]} \|S(t)x_0\|,
\]

for all \(\phi \in K_r\), and some \(x_0 \in D(A)\).

If \(0 \in A_0\), one can take \(x_0 = 0\) and observe that \(S(t)0 = 0\) in this case, so that \((H_4)\) would be satisfied provided that

\[
T \sup_{t \in [0, T], \phi \in K_r} \|f(t, \phi(t))\| + \sup_{\phi \in K_r} \|g(\phi)\| \leq r. \tag{2.4}
\]

**Proof of Theorem 2.4.** We will divide the proof into two steps.

First, set \(K_r(\delta) = \{\phi \in C([\delta, T]; X) : \phi(t) \in B_r, \forall t \in [\delta, T]\}\). For a fixed \(v \in K_r(\delta)\), we define a mapping \(F_v\) on \(K_r\) by \((F_v(\phi))(t) = u_\phi\) where \(u_\phi\) is the unique integral solution of

\[
(P_1) \begin{cases}
    u'_\phi(t) + Au_\phi(t) \ni f(t, \phi(t)), & t \in [0, T] \\
    u_\phi(0) = g(\tilde{v}),
\end{cases}
\]
with
\[ \tilde{v}(t) = \begin{cases} 
  v(t) & \text{if } t \in [\delta, T] \\
  v(\delta) & \text{if } t \in [0, \delta].
\end{cases} \]

1st step: \( F_v \) maps \( K_r \) into itself.

Indeed, from the definition of \( F_v \) and (2.3), we obtain
\[
\| (F_v \phi)(t) - S(t)g(\tilde{v})(t) \| \leq \int_0^t \| f(s, \phi(s)) \| ds \\
\leq T \sup_{t \in [0,T], \phi \in K_r} \| f(t, \phi(t)) \|, \forall t \in [0, T], \phi \in K_r.
\]

Combining the above inequality with condition \((H_4)\), we get
\[
\| (F_v \phi)(t) \| \leq tL(r) \max_{s \in [0,t]} \| \phi(s) - \psi(s) \|, \forall t \in [0, T], \phi, \psi \in K_r.
\]

This implies that \( F_v K_r \subset K_r \).

From (2.3) and condition \((H_2)\), it follows that
\[
\| (F_v \phi)(t) - (F_v \psi)(t) \| \leq tL(r) \max_{s \in [0,t]} \| \phi(s) - \psi(s) \|, \forall t \in [0, T], \phi, \psi \in K_r.
\]

Moreover, we deduce inductively that for \( n \in N \),
\[
\| (F_v^n \phi)(t) - (F_v^n \psi)(t) \| \leq \frac{(tL(r))^n}{n!} \max_{s \in [0,t]} \| \phi(s) - \psi(s) \|, \forall t \in [0, T], \phi, \psi \in K_r.
\]

Hence, we infer that for \( n \) large enough, the mapping \( F_v^n \) is a strict contraction.

Thus, by the Banach Contraction Mapping Principle, \( F_v \) has a unique fixed point \( \phi_v \in K_r \), which is the integral solution of
\[
(P_2) \quad \begin{cases} 
  \phi_v'(t) + A \phi_v(t) \ni f(t, \phi_v(t)), & t \in [0, T] \\
  \phi_v(0) = g(\tilde{v}).
\end{cases}
\]
2nd step: Problem (2.1) has at least one integral solution.

We define a mapping $\mathcal{G}$ from $K_r(\delta)$ into itself by $(\mathcal{G}v)(t) = \phi_v(t), t \in [\delta, T]$, where $\phi_v$ satisfies $(P_2)$. From the definition of $\mathcal{G}$, (2.3) and $(H_2)$, we get

$$\|(\mathcal{G}v_1)(t) - (\mathcal{G}v_2)(t)\| = \|\phi_v_1(t) - \phi_v_2(t)\|$$

$$\leq \|\phi_v_1(0) - \phi_v_2(0)\| + \int_0^t \|f(s, \phi_v_1(s)) - f(s, \phi_v_2(s))\| ds$$

$$\leq \|g(\bar{v}_1) - g(\bar{v}_2)\| + L(r) \int_0^t \|\phi_v_1(s) - \phi_v_2(s)\| ds, \forall t \in [0, T].$$

Using Gronwall’s inequality, we conclude that

$$\sup_{t \in [0, T]} \|\phi_v_1(t) - \phi_v_2(t)\| \leq e^{TL(r)} \|g(\bar{v}_1) - g(\bar{v}_2)\|.$$ 

Consequently, $\mathcal{G}$ is a continuous mapping on $K_r(\delta)$, since $g$ is continuous by $(H_3)$.

We next adapt some of the arguments of [117].

Let $t \in [\delta, T]$ be fixed, and $0 < \varepsilon < t$. Define

$$v_\varepsilon(s) = S(s - t + \varepsilon)\phi_v(t - \varepsilon), \forall s \in [t - \varepsilon, T]. \quad (2.5)$$

Invoking (2.3), we derive

$$\|\mathcal{G}v(s) - v_\varepsilon(s)\| = \|\phi_v(s) - v_\varepsilon(s)\| \leq \int_{s-\varepsilon}^s \|f(\tau, \phi_v(\tau))\| d\tau \leq M\varepsilon, \forall s \in [\delta, T],$$

(2.6)

where $M = \sup_{t \in [0, T], \phi \in K_r} \|f(t, \phi(t))\|$. Since $v_\varepsilon(t) = S(\varepsilon)\phi_v(t - \varepsilon)$, and $S(\varepsilon)$ is compact while $\phi_v \in K_r$, it follows that the set $\{v_\varepsilon(t) : v \in K_r(\delta)\}$ is relatively compact in $X$. Then (2.6) implies that the set $\{\mathcal{G}v(t) : v \in K_r(\delta)\}$ is relatively compact in $X$, as well.

Next, let us examine the equicontinuity of $\{\mathcal{G}v(t) : v \in K_r(\delta)\}$ on $[\delta, T]$. Since $S(t)$ is compact for $t > 0$, $\{S(t)u : u \in B\}$ is equicontinuous for all $t > 0$, and
any bounded $B \subset \overline{D(A)}$. Let $t \in [\delta, T]$ and $\varepsilon \in (0, t)$. Taking into account (2.5), we infer that \( \{v_\varepsilon(t) : v \in K_r(\delta)\} \) is equicontinuous at $t$. Therefore there exists $\gamma(t, \varepsilon) > 0$ such that

\[
\|v_\varepsilon(s) - v_\varepsilon(t)\| \leq M\varepsilon
\]  

(2.7)

for any $s \in [\delta, T]$ with $|s - t| \leq \gamma(t, \varepsilon)$, $\forall v \in K_r(\delta)$. Furthermore,

\[
\|Gv(s) - Gv(t)\| \leq \|Gv(s) - v_\varepsilon(s)\| + \|v_\varepsilon(s) - v_\varepsilon(t)\| + \|v_\varepsilon(t) - Gv(t)\|
\]  

(2.8)

Combining (2.6)-(2.8), we get $\|Gv(s) - Gv(t)\| \leq 3M\varepsilon$, for any $s \in [\delta, T]$ with $|s - t| \leq \gamma(t, \varepsilon)$, $\forall v \in K_r(\delta)$. Thus $\{Gv(\cdot) : v \in K_r(\delta)\}$ is equicontinuous on $[\delta, T]$. By Theorem 2.1, $G(K_r(\delta))$ is relatively compact in $C([\delta, T]; X)$.

We can now apply Theorem 2.2 to conclude that $G$ has at least one fixed point $v_* \in K_r(\delta)$. Let $u = \phi_{v_*}$. Then $u$ is an integral solution of

\[
\begin{cases}
    u'(t) + Au(t) \ni f(t, u(t)), & t \in [0, T] \\
    u(0) = g(\tilde{v}_*).
\end{cases}
\]

Inasmuch as $v_*(t) = (Gv_*)(t) = \phi_{v_*}(t) = u(t)$, $\forall t \in [\delta, T]$, we obtain $g(\tilde{v}_*) = g(u)$ by (H3). This implies that $u$ is an integral solution of (2.1), and completes the proof of Theorem 2.4.

In some mathematical models, the function $g$ takes the form

\[
g(u) = \sum_{i=1}^{p} c_i u(t_i)
\]  

(2.9)
where \( c_i \) are given constants, and \( 0 < t_1 < t_2 < \cdots < t_p \leq T \). For instance a more realistic model for diffusion of a small amount of gas in a transparent tube involves an initial condition of the form \( u(0) = g(u) \), with \( g \) given by (2.9). This allows measurements to be made at \( t = 0, t_1, t_2, \ldots, t_p \) rather than just at \( t = 0 \).

We remark that in this case, \( g \) obviously satisfies \((H_3)\) with \( \delta = t_1 \), provided that \( D(A) = X \).

The following corollaries are direct consequences of Theorem 2.4. (See also (2.4).)

**Corollary 2.5** Let \( A \) be an \( m\)–accretive operator in \( X \) with \( D(A) = X \) and \( A0 \ni 0 \), such that \((H_1)\) holds. Let \( f \) satisfy \((H_2)\) and \( g \) be given by (2.9). If also the condition

\[
T \sup_{t \in [0,T], \phi \in K_r} \|f(t, \phi(t))\| + \sum_{i=1}^p |c_i| \leq r.
\]

is satisfied, then the nonlinear Cauchy problem

\[
\begin{aligned}
&u'(t) + Au(t) \ni f(t, u(t)), \quad t \in [0,T] \\
u(0) = \sum_{i=1}^p c_i u(t_i)
\end{aligned}
\]

has at least one integral solution \( u \in C([0,T]; X) \).

**Corollary 2.6** Assume that the conditions \((H_1), (H_2)\) and \((H_3)\) of Theorem 2.4 hold for each \( r > 0 \). If

\[
\frac{\|f(t,u)\|}{\|u\|} \to 0, \quad \text{as} \quad \|u\| \to \infty, \quad \text{uniformly in} \quad t,
\]
\[
\frac{\|g(\varphi)\|}{\|\varphi\|_{C([0,T];X)}} \to 0, \quad \text{as} \quad \|\varphi\|_{C([0,T];X)} \to \infty,
\]

then (2.1) has at least one integral solution in \(C([0,T];X)\).

**Remark 2.2** Conditions (2.10) and (2.11) are satisfied if there are constants \(C_1 > 0, C_2 > 0\), and \(\alpha_1, \alpha_2 \in [0,1)\) such that

\[
\|f(t,u)\| \leq C_1(1 + \|u\|)^{\alpha_1}, \quad u \in X,
\]

(2.12)

\[
\|g(\varphi)\| \leq C_2(1 + \|\varphi\|_{C([0,T];X)})^{\alpha_2}, \quad \varphi \in C([0,T];X).
\]

(2.13)

**Remark 2.3** As compared to Theorem 3.5 in [7], we no longer require the compactness of \(g\). This enables us to cover the case when \(g\) is given by (2.9). On the other hand, we now have to impose a Lipschitz condition on \(f\).

**Example 2.1** Let \(\Omega\) be a bounded domain in \(\mathbb{R}^n (n \geq 1)\) with smooth boundary \(\Gamma\) and let \(\beta : D(\beta) \subseteq \mathbb{R} \to 2\mathbb{R}\) be \(m\)-accretive with \(0 \in \beta(0)\). Let \(p \in [2, \infty)\) and \(\lambda > 0\) be given. For each \(u \in W^{1,p}(\Omega)\), we define the pseudo-Laplacian operator by

\[
\Delta_{p}^\lambda u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( |\frac{\partial u}{\partial x_i}|^{p-2} \frac{\partial u}{\partial x_i} \right) - \lambda u |u|^{p-2},
\]

where the partial derivatives are taken in the sense of distributions over \(\Omega\). Let \(L_{p}^\lambda : D(L_{p}^\lambda) \subseteq L^2(\Omega) \to L^2(\Omega)\) be given by

\[
L_{p}^\lambda u = -\Delta_{p}^\lambda u, \quad \forall u \in D(L_{p}^\lambda),
\]

(2.14)

where

\[
D(L_{p}^\lambda) = \{ u \in W^{1,p}(\Omega) : \Delta_{p}^\lambda u \in L^2(\Omega), \quad -(\partial u/\partial \nu_p)(x) \in \beta(u(x)) \quad \text{a.e. on} \quad \Gamma \}.
\]
Here
\[
\frac{\partial u}{\partial \nu_p} = \sum_{i=1}^{n} |\frac{\partial u}{\partial x_i}|^{p-2} \frac{\partial u}{\partial x_i} \cos(n \cdot e_i),
\] (2.15)
where \( n \) is the outward unit normal to \( \Gamma \) and \( \{e_1, \ldots, e_n\} \) is the canonical basis of \( \mathbb{R}^n \).

It is well-known that \( L_p^\lambda \) is \( m \)-accretive in \( L^2(\Omega) \) and \( -L_p^\lambda \) generates a compact semigroup on \( D(L_p^\lambda) = L^2(\Omega) \) such that \( (H_1) \) holds (see, e.g., [118], pp. 18, 44).

Consider the nonlocal initial-boundary value problem:
\[
u(t, x) - \Delta_p^\lambda u(t, x) \ni c_0 \sin(u(t, x)), \quad \text{a.e. on } (0, T) \times \Omega,
\] (2.16)
\[
u(0, x) = \sum_{i=1}^{p} c_i \sqrt[p]{u(t_i, x)}, \quad \text{a.e. on } \Omega,
\] (2.17)
\[-\frac{\partial u}{\partial \nu_p} \in \beta(u(t, x)), \quad \text{a.e. on } (0, T) \times \Gamma,
\] (2.18)
where \( 0 < t_1 < t_2 < \cdots < t_p \leq T \) and \( c_i(i = 0, 1, \ldots, p) \) are constants.

Let \( X = L^2(\Omega) \) and \( A = L_p^\lambda \), where \( L_p^\lambda \) is defined by (2.14) and (2.15). Define \( f : [0, T] \times X \to X \) by \( f(t, u)(x) = c_0 \sin(u(t, x)) \), for all \( t \in [0, T] \) and almost all \( x \in \Omega \) (where \( u \in X \)), and \( g : C([0, T]; X) \to X \) by \( g(u)(x) = \sum_{i=1}^{p} c_i \sqrt[p]{u(t_i, x)} \), for any \( u \in X \), and almost all \( x \in \Omega \). It is easily seen that with these choices, all of the assumptions of Theorem 2.4 are satisfied. In particular, note that (2.10) and (2.13) hold. Applying Theorem 2.4(cf. also Corollary 2.6), we conclude that the problem (2.16)-(2.18) has at least one integral solution \( u \in C([0, T]; L^2(\Omega)) \).

**Remark 2.4** We may replace the initial condition (2.17) by
\[
u(0, x) = \int_{\delta}^{T} h(s) \log(1 + |u(s, x)|) ds,
\] (2.19)
for all \( u \in C([0, T]; L^2(\Omega)) \), and almost all \( x \in \Omega \),
where $\delta \in (0, T)$ and $h \in L^2(0, T; \mathbb{R})$. If we now define $g : C([0, T]; X) \rightarrow X$ by $g(u)(x) := \text{right-hand side of (2.19)}$, we can easily verify (2.11). Moreover, $(H_3)$ is clearly satisfied in this case. As a result, we derive the existence of an integral solution $u \in C([0, T]; L^2(\Omega))$ for the problem (2.16), (2.19), (2.18), as well.

### 2.2 The Nonautonomous Case

In this section we study the existence of weak solutions of nonlocal Cauchy problems for a nonlinear evolution equation in the time-dependent operator case:

$$
\begin{cases}
  u'(t) + A(t)u(t) \ni f(t, u(t)), & t \in [0, T] \\
  u(0) = g(u),
\end{cases}
$$

(2.20)

in a real Banach space $X$. Here, $A(t) : D(A(t)) \subset X \rightarrow 2^X$ is a nonlinear (possibly multivalued) $m$–accretive operator for each $t \in [0, T]$ such that $-A(t)$ generates an evolution operator $U(t, s)$ in the following sense:

(i) $U(t, s) : \overline{D(A(s)))} \rightarrow \overline{D(A(t))}, \ \ U(s, s) = I, \ 0 \leq s \leq t \leq T$,

(ii) $U(t, s)U(s, r) = U(t, r), \ \text{for every } 0 \leq r \leq s \leq t \leq T$,

(iii) The function $(t, s, x) \rightarrow U(t, s)x, \ (\text{with } t \geq s)$ is continuous,

(iv) $\|U(t, s)x - U(t, s)y\| \leq \|x - y\|, \ 0 \leq s \leq t \leq T, \ x, y \in \overline{D(A(s))}$,

and $f : [0, T] \times X \rightarrow X$ and $g : C([0, T]; X) \rightarrow \overline{D(A(t))}$, $t \in [0, T]$.

The study of abstract nonlocal time-dependent Cauchy problems was earlier done by a few authors. Aizicovici and Gao in [4] obtained existence and uniqueness results under Lipschitz conditions on both $f$ and $g$. In [7] Aizicovici and
McKibben established the existence of integral solutions under Carathéodory and some boundedness conditions on $f$, and continuity and compactness conditions on $g$. Byszewski in [28] showed the existence and uniqueness of weak solutions to some nonautonomous functional differential equations with nonlocal initial conditions under Lipschitz type assumptions on both $f$ and $g$. Here we use weaker conditions on $g$ and consider a general class of time dependent multivalued operators in a Banach space. Our approach relies on the theory of differential equations governed by $m$–accretive operators, compactness methods and fixed-point techniques.

Let $T$ be a fixed positive constant. We require the following conditions:

(A1) For each $t \in [0, T]$, $A(t) : D(A(t)) \subset X \rightarrow 2^X$ is $m$–accretive, i.e. $\|x_1 - x_2 + \lambda(y_1 - y_2)\| \geq \|x_1 - x_2\|$, for every $x_1, x_2 \in D(A(t))$, $[x_1, y_1] \in A(t)$, $[x_2, y_2] \in A(t)$ and $R(I + \lambda A(t)) = X$, for $\lambda > 0$,

(A2) There exist two continuous functions $h : [0, T] \rightarrow X, k : [0, \infty) \rightarrow [0, \infty)$ such that $<y_1 - y_2, x_1 - x_2>_s \geq -\|h(t) - h(s)\||x_1 - x_2||k(\max\{\|x_1\|, \|x_2\|\})$, for every $0 \leq s \leq t \leq T$, $[x_1, y_1] \in A(t)$ and $[x_2, y_2] \in A(s)$.

(A3) If $t_n \uparrow t, x_n \in D(A(t_n))$ and $x_n \rightarrow x$, then $x \in \overline{D(A(t))}$.

Under these three conditions, the set $\overline{D(A(t))}$ is independent of $t$ and so it is denoted by $\overline{D}$ (see, Lemma 3.1 in [40]). Note that condition (A2) (see, Pavel [94, 97]) contains some of the well-known conditions of Kato and Evans’ type:

$$\|A_\lambda(t)x - A_\lambda(s)x\| \leq \|h(t) - h(s)\|k(\|x\|)(1 + A_\lambda(t)x)),$$

where $J_\lambda(t) = (I + \lambda A(t))^{-1}$ and $A_\lambda(t) = (I - J_\lambda(t))/\lambda$. Here we recall that in [40] Evans studied:
by means of discrete approximation schemes of the form:

\[
\begin{cases}
\frac{u^n_k - u^n_{k-1}}{t^n_k - t^n_{k-1}} + A(t^n_k)u^n_k \ni f^n_k \\
u^n_0 = u_0,
\end{cases}
\]

Evans showed that the function \( u^n(t) \equiv u^n_k \) on \([t^n_{k-1}, t^n_k]\) solving the approximate problem converge uniformly to a continuous function \( u(t) \), when the step functions \( f^n(t) \equiv f^n_k \) on \([t^n_{k-1}, t^n_k]\) converge to \( f \) in \( L^1(0, T ; X) \). This limit function \( u(t) \) is called the weak solution of (2.21) It is well-known that (2.21) has a weak solution \( u \in C([0, T]; \overline{D}) \) (see, e.g., Theorem 1 and 3 in [40]). If \( u \) and \( v \) are two solutions of \( u' + A(t)u \ni f \) and \( v' + A(t)v \ni g \) respectively, with \( f, g \in L^1(0, T; X) \) then they satisfy Benilan’s inequality, i.e.

\[
\|u(t) - v(t)\| \leq \|u(s) - v(s)\| + \int_s^t \|u(\tau) - v(\tau), f(\tau) - g(\tau)\| \, d\tau, \quad \forall 0 \leq s \leq t \leq T.
\]

(2.22)

Since \( |<y, x>| \leq \|y\| \), we also have the weaker inequality:

\[
\|u(t) - v(t)\| \leq \|u(s) - v(s)\| + \int_s^t \|f(\tau) - g(\tau)\| \, d\tau, \quad \forall 0 \leq s < t \leq T.
\]

(2.23)

Let \( A(t) \) be an \( m \)-accretive operator in \( X \) with domain \( D(A(t)) \) for each \( t \in [0, T] \) such that \(-A(t)\) generates an evolution operator \( U(t, s) \) on \( \overline{D} \), which is given by the formula

\[
U(t, s)x = \lim_{n \to \infty} \prod_{i=1}^n (I + \frac{t - s}{n} A(s + i \frac{t - s}{n}))^{-1} x,
\]
for all \( x \in D \) and all \( 0 \leq s \leq t \leq T \). The evolution operator \( U(t, s) \) is said to be compact if \( U(t, s) \) maps bounded subsets of \( D \) into precompact subsets of \( D \), for all \( 0 \leq s < t \leq T \). Now we can give the definition of a weak solution of (2.20).

**Definition 2.7** A function \( u \in C([0, T]; D) \) is said to be a weak solution of (2.20) if \( u \) is a weak solution of \((2.21)\) in the sense of Evans [40] with \( f(t, u(t)) \) in place of \( f(t) \) and \( g(u) \) in place of \( u_0 \).

Let \( r, T \) be finite positive constants, and set \( B_r := \{ x \in X : \| x \| \leq r \} \), and \( K_r := \{ \phi \in C([0, T]; X) : \phi(t) \in B_r, \forall t \in [0, T] \} \).

**Theorem 2.8** Let \( A(t), t \in [0, T] \), satisfy conditions (A1)-(A3). Assume further that the evolution operator \( U(t, s), 0 \leq s < t \leq T \), generated by \( -A(t) \) is compact, and

(i) \( f : [0, T] \times X \to X \) is continuous in \( t \) on \([0, T]\) and there exists a constant \( L(r) > 0 \) such that \( \| f(t, u) - f(t, v) \| \leq L(r) \| u - v \|, \forall t \in [0, T], \forall u, v \in B_r; \)

(ii) \( g : C([0, T]; X) \to \overline{D} \) is a continuous mapping which maps \( K_r \) into a bounded set, and there is a \( \delta = \delta(r) \in (0, T) \) such that \( g(\phi) = g(\psi) \) for any \( \phi, \psi \in K_r \) with \( \phi(s) = \psi(s), s \in [\delta, T] \);

(iii) \( T \sup_{t \in [0, T], \phi \in K_r} \| f(t, \phi(t)) \| + \sup_{\phi \in K_r} \| U(t, 0)g(\phi) \| \leq r. \)

Then (2.20) has at least a weak solution, i.e. there exists a continuous function \( u \) from \([0, T]\) to \( X \) such that \( u \) is a weak solution of (2.20).

**Proof of Theorem 2.8** We will follow the same process as in the autonomous case except for replacing semigroup operators by evolution operators and Benilan’s theorem by Evans’ theorem.
First, set \( K_r(\delta) = \{ \phi \in C([\delta,T];X) : \phi(t) \in B_r, \forall t \in [\delta,T] \} \). For a fixed \( v \in K_r(\delta) \), from Theorem 1 and 3 in Evans \[40\] we define a mapping \( \mathcal{F}_v \) on \( K_r \) by \( (\mathcal{F}_v \phi)(t) = u_\phi \) where \( u_\phi \) is the weak solution of

\[
\begin{cases}
\phi'(t) + A(t)u_\phi(t) \ni f(t, \phi(t)), & t \in [0,T] \\
u_\phi(0) = g(\tilde{v}),
\end{cases}
\]

with

\[
\tilde{v}(t) = \begin{cases} 
  v(t) & \text{if } t \in [\delta,T] \\
  v(\delta) & \text{if } t \in [0,\delta].
\end{cases}
\]

1st step: \( \mathcal{F}_v \) maps \( K_r \) into itself.

Indeed, from the definition of \( \mathcal{F}_v \) and (2.23), we obtain

\[
\| (\mathcal{F}_v \phi)(t) - U(t,0)g(\tilde{v})(t) \| \leq \int_0^t \| f(s, \phi(s)) \| ds \\
\leq T \sup_{t \in [0,T], \phi \in K_r} \| f(t, \phi(t)) \|, \forall t \in [0,T], \phi \in K_r.
\]

Combining the above inequality with condition (iii), we get

\[
\| (\mathcal{F}_v \phi)(t) \| \leq \| (\mathcal{F}_v \phi)(t) - U(t,0)g(\tilde{v})(t) \| + \| U(t,0)g(\tilde{v})(t) \| \\
\leq T \sup_{t \in [0,T], \phi \in K_r} \| f(t, \phi(t)) \| + \sup_{\phi \in K_r} \| U(t,0)g(\tilde{v}) \| \\
\leq r, \forall t \in [0,T], \phi \in K_r.
\]

This implies that \( \mathcal{F}_v K_r \subset K_r \).

From (2.23) and condition (i), it follows that

\[
\| (\mathcal{F}_v \phi)(t) - (\mathcal{F}_v \psi)(t) \| \leq tL(r) \max_{s \in [0,t]} \| \phi(s) - \psi(s) \|, \forall t \in [0,T], \phi, \psi \in K_r.
\]

Moreover, we deduce inductively that for \( n \in N \),

\[
\| (\mathcal{F}_v^n \phi)(t) - (\mathcal{F}_v^n \psi)(t) \| \leq \frac{(tL(r))^n}{n!} \max_{s \in [0,t]} \| \phi(s) - \psi(s) \|, \forall t \in [0,T], \phi, \psi \in K_r.
\]
Hence, we infer that for \( n \) large enough, the mapping \( F^n \) is a strict contraction. Thus, by the Banach Contraction Mapping Principle, \( F \) has a unique fixed point \( \phi_v \in K_r \), which is the integral solution of

\[
\begin{align*}
\phi_v'(t) + A(t)\phi_v(t) &\ni f(t,\phi_v(t)), \quad t \in [0,T] \\
\phi_v(0) &= g(\tilde{v}).
\end{align*}
\tag{2.25}
\]

2nd step: Problem (2.20) has at least a weak solution.

We define a mapping \( \mathcal{G} \) from \( K_r(\delta) \) into itself by \((\mathcal{G}v)(t) = \phi_v(t), t \in [\delta,T], \) where \( \phi_v \) satisfies (2.25). From the definition of \( \mathcal{G} \), (2.23) and (i), we get

\[
\begin{align*}
\| (\mathcal{G}v_1)(t) - (\mathcal{G}v_2)(t) \| &= \| \phi_{v_1}(t) - \phi_{v_2}(t) \| \\
&\leq |\phi_{v_1}(0) - \phi_{v_2}(0)| + \int_0^t \| f(s,\phi_{v_1}(s)) - f(s,\phi_{v_2}(s)) \| ds \\
&\leq \| g(\tilde{v}_1) - g(\tilde{v}_2) \| + L(r) \int_0^t \| \phi_{v_1}(s) - \phi_{v_2}(s) \| ds, \quad \forall t \in [0,T].
\end{align*}
\]

Using Gronwall’s inequality, we conclude that

\[
\sup_{t \in [0,T]} \| \phi_{v_1}(t) - \phi_{v_2}(t) \| \leq e^{TL(r)}\| g(\tilde{v}_1) - g(\tilde{v}_2) \|.
\]

Consequently, \( \mathcal{G} \) is a continuous mapping on \( K_r(\delta) \), since \( g \) is continuous by (ii).

We next adapt some of the arguments of Pavel [97] and Kartsatos and Shin [64].

Let \( t \in [\delta,T] \) be fixed, and \( 0 < \varepsilon < t \). Define the function \( v_\varepsilon : [t-\varepsilon,t] \rightarrow X \) by

\[
v_\varepsilon(s) = U(s,t-\varepsilon)\phi_v(t-\varepsilon), \quad \forall s \in [t-\varepsilon,T].
\tag{2.26}
\]

Invoking (2.23), we derive

\[
\| \mathcal{G}v(s) - v_\varepsilon(s) \| = \| \phi_v(s) - v_\varepsilon(s) \| \leq \int_{s-\varepsilon}^s \| f(\tau,\phi_v(\tau)) \| d\tau \leq M\varepsilon, \quad \forall s \in [\delta,T].
\tag{2.27}
\]
where $M = \sup_{t \in [0, T], \phi \in K_r} \|f(t, \phi(t))\|$. Since $v_\varepsilon(t) = U(t, t-\varepsilon)\phi_v(t-\varepsilon)$, and $U(t, t-\varepsilon)$ is compact while $\phi_v \in K_r$, it follows that the set $\{v_\varepsilon(t) : v \in K_r(\delta)\}$ is relatively compact in $X$. Then (2.27) implies that the set $\{Gv(t) : v \in K_r(\delta)\}$ is relatively compact in $X$, as well.

Next, let us examine the equicontinuity of $\{Gv(t) : v \in K_r(\delta)\}$ on $[\delta, T]$. Since $U(t, s)$ is compact for $t > s > 0$, $\{U(t, s)u : u \in B\}$ is equicontinuous for all $t > s > 0$, and any bounded $B \subset \overline{D}$ (see, Theorem 1.1 in Pavel [98]). Let $t \in [\delta, T]$ and $\varepsilon \in (0, t)$. Taking into account (2.26), we infer that $\{v_\varepsilon(t) : v \in K_r(\delta)\}$ is equicontinuous at $t$. Therefore there exists $\gamma(t, \varepsilon) > 0$ such that

$$\|v_\varepsilon(s) - v_\varepsilon(t)\| \leq M\varepsilon$$

for any $s \in [\delta, T]$ with $|s - t| \leq \gamma(t, \varepsilon)$, $\forall v \in K_r(\delta)$. Furthermore,

$$\|Gv(s) - Gv(t)\| \leq \|Gv(s) - v_\varepsilon(s)\| + \|v_\varepsilon(s) - v_\varepsilon(t)\| + \|v_\varepsilon(t) - Gv(t)\|. \quad (2.29)$$

Combining (2.27), (2.26) and (2.29), we get $\|Gv(s) - Gv(t)\| \leq 3M\varepsilon$, for any $s \in [\delta, T]$ with $|s - t| \leq \gamma(t, \varepsilon)$, $\forall v \in K_r(\delta)$. Thus $\{Gv(\cdot) : v \in K_r(\delta)\}$ is equicontinuous on $[\delta, T]$. By Ascoli’s theorem, $G(K_r(\delta))$ is relatively compact in $C([\delta, T]; X)$.

We can now apply Schauder’s Fixed-Point Theorem to conclude that $G$ has at least one fixed point $v_\ast \in K_r(\delta)$. Let $u = \phi_{v_\ast}$. Then $u$ is a weak solution of

$$\begin{cases}
  u'(t) + A(t)u(t) \ni f(t, u(t)), & t \in [0, T] \\
  u(0) = g(\tilde{v}_\ast).
\end{cases}$$
Inasmuch as \( v_*(t) = (Gv_*)(t) = \phi_{v_*}(t) = u(t), \forall t \in [\delta, T] \), we obtain \( g(\tilde{v}_*) = g(u) \) by (iii). This implies that \( u \) is a weak solution of (2.20), and completes the proof of Theorem 2.8.

Now, let us consider an example concerning the nonlinear perturbed diffusion equation which includes as a specific case the porous medium equation.

**Example 2.2** (see, Vrabie [118], pp 70-74; Kartsatos [60]) Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n (n \geq 1) \) with smooth boundary \( \partial \Omega \). Let the following assumptions be satisfied:

1. \( \rho \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\}) \) is a nondecreasing continuous function with \( \rho(0) = 0 \), and for some constants \( C_1 > 0, \alpha \geq 1, \rho'(r) \geq C|r|^\alpha - 1 \), for each \( r \in \mathbb{R} \setminus \{0\} \),

2. \( \gamma : \mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous, \( m \)-accretive with respect to its third variable and such that \( |\gamma(t, x, u)| \leq q(t, x) + q_1(t)|u| \), where \( q : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+ \) is in \( L^1(\Omega) \), for each \( t \in [0, T] \), and \( q_1 : [0, T] \rightarrow \mathbb{R}_+ \),

3. there exists a continuous function \( h : [0, T] \rightarrow \mathbb{R} \) which is of bounded variation with respect to \( t \in [0, T] \) and any \( u \in \mathbb{R} \) we have \( |\gamma(t, x, u) - \gamma(s, x, u)| \leq |h(t) - h(s)|(1 + |h_1(x)| + |u|), \) a.e. in \( x \in \Omega \).

4. there exists a constant \( C_2 > 0 \) such that \( |\gamma(t, x, u) - \gamma(t, x, v)| \leq C_2|u - v| \), for all \( (t, x, u, v) \in [0, T] \times \overline{\Omega} \times \mathbb{R}^2 \),

5. the operator \( A : D(A) \subset L^1(\Omega) \rightarrow L^1(\Omega) \) is defined by \( (Au)(x) = -\Delta \rho(u(x)) \) where \( D(A) = \{ u \in L^1(\Omega); \rho(u) \in W_0^{1,1}(\Omega), \Delta \rho(u) \in L^1(\Omega) \} \).

The operators \( B(t) \) are defined by \( (B(t)u)(x) = \gamma(t, x, u(x)) \).
It is well-known that $A = -\Delta \rho$ is $m$-accretive in $L^1(\Omega)$ and $-A$ generates a compact semigroup on $D(A) = L^1(\Omega)$. It is also known that $B(t)$ is $m$-accretive, continuous and bounded on $L^1(\Omega)$ for each $t \in [0, T]$. Kartsatos in [60] showed that $\|\gamma(t, \cdot, u(\cdot)) - \gamma(s, \cdot, u(\cdot))\| \leq \|h_B(t) - h_B(s)\|_{L^1} k_B(\|u\|_{L^1})$, which implies condition (A2). Since $A$ and $B(t)$ satisfy all of the assumptions of Theorem 1 in [60], the time-dependent $m$-accretive operator $A + B(t)$ generates a compact evolution operator on $L^1(\Omega)$.

Now consider the nonlinear nonlocal perturbed diffusion equation:

$$
\begin{align*}
\frac{\partial u(t,x)}{\partial t} - \Delta \rho(u(t,x)) + \gamma(t,x,u(t,x)) &= c_0 \sin(u(t,x)), \text{ a.e. on } (0, T) \times \Omega, \\
\rho(u(t,x)) &= 0 \text{ a.e. on } (0, T) \times \partial \Omega, \\
u(0, x) &= \sum_{i=1}^{p} c_i \sqrt[3]{u(t_i,x)}, \text{ a.e. on } \Omega.
\end{align*}
$$

(2.30)

where $0 < t_1 < t_2 < \cdots < t_p \leq T$ and $c_i (i = 0, 1, \ldots, p)$ are constants.

Let $X = L^1(\Omega)$ and $A(t) = A + B(t)$ with $A = -\Delta \rho$ and $B(t)u = \gamma(t, \cdot, u)$. Define $f : [0, T] \times X \to X$ by $f(t,u)(x) = c_0 \sin(u(t,x))$, for all $t \in [0, T]$ and almost all $x \in \Omega$ (where $u \in X$), and $g : C([0, T]; X) \to X$ by $g(u)(x) = \sum_{i=1}^{p} c_i \sqrt[3]{u(t_i, x)}$, for any $u \in X$, and almost all $x \in \Omega$. It is easily seen that the $m$-accretive operator $A(t)$ satisfies conditions (A1)-(A3) and $-A(t)$ generates a compact evolution operator on $X$. Thus all of the assumptions of Theorem 2.8 are satisfied. Applying Theorem 2.8, we conclude that the problem (2.30) has at least a weak solution $u \in C([0, T]; L^1(\Omega))$. 
Remark 2.5  We may give a more concrete example of a function $\gamma(t, x, u)$ in Example 2.2.1 as follows:

$$\gamma(t, x, u) \equiv (a(t) + \|x\|) \sin u + b(t)u,$$

where the functions $a : [0, T] \to \mathbb{R}$, $b : [0, T] \to \mathbb{R}_+ \setminus \{0\}$ are continuous and of bounded variation on $[0, T]$. 
2.3 The Multivalued Case

In this section we study the existence of integral solutions of nonlocal Cauchy problems for the nonlinear evolution equation with a nonconvex multivalued perturbation:

\[
\begin{cases}
    u'(t) \in -Au(t) + F(t, u(t)), & t \in [0, T] \\
    u(0) = g(u),
\end{cases}
\]  

(2.31)

in a real Banach space \(X\). Here, \(A : D(A) \subset X \to 2^X\) is a nonlinear \(m\)-accretive operator such that \(-A\) generates a compact semigroup \(S(t)\) \((t > 0)\), \(F : [0, T] \times X \to 2^X \setminus \{\phi\}\) and \(g : C([0, T]; X) \to \overline{D(A)}\). The study of semilinear evolution inclusions with nonlocal conditions was earlier done in [18]-[21]. It has a significant application in controllability testing (see, e.g., [19]). Aizicovici and McKibben [7] obtained the existence of integral solutions of (2.31) with \(A\) nonlinear, \(F\) lower semicontinuous, and \(g\) of Lipschitz type. In [9] Aizicovici and Staicu have considered the case when \(F\) is upper semicontinuous and convex valued. Here we obtain the existence of integral solutions to (2.31) in the case when \(F\) is a closed valued, lower semicontinuous, multifunction, under weaker conditions than those of Theorem 3.8 [7].

Let \(X\) be a real Banach space of norm \(\| \cdot \|\) and dual \((X^*, \| \cdot \|_*)\) and \(I = [0, T]\), where \(0 < T < \infty\). We denote by \(C(I, X)\) (resp. \(L^1(I, X)\)) the Banach space of all continuous (resp. Bochner integrable) functions \(u : I \to X\) with norm \(\|u\|_{\infty} = \sup_{t \in I} \|u(t)\|\) (resp. \(\|u\|_1 = \int_0^T \|u(t)\|dt\)). As usual, \(2^X\) stands for the family of all subsets of \(X\) and \(\overline{\Omega}\) denotes the closure of a set \(\Omega \in 2^X\). We use the
notations $P(X) = 2^X \setminus \{\phi\}$, $P_c(X) = \{Y \in P(X) : Y \text{ closed}\}$. A multivalued map $G : X \to P(X)$ is closed valued if $G(x)$ is closed for all $x \in X$.

Let $Z$ be another real Banach space. In addition, let $(X, d)$ and $(Z, d)$ be the metric space induced from the normed space $(X, \| \cdot \|)$ and $(Z, \| \cdot \|)$, respectively. Consider the Hausdorff distance $d_H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$, given by

$$d_H(A, B) := \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

where $d(A, b) = \inf_{a \in A} d(a, b), d(a, B) = \inf_{b \in B} d(a, b)$. A multivalued map $G : X \to P_c(Z)$ is said to be measurable, if for any $C \in P_c(Z), \{x \in X : G(x) \cap C \neq \phi\}$ is measurable, or equivalently, if for every $z \in Z$, the function $x \mapsto d(z, G(x)) = \inf\{\|z - v\| : v \in G(x)\}$ is measurable where $d$ is the metric induced by the Banach space $Z$. $G : X \to P(Z)$ is called lower semicontinuous (l.s.c.) on $X$ if $G^{-1}(V) = \{x \in X : G(x) \cap V \neq \phi\}$ is open in $X$ whenever $V$ is open in $Z$, or equivalently, if $\{x \in X : G(x) \subset C\}$ is closed in $X$ whenever $C$ is closed in $Z$. $G : X \to P(Z)$ is called upper semicontinuous (u.s.c.) on $X$ if $G^{-1}(C) = \{x \in X : G(x) \cap C \neq \phi\}$ is closed in $X$ whenever $C$ is closed in $Z$, or equivalently, if $\{x \in X : G(x) \subset V\}$ is open in $X$ whenever $V$ is open in $Z$.

$G : X \to P_c(Z)$ is l.s.c. if and only if the function $x \mapsto d(z, G(x))$ is u.s.c., for every $z \in Z$ ([55] p. 45). A function $\varphi : \Omega \subset X \to Z$ that satisfies $\varphi(\omega) \in G(\Omega)$ a.e. on $\Omega$, is called a selection of $G$. We define $S^1_G$ as the set of all selections of $G$ that belong to $L^1(\Omega, Z)$. For further background and details we refer the reader to Hu and Papageorgiou [55] and Deimling [37, 38].

Let $K$ be a subset of $I \times X$. We say that $K$ is $L \otimes B$ measurable if $K$ belongs to the $\sigma$-algebra generated by all sets of the form $L \times B$ where $L$ is Lebesgue
measurable in $I$ and $B$ is Borel measurable in $X$. A subset $K$ of $L^1(I,X)$ is called decomposable if, for all $u,v \in K$, and all measurable subsets $E$ of $I$, the function $u\chi_E + v\chi_{I\setminus E} \in K$, where $\chi_E$ denotes the characteristic function of $E$. A multi-valued function $G : X \to P_d(L^1(I,X))$ is said to be decomposable if $G(x)$ is a decomposable subset of $L^1(I,X)$, for all $x \in X$.

In the proof of our main theorem we will need the following auxiliary results.

**Theorem 2.9** (Mazur, [55], p. 915) The closed convex hull of a strongly compact set is strongly compact.

**Theorem 2.10** (Baras, [118], p. 47) Let $A : D(A) \subset X \to P(X)$ be an $m$–accretive operator such that $-A$ generates a compact contraction semigroup on $\overline{D(A)}$, and $V \subseteq L^1(I,X)$ is uniformly integrable. Then the set $\{u : u$ is an integral solution to (2.2) for some $u_0 \in \overline{D(A)}, f \in V\}$ is relatively compact in $C(I,X)$.

The following well-known fixed-point theorem plays a key role in the proof of our main theorem.

**Theorem 2.11** (Tychonoff, [121] I, p. 452) Let $F : K \subset X \to K$ be continuous and let $K$ be a nonempty compact convex set in $X$. Then $F$ has a fixed point.

Throughout the remainder of this section we will assume that $X$ is a separable Banach space.

**Theorem 2.12** ([55], p. 238) Let a multivalued map $F : I \times X \to P_d(Z)$ satisfy the following assumptions:
(A₁) \((t, x) \mapsto F(t, x)\) is \(\mathcal{L} \otimes \mathcal{B}\) measurable,

(A₂) for every \(t \in I, x \mapsto F(t, x)\) is lower semicontinuous (l.s.c.),

(A₃) \(|F(t, x)| := \sup\{\|w\| : w \in F(t, x)\} \leq a(t) + c\|x\|, \text{ a.e. } t \in I\) with \(a \in L^1(I, \mathbb{R}^+), c \geq 0\).

Then the map \(u \mapsto S^1_{F(\cdot, u(\cdot))}\) is l.s.c, viewed as a multifunction from \(C(I, X)\) into \(L^1(I, X)\), where \(S^1_{F(\cdot, u(\cdot))} = \{f \in L^1(I, X) : f(t) \in F(t, u(t)) \text{ for a.e. } t \in I\}\).

Next we state the Bressan-Colombo theorem.

**Theorem 2.13** ([25], [55], p. 245) If \(Y\) is a separable metric space and the multivalued map \(F : Y \to P_{\text{cl}}(L^1(I, X))\) is l.s.c. with decomposable values, then \(F\) has a continuous selection, i.e. there exists a continuous function (single-valued) \(f : Y \to L^1(I, X)\) such that \(f(y) \in F(y)\) for every \(y \in Y\).

Here, we make precise what we mean by a solution of (2.31) when \(F\) is multivalued.

**Definition 2.14** A function \(u \in C(I, D(A))\) is called an integral solution of the problem (2.31) if there exists \(f \in L^1(I, X)\) with \(f(t) \in F(t, u(t))\), a.e. on \(I\), such that \(u\) is an integral solution of (2.2) in the sense of Benilan with \(g(u)\) in place of \(u_0\).

Now, we are able to formulate our main result:

**Theorem 2.15** Assume that following conditions are satisfied:

\((N_1) A : D(A) \subset X \to P(X)\) is \(m\)--accretive such that \(-A\) generates a compact semigroup \(S(t), t > 0,\)
\((N_2)\) \(g : C(I, X) \to \overline{D(A)}\) is such that
\[
\|g(u) - g(v)\| \leq m\|u - v\|_\infty, \forall u, v \in C(I, X),
\]
for some constant \(m > 0\).

\((N_3)\) \(F : I \times X \to P_{cl}(X)\), satisfies

(i) \((t, x) \mapsto F(t, x)\) is \(\mathcal{L} \otimes \mathcal{B}\) measurable,

(ii) \(x \mapsto F(t, x)\) is lower semicontinuous (l.s.c.) for a.e. \(t \in I\),

(iii) there exists a function \(\gamma : I \times \mathbb{R}^+ \to \mathbb{R}^+\) such that \(\gamma(\cdot, r) \in L^1(I, \mathbb{R})\) for every \(r \in \mathbb{R}^+\), \(\gamma(t, \cdot)\) is continuous and nondecreasing for a.a. \(t \in I\), and
\[
\lim \sup_{r \to \infty} \frac{1}{r} \int_0^T \gamma(t, r)dt < 1 - m
\]
where \(m\) is the same as in condition \((N_2)\) with the additional property that
\[
|F(t, x)| := \sup\{\|w\| : w \in F(t, x)\} \leq \gamma(t, \|x\|_\infty),
\]
for a.e. \(t \in I\), and \(x \in \overline{D(A)}\).

Then (2.31) has at least one integral solution.

**Proof of Theorem 2.15** Without loss of generality, we assume that \(0 \in D(A)\).

We start with the inhomogeneous nonlocal Cauchy problem
\[
u'(t) \in -Au(t) + f(t), t \in I; u(0) = g(u),
\]\(2.34\)
where \(f \in L^1(I, X)\), \(A\) is \(m\)-accretive in \(X\), and \(g\) satisfies \((N_2)\). An integral solution to (2.34) means a solution \(u \in C(I, \overline{D(A)})\) of (2.2) in the sense of Benilan.
with \( g(u) \) in place of \( u_0 \). Since \( g \) satisfies a contraction condition, from Benilan’s Inequality and the Contraction Mapping Principle, we can easily show that (2.34) has a unique integral solution (see [9]).

Next, we obtain an a-priori bound for all possible solutions of (2.31). Let \( u \) be an integral solution of (2.31) in the sense of Definition 2.14. From the definition of \( u \) there exists \( f_1 \in L^1(I, X) \) with \( f_1(t) \in F(t, u(t)) \), a.e. on \( I \), such that \( u \) is an integral solution of the related problem:

\[
\begin{align*}
u'(t) &\in -Au(t) + f_1(t), \quad t \in I; u(0) = g(u), \\
\end{align*}
\]

where \( f_1 \in L^1(I, X) \). From Benilan’s Inequality (2.3) we deduce that

\[
\|u(t) - x\| \leq \|g(u) - x\| + \int_0^t \|f_1(s) - y\| \, ds, \quad \forall t \in I.
\]

for a fixed \( x \in D(A) \) and \( y \in Ax \). From (2.36) and the triangle inequality it follows that

\[
\|u(t)\| \leq 2\|x\| + \|g(0)\| + T\|y\| + \|g(u) - g(0)\| + \int_0^T \|f_1(s)\| \, ds.
\]

Using \((N_2)\) and \((N_3)(iii)\) in (2.37) we obtain

\[
(1 - m)\|u\|_\infty \leq C + \int_0^T \gamma(s, \|u\|_\infty) \, ds.
\]

where \( C = 2\|x\| + \|g(0)\| + T\|y\| \). Combining (2.32) and (2.38) implies that there exists a finite positive constant \( M \) (which is independent of \( u \)) such that

\[
\|u\|_\infty \leq M.
\]

Indeed, if this is not the case, we can construct a sequence \( \{u_n\}_{n \in \mathbb{N}} \) of integral solutions of (2.31) such that \( \|u_n\|_\infty \to \infty \), as \( n \to \infty \). Then (2.38) leads to

\[
(1 - m) \leq \limsup_{n \to \infty} \frac{\int_0^T \gamma(s, \|u_n\|_\infty) \, ds}{\|u_n\|_\infty} \leq \limsup_{n \to \infty} \frac{1}{r} \int_0^T \gamma(t, r) \, dt,
\]

with \( r \).
which is a contradiction to (2.32). Hence, there must exist a constant $M$ such that (2.39) is satisfied by all integral solutions $u$ of (2.31). Let

$$\varphi(t) := \gamma(t, M)$$

(2.40)

and remark that $\varphi \in L^1(I, \mathbb{R})$. Combining (2.33), (2.39) and (2.40), we may assume

$$|F(t, x)| \leq \varphi(t), \text{ a.e. on } I.$$  (2.41)

Otherwise we replace $F(t, x)$ by $\tilde{F}(t, x) = F(t, R_M(x))$, where the retraction map $R_M : X \to X$ is given by

$$R_M(x) = \begin{cases} x & \text{if } \|x\| \leq M \\ \frac{x}{\|x\|} & \text{if } \|x\| > M, \end{cases}$$

(2.42)

with $M$ as in (2.39). It is easily seen that all conditions in $(N_3)$ are satisfied with $\tilde{F}$ in place of $F$(see, [9]). Let $V = \{h \in L^1(I, X) : \|h(t)\| \leq \varphi(t) \text{ a.e. on } I\}$. Clearly, $V$ is nonempty, closed and convex. In addition, $V$ is uniformly integrable.

Next, let’s consider the solution set of the problem (2.34), i.e. \{$u_f : u_f$ is an integral solution to (2.34) for some $f \in V$\}. We now follow [9] to show that \{$u_f$\}_f\in V is relatively compact in $C(I, X)$. Define the operator $L : C(I, X) \to C(I, X)$ given by

$$(Lw)(t) = w(t) - S(t)g(w), \forall t \in T, w \in C(I, X),$$

(2.43)

where \{S(t) : t \geq 0\} denotes the contraction semigroup generated by $-A$. By $(N_2)$ it is easily verified that $L$ is one-to-one and onto, and $L^{-1}$ is Lipschitz continuous on $C(I, X)$, with a Lipschitz constant $(1 - m)^{-1}$(see, [120] Lemma 2.5). Accordingly, we write

$$u_f(t) = L^{-1}(w_f(t)), \forall t \in T, f \in V,$$

(2.44)
where

\[ w_f(t) = u_f(t) - S(t)g(u_f), \quad \text{by (2.43).} \]  

(2.45)

From Theorem 2.10 with \((N_1)\), we deduce that \( \{w_f(\cdot)\}_{f \in V} \) is relatively compact in \( C([\delta, T], X) \), for any \( 0 < \delta < T \). Next, since by (2.45), \( w_f(0) = g(w_f) - g(w_f) = 0, \forall f \in V \), the set \( \{w_f(0)\}_{f \in V} \) is trivially compact in \( X \). To prove the equicontinuity of \( \{w_f(\cdot)\}_{f \in V} \) at \( t = 0 \), remark that \( z_f(\cdot) = S(\cdot)g(u_f) \) can be viewed as an integral solution of \( z'(t) \in -A z(t), t \in I, z(0) = g(u_f) \). Applying Benilan’s Inequality, we obtain

\[ \|w_f(t) - w_f(0)\| = \|u_f(t) - z_f(t)\| \leq \int_0^t \|f(s)\| ds, \forall t \in I. \]

Recalling that \( f \in V \), we conclude that

\[ \|w_f(t) - w_f(0)\| \leq \int_0^t \varphi(s) ds, \forall t \in I. \]

This implies that \( \{w_f(\cdot)\}_{f \in V} \) equicontinuous at \( t = 0 \), as desired. By Ascoli’s Theorem it follows that \( \{w_f(\cdot)\}_{f \in V} \) is relatively compact in \( C(I, X) \). From (2.44) and the continuity of \( L^{-1} \) on \( C(I, X) \), we have that \( \{u_f\}_{f \in V} \) is relatively compact in \( C(I, X) \), as desired. Therefore from Theorem 2.9 we obtain that \( K = \overline{\text{conv}}\{u : u \text{ is an integral solution to (2.34) for some } f \in V\} \) is nonempty, compact and convex in \( C(I, X) \). Define \( \mathcal{F} : K \to P_{\text{cl}}(L^1(I, X)) \) by \( \mathcal{F}(u) = S^1_{\frac{1}{2}}(\cdot, u(\cdot)). \) From Theorem 2.12 we infer that \( \mathcal{F} \) is l.s.c. and clearly, it has decomposable values. Applying the selection theorem 2.13 we have a single-valued continuous function \( f : K \to L^1(I, X) \) such that \( f(u) \in \mathcal{F}(u) \), for all \( u \in K \).

Next, we consider the problem

\[
\begin{cases}
    u'(t) \in -Au(t) + f(u)(t), & t \in I \\
    u(0) = g(u),
\end{cases}
\]  

(2.46)
We remark that if $u \in C(I, X)$ is a solution of (2.46), then $u$ is a solution to problem (2.31). Let’s transform problem (2.46) into a fixed-point problem by considering the operator $\mathcal{G}$ on $K$, given by $(\mathcal{G}(v))(t) = u_v(t), \ t \in I$, where $u_v$ satisfies

$$
\begin{cases}
    u'_v(t) \in -Au_v(t) + f(v)(t), \ t \in I \\
    u_v(0) = g(u_v),
\end{cases}
$$

for a fixed $v \in K$. In view of (2.40) it is easy to show that $\mathcal{G}(K) \subset K$. Let $\{v_n\}$ be a sequence in $K$ such that $v_n \to v$. Then we have from (2.47) and Benilan’s inequality that

$$
\| (\mathcal{G}(v_n))(t) - (\mathcal{G}(v))(t) \|
\leq \|g(v_n) - g(v)\| + \int_0^t \|f(v_n)(s) - f(v)(s)\| ds, \ \forall t \in I.
$$

Since the functions $f$ and $g$ are continuous, we obtain that $\|\mathcal{G}(v_n) - \mathcal{G}(v)\|_\infty \to 0$, as $n \to \infty$, which implies the continuity of $\mathcal{G}$. Since $K$ is a nonempty compact convex subset of $C(I, X)$, we conclude from Tychonoff’s Theorem 2.11 that $\mathcal{G}$ has a fixed point in $K$ which is an integral solution of (2.31). This completes the proof of Theorem 2.15.

Now, we consider a control problem for a distributed parameter system to show the applicability of our work to multivalued evolution equations in the nonconvex valued case (see, e.g. [91, 92]).

**Example 2.3** Let $I = [0, T]$ and $\Omega \subseteq \mathbb{R}^n$ a bounded domain, with smooth boundary $\partial \Omega$. Consider the following parabolic control system with the pseudo-
The hypotheses on the data of (2.48) are the following:

\((H_f)\) \(f : I \times \Omega \times \mathbb{R} \to \mathbb{R}\) is a function such that

(i) \((t, x) \mapsto f(t, x, r)\) is \(\mathcal{L} \otimes \mathcal{B}\) measurable for each \(r\) in \(\mathbb{R}\),

(ii) \(|f(t, x, u) - f(t, x, v)| \leq k_1(t, x)\|u - v\|\) a.e. with \(k_1(t, x) \geq 0\),

\(k_1 \in L^1(I, L^2(\Omega))\),

(iii) \(|f(t, x, u)| \leq a(t, x) + b(t)\|u\|\) a.e. with \(a(t, x), b(t) \geq 0\),

\(a \in L^1(I, L^2(\Omega)), b \in L^1(I, \mathbb{R})\)

\((H_h)\) \(h : I \times \mathbb{R} \to \mathbb{R}\) is a function such that

(i) \(h(\cdot, r)\) is measurable for each \(r \in \mathbb{R}\) and \(h(\cdot, 0) \in L^1(I, \mathbb{R})\),

(ii) \(h(t, \cdot)\) is continuous for a.a. \(t \in I\),

(iii) \(|h(t, r_1) - h(t, r_2)| \leq k_2(t)|r_1 - r_2|\) a.e. \(t \in I\) and all \(r_1, r_2 \in \mathbb{R}\), with \(k_2(t) \in L^1(I, \mathbb{R}^+)\),

\((H_Z)\) \(Z = Z_0 \cap Z_1\), where \(Z_0\) is a closed bounded subset of \(L^2(\Omega)\) and \(Z_1 = \{z \in L^\infty(\Omega) : |z(x)| \leq 1, \text{ a.e. on } \Omega\}, Z \neq \emptyset\).

Let \(X = L^2(\Omega, \mathbb{R})\), with the norm \(\| \cdot \|\). This is a separable Hilbert space. Let \(A = L^\lambda_p\), where \(L^\lambda_p\) is defined by (2.14) and (2.15). Then it is well-known that \(A\)
is $m-$accretive in $X$ and $-A$ generates a compact contraction semigroup on $X$.

Next we introduce the functional $g : C(I, X) \to X$ by

$$g(u)(x) = \int_0^T h(s, u(s, x))ds, \ \forall u \in C(I, X), \ \text{a.e. on } \Omega.$$  

From $(H_h)$ it follows that $g$ is well defined and

$$\| g(u) - g(v) \|_2 \leq \| k_2 \|_1 \| u - v \|_\infty, \ \forall u, v \in C(I, X).$$

Now we define $F : I \times X \to P_{cl}(X)$, given by

$$F(t, u) = \tilde{f}(t, u)Z$$

with $\tilde{f}(t, u)(\cdot) = f(t, \cdot, u(\cdot)) \in X$. From the definition of $F$ and the hypotheses $(H_f)$ and $(H_Z)$, we have

$$d_H(F(t, u), F(t, v)) \leq \| \tilde{f}(t, u) - \tilde{f}(t, v) \|_2 \leq l(t) \| u - v \|_2,$$

where $l(t) = \| k_1(t, \cdot) \|_2$ and

$$|F(t, u)| := \sup \{ \| w \| : w \in F(t, u) \} \leq \| a(t, \cdot) \|_2 + b(t) \| u \|_2.$$

From Aumann’s selection theorem (see, [14], [55] Sec 2.5.0), system (2.48) is equivalent to the following nonlocal multivalued Cauchy problem:

$$\begin{cases} u_t(t, x) \in -Au(t, x) + F(t, u(t, x)), \ \text{a.e. } t \in I \\ u(0, x) = \int_0^T h(s, u(s, x))ds, \ \text{on } \Omega, \end{cases}$$

If $\| k_2 \|_1 + \| b \|_1 < 1$, then it is easy to show that assumptions $(N_1)$-$(N_3)$ of Theorem 2.15 are satisfied with $m = \| k_2 \|_1$ and $\gamma(t, r) = \| a(t, \cdot) \|_2 + b(t)r$. A direct application of Theorem 2.15 yields:
**Theorem 2.16** Assume that conditions $(H_f)$, $(H_h)$ and $(H_Z)$ are satisfied, and in addition, $\|k_2\|_1 + \|b\|_1 < 1$. Then the parabolic control system (2.48) has an integral solution $u \in C(I, L^2(\Omega))$. 
3 Optimization Problems

3.1 Introduction

In this chapter we obtain some higher order necessary conditions of optimality for nonlinear programming problems in a Banach space by using the higher order tangential cones, which were introduced in [100]. We improve and extend the result of [101]. We first present the generalization of the result in [102] and then give higher order necessary conditions for a local minimum. We also give significant examples to show the applicability of these results. Finally we present some higher order sufficient conditions for a local minimum in a finite dimensional space and then give counterexamples to show that these results are not applicable in an infinite dimensional space. Note that throughout this paper, differentiability means Fréchet differentiability. There is a natural overlap between the general results of this paper and of the paper [31] written also under the influence of the paper [101] by Pavel and Potra. However, our examples here are different from those in [31]. Moreover, this paper provides important counterexamples to point out that the sufficient conditions for optimality may not remain valid in infinite dimensional spaces.

3.2 Auxiliary Results

For further background knowledge and details of this section, we refer the reader to [10, 24, 85]. We need to generalize the results in [102] for using in later sections.
Let $\alpha_k = (i_1, i_2, \ldots, i_k)$ be a multi-index with positive integer components. Then $|\alpha_k| = i_1 + i_2 + \cdots + i_k$, $\alpha_k! = i_1!i_2!\cdots i_k!$, $(v_{\alpha_k}) = (v_{i_1})(v_{i_2})\cdots (v_{i_k})$.

**Definition 3.1** Let $X$ be a normed space, of norm $\| \cdot \|$ and let $D$ be a nonempty subset of $X$. Then a vector $v_1 \in X$ is said to be "tangent" to $D$ at $x \in D$, if

$$\lim_{t \downarrow 0} \frac{1}{t} d[x + tv_1; D] = 0,$$

(3.1)

where $d[z; D]$ stands for the distance from $z \in X$ to $D$. It is well-known that (3.1) is equivalent to:

$$x + tv_1 \in D, \text{ for some } r_1(t) \to 0 \text{ as } t \downarrow 0.$$

(3.2)

Denote by $T_x D$ the set of all $v_1 \in X$ satisfying (3.1) or (3.2).

The generalization is straightforward:

**Definition 3.2** For a given $n \in \mathbb{N}$, a vector $v_n \in X$ is said to be a "$n$-th order tangent vector" to $D$ at $x \in D$, if there exist $v_i \in X, i = 1, 2, \ldots, n - 1$, such that

$$\lim_{t \downarrow 0} \frac{1}{t^n} d[x + tv_1 + \frac{t^2}{2!} v_2 + \cdots + \frac{t^n}{n!} v_n; D] = 0$$

(3.3)

The vectors $v_i \in X, i = 1, 2, \ldots, n - 1$, in (3.3) are said to be "associated" with $v_n$. It follows that (3.3) is equivalent to:

$$x + tv_1 + \frac{t^2}{2!} v_2 + \cdots + \frac{t^{n-1}}{(n-1)!} v_{n-1} + \frac{t^n}{n!} (v_n + r_n(t)) \in D,$$

for some $r_n(t) \to 0$ as $t \downarrow 0$.

(3.4)

Denote by $T^n_x D$ the set of all $n$-th order tangent vectors $v_n$ to $D$ at $x$. Equivalently,

$$T^n_x D = \{ v_n \in X; \text{ there exist } v_i \in X, i = 1, 2, \ldots, n - 1, \text{ such that (3.3) or (3.4) holds } \}$$

(3.5)
with \( T^1_x D = T_x D \) (by convention). It is obvious that \( v_n \in T^n_x D \) implies \( v_i \in T^i_x D, i = 1, 2, \ldots, n - 1. \)

Let \( E \) be a nonempty subset of real normed space \( Y, A \subset X \) an open subset of \( X \) and \( G : A \to Y \) a Frechet differentiable function at \( x \in A \).

We now introduce the condition:

\[
\lim_{t \to 0} \frac{1}{t} d[G(x) + t\dot{G}(x)(v_1); E] = 0, \tag{3.6}
\]

**Theorem 3.3** (Pavel and Ursescu [102]) If \( D = G^{-1}(E) \) and \( G \) is differentiable at \( x \in A \), then (3.1) implies (3.6). If in addition to the above hypothesis we assume that \( G \) is continuous on \( A \), \( \dot{G}(x) : X \to Y \) is onto and \( \dim(Y) < \infty \), then (3.6) implies (3.1).

**Corollary 3.4** ([102]) Assume that in addition \( D = D_G \), i.e. \( D_G = \{ x \in X | G(x) = 0 \} \). Then the following conditions are equivalent:

(i). \( \lim_{t \to 0} \frac{1}{t} d[x + tv_1; D_G] = 0, v_1 \in X, \)

(ii). \( G(x) = 0, \dot{G}(x)(v_1) = 0. \)

We have the generalization of the above theorem as follows. Assume that \( G \in C^n(A, Y) \). Denote by \( G^{(n)}(x)(v)^n \), the \( n \)-th order derivative of \( G \) in the direction of \( v \) at \( x \), and \( (v)^n = (v)(v) \cdots (v) \). By the Taylor’s formula, we get that

\[
\lim_{v \to 0, v \neq 0} \frac{1}{\|v\|^n} \|G(x + v) - G(x) - \dot{G}(x)(v) - \ddot{G}(x)(v)^2 - \cdots - \frac{1}{n!} G^{(n)}(x)(v)^n\| = 0, \tag{3.7}
\]
Replacing $v$ in (3.7) by $t v_1 + \frac{t^2}{2!} v_2 + \cdots + \frac{t^n}{n!} v_n$ and taking into account $G^{(n)}(x) \in L \left( \underset{n \text{-times}}{L(X, L(X, \ldots, L(X, Y) \cdots))} \right) \equiv L^n(X, Y)$, we get

\[
\lim_{t \downarrow 0} \frac{n!}{t^n} \left[ G(x + tv_1 + \frac{t^2}{2!} v_2 + \cdots + \frac{t^n}{n!} v_n) - G(x) - t \dot{G}(x)(v_1) \right. \\
\left. - \frac{t^2}{2!} (\ddot{G}(x)(v_1)^2 + \dot{G}(x)(v_2)) - \cdots \right. \\
- \frac{t^n}{n!} \sum_{|\alpha_k|=n} \frac{n!}{k! \alpha_k!} G^{(k)}(x)(v_{\alpha_k}) = 0, \tag{3.8}
\]

uniformly with respect to $(v_1, v_2, \ldots, v_n)$ on bounded subsets of $\prod_{i=1}^{n} X$. In (3.8), if we replace $v_n$ by $v_n + r$ and we have in mind $\dot{G}(x) \in L(X, Y)$, we obtain that

\[
\lim_{t \uparrow 0, r \to 0} \frac{n!}{t^n} \left[ G(x + tv_1 + \frac{t^2}{2!} v_2 + \cdots + \frac{t^n}{n!} (v_n + r)) - G(x) - t \dot{G}(x)(v_1) \right. \\
\left. - \frac{t^2}{2!} (\ddot{G}(x)(v_1)^2 + \dot{G}(x)(v_2)) - \cdots \right. \\
- \frac{t^n}{n!} \sum_{|\alpha_k|=n} \frac{n!}{k! \alpha_k!} G^{(k)}(x)(v_{\alpha_k}) + \dot{G}(x)(r)) = 0. \tag{3.9}
\]

**Lemma 3.5** (see [102]) Let $L : X \to Y$ be a linear onto operator and $\dim(Y) < \infty$. Then there is a linear continuous operator $R : Y \to X$ such that $L(R(y)) = y$ for $\forall y \in Y$.

The lemma below establishes a reciprocal relation to (3.9):

**Lemma 3.6** Assume that $G$ is continuous on $A$, $G$ is $n$-times differentiable at $x \in A$, $\dot{G}(x) : X \to Y$ is onto and $\dim(Y) < \infty$. Then for every $\varepsilon > 0$, there is $\delta > 0$ such that for each $t \in (0, \delta)$ and $z \in Y$ with $\|z\| < \delta$, there is $r \in X$ with
the properties

\[ \| r \| < \varepsilon, \quad x + tv_1 + \frac{t^2}{2!}v_2 + \cdots + \frac{t^n}{n!}(v_n + r) \in A, \]

\[ G(x + tv_1 + \frac{t^2}{2!}v_2 + \cdots + \frac{t^n}{n!}(v_n + r)) = G(x) + t\dot{G}(x)(v_1) + \frac{t^2}{2!}(\ddot{G}(x)(v_1)^2 + \dot{G}(x)(v_2)) + \cdots \]

\[ + \frac{t^n}{n!} \left( \sum_{|\alpha_k| = n} \frac{n!}{k!\alpha_k!} G^{(k)}(x)(v_{\alpha_k}) + z \right). \]

**Proof.** Since \( \dot{G}(x) \) is onto, by Lemma 3.5 with \( L = \dot{G}(x) \) there is a linear continuous operator \( R : Y \rightarrow X \), such that

\[ \dot{G}(x)(R(w)) = w, \quad \text{for } \forall w \in Y. \quad (3.10) \]

Let \( \varepsilon > 0 \) be arbitrary. Since \( R(0) = 0 \) and \( R \) is continuous at 0, there is \( r > 0 \) such that \( \| R(w) \| < \varepsilon \), for \( \forall w \in B(\rho) = \{ w \in Y : \| w \| < \rho \} \). Since \( A \) is open and (3.8) holds with \( v_n + R(w) \) instead of \( v_n \), it follows that there is \( \delta > 0 \) such that

\[ x + tv_1 + \frac{t^2}{2!}v_2 + \cdots + \frac{t^n}{n!}(v_n + r) \in A, \quad (3.11) \]

and

\[ \frac{n!}{t^n} \| G(x + tv_1 + \frac{t^2}{2!}v_2 + \cdots + \frac{t^n}{n!}(v_n + R(w))) \]

\[ - G(x) - t\dot{G}(x)(v_1) - \frac{t^2}{2!}(\ddot{G}(x)(v_1)^2 + \dot{G}(x)(v_2)) - \cdots \]

\[ - \frac{t^n}{n!} \left( \sum_{|\alpha_k| = n} \frac{n!}{k!\alpha_k!} G^{(k)}(x)(v_{\alpha_k}) + \dot{G}(x)(R(w))) \right) \| \leq \frac{\rho}{2}, \quad (3.12) \]

for all \( t \in (0, \delta) \) and \( w \in B(\rho) \). Without loss of generality, we may assume that \( \delta \leq \frac{\rho}{2} \). Let us show that this \( \delta \) satisfies the condition required by our lemma. Take an arbitrary \( t \in (0, \delta) \) and \( z \in Y \) and \( \| z \| < \delta \) and denote by \( F : B(\rho) \rightarrow Y \) the
function

\[ F(w) = \frac{n!}{t^n} [-G(x + tv_1 + \frac{t^2}{2!}v_2 + \cdots + \frac{t^n}{n!}(v_n + R(w))] \]

\[ + G(x) + t\dot{G}(x)(v_1) + \frac{t^2}{2!}(\ddot{G}(x)(v_1)^2 + \dot{G}(x)(v_2)) + \cdots \]

\[ + \frac{t^n}{n!} \left( \sum_{|\alpha_k| = n} \frac{n!}{k!\alpha_k!} G^{(k)}(x)(v_{\alpha_k}) + \dot{G}(x)(R(w)) \right) + z, \]  

(3.13)

The linearity of \( \dot{G}(x) \) and (3.10) and (3.13) implies

\[ G(x + tv_1 + \frac{t^2}{2!}v_2 + \cdots + \frac{t^n}{n!}(v_n + R(w))) = G(x) + t\dot{G}(x)(v_1) + \frac{t^2}{2!}(\ddot{G}(x)(v_1)^2 + \dot{G}(x)(v_2)) + \cdots \]

\[ + \frac{t^n}{n!} \left( \sum_{|\alpha_k| = n} \frac{n!}{k!\alpha_k!} G^{(k)}(x)(v_{\alpha_k}) + \dot{G}(x)(R(w)) \right) + z + w - F(w). \]  

(3.14)

Clearly (3.12) and (3.13) implies \( \|F(w)\| \leq \rho \) for \( \forall w \in B(\rho) \), that means \( F : B(\rho) \to B(\rho) \). Since \( F \) is continuous on \( B(\rho) \), by Brower fixed point theorem there is \( w \in B(\rho) \) such that \( F(w) = w \). With this \( w \) and \( r = R(w) \), (3.10), (3.11) and (3.14) conclude the proof of the lemma.

We now introduce the generalized condition of (3.6):

\[ \lim_{t \to 0} \frac{1}{t^n} \left[ d[G(x) + t\dot{G}(x)(v_1) + \frac{t^2}{2!}(\ddot{G}(x)(v_1)^2 + \dot{G}(x)(v_2)) + \cdots \right. \]

\[ \left. + \frac{t^n}{n!} \sum_{|\alpha_k| = n} \frac{n!}{k!\alpha_k!} G^{(k)}(x)(v_{\alpha_k}); E \right] = 0, \]

(3.15)

**Theorem 3.7** If \( D = G^{-1}(E) \) and \( G \) is \( n \)-times differentiable at \( x \in A \), then (3.3) implies (3.15). If in addition we assume that \( G \) is continuous on \( A \), \( \dot{G}(x) : X \to Y \) is onto and \( \dim(Y) < \infty \), then (3.15) is equivalent to (3.3).

**Proof.** Let \( D = G^{-1}(E) \) and \( G : A \to Y \) be \( n \)-times differentiable at \( x \in A \) and assume that (3.3) holds. Then there is \( r(t) \in X \) such that \( r(t) \to 0 \) as \( t \downarrow 0 \)
and \( x + tv_1 + \frac{t^2}{2!}v_2 + \cdots + \frac{t^n}{n!}(v_n + r(t)) \in D \) which gives \( G(x + tv_1 + \frac{t^2}{2!}v_2 + \cdots + \frac{t^n}{n!}(v_n + r(t))) \in E \), for \( \forall t > 0 \). Therefore

\[
\frac{1}{t^n}d[G(x) + t\dot{G}(x)(v_1) + \frac{t^2}{2!}(\ddot{G}(x)(v_1)^2 + \dot{G}(x)(v_2)) + \cdots + \frac{t^n}{n!}\sum_{|\alpha_k|=n} \frac{n!}{k!\alpha_k!}G^{(k)}(x)(v_{\alpha_k}); E] \\
\leq \frac{1}{t^n}\|G(x) + t\dot{G}(x)(v_1) + \frac{t^2}{2!}(\ddot{G}(x)(v_1)^2 + \dot{G}(x)(v_2)) + \cdots + \frac{t^n}{n!}\sum_{|\alpha_k|=n} \frac{n!}{k!\alpha_k!}G^{(k)}(x)(v_{\alpha_k}) \\\n- G(x + tv_1 + \frac{t^2}{2!}v_2 + \cdots + \frac{t^n}{n!}(v_n + r(t)))\|
\]

which implies (3.15) according to (3.9). Assume now in addition that \( \dot{G}(x) : X \to Y \) is onto, \( \text{dim}(Y) < \infty \) and (3.15) holds. Then we get that for each \( t > 0 \), there is \( r(t) \in X \) such that \( r(t) \to 0 \) as \( t \downarrow 0 \) and \( x + tv_1 + \frac{t^2}{2!}v_2 + \cdots + \frac{t^n}{n!}(v_n + r(t)) \in D \).

So, there is \( r_G(t) \in Y \) such that \( r_G(t) \to 0 \) as \( t \downarrow 0 \) and

\[
G(x) + t\dot{G}(x)(v_1) + \frac{t^2}{2!}(\ddot{G}(x)(v_1)^2 + \dot{G}(x)(v_2)) + \cdots + \frac{t^n}{n!}\sum_{|\alpha_k|=n} \frac{n!}{k!\alpha_k!}G^{(k)}(x)(v_{\alpha_k}) + r_G(t) \in E
\]

(3.16)

for \( \forall t > 0 \).

For \( \varepsilon > 0 \), let \( \delta > 0 \) be a number with the property given by Lemma 3.6. Since \( r_G(t) \to 0 \) as \( t \downarrow 0 \), there is \( \delta_1 = \delta_1(\varepsilon) \in (0, \delta) \) such that \( \|r_G(t)\| < \delta \), for \( \forall t \in (0, \delta_1) \). Since \( 0 < \delta_1 < \delta \), according to Lemma 3.6 we get that for \( \forall t \in (0, \delta_1) \) there
is $r \in X$ with the properties

$$\|r\| < \varepsilon, \quad x + tv_1 + \frac{t^2}{2!}v_2 + \cdots + \frac{t^n}{n!}(v_n + r) \in A,$$

$$G(x + tv_1 + \frac{t^2}{2!}v_2 + \cdots + \frac{t^n}{n!}(v_n + r))$$

$$= G(x) + t\dot{G}(x)(v_1) + \frac{t^2}{2!}(\ddot{G}(x)(v_1)^2 + \ddot{G}(x)(v_2)) + \cdots$$

$$+ \frac{t^n}{n!} \left( \sum_{|\alpha_k|=n} \frac{n!}{k!\alpha_k!} G^{(k)}(x)(v_{\alpha_k}) + rG(t) \right)$$

which implies (using (3.16)) $x + tv_1 + \frac{t^2}{2!}v_2 + \cdots + \frac{t^n}{n!}(v_n + r) \in G^{-1}(E) = D$. This implies (3.3) (i.e. $\lim_{t\downarrow 0} \frac{1}{t} d[x + tv_1 + \frac{t^2}{2!}v_2 + \cdots + \frac{t^n}{n!}v_n; D] = 0$).

**Corollary 3.8** Assume that in addition to the above hypothesis $D = D_G$, i.e. $D_G = \{x \in X | G(x) = 0\}$. Then the following conditions are equivalent:

(i). $\lim_{t_i \downarrow 0} \frac{1}{t_i} d[x + tv_1 + \frac{t^2}{2!}v_2 + \cdots + \frac{t^n}{n!}v_n; D_G] = 0, \quad v_i \in X, \quad \forall i = 1, 2, \ldots, n$,

(ii). $G(x) = 0, \quad \sum_{|\alpha_k|=l} \frac{n!}{k!\alpha_k!} G^{(k)}(x)(v_{\alpha_k}) = 0, \quad \forall l = 1, 2, \ldots, n$.

**Proof.** Set $E = \{0\}$. Then $D_G = G^{-1}(0) = \{x \in X | G(x) = 0\}$. In view of Theorem 3.7, the condition (i) is equivalent to

$$\lim_{t_i \downarrow 0} \frac{1}{t_i^n} d[G(x) + t\dot{G}(x)(v_1) + \frac{t^2}{2!}(\ddot{G}(x)(v_1)^2 + \ddot{G}(x)(v_2)) + \cdots$$

$$+ \frac{t^n}{n!} \sum_{|\alpha_k|=n} \frac{n!}{k!\alpha_k!} G^{(k)}(x)(v_{\alpha_k}); \{0\}] = 0,$$

(3.17)

Obviously (3.17) holds iff the condition (ii) holds.

### 3.3 Necessary Conditions for Optimization

With the previous results we present the main result of this paper as follows.
Theorem 3.9 (Minimum Theorem) Let $D$ be a nonempty subset of a Banach space $X$ and let
\[ F(x) = \text{loc min } F(z), \text{ subject to } z \in D, \] (3.18)
where $F : X \rightarrow \mathbb{R}$ is a sufficiently smooth functional.

Then there exists an even integer $n$ such that for all $(v_1, v_2, ..., v_n) \in \prod_{i=1}^{n} T_x D$ satisfying
\[ x + tv_1 + \frac{t^2}{2!} v_2 + \frac{t^3}{3!} v_3 + \cdots + \frac{t^{n-1}}{(n-1)!} v_{n-1} + \frac{t^n}{n!} (v_n + r(t)) \in D, \]
for some $r(t) \to 0$ as $t \to 0$, and
\[ \sum_{|\alpha_k|=l} \frac{n!}{k! \alpha_k!} F^{(k)}(x)(v_{\alpha_k}) = 0, \forall l = 1, 2, ..., n-1, \]
\[ \sum_{|\alpha_k|=n} \frac{n!}{k! \alpha_k!} F^{(k)}(x)(v_{\alpha_k}) \geq 0. \]

Remark 3.1 When $n = 4$, the necessary conditions of the above theorem for minimum can be expressed as: For all $(v_1, v_2, v_3, v_4) \in \prod_{i=1}^{4} T_x D$ satisfying
\[ x + tv_1 + \frac{t^2}{2!} v_2 + \frac{t^3}{3!} v_3 + \frac{t^4}{4!} (v_4 + r(t)) \in D, \]
for some $r(t) \to 0$ as $t \to 0$, and
\[ \begin{cases} \dot{F}(x)(v_1) = 0, \\ \ddot{F}(x)(v_1)(v_1) + \dot{F}(x)(v_2) = 0, \\ \dddot{F}(x)(v_1)(v_1)(v_1) + 3 \dddot{F}(x)(v_1)(v_2) + \dot{F}(x)(v_3) = 0, \\ F^{(4)}(x)(v_1)(v_1)(v_1)(v_1) + 6 \dddot{F}(x)(v_1)(v_1)(v_2) + 3 \dddot{F}(x)(v_2)(v_2) + 4 \dddot{F}(x)(v_1)(v_3) + \dot{F}(x)(v_4) \geq 0. \end{cases} \]
Proof. Since $F$ is sufficiently smooth, we get, from the Taylor’s Formula for functions of several variables,

$$F(x + z) - F(x) = \sum_{l=1}^{n} \frac{F^{(l)}(x)}{l!} z^l + \|z\|^n \alpha(z), \tag{3.19}$$

with some $\alpha(z) \to 0$ as $z \to 0$. Replacing $z = tv_1 + \frac{t^2}{2!} v_2 + \frac{t^3}{3!} v_3 + \cdots + \frac{t^{n-1}}{(n-1)!} v_{n-1} + \frac{t^n}{n!} (v_n + r(t))$ into (3.19), and taking $F(x) = \text{loc min} F(z)$, subject to $z \in D$, we get

$$0 \leq F(x + tv_1 + \frac{t^2}{2!} v_2 + \frac{t^3}{3!} v_3 + \cdots + \frac{t^{n-1}}{(n-1)!} v_{n-1} + \frac{t^n}{n!} (v_n + r(t))) - F(x)$$

$$= \sum_{l=1}^{n} \frac{F^{(l)}(x)}{l!} (tv_1 + \frac{t^2}{2!} v_2 + \frac{t^3}{3!} v_3 + \cdots + \frac{t^{n-1}}{(n-1)!} v_{n-1} + \frac{t^n}{n!} (v_n + r(t)))^l + t^n \gamma(t),$$

$(\gamma(t) \to 0$ as $t \to 0)$

$$= \sum_{l=1}^{n} \frac{n!}{l!} (\sum_{|\alpha_k|=l} \frac{n!}{k! \alpha_k!} F^{(k)}(x)(v_{\alpha_k})) + t^n \beta(t), \quad (\beta(t) \to 0$ as $t \to 0)$

$$= \sum_{|\alpha_k|=n} \frac{n!}{k! \alpha_k!} F^{(k)}(x)(v_{\alpha_k}) + t^n \beta(t), \quad (\beta(t) \to 0$ as $t \to 0).$$

Dividing by $\frac{n!}{n!}$ and then letting $t \to 0$, we obtain

$$\sum_{|\alpha_k|=n} \frac{n!}{k! \alpha_k!} F^{(k)}(x)(v_{\alpha_k}) \geq 0.$$

This completes the proof.

Corollary 3.10 Suppose the hypotheses of theorem hold and $D = D_G(= \{x \in X|G(x) = 0\}$, i.e.

$$F(x) = \text{loc min} F(z), \text{ subject to } G(z) = 0,$$

Then there exists an even integer $n$ such that for all $(v_1, v_2, ..., v_n) \in \prod_{i=1}^{n} T_x D$ satisfying

$$G(x + tv_1 + \frac{t^2}{2!} v_2 + \frac{t^3}{3!} v_3 + \cdots + \frac{t^{n-1}}{(n-1)!} v_{n-1} + \frac{t^n}{n!} (v_n + r(t))) = 0,$$
for some \( r(t) \to 0 \) as \( t \to 0 \), and
\[
\sum_{|\alpha_k| = l} \frac{n!}{k!\alpha_k!} F^{(k)}(x)(v_{\alpha_k}) = 0, \quad \forall l = 1, 2, ..., n - 1,
\]
\[
\sum_{|\alpha_k| = n} \frac{n!}{k!\alpha_k!} \sum_{\alpha_k} \left| \alpha_k \right| F^{(k)}(x)(v_{\alpha_k}) \geq 0.
\]

**Proof.** This is the direct consequence of the above theorem and the definition of \( D_G \).

**Corollary 3.11** Suppose in addition that \( \dot{G}(x) \) of \( G \) at \( x \) is onto from \( X \) into \( Y(\dim(Y) < \infty) \), and \( G \) is sufficiently smooth. Then there exists an even integer \( n \) such that for all \((v_1, v_2, ..., v_n) \in \prod_{i=1}^{n} T^x_i D\) satisfying
\[
\sum_{|\alpha_k| = l} \frac{n!}{k!\alpha_k!} \sum_{\alpha_k} \left| \alpha_k \right| F^{(k)}(x)(v_{\alpha_k}) = 0, \quad \forall l = 1, 2, ..., n, \quad \forall \sum_{|\alpha_k| = n} \frac{n!}{k!\alpha_k!} \sum_{\alpha_k} \left| \alpha_k \right| F^{(k)}(x)(v_{\alpha_k}) = 0,
\]
\[
\forall l = 1, 2, ..., n - 1,
\]
\[
\sum_{|\alpha_k| = n} \frac{n!}{k!\alpha_k!} \sum_{\alpha_k} \left| \alpha_k \right| F^{(k)}(x)(v_{\alpha_k}) \geq 0.
\]

**Proof.** The above corollary and Corollary 3.8 imply the result.

**Remark 3.2** If ”\( \geq \)” is replaced by ”\( \leq \)”, we can get the corresponding maximum theorem and corollaries.

**Example 3.1** For \( k = 1, 2, ..., \) consider the behavior of \( f \) near \((0,0)\), given by:
\[
f(x, y) = y^2 - (p + q)yx^k + pqx^{2k} = (y - px^k)(y - qx^k), \quad (0 < p < q).
\]

Notice that \((0,0)\) is a local minimum of \( f(x, y) \) along every curve which belongs to the class of \( \{ y = ax^n | a \text{ is a constant}, \forall n \in \mathbb{N} - \{k\} \} \) that passes through the origin, but \((0,0)\) is a local maximum of \( f(x, y) \) along every curve between \( y = px^k \) and \( y = qx^k \) that passes through the origin. So \((0,0)\) is neither a local minimum point nor a local maximum point. But so far there has been no
general criterion to determinate the behavior of $f(x, y)$ at the origin $(0, 0)$ except for ad hoc methods. By applying the above theorems of $2k$-th order necessary conditions for Minimum and Maximum, we can conclude that $(0, 0)$ is a saddle point of $f(x, y)$, which means that the behavior of $f(x, y)$ can be determinated completely at the origin $(0, 0)$. In fact, we here need to apply the contrapositive statement of those theorems.

When the constraint function $g(x, y) = y - rx^k(r < p \mathrm{ or } r > q)$ is given, these $f$ and $g$ satisfy the necessary conditions of the Corollary 3.11 following the minimum theorem. In other hand, when $g(x, y) = y - rx^k(p < r < q)$ is given, these $f$ and $g$ satisfy the necessary conditions of the corresponding maximum theorem.

This kind of functions has the following background story:

(i) $k = 1$: $f(x, y) = y^2 - (p + q)xy + pqx^2$,

$f_x(0, 0) = f_y(0, 0) = 0$,

$f_{xx}(0, 0) > 0$, $f_{xx}f_{yy} - f_{xy}^2 = 4pq - (p + q)^2 = -(p - q)^2 < 0$.

So, by the second order derivative test for the functions with two variables, we get that $(0, 0)$ is a saddle point of $f(x, y)$.

(ii) $k = 2$: $f(x, y) = (y - px^2)(y - qx^2)$,

This is the Peano’s classic example(see Hancock [53], Pierpont [105]). Since $\det Hf(0, 0) = 0$, the second order derivative test can’t determinate the behavior of $f(x, y)$ at $(0, 0)$. Notice that $(0, 0)$ is a local minimum of $f(x, y)$ along every straight line that passes through the origin, and $(0, 0)$ is a local maximum of $f(x, y)$ along every parabola between $y = px^2$ and $y = qx^2$ that passes through the origin. We can conclude that $(0, 0)$ is a saddle point of $f(x, y)$. Our theorems of this section also gives us the same result.
Example 3.2  For $k > 1$, consider the behavior of $f_1$ and $f_2$ near $(0, 0)$, given by:

$$f_1(x, y) = m(x - y)^2 + ax^{2k} + by^{2k},$$
$$f_2(x, y) = mx^2 + ax^{2k} + by^{2k},$$

where $m, a$ and $b$ are constants. Since $\det H f_1(0, 0) = 0$ and $\det H f_2(0, 0) = 0$, we can’t apply the second order derivative test to determinate the behavior of $f_1(x, y)$ and $f_2(x, y)$ at $(0, 0).$ (cf. Spring [111]) But by applying our theorems with $2k$-th order necessary conditions for Minimum and Maximum, we obtain the following result:

(i) $f_1(x, y) = m(x - y)^2 + ax^{2k} + by^{2k}$ is saddle at $(0, 0)$, if $(a + b)m < 0$.
(ii) $f_2(x, y) = mx^2 + ax^{2k} + by^{2k}$ is saddle at $(0, 0)$, if $bm < 0$.

Remark 3.3 Consider the behavior of the following function $f$ near $(0, 0)$:

$$f(x, y) = (y - pe^{-\frac{x}{x^2}})(y - qe^{-\frac{x}{x^2}}), \quad (0 < p < q).$$

Since the absolute value of $e^{-\frac{1}{x^2}}$ is far smaller than any arbitrary power $ax^k$ in the sufficiently small neighborhood of $x = 0$, $(0, 0)$ is a local minimum of $f(x, y)$ along every polynomial functions passing through the origin. (see Hancock [53])

But $(0, 0)$ is a local maximum of $f(x, y)$ along every curve $(y = re^{-\frac{1}{x^2}}, p < r < q)$ between $y = pe^{-\frac{1}{x^2}}$ and $y = qe^{-\frac{1}{x^2}}$ that passes through the origin. So $(0, 0)$ is a saddle point of $f(x, y)$. But here we can’t apply our theorem to this function as we can’t determinate the behavior of the Cauchy function $y = e^{-\frac{1}{x^2}}$ at the origin by the higher order derivative test for the functions with one variable.
Theorem 3.12 Let $D$ be a nonempty convex subset of $\mathbb{R}^m$, $x \in D$, and $F : D \to \mathbb{R}$ of class $C^1$ such that $\dot{F}(x)(v) > 0$, for all nonzero $v \in T_xD$. Then $F(x)$ is a strict local minimum of $F$ on $D$. That is, there are $\alpha > 0$ and $\delta > 0$ such that $F(z) \geq F(x) + \alpha \|z - x\|$, for $\forall z \in D, \|z - x\| < \delta$.

Proof. Suppose by contradiction that the conclusion of this theorem is not true, so that there is a sequence $\{z_n\} \subset D$ with $z_n \to x$ such that $F(z_n) < F(x) + \frac{1}{n}\|z_n - x\|$, for all $n$. Let us write $z_n = x + \delta_n v_n$, where $\delta_n = \|z_n - x\|$, $v_n = \frac{z_n - x}{\|z_n - x\|}$. Since $D$ is convex, $z_n - x = \delta_n v_n \in T_xD$. So $v_n \in T_xD$. The sequence $\{v_n\}$ is bounded, so it must have a subsequence converging to some $v$ with $\|v\| = 1$. Without loss of generality, we assume that the whole sequence $\{v_n\}$ converges to $v$. From the Taylor formula, we get that $\frac{1}{n}\|z_n - x\| > F(z_n) - F(x) = \delta_n \dot{F}(x)(v_n) + \delta_n r_n$ with some $r_n \to 0$ as $n \to \infty$. Dividing by $\delta_n$ and then letting $n \to \infty$ we get $0 \geq \dot{F}(x)(v)$ with contradicts the hypothesis. This completes the proof.

Example 3.3 Let $D$ be the triangle of vertices $(0,0)$, $(2,0)$, $(2,1)$ in $\mathbb{R}^2$ and $F(x_1, x_2) = x_1^3 + x_1^2 + x_1 - x_2^3 - x_2 + x_2$ with $x = (x_1, x_2)$. Then $\dot{F}(x) = (5x_1^4 + 2x_1 + 1, 5x_2^4 - 2x_2 + 1)$ and so $\dot{F}(0,0) = (1,1)$. It is also easy to check that at $x = (0,0)$,

$$T_xD = \{v = (v_1, v_2) | v_1 \geq 2v_2 \geq 0, \text{ for } \forall v_1 \neq 0\} \cup \{(0,0)\},$$
so \( v \neq 0 \) implies \( v_1 \neq 0 \). Therefore \( \dot{F}(0,0)(v) = v_1 + v_2 > 0 \), for all \( v \in T_xD \), with \( v \neq 0 \). According to the above theorem, \( F(x_1,x_2) \) has a strict local minimum at \( x = (0,0) \).

**Remark 3.4** The Hessian matrix \( \tilde{F}(0,0) = 
\begin{bmatrix}
2 & 0 \\
0 & -2
\end{bmatrix}
\)

So, \( \tilde{F}(0,0)(v)(v) = 2(v_1^2 - v_2^2) \) is not positive definite on \( \mathbb{R}^2 \). Since \( \det \tilde{F}(0,0) = -4 < 0 \), the classical criterion can only say that \( x = (0,0) \) is a saddle point of \( F \) on \( \mathbb{R}^2 \). But the above theorem allows us to see much more, that is, the functional \( F \) can have a strict local minimum on the restricted domain.

The example below shows that Theorem 3.12 may not be true in infinite dimensional spaces:

**Example 3.4** (Counterexample in case of \( \dim X = \infty \), cf. [81]) Let \( X = l_2 = \{x = \{x_n\}||x||^2 = \sum x_n^2 < \infty \} \) and \( D = \{x = \{x_n\}|x_n \geq 0, \forall n\} \). Then it is easy to verify that \( T_0D = D \). Define \( f : X \to \mathbb{R} \) by \( f(x) = < a - x, x > \) for \( a = (1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots) \). Then \( f'(0)(v) = < a, v > > 0 \), for all nonzero \( v \in T_0D = D \). But \( x = 0 \) is not a local minimal point. Indeed, let us consider a sequence \( \{x^k\} \subset D \) with \( x^k = \frac{2}{k}e_k \). Then \( x^k \in D \) for \( \forall k \), and \( x^k \to 0 \) as \( k \to \infty \).

But \( f(x^k) = \frac{2}{k^2} - \frac{4}{k^2} = -\frac{2}{k^2} < 0 = f(0) \), for all \( k \).

**Theorem 3.13** Let \( D \) be a nonempty convex subset of \( \mathbb{R}^m \), \( x \in D \), and \( F : D \to \mathbb{R} \) a sufficiently smooth functional such that:

(i) \( F^{(k)}(x)(v)^k = 0 \), for \( \forall v \in T_xD \) and \( \forall k = 1, 2, \cdots, n - 1 \).

(ii) \( F^{(n)}(x)(v)^n > 0 \), for all nonzero \( v \in T_xD \).
Then $F(x)$ is a strict local minimum of $F$ on $D$. That is, there are $\alpha > 0$ and $\delta > 0$ such that $F(z) \geq F(x) + \alpha \|z - x\|^n$, for $\forall z \in D, \|z - x\| < \delta$.

**Proof.** Suppose by contradiction that the conclusion of this theorem is not true, so that there is a sequence $\{z_p\} \subset D$ with $z_p \rightarrow x$ such that $F(z_p) < F(x) + \frac{1}{p}\|z_p - x\|^n$, for all $p$. Let us write $z_p = x + \delta_p v_p$, where $\delta_p = \|z_p - x\|$, $v_p = \frac{z_p - x}{\|z_p - x\|}$. Since $D$ is convex, $z_p - x = \delta_p v_p \in T_x D$. So $v_p \in T_x D$. The sequence $\{v_p\}$ is bounded, so it must have a subsequence converging to some $v$ with $\|v\| = 1$. Without loss of generality, we assume that the whole sequence $\{v_p\}$ converges to $v$. On the basis of Taylor formula and the condition (i), we get that $\frac{1}{p}\|z_p - x\| > F(z_p) - F(x) = \frac{1}{n!} \delta_p^n F^{(n)}(x)(v_p)^n + \delta_p^n r_p$ with some $r_p \rightarrow 0$ as $p \rightarrow \infty$. Dividing by $\delta_p$ and then letting $p \rightarrow \infty$ we get $0 \geq F^{(n)}(x)(v)^n$ with contradicts the hypothesis. This completes the proof.

The example below shows that Theorem 3.13 may not be true in infinite dimensional spaces:

**Example 3.5** (Counterexample in case of dim $X = \infty$, cf. [22], [10] p.157)

Let $X = l_2$ be the same as the above example and $D = X = l_2$. Define $f : X \rightarrow \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} \left(\frac{x_n^2}{n^2} - x_n^4\right)$. Then $f'(0) = 0$, i.e. $x = 0$ is a critical point. $f''(0)(v)(v) = 2 \sum_{n=1}^{\infty} \frac{v_n^2}{n^2} > 0, \forall v \neq 0$. But $x = 0$ is not a local minimal point.

Consider a sequence $\{x^k\} \subset D$ with $x^k = \frac{2}{k} e_k$. Then $x^k \in D$ for $\forall k$, and $x^k \rightarrow 0$ as $k \rightarrow \infty$. But $f(x^k) = \frac{4}{k^2} - \frac{16}{k^4} = -\frac{12}{k^2} < 0 = f(0)$, for all $k$. 
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