Optimization and Flow Invariance via High Order Tangent Cones

A dissertation presented to
the faculty of
the College of Arts and Sciences

In partial fulfillment
of the requirements for the degree
Doctor of Philosophy

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June 2005
This dissertation entitled

**OPTIMIZATION AND FLOW INVARIANCE VIA HIGH ORDER TANGENT CONES**

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Optimization and Flow Invariance via High Order Tangent Cones (74pp.)
Director of dissertation: Nicolai H. Pavel

The goals of this dissertation are: 1) to present some results on the flow-invariance of a closed set $S$ of a Banach space with respect to a differential equation, and to discuss optimization problems on $S$, as well; 2) to point out their unifying effect in the theory of differential equations and optimization.

For the following optimization problem, one establishes necessary conditions of extremum in terms of the high order tangential directions to the constraint set at the extremum point:

$$F(x_0) = \text{Local Minimum } F(x), \text{ subject to } x \in S,$$

where $X$ is a normed space, $F : X \rightarrow \mathbb{R}$ is a function of class $C^p$ in a neighborhood of $x_0 \in S \subseteq X, S \neq \emptyset, p \geq 1$.

It is analyzed in detail the case when $S$ is the kernel $D_G$ of a function $G : X \rightarrow \mathbb{R}^m, m \geq 1$. To this aim, one describes the high order tangent cones to the set $D_G$ at $x \in D_G$, and then derives some sufficient conditions for the optimality of $F$ on $D_G$.

The characterizations of the high order tangent cones are also used to obtain some necessary and sufficient conditions for the flow-invariance of a subset $D_G = \{x \in X; G(x) = 0\}$ of a Banach space $X$ with respect to the differential equation $u^{(n)}(t) = F(u(t)), t \geq 0$, where $G : U \rightarrow \mathbb{R}^m, m \geq 1$, is a $n$-times Fréchet differentiable mapping on an open subset $U$ of $X, n \geq 3$, and $F : U \rightarrow X$ is locally Lipschitz.

The examples discussed illustrate some applications of the results presented.

Approved: Nicolai H. Pavel
Professor of Mathematics
To my parents
I would like to thank Dr. Pavel for his support and guidance.

Also, I would like to express my gratitude to all my professors here at Ohio University.
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List of Symbols

\(B(0, r)\) - the ball of radius \(r\) centered at the origin
\(D_G = \{x \in X; G(x) = 0\}\) - the kernel of \(G\)
\(<, >\) - the inner product in a Hilbert space
\(F : X \to \mathbb{R}\) - a function of class \(C^p\) - pp. 35
\(G : X \to \mathbb{R}^m\) - a Fréchet differentiable function - pp. 14
\(G'(x)\) - the Fréchet derivative of \(G\) at \(x\)
\(F^n(x), G^n(x)\) - the \(n\)-th Fréchet derivatives of \(F, G\) at \(x\) - pp. 14
\(\gamma : (0, \infty) \to X\)
i.e., - that is
\(L\) - a linear continuous operator
\(|| \cdot ||\) - the norm
\(\mathbb{R}\) - the set of real numbers
\(\nabla\) - the gradient
\(S\) - a subset of \(X\)
\(T^n_xS\) - the \(n\)-th order tangent cone to \(S\) at \(x \in S\) - pp. 11
\(v_n\) - a \(n\)-th order tangent vector to \(S\) at \(x \in S\) - pp. 11
\(X\) - a normed space
Chapter 1

Introduction

1.1 Preliminaries

The aim of this dissertation is to provide high order necessary and sufficient conditions of optimality for a class of abstract extremum problems and for the flow-invariance of a set given in operator form with respect to a high order autonomous differential equation, using our characterizations of the high order tangent cones.

The plan of the thesis is as follows.

In Chapter 1 we present some preliminary classical notions and results needed for the rest of the thesis.

The main tools in stating and proving our results concerning both the optimization problem and the flow-invariance problem considered are Bouligand’s tangent vector (see (8)) and the high order tangent vectors defined by N.H. Pavel (see (32)). Therefore, Chapter 2 comprises characterizations of the \( n \)-th order, \( n \geq 2 \), tangent cones to the set \( D_G = \{ x \in X, G(x) = 0 \} \) at \( x \in D_G \) in the cases when \( G'(x) \) is onto and, respectively, when \( G'(x) = 0 \) and \( G \) has a regularity property introduced by Tret’yakov (see (14) and (33)).
In Chapter 3 we are concerned with the optimization problem

\[ F(x_0) = \text{Local Minimum } F(x), \text{subject to } x \in S, \]

where \( X \) is a normed space and \( F : X \to \mathbb{R} \) is a function of class \( C^p \) in a neighborhood of \( x_0 \in S \subseteq X, \ p \geq 3 \). In Section 3.1 we extend a result of Pavel et al. (see (32)) by establishing necessary conditions satisfied by a local minimum point \( x_0 \) of \( F \) on \( S \), involving the \( n \)-th order tangential directions to \( S \) at \( x_0 \in S, \ n \geq 1 \), defined as in (8) and (32). In the case of an extremum problem with an equality constraint, our high order necessary conditions are useful when both Pavel’s second order necessary conditions and the second derivative test for the lagrangian function do not provide any information.

In Section 3.2 we formulate some sufficient conditions for the optimality of \( F \) on \( D_G \) when \( X \) is finite dimensional (see (14)) using some techniques due to Bertsekas (see (6)).

Section 3.3 consists of various examples which illustrate how our results can be useful for analyzing classes of problems that could not be investigated before. We present applications of our necessary conditions to ODE’s by showing examples of functionals that must take positive values on a set of constraints given by a differential equation, utilizing some examples of tangent sets studied by Girsanov (see (18)). We illustrate how our sufficient conditions can be applied to analyzing if the critical points of a functional on a set of constraints \( D_G \) given in operator form as \( D_G = \{ x \in X, G(x) = 0 \} \), are extremum points, and how our necessary conditions can help us decide the nature of the extremum points.

In the last chapter we describe the sets \( S = D_G = \{ x \in X; G(x) = 0 \} \), that are flow-invariant with respect to the \( n \)-th order autonomous differential equation

\[ u^{(n)}(t) = F(u(t)), t \geq 0, \]

where \( G : U \to \mathbb{R}^m, \ m \geq 1 \) is a \( n \)-times Fréchet differentiable mapping on an open subset \( U \) of a Banach space \( X, \ n \geq 3 \), and \( F : U \to X \) is a locally Lipschitz mapping.

The invariant sets for the first order differential equations were studied by H.
Brézis (9), M.G. Crandall (13), R.H. Jr. Martin (23), N.H. Pavel and F. Iacob (28), and many other authors.

In (30), N.H. Pavel and C. Ursescu treated the problem of flow-invariance of a set with respect to the second order differential equation $u''(t) = F(u(t)), t \geq 0$, using the theory of `tangent sets’.

The purpose of this chapter is to generalize their result Theorem 2.6, (30).

The proofs of our results are based on our description of the high order tangent cones (Corollary 2.1) and some representations of a flow-invariant set given in (9) and (30).

As an application, we derive explicit conditions for the flow-invariance of the sphere $S(r) = \{ x \in H, \| x \| = r \}, r > 0$, of a Hilbert space $H$, with respect to the equation $u'''(t) = F(u(t)), t \geq 0$.

Theorem 4.3 has the following geometrical interpretation: suppose a mass particle is launched from a point $x$ in a set $D_G$, at the initial moment it has a certain velocity $v_1$ and a certain acceleration $v_2$, and its trajectory satisfies the equation $u'''(t) = F(u(t)), t \geq 0$. If our sufficient condition for flow-invariance is verified, then the trajectory of the particle remains in the set $D_G$.

Most of the results in chapters 2 and 3 are included in (14), while the results in chapter 4 are contained in (15).
1.2 Tangency in Normed Spaces

The following notion of \( n \)-th order tangent vector to an arbitrary set will play a basic role in the next chapters.

**Definition 1.2.1.** Let \( S \) be a nonempty subset of \( X \) and let \( x \in S \) be a given point.

i) (Bouligand (8)) An element \( v \in X \) is called a tangent vector to \( S \) at \( x \) if
\[
\lim_{t \downarrow 0} \frac{1}{t} d(x + tv; S) = 0.
\]

ii) (Pavel (32)) An element \( v_n \in X \) is called a \( n \)-th order tangent vector to \( S \) at \( x \in S, n \geq 2 \), if there are some \( v_i \in T^i_x S, i = 1, 2, ..., n-1 \) such that
\[
\lim_{t \downarrow 0} \frac{1}{t^n} d(x + t v_1 + \frac{t^2}{2!} v_2 + \frac{t^3}{3!} v_3 + ... + \frac{t^n}{n!} v_n; S) = 0,
\]
where \( d(x; D) \) stands for the distance from \( x \) to \( D \), i.e., \( d(x; D) = \inf\{||x - y||; y \in D\} \).

The vectors \( v_i, i = 1, 2, ...n - 1 \) are said to be the correspondent vectors of \( v_n \) (or associated with \( v_n \)).

The set of all \( n \)-th order tangent vectors to \( S \) at \( x \in S \) is denoted by \( T^n_x S \).

It is obvious that if \( x \) belongs to the interior of \( S \), then
\[
X = T_x S = T^n_x S.
\]

The first aim is to make precise the structure of \( T^n_x S \).

**Proposition 1.2.1**

i) \( T_x S \) is a closed cone in \( X \).

ii) \( T^n_x S, n \geq 2 \), is a cone in \( X \).

**Proof.** Given \( v_1 \in T_x S \) and \( \lambda > 0 \) from (1.2.1) one sees that
\[
\lim_{t \downarrow 0} \frac{1}{t} d(x + t \lambda v_1; S) = \lambda \lim_{\tau \downarrow 0} \frac{1}{\tau} d(x + \tau v; S) = 0.
\]
so \( \lambda v \in T_x S \).

Take a sequence \( v_{1n} \subset T_x S, n \geq 1 \), with \( v_{1n} \to v_1 \) in \( X \). Then, for every \( \epsilon > 0 \),
one can choose some $v_{1n}$ and $\delta > 0$ such that $||v_{1n} - v_1|| < \epsilon/2$, and

$$\frac{1}{t} d(x + t\lambda v_{1n}; S) < \frac{\epsilon}{2}, \forall t \in (0, \delta).$$

It follows that

$$\frac{1}{t} d(x + t\lambda v_1; S) \leq \frac{1}{t} d(x + t\lambda v_{1n}; S) + ||v_{1n} - v_1|| < \epsilon, \forall t \in (0, \delta).$$

This expresses that $T_xS$ is closed in $X$.

ii) Let $v_n \in T^n_xS$ and $\lambda > 0$. We infer from (1.2.1) that

$$\lim_{t \downarrow 0} \frac{1}{t^n} d(x + t\sqrt[n]{\lambda}v_1 + \frac{t^2(\sqrt[n]{\lambda})^2}{2!}v_2 + \frac{t^3(\sqrt[n]{\lambda})^3}{3!}v_3 + \ldots + \frac{t^n}{n!}\lambda v_n; S) =$$

$$= \lambda \lim_{t \downarrow 0} \frac{1}{(t\sqrt[n]{\lambda})^n} d(x + t\sqrt[n]{\lambda}v_1 + \frac{(t\sqrt[n]{\lambda})^2}{2!}v_2 + \frac{(t\sqrt[n]{\lambda})^3}{3!}v_3 + \ldots + \frac{(t\sqrt[n]{\lambda})^n}{n!}v_n; S) = 0,$$

which shows that $\lambda v_n \in T^n_xS$.

The $n$-th order tangent set $T^n_xS$ is nonempty, containing at least $v_n = 0$ (where one can take some arbitrary $v_i \in T^i_xS$, $i = 1, 2, \ldots, n - 1$).

It is known the following characterization of a $n$-th order tangent vector.

**Proposition 1.2.2.** The fact that $v_n$ belongs to $T^n_xS$ with the corespondent vectors $v_i \in X$, $i = 1, 2, \ldots, n - 1$ as in (1) is equivalent to the existence of a function $\gamma_n : (0, \infty) \rightarrow X$ with $\gamma_n(t) \rightarrow 0$ as $t \downarrow 0$ and

$$x + tv_1 + \frac{t^2}{2}v_2 + \ldots + \frac{t^n}{n!}(v_n + \gamma_n(t)) \in S, \forall t > 0. \quad (1.2.2)$$

**Proof.** It suffices to note that (1.2.1) holds if there exists a map $u : (0, \infty) \rightarrow S$ such that

$$\frac{1}{t^n} \left\| x + tv_1 + \frac{t^2}{2}v_2 + \ldots + \frac{t^n}{n!}v_n - u(t) \right\| \rightarrow 0 \text{ as } t \downarrow 0.$$
Indeed, there is \( u : (0, \infty) \to S \) with the property that
\[
d(x + tv_1 + \frac{t^2}{2!}v_2 + \frac{t^3}{3!}v_3 + \ldots + \frac{t^n}{n!}v_n; S) \leq \left\| x + tv_1 + \frac{t^2}{2}v_2 + \ldots + \frac{t^n}{n!}v_n - u(t) \right\| < \\
< d(x + tv_1 + \frac{t^2}{2}v_2 + \ldots + \frac{t^n}{n!}v_n; S) + t^{n+1}.
\]
Then the function \( \gamma_n \) below satisfies condition (1.2.2)
\[
\gamma_n(t) = -\frac{n!}{t^n}(x + tv_1 + \frac{t^2}{2}v_2 + \ldots + \frac{t^n}{n!}v_n - u(t)), t > 0.
\]

We point out that in the definition given by Pavel et al. (see (32)), it is required \( v_i \in T_x^iS, i = 1, 2, \ldots, n - 1 \) while, here, one shows that these assumptions are redundant.

**Proposition 1.2.3.** If \( v_n \in T_x^nS \) then its associated vectors \( v_i, 1 \leq i \leq n - 1 \) belong to \( T_x^iS \), respectively.

**Proof.** According to the previous proposition \( v_n \) belongs to \( T_x^nS \) with the associated vectors \( v_i \in X \), if and only if there exists a map \( \gamma_n : (0, \infty) \to X \) with \( \gamma_n(t) \to 0 \) as \( t \to 0 \), and (1.2.2) holds. But then, considering the functions
\[
r_i(t) = \frac{i!}{(i + 1)!}v_{i+1} + \frac{t^2}{(i + 2)!}v_{i+2} + \ldots + \frac{t^{n-i}}{n!}v_n + \frac{t^{n-i}}{n!}\gamma_n(t)
\]
for every \( t > 0, 1 \leq i \leq n - 1 \), which obviously approach zero when \( t \to 0 \), we get that
\[
x + tv_1 + \frac{t^2}{2}v_2 + \ldots + \frac{t^n}{n!}(v_i + r_i(t)) \in S, \forall t > 0, 1 \leq i \leq n - 1,
\]
which is equivalent to \( v_i \in T_x^iS \).

Let \( X \) and \( Y \) be normed linear spaces, let \( U \) be a neighborhood of a point \( x \) in \( X \), and let \( G \) be a mapping from \( U \) into \( Y \).

**Definition 1.2.2.** A mapping \( G \) is said to be Fréchet differentiable at the point \( x \) if it exists \( \Lambda \in \mathcal{L}(X, Y) \) \( \lim \|\alpha(h)\| = \|\alpha(0)\| = 0 \) such that it is possible to represent \( G \) in a neighborhood of \( x \) in the form
\[
G(x + h) = G(x) + \Lambda h + \alpha(h)\|h\|.
\]
The operator $\Lambda$ is called the Fréchet derivative (or simply the derivative) of the mapping $G$ at the point $x$ and is denoted by $G'(x)$.

Finally, in terms of the $\epsilon, \delta$ formalism the previous two relations can be stated thus: given an arbitrary $\epsilon > 0$ there is $\delta > 0$ for which the inequality

$$
||G(x + h) - G(x) - \Lambda h|| \leq \epsilon ||h||
$$

holds for all $h$ such that $||h|| < \delta$.

If a mapping $G : U \rightarrow Y$ is Fréchet differentiable at each point $x \in U$, then the mapping $G'(x) : U \rightarrow \mathcal{L}(X,Y)$ is defined. Since $\mathcal{L}(X,Y)$ is a normed space, the question of the existence of the second derivative $G''(x) = (G')'(x) \in \mathcal{L}(X, \mathcal{L}(X,Y))$ can be posed. The higher order derivatives are defined by induction: if $G^{(n-1)}(x)$ has already been defined in $U$, then

$$
G^{(n)}(x) = (G'(x))' \in \mathcal{L}(X, \ldots, \mathcal{L}(X,Y), \ldots).
$$

**Definition 1.2.3.** Let $G : U \rightarrow Y$. We shall say that $G^{(n)}$ exists at a point $x \in U$ if $G'(u), G''(u), G^{(n-1)}(u)$ exist in a neighborhood of the point $x$ and $G^{(n)}(x)$ exists.

If $G^{(n)}(x)$ exists at each point $x \in U$ and the mapping $G^{(n)}(x)$ is continuous in the uniform topology of the space $\mathcal{L}(X, \ldots, \mathcal{L}(X,Y), \ldots)$ (generated by the norm), then $G$ is said to be a mapping of class $C^n(U)$.

**Theorem 1.2.1** (Lyusternik). Let $X, Y$ be two Banach spaces, $G$ an operator mapping $X$ into $Y$, Fréchet differentiable in a neighborhood of $x$, $G(x) = 0$. Let $G'$ be continuous in a neighborhood of $x$, and suppose that $G'(x)$ maps $X$ onto $Y$. Then the set of tangent directions $T_xS$ to the set $D_G = \{ z \in X, G(z) = 0 \}$ at the point $x$ is the subspace $\text{Ker}G'(x) = \{ h \in X, G'(x)h = 0 \}$.

Next we introduce the notation

$$
\text{Ker}G^{(p)}(x) = \{ h \in X, G^{(p)}(x)[h]^p = 0 \}.
$$
Definition 1.2.4. Let $X$ and $Y$ be normed spaces. The mapping $G : X \rightarrow Y$ is said to be $p$-regular at the point $x$ if $G^{(p)}(x)[h]^{p-1}X = Y$, for any $h \in KerG^{(p)}(x)$, $h \neq 0$.

It is known the following result of A.A. Tret’yakov (33).

Theorem 1.2.2. Let $X$ and $Y$ be Banach spaces, $U$ a neighborhood of the point $x \in X$, $G$ a Fréchet differentiable up to and including $p$-th order mapping of set $U$ into $Y$, satisfying

$$G^{(r)}(x) = 0, \quad r = 0, 1, ..., p - 1.$$ 

Let $G$ be $p$-regular at the point $x$ and let its $p$-th derivative be continuous at this point (in the uniform operator topology of space $\mathcal{L}((X, ..., X), Y)$). Then the tangent cone to the set $D_G(x) = \{z \in U; G(z) = 0\}$ at the point $x \in D_G$, is the kernel of the operator $G^{(p)}(x)$:

$$T_xD_G = KerG^{(p)}(x). \quad (1.2.3)$$

Proof. Let the vector $v$ belong to $T_xD_G$. Let us show that $v \in KerG^{(p)}(x)$. In fact, $x + tv + tr(t) \in D_G$, $r(t) \rightarrow 0$, $t \downarrow 0$, or

$$G(x + tv + tr(t)) = G(x).$$ \quad (1.2.4)

On the other hand,

$$G(x + v + tr(t)) = G(x) + \frac{G^{(p)}(x)}{p!}[tv + tr(t)]^p + t^p\omega(t), \quad ||\omega(t)|| \rightarrow 0, \quad (1.2.5)$$

in view of (1.2.4). Equating the right-hand sides of (1.2.4) and (1.2.5), we have

$$G(x) = G(x) + \frac{G^{(p)}(x)}{p!}[tv + tr(t)]^p + t^p\omega(t),$$

or

$$\frac{t^p}{p!}G^{(p)}(x)[v + r(t)]^p + t^p\omega(t) = 0. \quad (1.2.6)$$
Dividing (1.2.6) by $t^p$, we obtain

$$\frac{1}{p!} G(x)^{(p)}[v + r(t)]^p + \omega(t) = 0. \tag{1.2.7}$$

But the left-hand side of (1.2.7) is

$$G^{(p)}(x)[v]^p + C_p^1 G^{(p)}(x)(r(t), \overrightarrow{v}, \ldots, \overrightarrow{v}) + \ldots + G^{(p)}(x)[r(t)]^p + \omega(t)p! = 0, \tag{1.2.8}$$

Here we have used the fact that $G^{(p)}(x)(v_1, \ldots, v_p)$ is a symmetric multilinear form. Since the operator $G^{(p)}(x)$ is continuous, and using condition (1.2.3), we finally find from (1.2.8) that, as $t \downarrow 0$, we have $G^{(p)}(x)[v]^p = 0$, i.e., $v \in KerG^{(p)}(x)$ or $T_x D G \subset KerG^{(p)}(x)$.

Henceforth we can assume without loss of generality that $x = 0$ and $G(0) = 0$.

Now let $h \in KerG^{(p)}(0)$, i.e., $G^{(p)}(0)[v]^p = 0$. It can be assumed without loss of generality that $||h|| = 1$. Put $x_\alpha = \alpha h, \alpha > 0$. Denote by $U_r(z) = \{ u \in X; ||u - z|| < r \}$ the neighborhood of the point $z$ of radius $r$, and by $h(A, B)$ the Haussdorff distance between the sets $A$ and $B$. Consider the many-valued mapping

$$\phi_{x_\alpha}(z) = z - (p - 1)! \{G^{(p)}(0)[x_\alpha]^{p-1}\}^{-1} G(x_\alpha + z),$$

where $G^{(p)}(0)[x_\alpha]^{p-1} \in L(X, Y)$ is a continuous linear operator for every fixed $\alpha > 0$. Let us show that, for any sufficiently small $\alpha > 0$, the operator $\phi_{x_\alpha}(z)$ satisfies the relations

$$h(\phi_{x_\alpha}(x_1), \phi_{x_\alpha}(x_2)) \leq \gamma ||x_1 - x_2||, \ x_1, x_2 \in U_{r(\alpha)}(0), \gamma \in (0, 1) \tag{1.2.9a}$$

$$\rho(0, \phi_{x_\alpha}(0)) \leq r(\alpha)(1 - \gamma). \tag{1.2.9b}$$

In turn, this will mean that the conditions of the generalized principle of contraction mappings (see (2)) are satisfied for $\phi_{x_\alpha}(z)$, i.e., there exists a point $z(\alpha)$ such that $z(\alpha) \in \phi_{x_\alpha}(z(\alpha))$, $\rho(z(\alpha), 0) \leq 2\rho(0, \phi_{x_\alpha}(0))/(1 - \gamma)$. We shall first show that

$$\rho(0, \phi_{x_\alpha}(0)) = o(\alpha) \tag{1.2.10}$$

for sufficiently small $\alpha > 0$. By Banach’s theorem on an open linear mapping, we have

$$m_\alpha = \sup_{y \in Y}(||y||^{-1}\inf\{||z||; \ z \in X, \ G^{(p)}(0)[h]^{p-1}z = y\}) < \infty$$
and hence
\[ m_\alpha = \sup_{y \in Y} (|y|^{-1} \inf \{|z|; z \in X, G^{(p)}(0)[\alpha h]^{p-1} z = y\}) = \frac{m}{\alpha^{p-1}}, \]
\[ G^{(p)}(0)[\alpha h]^{p-1} z = \alpha^{p-1} G^{(p)}(0)[h]^{p-1} z. \]

Since \( h \in \ker G^{(p)}(0) \), we have
\[ \rho(0, \phi_{x_\alpha}(0)) = ||(p-1)!\{G^{(p)}(0)[\alpha h]^{p-1}\}^{-1}G(\alpha h)|| \leq \]
\[ \leq (p-1)!m||G(\alpha h)||\alpha^{-p} = (p-1)!m\alpha^{-p}||G(0) + G'(0)[\alpha h] + \ldots + \]
\[ + \frac{1}{(p-1)!}G^{(p)}(0)[\alpha h] + \omega(\alpha h)|| = (p-1)\alpha^{-p}m||\omega(\alpha h)||, \]
where ||\omega(\alpha h)|| = o(\alpha^p). Hence
\[ \rho(0, \phi_{x_\alpha}(0)) \leq (p-1)!\alpha^{-p}m||\omega(\alpha h)|| = (p-1)! m o(\alpha) \] (1.2.11)
and relation (1.2.10) is satisfied. We will now show that the condition (1.2.9a) holds, i.e., given any sufficiently small \( \alpha > 0 \), the operator \( \phi_{x_\alpha}(z) \) is contracting in the sphere of center \( 0 \) and radius \( r(\alpha) \). It will be shown below that \( r(\alpha) = \alpha/R, R > 1 \) is a constant for all sufficiently small \( \alpha > 0 \). In turn, in conjunction with (1.2.11), this means that the condition (1.2.9b) holds. We have
\[ h(\phi_{x_\alpha}(x_1), \phi_{x_\alpha}(x_2)) = \inf \{|z_1 - z_2|; z_i \in \phi_{x_\alpha}(x_i), i = 1, 2\} = \]
\[ = \inf \{ \frac{1}{(p-1)!}G^{(p)}(0)[\alpha h]^{p-1} z_i = \]
\[ = \frac{1}{(p-1)!}G^{(p)}(0)[\alpha h]^{p-1} x_i - G(\alpha h + x_i), i = 1, 2\} \leq \]
\[ \leq \frac{m}{(p-1)!}||G(\alpha h + x_1) - G(\alpha h + x_2) - \frac{1}{(p-1)!}G^{(p)}(0)[\alpha h]^{p-1}(x_2 - x_1)||. \]
By the mean value theorem (see (2)),
\[ ||G(\alpha h + x_1) - G(\alpha h + x_2)|| \leq \sup_{\theta \in [0,1]} ||G'((\alpha h + x_1 + \theta(x_2 - x_1)))|| \cdot ||x_2 - x_1||, \] and
\[ ||G(\alpha h + x_1) - G(\alpha h + x_2) - \frac{1}{(p-1)!}G^{(p)}(0)[\alpha h]^{p-1}(x_2 - x_1)|| \leq \]
\[ \leq \sup_{\theta \in [0,1]} ||G'(\alpha h + x_1 + \theta(x_2 - x_1)) - \frac{1}{(p-1)!}G^{(p)}(0)[\alpha h]^{p-1}|| \cdot ||x_2 - x_1||. \quad (1.2.12) \]

We assume that \( ||x_1|| \leq \alpha/R, R > 1 \), where \( R \) is chosen so that relations (1.2.9a) hold simultaneously for all sufficiently small \( \alpha > 0 \). The feasibility of doing this, and the value of \( R \), are obtained below. Using Taylor’s formula we have

\[ G'(\alpha h + x_1 + \theta(x_2 - x_1)) = G'(0) + G''(0)[\alpha h + x_1 + \theta(x_2 - x_1)] + \ldots \]
\[ + \frac{1}{(p-1)!}G^{(p)}(0)[\alpha h + x_1 + \theta(x_2 - x_1)]^{p-1} + \epsilon(\alpha), \quad (1.2.13) \]

where \( ||\epsilon(\alpha)|| = o(\alpha^{p-1}) \), since, from (1.2.13), \( ||\alpha h + x_1 + \theta(x_2 - x_1)|| \leq 4\alpha \). The first \( p - 1 \) terms in (1.2.13) are zero, so

\[ G'(\alpha h + x_1 + \theta(x_2 - x_1)) = \frac{1}{(p-1)!}G^{(p)}(0)[\alpha h + x_1 + \theta(x_2 - x_1)]^{p-1} + \epsilon(\alpha) = \]
\[ = \frac{1}{(p-1)!}G^{(p)}(0)[\alpha h]^{p-1} + \frac{1}{(p-1)!} \sum_{i=1}^{p-1} C_{p-1}^{i} G^{(p)}(0)[\alpha h]^{p-1-i}[x_1 + \theta(x_2 - x_1)]^i + \epsilon(\alpha). \quad (1.2.14) \]

We now find \( R > 1 \), (\( ||x_1|| \leq \alpha/R, ||x_2|| \leq \alpha/R \)) such that the norm of the second term on the right of (1.2.14) is less than \( \alpha^{p-1}/4m \). This can be done, since

\[ \left| \frac{1}{(p-1)!} \sum_{i=1}^{p-1} C_{p-1}^{i} G^{(p)}(0)[\alpha h]^{p-1-i}[x_1 + \theta(x_2 - x_1)]^i \right| \leq \]
\[ \leq \frac{1}{(p-1)!} ||G^{(p)}(0)|| \sum_{i=1}^{p-1} C_{p-1}^{i} \alpha^{p-1-i} ||x_1 + \theta(x_2 - x_1)||^i \leq \]
\[ \leq \frac{4}{(p-1)!} ||G^{(p)}(0)|| \sum_{i=1}^{p-1} C_{p-1}^{i} \alpha^{p-1-i} \frac{\alpha^{p-1-i}}{R} \leq \frac{1}{(p-1)!} \frac{4\alpha^{p-1}}{R} ||G^{(p)}(0)|| \sum_{i=1}^{p-1} C_{p-1}^{i} \frac{\alpha^{p-1-i}}{R}. \]

Taking \( R \geq 2^{p+3} ||G^{(p)}(0)|| m/(p-1)! \), we obtain

\[ \left| \frac{1}{(p-1)!} \sum_{i=1}^{p-1} C_{p-1}^{i} G^{(p)}(0)[\alpha h]^{p-1-i}[x_1 + \theta(x_2 - x_1)]^i \right| \leq \frac{\alpha^{p-1}}{4m}. \quad (1.2.15) \]
On the basis of (1.2.14) and (1.2.15), we estimate the right-hand side of (1.2.12):

\[
\sup_{\theta \in [0,1]} ||G'(ah + x_1 + \theta(x_2 - x_1)) - \frac{1}{(p-1)!}G^{(p)}(0)[ah]^{p-1}|| \cdot ||x_2 - x_1|| \leq 
\]

\[
\leq \sup_{\theta \in [0,1]} \left\| \frac{1}{(p-1)!} \sum_{i=1}^{p-1} C^i_{p-1} G^{(p)}(0)[ah]^{p-1-i}[x_1 + \theta(x_2 - x_1)]^i \right\| \cdot ||x_2 - x_1|| \leq 
\]

\[
\leq \frac{\alpha^{p-1}}{4m} ||x_2 - x_1|| + ||\epsilon(\alpha)|| \cdot ||x_2 - x_1|| \leq \frac{\alpha^{p-1}}{2m} ||x_2 - x_1||
\]

for all sufficiently small \( \alpha > 0 \). This implies in turn that

\[
h(\phi_{ah}(x_1), \phi_{ah}(x_2)) \leq \frac{m}{\alpha^{p-1}} ||G(ah + x_1) - G(ah + x_2) - 
\]

\[
- \frac{1}{(p-1)!} G^{(p)}(0)[ah]^{p-1}(x_2 - x_1)|| \leq \frac{1}{2} ||x_2 - x_1||, \; x_1, x_2 \in U_{r(\alpha)}(0),
\]

i.e., condition (1.2.9a) holds with \( \gamma = \frac{1}{2} \) and \( R \geq 2^{p+3}||G^{(p)}(0)||m/(p-1)! \). We have proved conditions (1.2.9), and hence, for any sufficiently small \( \alpha > 0 \), there is a point

\[
z(\alpha) \in \phi_{x_1}(z(\alpha)) \tag{1.2.16}
\]

such that

\[
\rho(z(\alpha), 0) = ||z(\alpha)|| \leq 4\rho(0, \phi_{x_1}) = o(\alpha). \tag{1.2.17}
\]

It follows from (1.2.16) that \( G(\alpha h + z(\alpha)) = 0 \) or \( \alpha h + z(\alpha) \in D_G \). In conjunction with (1.2.17), this means that \( h \in T_0D_G \), i.e., we have proved that \( KerG^{(p)}(0) \subset T_0D_G \), and also proved the theorem.

In Chapter 2, we are going to generalize the following two theorems (Theorem 1.5 and Theorem 1.10, (24)).

**Theorem 1.2.3** (Pavel & Ursescu). Let \( G : A \rightarrow Y \) be a mapping from an open subset \( A \) of a normed space \( X \) into a normed space \( Y \) that is Fréchet differentiable at a point \( x \in S = G^{-1}(E) \), where \( E \) is a given subset of \( Y \).
Then the inclusion below holds:

\[ G'(x)(T_xS) \subset T_{G(x)}S. \]

Conversely, if we assume in addition to the differentiability of \( G \) at \( x \in S \) that \( G'(x) \) is onto and \( Y \) is finite dimensional, then the relation \( G'(x)v \in T_{G(x)}E \) with \( v \in X \) implies \( v \in T_xS \). Consequently, under these hypotheses one has

\[ T_xS = G'(x)^{-1}(T_{G(x)}E). \]

**Theorem 1.2.4** (Pavel & Ursescu) Let \( G : A \to Y \) be a mapping from an open subset \( A \) of a normed space \( X \) into a normed space \( Y \) and let \( E \) be a nonempty subset of \( Y \). Assume that the mapping \( G \) is twice Fréchet differentiable at a given point \( x \in S = G^{-1}(E) \). Then one has the relation

\[ G'(x)(v_2) + G''(x)(v_1)(v_1) \in T^2_{G(x)}E \quad (1.2.18) \]

whose associated vector is \( G'(x)v \in T_{G(x)}E \) for each \( v_2 \in T^2_xS \) with the correspondent vector \( v_1 \in T_xS \).

Moreover, under the additional hypotheses that the differential \( G'(x) : X \to Y \) is onto and \( Y \) is finite dimensional, one obtains that (1.2.18), together with the fact that \( G'(x)(v_1) \in T_{G(x)}E \) is the associated vector of \( G'(x)(v_2) + G''(x)(v_1)(v_1) \in T^2_{G(x)}E \) imply that \( v_2 \in T^2_xS \) with correspondent vector \( v_1 \in T_xS \).

The proofs of the previous two theorems as well as our results (Theorem 2.1 and Theorem 2.3) make use of Brouwer’s theorem.

**Theorem 1.2.5** (Brouwer). Let \( B \) be the unit closed ball of \( \mathbb{R}^p \) and \( f : B \to B \). Then there exists \( x \in B \) such that \( f(x) = x \).

The proof of the theorem is based on the following lemma.
Lemma 1.2.1. Let \( U = \{ x \in \mathbb{R}^p ; ||x|| = 1 \} \). Then there is no continuous function \( f : B \to U, \forall x \in U \).

Proof. Assume by contradiction that there exists a function \( f \) with the indicated properties. We extend \( f \) to a continuous function \( f_1 : \mathbb{R}^p \to U \) by

\[
f_1 = \frac{x}{||x||}, \forall ||x|| > 1.
\]

Applying Weierstrass’ approximation theorem, there exists \( f_2 : \mathbb{R}^p \to \mathbb{R}^p \) of class \( C^1 \) such that \( ||f_2(x) - f_1(x)|| < 1 \) for \( ||x|| \leq 2 \).

We consider the function \( f_3 : \mathbb{R}^p \to \mathbb{R}^p \),

\[
f_3(x) = (1 - \phi(||x||))f_1(x) + \phi(||x||)f_2(x),
\]

\( \phi : \mathbb{R} \to \mathbb{R} \) is a function of class \( C^1 \) verifying the conditions below:

\[
0 \leq \phi(t) \leq 1, \quad \phi(t) = 1 \quad \forall \, t \leq \frac{3}{2}, \quad \phi(t) = 0 \quad \forall \, t \geq 2.
\]

One can take for example

\[
\phi(t) = \frac{1}{16}t^3 - \frac{21}{64}t^2 + \frac{9}{16}t - \frac{5}{16}, \text{ for } t \in \left(\frac{3}{2}, 2\right).
\]

It is easy to see that \( f_3 \) is of class \( C^1 \),

\[
f_3(x) = \frac{x}{||x||}, \forall ||x|| \geq 2 \text{ and } \quad f_3(x) \geq ||f_1(x)|| - \phi(||x||)||f_2(x) - f_1(x)|| > 0
\]

\[
> ||f_2(x) - f_1(x)||(1 - \phi(||x||)) \geq 0,
\]

for \( ||x|| < 2 \), as \( 1 = ||f_1(x)|| > ||f_2(x) - f_1(x)||. \)

We can define a function of class \( C^1 \), \( f_4 : \mathbb{R}^p \to \mathbb{R}^p \), by

\[
f_4(x) = \frac{f_3(2x)}{||f_3(2x)||}.
\]
We notice that \( ||f_4(x)|| = 1 \) for any \( x \in \mathbb{R}^p \) and
\[
f_4(x) = \frac{x}{||x||}, \quad \forall \; ||x|| \geq 1.
\]

Clearly \( f_4 \) is a Lipschitz function on \( B \) of Lipschitz constant \( k \) and let \( t \in [0, 1/k) \).
Next we will show that
\[
(I + tf_4)B = (1 + t)B,
\]
where \( I \) is the identity mapping.

Indeed, for \( x \in B \) we have
\[
||x + tf_4(x)|| \leq ||x|| + t||f_4(x)|| \leq 1 + t,
\]
and for \( x \notin B(0, 1) \)
\[
||x + tf_4(x)|| = ||x + t\frac{x}{||x||}|| = ||x|| + t > 1 + t.
\]

On the other hand, \( t \in [0, 1/k) \) implies that \( t||f_4'(x)|| < 1 \) for \( x \in B \) and therefore \( I + tf_4'(x) \) is an isomorphism and \( \det(I + tf_4'(x)) > 0 \) for \( x \in B \). Applying the inverse mapping theorem we conclude that the set \( (I + tf_4(x))(B) \) is open in \( B(0, 1 + t) \). It is closed too, being compact. Since \( B(0, 1 + t) \) is connected, we obtain (1.2.19). So the function \( I + tf_4(x) : B(0, 1) \to B(0, 1 + t) \) is onto. It is also one-to-one because if \( I + tf_4(x) = I + tf_4(y) \), we have
\[
||x - y|| = t||f_4(x) - f_4(y)|| \leq tk||x - y||.
\]
Since \( tk < 1 \) we get \( x = y \).

We use now the change of variable formula for the integral in \( \mathbb{R}^p \) and we deduce
\[
(1 + t)^p \int_{B(0,1)} dy = \int_{B(0,1+t)} dy = \int_{(I+tf_4)(B(0,1))} dy = \int_B \det(I + tf_4'(x))dx =
\]
\[
= t^p \int_B \det f_4'(x)dx + P_{n-1}(t),
\]
where \( P_{n-1}(t) \) is a polynomial function of at most \( n - 1 \) degree.
By identification we obtain
\[
\int_B \det f_4'(x)dx = \int_{B(0,1)} dy \neq 0. \tag{1.2.20}
\]
On the other hand, $||f_4(x)||^2 = 1$ implies $(f'_4(x))^*(f_4(x)) = 0$, where $(f'_4(x))^*$ is the adjoint of the linear operator $f'_4(x)$. Since $f_4(x) \neq 0$ for any $x \in \mathbb{R}^p$ we conclude that $\det f'_4(x) = 0$, $\forall x \in \mathbb{R}^p$, which contradicts (1.2.20).

**Proof of Theorem 1.2.1.** Assume by contradiction that $f(x) \neq x$, $\forall x \in B$. One can see easily that there exists a continuous function $h : B \to [1, \infty)$ such that the function defined by

$$
g(x) = f(x) + h(x)(x - f(x))$$

for $x \in B$ has the property $||g(x)|| = 1$. More precisely, $g(x)$ is the point where the semiline with the origin at $f(x)$ and which contains $x$, intersects $U$. Applying the previous lemma, our assumption is false.
Chapter 2

Characterizations of High Order Tangent Cones

In this chapter one describes the tangent cones to the kernel $D_G$ of a mapping $G$ at $x \in D_G$ when $G$ has different properties.

In (30), N.H. Pavel and C. Ursescu characterized the second order tangent cone to $D_G$ when the mapping $G$ is real-valued and we generalize their result as follows.

**Theorem 2.1.** Let $G : A \rightarrow Y$ be a mapping from an open subset $A$ of a normed space $X$ into a normed space $Y$ that is four times Fréchet differentiable at a point $x \in S = G^{-1}(E)$, where $E$ is a given subset of $Y$ and $G$ is continuous near $x$. Then

\[
G^{(4)}(x)(v_1)(v_1)(v_1) + 6G''''(x)(v_1)(v_1)(v_2) + 4G'''(x)(v_1)(v_3) + \\
+3G''(x)(v_2)(v_2) + G'(x)(v_4) \in T^4_{G(x)}E
\]

(2.1)

with associated vectors

\[
G'(x)(v_1) \in T_{G(x)}E, \ G'(x)(v_2) + G''(x)(v_1)(v_1) \in T^2_{G(x)}E, \text{ and}
\]

\[
G''''(x)(v_1)(v_1) + 3G''''(x)(v_1)(v_2) + G'(x)(v_4) \in T^3_{G(x)}E,
\]

for each $v_4 \in T^4_xS$ with correspondent vectors $v_i \in T^i_xS$, $1 \leq i \leq 3$.

Moreover, under the additional hypotheses that the differential
$G'(x) : X \to Y$ is onto and $Y$ is finite dimensional, one obtains that (2.1), together with the fact that

$$G'(x)(v_1) \in T_{G(x)}E, \ G'(x)(v_2) + G''(x)(v_1)(v_1) \in T^2_{G(x)}E, \ \text{and}$$

$$G''(x)(v_1)(v_1)(v_1) + 3G''(x)(v_1)(v_2) + G'(x)(v_3) \in T^3_{G(x)}E$$

are associated vectors of

$$G^{(4)}(x)(v_1)(v_1)(v_1) + 6G''(x)(v_1)(v_1)(v_2) + 4G''(x)(v_1)(v_3) +$$

$$+3G''(x)(v_2)(v_2) + G'(x)(v_4),$$

imply that $v_4 \in T^4_xS$, with correspondent vectors $v_i \in T^i_xS$, $1 \leq i \leq 3$.

**Proof.** Let $v_4 \in T^4_xS$. Then, Proposition 1.2.2 supplies a mapping

$$\gamma : (0, \infty) \to X \ \text{with} \ \gamma(t) \to 0 \ \text{as} \ t \downarrow 0 \ \text{and}$$

$$G(x + tv_1 + t^2v_2 + t^3v_3 + \frac{t^4}{4!}(v_4 + \gamma(t))) \in E, \ \forall \ t > 0.$$  

It follows that

$$\frac{1}{t^4} d(G(x) + tG'(x)v_1 + \frac{t^2}{2}(G''(x)(v_1)(v_1) + G'(x)v_2) + \frac{t^3}{3!}(G'''(x)(v_1)(v_1)(v_1) +$$

$$+3G''(v_1)(v_2) + G'(x)(v_3)) + \frac{t^4}{4!}(G^{(4)}(x))(v_1)(v_1)(v_1) + 6G''(x)(v_1)(v_1)(v_2) +$$

$$+4G''(x)(v_1)(v_3) + 3G''(x)(v_2)(v_2) + G'(x)(v_4)); E) \leq$$

$$\leq \frac{1}{t^4} ||G(x) + tG'(x)v_1 + \frac{t^2}{2}(G''(x)(v_1)(v_1) + G'(x)v_2) + \frac{t^3}{3!}(G'''(x)(v_1)(v_1)(v_1) +$$

$$+3G''(v_1)(v_2)) + G'(x)(v_3)) + \frac{t^4}{4!}(G^{(4)}(x))(v_1)(v_1)(v_1) +$$

$$+6G''(x)(v_1)(v_1)(v_2) + 4G''(x)(v_1)(v_3) + 3G''(x)(v_2)(v_2) + G'(x)(v_4)) -$$

$$-G(x + tv_1 + \frac{t^2}{2}v_2 + \frac{t^3}{3!}v_3 + \frac{t^4}{4!}(v_4 + \gamma(t))))||$$

The definition of four times Fréchet differentiability of $F$ at $x$, guarantees that the right hand side of the above inequality goes to zero as $t \downarrow 0$. So, (2.1) holds.

Conversely, suppose that $G'(x)$ is onto, $Y$ is finitely dimensional and the tangential condition (2.1) is verified, with associated vectors

$$G'(x)(v_1) \in T_{G(x)}E, \ G'(x)(v_2) + G''(x)(v_1)(v_1) \in T^2_{G(x)}E, \ \text{and}$$

$$G''(x)(v_1)(v_1) + 3G''(x)(v_1)(v_2) + G'(x)(v_3) \in T^3_{G(x)}E.$$  

Then, there exists a map $\gamma_1 : (0, \infty) \to Y$ with $\gamma_1(t) \to 0$ as $t \downarrow 0$, and
\[ G(x) + tG'(x)v_1 + \frac{t^2}{2}(G''(x)(v_1)(v_1)) + G'(x)v_2 + \frac{t^3}{3!}(G''(x)v_3 + 3G''(x)(v_1)(v_2) + \\
+ G''(x)(v_1)(v_1)(v_1)) + \frac{t^4}{4!}(G^{(4)}(x)(v_1)(v_1)(v_1)(v_1)) + 6G''(x)(v_1)(v_1)(v_2) + \\
+ 4G''(x)(v_1)(v_2)(v_2) + G'(x)(v_4) + \gamma_1(t)) \in E, \forall t > 0 \] (2.2)

By hypothesis, \( G \) is four times differentiable at \( x \), so

\[
\frac{1}{t^4}||G(x + ta) - G(x) - tG'(x)(a) - \frac{t^2}{2}G''(x)(a)(a) - \frac{t^3}{3!}G'''(x)(a)(a)(a) - \\
- \frac{t^4}{4!}G^{(4)}(x)(a)(a)(a)(a)|| \to 0
\] (2.3)

as \( t \downarrow 0 \), uniformly with respect to \( a \) in bounded sets of \( X \).

For \( a = v_1 + \frac{t}{2}v_2 + \frac{t^2}{3!}v_3 + \frac{t^3}{4!}(v_4 + u) \), (2.3) shows that

\[
\frac{1}{t^4}\{G(x + tv_1 + \frac{t^2}{2}v_2 + \frac{t^3}{3!}v_3 + \frac{t^4}{4!}(v_4 + u)) - G(x) - tG'(x)(v_1) - \frac{t^2}{2}G''(x)(v_1) + \\
+ \frac{t^3}{3!}(v_4 + u)\} - \frac{t^2}{2}G''(x)(v_1) + \frac{t^3}{3!}v_3 + \frac{t^4}{4!}(v_4 + u)^2 - \frac{t^3}{3!}G'''(x)(v_1) + \\
+ \frac{t^2}{3!}v_3 + \frac{t^3}{4!}(v_4 + u)^3 - \frac{t^4}{4!}G^{(4)}(x)(v_1) + \frac{t^2}{2}v_2 + \frac{t^3}{3!}v_3 + \frac{t^4}{4!}(v_4 + u)^4 \to 0
\] (2.4)

as \( t \downarrow 0 \), uniformly with respect to \( u \) on bounded sets.

Precisely, (2.4) says that

\[
\frac{1}{t^4}\{G(x + tv_1 + \frac{t^2}{2}v_2 + \frac{t^3}{3!}v_3 + \frac{t^4}{4!}(v_4 + u)) - G(x) - tG'(x)(v_1) - \frac{t^2}{2}(G''(x)(v_1)(v_1) + \\
+ G''(x)(v_2)) - \frac{t^3}{3!}(G'''(x)(v_1)(v_1)(v_1)) + G'(x)(v_3) + 3G''(x)(v_1)(v_2) - \\
- \frac{t^4}{4!}(G^{(4)}(x)(v_1)(v_1)(v_1)(v_1)) + 6G''(x)(v_1)(v_1)(v_2) + 4G''(x)(v_1)(v_3) + \\
+ 3G''(x)(v_2)(v_2) + G'(x)(v_4 + u))\} \to 0
\] (2.5)

as \( t \downarrow 0 \), uniformly with respect to \( u \) on bounded sets.

This suggests that the following “inverse Taylor formula” holds:

For every \( \eta > 0 \), there exists \( \delta > 0 \) such that for any \( t \in (0, \delta) \), and \( q \in Y \) with \( ||q|| < \delta \), there is \( u \in X \) satisfying \( ||u|| < \eta \), and

\[
G(x + tv_1 + \frac{t^2}{2}v_2 + \frac{t^3}{3!}v_3 + \frac{t^4}{4!}(v_4 + u)) = G(x) + tG'(x)(v_1) + \\
+ \frac{t^2}{2}(G''(x)(v_1)(v_1)) + G'(x)(v_2) + \frac{t^3}{3!}(G'''(x)(v_1)(v_1)(v_1)) + \\
+ 3G''(x)(v_1)(v_2) + G'(x)(v_3) + \frac{t^4}{4!}(G^{(4)}(x)(v_1)(v_1)(v_1)(v_1)) + \\
+ 6G''(x)(v_1)(v_1)(v_2) + 4G''(x)(v_1)(v_3) + 3G''(x)(v_2)(v_2) + G'(x)(v_4 + q).\] (2.6)
To this aim, let $L : Y \to X$ be an inverse of $G'(x) : X \to Y$, i.e.,

$$G'(x)L(y) = y, \forall y \in Y.$$  \hspace{2cm} (2.7)

To prove the existence of a linear continuous operator $L$ with this property, let

$\{e_1, e_2, ..., e_n\}$ be a basis for $Y$. $G'(x)$ is onto, so one can choose $x_i \in X$ such that $G'(x)(x_i) = e_i$, $1 \leq i \leq n$. If $y \in Y$, we have $y = \sum_{i=1}^{n} a_i e_i$ for some $a_i \in \mathbb{R}$. Then (2.7) holds with $L(y) = \sum_{i=1}^{n} a_i x_i$.

Since $L$ is continuous and $L(0) = 0$, there exists $\rho > 0$ such that $\|Ly\| < \eta$ for all $y \in Y$ with $\|y\| < \rho$.

In view of (2.5), there is $\delta > 0$ such that

$$\frac{4!}{t^4} |G(x + tv_1 + \frac{t^2}{2} v_2 + \frac{t^3}{3!} v_3 + \frac{t^4}{4!} (v_4 + Ly)) - G(x) - tG'(x)(v_1) -$$

$$- \frac{t^2}{2} (G''(x)(v_1)(v_1) + G'(x)(v_2)) - \frac{t^3}{3!} (G'''(x)(v_1)(v_1)(v_1) + 3G''(x)(v)(v_2) +$$

$$+ G'(x)(v_3)) - \frac{t^4}{4!} (G^{(4)}(x)(v_1)(v_1)(v_1)(v_1) + 6G'''(x)(v_1)(v_1)(v_2) +$$

$$+ 4G''(x)(v_1)(v_3) + 3G''(x)(v_2)(v_2) + G'(x)(v_4 + Ly))| < \frac{\rho}{2}.$$ \hspace{2cm} (2.8)

for all $t \in (0, \delta), \|y\| < \rho$, i.e., $y \in B_\rho$, the ball of radius $\rho$.

Next, one defines the mapping $F : B_\rho \to Y$ for $0 < t < \delta, y \in B_\rho, q \in Y, \|q\| < \delta < \frac{\rho}{2}$ by

$$F(y) = \frac{4!}{t^4} [-G(x + tv_1 + \frac{t^2}{2} v_2 + \frac{t^3}{3!} v_3 + \frac{t^4}{4!} (v_4 + Ly)) + G(x) + tG'(x)(v_1) +$$

$$+ \frac{t^2}{2} (G''(x)(v_1)(v_1) + G'(x)(v_2)) + \frac{t^3}{3!} (G'''(x)(v_1)(v_1)(v_1) + 3G''(x)(v_1)(v_2) +$$

$$+ G'(x)(v_3)) + \frac{t^4}{4!} (G^{(4)}(x)(v_1)(v_1)(v_1)(v_1) + 6G'''(x)(v_1)(v_1)(v_2) +$$

$$+ 4G''(x)(v_1)(v_3) + 3G''(x)(v_2)(v_2) + G'(x)(v_4 + Ly)) + q.$$ \hspace{2cm} (2.9)

Clearly, $F$ maps $B_\rho$ into $B_\rho$ and is continuous, so it has a fixed point $y_0$ (by Brouwer’s fixed point theorem in $Y$, which is finite dimensional).

For $y = y_0$, (2.9) can be rewritten as

$$G(x + tv_1 + \frac{t^2}{2} v_2 + \frac{t^3}{3!} v_3 + \frac{t^4}{4!} (v_4 + Ly_0)) = G(x) + tG'(x)(v_1) +$$
for each with associated vectors since $G$ is a given subset of $S$, where $y_0 = F(y_0)$ is the fixed point of $F$.

Further, let $\eta > 0$ be arbitrary and let $\delta > 0$ be a number as in (2.6). For the function $\gamma_1(t)$ in (2.2), there is $0 < \varepsilon < \delta$ such that $\|\gamma_1(t)\| < \delta$ if $t \in (0, \varepsilon)$. Then (2.6) enables us to find a map $u : (0, \varepsilon) \to X$ with

$$
\|u(t)\| < \eta, \quad x + tv_1 + \frac{t^2}{2} v_2 + \frac{t^3}{3!} v_3 + \frac{t^4}{4!} (v_4 + u(t)) \in A \\
G(x + tv_1 + \frac{t^2}{2} v_2 + \frac{t^3}{3!} v_3 + \frac{t^4}{4!} (v_4 + u(t))) = G(x) + tG'(x)(v_1) + \\
+ \frac{t^2}{2} (G''(x)(v_1)(v_1) + G'(x)(v_2)) + \frac{t^3}{3!} (G'''(x)(v_1)(v_1)(v_1) + 3G''(x)(v_1)(v_2) + \\
+ G'(x)(v_3)) + \frac{t^4}{4!} (G^{(4)}(x)(v_1)(v_1)(v_1)(v_1) + 6G'''(x)(v_1)(v_1)(v_2) + \\
+ 4G''(x)(v_1)(v_3) + 3G''(x)(v_2)(v_2) + G'(x)(v_4) + \gamma_1(t)), \forall t \in (0, \varepsilon)
$$

Since by (2.2), the right-hand side of the above equality belongs to $E$, and $u(t) \to 0$ as $t \downarrow 0$, one concludes that $v_4 \in T^4_x S$, and $v_i \in T^i_x S$, $i \in \{1, 2, 3\}$, are its correspondent vectors. This completes the proof.

We formulate now a generalization of Theorem 2.1.

**Theorem 2.2.** Let $G : A \to Y$ be a mapping from an open subset $A$ of a normed space $X$ into a normed space $Y$ that is $p$ times Fréchet differentiable at a point $x \in S = G^{-1}(E)$, where $E$ is a given subset of $Y$ and $G$ is continuous near $x$. Then

$$
S^G_p(x) \in T^p_{G(x)} E
$$

with associated vectors $S^G_n(x) \in T^p_{G(x)} E$, $1 \leq n \leq p - 1$,

for each $v_p \in T^p_x S$ with correspondent vectors $v_n \in T^n_x S$, $1 \leq n \leq p - 1$.

Moreover, under the additional hypotheses that the differential
$G(x) : X \to Y$ is onto and $Y$ is finite dimensional, one obtains that (2.10), together with the fact that

$$S_n^G(x) \in T_{G(x)}^n E, \ 1 \leq n \leq p - 1$$

are associated vectors of $S_p^G(x) \in T_{G(x)}^p E$,

imply that $v_p \in T_x^p S$ with correspondent vectors $v_n \in T_x^n S, \ 1 \leq n \leq p - 1$.

Here, for every positive integer $m$, $S_m^G(x)$ denotes the expression

$$S_m^G(x) = \sum_{k=1}^{m} \frac{m!}{k!} \sum_{i_1, \ldots, i_k \in \{1, \ldots, m\}} \frac{1}{i_1! i_2! \ldots i_k!} G^{(k)}(x)(v_{i_1}) \ldots (v_{i_k}).$$

More precisely,

for $m = 2$, $S_2^G(x) = G'(x)(v_2) + G''(x)(v_1)(v_1),$

for $m = 3$, $S_3^G(x) = G''(x)(v_1)(v_1)(v_1) + 3G''(x)(v_1)(v_2) + G'(x)(v_3),$

for $m = 4$, $S_4^G(x_0) = G^{(4)}(x)(v_1)(v_1)(v_1)(v_1) + 6G''(x)(v_1)(v_1)(v_2) +$

$$+ 4G''(x)(v_1)(v_3) + 3G''(x)(v_2)(v_2) + G'(x)(v_4).$$

We specialize Theorem 2.2 to the case of the null-sets for vector-valued functions.

**Corollary 2.1.** Assume that $G : X \to \mathbb{R}^m$ is $p$ times Fréchet differentiable at $x \in X$ with $G(x) = 0$, $G$ continuous near $x$, and $G'(x) : X \to \mathbb{R}^m$ is onto, $p \geq 1$.

Then $v_p \in T_x^p D_G$ with the associated vectors $v_n \in T_x^n D_G, \ n = 1, \ldots, p - 1$, if and only if

$$S_n^G(x) = 0, \ \forall 1 \leq n \leq p,$$

where $D_G = \{z \in X, G(z) = 0\}$ is nonempty.

**Proof.** The necessity part does not require $G'(x)$ to be onto.

Since $v_p \in T_x^p D_G$ with associated vectors $v_n \in T_x^n D_G, \ n = 1, \ldots, p - 1$, there exists $\gamma(t) \to 0$ as $t \downarrow 0$ such that

$$x + tv_1 + \frac{t^2}{2}v_2 + \ldots + \frac{t^p}{p!}(v_p + \gamma(t)) \in D_G, \text{ i.e.},$$

$$0 = G(x + tv_1 + \frac{t^2}{2}v_2 + \ldots + \frac{t^p}{p!}(v_p + \gamma(t))) - G(x) =$$
\[ G'(x)(tv_1 + \frac{t^2}{2}v_2 + \ldots + \frac{t^p}{p!}(v_p + \gamma(t))) + \]
\[ + \frac{1}{2} G''(x)(tv_1 + \frac{t^2}{2}v_2 + \ldots + \frac{t^p}{p!}(v_p + \gamma(t)))(tv_1 + \frac{t^2}{2}v_2 + \ldots + \frac{t^p}{p!}(v_p + \gamma(t))) + \]
\[ + \frac{1}{3!} G'''(x)[tv_1 + \frac{t^2}{2}v_2 + \ldots + \frac{t^p}{p!}(v_p + \gamma(t))]^3 + \ldots + \frac{1}{p!} G^{(p)}(x)[tv_1 + \frac{t^2}{2}v_2 + \ldots + \frac{t^p}{p!}(v_p + \gamma(t))]^p + t^p \psi(t), \psi(t) \rightarrow 0, t \downarrow 0 \]  
(2.12)

If one divides (2.12) by $t$ and then passes to limit for $t \downarrow 0$, one gets

\[ G'(x)(v_1) = 0. \]

Dividing by $t^2$ and then letting $t \downarrow 0$ leads to

\[ G'(x)(v_1) + G'(x)(v_2) = 0. \]

Dividing (2.12) by $t^3$ and letting $t \downarrow 0$ yields

\[ G''(x)(v_1)(v_1) + 3G''(x)(v_1)(v_2) + G'(x)(v_3) = 0. \]

Similarly, one can obtain

\[ G^{(4)}(x)(v_1)(v_1)(v_1)(v_1) + 6G''''(x)(v_1)(v_1)(v_2) + 4G''(x)(v_1)(v_3) + 3G''(x)(v_2)(v_2) + G'(x)(v_4) = 0 \text{ and} \]

\[ S^C_n(x) = 0, 5 \leq n \leq p. \]

Conversely, assume that (2.11) holds and $G'(x)$ is onto. Then, the fact that $v_p \in T^n x \mathcal{D}_G$ with associated vectors $v_n \in T^n x \mathcal{D}_G$, $1 \leq n \leq p-1$, is a direct consequence of the second part of Theorem 2.2, with $S = D_G$ and $E = \{0\}$.

In (30) and (31), N.H. Pavel and C. Ursescu characterized the second order tangent vectors to $D_G$ at $x \in D_G$ when $G : X \rightarrow \mathbb{R}$ is twice Fréchet differentiable at $x$ and $G'(x)$ is onto. Next, one describes the second order tangent cone to $D_G$ at $x \in D_G$ when $G'(x) = 0$ and $G$ has some additional properties.
Theorem 2.3. Assume that $G : X \to \mathbb{R}^m$ is three times Fréchet differentiable at $x \in D_G$ with its third derivative $G'''(x)$ continuous at $x$, $G'(x) = 0$, $G$ is continuous near $x$, and 2-regular at $x$.

Then $v_2 \in T_x^2 D_G$ with associated vector $v_1 \in T_x D_G$ if and only if

$$G''(x)(v_1)(v_1) = 0 \quad \text{and} \quad G'''(x)(v_1)(v_1) + 3G''(x)(v_1)(v_2) = 0. \quad (2.13)$$

$$G'''(x)(v_1)(v_1) + 3G''(x)(v_1)(v_2) = 0. \quad (2.14)$$

Proof. For the necessary implication the assumptions that $G$ is 2-regular at $x$ will not be used.

The fact that $v_2 \in T_x^2 D_G$ with associated vector $v_1 \in T_x D_G$ is equivalent to the existence of a mapping $\gamma(t) \to 0$ as $t \downarrow 0$ such that

$$x + tv_1 + \frac{t^2}{2!}(v_2 + \gamma(t)) \in D_G, \text{ i.e.,}$$

$$0 = G(x + tv_1 + \frac{t^2}{2}(v_2 + \gamma(t))) - G(x) =$$

$$G'(x)(tv_1 + \frac{t^2}{2}(v_2 + \gamma(t))) + \frac{1}{2!}G''(x)(tv_1 + \frac{t^2}{2}(v_2 + \gamma(t)))(tv_1 + \frac{t^2}{2}(v_2 + \gamma(t))) +$$

$$+ \frac{1}{3!}G'''(x)(tv_1 + \frac{t^2}{2}(v_2 + \gamma(t)))(tv_1 + \frac{t^2}{2}(v_2 + \gamma(t)))(tv_1 + \frac{t^2}{2}(v_2 + \gamma(t))) + t^3\psi(t),$$

$\psi(t) \to 0, t \downarrow 0.$

Since $G'(x) = 0$, dividing by $t$ and then passing to limit when $t \downarrow 0$, yields $G''(x)(v_1)(v_1) = 0$, while after dividing by $t^3$ and letting $t \downarrow 0$, we get

$$G'''(x)(v_1)(v_1) + 3G''(x)(v_1)(v_2) = 0. \quad (2.15)$$

Conversely, assume that (2.13) and (2.14) are true and $G$ is 2-regular at $x$.

We show that the following formula holds:

For every $\eta > 0$, there exists $\delta > 0$ such that for any $t \in (0, \delta)$ and $q \in Y$, $\|q\| < \delta$, there exists $u \in X$, $\|u\| < \eta$ satisfying

$$G(x + tv_1 + \frac{t^2}{2!}(v_2 + u)) - G(x) = \frac{t^2}{2!}G''(x)(v_1)(v_1) +$$

$$+ \frac{t^3}{3!}(G'''(x)(v_1)(v_1) + 3G''(x)(v_1)(v_2) + q). \quad (2.15)$$

The hypothesis $G''(x)(v_1)$ is onto for each $v_1$ satisfying (2.13), implies the existence of a linear continuous operator $L : \mathbb{R}^m \to X$, with the property $G''(x)(v_1)L(y) = y$, $\forall y \in \mathbb{R}^m$. 


Let us fix $\eta > 0$. Because $L$ is continuous and $L(0) = 0$, there exists $\rho > 0$ such that $||Ly|| < \eta$ for all $y \in Y$ with $||y|| < \rho$.

$G$ is three times Fréchet differentiable at $x$, so

$$
\frac{1}{t^3}||G(x + ta) - G(x) - tG'(x)(a) - \frac{t^2}{2!}G''(x)[a]^2 - \frac{t^3}{3!}G'''(x)[a]^3|| \to 0
$$

(2.16)
as $t \downarrow 0$, uniformly with respect to $a$ on bounded sets of $X$.

For $a = v_1 + \frac{t}{2}(v_2 + u)$, (2.16) can be rewritten as

$$
\frac{1}{t^3}[G(x + tv_1 + \frac{t^2}{2}(v_2 + u)) - G(x) - tG'(x)(v_1 + \frac{t}{2}(v_2 + u)) - \\
\frac{t^2}{2!}G''(x)[v_1 + \frac{t}{2}(v_2 + u)]^2 - \frac{t^3}{3!}G'''(x)[v_1 + \frac{t}{2}(v_2 + u)]^3]\to 0
$$

(2.17)
as $t \downarrow 0$, uniformly with respect to $u$ on bounded sets.

So, (2.17) says that there is $\delta > 0$ such that

$$
\frac{2t}{t^3} || - G(x + tv_1 + \frac{t^2}{2}(v_2 + Ly)) + G(x) + \frac{t^2}{2!}G''(x)(v_1)(v_1) + \\
+ \frac{t^3}{3!}(G'''(x)(v_1)(v_1) + 3G''(x)(v_1)(v_2 + Ly))|| < \frac{\rho}{2}, \forall t \in (0, \delta), \forall y \in B_\rho.
$$

Next, one constructs the function $F : B_\rho \to \mathbb{R}^m$

$$
F(y) = \frac{2t}{t^3}[-G(x + tv_1 + \frac{t^2}{2}(v_2 + Ly)) + G(x) + \frac{t^2}{2!}G''(x)(v_1)(v_1) + \\
+ \frac{t^3}{3!}(G'''(x)(v_1)(v_1) + 3G''(x)(v_1)(v_2 + Ly))] + \frac{q}{3},
$$

(2.18)
for $y \in B_\rho$, $0 < t < \delta$, $q \in \mathbb{R}^m$, $||q|| < \delta < \frac{3\rho}{2}$.

Since $F$ is continuous and maps $B_\rho$ into itself, it has a fixed point $y_0$ (by Brouwer’s fixed point theorem in $\mathbb{R}^m$).

Taking into account that $G''(x)(v_1)(Ly_0) = y_0$, for $y = y_0$, (2.18) becomes

$$
G(x + tv_1 + \frac{t^2}{2}(v_2 + Ly_0)) = G(x) + \frac{t^2}{2!}G''(x)(v_1)(v_1) + \\
+ \frac{t^3}{3!}(G'''(x)(v_1)(v_1) + 3G''(x)(v_1)(v_2) + \frac{t^3}{2!}(y_0 - F(y_0)) + \frac{t^3}{3!}q,
$$

Therefore, (2.15) holds with $u = L(y_0)$.

Further, let $\eta > 0$ be arbitrary. Then, by taking $q = 0$ in (2.15), it is assured the existence of a mapping $u : (0, \varepsilon) \to X$, for some $\varepsilon > 0$, with $||u(t)|| < \eta$ and
\[ G(x + tv_1 + \frac{t^2}{2!}(v_2 + u(t))) = G(x) + \frac{t^2}{2!}G''(x)(v_1)(v_1) + \frac{t^3}{3!}(G'''(x)(v_1)(v_1) + 3G''(x)(v_1)(v_2)), \forall t \in (0, \varepsilon). \]

Since the right-hand side of the above equality is equal to 0 and \( u(t) \to 0 \) as \( t \downarrow 0 \), we draw the conclusion that \( v_2 \in T^2_xD_G \) with corespondent vector \( v_1 \in T_xD_G \).

Theorem 2.3 can be generalized as follows.

**Theorem 2.4.** Suppose that \( G : X \to \mathbb{R}^m \) is \((p + 1)\)-times Fréchet differentiable at \( x \in X \), with \( G(x) = G'(x) = \cdots = G^{(n-1)}(x) = 0, 1 < n < p \), and its \( n \)-th derivative continuous at \( x \), and \( G \) is continuous near \( x \) and \( n \)-regular at \( x \).

Then \( v_p \in T^p_xD_G \) with associated vectors \( v_i \in T^i_xD_G, 1 \leq i \leq p - 1 \), if and only if \( E_{nl}^G(x) = 0, \forall n \leq l \leq p + 1 \).

Here, \( E_{nl}^G(x) \) denotes the expression
\[ E_{nl}^G(x) = \sum_{k=n}^{l} \frac{l!}{k!} \sum_{1 \leq i_1, \ldots, i_k \leq l} \frac{1}{i_1! \cdots i_k!} G^{(k)}(x)(v_{i_1}) \cdots (v_{i_k}) \]

More precisely,
for \( l = 3, n = 2, E_{32}^G(x) = G'''(x)(v_1)(v_1)(v_1) + 3G''(x)(v_1)(v_2) \), and
for \( l = 4, n = 2, E_{42}^G(x) = G^{(4)}(x)(v_1)(v_1)(v_1)(v_1) + 6G'''(x)(v_1)(v_1)(v_2) + 4G''(x)(v_1)(v_2) + 3G'(x)(v_2)(v_2) \),
for \( l = n, E_{nn}^G(x) = G^{(n)}(x)[v_1]^n \).

**Proof.** The proof of the necessary conditions follows the same lines as in the proof of Theorem 2.3, while for the sufficient part, it can be demonstrated the following formula:

For \( \eta > 0 \), there exists \( \delta > 0 \) such that for any \( t \in (0, \delta) \) and \( q \in \mathbb{R}^m, \|q\| < \delta \), there exists \( u \in X, \|u\| < \eta \) satisfying
\[ G(x + tv_1 + \frac{t^2}{2!}v_2 + \cdots + \frac{t^p}{p!}(v_p + u)) = G(x) + \sum_{l=n}^{n+1} \frac{l!}{l!} E_{nl}^G(x) + \frac{t^{l+1}}{(p+1)!} q. \]

Indeed, the above formula holds with \( u = Ly_0 \), where \( y_0 \) is the fixed point of the continuous function \( F : B_{\rho} \to B_{\rho} \), given by
\[ F(y) = \frac{p!}{t^{p+1}} \left( -G(x + tv_1 + \frac{t^2}{2!}v_2 + \ldots + \frac{tp}{p!}(v_p + u) \right) + G(x) + \]
\[ + \sum_{l=n}^{p+1} \frac{t^l}{l!} E_{n}^{G}(x) \right) + \frac{q}{p + 1}, \|q\| < \delta < \frac{p(p + 1)}{2}, \text{ and } L \text{ is the inverse of the onto mapping} \]
\[ G^{(n)}(x)[v_1]^{n-1} : X \rightarrow \mathbb{R}^m, G^{(n)}(x)[v_1]^n = 0, v_1 \neq 0. \]
Chapter 3

Optimality Conditions

3.1 High Order Necessary Conditions

In this chapter we deal with the following optimization problem

\[
\inf_{x \in S} F(x) = F(x_0)
\]  \hspace{1cm} (P)

We will formulate necessary conditions for the existence of the minimum point when \( F : X \to \mathbb{R} \) is of class \( C^p \) in a neighborhood of \( x_0 \in S \subseteq X, \; S \neq \emptyset, \; p \geq 1 \).

N.H. Pavel et al. (32) stated some necessary conditions for extremum in terms of first and second order tangential directions to \( S \) at \( x \in S \), in the case when \( S \) is the kernel of a real-valued function.

**Theorem 3.1.1.** Let \( G \) and \( F \) be two functions from a normed space \( X \) into \( \mathbb{R} \) and let \( x_0 \) be a minimum point of \( F \) on \( D_G = G^{-1}(0) \), i.e.,

\[
\inf_{x \in D_G} F(x) = F(x_0)
\]  \hspace{1cm} (P')

Suppose \( F \) is of class \( C^2 \) in a neighborhood of \( x_0 \) (\( F \) not constant) and \( G \) is continuous in a neighborhood of \( x_0 \) and Fréchet differentiable at \( x_0 \) with \( G'(x_0) \) onto. Then (necessarily)
\[ F'(x_0)(v_1) = 0, \forall v_1 \in X \text{ with } G'(x_0)(v_1) = 0 \]  
(3.1.1)

(i.e., \( F'(x_0) = \lambda G'(x_0) \) for some \( \lambda \neq 0 \)),

\[ F'(x_0)(v_2) + F''(x_0)(v_1)(v_1) \geq 0 \]  
(3.1.2)

for all \((v_1, v_2)\) satisfying

\[ G'(x_0)(v_1) = 0, G'(x_0)(v_2) + G''(x_0)(v_1)(v_1) = 0. \]  
(3.1.3)

According to Theorem 1.2.2 and Theorem 1.2.3, \( T_{x_0}D_G = \text{Ker}G'(x_0) \) and

\[ T_{x_0}^2D_G = \{ v_2 \in X; \exists v_1 \text{ such that (3.1.3) holds} \}. \]

**Definition 3.1.1.** We say that a vector \( x_0 \in S \) is a local minimum for (P) if there exists an \( \varepsilon > 0 \) such that \( f(x_0) \leq f(x), \forall x \in S, ||x - x_0|| \leq \varepsilon \). It is a strict local minimum if there exists an \( \varepsilon > 0 \) such that \( f(x_0) < f(x), \forall x \in S, ||x - x_0|| \leq \varepsilon, x \neq x_0 \).

Next one derives some new necessary conditions of optimization for problem (P), involving higher order tangential directions.

**Theorem 3.1.2.** Let \( F : X \to \mathbb{R} \) be a function of class \( C^p \) in a neighborhood of \( x_0 \) which is a local minimum point of \( F \) on \( S \subset X \), i.e.,

\[ F(x_0) = \text{Local Minimum } F(x), \text{ subject to } x \in S. \]  
(P)

Then

i) \[ F'(x_0)(v_1) \geq 0, \forall v_1 \in T_{x_0}S. \]  
(3.1.4)

Also, one has the relations

ii) \[ F'(x_0)(v_2) + F''(x_0)(v_1)(v_1) \geq 0, \]  
(3.1.5)

\( \forall v_2 \in T_{x_0}^2S \) with the associated vector \( v_1 \in T_{x_0}S \) such that \( F'(x_0)(v_1) = 0 \),
iii) \[ F'''(x_0)(v_1)(v_1)(v_1) + 3F''(x_0)(v_1)(v_2) + F'(x_0)(v_3) \geq 0, \]  
(3.1.6)

\[ \forall v_3 \in T^3_{x_0}S \text{ with associated vectors } v_1 \in T_{x_0}S \text{ and } v_2 \in T^2_{x_0}S, \text{ having the property } \]
\[ F'(x_0)(v_1) = 0 \text{ and } F'(x_0)(v_2) + F''(x_0)(v_1)(v_1) = 0, \]  
(3.1.7)

iv) \[ F^{(4)}(x_0)(v_1)(v_1)(v_1)(v_1) + 6F'''(x_0)(v_1)(v_1)(v_2) + 4F''(x_0)(v_1)(v_3) + \]
\[ + 3F''(x_0)(v_2)(v_2) + F'(x_0)(v_4) \geq 0 \]  
(3.1.8)

whenever \( v_4 \in T^4_{x_0}S \) with the associated vectors \( v_i \in T^i_{x_0}S, 1 \leq i \leq 3 \), satisfying (3.1.7) and

\[ F'''(x_0)(v_1)(v_1) + 3F''(x_0)(v_1)(v_2) + F'(x_0)(v_3) = 0. \]

In general,

\[ S^F_p(x_0) \geq 0, \forall v_1, v_2, ..., v_p \in X \text{ such that } \]
\[ x_0 + tv_1 + \frac{t^2}{2}v_2 + ... + \frac{t^p}{p!}(v_p + \gamma(t)) \in S, \forall t > 0, \gamma(t) \rightarrow 0 \text{ as } t \downarrow 0 \text{ and } \]
\[ S^F_p(x_0) = 0, \text{ for every } n, 0 < n \leq p - 1. \]

The expression \( S^F_m(x_0) \) is defined in the same way as \( S^G_m(x) \), \( m \geq 1 \).

\[ S^F_m(x_0) = \sum_{k=1}^{m} \frac{m!}{k!} \sum_{i_1, ..., i_k \in \{1, ..., m\}} \frac{1}{i_1!i_2!...i_k!} F^{(k)}(x_0)(v_{i_1}...v_{i_k}). \]

**Proof.** We use the fact that \( v_4 \in T^4_{x_0}S \) and \( v_i \in T^i_{x_0}S, 1 \leq i \leq 3 \) are its associated vectors, is equivalent to the existence of a map \( \gamma_1(t) \rightarrow 0 \) as \( t \downarrow 0 \) and

\[ x_0 + tv_1 + \frac{t^2}{2}v_2 + \frac{t^3}{3!}v_3 + \frac{t^4}{4!}(v_4 + \gamma_1(t)) \in S, \forall t > 0. \]

Since \( x_0 \) is a minimum point of \( F \) on \( S \), there is \( \varepsilon > 0 \) such that

\[ F(x_0 + tv_1 + \frac{t^2}{2}v_2 + \frac{t^3}{3!}v_3 + \frac{t^4}{4!}(v_4 + \gamma_1(t))) - F(x_0) \geq 0, \forall 0 < t \leq \varepsilon. \]

In view of Taylor’s formula, the right-hand side can be written as
For all $\alpha$ such that $0 < \alpha$, for $\alpha > 0$ we can give now some sufficient conditions for an extremum point to be a maximum or a minimum.

Let $F$ be as in the hypothesis of Theorem 3.1.2 and $x_0$ a local extremum point of $F$ on $S$.

Then, Theorem 3.1.2 leads to the following conclusions:

i) If $F^{(4)}(x_0)(v_1)(v_1)(v_1)(v_1) + 6F''(x_0)(v_1)(v_1)(v_2) + 4F''(x_0)(v_1)(v_3) + 3F''(x_0)(v_1)(v_1) + 6F''(x_0)(v_1)(v_1)(v_2) + 4F''(x_0)(v_1)(v_3) + 3F''(x_0)(v_1)(v_3) + 3F''(x_0)(v_3)(v_2) + F'(x_0)(v_4)) > 0$, $0 < t \leq \epsilon$,

where $\alpha(t), \alpha_1(t) \to 0$ as $t \downarrow 0$.

Dividing by $t$ and letting $t$ go to zero through positive values, one gets $F'(x_0)(v_1) \geq 0$, for $v_1 \in T_{x_0}S$. If $F'(x_0)(v_1) = 0$, one divides by $t^2$ and obtains (3.1.5) when $t \downarrow 0$.

Taking into account that (3.1.7) holds, after dividing by $t^3$ and letting $t$ go to 0, it can be seen that (3.1.6) occurs too. Now, (3.1.8) follows in the same way as (3.1.5) and also the general case can be proved analogously. This ends the proof.

We can give now some sufficient conditions for an extremum point to be a maximum or a minimum.

Let $F$ be as in the hypothesis of Theorem 3.1.2 and $x_0$ a local extremum point of $F$ on $S$.

Then, Theorem 3.1.2 leads to the following conclusions:

i) If $F^{(4)}(x_0)(v_1)(v_1)(v_1)(v_1) + 6F''(x_0)(v_1)(v_1)(v_2) + 4F''(x_0)(v_1)(v_3) + 3F''(x_0)(v_1)(v_1) + 6F''(x_0)(v_1)(v_1)(v_2) + 4F''(x_0)(v_1)(v_3) + 3F''(x_0)(v_1)(v_3) + 3F''(x_0)(v_3)(v_2) + F'(x_0)(v_4)) > 0$, $0 < t \leq \epsilon$,

for all $v_4 \in T_{x_0}S$ with the associated vectors $v_i \in T_{x_0}S, 1 \leq i \leq 3$, $(v_1, \ldots, v_4) \neq 0$ such that

$F'(x_0)(v_1) = 0$, $F'(x_0)(v_2) + F'(x_0)(v_1)(v_1) = 0$, and
\[ F'''(x_0)(v_1)(v_1)(v_1) + 3F''(x_0)(v_1)(v_2) + F'(x_0)(v_3) = 0, \]
then \( x_0 \) is a local maximum point (this conclusion follows by contradiction on the basis of (3.1.8)).

ii) If \( F^{(4)}(x_0)(v_1)(v_1)(v_1)(v_1) + 6F'''(x_0)(v_1)(v_1)(v_2) + 4F''(x_0)(v_1)(v_3) + 3F''(x_0)(v_2)(v_2) + F'(x_0)(v_4) > 0, \)
for all \( v_i \) as above, then \( x_0 \) is a local minimum point, due to the fact that if it was a local maximum point of \( F \) on \( S \), then it could be shown analogously to the proof of Theorem 3.1.2 that this expression should be less than or equal to zero.

A general result can be deduced in the same way.

**Corollary 3.1.1.** Let \( F : X \to \mathbb{R} \) be a function which is of class \( C^p \) in a neighborhood of its local extremum point \( x_0 \) on \( S \subseteq X, S \neq \emptyset \).

i) If \( S^F_p(x_0) < 0 \), for all \( v_1, \ldots, v_p \in X, (v_1, \ldots, v_p) \neq 0 \) such that
\[ x_0 + tv_1 + \frac{t^2}{2}v_2 + \ldots + \frac{p^p}{p!}(v_p + \gamma(t)) \in S, \forall t > 0, \gamma(t) \to 0 \text{ as } t \downarrow 0, \text{ and } \]
\[ S^F_j(x_0) = 0, \forall 1 \leq j \leq p - 1, \]
then \( x_0 \) is a local maximum point for \( F \) on \( S \).

ii) If \( S^F_p(x_0) > 0 \), for all \( v_1, \ldots, v_p \) as above,
then \( x_0 \) is a local minimum point for \( F \) on \( S \).

Considering the characterization of the \( p \)-th order tangent cone to \( D_G \), as developed in Corollary 2.1, in the case of the problem \( (P') \), Theorem 3.1.2 becomes

**Corollary 3.1.2.** Let \( F \) and \( G \) be two functions on a normed space \( X, F : X \to \mathbb{R}, G : X \to \mathbb{R}^m \) and let \( x_0 \) be a minimum point of \( F \) on \( D_G = \{x \in X, G(x) = 0\} \), i.e.,
\[ F(x_0) = \text{Local Minimum } F(x) : \text{ subject to } x \in D_G \]
(\( P' \))
If $F$ is of class $C^4$ in a neighborhood of $x_0$ and $G$ is continuous near $x_0$, four times Frechét differentiable at $x_0$, $G'(x_0)$ is onto, and
\[
F'(x_0)(v_1) = 0, \quad F''(x_0)(v_2) + F'''(x_0)(v_1)(v_1) = 0, \quad (3.1.9)
\]
\[
F'''(x_0)(v_1)(v_1) + 3F''(x_0)(v_1)(v_2) + F'(x_0)(v_3) = 0, \quad (3.1.10)
\]
for all $v_1, v_2$ and $v_3$ for which
\[
G'(x_0)(v_1) = 0, \quad G'(x_0)(v_2) + G''(x_0)(v_1)(v_1) = 0, \quad \text{and} \quad (3.1.11)
\]
\[
G''(x_0)(v_1)(v_1) + 3G''(x_0)(v_1)(v_2) + G'(x_0)(v_3) = 0 \quad (3.1.12)
\]
then
\[
F^{(4)}(x_0)(v_1)(v_1)(v_1) + 6F'''(x_0)(v_1)(v_1)(v_2) + 4F''(x_0)(v_1)(v_3) + 3F''(x_0)(v_2)(v_2) + F'(x_0)(v_4) \geq 0 \quad (3.1.13)
\]
whenever $v_1, v_2, v_3, v_4$ satisfy
\[
G^{(4)}(x_0)(v_1)(v_1)(v_1) + 6G'''(x_0)(v_1)(v_1)(v_2) + 4G''(x_0)(v_1)(v_3) + 3G''(x_0)(v_2)(v_2) + G'(x_0)(v_4) = 0, \quad (3.1.14)
\]
besides (3.1.11) and (3.1.12).

In general, if $F$ is $p$ times Frechét differentiable in a neighborhood of $x_0$ and
\[
S^F_n(x_0) = 0,
\]
for every $n, 1 \leq n \leq p - 1$, and for all $v_1, v_2, \ldots, v_{p-1} \in X$ such that
\[
S^G_n(x_0) = 0, \quad 1 \leq n \leq p - 1,
\]
then
\[
S^F_p(x_0) \geq 0, \quad \forall \; v_1, v_2, \ldots, v_p \in X \text{ such that } \quad S^G_n(x_0) = 0, \quad 1 \leq n \leq p.
\]
Proof. The proof follows from the previous theorem considering the structure of the $p$-th order tangent cone $T_{x_0}^p D_G$, provided by Corollary 2.1.

Remark 3.1.1. If $x_0$ is a local maximum point of $F$ on $S$ then similar conclusions to those of theorems 3.1.2 and 3.1.3 can be obtained, the inequalities involved being reversed.
3.2 High Order Sufficient Conditions

In this chapter we give some sufficient conditions for optimality, in the case of problem (P'), when $F$ and $G$ are defined on a finite dimensional normed space.

**Theorem 3.2.1.** Let $F$ and $G$ be two functions on a finite dimensional normed space $X$, $F : X \to \mathbb{R}, G : X \to \mathbb{R}^m$.

If $F$ and $G$ are of class $C^p$ in a neighborhood of $x_0 \in D_G = \{x \in X; G(x) = 0\}$, $p \geq 1$,

$$F^{(l)}(x_0) = G^{(l)}(x_0) = 0, \forall 0 \leq l \leq n - 1,$$

$G^{(n)}(x_0)$ is not identically zero, and there exists $\lambda \in \mathbb{R}^m$ such that

$$[F^{(l)}(x_0) - \lambda G^{(l)}(x_0)] = 0, \forall n \leq l \leq p - 1,$$

$$[F^{(p)}(x_0) - \lambda G^{(p)}(x_0)][y]^p > 0,$$

for every $y \neq 0$ such that

$$G^{(n)}(x_0)[y]^n = 0,$$

then $x_0$ is a strict local minimum point of $F$ on $D_G$.

It can be assumed without loss of generality that $F(x_0) = 0$.

**Proof.** We can show that $F(x) - F(x_0) \geq \alpha ||x - x_0||^p$, for some $\alpha > 0$ and for all $x \in D_G$ in a neighborhood of $x_0$.

Assume by contradiction, that there exists a sequence $\{x_k\}_{k \geq 1}$ such that $x_k \to x_0$, and for all $k$, $x_k \neq x_0$, $G(x_k) = 0$ and

$$F(x_k) < F(x_0) + \frac{1}{k}||x_k - x_0||^p.$$
Let us write $x_k - x_0 = \delta_k y_k$, $y_k = \frac{x_k - x_0}{||x_k - x_0||}$, $\delta_k = ||x_k - x_0|| \to 0$ as $k \to \infty$. The sequence $\{y_k\}_{k \geq 1}$ is bounded, so it must have a subsequence converging to some $y$ with $||y|| = 1$. Without loss of generality, we assume that the whole sequence $\{y_k\}_{k \geq 1}$ converges to $y$.

Using Taylor’s formula, we get the following two relations

\[
\frac{1}{k}||x_k - x_0||^p > F(x_k) - F(x_0) = \delta_k F'(x_0)(y_k) + \frac{\delta_k^2}{2!} F''(x_0)(y_k)(y_k) + \\
+ \frac{\delta_k^3}{3!} F'''(x_0)(y_k)(y_k)(y_k) + \ldots + \frac{\delta_k^p}{p!} F^{(p)}(x_0)[y_k]^p + \delta_k^p \tilde{r}_k
\]

(3.2.1)

\[
0 = G(x_k) - G(x_0) = \delta_k G'(x_0)(y_k) + \frac{\delta_k^2}{2} G''(x_0)(y_k)(y_k) + \\
+ \frac{\delta_k^3}{3!} G'''(x_0)(y_k)(y_k)(y_k) + \ldots + \frac{\delta_k^p}{n!} G^{(n)}(x_0)[y_k]^n + \delta_k^p \tau_k
\]

(3.2.2)

where $\tilde{r}_k, \tau_k \to 0$ as $k \to \infty$.

By taking the limit as $\delta_k \to 0$ in the relation

\[
0 = \frac{G(x_k) - G(x_0)}{\delta_k^n} = G^{(n)}(x_0)[y_k]^n + \tilde{r}_k, \tau_k \to 0, k \to \infty,
\]

we obtain $G^{(n)}(x_0)[y]^n = 0$.

We subtract (3.2.2) multiplied by $\lambda$ from (3.2.1), to get

\[
\frac{1}{k}||x_k - x_0||^p > \delta_k (F'(x_0) - \lambda G'(x_0))(y_k) + \\
+ \frac{\delta_k^2}{2!} (F''(x_0)(y_k)(y_k) - \lambda G''(x_0)(y_k)(y_k)) + \\
+ \frac{\delta_k^3}{3!} (F'''(x_0)(y_k)(y_k)(y_k) - \lambda G'''(x_0)(y_k)(y_k)(y_k)) + \ldots \\
+ \frac{\delta_k^p}{p!} (F^{(p)}(x_0)[y_k]^p - \lambda G^{(p)}(x_0)[y_k]^p) + \delta_k^p \tilde{r}_k, \tilde{r}_k \to 0, k \to \infty.
\]

Since $F^{(n)}(x_0) - \lambda G^{(n)}(x_0), \ldots, F^{(p)}(x_0) - \lambda G^{(p)}(x_0), F'(x_0), G'(x_0), \ldots, F^{(n-1)}(x_0), G^{(n-1)}(x_0)$ are identically zero, after dividing by $\delta_k^p$ and letting $k$ go to infinity, we have

\[
0 \geq [F^{(p)}(x_0) - \lambda G^{(p)}(x_0)][y]^p,
\]

On the other hand, by hypothesis $[F^{(p)}(x_0) - \lambda G^{(p)}(x_0)][y]^p$ is strictly positive because $G^{(n)}(x_0)[y]^n = 0$ and $y \neq 0$ as $||y|| = 1$. This contradiction shows that $x_0$ is a strict local minimum point for $F$ on $D_G$. 
Theorem 3.2.2. Let $F$ and $G$ be two functions on a finite dimensional normed space $X$, $F : X \to \mathbb{R}$, $G : X \to Z$, $Z$ being a linear topological vector space.

If $F$ is of class $C^p$ in a neighborhood of $x_0 \in G^{-1}(M) = \{x \in X; G(x) \in M\}$, where $M$ is a closed, linear subspace of $Z$, $G$ is of class $C^n$ in a neighborhood of $x_0$, $p, n \geq 1$,

$$G'(x_0) = \ldots = G^{(n-1)}(x_0) = F'(x_0) = \ldots = F^{(p-1)}(x_0) = 0,$$

$G^{(n)}(x_0)$ is not identically zero and

$$F^{(p)}(x_0)[y]^p > 0,$$

for every $y \neq 0$ such that $G^{(n)}(x_0)[y]^n \in M,$

then $x_0$ is a strict local minimum point of $F$ on $G^{-1}(M)$.

Proof. We prove that $F(x) - F(x_0) \geq \alpha ||x - x_0||^p$, for some $\alpha > 0$ and for all $x \in G^{-1}(M)$ in a neighborhood of $x_0$.

Let us assume by contradiction that there exist some sequences $x_k, y_k$ and $\delta_k$ defined as in the previous theorem.

According to Taylor’s formula

$$G(x_k) - G(x_0) = \delta_k G'(x_0)(y_k) + \frac{\delta_k^2}{2!} G''(x_0)(y_k)(y_k) + \ldots + \frac{\delta_k^n}{n!} G^{(n)}(x_0)[y_k]^n + \delta_k^n r_k \in M,$$

where $r_k \to 0$ as $k \to \infty$.

Taking into account that $G'(x_0) = \ldots = G^{(n-1)}(x_0) = 0$, after dividing by $\delta_k^p$ and letting $k$ tend to $\infty$, we get $G^{(n)}(x_0)[y]^n \in M$, since $M$ is a closed, linear subspace of $Z$.

Next, we divide the following inequality by $\delta_k^p$

$$\frac{1}{k} ||x_k - x_0||^p > F(x_k) - F(x_0) = \delta_k F'(x_0)(y_k) + \frac{\delta_k^2}{2!} F''(x_0)(y_k)(y_k) + \ldots + \frac{\delta_k^n}{n!} F^{(n)}(x_0)[y_k]^n + \delta_k^n \tilde{r}_k,$$

where $\tilde{r}_k \to 0$ as $k \to \infty$.

Since $F'(x_0), F''(x_0), \ldots, F^{(p-1)}(x_0)$ are identically zero, when $k \to \infty$, we obtain

$$0 \geq F^{(p)}(x_0)[y]^p.$$
We arrive at a contradiction because, by hypothesis,

\[ F^{(p)}(x_0)[y]^p > 0, \text{ for any } y \neq 0 \text{ with } G^{(n)}(x_0)[y]^n \in M. \]
3.3 Applications

In this chapter we give several examples which illustrate how theorems 3.1.2, 3.2.1, 3.2.2 can be applied to analyze wide classes of problems that could not be investigated before.

The first example deals with an application of our fourth order necessary conditions when the gradient of the function $G$ is zero and of our sufficient conditions given by Theorem 3.1.2.

Example 3.3.1. Let us analyze the critical point $(0, 0)$ of the function $F(x_1, x_2) = x_1^4 + x_2^5$, subject to $G(x_1, x_2) = x_1^3 + x_1 x_2 + x_2^2 = 0$, $F, G : \mathbb{R}^2 \to \mathbb{R}$.

First we will determine the explicit form of the derivatives of $F$.

Denoting by $\varphi$ the function $t \to F'(x + tv)$, where $x = (x_1, x_2)$, $v = (v_1, v_2)$, we have

$$F''(x)(v) = \varphi'(0) = \begin{pmatrix} F_{x_1x_1}(x)v_1 + F_{x_1x_2}(x)v_2 \\ F_{x_2x_1}(x)v_1 + F_{x_2x_2}(x)v_2 \end{pmatrix}, \text{ and}$$

$$F''(x)(v)(v) = F_{x_1x_1}(x)v_1^2 + 2F_{x_1x_2}(x)v_1v_2 + F_{x_2x_2}(x)v_2^2$$

and, similarly,

$$F''(x)(v)(w) = F_{x_1x_1}(x)v_1w_1 + F_{x_1x_2}(x)v_1w_2 + F_{x_2x_2}(x)v_2w_2.$$

Analogously, we get that

$$F'''(x)(v)(v)(v) = F_{x_1x_1x_1}(x)v_1^3 + 3F_{x_1x_1x_2}(x)v_1^2v_2 + 3F_{x_1x_2x_2}(x)v_1v_2^2 + F_{x_2x_2x_2}(x)v_2^3,$$

$$F'''(x)(v)(v)(w) = F_{x_1x_1x_1}(x)v_1^3w_1 + 2F_{x_1x_1x_2}(x)v_1^2v_2w_1 + F_{x_1x_2x_2}(x)v_1^2v_2^2w_1 + F_{x_2x_2x_2}(x)v_1v_2^3w_1 + F_{x_2x_2x_2}(x)v_2^3w_2,$$

$$F^{(4)}(x)(v)(v)(v)(v) = F_{x_1x_1x_1x_1}(x)v_1^4 + 4F_{x_1x_1x_1x_2}(x)v_1^3v_2 + 6F_{x_1x_1x_2x_2}(x)v_1^2v_2^2 + 4F_{x_1x_2x_2x_2}(x)v_1v_2^3 + F_{x_2x_2x_2x_2}(x)v_2^4.$$
Let us return to our example.

Here \( F'(x_1, x_2) = \left( 4x_1^3, 5x_2^4 \right) \), \( F_{x_1x_1}(x) = 12x_1^2, F_{x_1x_2}(x) = 0, \)
\( F_{x_2x_2}(x) = 20x_2^3, F_{x_1x_1x_1}(x) = 24x_1, F_{x_1x_2x_2}(x) = 0, F_{x_1x_2x_1}(x) = 0, \)
\( F_{x_2x_2x_2}(x) = 60x_2^2, F_{x_1x_1x_1x_1}(x) = 24, F_{x_1x_1x_1x_2}(x) = 0, F_{x_1x_1x_2x_2}(x) = 0, \)
\( F_{x_1x_2x_2x_2}(x) = 0, F_{x_2x_2x_2x_2}(x) = 120x_2, G'(x_1, x_2) = (3x_1^2 + x_2, x_1 + 2x_2), \forall x = (x_1, x_2) \in \mathbb{R}^2. \) We notice that the point \((0, 0)\) verifies the constraint \(G(x_1, x_2) = 0\) and also it is a solution of the equation \( F'(x_1, x_2) = 0. \) Therefore \((0, 0)\) is a candidate for an extremum point.

We have that
\[
T_{(0, 0)} DG \subset \{(v_{11}, v_{12}) \in \mathbb{R}^2, v_{11}v_{12} + v_{12}^2 = 0\}.
\]

Indeed, if \((v_{11}, v_{12}) \in T_{(0, 0)} DG\) then there exists a mapping \(r(t) = (r_1(t), r_2(t)), \)
\( r(t) \to 0 \) as \( t \downarrow 0 \) such that \(tv_1 + tr(t) \in D_G \forall t > 0, \) which means that
\[
(tv_{11} + tr_1(t))^3 + (tv_{12} + tr_2(t))^2 + (tv_{11} + tr_1(t))(tv_{12} + tr_2(t)) = 0, \forall t > 0.
\]

After dividing by \( t^2 \) and letting \( t \) go to \( 0, \) we get \( v_1v_2 + v_2^2 = 0. \)

We find that
\[
F'(0, 0)(v_2) + F''(0, 0)(v_1)(v_1) = 0,
\]
\forall v_2 \in T_{(0, 0)}^2 DG with correspondent vector \( v_1 \in T_{(0, 0)} DG, \) as \( F_{x_1x_1}(0, 0) = F_{x_1x_2}(0, 0) = F_{x_2x_2}(0, 0) = 0, \)
\( F''(0, 0)(v_1)(v_1) + 3F''(0, 0)(v_1)(v_2) + F'(0, 0)(v_3) = 0, \)
since all the third order partial derivatives of \( F \) at \((0, 0)\) are equal to zero and, finally
\[
F^{(4)}(0, 0)(v_1)(v_1)(v_1)(v_1) + 6F''(0, 0)(v_1)(v_2) + 4F'(0, 0)(v_3) + 3F''(0, 0)(v_2)(v_2) + F'(0, 0)(v_4) = 24v_{11}^4,
\]
because all the derivatives involved are equal to zero at \((0, 0)\), except for \( F_{x_1x_1x_1x_1}(0, 0) = 24. \)
The result obtained is strictly positive for \( v_1 \neq 0 \), because if we assume that there exists \( v_1 \in T_{(0,0)}D_G \) with \( v_{12} = 0 \), then the relation \( v_{11}v_2 + v_{22}^2 = 0 \) would imply \( v_{12} = 0 \) too, which contradicts the fact that \( v_1 \neq 0 \).

Corollary 3.1.1 ensures us that \((0,0)\) is a candidate for a local minimum point of \( F \) on \( D_G \).

Moreover, the hypothesis of Theorem 3.2.2 are satisfied. Indeed,
\[
F'(0,0) = F''(0,0) = F'''(0,0) = 0 \text{ and } F^{(4)}(0,0)(y)(y)(y) = 24y_1^4 > 0 \text{ for every } y = (y_1, y_2), y \neq 0 \text{ such that } G''(0,0)(y)(y) = 0, \text{ i.e., for all } y \neq 0 \text{ with the property that } y_1y_2 + y_2^2 = 0.
\]

By virtue of Theorem 3.2.2, we conclude that \((0,0)\) is a strict local minimum point of \( F \) on \( D_G \) (here \( M = \{0\} \)).

In this example, the first derivatives of the functions \( F \) and \( G \) at \((0,0)\) are identically zero. For any \( v_1 \in T_{(0,0)}D_G \), \( G''(0,0)(v_1) \) is onto so, by Theorem 1.2.2, \( T_{(0,0)}D_G = \text{Ker } G''(0,0) = \{(v_{11}, v_{12}) \in \mathbb{R}^2, v_{11}v_{12} + v_{22}^2 = 0\} \), and, also, the higher order tangent cones can be determined. For example,
\[
T^2_{(0,0)}D_G = \{(v_{21}, v_{22}) \in \mathbb{R}^2, \text{ for which there exists } (v_{11}, v_{12}) \neq (0,0) \text{ with } v_{11}v_{12} + v_{22}^2 = 0, \text{ such that } 2v_{11}^3 + v_{11}v_{22} + v_{12}v_{21} + 2v_{12}v_{22} = 0\}.
\]

In the sequel, we present an example which demonstrates the applicability of Corollary 3.1.2. We point out that in this example the tangent cones can be characterized explicitly with the aid of Corollary 2.1, as \( G'(0,0) \) is onto.

**Example 3.3.2.** Let us consider the function \( F(x_1, x_2) = x_1^4 + 4x_1 + 4x_2 \), subject to \( G(x_1, x_2) = x_1^5 - x_2^5 + x_1 + x_2 = 0 \), \( F, G : \mathbb{R}^2 \rightarrow \mathbb{R} \).

We notice that the point \((0,0)\) verifies the constraint \( G(x_1, x_2) = 0 \) and also \((x_1, x_2, \lambda) = (0, 0, 4)\) is a solution of the equation \( F'(x_1, x_2) = \lambda G'(x_1, x_2) \), i.e., of the system
\[
G'(x_1, x_2) = (5x_1^4 + 1, -5x_2^4 + 1), \forall (x_1, x_2) \in \mathbb{R}^2.
\]
Therefore, \((0, 0)\) is a candidate for an extremum point.

As the Hessian matrices \(F''(0, 0)\) and \(G''(0, 0)\) have all the entries zero, the classical criteria are not applicable to this example.

Next we will apply Corollary 3.1.2 to the critical point \((0, 0)\).

Clearly,

\[
F_{x_1x_1}(x_1, x_2) = 12x_1^2, \quad F_{x_1x_1x_1}(x_1, x_2) = 24x_1, \quad F_{x_1x_1x_1x_1}(x_1, x_2) = 24,
\]

and all the other partial derivatives of \(F\) evaluated at any \((x_1, x_2)\) \(\in \mathbb{R}^2\) are equal to zero. In particular, the only nonzero partial derivatives of \(F\) at \((0, 0)\) are \(F_{x_2}(0, 0) = 4\) and \(F_{x_1x_1x_1x_1}(0, 0) = 24\).

Also, \(G_{x_1x_1}(x_1, x_2) = 20x_1^3, \quad G_{x_2x_2}(x_1, x_2) = -20x_2^3, \quad G_{x_1x_1x_1}(x_1, x_2) = 60x_1^2, \quad G_{x_2x_2x_2}(x_1, x_2) = -60x_2^2, \quad G_{x_1x_1x_1x_1}(x_1, x_2) = 120x_1, \quad G_{x_2x_2x_2x_2}(x_1, x_2) = -120x_2\) and all the other second, third and fourth order partial derivatives of \(G\) at \((0, 0)\) are equal to zero. Obviously, \(G_{x_1}(0, 0) = G_{x_2}(0, 0) = 1\).

The first and second order tangent cones to \(D_G\) at \((0, 0)\) are expressed by (3.1.11), which in our case reduces to

\[
T_{(0,0)}D_G = \{v_1 = (v_{11}, v_{12}), \quad v_{11} + v_{12} = 0\}, \quad \text{and}
\]

\[
T^2_{(0,0)}D_G = \{v_2 = (v_{21}, v_{22}), \quad v_{21} + v_{22} = 0\}.
\]

In this situation,

\[
F'(0, 0)(v_1) = 4(v_{11} + v_{12}) = 0, \quad \forall v_1 \in T_{(0,0)}D_G, \quad \text{and}
\]

\[
F'(0, 0)(v_2) + F''(0, 0)(v_1)(v_1) = 4(v_{21} + v_{22}) = 0,
\]

for any \(v_2 \in T^2_{(0,0)}D_G\), with associated vector \(v_1 \in T_{(0,0)}D_G\).

We refer to Example 3.3.1 for the explicit form of the derivatives of \(F\).

Taking into account that \(v_1, v_2, v_3 = (v_{31}, v_{32}), v_4 = (v_{41}, v_{42})\) given by (3.1.11), (3.1.12) and (3.1.14) are characterized by the relations \(v_{11} + v_{12} = 0, \quad v_{21} + v_{22} = 0, \quad v_{31} + v_{32} = 0, \quad v_{41} + v_{42} = 0, \quad \text{the expression in (3.1.10) becomes}

\[
F'''(0, 0)(v_1)(v_1)(v_1) + 3F''(0, 0)(v_1)(v_2) + F'(0, 0)(v_3) = 4(v_{31} + v_{32}) = 0,
\]
whenever \( v_3 \in T^3_{(0,0)} D_G \) with correspondent vectors \( v_2 \in T^2_{(0,0)} D_G \) and \( v_1 \in T_{(0,0)} D_G \), while the expression in (3.1.13) can be simplified to
\[
F^{(4)}(0,0)(v_1)(v_1)(v_1)(v_1) + 6F''''(0,0)(v_1)(v_1) + 4F''''(0,0)(v_1) + 3F''(0,0)(v_2)(v_2) + F'(0,0)(v_4) = 24v_{11}^2 + 4(v_{41} + v_{42}) = 24v_{11}^2 > 0,
\]
for all \( v_1, v_2, v_3 \) and \( v_4 \) as above, \( v_1 \neq 0 \), as \( v_{11} = 0 \) would yield \( v_{12} = 0 \), too.

This ensures us that \((0,0)\) is a candidate for a local minimum point of \( F \) on \( D_G \).

**Remark 3.2.1.** Making use of Theorem 3.1.2 and Corollary 2.1, it can be shown that the critical point \((0,0)\) of the function \( F(x_1, x_2) = x_1^2 + 4x_1 + 4x_2 \), subject to the constraint \( G(x_1, x_2) = x_1^{m+1} + x_2^{m+1} + x_1 + x_2 = 0 \), \((x_1, x_2) \in \mathbb{R}^2, m \geq 3\), is a candidate for a local minimum point. Theorem 3.1.2 can confirm that \((0,0)\) is a strict local minimum point for \( F \) on \( D_G \).

In the following example we show how our necessary conditions allow us to exclude some of the critical points which cannot be extremum points. We determine the first order tangent cone using Theorem 1.2.2 due to Tret’yakov and the second order tangent cone to \( D_G \) using our Theorem 2.3.

**Example 3.3.3.** Let us minimize the function \( F(x_1, x_2) = x_1^4 + x_1 x_2^2 + x_2^3 \) subject to \( G(x_1, x_2) = x_1^3 + x_1 x_2 + x_2^2 = 0 \), \( F, G : \mathbb{R}^2 \rightarrow \mathbb{R} \).

The only critical point, i.e., the only solution of the equation \( F'(x_1, x_2) = 0 \), which satisfies the constraint is \((0,0)\).

In Example 3.3.1, we have found the first and second order tangent cones to \( D_G \), namely,
\[
T_{(0,0)} D_G = Ker \ G''(0,0) = \{ (v_{11}, v_{12}) \in \mathbb{R}^2, v_{11}v_{12} + v_{12}^2 = 0 \},
\]
The relations that hold between the components of that vectors $v_1$ and $v_2$ allow us to conclude that the fourth order expression does not have constant sign for all $v_2 \in T_{(0,0)}^2 D_G$ with correspondent $v_1 \in T_{(0,0)} D_G$, which by Theorem 3.1.2 means that $(0,0)$ is not either a minimum or a maximum of $F$ on $D_G$.

**Example 3.3.4.** We consider the integral $F(x) = \int_0^1 x(t)(x'(t))^4 dt$, where

$$x \in S = \{x \in C^1[0,1], x(0) = x(1) = 1\}$$

and

$$||x||_{C^1[0,1]} = \sup_{t \in [0,1]} |x(t)| + \sup_{t \in [0,1]} |x'(t)|.$$
First, we will characterize the tangent cone to $S$ at an element $x \in S$.

We show that

$$ T_x S = \{ v \in C^1[0, 1], v(0) = v(1) = 0 \} := M. $$

The inclusion $M \subset T_x S$ is almost trivial.

Indeed, if $v \in M$ then $x + hv \in S$, $h > 0$ because $(x + hv)(0) = x(0) + hv(0) = 1$, $(x + hv)(1) = 1$, and $x + hv$ belongs to $C^1[0, 1]$ since $x, v \in C^1[0, 1]$.

Conversely, if $v \in T_x D$ then $x + hv + hr(h) \in S$ for some mapping $r : [0, 1] \to \mathbb{R}$, $r(h) \to 0$ as $h \downarrow 0$, so we have $(x + hv + hr(h))(0) = 1$, which together with $x(0) = 1$, gives $hv(0) + hr(h)(0) = 0$. After dividing by $h$ and then passing to limit when $h \downarrow 0$ in $v(0) + r(h)(0) = 0$, we get $v(0) = 0$ because $|r(h)||_{C^1[0, 1]} \to 0$, i.e.,

$$ \sup_{0 \leq t \leq 1} |r(h)(t)| \to 0 \text{ as } h \downarrow 0, \forall t \in [0, 1]. $$

Similarly, it can be obtained that $v(1) = 0$.

Next, we will turn our attention to the derivatives of

$$ F(x) = \int_0^1 f(x(t), x'(t))dt, \text{ where } f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ is of class } C^4 \text{ on } [0, 1]. $$

We denote by $\varphi$ the function $\lambda \to F(x + \lambda v)$ so

$$ \varphi(\lambda) = F(x + \lambda v) = \int_0^1 f(x(t) + \lambda v(t), x'(t) + \lambda v'(t))dt \geq F(x) = \varphi(0). $$

$$ F'(x)(v) = \lim_{\lambda \to 0} \frac{F(x + \lambda v) - F(x)}{\lambda} = \lim_{\lambda \to 0} \frac{\varphi(\lambda) - \varphi(0)}{\lambda} = \varphi'(0) \text{ and } $$

$$ \varphi'(\lambda) = \int_0^1 [f_{x + \lambda v}(x + \lambda v, x' + \lambda v')v + f_{x' + \lambda v'}(x + \lambda v, x' + \lambda v')v']dt $$

By virtue of Fermat’s theorem, a necessary condition for $\varphi(\lambda)$ to have a minimum when $\lambda = 0$ is the vanishing of the derivative $\varphi'(\lambda)$ when $\lambda = 0$,

$$ F'(x)(v) = \varphi'(0) = 0 = \int_0^1 (f_x v + f_{x'} v')dt. $$

Obviously, $F'(x_1) = 0$, where $x_1(t) = 1, \forall t \in [0, 1]$, and $x_1 \in S$, too.

In the above equation the derivative $v'(t)$ appears along with the function $v(t)$.

We can eliminate $v'(t)$ by integrating the last term by parts, which gives

$$ \int_0^1 f_x v'dt = f_x v'_0 - \int_0^1 \frac{d}{dt}(f_x'v)dt = -\int_0^1 \frac{d}{dt}(f_x)vdt $$
We conclude that for an arbitrary function $v(\cdot) \in C^1[0, 1]$ such that $v(0) = v(1) = 0$, the equality $\int_0^1 [f_x - \frac{d}{dt}(f_{x'})]v dt = 0$ takes place. The fundamental lemma of the classical calculus of variations yields

$$f_x - \frac{d}{dt}(f_{x'}) = 0,$$

where $x \in C^1[0, 1]$, $x(0) = x(1) = 1$.

In particular, recalling that $T_xS = M$, we can write

$$F'(x)(v) = \int_0^1 [f_x - \frac{d}{dt}(f_{x'})]v dt = 0, \forall v \in T_xS.$$

In our example, $f(x, x') = x(x')^4$. Clearly, $f_x(x, x') = (x')^4$,

$$f_{x'}(x, x') = 4x(x')^3, \quad f_{xx}(x, x') = 0, \quad f_{xx'}(x, x') = 4(x')^3, \quad f_{xxx}(x, x') = 12x(x')^2,$$

$$f_{xxx'}(x, x') = 0, \quad f_{xxx''}(x, x') = 12(x')^2, \quad f_{xxx'''}(x, x') = 0, \quad f_{xxx'''}(x, x') = 24x,$$

$$f_{xxx''''}(x, x') = 24x.$$

Next we will analyze how Theorem 3.1.2 works for the only solution $x_1(t) = 1$, $\forall t \in [0, 1]$.

For this, we need the explicit form of the derivatives of $F$.

We consider the function $\lambda \to F'(x + \lambda v)(v)$ denoted by $\varphi_1$.

$$\varphi_1(\lambda) = F'(x + \lambda v)(v) = \int_0^1 [f_{xx}v + f_{xx'}v']v + f_{x'}(x + \lambda v, x' + \lambda v')v'\Big|_0^1 dt$$

Hence, if we differentiate under the integral sign

$$\varphi_1'(\lambda) = \int_0^1 [(f_{xx}v + f_{xx'}v')v + (f_{x'} + \lambda v')v']dt$$

It follows that

$$F''(x)(v)(v) = \varphi_1'(0) = \int_0^1 [(f_{xx}v + f_{xx'}v')v + (f_{x'} + \lambda v')v']dt =$$

$$= \int_0^1 [(f_{xx}v^2 + 2f_{xx'}v' + f_{xx''}v'^2)]dt.$$

Analogously, we set $\varphi_2(\lambda) = F'(x + \lambda v)(w)$, and we obtain that

$$F''(x)(v)(w) = \varphi_2'(0) = \int_0^1 [(f_{xx}vw + f_{xx'}v'w + f_{x'}vw')dt.$$
Also,

\[ F'''(x)(v)(v) = \varphi'_3(0) = \int_0^1 \left[ f_{xxx}v^3 + 3f_{xxx}v^2v' + 
+ 3f_{xx'x'}v(v')^2 + f_{x'x'}(v')^3 \right] dt \]

with \( \varphi_3 \) defined by \( \varphi_3(\lambda) = F''(x + \lambda v)(v), \forall \lambda \in \mathbb{R} \),

\[ F^\prime\prime\prime(x)(v)(v)(w) = \varphi'_4(0) = \int_0^1 \left[ f_{xxx}v^3 + 4f_{xxx}v^2v' + 
+ 2f_{xxx}vv'v' + 2f_{xx'x'}v'v'w' + f_{xx'x'}(v')^2w' + f_{x'x'}(v')^2w' \right] dt \]

where \( \varphi_4(\lambda) = F''(x + \lambda w)(v), \forall \lambda \in \mathbb{R} \),

\[ F^{(4)}(x)(v)(v)(v)(w) = \varphi'_5(0) = \int_0^1 \left[ f_{xxx}v^3 + 4f_{xxx}v^2v' + 
+ 6f_{xxx}v^2(v')^2 + 
+ 4f_{xx'x'}v(v')^3 + f_{x'x'x'}v(v')^3 \right] dt \]

where \( \varphi_5(\lambda) = F''''(x + \lambda v)(v)(v), \forall \lambda \in \mathbb{R} \).

Taking into account the expressions of the partial derivatives of \( F \) at \( x_1 \), it is clear that

\[ F'(x_1)(v) = 0, \forall v \in T_{x_1}S, \]
\[ F''(x_1)(v)(v) + F'(x_1)(w) = 0 \forall w \in T_{x_1}^2S \text{ with associated } v \in T_{x_1}S, \]
\[ F'''(x_1)(v)(v)(v) + 3F''(x_1)(v)(w) + F'(x_1)(z) = 0, \forall z \in T_{x_1}^3S \text{ with correspondent } \]
\text{vectors } w \in T_{x_1}^2S, v \in T_{x_1}S, \text{ and } \]
\[ F^{(4)}(x_1)(v)(v)(v)(v)(w) + 6F''(x_1)(v)(v)(w) + 4F''(x_1)(v)(w)z + 3F''(x_1)(w)(w) + 
+ F'(x_1)(u) = \int_0^1 24(v')^3 dt, \forall u \in T_{x_1}^1S \text{ with associated } z \in T_{x_1}^3S, w \in T_{x_1}^2S, \]
\text{v} \in T_{x_1}S. \]

The last expression is strictly positive for \( v \neq 0 \) as \( v'(t) = 0, \forall t \in [0,1] \) would lead us to the conclusion that \( v \) is a constant function, namely the null function since \( v(0) = v(1) = 0 \), and hence, by Theorem 3.1.2, \( x_1(t) = 1, \forall t \in [0,1] \) is a candidate for a local minimum point of the function \( F \) on \( S \).

In the sequel we present two examples of integral functionals which can take positive values on a constraint set given by an integral equation. In the first example we determine the first order tangent cone to the constraint set, while in the second
example it is sufficient only to find relations among the components of the first, second, third, and fourth order tangent vectors to the constraint set.

**Example 3.3.5.** Let us apply Theorem 3.1.2 to the functional

$$F(x, u) = \int_0^T (x^4(\tau) + u^5(\tau)) \, d\tau,$$

where $(x, u) \in S = \{(x, u) \in E = C(0, T) \times L^\infty(0, T) : x(t) = \int_0^t (x^3(\tau) + u(\tau)) \, d\tau\}$. (in other words, $x$ and $u$ satisfy the differential equation

$$\frac{dx(t)}{dt} = x^3(t) + u(t) \text{ and } x(0) = 0)$$

We notice that the pair $(x, u) = (0, 0)$ is a critical point for $F$ as

$$F'(x, u)(x_1, u_1) = \int_0^T (4x^3(\tau)x_1(\tau) + 5u^4(\tau)u_1(\tau)) \, d\tau,$$

for every $(x, u), (x_1, u_1) \in E$.

Since the first, the second and the third order partial derivatives of $f(x, u) = x^4 + u^5$ with respect to $x$ and $u$ are equal to zero at $(0, 0)$, the expressions

$$F'(0, 0)(x_1, u_1), F'(0, 0)(x_2, u_2) + F'(0, 0)(x_1, u_1)(x_1, u_1), \text{ and}$$

$$F''(0, 0)(x_1, u_1)(x_1, u_1)(x_1, u_1) + 3F''(0, 0)(x_1, u_1)(x_2, u_2) + F'(0, 0)(x_3, u_3)$$

are also zero, for any $(x_3, u_3) \in T^3_{(0, 0)}S$ with associated vectors $(x_2, u_2) \in T^2_{(0, 0)}S, (x_1, u_1) \in T_{(0, 0)}S$.

The fourth order expression becomes

$$F'(0, 0)(x_4, u_4) + 3F''(0, 0)(x_2, u_2)(x_2, u_2) + 4F'''(0, 0)(x_1, u_1)(x_3, u_3) +$$

$$6F''''(0, 0)(x_1, u_1)(x_1, u_1)(x_2, u_2) + F^{(4)}(0, 0) [(x_1, u_1)]^4 = 4! \int_0^T x_1^4(\tau) \, d\tau.$$ 

This means that we have to determine only the tangent cone $T_{(0, 0)}S$ at $(0, 0)$ to the given constraint.

We first consider a more general case when

$$S = \{(x, u) \in E : x(t) = \int_0^t \varphi(x(\tau), u(\tau), \tau) \, d\tau, 0 \leq t \leq T\}.$$
Assume that, for all bounded \(x, u\) and all \(0 \leq t \leq T\), the derivatives \(\varphi_x(x, u, t)\), \(\varphi_u(x, u, t)\) exist, are continuous in \(x, u\), measurable in \(t\) and bounded.

We introduce the operator

\[
P(x, u)(t) = x(t) - \int_0^t \varphi(x(\tau), u(\tau), \tau) \, d\tau,
\]

which maps \(E\) into \(C(0, T)\).

Then \(S = \{(x, u) \in E : P(x, u) = 0\}\). Furthermore,

\[
P(x + x_1, u + u_1) - P(x, u) = x_1(t) - \int_0^t [\varphi_x(x, u, \tau)x_1(\tau) + \varphi_u(x, u, \tau)u_1(\tau)] \, d\tau + \delta,
\]

where \(\delta\) is a remainder term for which an estimate can be easily found

\[
\delta = o\left(\sqrt{||x_1||_C^2 + ||u_1||_{L^\infty}}\right),
\]

and the first term on the right-hand side is a linear operator of \((x_1, u_1)\).

Therefore, \(P(x, u)\) is differentiable,

\[
P'(x, u)(x_1, u_1) = x_1(t) - \int_0^t [\varphi_x(x, u, \tau)x_1(\tau) + \varphi_u(x, u, \tau)u_1(\tau)] \, d\tau,
\]

and \(P'(x, u)\) is continuous in a neighborhood of \((x, u)\).

We now show that \(P'(x, u)\) maps \(E\) onto \(C(0, T)\), i.e., the equation

\[
x_1(t) - \int_0^t [\varphi_x(x, u, \tau)x_1(\tau) + \varphi_u(x, u, \tau)u_1(\tau)] \, d\tau = a(t)
\]

has a solution \((x, u)\) for any \(a \in C(0, T)\).

Set \(u_1(t) \equiv 0\). Then the equation becomes

\[
x_1(t) = a(t) + \int_0^t \varphi_x(x, u, \tau)x_1(\tau) \, d\tau.
\]

This integral equation has a solution \(x_1 \in C(0, T)\) for any \(a \in C(0, T)\), because it is a linear Volterra equation of the second kind.

Thus, all the assumptions of Lyusternik’s theorem hold for the operator \(P(x, u)\) and therefore the tangent cone \(T_{(x,u)}S\) coincides with the kernel of the operator \(P(x, u)\), i.e., consists of all pairs \((x_1, u_1)\), which satisfy the linear differential equation.
\[
\frac{dx_1(t)}{dt} = \varphi(x, u, t) x_1(t) + \varphi_u(x, u, t) u_1(t), \quad x_1(0) = 0.
\]

Let us return to our example.

In our case \(\varphi(x, u, t) = x^3(t) + u(t)\), so

\[
T_{(0,0)}S = \{ (x_1, u_1) \in E; \frac{dx_1(t)}{dt} = u_1(t), \quad x_1(0) = 0 \}.
\]

It is easy now to see that the fourth order expression is strictly positive for any pair \((x_1, u_1) \neq (0, 0)\) in the tangent subspace, because if it was zero for some \((x_1, u_1)\), then \(\int_0^T x_1^4(\tau) d\tau = 0\), thus \(x_1(t) = 0, \forall t \in [0, T]\), and consequently, \(u_1(t) = \frac{dx_1(t)}{dt} = 0, \forall t \in [0, T]\). In conclusion, the critical point \((0, 0)\), i.e., the pair \((x, u)\) consisting of the functions identically zero on \([0, T]\), is a candidate for a minimum point of \(F\) on \(S\), i.e., it can not be a maximum point of \(F\) on \(S\), which means that the functional \(F\) can take positive values on \(S\).

**Example 3.3.6.** Let us apply Theorem 3.1.2 to the functional

\[
F(x, u) = \int_0^T (x^4(\tau) - x(\tau) u(\tau) + u^5(\tau)) d\tau,
\]

where

\[
(x, u) \in S = \{ (x, u) \in E = C(0,T) \times L^\infty(0,T) : x(T) = 0, x(t) = \int_0^t (x^7(\tau) + u(\tau)) d\tau, \}
\]

(in other words, \(x\) and \(u\) satisfy the differential equation

\[
\frac{dx(t)}{dt} = x^7(t) + u(t) \text{ and } x(0) = x(T) = 0.
\]

We can see that the pair \((x, u) = (0, 0)\) is a critical point for \(F\) as

\[
F'(x, u)(x_1, u_1) = \int_0^T ((4x^3(\tau) - u(\tau))x_1(\tau) + (-x(\tau) + 5u^4(\tau)) u_1(\tau)) d\tau,
\]

for every \((x, u), (x_1, u_1) \in E\).

Let us find a relation between the components of a tangent vector \((x_1, u_1)\) at \((x, u)\) to the set

\[
S = \{ (x, u) \in E : x(t) = \int_0^t \varphi(x(\tau), u(\tau), \tau) d\tau, 0 \leq t \leq T, x(T) = 0 \}.
\]
Assume that, for all bounded $x, u$ and all $0 \leq t \leq T$, the partial derivatives up to the fourth order of $\varphi(x, u, t)$ with respect to $x$ and $u$ exist, are continuous in $x, u$, measurable in $t$ and bounded.

We introduce the operator:

$$P(x, u)(t) = (x(t) - \int_0^t \varphi(x(\tau), u(\tau), \tau) d\tau, x(T)),$$

which maps $E$ into $C(0, T) \times \mathbb{R}$. 

Then $S = \{(x, u) \in E : P(x, u) = 0\}$. Furthermore,

$$P(x + x_1, u + u_1)(t) - P(x, u)(t) =
(x_1(t) - \int_0^t [\varphi_x(x, u, \tau) x_1(\tau) + \varphi_u(x, u, \tau) u_1(\tau)] d\tau + \delta, x_1(T)),$$

where $\delta$ is a remainder term for which an estimate can be easily found

$$\delta = o\left(\sqrt{||x_1||_C^2 + ||u_1||_{L^\infty}^2}\right),$$

and the first term of the first component on the right-hand side is a linear operator of $(x_1, u_1)$.

Therefore, $P(x, u)$ is differentiable,

$$P'(x, u)(x_1, u_1)(t) = (x_1(t) - \int_0^t [\varphi_x(x, u, \tau) x_1(\tau) + \varphi_u(x, u, \tau) u_1(\tau)] d\tau, x_1(T)).$$

The first part of Corollary 2.1 states that

$$T_{(x,u)}S \subset \{(x_1, u_1) \in E : P'(x, u)(x_1, u_1) = 0\},$$

i.e., any vector $(x_1, u_1) \in E$ in the tangent cone $T_{(x,u)}S$, belongs also to the set

$$K = \{ (\overline{x}, \overline{u}) \in E : \overline{x}(t) = \int_0^t [\varphi_x(x, u, \tau) \overline{x}(\tau) + \varphi_u(x, u, \tau) \overline{u}(\tau)] d\tau, 0 \leq t \leq T, \overline{x}(T) = 0 \}.$$

Alternatively, in terms of differential equations,

$$K = \{ (\overline{x}, \overline{u}) \in E : \overline{x}(t) = \varphi_x(x, u, t) \overline{x}(t) + \varphi_u(x, u, t) \overline{u}(t), 0 \leq t \leq T, \overline{x}(0) = \overline{x}(T) = 0 \},$$

and, because in our example, $\varphi(x, u, t) = x^7(t) + u(t), \forall (x, u) \in E$, $K$ reduces to
\[ K = \{(x, u) \in E : \frac{d\bar{x}}{dt}(t) = \bar{u}(t), \ 0 \leq t \leq T, \ 0 = \bar{x}(0) = \bar{x}(T) = 0\}. \]

Since all the second, third and fourth order partial derivatives with respect with \( x \) and \( u \) of \( \varphi(x, u, t) = x^7(t) + u(t) \) are equal to zero, we get that the second, third and fourth Fréchet derivatives of the operator \( P(x, u) \) at \((0, 0)\) are identically zero because

\[
P''(x, u)(x_1, u_1)(x_2, u_2) = (-\int_0^1 (\varphi_{xx}(x, u, \tau) x_1(\tau) x_2(\tau) + \\
+ \varphi_{xu}(x, u, \tau) x_1(\tau) u_2(\tau) + \varphi_{ux}(x, u, \tau) x_2(\tau) u_1(\tau) + \\
+ \varphi_{uu}(x, u, \tau) u_1(\tau) u_2(\tau)) d\tau, \forall (x_1, u_1), (x_2, u_2) \in E. \]

\[
P''(x, u)(x_1, u_1)(x_3, u_3) = (-\int_0^1 (\varphi_{xxxx}x_1^2x_3^2 + \varphi_{xxu}x_1^2u_3 + \\
+ 2\varphi_{xxx}x_1x_3u_1 + 2\varphi_{xuu}x_1u_1u_3 + \varphi_{xuu}x_1u_3^2 + \\
+ \varphi_{uuu}u_1^2u_3 d\tau, \forall (x_1, u_1), (x_3, u_3) \in E. \]

By Corollary 2.1, we have that

\[
T^2_{(x,u)}S \subset \{(x_2, u_2) \in E, \text{ for which there exists } (x_1, u_1) \in T_{(x,u)}S, \text{ such that } \\
P'(x, u)(x_1, u_1)(x_2, u_2) = 0\}, 
\]

\[
T^3_{(x,u)}S \subset \{(x_3, u_3) \in E, \text{ for which there exist } (x_2, u_2) \in T^2_{(x,u)}S \text{ and } \\
(x_1, u_1) \in T_{(x,u)}S, \text{ such that } \\
P''(x, u)[(x_1, u_1)]^3 + 3P'(x, u)(x_1, u_1)(x_2, u_2) + P'(x, u)(x_3, u_3) = 0, 
\]

\[
T^4_{(x,u)}S \subset \{(x_4, u_4) \in E, \text{ for which there exist } (x_3, u_3) \in T^3_{(x,u)}S, (x_2, u_2) \in T^2_{(x,u)}S \text{ and } \\
(x_1, u_1) \in T_{(x,u)}S, \text{ such that } \\
P'(x, u)(x_4, u_4) + 3P''(x, u)(x_2, u_2)(x_2, u_2) + 4P''(x, u)(x_1, u_1)(x_3, u_3) + \\
6P''(x, u)(x_1, u_1)(x_1, u_1)(x_2, u_2) + P^{(4)}(x, u)[(x_1, u_1)]^4 = 0. \]
Thus, $\frac{dx_i(t)}{dt} = u_i(t), \ 0 \leq t \leq T, x_i(0) = x_i(T) = 0, \forall (x_i, u_i) \in T^{i}_t(x, u) S, i = 2, 3, 4$. The derivatives of $F$ are given explicitly by

$$F''(x, u)(x_1, u_1)(x_2, u_2) = \int_0^T (f_{xx} x_1^2 x_2 + f_{xu} x_1 x_2 u_1 + f_{uu} u_1^2) dt,$$

$$F''(x, u)(x_1, u_1)(x_2, u_2) = \int_0^T (f_{xxx} x_1^3 x_2 + f_{xuu} x_1^3 u_1 + f_{uuu} u_1^3) dt,$$

$$F''(x, u)(x_1, u_1)(x_2, u_2) = \int_0^T (f_{xxxx} x_1^4 + f_{xxxx} x_1^4 u_1 + f_{xxxx} x_1^4 u_1^3) dt, \ \forall (x_1, u_1), (x_2, u_2) \in E,$$

where $f(x, u) = x^4 + xu + u^5, f : C(0, T) \times L^\infty (0, T) \rightarrow L^\infty (0, T)$.

Since $F'''(0, 0) = F'(0, 0) = 0$ and any $(x_i, u_i) \in T_{(0,0)} S, i = 1, 2, 3, 4$ has the property that $x_i(T) = x_i(0) = 0$ and $x'_i(\tau) = u_i(\tau), \ \forall \tau \in [0, T]$, we obtain successively

$$F'(0, 0)(x_2, u_2) + F''(0, 0)(x_1, u_1)^2 = -2 \int_0^T x_1(\tau) u_1(\tau) d\tau = -2 \int_0^T x_1(\tau) x'_1(\tau) d\tau = -\left[ x_1^2(T) - x_1^2(0) \right] = 0,$$

$\forall (x_2, u_2) \in T^2_{(0,0)} S$ with associated $(x_1, u_1) \in T_{(0,0)} S$,

$$F'''(0, 0)[(x_1, u_1)^3 + 3F''(0, 0)(x_1, u_1)(x_2, u_2) + F'(0, 0)(x_3, u_3) = -3 \int_0^T [x_1(\tau) x_2(\tau) + x_2(\tau) x_1(\tau)] d\tau = -3 \int_0^T [x_1(\tau) x_2(\tau) + x_2(\tau) x_1(\tau)] d\tau = -3 \left[ (x_1x_2)(T) - (x_1x_2)(0) \right] = 0,$$

$\forall (x_3, u_3) \in T^3_{(0,0)} S$ with associated vectors $(x_2, u_2) \in T^2_{(0,0)} S$ and $(x_1, u_1) \in T_{(0,0)} S$,

$$F'(0, 0)(x_4, u_4) + 3F''(0, 0)(x_2, u_2)(x_2, u_2) + 4F''(0, 0)(x_1, u_1)(x_3, u_3) + 6F'''(0, 0)(x_1, u_1)(x_2, u_2) + F^{(4)}(0, 0)[(x_1, u_1)^4 = \int_0^T [24x_1^4(\tau) - 4(x_1(\tau) u_3(\tau) - x_3(\tau) u_1(\tau)) - 6x_2(\tau) u_2(\tau)] d\tau = 24 \int_0^T x_1^4(\tau) d\tau - 4(x_1x_3)^T_0 - 3x_2^2T_0 = 24 \int_0^T x_1^4(\tau) d\tau,$$

$\forall (x_4, u_4) \in T^4_{(0,0)} S$ with associated vectors $(x_1, u_1) \in T^4_{(0,0)} S, i = 1, 2, 3.$
The last expression is strictly positive for any nonzero pair \((x_1, u_1)\) in the tangent cone \(T_{(0,0)}S\), as \(\int_0^T x_1^4(\tau)\ d\tau = 0\) would yield \(x_1(\tau) = 0, \forall \tau \in [0, T]\), and consequently \(u_1(\tau) = x_1'(\tau) = 0, \forall \tau \in [0, T]\). Thus, by Theorem 3.1.2, the pair of null functions \((0, 0)\) is a candidate for a minimum point for \(F\) on \(S\). This means that \(F\) can take positive values on \(S\).

The condition \(G'(x_0)\) is onto can not be omitted in the statement of Corollary 2.1, as it is shown by the following example in which the second order necessary conditions are not verified although the point \(x_0\) is a minimum point. At the same time this is an example of an objective function \(F\) and a function \(G\) whose kernel is the constraint set, for which there are no Lagrange multipliers.

**Example 3.3.7.** Let us find \(x_0 = (x_0, y_0)\) such that

\[
\min \{x^2 + y^2 + z^2, \text{subject to } x - y + z = 1, 4(xy + yz) + 1 = 0\} = x_0^2 + y_0^2 + z_0^2.
\]

Here \(X = \mathbb{R}^3\), \(F(x, y, z) = x^2 + y^2 + z^2\), \(G(x, y, z) = (g_1(x, y, z), g_2(x, y, z))\),
\[
g_1(x, y, z) = x - y + z - 1 = 0 \quad \text{and} \quad g_2(x, y, z) = 4xy + 4yz + 1 = 0.
\]

Clearly, \(g_1'(x, y, z) = (1 \quad -1 \quad 1)\) and \(g_2'(x, y, z) = (4y \quad 4(x + z) \quad 4y)\),
\[
g_1''(x, y, z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g_2''(x, y, z) = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix},
\]
\[
F'(x, y, z) = \begin{pmatrix} 2x & 2y & 2z \end{pmatrix}, \quad F''(x, y, z) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
\]

The points that verify the constraints have the form
\[
\begin{cases} 
  x - y + z - 1 = 0 \\
  4xy + 4yz + 1 = 0
\end{cases}
\]

\[
4y(y + 1) + 1 = 0, 4y^2 + 4y + 1 = 0, (2y + 1)^2 = 0, \quad y = -\frac{1}{2}, \ x + z = \frac{1}{2}, \ z = \frac{1}{2} - x.
\]

Thus, we have to minimize
\[
F(x, -\frac{1}{2}, \frac{1}{2} - x) = x^2 + (-\frac{1}{2})^2 + (\frac{1}{2} - x)^2 = 2x^2 - x + \frac{1}{2}.
\]
It is obvious that $F$ has a minimum for $x_0 = \frac{1}{4}$, namely $\pi_0 = (\frac{1}{4}, -\frac{1}{2}, \frac{1}{4})$.

We get that $g_2'({\pi_0}) = (-2 \ 2 \ -2)$, so $g_1'({\pi_0})$ and $g_2'({\pi_0})$ are linearly dependent, which is equivalent to the fact that $G'({\pi_0})$ is not onto.

Obviously, $T_{\pi_0}D_G \subseteq Ker G'({\pi_0})$.

Suppose by contradiction that $Ker G'({\pi_0}) = Ker g_1'({\pi_0}) \subseteq T_{\pi_0}D_G$. Then for any $y \in Ker g_1'({\pi_0})$, i.e., $g_1'({\pi_0})(y) = 0$ we have $g_1'({\pi_0})(-y) = 0$ too, as $g_1'({\pi_0})$ is linear and thus both $y$ and $-y$ belong to $T_{\pi_0}D_G$.

Since $F'({\pi_0})(v) \geq 0$ for any $v \in T_{\pi_0}D_G$, we obtain that $F'({\pi_0})(y) = 0$ for any $y \in Ker g_1'({\pi_0})$. This implies that there exists $\lambda \neq 0$ with the property that $F'({\pi_0}) = \lambda g_1'({\pi_0})$, which is false as $F'({\pi_0}) = (\frac{1}{2} -1 \ \frac{1}{2})$ and $g_1'({\pi_0}) = (1 \ -1 \ 1)$.

Also, in this example, although $\pi_0$ is a minimum point of $F$ on $D_G$, there are no Lagrange multipliers $\lambda_1, \lambda_2$ because the equation

$$F'({\pi_0}) = \lambda_1 g_1'({\pi_0}) + \lambda_2 g_2'({\pi_0}),$$

or

$$(\frac{1}{2} \ -1 \ \frac{1}{2}) = \lambda_1(1 \ -1 \ 1) + \lambda_2(-2 \ 2 \ -2),$$

reduces to the system

$$\begin{cases} 
\lambda_1 - 2\lambda_2 = \frac{1}{2} \\
-\lambda_1 + 2\lambda_2 = -1
\end{cases}$$

which is incompatible.

Let us examine the expression $F''({\pi_0})(v)(v) + F'({\pi_0})(w)$ for any $v$ and $w$ satisfying

$g_1'({\pi_0})v = 0$, i.e., $v_1 - v_2 + v_3 = 0$,

$g_2'({\pi_0})v = 0$,

$g_1''({\pi_0})(v)(v) + g_1'({\pi_0})(w) = 0$, i.e., $w_1 - w_2 + w_3 = 0$,

$g_2''({\pi_0})(v)(v) + g_2'({\pi_0})(w) = 0$, i.e., $8v_1v_2 + 8v_2v_3 - 2w_1 + 2w_2 - 2w_3 = 0$,

so $8(v_1 + v_3)v_2 = 0$ and since $v_2 = v_1 + v_3$, $v_2 = v_1 + v_3 = 0$.

Now, the above expression becomes

$$F''({\pi_0})(v)(v) + F'({\pi_0})(w) = 2v_1^2 + 2v_2^2 + 2v_3^2 + \frac{1}{2}w_1 - w_2 + \frac{1}{2}w_3 = 4v_1^2 - \frac{w_2}{2},$$

which can not be positive for all $v, w \neq 0$, i.e., $\forall v_1 \neq 0, \forall w_2$.

**Example 3.3.8.** Let us consider the function $F_1(x, y, z) = x^2 + y^4 + z^2$ subject to the same constraint as in the previous example. The point $\pi_0 = (\frac{1}{4}, -\frac{1}{2}, \frac{1}{4})$ is a minimum point for this function on $D_G$, as well.
The functional $G'(x_0)$ is not onto as we have seen in example 3.3.7, but in this case there are Lagrange multipliers $\lambda_1 = \frac{1}{4}$ and $\lambda_2 = -\frac{1}{8}$ such that the equation $F_1(x, y, z) = \lambda_1 g'_1(x, y, z) + \lambda_2 g'_2(x, y, z)$, i.e.,

$$(2x, 4y^3, 2z) = \lambda_1 (1, -1, 1) + \lambda_2 (4y, 4(x + z), 4y)$$

has the solution $x = \frac{1}{4}, y = -\frac{1}{2}, z = \frac{1}{4}$. 
Chapter 4

Flow-Invariant Sets

In this chapter we provide a characterization of the sets $S = D_G = \{ x \in X; G(x) = 0 \}$, that are flow-invariant with respect to the $n$-th order autonomous differential equation

$$u^{(n)}(t) = F(u(t)), \quad t \geq 0,$$

(4.1)

where $G : U \to \mathbb{R}^m$, $m \geq 1$ is a $n$-times Fréchet differentiable mapping on an open subset $U$ of a Banach space $X$, $n \geq 3$, and $F : U \to X$ is a locally Lipschitz mapping.

We recall the definition of a flow-invariant set with respect to the autonomous $n$-th order differential equation (4.1) (see Definition 1.9, (24)).

**Definition 4.1.** The nonempty set $S \subset U$ is said to be (right-hand) flow-invariant with respect to the $n$-th order differential equation (4.1) if the solution $u : [0, T) \to X$ to the Cauchy problem (4.1) determined by the initial conditions

$$u(0) = x, \quad u'(0) = v_1, \ldots, \quad u^{(n-1)}(0) = v_{n-1},$$

(4.2)

with $x \in S$, $v_1 \in T_x S, \ldots, v_{n-1} \in T_{x}^{n-1}S$, $F(x) \in T_x^n S$ having correspondent vectors $v_1, \ldots, v_{n-1}$, satisfies

$$u(t) \in S, \quad \forall \ t \geq 0, \ t \in \text{dom}u.$$

(4.3)

The constraints imposed to $(x, v_1, \ldots, v_{n-1})$ are necessary conditions to have the invariance property (4.3).
In (30), N.H. Pavel and C. Ursescu introduced the set
\[ M_S^{(n)} = \{(x, v_1, \ldots, v_{n-1}) \in S \times X^{n-1} : v_i \in T_x S, i = 1, \ldots, n-1, \] 
and they expressed the choice in (4.2) for the initial conditions by means of (4.4) as follows
\[ (u(0), u'(0), \ldots, u^{(n-1)}(0)) = (x, v_1, \ldots, v_{n-1}) \in M_S^{(n)} \]
This was justified by the following result.

**Theorem 4.1.** The closed set \( S \subset U \) is flow-invariant for equation (4.1) iff
\[ (u(t), u'(t), \ldots, u^{(n-1)}(t)) \in M_S^{(n)}, \forall t \in [0, T), \] 
for any solution \( u : [0, T) \to X \) of equation (4.1) whose initial data \((u(0), u'(0), \ldots, u^{(n-1)}(0))\) is in \( M_S^{(n)} \).

N.H. Pavel and C. Ursescu reduced the problem of invariant sets for (4.1) to a similar problem for a first order differential equation, fact that allowed them to utilize a theorem proved by M. Nagumo (25) and, independently, by H. Brézis (9), in order to obtain a characterization of flow-invariant sets \( S \subset U \) with respect to the \( n \)-th order differential equation (4.1), which we will use for deriving the main results of this chapter (see (24) and (30)).

**Theorem 4.2.** Assume that \( M_S^{(n)} \) is a nonempty closed subset of \( U \times X^{n-1} \), for a closed subset \( S \) of \( U \), \( n \geq 2 \). Then \( S \subset U \) is a flow-invariant set with respect with the \( n \)-th order differential equation (4.1) if and only if \((v_1, \ldots, v_{n-1}, F(x)) \in X^n \) is a tangent vector to \( M_S^{(n)} \) at \((x, v_1, \ldots, v_{n-1}) \in M_S^{(n)} \), for any such \( n \)-tuples, i.e.,
\[ \lim_{t \to 0} t^{-1}d((x, v_1, \ldots, v_{n-1}) + t(v_1, \ldots, v_{n-1}, F(x)); M_S^{(n)}) = 0. \]
Next we determine $M_S^{(n)}$, $n \geq 3$, when $S = D_G$ for a mapping $G : U \to \mathbb{R}^m$, $m \geq 1$ and we provide an explicit description for those sets of this form that are flow-invariant with respect to (4.1).

First we analyze the case $n = 3$. We mention that the case $n = 2$ has been considered in (30).

**Theorem 4.3.** Assume that $G : U \to \mathbb{R}^m$, $m \geq 1$ is three times Fréchet differentiable and its first Fréchet derivative $G'(x) : X \to \mathbb{R}^m$ is onto for each $x \in S = D_G$. Then $M_S^{(3)}$ is given by

$$M_S^{(3)} = \{(x, v_1, v_2) \in U \times X \times X : G(x) = 0, G'(x)(v_1) = 0, G''(x)(v_1)(v_1) + G'(x)(v_2) = 0, G''(x)(v_1)(v_1)(v_1) + 3G''(x)(v_1)(v_2) + G'(x)(F(x)) = 0\}.$$  

(4.6)

Suppose further that $G$ is four times Fréchet differentiable on $U$, the function $h : U \to \mathbb{R}^m$ given by

$$h(x) = G'(x)(F(x)), \forall x \in U,$$  

(4.7)

is Fréchet differentiable, $M_S^{(3)}$ is nonempty and the mapping

$$(G'(x)(\cdot), G''(x)(v_1)(\cdot)) : X \to \mathbb{R}^m \times \mathbb{R}^m$$

is onto for every $(x, v_1, v_2) \in M_S^{(3)}$.

Then $S = D_G$ is flow-invariant with respect to the differential equation $u'''(t) = F(u(t)), t \geq 0$ if and only if

$$G^{(4)}(x)(v_1)(v_1)(v_1) + 6G''(x)(v_1)(v_1)(v_2) + 3G''(x)(v_2)(v_2) + 3G''(x)(v_1)(v_1)(F(x)) + h'(x)(v_1) = 0.$$  

(4.8)

**Proof.** Formula (4.6) follows directly from Corollary 2.1.

To prove the second part, we notice that, due to (4.6), $M_S^{(3)}$ can be rewritten as

$$M_S^{(3)} = g^{-1}(0),$$

where $g : U \times X \times X \to \mathbb{R}^{4m}$ is defined by

$$g(x, v_1, v_2) = (G(x), G'(x)(v_1), G''(x)(v_1)(v_1) + G'(x)(v_2),$$
\[ G'''(x)(v_1)(v_1) + 3G''(x)(v_1)(v_2) + G'(x)(F(x)) \]

for all \((x, v_1, v_2) \in U \times X \times X \).

It can be easily seen that under the hypothesis of the theorem, \(g\) is Fréchet differentiable and its Fréchet derivative is determined by the relation

\[
g'(x, v_1, v_2)(u, y_1, y_2) = (G'(x)(u), G''(x)(u)(v_1) + G'(x)(y_1),
\]

\[
G'''(x)(u)(v_1)(v_1) + 2G''(x)(v_1)(y_1) + G''(x)(u)(v_2) + G'(x)(y_2),
\]

\[
G^{(4)}(x)(u)(v_1)(v_1)(v_1) + 3G'''(x)(v_1)(v_1)(y_1) + 3G''(x)(u)(v_1)(v_2) +
\]

\[ + 3G''(x)(y_1)(v_2) + 3G''(x)(v_1)(y_2) + h'(x)(u)). \]

We now show that \(g'(x, v_1, v_2) : X \times X \times X \to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \) is onto for each \((x, v_1, v_2) \in M_S^{(3)}, \) i.e., the equation \(g'(x, v_1, v_2)(u, y_1, y_2) = (z_1, z_2, z_3, z_4)\) has a solution \((u, y_1, y_2) \in X \times X \times X\) for any \((z_1, z_2, z_3, z_4) \in \mathbb{R}^{4m}.\) Since \(G'(x)\) is onto, there are \(u \in X\) such that \(G'(x)(u) = z_1\) and \(y_1 \in X\) such that \(G''(x)(u)(v_1) + G'(x)(y_1) = z_2.\) Then the element \(y_2 \in X\) can be obtained using the fact that the mapping \((G'(x)(\cdot), G''(x)(v_1)(\cdot)) : X \to \mathbb{R}^m \times \mathbb{R}^m\) is onto.

Thus, \(T_{(x, v_1, v_2)}M^{(3)}_S = g'(x, v_1, v_2)^{-1}(0).\)

Finally, Theorem 4.2 completes the proof.

The following result represents a generalization of the previous theorem for the case of equation (4.1).

**Theorem 4.4.** Assume that \(G : U \to \mathbb{R}^m\) is \(n\) times Fréchet differentiable and its first Fréchet derivative \(G'(x) : X \to \mathbb{R}^n\) is onto for each \(x \in S = D_G.\) Then \(M^{(n)}_S\) is given by

\[
M^{(n)}_S = \left\{ (x, v_1, \ldots, v_{n-1}) \in U \times X^{n-1} : G(x) = 0,
\right.

\[
S_j^G(x, v_1, \ldots, v_j) = 0, 1 \leq j \leq n - 1,
\]

\[
G'(x)(F(x)) + \sum_{k=2}^{n} \frac{n!}{k!} \left[ \sum_{i_1+\cdots+i_k=n} \frac{1}{i_1! \cdots i_k!} G^{(k)}(x)(v_{i_1}) \cdots (v_{i_k}) \right] = 0. \quad (4.9)
\]
Suppose further that $G$ is $n + 1$ times Fréchet differentiable on $U$, the function $h : U \to \mathbb{R}^m$ given by
\[ h(x) = G'(x)(F(x)), \forall x \in U, \]
is Fréchet differentiable, $M^{(n)}_S$ is nonempty and the mapping
\[ (G'(x)\cdot, G''(x)(v_1)(\cdot)) : X \to \mathbb{R}^m \times \mathbb{R}^m \]
is onto for any $(x, v_1, \ldots, v_{n-1}) \in M^{(n)}_S$.

Then $S = D_G$ is flow-invariant with respect to the differential equation $u^{(n)}(t) = F(u(t))$, $t \geq 0$ if and only if
\[ h'(x)(v_1) + \sum_{k=3}^{n} \frac{n!}{k!} \left\{ \sum_{i_1, \ldots, i_k \in \{1, \ldots, n-2\}} \frac{1}{i_1! \cdots i_k!} \left[ G^{(k+1)}(x)(v_1)(v_{i_1}) \cdots (v_{i_k}) + \right. \right. \]
\[ + G^{(k)}(x)(v_{i_1+1})(v_{i_2}) \cdots (v_{i_k}) + \ldots + G^{(k)}(x)(v_{i_1})(v_{i_2+1}) \cdots (v_{i_k}) \left. \right] \right\} + \]
\[ + nG''(x)(v_1)(v_{n-1}) + nG''(x)(v_2)(v_{n-1}) + nG''(x)(v_1)(F(x)) = 0 \quad (4.10) \]
or, equivalently
\[ h'(x)(v_1) + \sum_{k=2}^{n} \frac{n!}{k!} \left\{ \sum_{i_1, \ldots, i_k \in \{1, \ldots, n-1\}} \frac{1}{i_1! i_2! \cdots i_k!} \left[ G^{(k+1)}(x)(v_1)(v_{i_1}) \cdots (v_{i_k}) + \right. \right. \]
\[ + G^{(k)}(x)(v_{i_1+1})(v_{i_2}) \cdots (v_{i_k}) + \ldots + G^{(k)}(x)(v_{i_1})(v_{i_2+1}) \cdots (v_{i_k}) \left. \right] \right\} = 0, \quad (4.10') \]
for any $(x, v_1, \ldots, v_{n-1}) \in M^{(n)}_S$, $v_n = F(x)$.

Here, $S^G_j(x, v_1, \ldots, v_j)$, $j \geq 1$ denotes the expression
\[ S^G_j(x, v_1, \ldots, v_j) = \sum_{k=1}^{j} \frac{j!}{k!} \left[ \sum_{i_1 + \ldots + i_k = j} \frac{1}{i_1! \cdots i_k!} G^{(k)}(x)(v_{i_1}) \cdots (v_{i_k}) \right]. \]

**Proof.** This result can be proved analogously to the previous theorem.

We have $M^{(n)}_S = g^{-1}(0)$, where $g : U \times X^{n-1} \to \mathbb{R}^{n+1}$ is equal to
\[ g(x, v_1, \ldots, v_{n-1}) = \left( G(x), S^G_1(x, v_1), \ldots, S^G_{n-1}(x, v_1, \ldots, v_{n-1}) \right), \]
\[ G'(x)(F(x)) + \sum_{k=2}^{n} \frac{n!}{k!} \left[ \sum_{i_1 + \ldots + i_k = n \atop i_1, \ldots, i_k \in \{1, \ldots, n-1\}} \frac{1}{i_1! \ldots i_k!} G^{(k)}(x)(v_{i_1}) \ldots (v_{i_k}) \right], \]

it is Fréchet differentiable and its first Fréchet derivative is

\[ g'(x, v_1, \ldots, v_{n-1})(y_1, \ldots, y_n) = \left( G'(x)(y_1), [S^G_{n-1}'](x, v_1)(y_1, y_2), \ldots, [S^G_{n-1}'](x, v_{n-1})(y_1, \ldots, y_n) \right), \]

\[ h'(x)(y_1) + \sum_{k=2}^{n} \frac{n!}{k!} \left\{ \sum_{i_1 + \ldots + i_k = n \atop i_1, \ldots, i_k \in \{1, \ldots, n-1\}} \frac{1}{i_1! \ldots i_k!} \right. \]

\[ \left. G^{(k+1)}(x)(y_1)(v_{i_1}) \ldots (v_{i_k}) \right. \]

\[ + G^{(k)}(x)(y_{i_1+1})(v_{i_2}) \ldots (v_{i_k}) + \ldots + G^{(k)}(x)(v_{i_1}) \ldots (v_{i_{k-1}})(y_{i_k+1}) \right\}, \]

with \([S^G_{j}'](x, v_1, \ldots, v_j)(y_1, \ldots, y_{j+1}), 1 \leq j \leq n - 1\), as below

\[ [S^G_{j}'](x, v_1, \ldots, v_j)(y_1, \ldots, y_{j+1}) = \]

\[ \sum_{k=1}^{j} \frac{j!}{k!} \left\{ \sum_{i_1 + \ldots + i_k = j \atop i_1, \ldots, i_k \in \{1, \ldots, j\}} \frac{1}{i_1! \ldots i_k!} \right. \]

\[ \left. G^{(k+1)}(x)(y_1)(v_{i_1}) \ldots (v_{i_k}) \right. \]

\[ + G^{(k)}(x)(y_{i_1+1})(v_{i_2}) \ldots (v_{i_k}) + \ldots + G^{(k)}(x)(v_{i_1}) \ldots (v_{i_{k-1}})(y_{i_k+1}) \right\}. \]

Since \((G'(x)(\cdot), G''(x)(v_1)(\cdot)) : X \to \mathbb{R}^m \times \mathbb{R}^m\) is onto for every \((x, v_1, \ldots, v_n) \in M^{(n)}_S\), we can see that \(g'(x, v_1, \ldots, v_{n-1})\) is onto as well, because the derivative of the last component of \(g\) in the direction \((y_1, \ldots, y_n)\), can be written more explicitly as

\[ h'(x)(y_1) + \sum_{k=3}^{n} \frac{n!}{k!} \left\{ \sum_{i_1 + \ldots + i_k = n \atop i_1, \ldots, i_k \in \{1, \ldots, n-2\}} \frac{1}{i_1! \ldots i_k!} \right. \]

\[ \left. G^{(k+1)}(x)(y_1)(v_{i_1}) \ldots (v_{i_k}) \right. \]

\[ + G^{(k)}(x)(y_{i_1+1})(v_{i_2}) \ldots (v_{i_k}) + \ldots + G^{(k)}(x)(v_{i_1}) \ldots (v_{i_{k-1}})(y_{i_k+1}) \right\} + \]

\[ + nG''(x)(y_1)(v_{n-1}) + nG''(x)(y_2)(v_{n-1}) + nG''(x)(v_1)(y_n), \]

and

\[ [S^G_{n-1}'](x, v_1, \ldots, v_{n-1})(y_1, \ldots, y_n) = \]

\[ = \sum_{k=2}^{n-1} \frac{(n - 1)!}{k!} \left\{ \sum_{i_1 + \ldots + i_k = n-1 \atop i_1, \ldots, i_k \in \{1, \ldots, n-1\}} \frac{1}{i_1! \ldots i_k!} \right. \]

\[ \left. G^{(k+1)}(x)(y_1)(v_{i_1}) \ldots (v_{i_k}) \right. \]

\[ + G^{(k)}(x)(y_{i_1+1})(v_{i_2}) \ldots (v_{i_k}) + \ldots + G^{(k)}(x)(v_{i_1}) \ldots (v_{i_{k-1}})(y_{i_k+1}) \right\}. \]
\[ + G^{(k)}(x)(y_{i+1})(v_{i+2}) \ldots (v_k) + \ldots + G^{(k)}(x)(v_1) \ldots (v_{i-1})(y_{i+1}) + G''(x)(y_{i+1})(v_{i+2}) \ldots (v_k) + G'(x)(y_n). \]

We deduce that \( T_{(x,v_1,v_2,\ldots,v_{n-1})} M_S^{(n)} = g'(x,v_1,\ldots,v_{n-1})^{-1}(0) \) and, by Theorem 4.2, we obtain (4.10).

**Corollary 4.1.** Let \( H \) be a real Hilbert space of inner product \(<,>\) and norm \( \| \cdot \| \).

Then, in the case of the sphere \( S(r) = \{ x \in H, \| x \| = r \}, r > 0 \), the sets given by (4.6) and (4.9) become respectively

\[ M_S^{(3)}(r) = \{ (x,v_1,v_2) \in U \times H \times H, \| x \| = r, < x, v_1 > = 0, \| v_1 \|^2 + < x, v_2 > = 0, < x, F(x) > = 0, 3 \leq j \leq n-1, \]

\[ < x, v_j > + \frac{1}{2} \sum_{k=1}^{j-1} \binom{j}{k} < v_k, v_{j-k} > = 0, n > 3. \]

**Proof.** In this case \( S(r) = G^{-1}(0) \), with \( G(x) = \frac{1}{2}(\| x \|^2 - r^2) \), \( \forall x \in H \).

Then, from (4.6) and (4.9) we obtain (4.11) and (4.12), respectively, since

\[ G''(x)(y) = < x, y >, \forall y \in H, \]

\[ G''(x)(y)(v) = < y, v >, \forall y, v \in H, \]

and \( G^{(k)}(x), k \geq 3 \) are identically zero.

**Corollary 4.2.** Let \( U \subset H \) be an open subset of the Hilbert space \( H \), with \( S(r) \subset U \). Assume that \( F : U \rightarrow H \) is locally Lipschitz and the mapping \( h(x) = < x, F(x) > \) is Fréchet differentiable on \( U \).
Then $S(r)$ is a flow-invariant set with respect to the equation $u'''(t) = F(u(t)), t \geq 0$, if and only if

$$3\|v_2\|^2 + 3 < v_1, F(x) > + h'(x)(v_1) = 0, \forall (x, v_1, v_2) \in M^{(3)}_{S(r)},$$

and $S(r)$ is a flow-invariant set for (4.1) if and only if

$$n < v_2, v_{n-1} > + n < v_1, F(x) > + h'(x)(v_1) = 0, \forall (x, v_1, \ldots, v_{n-1}) \in M^{(n)}_{S(r)}.$$

**Proof.** These results follow straight forward from Theorems 4.2 and 4.3, taking into account the explicit form of the Fréchet derivatives of $G(x) = \frac{1}{2}(\|x\|^2 - r^2), \forall x \in H$, and the fact that $(G'(x)(\cdot), G''(x)(v_1)(\cdot)) : H \to \mathbb{R} \times \mathbb{R}$ is onto for every $(x, v_1, \ldots, v_{n-1}) \in M^{(n)}_{S(r)}$. Indeed, for any $(y_1, y_2) \in \mathbb{R} \times \mathbb{R}$, we can find $u = \frac{xy_1}{r^2} + \frac{v_1y_2}{\|v_1\|^2}$ such that $< x, u > = y_1$ and $< v_1, u > = y_2$, as $\|x\| = r$ and $< x, v_1 > = 0$. 
Bibliography


