NONLINEAR TRACKING BY TRAJECTORY REGULATION CONTROL USING BACKSTEPPING METHOD

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This thesis presents the development of nonlinear tracking by Trajectory Regulation Control using the Backstepping method. The purpose of this thesis is to perform a feasibility study on the Backstepping method. A brief background of the need for nonlinear tracking control techniques is presented with an overview on stabilization, based on control Lyapunov functions. The Backstepping method employed the design of a tracking controller for a benchmark nonlinear plant that is unstable and nonminimum phase. The design is implemented and tested in MATLAB/SIMULINK. The Backstepping controller is compared against Trajectory Linearization Control and Sliding Mode in tracking performance and robustness testing. Results of Backstepping model show an increased tracking performance and similar robustness when compared to Trajectory Linearization Control and Sliding Mode Control.

Approved:

J. Jim Zhu
Professor of Electrical Engineering
Acknowledgments

I dedicate this thesis to the following people who help bring me to this point in my career and will help continue my success for the coming years.

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<th>Description</th>
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<tr>
<td>AG&amp;C</td>
<td>Advanced Guidance and Control</td>
</tr>
<tr>
<td>BIBO</td>
<td>Bounded Input Bounded Output</td>
</tr>
<tr>
<td>BS</td>
<td>Backstepping</td>
</tr>
<tr>
<td>clf</td>
<td>Control Lyapunov Function</td>
</tr>
<tr>
<td>DOF</td>
<td>Degree-of-Freedom</td>
</tr>
<tr>
<td>DOA</td>
<td>Domain of Attraction</td>
</tr>
<tr>
<td>GARTEUR</td>
<td>Group for Aeronautical Research and Technology in Europe</td>
</tr>
<tr>
<td>GAS</td>
<td>Globally Asymptotically Stable</td>
</tr>
<tr>
<td>GES</td>
<td>Globally Exponentially Stable</td>
</tr>
<tr>
<td>I/O</td>
<td>Input/Output</td>
</tr>
<tr>
<td>ITAG&amp;C</td>
<td>Integrated Testing of Advanced Guidance and Control</td>
</tr>
<tr>
<td>LTI</td>
<td>Linear Time-Invariant</td>
</tr>
<tr>
<td>LTV</td>
<td>Linear Time-Varying</td>
</tr>
<tr>
<td>NASA</td>
<td>National Aeronautics and Space Administration</td>
</tr>
<tr>
<td>NL</td>
<td>Nonlinear</td>
</tr>
<tr>
<td>NLTI</td>
<td>Nonlinear Time-Invariant</td>
</tr>
<tr>
<td>NLTV</td>
<td>Nonlinear Time-Varying</td>
</tr>
<tr>
<td>RLV</td>
<td>Reusable Launch Vehicle</td>
</tr>
<tr>
<td>SISO</td>
<td>Single Input Single Output</td>
</tr>
<tr>
<td>s.t.</td>
<td>such that</td>
</tr>
<tr>
<td>TLC</td>
<td>Trajectory Linearization Control</td>
</tr>
<tr>
<td>TLO</td>
<td>Trajectory Linearization Observer</td>
</tr>
<tr>
<td>TRC</td>
<td>Trajectory Regulation Control</td>
</tr>
<tr>
<td>w.r.t.</td>
<td>with respect to</td>
</tr>
<tr>
<td>Xi</td>
<td>MATLAB Representation for $\xi$</td>
</tr>
</tbody>
</table>
∀ ................................................................................................................................. for all

∃ ................................................................................................................................. there exists

◊ ................................................................................................................................. end of a definition

∇ ................................................................................................................................. end of a theorem

\mathbb{R} ...................................................................................................................... set of real numbers
Chapter 1 Introduction

1.1 Background

On December 17, 1903, two men by the names of Wilber and Orville Wright designed and tested the first controlled powered air machine that maintained a sustained flight with one person on board. Since then, mankind's fascination with flight has advanced from single propeller airplanes to jet-powered passenger planes, modern fighters and space vehicles. As these modern aircraft take to the skies, new advanced methods in control techniques must be developed to ensure maximum performance and safety.

In order for these modern aircraft to achieve maximum performance and safety, there is a need for advanced nonlinear tracking control techniques. The need for advanced nonlinear tracking control techniques can be shown from various studies. Some of the studies are about the failures and challenges in Flight Control of Advanced Aerospace Vehicles written by Hanson. [14] This paper talks about how 41% of the launch vehicle failures that transpired since the 1990s could have been avoided by advance nonlinear control prior to actual hardware testing. Also, Croft [4] discusses how almost 50% of all aircraft failures were the result of loss of control as shown in Figure 1.1.1.

The need for advanced nonlinear tracking control techniques has brought about significant research effort in the last decade. The European Flight Control Challenge (GARTEUR) [19] was started in the 1990s, which was geared toward the improvement and optimization of aircraft flight control design using advance nonlinear control techniques. The GARTEUR Project ([7], [8]) used a simulation model of a commercial aircraft in MATLAB/SIMULINK as a benchmark test bed for flight controllers designed using advanced nonlinear techniques. This allowed various teams to develop and test nonlinear tracking control techniques comparing the benefits and drawbacks of each
other's methods. As for space exploration, the same type of comparison studies were created for reusable launch vehicles such as for NASA's X-33 Advanced Guidance and Control ([10], [11], [12]) and 2nd generation RLV Integrated Testing of Advanced Guidance and Control ([13], [14]). These two programs sponsored various teams to develop different types of nonlinear flight controllers based on reusable launch vehicle designs such as X-33. These research programs allow for new nonlinear tracking control techniques to be explored and determine their feasibility and advantage.

![Figure 1.1.1 Aviation Accident Causes](image)

Figure 1.1.1 Aviation Accident Causes

Since each of these nonlinear tracking control techniques have advantages as well as drawbacks, studies must be performed for each technique to determine their feasibility to a task. For this thesis, a feasibility study of the Trajectory Regulation Control (TRC) Nonlinear Tracking Controller using the Backstepping (BS) Method will be demonstrated. The true potential of Backstepping was discovered by Kokotovic [23]. With robust Backstepping, Kokotovic achieved global stabilization in the presence of disturbances. In order for this method to be successfully demonstrated, the tracking
performance and robustness of the TRC-BS needs to be determined as well as a comparison with other nonlinear tracking controller design techniques. Last, the pros and cons of the TRC-BS method need to be identified. The goal of the thesis is to gain a better understanding of the TRC-BS in terms of feasibility, tracking performance, robustness and the pros and cons.

In order to achieve the desired goals, a list of objectives that need to be completed is identified. The first objective is to demonstrate the feasibility of TRC-BS. This will entail research on the method of developing a Backstepping controller by surveying journals or conference articles that successfully implemented the Backstepping method into their design. With a successful research survey, the TRC-BS can then be designed for the benchmark problem used. The next objective is to determine tracking performance and robustness of the TRC-BS. The approach to this objective is to perform an experimental verification to determine the tracking performance and robustness against a benchmark problem that is highly nonlinear, unstable and nonminimum phase. The last objective is to identify the pros and cons of the TRC-BS. Comparing the TRC-BS with other design methods using the same benchmark will also accomplish this task.

The anticipated results of this thesis is to obtain improved tracking performance and robustness using the Backstepping Method compared to other control techniques being used today. In order to achieve this highly desired result, the proper research needs to be performed on the theory needed to implement such a design. Such further research includes Modern Control Theory ([17], [21], [27], [28]), Lyapunov theory ([2], [32], [24], [36]), Canonical Form Representation ([19], [22], [26], [35]), and previous Backstepping Design Methods that were implemented and tested [6], [9], [32], [36]). Once this theory that has been researched and then applied mathematically, the next step is to implement the Backstepping Method. Once the system has been implemented, then the Backstepping model can be tested against other control systems to determine if the Backstepping model has improved tracking performance and robustness.
Even though one would hope that the Backstepping Controller design does yield an improved robustness and performance, there are still drawbacks that need to be taken into consideration. The first drawback is due to the fact that the Backstepping Method does not ignore the nonlinear terms of the plant in controller design. This could then yield highly complex or unsolvable system of equations based on the type of system design being implemented. Also the Backstepping Method uses a Lyapunov function in order to take into account of the nonlinear terms. So based on the complexity of the nonlinear plant, the Lyapunov function may not be found within a reasonable amount of time. Also, based on the theory, there is no one Lyapunov function that can be used. So choosing one Lyapunov function that does yield a stabilizing tracking controller, there could be another Lyapunov function that yields a more robust stabilizing tracking controller. Therefore the use of a benchmark is a good practice to see how critical these drawbacks are to a system being designed.

With that said, the last topic of this background is the significance of the results that will be obtained. First of all, even with the drawbacks to the system design, we would like to see that the Backstepping Trajectory Regulation Control is feasible. With the TRC-BS being feasible, no gain scheduling or linear approximation of the plant is required. This will increase the systems overall tracking performance and robustness. In effect, the effort in designing and testing of this nonlinear system is worthwhile. By designing a TRC Backstepping controller on the benchmark platform, it has helped my understanding of nonlinear design techniques and would hopefully help others as well.

1.2 Overview of Thesis

This thesis consists of the following chapters. Chapter 1 is on the introduction to this thesis that gives a background on the advanced nonlinear tracking control techniques as well as the need for such advanced techniques.

In Chapter 2, the technical preliminaries will first include the concept of Lyapunov stability. Then the nonlinear tracking problem will be introduced. From the
NL tracking problem, the benchmark problem used for this thesis will be described and analyzed. Last in the technical preliminaries, the baseline designs will be presented for the Trajectory Linearization Control (TLC), Sliding Mode Control (SMC), Dynamic pseudo-inverse of the plant and a Trajectory Linearization Observer.

Chapter 3 presents the TRC-BS design technique, implementation and verification. At this stage the controller is designed for tracking a reference command with the baseline pseudo-inverse and observer. Also, the analytical design of the components of the system will be implemented into MATLAB/SIMULINK. Verification of this newly created Backstepping control system model will be performed for zero input response and zero-state response to some nominal tracking commands.

In Chapter 4, a comparison study will be performed to test the Backstepping design technique against other baseline advanced nonlinear tracking controllers, i.e. the TLC and SMC designs. The comparison method will be based on a simulation study with each nonlinear design technique chosen. The test matrix will compare each nonlinear design to various types of input responses such as step, ramp and staircase. The test results will analyze and compare each of the nonlinear systems tracking performance. A robustness study will include gain margin, delay margin, domain of stability and disturbances for comparison between design models.

Last, in Chapter 5, the conclusions of the thesis will be presented which will determine the feasibility of the Backstepping controller. Future work will also be explained in order to improve and further the Backstepping design.
Chapter 2 Technical Preliminaries

2.1 Lyapunov Stability of Motion

The main purpose of this thesis is to design nonlinear tracking controllers. Perfect tracking amounts to driving tracking error to zero asymptotically or, in the sense of Lyapunov, to stabilize the tracking error dynamics.

To analyze the stability of motion, based upon Lyapunov's principle, first consider the stability of solutions to nonautonomous systems of the form

\[ \dot{x} = f(t, x) \]  \hspace{1cm} 2.1.1

where \( t \) is the time and \( x \) is the state.

2.1.1 Equilibrium of Motion

In order to properly define the equilibrium state of motion, an equilibrium state is denoted as \( x_e \).

**Definition 2.1.1** A vector \( x_e \in \mathbb{R}^n \) is called an equilibrium state of a nonlinear dynamic system

\[ \dot{x} = f(t, x), \; x(t_0) = x_0, \; t \geq t_0 \]  \hspace{1cm} 2.1.2

if \( f(t, x_e) \equiv 0 \). Furthermore, an equilibrium point is said to be isolated if there exists a \( \delta > 0 \) such that for any other equilibrium point \( \bar{x}_e \) of 2.1.2, \( \|x_e - \bar{x}_e\| > \delta \).

This definition tells us that the operating point that is to be maintained is the equilibrium state of the closed-loop system. Therefore, if \( x \) in 2.1.2 represents the state tracking error of the nonlinear system, then \( x_e = 0 \) would be the equilibrium of the perfect tracking of the closed-loop system.
2.1.2 Stability of Motion

When discussing the stability of motion, this thesis will use the assumption that there is an isolated equilibrium point, \( x_e = 0 \). In order to discuss the stability of motion, the notion of attractiveness of an equilibrium point must be defined.

**Definition 2.1.2** An equilibrium \( x_e = 0 \) of Equation 2.1.1 is said to be attractive if for all \( t_0 \geq T_0 \), there exists an \( \eta = \eta(t_0) > 0 \) such that \( \|x_0\| < \eta(t_0) \) implies \( s(t, t_0, x_0) \to 0 \) as \( (t - t_0) \to \infty \).

**Definition 2.1.3** An equilibrium \( x_e = 0 \) of Equation 2.1.1 is said to be uniformly attractive if for all \( t_0 \geq T_0 \), there exists an \( \eta > 0 \), independent of \( t_0 \), such that \( \|x_0\| < \eta \) implies \( s(t, t_0, x_0) \to 0 \) as \( (t - t_0) \to \infty \), uniformly in \( t_0, x_0 \), i.e. given an \( \epsilon > 0 \), there exists a \( T = T(\epsilon) > 0 \), independent of \( t_0, x_0 \), such that \( \|s(t, t_0, x_0)\| < \epsilon \) whenever \( t > t_0 + T \).

This means that an equilibrium is attractive if all solutions within a norm bound \( \eta \) at \( t_0 \), eventually converge to the equilibrium as \( t \to \infty \). An equilibrium is considered uniformly attractive if all solutions nearby eventually converge to the equilibrium as \( t \to \infty \), with a domain of attraction and convergence rate that are independent of the initial time \( t_0 \). Now that the concept of attractiveness has been established, the following definition of the stability of an equilibrium point can be given.
Definition 2.1.4 The isolated equilibrium $x_e = 0$ of 2.1.2 is said to be

(a) **stable** if for all $t_0 \geq T_0$ and $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that $\|x_0\| < \delta$ implies that $x(t) = \|s(t, t_0, x_0)\| < \epsilon$, for all $t \geq t_0$. Otherwise it is said to be **unstable**.

(b) **uniformly stable** if it is stable and in addition, $\delta = \delta(\epsilon)$, independent of $t_0$.

(c) **asymptotically stable** if it is stable and attractive.

(d) **uniformly asymptotically stable** if it is uniformly stable and uniformly attractive.

(e) **(uniformly) exponentially stable** if there exists constants $\delta, k, \gamma > 0$ such that $\|x_0\| < \delta$ implies $x(t) = \|s(t, t_0, x_0)\| \leq k\|x_0\|e^{-\gamma(t-t_0)}$.

Thus by using part (a) of Definition 2.1.4, the implication of Lyapunov stability is that, given any prescribed tolerance $\epsilon$ of deviation of $x(t)$ from the operating point $x_e = 0$, an initial tolerance $\delta = \delta(t_0, \epsilon)$ for $x(t_0)$ can be specified such that the $\epsilon$ tolerance is satisfied for all $t \geq t_0$. When these conditions are not satisfied, then the equilibrium is said to be unstable. This concept of stability is illustrated in Figure 2.1.1.

![Figure 2.1.1 Lyapunov Stability of an Equilibrium](image)

For part (b) of the definition, the equilibrium is uniformly stable instead of just stable because there is an initial tolerance $\delta$, that is independent of $t_0$ for all $t \geq t_0$. Now
in order for the equilibrium to be considered nonuniformly stable, the initial tolerance $\delta$ need to be dependent on $t_0$, i.e., it must not shrink as $t_0 \to \infty$. Figure 2.1.2 shows the difference between uniformly stable and nonuniformly stable. Note that the difference between uniformly and nonuniformly is whether or not the domain of stability is uniform for all $t_0$ as $t_0 \to \infty$.

In part (c) of the definition above, for an equilibrium to be asymptotically stable, in addition to the state trajectories staying within the given tolerance as with stability, the equilibrium must now be attractive, i.e. the state trajectories $x(t)$ must converge to zero as $t \to \infty$.

Uniform asymptotic stability in part (d) defines uniform stability as well as uniform attractiveness. This means that not only does the state trajectories stay within the given tolerance as with stability, but all solutions nearby will eventually converge to the equilibrium as $t \to \infty$.

(a) Uniform domain of stability: DOS is independent of $t_0$
The final part of Definition 2.1.4 (e) defines exponential stability. From this definition, it can be seen that exponential stability is the strongest since it completely satisfies all the criteria of uniform asymptotic stability as well as converging to $x_e$ exponentially as $t \to \infty$. Table 2.1.1 shows the comparison between the stability notions from Definition 2.1.4.

<table>
<thead>
<tr>
<th>Type of System</th>
<th>Relation of Equilibrium Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{x} = Ax$</td>
<td>(e) $\iff$ (d) $\iff$ (c) $\iff$ (b) $\iff$ (a)</td>
</tr>
<tr>
<td>$\dot{x} = A(t)x, A(t + T) = A(t)$</td>
<td>(e) $\iff$ (d) $\iff$ (c) $\iff$ (b) $\iff$ (a)</td>
</tr>
<tr>
<td>$\dot{x} = A(t)x$</td>
<td>(e) $\iff$ (d) $\iff$ (c) $\iff$ (b) $\iff$ (a)</td>
</tr>
<tr>
<td>$\dot{x} = f(x)$</td>
<td>(e) $\iff$ (d) $\iff$ (c) $\iff$ (b) $\iff$ (a)</td>
</tr>
<tr>
<td>$\dot{x} = f(t, x)$</td>
<td>(e) $\iff$ (d) $\iff$ (c) $\iff$ (b) $\iff$ (a)</td>
</tr>
</tbody>
</table>

Table 2.1.1 Relationship Between Equilibrium Stability

Using the notions of stability of an isolated equilibrium, we can now define the stability of motion as the stability of a nominal trajectory.

**Definition 2.1.5** Let $\overline{x}(t) = \overline{x}(t, t_0, x_0)$ be a nominal (solution) trajectory of Equation 2.1.2, and let $x(t) = s(t, t_0, x_0)$ be any (solution) trajectory of Equation 2.1.2 in the...
operating region $\Omega$. Define the tracking error variable $x(t) = x(t) - \bar{x}(t)$. Then the tracking error dynamics governed by

$$\dot{x} = \dot{x} - \ddot{x} = f(t, \bar{x} + \ddot{x}) - f(t, \bar{x}) = g(t, \ddot{x}, \bar{x}(t)),$$

has an isolated equilibrium $\ddot{x}(\ddot{x_e}) \equiv 0$. The stability of the nominal trajectory $\bar{x}(t)$ is then defined by the stability of the null equilibrium $\ddot{x}(t) \equiv 0$.

The stability of a nominal trajectory will ensure that for any nominal path of $\bar{x}(t)$, as $t \to \infty$, $x(t)$ can be viewed as acquiring the nominal (commanded) trajectory, which is shown by Figure 2.1.3.

![Figure 2.1.3 Stability of a Nominal Trajectory [38]](image)

The last topic in the stability of motion is consider the input-output stability. Input-output stability is the stability of a forced system under perturbations of the input signal. An input-output system is said to be BIBO stable when a bound placed on the input resulting in a bound on the output as defined below.

**Definition 2.1.6** A nonlinear control system

$$\begin{align*}
\dot{x} &= f(t, x, u), \quad x(t_0) = x_0, \quad t \geq t_0 \\
y &= h(t, x, u)
\end{align*}$$

2.1.3
is said to be BIBO (or $L_\infty$) stable if for all bounded input $u(t)$, the output $y(t)$ is also bounded. That is, for all $x_0 \in \mathbb{R}^n, t \geq t_0$,

$$\|u(t)\| < M = M(x_0) \Rightarrow \|y(t)\| < N = n(x_0)$$  \hspace{1cm} 2.1.4

It is said to be finite gain BIBO (or $L_\infty$) stable if nonnegative constants $\gamma, \beta$ can be found such that the constants $M, N$ in 2.1.4 satisfy

$$N = \gamma M + \beta$$  \hspace{1cm} 2.1.5

The nonlinear control system 2.1.3 is said to be small signal (finite gain) BIBO stable if there exists an $r > 0$ such that 2.1.4 (resp. (2.1.5)) holds for all $\|u(t)\| < r$.

Table 2.1.2 shows the relationship between exponential stability of the null equilibrium, $(x, u) = (0, 0)$, and the BIBO stability, in which the conditions of $f$ and $h$ are defined as follows:

\begin{itemize}
  \item[(a)] $f$ is piecewise continuous in $t$ and locally Lipschitz in $(x, u)$.
  \item[(b)] $h$ is piecewise continuous in $t$ and Lipschitz continuous in $(x, u)$.
  \item[(c)] $f(t, x, u) = A(t)x + B(t)u$, $h(t, x, u) = C(t)x + D(t)u$, where $A(t), B(t), C(t)$ $D(t)$ are continuous and bounded.
\end{itemize}

\begin{tabular}{|c|c|c|}
  \hline
  Conditions on $f, h$ & Stability of $(x, u) = (0, 0)$ & Rel. & I/O Stability \\
  \hline
  (a) and (b) & local exponential & $\Rightarrow$ & small signal finite gain BIBO \\
  (a) and (b) & global exponential & $\Rightarrow$ & finite gain BIBO \\
  (a) and (b) & local uniform asymptotic & $\Rightarrow$ & small signal BIBO \\
  (a) and (b) & global uniform asymptotic & $\Rightarrow$ & BIBO \\
  (c) (LTV systems) & exponential & $\Rightarrow$ & finite gain BIBO \\
  \hline
\end{tabular}

Table 2.1.2 Relationship Between Equilibrium Stability and I/O Stability [38]
2.1.3 Lyapunov Stability Criteria

Now that the concepts of equilibrium of motion and stability of motion for a nonlinear dynamic system have been defined, the next topic is how to assess the stability of an equilibrium. For this thesis, the control design will attempt to achieve global exponential stability (GES) for the nominal design model. In order to achieve GES, Lyapunov’s First Method and Lyapunov’s Second Method can be used and are explained in the following sections.

2.1.3.1 First Method

In the indirect method, the Lyapunov Exponent, denoted by $\lambda$, was originally defined for linear time-varying systems $\dot{x} = A(t)x$ to measure the sensitivity of a system's behavior to initial conditions, thereby determining the stability.

**Definition 2.1.7 Lyapunov Exponent [2]**

\[ a) \text{Let } x(t,t_0) \text{ be a (real-valued) solution to } \dot{x} = A(t)x \text{ with a fixed initial time } t_0. \text{ Then, the Lyapunov characteristic exponent for } x(t,t_0) \text{ is defined by} \]

\[ \lambda(t_0) = \lambda(x(t,t_0)) = \lim_{t \to \infty} \sup \frac{\log \|x(t,t_0)\|}{t}, \quad 2.1.6 \]

where \[ \|x(t,t_0)\| = \sum_{i=1}^{n} |x_i(t,t_0)|. \]

\[ b) \text{A fundamental solution matrix} \]

\[ X(t,t_0) = [x_1(t,t_0) \ x_2(t,t_0) \ \cdots \ x_n(t,t_0)] \]

\[ Z(t,t_0) = [z_1(t,t_0) \ z_2(t,t_0) \ \cdots \ z_n(t,t_0)], \quad 2.1.9 \]

\[ \sum_{k=1}^{n} \lambda(x_k(t,t_0)) = \nu(X) \leq \nu(Z) = \sum_{k=1}^{n} \lambda(x_k(t,t_0)). \quad 2.1.10 \]

The set \( \{\lambda(x_k(t,t_0))\}_{k=1}^{n} \) is called a Lyapunov spectrum w.r.t. \( t_0 \).
c) An LTV system $\dot{x} = A(t)x$ with a fixed initial time $t_0$ is said to be regular at $t_0$ in the sense of the Lyapunov if any Lyapunov normal fundamental solution matrix $X(t, t_0)$ satisfies

$$\nu(X) = \limsup_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \text{tr} A(\tau) d\tau.$$  \hspace{1cm} 2.1.11

The Lyapunov Exponent can be used to access the exponential stability of the null equilibrium point. This is shown in part (a), when $\lambda$ is negative, points near the equilibrium point will converge to the equilibrium point as $t \to \infty$ yielding a stable response. When $\lambda$ is positive, points near the equilibrium point will diverge from the equilibrium point yielding an unstable response, which is represented by the following theorem.

**Theorem 2.1.1** An LTV system $\dot{x} = A(t)x$ is asymptotically stable at $t_0$ with an exponential convergence rate if all Lyapunov Characteristic Exponents in the Lyapunov spectrum are negative.

\vspace{1cm}

\[\nabla\]

While the concept of Lyapunov Exponents have been extended to nonlinear systems, it is most convenient to apply the first method to a nonlinear system via its linearization at the equilibrium point of concern per the following theorem.

**Theorem 2.1.2** Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x} = f(t, x),$$  \hspace{1cm} 2.1.12

where $f : [0, \infty) \times D \to \mathbb{R}^n$ is continuously differentiable, $D = \{x \in \mathbb{R}^n ||x|| < r\}$, and the Jacobian matrix $[\partial f / \partial x]$ is bounded and Lipschitz on $D$, uniformly in $t$. 
Let

\[ A = \frac{\partial f}{\partial x}(t, x)|_{x=0}. \quad 2.1.13 \]

Then, the origin is an exponentially stable equilibrium point for the nonlinear system if and only if it is an exponentially stable equilibrium point for the linear system

\[ \dot{x} = A(t)x. \quad 2.1.14 \]

\[ \nabla \]

2.1.3.2 Second Method

Lyapunov's second method uses a positive definite, continuously differentiable function, known as a Lyapunov function, in the stability assessment.

**Theorem 2.1.3 Lyapunov's second method (direct method):** Let \( x_e = 0 \) be an isolated equilibrium point for 2.1.1 and \( D \subset \mathbb{R}^n \) be a domain containing \( x_e \). Let \( V : [0, \infty] \times D \to \mathbb{R} \) be a continuously differentiable function s.t.

\[
\begin{align*}
  k_1\|x\|^a &\leq V(t, x) \leq k_2\|x\|^a \\
  \dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) &\leq -k_3\|x\|^a
\end{align*}
\]

\( \forall t \leq 0 \) and \( \forall x \in D \), where \( k_1, k_2, k_3, a \) are positive constants. Then, \( x_e \) is exponentially stable or globally exponentially stable if \( D = \mathbb{R} \).

\[ \nabla \]

A. M. Lyapunov's reasoning in using the scalar function \( V(x) \), is that \( V(x) \) can be considered as the energy contained in the system. So from this standpoint, in order to get the system to move toward the equilibrium point, the \( V(x) \) must be shown to continuously decrease in energy. For the purpose of this thesis, Lyapunov's direct method will be used over the indirect method.

2.2 Nonlinear Tracking Control

This section will discuss the general design process for tracking control of a nonlinear system. All of nonlinear control techniques have the same model scheme
illustrated by the following figure. Figure 2.2.1 shows the overall architecture of a nonlinear tracking control system.

![Figure 2.2.1 Overall System Design](image)

This design is comprised of the nonlinear plant, pseudo-inverse plant model and the error regulator. The error regulator will be broken down into the observer design and the stabilizing controller. The following sections of Chapter 2 will discuss the general scheme of what comprises the nonlinear tracking control. For each tracking controller that will be analyzed in the sequel, the pseudo-inverse and error state observer will be identical to aid in meaningful comparisons. In the last sections of Chapter 2, the benchmark problem that will be used for each nonlinear tracking design will be analyzed. Then the pseudo-inverse, observer for the benchmark problem as baselines for the subsequent studies as well as the Trajectory Linearization Control and Sliding Mode Control. Then in Chapter 3, the Backstepping error regulator will be introduced. That chapter will comprise of the design of the error regulator with the benchmark problem, its implementation with the pseudo-inverse and observer, and the verification process. Last,
in Chapter 4, a comparison study between all three nonlinear systems will be done to show the benefits and drawbacks of each.

2.2.1 Nonlinear Tracking Problem

In order to set up a stabilizing controller design, the nonlinear tracking system will be translated into a coordinate system centered on the nominal trajectories. This tracking scheme is the standard theme used by nonlinear systems, which was represented earlier in Figure 2.2.1. To start off, consider that the output tracking problem for a nonlinear dynamic system

\[
\dot{\xi}(t) = f(\xi(t), \mu(t), \theta(t)) \\
\eta(t) = h(\xi(t), \mu(t), \theta(t))
\]

where \(\xi(t) \in \mathbb{R}^n\), \(\mu(t) \in \mathbb{R}^l\), \(\eta(t) \in \mathbb{R}^m\), \(\theta \in \mathbb{R}^r\) are the state, input, output and time-varying parameters vectors, respectively. Next, we define the nominal state and output trajectories defined by \(\bar{\xi}(t), \bar{\eta}(t), \bar{\mu}(t)\) as follows

\[
\dot{\bar{\xi}}(t) = f(\bar{\xi}(t), \bar{\mu}(t), \theta(t)) \\
\dot{\bar{\eta}}(t) = h(\bar{\xi}(t), \bar{\mu}(t), \theta(t))
\]

From here, the next step is to define the state and output tracking errors and the tracking error control

\[
\tilde{\xi}(t) = \xi(t) - \bar{\xi}(t) \\
\tilde{\eta}(t) = \eta(t) - \bar{\eta}(t) \\
\tilde{\mu}(t) = \mu(t) - \bar{\mu}(t)
\]

These three error variables are needed to represent the difference between the actual state and the nominal state, and between the actual output and the desired output. From these equations, the nonlinear tracking error dynamics can be defined as

\[
\dot{\tilde{\xi}}(t) = f(\tilde{\xi}(t) + \tilde{\xi}(t), \tilde{\mu}(t) + \tilde{\mu}(t), \theta(t)) - f(\bar{\xi}(t), \bar{\mu}(t), \theta(t)) \\
= F(\tilde{\xi}(t), \tilde{\mu}(t), \theta(t), \bar{\xi}(t), \bar{\mu}(t)) \\
\dot{\tilde{\eta}}(t) = h(\tilde{\xi}(t) + \tilde{\xi}(t), \tilde{\mu}(t) + \tilde{\mu}(t), \theta(t)) - h(\bar{\xi}(t), \bar{\mu}(t), \theta(t)) \\
= H(\tilde{\xi}(t), \tilde{\mu}(t), \theta(t), \bar{\xi}(t), \bar{\mu}(t))
\]
where the nonlinear time-varying functions $F, H$ are representations of the original nonlinear system in the translated coordinate system centered on the nominal trajectories $\bar{\xi}(t), \bar{\mu}(t), \bar{\eta}(t)$. From the nonlinear tracking error dynamics, asymptotic tracking can be achieved by a 2 Degrees-of-Freedom (DOF) controller, which is shown in Figure 2.2.1. The first DOF is an open-loop controller, which consists of a dynamic inverse input/output mapping $\eta \mapsto \mu$ of the nonlinear plant model to compute the nominal control $\bar{\mu}(t)$ for any given nominal trajectory $\bar{\eta}(t)$. The second DOF is a closed-loop tracking error regulator, which outputs the tracking error stabilizing control law $\hat{\mu}(t)$. This control law will account for modeling uncertainties, disturbances, and initial conditions.

2.2.2 Nominal Control by NL Dynamic Pseudo-Inversion

This section will discuss the techniques of designing a stable pseudo-inverse for a nonlinear dynamic system. For this to happen, the inverse system must achieve (small signal) finite gain BIBO stability. With that in mind, it is necessary to design for exponential stability. The objective of the pseudo-inverse design is to minimize the error in a suitable norm within the operating envelope and bandwidth. To exemplify the design method we must first consider the SISO, affine, autonomous nonlinear system

$$\begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x) + d(x)u
\end{align*}$$

Next, an operation called the Lie derivative [36] that takes the partial derivative of a scalar function $h(x)$ with respect to a vector field $f(x)$, is defined as

$$L_f h(x) = \frac{\partial h(x)}{\partial x} \cdot f(x)$$

Now let $\Omega$ be a region in $\mathbb{R}^n$ containing the origin. The system 2.2.3 is said to have a well-defined relative order $r = 0$ in $\Omega$ if $d(x) \neq 0 \forall x \in \Omega$. Suppose that $\forall x \in \Omega$,

$$y^{(k)} = L_f^k h(x) + L_g L_f^{k-1} h(x)u, \quad L_g L_f^{k-1} h(x) \equiv 0, \quad k = 1, 2, \cdots, r - 1$$
where $L_j^0 h = h$, $L_j^k h = L_f L_j^{k-1} h$. The system 2.2.3 is said to have a well-defined relative order $r > 0$ in $\Omega$, if $L_g L_f^{r-1} h(x) \neq 0 \forall x \in \Omega$ in the equation

$$y^{(r)} = L_f^r h(x) + L_g L_f^{r-1} h(x) u$$ 2.2.4

The notion of zerodynamics plays a critical role for 2.2.3 if it has a well-defined relative degree $r < n$. In order to define the notion of zerodynamics, let $z = [\zeta_1 | \zeta_2]^T$, where

$$\zeta_1 = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_r \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(r-1)} \end{bmatrix}$$

$$\zeta_2 = \begin{bmatrix} z_{r+1} \\ \vdots \\ z_n \end{bmatrix}$$

It is now possible to find a nonlinear state coordinate transformation $z = \Phi(x)$, where $\Phi(\cdot)$ is a smooth invertible function, such that the normal form of the nonlinear system can be defined as

$$\dot{z} = \frac{\partial \Phi}{\partial x} \dot{x} = \begin{bmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{bmatrix}$$

$$y = z_1$$

where

$$\dot{\zeta}_1 = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{r-1} \\ \dot{z}_r \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_r \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} L_f^r h(\Phi^{-1}(z)) \\ L_g L_f^{r-1} h(\Phi^{-1}(z)) \end{bmatrix}$$ 2.2.5a

is in the nonlinear phase variable canonical form and

$$\dot{\zeta}_2 = \begin{bmatrix} \dot{z}_{r+1} \\ \vdots \\ \dot{z}_n \end{bmatrix} = \Psi(z) = \Psi(\zeta_1, \zeta_2)$$
defines an \( m = n - r \) dimensional submanifold where by inverting \( u \rightarrow y^{(r)} \) of the closed loop system yields

\[
u = \frac{-L_f^r h(x)}{L_g L_f^{-1} h(x)} + \frac{1}{L_g L_f^{-1} h(x)} v \quad \text{2.2.5b}
\]

When this state feedback control is applied to the plant, it results in a chain or \( r \) integrators. Equation 2.2.5b is the internal dynamics for the closed loop system. The zerodynamics of the nonlinear system is then defined by

\[
\dot{\zeta}_2 = \Psi(0, \zeta_2) \quad \text{2.2.6}
\]

Thus, 2.2.3 is said to be (exponentially) minimum phase if its zerodynamics 2.2.6 is (exponentially) asymptotically stable.

The next step after determining the internal dynamics of the closed-loop system is to find the nominal control and its inverse plant. To begin, we define

\[
\eta(x) = \frac{1}{L_g L_f^{-1} h(x)}
\]

where \( \eta(x) \) is well-defined in the region \( \Omega \). Since we already have an inverse mapping of \( y^{(r)} \rightarrow u \) from 2.3.4, we then need the inverse I/O mapping from the nominal \( \bar{y}^{(r)} \rightarrow \bar{u} \).

\[
\bar{u} = -\eta(\bar{x}) L_f^r h(\bar{x}) + \eta(\bar{x}) \bar{y}^{(r)}
\]

Then, by substituting 2.2.7 into the nonlinear system, 2.2.3 yields the equation for the nominal state variables defined as

\[
\dot{x} = f(\bar{x}) + g(\bar{x})[\psi(\bar{x}) + \eta(\bar{x}) \bar{y}^{(r)}] \quad \text{2.2.8}
\]

In order for the inverse system defined by 2.2.7 or 2.2.8 to be (small-signal) finite gain BIBO stable, the origin \( x = 0 \) must be a (locally) exponentially stable equilibrium state for 2.2.7.
To make sure that 2.2.7 is stable, we need to approximate \( \tilde{y}^{(r)} \) with \( \tilde{y}^{(r)}(x, \bar{y}) \). Next, we must check the zerodynamics of 2.2.3 to make sure it is exponentially stable.

When the zerodynamics of 2.2.3 is exponentially stable, let

\[
\tilde{y}^{(r)}(x, \bar{y}) = - \sum_{k=1}^{r} a_k z_k + a_1 \bar{y} \quad 2.2.9
\]

\[
= - \sum_{k=1}^{r} a_k L_j^{k-1} h(x) + a_1 \bar{y}
\]

where \( \sum_{k=0}^{r} a_{k+1} \lambda^k \) with \( a_{r+1} = 1 \) is a Hurwitz polynomial. Then the stable pseudo-inverse of 2.2.3 is stabilized from \( \bar{y} \rightarrow \hat{u} \) as

\[
\dot{x} = \left[ f(x) - \eta(x) \sum_{k=0}^{r} a_{k+1} L_j^k h(x) g(x) \right] + a_1 \eta(x) g(x) \bar{y} \quad 2.2.10
\]

\[
\hat{u} = - \eta(x) \sum_{k=0}^{r} a_{k+1} L_j^k h(x) + a_1 \eta(x) \bar{y}
\]

This design, when the zerodynamics are exponentially stable, is used for a minimum phase plant.

For a nonminimum phase plant for which the zerodynamics of 2.2.3 is not exponentially stable, let

\[
\hat{y}^{(r)}(x, \bar{y}) = k_1(x) - \sum_{k=1}^{r} a_k L_j^{k-1} h(x) + k_2(\bar{y}) \quad 2.2.11
\]

where \( k_1(x) \) is to exponentially stabilize the unstable modes of the inverse system corresponding to the unstable zerodynamic parts of 2.2.3 and \( k_2 \) is to equalize the DC gain of the corresponding pseudo-identity. Therefore the stable pseudo-inverse of 2.2.3 of the closed-loop inverse system is defined as
\[
\dot{x} = \left\{ f(x) + \eta(x) \left[ k(x) - \sum_{k=0}^{r} a_{k+1} L_j^k h(x) \right] g(x) \right\} + \eta(x) g(x) k_2(\tilde{y})
\]
\[
\hat{u} = \eta(x) \left[ k(x) - \sum_{k=0}^{r} a_{k+1} L_j^k h(x) \right] + \eta(x) k_2(\tilde{y})
\]

This will achieve exponential stability which yields the required (small input) finite gain BIBO stability. The nonminimum phase inverse plant design for this thesis is shown in Figure 2.2.2.

Figure 2.2.2 Stable Pseudo-Inverse \( \tilde{y} \rightarrow \bar{u} = \hat{u} \)

2.2.3 Error Regulator

The current state of the art nonlinear stability control techniques include Gain Scheduling, Time-Varying Linear Control, and Lyapunov 2\textsuperscript{nd} method based nonlinear control. Gain Scheduling, shown in Figure 2.2.3, is currently being widely used in the industry but has been shown to have limitations in controlling agile aircraft and reusable launch vehicles. ([3], [22], [37]) The reason why Gain Scheduling is widely used is due to its simple LTI techniques, which linearizes the nonlinear plant at frozen states. By linearizing at frozen states, there is a need to schedule the controllers for various
operating points to cover the entire flight envelope. In addition to using frozen states, Gain Scheduling also linearizes these frozen states with frozen time making them time invariant. With the combination of both the linearization at frozen states with frozen-time renders the Gain Scheduling control to a slow varying constraint. This means that gain scheduling control may yield poor performance or possibly lose stability when tracking fast varying trajectories and/or when the system has fast varying parameters. With more agile systems like the RLV, the tracking trajectory and the fast change in vehicle parameters such as the mass due to fuel consumption, require fast varying tracking.

On the other hand, the Trajectory Linearization Control (TLC) uses a linear time-varying regulator shown in Figure 2.2.4. [25]
Figure 2.2.4 Trajectory Linearization Control System Design

By utilizing a time-varying regulator, the slow-varying restrictions that exist in Gain Scheduling are eliminated. Therefore in systems like the RLV, as the systems vary in time, the TLC has an improved response in the tracking performance and stability robustness. Even though the TLC is an improvement over Gain Scheduling, there is still a restriction that can hamper tracking performance. The restriction is that the TLC still relies on a linear approximation of the nonlinear plant. The reason for linearizing the plant is to allow for simpler computation but the drawback is that by neglecting the nonlinearities, the domain of stability may be greatly reduced.

The last nonlinear tracking control technique that this thesis will discuss is the next step beyond TLC, Trajectory Regulation Control (TRC) [38]. The TRC, as shown in Figure 2.2.5, uses a nonlinear time-varying regulator to improve the tracking performance over Gain Scheduling and TLC.
As with the TLC, the TRC has no slowly varying restrictions. But unlike TLC, TRC has no linear approximations. Therefore all the nonlinearities are included in the controller design process, which then yields greater tracking performance and domain of stability. The downside for not neglecting the nonlinear terms is that the computations may be too complex to develop a TRC for complicated problems.

2.2.4 Error State Observer

An observer is needed to estimate the state of the plant for implementation of the state feedback control law when the states cannot be measured or it is not desirable to measure the states. For the nonlinear system given as

\[
\begin{align*}
\dot{x} &= f(x(t), u(t)) \\
\eta &= h(x(t), u(t))
\end{align*}
\]

a nonlinear observer may be designed by

\[
\begin{align*}
\dot{x} &= f(\hat{x}, u) + H(\cdot)(\hat{\eta} - \eta) \\
\hat{\eta} &= h(\hat{\eta}, u)
\end{align*}
\]

Figure 2.2.5 Trajectory Regulation Control System Design
where \( \hat{\xi} \) is the estimated state, \( \hat{\eta} \) is the estimate output, \( f(\cdot, \cdot) \) and \( h(\cdot) \) are the corresponding functions in 2.2.12 and \( H(\cdot) \) is the observer gain to be designed using the Trajectory Linearization method.

The reason for performing trajectory linearization is to drive the nonlinear observer error \( \hat{x} = \hat{\xi} - \xi \to 0 \) exponentially as \( t \to \infty \) by designing a linear time-varying observer gain \( H(\cdot) \). By linearizing the observer model along \( \xi(t) \), the nominal trajectory for the observer, the observer error dynamics is defined as

\[
\dot{\hat{x}} = A_0 \hat{x} + H \hat{y} = (A_0 + H(\xi)C_0)\hat{x},
\]
\[
\hat{y} = C_0 \hat{x}
\]

where \( \hat{y} = \hat{\eta} - \eta \) and

\[
A_0 = A(\xi, \mu) = \frac{\partial}{\partial \xi} f(\xi, \mu) |_{\xi, \mu}, \quad C_0 = C(\xi, \mu) = \frac{\partial}{\partial \xi} h(\xi, \mu) |_{\xi, \mu},
\]
\[
B_0 = B(\xi, \mu) = \frac{\partial}{\partial \mu} f(\xi, \mu) |_{\xi, \mu}
\]

In order for the observer error dynamics to be exponentially stable, the PD eigenvalues of \( A_0 + H(\xi)C_0 \) must be properly chosen to have extended mean value in the left hand side of the complex plane. Since this system design uses the plant state \( \xi(t) \) as the nominal trajectory, the observer gain \( H(\xi, \mu) \) cannot be implemented. The TLO design then uses \( \tilde{\xi} \) to approximate the system states instead of \( H(\xi, \mu) \). In order to use the TLO design, the substitution from \( H(\xi, \mu) \) to \( H(\tilde{\xi}, \mu) \) must be verified to ensure stability of the observer and the overall closed-loop system design. The justification is established [39]. Therefore by utilizing the Trajectory Linearization Observer, the observer still yields exponential stability even with the simpler linearized tracking error dynamics. Further detail development of the Trajectory Linearization Observer can be found in [39].
2.2.5 Transformation to LTV Canonical Form

The design of TLC and TLO can be facilitated using canonical representation of the linearized tracking error dynamics and the linearized observer error dynamics. To start out, the process for the controller canonical form will be exemplified with a SISO LTV system given as

\[ \dot{x} = A(t)x + B(t)u \]
\[ y = C(t)x \]

In order for this system to have a controller canonical form, the pair \( \{A(t), B(t)\} \) of the LTV system must be uniformly completely controllable. This means that the controllability matrix

\[ V_{cc}(t) = F(t)D(t) \quad \text{rank} \quad V_{cc} \equiv n \]

has a full rank

for all \( t \geq t_0 \), where \( P_A = [\frac{d}{dt} - A(t)] \), and \( P_A^k = P_A^k \). If the system is uniformly completely controllable, then it is possible to transform this LTV system into controller canonical form by first defining a state coordinate transformation

\[ x = L(t)z \]

such that

\[ \dot{z} = A_c(t)z + B_c(t)u \]
\[ y = C_c(t)z \]
is the controller canonical form given by
\[ A_c(t) = L^{-1}(t)A(t)L(t) - L^{-1}(t)\dot{L}(t) \quad B_c(t) = L^{-1}(t)B(t) \]
\[
C_c(t) = C(t)L(t)
\]
\[
= \left[ c_{c1}(t) \quad c_{c2}(t) \quad \cdots \quad c_{cn}(t) \right]
\]
where \( A_c(t), B_c(t) \) and \( C_c(t) \) are the controller canonical representation of \( A(t), B(t) \) and \( C(t) \). Also, \( \{a_i(t)\}_{i=1}^{n} \) are the left-hand side coefficients of the scalar LTV differential equation given by
\[
[\delta^n + a_n(t)\delta^{n-1} + \cdots + a_2(t)\delta + a_1(t)]z = [\beta_n(t)\delta^{n-1} + \cdots + \beta_2(t)\delta + \beta_1(t)]u
\]
where \( \delta = \frac{d}{dt} \) is the derivative operator and \( z = x_1 \).

Next we define the coordinate transformation matrix by using the following formula
\[
L(t) = \left[ \left[ p_n(t) \quad Q_A p_n(t) \quad \cdots \quad Q_A^{n-1} p_n(t) \right]^T \right]^{-1}
\]
where
\[
p_n^T(t) = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} C^{-1}(t)
\]
is the \( n \)th row of \( C^{-1}(t) \), \( Q_A = [\delta + A^T(t)] \), and \( Q_A^k = Q_A Q_A^{k-1} \). From this point we can compute the controller canonical form for the desired LTV plant.

The observer canonical form is the dual of the controller canonical form. In order for the LTV system (2.2.13) to have an observer canonical form, the pair \( \{C(t), A(t)\} \) of the LTV system must be uniformly completely observable. This means that the observability matrix
\[
O(t) = \left[ C^T(t) \quad (-Q_A) C^T(t) \quad \cdots \quad (-Q_A)^{n-1} C^T(t) \right]^T
\]
has a full rank

\[ \text{rank } \mathcal{O}(t) \equiv n \]

for all \( t \geq t_0 \), where \( \mathcal{Q}_A = \left[ \delta + A(t) \right] \), and \( \mathcal{Q}_A^k = Q_A Q_A^{k-1} \). If the system is uniformly completely observable, then the LTV system can be transformed into the observer canonical form by first defining a state transformation

\[ x = T(t)z \]

such that

\[
\begin{align*}
\dot{w} &= A_o(t)w + B_o(t)u \\
y &= C_o(t)w
\end{align*}
\]

Then the coordinate transformation matrix can be found using the following formula

\[ T(t) = \left[ q_n(t) \quad (-P_A)q_n(t) \quad \cdots \quad (-P_A^{n-1})q_n(t) \right] \]

where

\[ q_n(t) = \mathcal{O}^{-1}(t)[0 \quad \cdots \quad 0 \quad 1] \]

is the \( n^{th} \) column of \( \mathcal{O}^{-1}(t) \). Now that the coordinate transformation matrix has been defined, the next step is to define the observer canonical form given by

\[
\begin{align*}
A_o(t) &= T^{-1}(t)A(t)T(t) - T^{-1}(t)\dot{T}(t) \\
B_o(t) &= T^{-1}(t)B(t) \\
C_o(t) &= C(t)T(t)
\end{align*}
\]

\[
\begin{bmatrix}
0 & \cdots & 0 & -a_1(t) \\
& \ddots & \vdots & \vdots \\
& & -a_n(t) & \\
I_{n-1} & & & \\
& & & \\
& & & \vdots \\
& & & & b_{01}(t) \\
& & & & b_{02}(t) \\
& & & & \vdots \\
& & & & b_{0n}(t)
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & \cdots & 0 & 1
\end{bmatrix}
\]

where \( A_o(t) \), \( B_o(t) \), and \( C_o(t) \) are the canonical representation of \( A(t) \), \( B(t) \) and \( C(t) \).
Also, \( \{a_i(t)\}_{i=1}^n \) are the right-hand side coefficients of the scalar LTV differential equation given by

\[
[\delta^n + a_n(t)\delta^{n-1} + \cdots + a_2(t)\delta + a_1(t)]y = [\beta_n(t)\delta^{n-1} + \cdots + \beta_2(t)\delta + \beta_1(t)]u
\]

where \( \delta = \frac{d}{dt} \) is the derivative operator.

2.3 A Benchmark Problem

In this section, we now define the nonlinear test bed that will be used in all nonlinear control designs discussed in this thesis. The nonlinear test bed used is from Nonlinear Systems by Hassan K. Khalil [22]. The reason why this system was used versus a specific design like an aircraft model, is that if a controller design can stabilize the tracking error for this benchmark, then the same method will have the potential to stabilize other systems tracking error dynamics.

For this nonlinear test bed, the plant is set up such that the system is highly nonlinear unstable and nonminimum phase. Since the system is nonminimum phase, the zero dynamics of the plant are not asymptotically stable. So the goal is to use the Backstepping method to stabilize the system to compare with the tracking performance and robustness of current techniques being used such as Gain Scheduling or Trajectory Linearization. In order to start this design process, the first thing is to analyze the plant dynamics.

Consider the following SISO affine system representation for a nonlinear system is

\[
\begin{align*}
\dot{\xi} &= f(\xi) + g(\xi)\eta \\
\eta &= h(\xi) + d(\xi)\mu
\end{align*}
\] 2.3.1
where $\xi(t) \in \mathbb{R}^n$ is the state and $\mu \in \mathbb{R}$ is the control input. The nonlinear test bed that will be used is of this form and is given as

$$
\begin{align*}
\dot{\xi}_1 &= \tan(\xi_1) + \xi_2, \quad |\xi_1| < \frac{\pi}{2} \\
\dot{\xi}_2 &= \xi_1 + \mu \\
\eta &= \xi_2
\end{align*}
$$

which is in the nominal form of 2.3.2 with

$$
\begin{align*}
f(\xi) &= \begin{bmatrix} \tan(\xi_1) + \xi_2 \\ \xi_1 \end{bmatrix} \\
g(\xi) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
h(\xi) &= \xi_2 \\
d(\xi) &= 0
\end{align*}
$$

To analyze the stability of the nonlinear plant, assume zero control input. Therefore setting $\mu = 0$ of 2.3.2 yields

$$
\begin{align*}
\tan(\xi_1) + \xi_2 &= 0 \\
\xi_1 &= 0
\end{align*}
$$

with the only equilibrium solution at the origin $(0, 0)$. By linearizing 2.3.2 at the origin, the characteristic equation is found to be $s^2 - s - 1 = 0$. Since the roots of the characteristic equation are $s_1 = -0.6180$ and $s_2 = 1.6180$, the equilibrium is unstable.

For determining the stability of the zero dynamics of the nonlinear system, we first check what the relative degree $r$ is in order to perform the correct coordinate transformation. Since $d(\xi) = 0$, the relative degree $r$ is found as

$$
\dot{\eta} = \dot{\xi}_2 = \xi_1 + \mu
$$

This implies that the system has a well-defined relative order $r = 1$, $\forall \xi_1, \xi_2$. Since we have a well-defined relative order of $r = 1$, we can then find the normal coordinates, which are defined as

$$
\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \Phi(\xi) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \xi_2 \\ \xi_1 \end{bmatrix}
$$
From the normal coordinates, the normal form of the nonlinear equation can be determined.

\[
\begin{align*}
\dot{\zeta}_1 &= \zeta_2 + \mu \\
\dot{\zeta}_2 &= \tan(\zeta_2) + \zeta_1 = \Psi(\zeta_1, \zeta_2) \\
\eta &= \zeta_1
\end{align*}
\]

With the function \( \Psi(\zeta_1, \zeta_2) \), the zerodynamics can be found by setting \( \zeta_1 = 0 \). Therefore the zerodynamics are

\[
\dot{\zeta}_2 = \Psi(0, \zeta_2) = \tan(\zeta_2)
\]

Linearization at \( \zeta_2 = 0 \) yields \( \dot{\zeta}_2 = \zeta_2 \), thus \( \zeta_2 = 0 \) is unstable and the system is nonminimum phase.

The next thing that must be performed before the error regulator is developed is to derive the tracking error dynamics. To start out developing the tracking error dynamics, we first define the nominal trajectories for the nonlinear plant which are given as

\[
\begin{align*}
\dot{\xi}_1 &= \tan(\xi_1) + \xi_2 \\
\dot{\xi}_2 &= \xi_1 + \mu \\
\eta &= \xi_2
\end{align*}
\]

Then the error variables are defined as

\[
\begin{align*}
\tilde{\xi} &= \xi - \bar{\xi} \\
\tilde{\mu} &= \mu - \bar{\mu} \\
\tilde{\eta} &= \eta - \bar{\eta}
\end{align*}
\]

With the error variables defined, then the tracking error dynamics can be derived as

\[
\begin{align*}
\dot{\tilde{\xi}}_1 &= \tan(\tilde{\xi}_1 + \bar{\xi}_1) + \tilde{\xi}_2 + \bar{\xi}_2 - \tan(\bar{\xi}_1) - \bar{\xi}_2 \\
&= \tan(\tilde{\xi}_1 + \bar{\xi}_1) + \tilde{\xi}_2 - \tan(\bar{\xi}_1) \\
\dot{\tilde{\xi}}_2 &= \tilde{\xi}_1 + \bar{\xi}_1 + \tilde{\mu} + \bar{\mu} - (\bar{\xi}_1 + \tilde{\mu}) \\
&= \tilde{\xi}_1 + \tilde{\mu} \\
\tilde{\eta} &= \tilde{\xi}_2 + \bar{\xi}_2 - \bar{\xi}_2 \\
&= \tilde{\xi}_2
\end{align*}
\]
In order to begin the process of designing a tracking controller for the nonlinear plant, there are a few things left to do in order to assure that the controller being designed will actually stabilize the error dynamics. The first thing to do is to check to see whether or not the nonlinear plant is controllable and/or observable. If the plant is found to be not controllable or not observable, then designing the controller or the observer may not be able to stabilize the systems response. For the purpose of just testing to see whether or not the nonlinear plant is controllable and/or observable, a linearized version of the error dynamics will be used. The linearized error dynamics are defined as

\[
E(t) = \frac{\sec^2[\xi_1(t)]}{1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad B(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
C(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad D(t) = [0]
\]

Now we check if the system is uniformly completely controllable by the equation

\[
\text{rank } C(t) = \begin{bmatrix} B(t) & (-\mathcal{P}_A)B(t) \end{bmatrix} = \begin{bmatrix} B(t) & \dot{B}(t) - A(t)B(t) \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 2
\]

Since the rank \( C(t) \) equals the size of the controllability matrix, \( 2 \times 2 \), the system is uniformly completely controllable. Since we know that the nonlinear plant is controllable, then a state feedback controller can be designed to stabilize the error dynamics.

Now that we know that the nonlinear plant is controllable, the next step is to determine if the plant is observerable. This can be done by the following equation.

\[
\text{rank } \mathcal{O}(t) = \begin{bmatrix} C^T(t) & Q_A C^T(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2
\]

From 2.3.6, since the rank \( \mathcal{O}(t) \) equals the size of the observable matrix, \( \mathcal{O}(t) = 2 \times 2 \), then the nonlinear plant is found to be completely observable. Therefore an observer can be developed to facilitate state feedback control implementation.
2.4 Baseline Design

In this section, we now focus on the baseline designs for the benchmark problem. The baseline design consists of the pseudo-inverse, observer, Trajectory Linearization Control and Sliding Mode Control. The pseudo-inverse and observer will be used throughout each nonlinear design for meaningful comparisons between each controller. Since the plant is nonminimum phase, the pseudo-inverse is unstable. A backstepping stabilizing controller for the pseudo-differentiator was designed [38], which will be employed as the baseline design. In order to facilitate the exposition of the backstepping based pseudo-inverse, and the subsequent design of the tracking error stabilizing controller using backstepping, we begin this section with an introduction to the backstepping principle and design procedure.

2.4.1 Backstepping Principle and Procedure

The stability criterion based on Lyapunov's second method can be utilized to develop a stabilizing control design technique. Consider the system

\[ \dot{x} = f(t, x, u) \]  

with a state feedback control law \( u = k(t, x) \). The closed-loop system is given by

\[ \dot{x} = f(t, x, k(x)) \]

In order to achieve global exponential stability at the null equilibrium of the closed-loop system, we first define a candidate Lyapunov function \( V(t, x) \) such that

\[ c_1 \|x\|^2 \leq V(t, x) \leq c_2 \|x\|^2, \quad c_1, c_2 > 0 \]

If we can design \( u = k(t, x) \) such that

\[ \dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, k(x)) \leq -c_3 \|x\|^2, \quad c_3 > 0 \]

then the closed-loop exponential stability will follow from Theorem 2.1.3. Moreover, the exponential stability will be global if the above inequalities are satisfied for all \( x \). This stabilization control design approach leads to the next definition.
Definition 2.4.1 A smooth function $V(t, x)$ satisfying

$$c_1\|x\|^2 \leq V(t, x) \leq c_2\|x\|^2, \quad c_1, c_2 > 0$$

is called a control Lyapunov function (CLF) for 2.4.1 if for all $x \neq 0$,

$$\dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, k(x)) \leq -c_3\|x\|^2, \quad c_3 > 0$$  \hspace{1cm} 2.4.3

for some feedback control law $u = k(t, x)$.

It is important to note that the existence of a globally stabilizing control law is equivalent to the existence of a CLF. Therefore if a CLF can be found then a globally stabilizing control law can be designed. Stabilization control design methods using a CLF is known as Lyapunov design techniques.

The Backstepping design is a Lyapunov design method. It was first introduced by Kokotovic [23]. A problem with Lyapunov design methods is that it is usually very difficult to defining a CLF for an $n$th-order nonlinear plant. However, it is relatively easy to define a CLF for first-order system. The idea behind the Backstepping design is to break an $n$th-order system into $n$ first-order systems by the use of what is known as virtual controls in a recursive fashion. This approach requires that the nonlinear system can be transformed via a coordinate transformation into a triangular structure.

$$\dot{\xi}_i = f_i(\xi_1, \ldots, \xi_i) + g_i(\xi_1, \ldots, \xi_i)\xi_{i+1}, \quad i = 1, 2, \ldots, n - 1$$

$$\dot{\xi}_n = f_n(\xi_1, \ldots, \xi_n) + g_n(\xi_1, \ldots, \xi_n)\mu$$

Then the Backstepping design uses the state variable $\xi_{i+1}$ as a virtual control for the $\dot{\xi}_i$ equation to stabilize the states $\xi_1, \ldots, \xi_i$ using a CLF in a recursive fashion.

The demonstrate the idea, consider an autonomous nonlinear plant of order $n = 2$ in the triangular structure

$$\dot{\xi}_1 = f_1(\xi_1) + g_1(\xi_1)\xi_2$$

$$\dot{\xi}_2 = f_2(\xi_1, \xi_2) + g_2(\xi_1, \xi_2)\mu$$  \hspace{1cm} 2.4.4
The goal is to design a state feedback control law to stabilize the origin \((\xi_1 = 0, \xi_2 = 0)\). Suppose that the first component of 2.4.4 can be stabilized by a smooth state feedback control law \(\xi_2 = \phi_1(\xi_1)\), with \(\phi_1(0) = 0\), which serves as a virtual control. Then there exists a CLF \(V_1(\xi_1)\) satisfying the inequality
\[
c_{11}\|\xi_1\|^2 \leq V_1(\xi_1), \quad c_{11} > 0, \ \forall \xi_1
\]
such that
\[
\frac{\partial V_1}{\partial \xi_1}[f_1(\xi_1) + g_1(\xi_1)\phi_1(\xi_1)] \leq -c_{13}\|\xi_1\|^2, \quad c_{13} > 0, \ \forall \xi_1
\]
Then the origin \(\xi_1 = 0\) of
\[
\dot{\xi}_1 = f_1(\xi_1) + g_1(\xi_1)\phi_1(\xi_1)
\]
is exponentially stable. In order to implement the virtual control \(\phi_1(\xi_1)\) using \(\xi_2\), rewrite 2.4.3 as
\[
\begin{align*}
\dot{\xi}_1 &= [f_1(\xi_1) + g_1(\xi_1)\phi_1(\xi_1)] + g_1(\xi_1)[\xi_2 - \phi_1(\xi_1)] \\
\dot{\xi}_2 &= f_2(\xi_1, \xi_2) + g_2(\xi_1, \xi_2)\mu
\end{align*}
\]
Note that the virtual control will be implemented we can drive \(\xi_2 \to \phi_1(\xi_1)\) quickly. To this end, define a virtual control error variable \(z = \xi_2 - \phi_1(\xi_1)\). Then the error dynamics is given by
\[
\dot{z} = \dot{\xi}_2 - \dot{\phi}_1
\]
\[
= f_2(\xi_1, \xi_2) + g_2(\xi_1, \xi_2)\mu - \frac{\partial \phi_1}{\partial \xi_1}[f_1(\xi_1) + g_1(\xi_1)\xi_2]
\]
Clearly, if we can exponentially stabilize \(z = 0\) with a state feedback control \(\mu = \phi_2(\xi_1, \xi_2)\) designed using a CLF \(V_2(\xi_1, \xi_2)\), then \(\xi_2 \to \phi_1(\xi_1)\) exponentially, which would implement the virtual control \(\phi_1(\xi_1)\) that will in turn achieve exponential stability of \(\xi_1 = 0\). Moreover, by virtue that \(\phi_1(0) = 0\), both \(z(t) \to 0\) and \(\xi_1 \to 0\) exponentially imply that \(\xi_2 \to 0\) exponentially, which accomplished the design goal. This can be verified with the CLFs \(V_1(\xi_1)\) and \(V_2(\xi_1, \xi_2)\). In particular, design
which yields
\[ \dot{\xi}_1 = [f_1(\xi_1) + g_1(\xi_1)\phi(\xi_1)] + g_1(\xi_1)z \]
\[ \dot{z} = v \]

A CLF candidate can be defined as
\[ V_2(\xi_1, \xi_2) = V_1(\xi_1) + \frac{1}{2}z^2(\xi_1, \xi_2) > c_{21}||\xi||^2, \quad \text{for some } c_{21} > 0 \]

which has time derivative along the state trajectory
\[ \dot{V}_2(\xi_1, \xi_2) = \frac{\partial V_1}{\partial \xi_1} [f_1(\xi_1) + g_1(\xi_1)\phi(\xi_1)] + \frac{\partial V_1}{\partial \xi_1} g_1(\xi_1)z + zv \]
\[ \leq -c_{13}||\xi_1||^2 + \frac{\partial V}{\partial \xi} g(\xi)z + v \]

Choosing
\[ v = -\frac{\partial V_1}{\partial \xi_1} g_1(\xi_1) - kz, \quad k > 0 \]

yields
\[ \dot{V}_2(\xi_1, \xi_2) \leq -c_{13}||\xi_1||^2 - kz^2(\xi_1, \xi_2) \leq -c_{23}||\xi||^2, \quad \text{for some } c_{23} > 0 \]

which shows that the origin of \((\xi_1, \xi_2)\) is exponentially stable since \(\phi(0) = 0\). Therefore the state feedback control law can be defined as
\[ u = -\frac{1}{g_2(\xi_1, \xi_2)} \left\{ f_2(\xi_1, \xi_2) - \frac{\partial \phi_1}{\partial \xi_1} [f_1(\xi_1) + g_1(\xi_1)\xi_2] + \frac{\partial V_1}{\partial \xi_1} g_1(\xi_1) + k[\eta - \phi_1(\xi_1)] \right\} \]

It is noted that the above procedure is applicable to cases where \(\xi_1\) and \(\xi_2\) are vectors, provided that the CLFs \(V_1(\xi_1)\) and \(V_2(\xi_1, \xi_2)\) yields stabilizing (virtual) control laws at each stage.
2.4.2 Nonlinear Dynamic Pseudo-Inversion

Because the plant is nonminimum phase, its inverse is unstable. In order to stabilize the inverse, the same Backstepping method that will be used in designing the error regulator was used in designing the stability controller for the pseudo-inverse.

Since we already know that the benchmark is unstable and nonminimum phase, the nominal control can be calculated as

\[
\hat{\eta} = \bar{\xi}_1 + \bar{\mu} \\
\bar{\mu} = -\bar{\xi}_1 + \hat{\eta}
\]
yielding the unstable nominal inverse plant

\[
\hat{\bar{\xi}}_1 = \tan(\bar{\xi}_1) + \bar{\xi}_2 \\
\hat{\bar{\xi}}_2 = \hat{\eta} \\
\bar{\mu} = -\bar{\xi}_1 + \hat{\eta}
\]

Since the inverse plant is unstable, the Backstepping method is used to yield a globally stabilizing nonlinear control law [36]. To start off designing the Backstepping pseudo-inverse, the first thing to do is find a state feedback control law \( \phi(\bar{\xi}_1) \) to exponentially stabilize \( \bar{\xi}_1 = 0 \) for

\[
\hat{\bar{\xi}}_1 = f(\bar{\xi}_1) + g(\bar{\xi}_1)\phi(\bar{\xi}_1)
\]

Now designing \( \phi(\bar{\xi}_1) \)

\[
\phi(\bar{\xi}_1) = -\tan(\bar{\xi}_1) - a_1\bar{\xi}_1, \quad a_1 > 0
\]
yields

\[
\hat{\bar{\xi}}_1 = -a_1\bar{\xi}_1
\]

A control Lyapunov function (clf) for 2.4.6 is then defined as

\[
V_1(\bar{\xi}_1) = \frac{1}{2}\bar{\xi}_1^2 > 0 \\
V_1(\bar{\xi}_1) = \bar{\xi}_1\dot{\bar{\xi}}_1 = -a_1\bar{\xi}_1^2 < 0
\]
The next thing is to define a coordinate transformation by using a change of variables.

Defining the change of variables, \( z(\xi_1, \xi_2) = \xi_2 - \phi(\xi_1) \) and write

\[
\dot{\xi}_1 = f(\xi_1) + g(\xi_1)\phi(\xi_1) + \xi_2 - g(\xi_1)\phi(\xi_1) \\
= -a_1\xi_1 + g(\xi_1)z \\
\dot{z} = \xi_2 - \phi(\xi_1) \\
= v - \phi(\xi_1)
\]

Then a Lyapunov function candidate can be solved

\[
V_2(\xi_1, z) = V_1(\xi_1) + \frac{1}{2}z^2 > 0 \\
\dot{V}_2(\xi_1, z) = \xi_1\dot{\xi}_1 + z\dot{z} \\
= [-a_1\xi_1^2 + \xi_1g(\xi_1)z] + z[v - \phi(\xi_1)]
\]

Next we need to determine a value for \( v \) such that \( \dot{V}_2(\xi_1, z) < 0 \). Choosing

\[
v = -\xi_1g(\xi_1) - b_2z + \phi(\xi_1) \\
b_2 > 0
\]

yields

\[
\dot{V}_2(\xi_1, z) = -a_1\xi_1^2 - a_2z^2 < 0
\]

Therefore the \( \dot{V}_2(\xi_1, z) \) satisfies the requirements to be a Lyapunov function since given a \( V_2(\xi_1, z) > 0 \), a \( \dot{V}_2(\xi_1, z) \) was found such that \( \dot{V}_2(\xi_1, z) < 0 \). Therefore the stabilizing control law is given as

\[
v = \frac{\partial V_1}{\partial \xi_1}g(\xi_1) - a_2z + \phi(\xi_1) \\
= -[a_1 + a_2 + \sec^2(\xi_1)][\tan(\xi_1) + \xi_2] - (1 + a_1a_2)\xi_1
\]

Now that we have a stabilizing control law, the next thing to do is to determine \( k(\cdot) \) which will equalize the DC gain of the pseudo-identity. To find this \( k(\cdot) \), we first approximate \( \hat{\eta} \) by

\[
\hat{\eta} = v + k(\eta)
\]
where the nominal control is

\[ \hat{\mu} = -\xi_1 + \hat{\eta} \]
\[ = -\xi_1 + v + k(\eta) \]

with a constant \( \eta \) to the plant, and setting \( \dot{\xi}_1 = 0, \dot{\xi}_2 = 0, \eta = \bar{\eta} \) yields

\[ \tan(\xi_1) + \bar{\eta} = 0 \]
\[ v + k(\eta) = 0 \]

Solving these equations for \( k(\eta) \) gives

\[ k(\eta) = (1 + a_1 a_2) \tan^{-1}(\bar{\eta}) \]

Therefore, the complete nonlinear stabilizing control law for the inverse system is given by

\[ \dot{\eta} = -[a_1 + a_2 + \sec^2(\xi_1)] [\tan(\xi_1) + \bar{\xi}_2] - (1 + a_1 a_2) [\xi_1 + \tan^{-1}(\bar{\eta})] \]

where \( a_1, a_2 > 0 \) are design parameters. Therefore using the nonlinear stabilizing control law, the stable pseudo-inverse is found to be

\[ \dot{\xi}_1 = \tan(\xi_1) + \bar{\xi}_2 \]
\[ \dot{\xi}_2 = -[a_1 + a_2 + \sec^2(\xi_1)] [\tan(\xi_1) + \bar{\xi}_2] - (1 + a_1 a_2) [\xi_1 + \tan^{-1}(\bar{\eta})] \]
\[ \hat{\mu} = -\xi_1 - [a_1 + a_2 + \sec^2(\xi_1)] [\tan(\xi_1) + \bar{\xi}_2] - (1 + a_1 a_2) [\xi_1 + \tan^{-1}(\bar{\eta})] \]

To ensure a valid comparative study of the Backstepping Controller with other baseline controllers, this pseudo-inverse will be used in all tracking controller designs.

2.4.3 Trajectory Linearization Observer

The purpose of the observer is to estimate the state from the nonlinear plant. These estimated states are then fed into the state feedback controller, which are used to stabilize the tracking error dynamics. For this thesis, a TL Observer was chosen to estimate the nonlinear plant states, which uses the linearized observer error dynamics. Then the system of equations for the TL Observer is defined as

\[ \dot{x} = A(t)x + B(t)u \]
\[ y = C(t)x \]
where $A(t)$, $B(t)$ and $C(t)$ are the linearized tracking error dynamics and the observer is defined as

$$
\begin{align*}
\dot{\xi} &= f(\hat{\xi}) + g(\hat{\xi})\mu + H(\hat{\xi} - \xi) \\
\hat{\eta} &= h(\hat{\xi})
\end{align*}
$$

The next thing that needs to be defined is the canonical form representation. In order to do this, first the State Coordinate Transformation needs to be defined as

$$
X(t) = \begin{bmatrix}
q_n(t) \\
-P_A q_n(t)
\end{bmatrix} = \begin{bmatrix}
q_n(t) \\
-\dot{q}_n(t) + A(t)q_n(t)
\end{bmatrix}
$$

Then solving for Observer Canonical Form realization yields

$$
\begin{align*}
A_o &= T^{-1}(t)A(t)T(t) - T^{-1}(t)\dot{T}(t) \\
&= \begin{bmatrix} 0 & 1 - 2\hat{\xi}_1\sec^2(\hat{\xi}_1)\tan(\hat{\xi}_1) \\
1 & \sec^2(\hat{\xi}_1)
\end{bmatrix}
\end{align*}
$$

$$
B_o = T^{-1}(t)B(t) \quad C_o = C(t)T(t)
$$

$$
B_o = \begin{bmatrix} -\sec^2(\hat{\xi}_1) \\
1
\end{bmatrix} \quad C_o = \begin{bmatrix} 0 & 1
\end{bmatrix}
$$

Now that we have an Observer Canonical Form realization, the next thing is to do eigenvalue placement by what is called PD-Spectrum Assignment. What PD-Spectrum Assignment does is to use desired PD-eigenvalues for our system realization that are stable and assign them to replace the actual eigenvalues. So from our observer canonical form, the last row of $A_o$ are the coefficients denoted as $\alpha_{1,2}$. The desired coefficients are
denoted as $\beta_{1,2}$.

$$
\begin{align*}
\alpha_1 &= -1 + 2\dot{\xi}_1 \sec^2(\xi_1) \tan(\xi_1) \\
\alpha_2 &= -\sec^2(\xi_1) \\
D_\beta &= \zeta^2 + \beta_2(t)\zeta + \beta_1(t)
\end{align*}
$$

The chosen desired eigenvalues were determined to be $-\frac{3}{2} \pm 3\frac{\sqrt{3}}{2}i$, which yields our $D_\beta$ characteristic equation to be $s^2 + 3s + 9$. Therefore our desired values represented in the canonical form are $\beta_1 = 9$ and $\beta_2 = 3$. After that we can define the observer gain $H(t)$

$$
H_o = \begin{bmatrix}
\alpha_1(t) - \beta_1(t) \\
\alpha_2(t) - \beta_2(t)
\end{bmatrix} = 
\begin{bmatrix}
-10 + 2\dot{\xi}_1 \sec^2(\xi_1) \tan(\xi_1) \\
-\sec^2(\xi_1) - 3
\end{bmatrix}
$$

Then the Linearized closed-loop observer error dynamics is defined by

$$
A - HC = 
\begin{bmatrix}
\sec^2(\xi_1) & 11 - (2\dot{\xi}_1 \tan(\xi_1) - \sec^2(\xi_1) + 3)\sec^2(\xi_1) \\
1 & -\sec^2(\xi_1) + 3
\end{bmatrix}
$$

which will output the estimated states of the nonlinear plant for the Backstepping controller to use to stabilize the tracking error response. This observer design will be used for all controller designs.

2.4.4 Trajectory Linearization Control

The Trajectory Linearization Tracking Controller uses a linear time-varying (LTV) stabilization controller. This means that the nonlinear functions $f$ and $h$, described in the previous section, are linearized along the nominal trajectories $\bar{\xi}(t)$ and $\bar{\mu}(t)$. If the nonlinear functions $f$, $h$ are continuously differentiable with respect to $\xi$, $\mu$, then linearizing the tracking error dynamics will yield

$$
\begin{align*}
\dot{x} &= A(t)x + B(t)u \\
y &= C(t)x + D(t)u
\end{align*}
$$
Therefore \( A(t), B(t), C(t), D(t) \) constitute a matrix representation of a linearized version of the nonlinear dynamic system 2.2.3. Since the matrix equations, \( A(t), B(t), C(t), D(t), \) are Lipschitz and uniformly bounded on the operating region \( \Omega \), the nonlinear error dynamics are exponentially stable if and only if the linearized error dynamics are also exponentially stable. Therefore the resulting Trajectory Linearization Control provides exponential stability as well as provides small signal, finite gain BIBO stability and inherent robustness to parametric perturbations [21].

Even though trajectory linearization provides exponential stability, one thing that can hamper stability is the LTV control law defined as \( \hat{\mu} = K(t)\hat{\xi}(t) \). In order for the stabilizing controller to operate correctly, the parallel differential (PD) eigenvalue assignment, when calculating the canonical form, will be used to define the LTV control law for the stabilizing controller.

The PD-eigenvalue assignment is defined by

\[
D_\beta = \beta_n(t)\delta^{n-1} + \cdots + \beta_2(t)\delta + \beta_1(t)
\]

where \( \beta_i(t), i = 1, \ldots n, \) are synthesized from the desired PD-spectrum \( \rho_i(t) \) [36]

\[
D_\beta = (\delta - \rho_n(t))\cdots(\delta - \rho_2(t))(\delta - \rho_1(t))
\]

The PD-eigenvalues are chosen as \( \rho_i(t) = \rho_i \omega_n(t) \omega_n(t) > 0, \forall t \) and where \( \rho_i \) is a constant whose real part is negative and any complex values are chosen as a complex conjugate pair. The PD-eigenvalues chosen for the TLC design are \( \rho_{1,2} = -0.5 \pm \frac{\sqrt{3}}{2} \). The PD-eigenvalue assignment can be used for both the controller and observer if they are both uniformly completely controllable/observable.
For this thesis, the Trajectory Linearization Control law $\hat{\mu}(t)$ is defined as follows

$$\hat{\mu}(t) = \bar{\mu}(t) + K(t)\hat{\xi} = \bar{\mu}(t) + [k_1, k_2]\hat{\xi}$$

where

$$\bar{\mu}(t) = -\xi_1 - [a_1 + a_2 + \sec^2(\xi_1)][\tan(\xi_1) + \xi_2] - (1 + a_1a_2)[\xi_1 + \tan^{-1}(\eta)]$$

$$k_1 = [2\sec^2(\xi_1)\tan^2(\bar{\xi}_1) + \tan(\bar{\xi}_1)(\bar{\xi}_2) + (\sec(\bar{\xi}_1) - 1)\sec^2(\bar{\xi}_1)$$

$$k_2 = \sec^2(\bar{\xi}_1) - 1$$

### 2.4.5 Sliding Mode Control

Another nonlinear tracking control design, similar to Backstepping, which does not linearize the tracking dynamics is called Sliding Mode Control (SMC). What SMC does is that it designs a control law such that it constrains the motion of the system to an exponentially stable, $(n - 1)$ – dimensional manifold or surface with a first-order high-gain control law design using a control Lyapunov function. An example from [22] is used to further elaborate this idea of Sliding Mode Control.

Consider the SISO, autonomous affine NL nominal system

$$\dot{\xi} = f(\xi) + g(\xi)u$$

$$y = h(\xi)$$

where \{f, g\} is full-state linearizable via a coordinate transformation

$$z = \Phi(\xi), \xi = \Phi^{-1}(z)$$

In order to control a regularly perturbed plant given by

$$\dot{\xi} = f(\xi) + \Delta_1(\xi) + (1 + \Delta_2(\xi))g(x)u$$

where

$$f(\xi) = \begin{bmatrix} \tan(\bar{\xi}_1 + \bar{\xi}_1) - \tan(\xi_1) + \bar{\xi}_2 \\ \bar{\xi}_1 \\ \bar{\xi}_2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
and the unknown functions $\triangle_1(\xi)$ and $\triangle_2(\xi)$ satisfy

\[
0 \leq \|\triangle_1(\xi)\| \leq b_1 \\
0 \leq \|\triangle_2(\xi)\| \leq b_2 < 1
\]

The plant is represented in normal form of the perturbed plant as

\[
\begin{align*}
\dot{x}_1 &= z_2 \\
\dot{x}_2 &= z_3 \\
\dot{x}_3 &= \phi(z) + \tilde{\triangle}_1(z) + (\psi(z) + \tilde{\triangle}_2(z))\mu
\end{align*}
\]

where

\[
\begin{align*}
\phi(z) &= L_f^3 h_d(\xi) = 2z_3(z_2 + \xi_2)\tan(\xi_1 + \xi_1) + (z_3 + \xi_1)\sec^2(\xi_1 + \xi_1) \\
\psi(z) &= L_g L_f^7 h_d(\xi) = \sec^2(\xi_1 + \xi_1) \\
\tilde{\triangle}_1(z) &= L_d L_f^{n-1} h_d(\Phi^{-1}(z)) \\
\tilde{\triangle}_2(z) &= L_g L_f^{n-1} h_d(\Phi^{-1}(z))\delta_2(z)
\end{align*}
\]

and

\[
0 \leq |\tilde{\triangle}_1(z)| \leq c_1 \\
0 \leq |\tilde{\triangle}_2(z)| \leq c_2 < 1
\]

Now $z_3$ can be used as a virtual control to stabilize $[z_1 \; z_2]^T$ to the origin of the 2-dimensional space by using the LTI control law

\[
z_3 = -a_1 z_1 - a_2 z_2
\]

where $a_1, a_2 > 0$ are coefficients of a Hurwitz polynomial. The "free" state $z_3$ must then be driven to the sliding manifold defined by

\[
s = z_3 - (-a_1 z_1 - a_2 z_2) = 0
\]

The task is thus to stabilize the 1st-order dynamics

\[
\begin{align*}
\dot{s} &= \dot{z}_3 + a_1 \dot{z}_1 + a_2 \dot{z}_2 \\
&= \phi(z) + \tilde{\triangle}_1(z) + (\psi(z) + \tilde{\triangle}_2(z))\mu + \gamma(z)
\end{align*}
\]

where

\[
\gamma(z) = a_1 \dot{z}_1 + a_2 \dot{z}_2
\]
The control can then be designed as \( u = u_{eq} + u_{smc} \) where

\[
u_{eq} = \frac{1}{\psi(z)} + [-\phi(z) - \gamma(z)]\]

Applying to 2.4.7 gives

\[
\dot{s} = \tilde{\Delta}_1(z) + \tilde{\Delta}_2(z)[-\phi(z) - \gamma(z)] + (1 + \tilde{\Delta}_2(z))\psi(z)u_{smc}
\]

Choosing the Lyapunov design function

\[
V = \frac{1}{2} s^2 \\
\dot{V} = s \dot{s} \\
= s [\tilde{\Delta}_1(z) + \tilde{\Delta}_2(x)[-\phi(z) - \gamma(z)] + (1 + \tilde{\Delta}_2(z))\psi(z)u_{smc}
\]

Assuming

\[
\frac{\tilde{\Delta}_1(z) + \tilde{\Delta}_2(x)[-\phi(z) - \gamma(z)]}{1 + \tilde{\Delta}_2(z)} < \Delta(z)
\]

for some chosen upper bound \( \Delta(z) \), then

\[
\dot{V} \leq s[(1 + c_2)(\Delta(z) + \psi(z)u_{smc})]
\]

and \( u_{smc} \) can be designed as

\[
u_{smc} = \frac{1}{\psi(z)} [-\Delta(z) - k\text{sgn}(s)] \tag{2.4.8}
\]

to give

\[
\dot{V} = -(1 + c_2)k\text{sgn}(s) < 0, \forall z
\]

where \( k \) is a design parameter. With the sliding mode control law defined by 2.4.8, it may not be desirable to use the signum function due to its "chattering" effect when \( s \) crosses zero. Instead, the saturation function is used which is defined by

\[
\text{sat}(s) = \begin{cases} 
  s, & \text{if } |s| \leq 1 \\
  \text{sgn}(s), & \text{if } |s| > 1
\end{cases} \tag{2.4.9}
\]
Using the saturation function allows the Sliding Mode Controller to proportionally drive the signal back to the origin when it diverges, therefore reducing chattering. The signum function, unlike the saturation function, drives the signal back to the origin no matter what distance the signal trails from the origin, which can cause chattering.

For this thesis, the Sliding Mode Control law \( \hat{\mu}(t) \) is defined as follows

\[
\hat{\mu} = u_{eq} + u_{smc} = \frac{1}{\psi(z)} [-\phi(z) - \gamma(z)] + \frac{1}{\psi(z)} \Delta(z) - k \text{ sat}(s)
\]
Chapter 3 Trajectory Regulation Control by Backstepping

3.1 Backstepping Error Regulator for Benchmark Problem

For the design of the Backstepping error regulator, the benchmark problem will be used once again. In order to clearly understand the Backstepping process, first the benchmark problem is redefined below

\[
\begin{align*}
\dot{\xi}_1 &= \tan(\xi_1) + \xi_2, \quad |\xi_1| < \frac{\pi}{2} \\
\dot{\xi}_2 &= \xi_1 + \mu \\
\eta &= \xi_2
\end{align*}
\]

Once again the tracking error dynamics will be utilized in designing the Backstepping error regulator.

\[
\begin{align*}
\dot{\tilde{\xi}}_1 &= \tan(\tilde{\xi}_1 + \tilde{\xi}_1) - \tan(\tilde{\xi}_1) + \tilde{\xi}_2 \\
\dot{\tilde{\xi}}_2 &= \tilde{\xi}_1 + \tilde{\mu} \\
\tilde{\eta} &= \tilde{\xi}_2
\end{align*}
\]

Then to start off designing the Backstepping error regulator, \(\xi_2\) will be used as a virtual control to stabilize \(\xi_1\) with nonlinear time-varying cancellation and replace with exponentially stable linear dynamics. To do this, first define the first order system

\[
\dot{\bar{\xi}}_1 = f(\bar{\xi}_1) + g(\bar{\xi}_1)\phi(\bar{\xi}_1)
\]

where \(f(\bar{\xi}_1) = \tan(\bar{\xi}_1 + \bar{\xi}_1) - \tan(\bar{\xi}_1)\) and \(g(\bar{\xi}_1) = 1\) of the first state in the tracking error dynamics. Then designing the virtual feedback control law \(\phi(\bar{\xi}_1)\) to cancel the NLTV terms

\[
\phi(\bar{\xi}_1) = -\tan(\bar{\xi}_1 + \bar{\xi}_1) + \tan(\bar{\xi}_1) - b_1(\bar{\xi}_1), \quad b_1 > 0
\]

yields

\[
\dot{\bar{\xi}}_1 = -b_1(\bar{\xi}_1)
\]
where \( b_1 > 0 \) is the desired closed-loop dynamics for \( \xi_1 \), and \( \phi(\xi_1) \) is the virtual control to be implemented with \( \xi_2 \).

A control Lyapunov function (clf) for 3.1.10 is then chosen as

\[
V_1(\xi_1) = \frac{1}{2} \xi_1^2 > 0
\]

\[
\dot{V}_1(\xi_1) = \xi_1 \dot{\xi}_1 = -b_1 \xi_1^2 < 0
\]

which affirms exponential stability of \( \xi_1 = 0 \). The next thing is to define a coordinate transformation by using a change of variables \( \tilde{z}(\xi_1, \xi_2) = \xi_2 - \phi(\xi_1) \) and write

\[
\dot{\xi}_1 = f(\xi_1) + g(\xi_1) \phi(\xi_1) + g(\xi_1) \xi_2 - g(\xi_1) \phi(\xi_1)
= -b_1(\xi_1) + g(\xi_1) z
\]

\[
\dot{\tilde{z}} = \xi_2 - \dot{\phi}(\xi_1)
= w - \dot{\phi}(\xi_1)
\]

Therefore when \( z \to 0 \), this implies that \( \xi_2 \to \phi \). Then a Lyapunov function candidate can be solved

\[
V_2(\xi_1, \tilde{z}) = V_1(\xi_1) + \frac{1}{2} \tilde{z}^2 > 0
\]

\[
\dot{V}_2(\xi_1, \tilde{z}) = \xi_1 \dot{\xi}_1 + \dot{\tilde{z}} \tilde{z}
= [-b_1 \xi_1^2 + \xi_1 g(\xi_1) \tilde{z}] + \tilde{z} [w - \dot{\phi}(\xi_1)]
\]

Next we need to determine a value for \( w \) such that \( \dot{V}_2(\xi_1, z) < 0 \). Choosing

\[
w = -\xi_1 g(\xi_1) - b_2 \tilde{z} + \dot{\phi}(\xi_1)
\]

which is the virtual stabilizing control law for \( z \) to be implemented with \( \mu(t) \). This yields

\[
\dot{V}_2(\xi_1, z) = -b_1 \xi_1^2 - b_2 \tilde{z}^2 < 0
\]

Therefore the \( \dot{V}_2(\xi_1, z) \) satisfies the requirements to be a Lyapunov function. Now the State Feedback Control Law is obtained by first solving for \( w \).

\[
w = \frac{\partial \dot{\phi}}{\partial \xi_1} [f(\xi_1) + g(\xi_1) \eta] - \frac{\partial V_1}{\partial \xi_1} g(\xi_1) - b_1 b_2 [\eta - \phi(\xi_1)]
= (-\sec^2(\xi_1 + \xi_1) - b_1 - b_2) [\tan(\xi_1 + \xi_1) + \tilde{z} - \tan(\xi_1)] - b_1 b_2 \xi_1
Plugging $w$ into $\xi_2$, in equation 3.1.2, yields the feedback control law

$$
\mu = (-\sec^2(\xi_1 + \overline{\xi}_1) - b_1 - b_2)[\tan(\overline{\xi}_1 + \overline{\xi}_1) + \overline{\xi}_2 - \tan(\overline{\xi}_1)] - \overline{\xi}_1 - b_1 b_2 \overline{\xi}_1
$$

For good performance of the feedback control law, the controller gains should be chosen such that $b_2 \gg b_1$. The controller gains used for the Backstepping feedback control law are chosen as $b_1 = 1, b_2 = 2.5$.

Now that we have solved the feedback control law, we know from the theory of Lyapunov functions in the technical preliminaries that we now have a control law that is globally exponentially stable. By having a GES controller, we hope to see a more robust design with high tracking performance compared to other linearized designs. Even though the system may be GES in theory, the Backstepping controller does utilize a nonlinear cancellation scheme. Non ideal cancellation will result in non vanishing perturbations, which are expected to cause tracking performance degradation and can reduce domain of attraction. So with that in mind, the next step in designing our system is to implement the Backstepping system design into MATLAB/SIMULINK.

### 3.2 Implementation

This section of the thesis will talk about the techniques for the implementation of the overall system design into MATLAB/SIMULINK. MATLAB is a powerful mathematical program, and SIMULINK is a component of MATLAB that builds simulations with a graphical interface. For this thesis, SIMULINK is used to implement the nonlinear system which in turn calls MATLAB to run mathematical calculation scripts and display output plots of the system responses.

The order that these components were implemented was the nonlinear plant, pseudo-inverse, controller and observer. Also, after each component was implemented, the component was verified. This was done to make sure that there was no connection errors and that the proper signals were going through the proper blocks. This was done instead of just putting the whole design together at once in order to aid in the debugging
process. With that said, let us go over the implementation of this nonlinear system design.

Figure 3.2.1 shows the overall SIMULINK system representation. The system representation is broken down into four main components. These components consist of the Nonlinear Plant, Pseudo-Inverse Plant, Backstepping Error Regulator and Observer Design. The flow of the system representation is similar to the overall system design, which was shown in Figure 2.2.1. The inputs into the individual components are represented as the oval on the left side labeled as "In1". Some of the types of inputs into the subsystem that were used to test this system are step, ramp and staircase inputs. The output of a subsystem is labeled as "Out1" and this is where the data is outputted to show how the system stabilizes to the input command. The common things that can be attached to this output terminal are sending data out to the MATLAB workspace or just using a scope to get a plot of the data. Now that the basic design idea for the overall system realization has been established, the next step is to examine all the various components that make up the implementation.

![Figure 3.2.1 Overall SIMULINK System Representation](image-url)
The first component of the nonlinear system developed is the nonlinear plant represented in SIMULINK as Figure 3.2.2. The mathematical representation of the nonlinear plant is given as

\[
\begin{align*}
\dot{\xi}_1 &= \tan(\xi_1) + \xi_2 \\
\dot{\xi}_2 &= \xi_1 + \mu \\
\eta &= \xi_2
\end{align*}
\]

where the input to the plant is \( \mu \) and the plant outputs \( \xi_2 \), which the observer uses as its input. A MATLAB function block is used where a MATLAB script calculates the nonlinear part of the plant, which is function \( f(\cdot) \) and sends that signal into a summing block with function \( g(\cdot) \). This signal then goes into the integrator and loops back into the MATLAB function block as well as to the function \( h(\cdot) \), which outputs to the overall system.

![Figure 3.2.2 Nonlinear Plant SIMULINK System Representation](image)

The next component that we would like to look at is the Pseudo-Inverse, which is represented by Figure 3.2.3. The mathematical version of the SIMULINK design is represented in Chapter 3 but is shown below for easy reference.

\[
\begin{align*}
\dot{\xi}_1 &= \tan(\xi_1) + \xi_2 \\
\dot{\xi}_2 &= -[a_1 + a_2 + \sec^2(\xi_1)][\tan(\xi_1) + \xi_2] - (1 + a_1a_2)[\xi_1 + \tan^{-1}(\eta)] \\
\tilde{\mu} &= -\xi_1 - [a_1 + a_2 + \sec^2(\xi_1)][\tan(\xi_1) + \xi_2] - (1 + a_1a_2)[\xi_1 + \tan^{-1}(\eta)]
\end{align*}
\]
As you can see from both the SIMULINK and the mathematical version, the input to the pseudo-inverse is $\eta$ while the outputs are $\bar{\xi}$ as well as $\bar{\mu}$. The black vertical bars represent multiplexers, which combine multiple lines together into one line by lining them up into an array. This is very helpful in eliminating clutter in the design. The three MATLAB function blocks are scripts that take the input signals and perform calculations on them. The calculations that are being performed are the three equations for the pseudo-inverse that is shown in equation 3.2.1. After the signals are passed through each of the MATLAB blocks, the $\bar{\mu}$ gets outputted while the $\bar{\xi}_1$ and $\bar{\xi}_2$ go through a pseudo-differentiator to yield $\dot{\bar{\xi}}_1$ and $\dot{\bar{\xi}}_2$. The system gains and constants are pulled out of the MATLAB equation in order to allow for easier tuning for when the system implementation is completed. The next component to look at is the Backstepping controller.

![Figure 3.2.3 Pseudo-Inverse SIMULINK System Representation](image)

The Backstepping Error Regulator Controller is quite simple in implementation due to the fact that most of the calculations are done from within the MATLAB function blocks. The inputs to the controller are the $\xi$ nominal and error values with our constant
values taking out of the Backstepping controller equation for tuning purposes. The controller mathematical parameters are shown below for easy reference.

\[
\hat{\mu} = (-\sec^2(\xi_1 + \bar{\xi}_1) - b_1 - b_2)\left[\tan(\xi_1 + \bar{\xi}_1) + \tilde{\xi}_2 - \tan(\bar{\xi}_1)\right] - \tilde{\xi}_1 - b_1 b_2 \tilde{\xi}_1
\]

where \( b_1 = 1 \) and \( b_2 = 2.5 \) are design parameters. So therefore the output of this controller is found to be \( \hat{\mu} \), which is the feedback control law.

![Figure 3.2.4 Backstepping Error Regulator SIMULINK System Representation](image)

The last component that was design for the overall system realization is the observer design. The observer is designed to estimate the states of the nonlinear plant for the controller to use in its stabilization feedback. The general mathematical equation for the observer design is given as

\[
\begin{align*}
\dot{x} &= f(\bar{x}) + g(\bar{x})u + H(t)(y - \tilde{y}) \\
\tilde{y} &= h(\bar{x})
\end{align*}
\]

where

\[
H(t) = \begin{bmatrix} -10 + (2\dot{\xi}_1 \tan(\xi_1) - \sec^2(\xi_1) + 3)\sec^2(\xi_1) \\ -\sec^2(\xi_1) - 3 \end{bmatrix}
\]

From this equation and the design, it is clear to see that the inputs of the observer comes from the output variables of the nonlinear plant as well as the input into the nonlinear plant. For the observer design, the use of two function blocks aid in the mathematical
calculations that need to be performed before entering into the integrator. After all calculations are performed, the output signal is then outputted to the controller as the estimated states of the nonlinear plant.

Figure 3.2.5 Observer SIMULINK System Representation

3.3 Verification

Now that the implementation phase of this thesis is complete the next section is verifying that the design that has been implemented, actually stabilizes the nonlinear tracking error dynamics. In the verification phase, once the system is verified to have no errors that were made in the design phase, the system response is placed against various control inputs. Input commands such as step, ramp and staircase are used to test its tracking performance. The system response is displayed in the form of a graph showing the nominal control input with the corresponding system output response. Each of the input reference commands tested the tracking performance of the system's response with input commands of a minimum, medium and maximum value. The reasoning for verifying these three different values for each reference command was to determine the
limits of stable tracking for the overall system design. For the verification process, the step, ramp and staircase input commands are shown in the following figures.

Upon initial verification of the system response, an input reference command of 0.0 was inputted into the system with nonzero initial conditions. This showed that the system had tracking stability by having exponential convergence to the zero reference command. Also, the simulation time was set for 100 seconds due to the fact that the response from $\xi_2$, from the plant, may take as long as 40 seconds to show noticeable departure from the equilibrium point when the input exceeds the system's threshold of input-output stability. This is very useful to know because a system may appear stable during entire verification process if a short simulation time is used even though the system is still unstable due to errors in the system design or implementation. This issue was discovered when an incorrect sign was placed at a summing junction. During the initial verification process, there was no input during a simulation time of only 10 seconds. The system response showed no divergence from the zero reference input. When the simulation time was extended to 100 seconds, the simulation diverged from the referenced or nominal input and lost tracking by 60 seconds. The sign error was later fixed and verified to be correct.

The next input command was a step command with a magnitude of 0.1 for 100 seconds. The reason for such a small number was the fact the nonlinear plant was highly nonlinear and unstable that giving an input command of a value of 1 may yield instability. The results are shown in Figure 3.3.1 below.
It can be seen that the initial commanded input is set at zero for 5 seconds then steps up to a value of 0.1. During that jump from 0.0 to 0.1, the response is to initially drop down to a value of −0.04 and then stabilizes to 0.1 with a settling time of around 5 seconds and a slight overshoot. The initial drop down to −0.04 is due to the fact that the nonlinear system is nonminimum phase. The system response will always have some negative undershoot before it settles to a positive step nominal command. By tuning the gains of the overall system, it can have an outcome of increasing the percent overshoot while decreasing the initial downward spike which in turn will also affect the settling time.

In this next Figure 3.3.2, the step command is increased to a value of 1.0. As you can see when the nominal command was increased from 0.1 to 1.0, the initial downward spike also increased from 0.04 to almost 0.4. The next is to see how far the system response will go before losing stability.
In Figure 3.3.3, the step input is increased to a value of 1.5 which brings the system response to its upper tracking limit and hence loses tracking. It can be seen from the figure that initially the system tracks to the reference input at 1.5 but then loses tracking after a simulation time of 75 seconds. Therefore, once the response changes due to the reference input, there needs to be a long enough simulation time for the full effect of the tracking response to be shown by both $\xi_1$ and $\xi_2$. With that said, let's now take a look at how the system response behaves due to a staircase reference input command.
The next reference input command that was used was a staircase command. The staircase is generated using a combination of the ramp reference input and a quantizer. Figure 3.3.4 shows the stable system response to the nominal input. During each increase in the staircase, there is an initial downward spike followed by an overshoot which settles down to the nominal command until the next staircase jump.
Figure 3.3.5 shows maximum tracking with the staircase input response. As can be seen, the response loses tracking just after the jump from a nominal commanded value from 2.4 to 3.2. This shows that the greatest tracking response possible for this staircase sequence is 2.4. Even though the tracking response does not lose tracking until the staircase command hits a value of 2.4, this does not mean that the maximum tracking value is 2.4 due to the delay time of $\xi_2$ in departing from the equilibrium point as the system exceeds the threshold of input-output stability.
In order to determine the maximum and minimum tracking responses, the output response value is gradually increased until the system loses tracking. Figures 3.3.6 and 3.3.7 show the upper and lower tracking limits for the Backstepping controller. As it can be seen from each scenario, the maximum tracking limit is 1.42 with a minimum tracking limit of −1.42, which are the values just before loss of tracking. Also, the Backstepping controller can handle various slopes for the ramp input response where each ramp saturates at 1.42. If each of the ramp slope were allowed to track past 1.42, the tracking response would lose tracking shortly after. Therefore, the Backstepping controller can tracking various types of input response as long as the input response stays with ±1.42.
Figure 3.3.6 Upper Limit Tracking Stability for TRC-BS
Figure 3.3.7 Lower Limit Tracking Stability for TRC-BS

For different types of reference command signals, the system response loses tracking at roughly the same values. For the ramp response, it can be seen that no matter whether the ramp is slow or fast, the Backstepping controller can still track the input response.
Since the Backstepping model utilizes individual components to track a reference signal, (i.e. pseudo-inverse, error regulator, and TLO observer), each components signals can have an impact on the overall tracking response to the reference signal. To start off, consider the staircase input signal to the Backstepping control system. The staircase reference signal and the associated tracking response is shown in Figure 3.3.8.

![Figure 3.3.8 Tracking Response to Staircase Reference Command](image)

For the Backstepping error regulator, the main signals that will be discussed are the output response and tracking error response. Each one of these responses is based upon the inputted reference signal which is shown in Figure 3.3.8. By discussing the internal responses of the Backstepping model, instead of just the output, the objective is to determine what effect each subsystem has on the overall design in order to understand
and improve the overall design. Of the many responses in the Backstepping model, two responses of the error regulator is shown. The first response is shown in Figure 3.3.9.

![Figure 3.3.9 Backstepping Error Regulator Response](image)

In Figure 3.3.9, the output response of the error regulator is an error correction control value \( \tilde{\mu}(t) \) which is summed with the nominal input \( \bar{\mu}(t) \) to the nonlinear plant. The output of the error regulator does in essence, add a correction to the input signal of the nonlinear plant whenever the output signal of the nonlinear plant falls off tracking from the reference commanded signal. Therefore, during the time when the output signal of the nonlinear plant, \( \eta \), is tracking the reference command, \( \bar{\eta} \), the output response of the error regulator, \( \tilde{\mu} \), is zero. When the output signal of the nonlinear plant falls off from tracking the reference command, the output error signal from the error regulator varies in order to counteract the loss of tracking. In Figure 3.3.9, the loss of tracking is easily
shown which corresponds to the step change of the reference command. With close tracking to the nominal input, the output response of the error regulator approaches back to zero. The Backstepping error regulator can be said to have good error correction due to the small signal magnitude in the output as the reference command increases.

The next signal to observe is the tracking error response, which is shown in Figure 3.3.10.

![Tracking Error Response](image)

Figure 3.3.10 Tracking Error Response

The tracking error response is the difference between the output of the nonlinear plant, $\xi_{1,2}$, and the output signals from the pseudo-inverse, $\bar{\xi}_{1,2}$. When the tracking error response is zero, the $\xi$ and $\bar{\xi}$ are the same. As we expect to see with tracking error responses, Figure 3.3.10 shows the largest tracking error signal spikes during the transition phase when a new step signal is introduced by the staircase input command.
With the tracking error response having a small signal value, this shows that the system response has tight tracking to the reference commanded input. Now that the characteristics of the Backstepping design has been demonstrated, the next step is to see how this nonlinear tracking controller compares to other design techniques.
Chapter 4 Comparison Study

4.1 Test Purpose

In this chapter, the main focus is to show how the TRC Backstepping design compares to other nonlinear design techniques. The purpose of performing various comparison tests is to determine the advantages and disadvantages in tracking performance and robustness of each nonlinear design techniques used. The nonlinear design techniques that will be compared with the Backstepping in this study are Trajectory Linearization Control and Sliding Mode Control. After the tracking performance has been evaluated, a robustness test will be performed on both the Backstepping design and TLC design.

4.2 Test Method and Test Matrix

In order to provide an accurate comparison between the stabilizing controllers, the following two subsections will compare separately the performances with direct state feedback and state estimate, or output feedback. The direct state feedback is where the nonlinear plant states are directly inputted into the stabilizing controller, thus no observer is present. The state estimate feedback is where the nonlinear plant states are estimated by the observer from the output of the plant and the estimated states are inputted into the stabilizing controller. For the comparison tests of each stabilizing controllers, the tracking responses to step, ramp and staircase will be evaluated. The robustness test will compare both designs on their domain of stability and how they handle singular perturbations, regular perturbations and disturbances.

4.2.1 Direct State Feedback Comparison

The first three figures, Figures 4.2.1 – 4.2.3, will demonstrate the system response due to a step, ramp and staircase input. The Trajectory Linearization Controller is denoted as a blue solid line, Sliding Mode Controller as a green dotted line and the Backstepping controller as a red hashed line.
Figure 4.2.1 Step Input Comparison with State Feedback

Figure 4.2.2 Ramp Input Comparison with State Feedback
As it can be seen from the results of the figures above, each of the stabilizing controllers yield desirable responses. It can also be seen that the Sliding Mode Controller does have slightly less overshoot and undershoot versus the Backstepping Controller and Trajectory Linearization Controller. Adjusting the gains for each controller can also yield various settling time and overshoot which allows for some flexibility to tune the controller to a desired response. Not only is the overshoot and settling time of relevant importance, but also the stabilizing controller's maximum tracking response could be of more desire. The maximum tracking response for each system, which is compared using a staircase input response is shown in Figure 4.2.4.
Figure 4.2.4 Maximum Response Comparison with TLC

As can be seen from Figure 4.2.4, the Backstepping controller falls off tracking at after the step increment to a value of 2.0 with a simulation time of 68 seconds. The Sliding Mode controller falls off tracking during the jump from 1.5 to 2.0. Last the Trajectory Linearization Control losses tracking after the step increment to a value of 1.5 with a simulation time of 55 seconds. With these comparison results shown, it can be concluded that with direct state feedback realization, the Backstepping controller has the longest tracking before departure, with the Sliding Mode Control in a close second and TLC the last. These longer tracking times for the Backstepping and Sliding Mode controller may be contributed to the fact that these methods do not depend on linearize approximation of the tracking error dynamics. The next step is to implement the TLO Observer in conjunction with each stabilization controller and verify the results.
4.2.2 State Estimate Comparison

As with the direct state feedback comparison section, this section will follow the same approach only that the TLO Observer is added to each nonlinear controller. Once the TLO Observer was added into each nonlinear system, each system was tested with no input to verify tracking stability at the equilibrium. Unlike the TLC and Backstepping controllers, the Sliding Mode Controller was unable to track the zero input, let alone any input. With various attempts to adjust controller and observer gain as well as verify all design equations, the output response still had no tracking to zero input, which is shown in Figure 4.2.5.

Considering that the TLO Observer can be considered as a singular perturbation to the direct feedback controller, it was conjected that the problem was caused by the poor robustness of the Sliding Mode Control to singular perturbations. In order to test
this theory, a gain and delay margin test was performed in order to ascertain the Sliding Mode Controller's performance to perturbations. If a controller design provides a high gain margin, that means that the control system can handle larger regular perturbations, i.e. larger parametric modeling errors, parameter variations, adverse operating conditions and vehicle performance deterioration and faults. If the control system has a high delay margin, that means that the system can handle larger singular perturbations, i.e. dynamic modeling errors, parasitic dynamic modes such as actuator dynamics, unmodeled structural modes, etc. The gain margin and delay margin assessment of each stabilizing controller is shown in Table 4.2.1. The setup for the gain margin test is an initial condition placed on $\xi_1$ of the nonlinear plant, with a value of 0.1 with no reference input. For the delay margin, an initial condition of $\xi_1(0) = 0.1$ was inputted into the nonlinear plant.

<table>
<thead>
<tr>
<th></th>
<th>Gain Margin (No Observer)</th>
<th>Delay Margin (No Observer)</th>
</tr>
</thead>
<tbody>
<tr>
<td>TLC</td>
<td>23.64 dB</td>
<td>0.25 Seconds</td>
</tr>
<tr>
<td>Backstepping</td>
<td>26.65 dB</td>
<td>0.27 Seconds</td>
</tr>
<tr>
<td>SMC</td>
<td>6.53 dB</td>
<td>0.12 Seconds</td>
</tr>
</tbody>
</table>

Table 4.2.1 Upper Gain and Delay Margin Comparison

As it can be seen from Table 4.2.1, the Sliding Mode Controller's gain margin is far less than the TLC and the Backstepping controller when there is no observer present. The lower gain margin value of the Sliding mode Control shows that it is more susceptible to regular perturbations. The Sliding Mode Control also has a lower delay margin time compared to the TLC and Backstepping design. Therefore the combination of both perturbations may have played a role on why the TLO Observer caused a loss of tracking control for Sliding mode. It is noted that the SMC includes an integral feedback, which are not present in the TLC and BS controllers. It is known that integral feedback generally increases the steady state tracking accuracy, at the cost of reduced robustness to
singular perturbations. Due to time constraints and the need for research on observers for Sliding Mode Control, the addition of an observer for the Sliding Mode Control will be left as future work.

Thus, only the TLC and Backstepping controller with the TLO Observer will be compared. The standard responses are shown below in the following figures.

![Figure 4.2.6 Step Input Comparison with TLC](image-url)
Figure 4.2.7 Ramp Input Comparison with TLC

Figure 4.2.8 Staircase Input Comparison with TLC
From Figures 4.2.6 to 4.2.8, it can be seen that the Trajectory Linearization Control controller has the same response as the Backstepping controller where the TLC controller is denoted as the blue line and the Backstepping controller is denotes as the red dashed line. It can then be said that the Backstepping controller tracks the reference input just as well as the TLC controller in these three system response cases. This is highly desirable since the Backstepping design method takes into account the nonlinear terms and can use these nonlinear terms to yield the desired response that the TLC controller gives.

Now that it has been shown that the Backstepping controller can at least perform just as well as the TLC controller, the next thing is to show whether or not the Backstepping controller can out perform the TLC controller. In order to demonstrate this comparison the staircase method is used to test the system response. Each controller will operate independently from each other and the input response will increase over time until the system loses tracking. The following figure, Figure 4.2.9, shows each controllers maximum response to the staircase input.

From Figure 4.2.9, it can be seen that the Backstepping response loses tracking at a magnitude of 2.0 while the TLC response departs at 1.5. Therefore the Backstepping controller has higher tracking performance than the TLC controller. This is due to the fact that the nonlinear terms have been taken into account in the controller design. It can also be attributed to the Lyapunov functions that were used in the Backstepping design process for both the pseudo-inverse and the controller that yielded global exponential stability (GES) for the nominal models. Even though the Backstepping controller and Pseudo-inverse are GES, the overall tracking response stability is only local. This is due to regular and singular perturbations to the nominal mathematical model used in the BS controller design and the TLO observer, which is only locally stable since it is based on the trajectory linearization method.
4.3 Robustness Testing with Observer

In order to perform a robustness test on the TRC Backstepping design, certain tests must be done. To determine robustness, the tests that will be run are gain margin, delay margin, domain of stability and disturbance. The main objective in the robustness testing is to have high gain and delay margins. Each of these tests will be performed on the TLC design and compared to the Backstepping design in order to evaluate, which design is more robust.

When comparing the TLC with the Backstepping method, using a step response of 0.5 was tested for the robustness. In testing the gain margin for the two nonlinear systems the maximum gain was determined by increasing the gain of each system until the response loses tracking. A higher gain margin would enable the control system to tolerate larger parametric modeling errors, parameter variations, adverse flight conditions and vehicle performance deteriorations and faults. So when testing the upper gain
margin, the Backstepping tracking performance had a higher gain of 2.10 dB while the TLC had a gain of 1.05 dB. When testing the lower gain margin, the Backstepping controller also had a higher gain of $-12.04$ dB while the TLC had a gain of $-1.41$ dB. The lower gain margin response is shown in Figure 4.3.1.

In the next figure, Figure 4.3.2, the time-delay margin is compared. A higher delay margin would allow the control system to accommodate dynamic modeling errors, singular perturbations, and parasitic dynamic modes such as spin departure, rocking and flutter. Here it can be seen that the Backstepping controller could tolerate a 0.27 second time-delay while the TLC controller had a shorter time-delay of 0.25 seconds. With the Backstepping Controller allows for a slightly longer time-delay in the loop than the Trajectory Linearization Controller, it will allow for the Backstepping Controller to handle slightly more singular perturbations than the Trajectory Linearization Controller.

Figure 4.3.1 Step Response at Lower Gain Margin, $-1.41$ dB for TLC vs. $-12.04$ dB for TRC-BS
In the next figure, Figure 4.3.3, the domain of stability is tested for each nonlinear tracking controller design. In order to test the domain of stability, both controllers' initial conditions were set to $\xi_1(0) = 0.1$, $\xi_2(0) = 0$. Both controllers exhibit similar tracking performance with the step response in terms of settling time and overshoot. The only difference between the two controllers is that the Backstepping controller had a lower initial undershoot. When the initial conditions were set to their maximum value, the Backstepping had the highest value of $\xi_1(0) = 0.52$. While the TLC has a maximum $\xi_1(0) = 0.37$ and Sliding Mode, with no observer, had a maximum $\xi_1(0) = 0.20$. The limits of each controller is shown in Figure 4.3.4 below.
Figure 4.3.3 Step Response with $\xi_1(0) = 0.1$; TLC vs. TRC-BS
In order to determine the step disturbance for the Backstepping controller a step input was placed in the system design just in front of the nonlinear plant shown in Figure 4.3.5.
In Figure 4.3.6, a step disturbance of 0.5 was added to the plant input for each controller at 20 seconds in the simulation time. It can be seen that both the TLC and Backstepping controllers stabilized when the step disturbance was active and they both stay stable with even higher step disturbance sizes. The difference is that both the TLC controller and Backstepping controller do not track the reference command, which stabilized at 0.25 instead of 0.5. This is due to the low values of the effective "gain" for Backstepping and TLC, where the trade off is a higher delay margin.
4.4 Test Results

The last section of the comparison study is to evaluate how the Backstepping controller design performed with the test results collected. The first set of test results came from the comparison of the Backstepping controller with the TLC controller and the Sliding Mode controller. From the figures that compared the tracking performance due to various types of tracking commands, in every single comparison, Backstepping versus TLC with TLO Observer, the Backstepping controller tracked the reference command as well as the TLC controller. This shows that based on tracking performance, the nonlinear Backstepping design can handle each reference command scenario as the TLC controller which uses linearized tracking dynamics. What separates the Backstepping controller from the TLC controller is that the Backstepping controller can handle much higher reference command magnitude when tracking the input reference
command. Table 4.4.1 summarizes the robustness test results of the TLC and Backstepping controller.

<table>
<thead>
<tr>
<th></th>
<th>Delay Margin (Seconds)</th>
<th>Gain Margin (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>TLC with Observer</td>
<td>0.25</td>
<td>[−1.41, 1.05]</td>
</tr>
<tr>
<td>TRC-BS with Observer</td>
<td>0.27</td>
<td>[−12.04, 2.10]</td>
</tr>
</tbody>
</table>

Table 4.4.1 Robustness Test for TLC and TRC-BS Design

When looking at the delay margin, the Backstepping controller had a slightly longer delay time than the TLC controller. This means that the Backstepping controller is slightly more robust than the TLC controller when handling singular perturbations. For the gain margin, each nonlinear system had a step response of 0.5. The results are displayed where the lower gain margin is displayed first in the brackets and the upper gain margin follows. The gain margin has no effect when set to a gain value of 1 which is 0 dB. Therefore by the robustness table, the Backstepping controller has both higher upper and lower gain margins. Since the Backstepping controller has a higher gain margin than the TLC controller, the Backstepping controller has a higher robustness to regular perturbations.

As with the robustness tests on the domain of stability and disturbance, the Backstepping controller had a higher performance than the TLC. With a step disturbance of 0.5 the TLC controller tracking response had a higher percent overshoot and settling time. Also, the TLC did not return to tracking the reference input, which was set at 0.5. Instead the TLC tracking stabilized at approximately 0.25. The Backstepping controller with the same step disturbance was able to stay within tracking of the reference input but did fall off tracking slightly where the tracking stabilized at approximately 0.49. With additional tuning of the controllers and the addition of integral control, the difference in results would be eliminated.
The overall analysis of the test results between the TLC and Backstepping show that the Backstepping controller performs just as well as the TLC controller. As for tracking performance alone, the Backstepping controller had improved tracking performance over the TLC. For the robustness analysis, the Backstepping controller had similar robustness in all the tests against the TLC. In the delay margin, the TLC could only keep tracking stability with small time delays of less than 0.25 seconds with the Backstepping controller having slightly better tracking of 0.27 seconds.
5.1 Conclusions

This thesis gave a step-by-step procedure for designing, implementing and verifying the Backstepping controller using MATLAB/SIMULINK simulation software for a benchmark nonlinear plant that is unstable and nonminimum phase, with a comparison studying against the TLC and Sliding Mode Control designs. It has shown that the Backstepping error regulation controller has the highest tracking performance and good robustness compared to the Trajectory Linearization Controller and Sliding Mode controller.

Some reasons for the improvement in the tracking performance can be attributed to the inclusion of the nonlinearity in controller design. By utilizing a Lyapunov function the Backstepping controller had increased tracking performance compared to the Trajectory Linearization Controller in terms of the maximum tracking command. For the Sliding Mode Controller, which also utilizes the nonlinearity of the plant in design, it still had a lower tracking performance when compared to the Backstepping Controller which could be contributed to integral feedback. The decrease in the tracking performance may be attributed to the constraint of the sliding manifold. Also, the Sliding Mode had an improved overshoot and settling time over the Backstepping design. By adjusting the manifold or relaxing the overshoot and settling, may yield a desired results closer to the Backstepping controller but may not be feasible. Also, the Sliding Mode Controller failed to track the command when the TLO Observer was introduced. One possible reason for this loss of tracking is the adverse effects of singular and regular perturbations caused by the TLO Observer due to the small gain and delay margins that may be caused by the integral feedback.

For the robustness testing, the Backstepping controller when compared to the TLC controller yielded interesting results. The parts of the robustness test that showed...
increase improvement was noticeably the gain margin test. The Backstepping controller had an upper gain margin result of 2.10 dB while the TLC only had an upper gain margin of 1.05 dB. With the increase in the gain margin, the Backstepping controller has an advantage of handling regular perturbations. When comparing the domain of stability, the responses to small initial conditions show very similar tracking responses. The difference between the TLC and Backstepping controller was that the Backstepping controller could handle much higher initial conditions, $\xi_1(0) = 0.51$ vs. $\xi_1(0) = 0.37$.

In response to the delay margin, the TLC controller had a much higher delay of 0.25 seconds while the Backstepping controller had a delay of 0.27 seconds. This means that the Backstepping controller can handle slightly more singular perturbations than the TLC controller. Last, for the step disturbance, the Backstepping controller had as much tracking as the TLC controller but both controller still did not track the command trajectory very close. When the step size was increased for the Backstepping controller, it demonstrated a similar loss of tracking.

When analyzing the results, it can be concluded that there are positives to the Backstepping controller as well as negatives. First, the tracking performance was much improved over the TLC controller as well as the Sliding Mode controller. When comparing with the TLC controller, the Backstepping controller had an increased robustness in the gain margin but with robustness as good as TLC in the delay margin. With the addition of an integral controller, it is likely to see a decrease in the delay time as well as a decrease in the overall robustness for each nonlinear system. The drawbacks to this benchmark system did not hamper the tracking performance and robustness to make it unfavorable to TLC or Sliding Mode designs. The determination of a Lyapunov function was feasible with the benchmark design but there still may exist a Lyapunov function that may be better.

To conclude, the Backstepping based TRC was determined feasible. By utilizing the Backstepping Error Regulator, it allowed for an increase in tracking performance and
robustness while not requiring linear approximation or gain scheduling. Designing and implementing the Backstepping controller was proved to be a worthwhile effort, which greatly increase my understanding of nonlinear design techniques.

5.2 Future Work

There is much future work that can be performed on this nonlinear system, which is mainly due to the fact that advanced nonlinear control theory has only been around since the 1980s. Also, many linearized design techniques are preferred over nonlinear designs due to the fact that they have been proven to work for a large class of nonlinear systems for decades. As technology grows in sophistication, more nonlinear design techniques will need to be matured and developed. Hopefully this thesis, even though the nonlinear system is a textbook benchmark problem, will lead to future nonlinear designs that model real-life devices such as fighter planes and spacecraft.

In order to improve the Backstepping controller design for the thesis, the first thing that should be added is integral control. By adding integral control, an additional state is added. When the Backstepping controller is re-implemented with the additional state, the likely outcome is an increase in steady state tracking performance and rejection of constant disturbances, at the cost of reduced robustness to singular perturbations. After the integral control is added, the TLC controller will also need integral control. This will lead to an accurate comparison between TLC, Backstepping and Sliding Mode Control.

The second thing that can be done is trying to design a better Lyapunov function for the Backstepping error regulator. Based upon the Lyapunov theory, there may exist many different types of Lyapunov functions that will globally stabilize the system. Therefore just because a Lyapunov candidate is found which yields a Lyapunov function, does not mean that there is not another Lyapunov candidate that yields a Lyapunov
function that increases stability and robustness. The downside is that sometimes in order to find another Lyapunov function, the complexity in solving for it may be too great for any practical use.

The third thing left for future work is to implement an Observer into the Sliding Mode Control. This could entail making adjustments to the gain of TLO Observer or possibly designing a new observer where the nonlinear states are not linearized. Also, since the Sliding Mode Control has integral control, this could also be the reason for the decrease response in singular perturbations. It should be noted that if a new observer is added to the Sliding Mode Control, the following implementations should also be added to the Backstepping controller or other nonlinear control methods that are being compared for accurate comparisons.

The fourth improvement is to replace the trajectory linearization observer with a trajectory regulation observer. This will hopefully increase the accuracy of the estimated states as well as increase beyond local stability. The new TR Observer could then be used on each of the tracking controllers in order to increase the overall tracking response and robustness.

The last thing that can be done in the future is to take the current model that was used in this thesis and extend it a known design like an aircraft, spacecraft, or any other system that uses control design. For the purpose of continuing the work on the NASA reusable launch vehicle project, the Backstepping design can be implemented on the X-33 benchmark. This would differ significantly from the present design, but the theory and design steps should stay relatively the same. Once the controller has been implemented into the X-33 overall system, simulation tests can compare the Backstepping controller design with the old TLC controller design. Hopefully, one day implementing this Backstepping controller design into the X-33, it could yield a more robust system design to handle more flight conditions than the current TLC design.
References


%FileName: load_vars.m
%Description: Loads Eigenvalues and Gains for Backstepping, Sliding Mode and TLC Controller

clear
close all

%Controller Eigenvalues
pc1 = 2*(-.5+j*sqrt(3)/2);
pc2 = 2*(-.5-j*sqrt(3)/2);
k1 = pc1*pc2;
k2 = -pc1 - pc2;
K = [k1,k2];

%Observer Eigenvalues
po1 = -3/2+j*3*sqrt(3)/2;
po2 = -3/2-j*3*sqrt(3)/2;
betao(1) = po1*po2;
betao(2) = -po1-po2;

k = 7;

a1 = 1;
a2 = 1;
Xi0 = [.01 .01];

alpha = [.8,.2];
beta = [.5,.2];

ks = [20]
satfunc =1;
reg_perturb = 1;

%Controller Eigenvalues
%TLC
pc1 = -.5+j*sqrt(3)/2;
pc2 = -.5-j*sqrt(3)/2;
betac(1) = pc1*pc2;
betac(2) = -pc1-pc2;
alpaha1 = 1+2*(sec(xi1))^2*((tan(xi1))^2+tan(xi1)*xi2);
alpaha2 = (sec(xi1))^2;
R = [1 0; (sec(xi1))^2 1];
K1 = alpha1-betac(1);
K2 = alpha2-betac(2);
K = [K1 K2]*R

fprintf('Variables Loaded')

%End of File

%FileName: nonlinear.m
%Description: Performs nonlinear state computation
%Nonlinear state computation

function x = nonlinear(x1,x2)

x1_dot = tan(x1) + x2;

x2_dot = x1;

x = [x1_dot,x2_dot];

%End of File

%FileName: observerh.m
%Description: Calculates H Matrices

function x = observerh(xi1,xi2,beta1,beta2)

xi1_dot = tan(xi1)+xi2;
H1 = -1-beta1+2*xi1_dot*(1+tan(xi1)*tan(xi1))*tan(xi1)-(1+tan(xi1)*tan(xi1)*(-
(1+tan(xi1)*tan(xi1)))-beta2;
H2 = (1+tan(xi1)*tan(xi1))-beta2;

x = [H1,H2];

%End of File

%FileName: project.m
%Description: Determines canonical realizations for LTV controller and LTV observer

clear

%LTV Controller
syms x1 x2 u n xi1_dot Kc1 Kc2 Kc3 xi1_err xi2_err xi3_err xi1_nom xi2_nom mu_err mu_nom v k1 k2 k1_ic k2_ic k3_ic real
x1_dot = tan(x1)+x2;
x2_dot = x1+u;
n = x2;

A = [(sec(x1))^2 1;1 0];
B = [0;1];
C = [0 1];

CC = [0 -1;1 0];
PnT = [0 1]*inv(CC);
Pn = PnT';
L = inv([1 0; (sec(x1))^2 1]);
L_dot = [0 0; 2*sec(x1)^2*tan(x1)*x1_dot 0];

Ac = simple(inv(L)*A*L-inv(L)*L_dot);
Bc = simple(inv(L)*B);
Cc = simple(C*L);

%LTV Observer
OC = [0 1;1 0];
qn = (inv(OC)*[0;1])
T = [qn (A)*qn]
T_dot = simple(diff(T)*xi1_dot)
Ao = inv(T)*A*T-inv(T)*T_dot
Bo = inv(T)*B
Co = C*T
alpha1 = 2*(sec(x1))^2*(tan(x1))*x1_dot - 10;
alpha2 = -(sec(x1))^2 - 3;
Kp = [alpha1; alpha2];
Ko = inv(T)*Kp
Ho = [-10+2*xi1_dot*(sec(x1))^2*tan(x1);-(sec(x1))^2-3]
H = simple(inv(T)*Ho)

%End of File