DEVELOPMENT OF A BLOCK PROCESSING CARRIER TO NOISE RATIO ESTIMATOR FOR THE GLOBAL POSITIONING SYSTEM

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This thesis entitled
DEVELOPMENT OF A BLOCK PROCESSING CARRIER TO NOISE RATIO
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Existing carrier to noise ratio, C/No, estimators accumulate wide and narrow band powers to determine C/No. This thesis presents a complete mathematical derivation for such a C/No estimator. This estimator is biased by approximately –0.22 dB and has an accuracy of better than 0.5 dB (95%) for C/No’s between 40 and 56 dB-Hz averaged over 1 s. Next, a block processing C/No estimator is developed that maximizes the observability of the C/No using frequency domain techniques. The block processing C/No estimator is unbiased and has an accuracy of better than 0.09 dB (97.5%) for C/No’s between 40 and 56 dB-Hz averaged over 1 s.

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1 INTRODUCTION

The Global Positioning System, GPS, provides position, velocity, and time information to a GPS user. A GPS receiver will first receive the GPS signal and then filter and amplify the received signal. There will be some noise that will accompany the signal into the receiver, but the power of the thermal noise produced by the filters and amplifiers will be larger than the power of both the external noise and the GPS signal. However, the structure of the GPS signal is such that the receiver may still detect the signal. GPS and the GPS signal structure are discussed further in chapter 2.

A typical GPS receiver may have a minimum bandwidth of about 2.046 MHz. Since a larger receiver bandwidth, $B$, introduces a greater amount of noise, $B$ is partially responsible for the value of $\frac{S}{No}$. Carrier to noise ratio, C/No, is the signal to noise ratio measured over a bandwidth of 1 Hz. Typical values of C/No for GPS range between 40 and 50 decibel-Hertz, or dB-Hz.

For the purposes of this research, assume a stationary receiver and a well-behaved reference oscillator such that the uncertainty in the frequency change is less than 5 Hz over 1 s. Let $\lambda$ equal the wavelength of the GPS signal, approximately 0.19 m. For this case, the phase of the signal changes less than $0.005\lambda$ over 1 ms. This will be referred to as fine tracking.
There are several methods by which a GPS receiver may determine a C/No estimate for a satellite that the receiver is tracking. One such method utilizes wide and narrow band powers of the sampled GPS signal and noise. This method may be used in a receiver that performs sequential processing. Sequential processing means that the receiver samples the signal and noise, processes each sample individually, and then determines the sum of the desired quantities over a period of time. A further discussion on sequential processing and this method of C/No estimation is presented in Chapter 3.

A new method of C/No estimation was created for the Software Receiver developed by the Avionics Engineering Center at Ohio University. The Software Receiver utilizes a processing technique known as block processing. Block processing means that multiple samples of signal plus noise are processed at once. This processing technique allows for a new C/No estimation method based upon average power of processed signal plus noise samples. A further discussion of this processing technique and the corresponding C/No estimation method are presented in Chapter 4. Finally, a comparison of the results obtained from this method and from the method described in Chapter 3 is presented in Chapter 5.
2 THE GLOBAL POSITIONING SYSTEM

By the early 1970’s, the United States Air Force and Navy had experimented with satellite based navigation systems. In 1973, a small group of armed forces officers and civilians created a plan for what was later termed the Global Positioning System, GPS [Spilker 1996]. In 1978, the first GPS satellite was launched, and, in 1994, the Federal Aviation Administration, FAA, declared GPS ready for use in civilian aviation [Tsui 2000].

GPS consists of three segments: the control segment, the space segment, and the user segment. The control segment tracks each satellite; periodically, the control segment will upload predictions of future satellite positions and satellite clock corrections to each individual satellite [Van Dierendonck 1978]. The space segment consists of the GPS satellites that continuously transmit signals to the user segment. These signals contain the navigation message. By receiving and processing the satellites’ signals, the user segment determines three-dimensional position and local time. [Spilker 1996].

If the user is near the surface of the Earth, a GPS receiver typically receives between 4 and 11 signals simultaneously. A minimum of 4 satellites is required to determine the three–dimensional user position. The GPS military specifications state that GPS must provide military users with a 10 m, or less, root mean square position error [Spilker 1996].
2.1 GPS Signal Structure

Each GPS satellite transmits a right-hand circularly polarized signal with two frequency components: Link 1, L1, and Link 2, L2 [Tsui 2000]. The center frequency of L1 is 1575.42 MHz, and the center frequency of L2 is 1227.6 MHz. The clock frequency of each GPS satellite is approximately 10.23 MHz. The center frequency of L1 is 154 times the clock frequency, and the center frequency of L2 is 120 times the clock frequency [Spilker 1978].

The actual frequency of a satellite clock is 10.23 MHz minus an offset of approximately 4.57 mHz [Spilker 1996]. As the signal approaches the Earth, the frequency of the signal increases by approximately the same value as the offset clock frequency due to relativity [Spilker 1996]. Thus, to a stationary user, the clock frequency of the satellite appears to be 10.23 MHz.

The use of L-band provides acceptable received signal power with reasonable satellite transmit power levels and Earth coverage satellite antenna patterns [Spilker 1996]. The GPS satellite’s transmitter has a power amplifier equal to 50 W which is equal to 17 dBW; however, the input to the transmitting antenna is only 14.3 dBW [Tsui 2000]. The minimum specified received signal strength at L1 is -160 dBW for the C/A code and -163 dBW for the P code [Spilker 1996].
The GPS signal at L1 contains the Coarse/acquisition, C/A, code and the Precision, P, code. The GPS signal at L2 contains only the P code. The P code is encrypted, and the resulting code is referred to as the Y code. The focus of this research is C/A code processing. Therefore, L2 will not be discussed further. The signal for satellite $i$ at L1 is given by the following:

$$s_i(t) = AC(t)D(t)\cos(\omega t + \Delta \theta) + AP(t)D(t)\sin(\omega t + \Delta \theta)$$  \hspace{1cm} [2.1]

where $C(t)$, $P(t)$, and $D(t)$ are the C/A code, P code, and navigation message at time $t$, respectively. The navigation message at time $t$ represents the navigation data bit at time $t$. $A$ is the amplitude of the C/A code component of the signal, and $A_P$ is the amplitude of the P code component of the signal. $\Delta \theta$ represents a small phase noise and oscillator drift, and $\omega$ is equal to $2\pi$ times the L1 frequency [Spilker 1996].

Before the signal reaches the receiver, the carrier undergoes a Doppler shift. The Doppler shift for a stationary user is caused by the satellite’s velocity component toward the user in combination with the velocity component due to earth rotation at the user [Tsui 2000]. The maximum Doppler frequency shift of L1 for a stationary user is approximately 5 kHz. If a receiver is used for a high-speed vehicle, the maximum Doppler shift is equal to 10 kHz. In order for the vehicle alone to produce a Doppler shift of 5 kHz, the vehicle must move toward the satellite at a rate of 2,078 miles per hour [Tsui 2000].
The C/A code, P code, and navigation message are phase-modulated signals with angle, \( \phi \), equal to \( \pm \pi \) [Tsui 2000]. This type of phase modulation is known as binary phase shift keying, BPSK. For the C/A and P codes, the phase shift rate is known as the chip rate. For the navigation message, the phase shift rate is known as the bit rate. The chip rate of the C/A code is 1,023 chips per ms, chips/ms; the chip rate of the P code is 10,230 chips/ms. The bit rate of the navigation message is 50 bits per s, bps. The C/A code, P code, and navigation message are always equal to either \( \pm 1 \).

The GPS signal at L1 is a code division multiple access, CDMA, signal. CDMA is a multiple access technique that allows multiple spread spectrum signals to be broadcast at the same frequency [Couch 1997]. There are two requirements for a signal to be considered a spread spectrum signal. The first is that a spread spectrum signal must have a bandwidth much larger than the bandwidth of the message [Couch 1997]. The second is that this wider bandwidth must be due to an independent modulating waveform, known as the spreading signal; the spreading signal must be known by the receiver for the message to be decoded [Couch 1997]. Thus, the navigation message may be found using only the C/A code component of equation [2.1]. In fact, the P code is primarily used by the military, and it will not be discussed further.

The C/A code belongs to a family of pseudorandom noise, PRN, codes known as Gold codes. The C/A code is generated from the product of two PRN codes, G1 and G2, which have length equal to 1,023 chips and a chip rate of 1,023 chips/ms [Tsui 2000].
Thus, the C/A code has length equal to 1,023 chips and duration equal to 1 ms; the
duration of a navigation data bit is equal to 20 times the duration of a C/A code.
Maximum-length linear shift registers of 10 stages generate G1 and G2 [Tsui 2000].
There is a shift that occurs in G2 before the product of G1 and G2, or the C/A code, is
formed. The number of chips that are shifted is unique to each individual satellite and
causes each satellite to have a unique C/A code. The shifts that produce the best
autocorrelation and cross-correlation results for the C/A codes were chosen for the GPS
satellites [Tsui 2000].

2.2 Autocorrelation Properties of the C/A Code

$R(\tau)$ is equal to the autocorrelation of the C/A code, $C(t)$, and $C(t + \tau)$ where $\tau$ is a
time delay. When $\tau$ is equal to 0, $R(\tau)$ is equal to the maximum autocorrelation peak;
the autocorrelation values for the absolute value of $\tau$, $|\tau|$, greater than 1 μs are smaller
than the maximum autocorrelation peak value. This characteristic of $R(\tau)$ allows the
time offset of the C/A code transmitted by the satellite to be determined by the receiver.

Consider the case where $R(\tau)$ is normalized and the sampling frequency is 1,023 samples
per ms, samples/ms. Thus, the maximum autocorrelation peak is equal to 1. $R(\tau)$ for $|\tau|$ greater than 1 μs is equal to $-\frac{1}{1,023}$ for approximately 75% of $\tau$ [Tsui 2000]. This is
due to the fact that there is always one more negative chip than positive chips in a C/A
code. However, there are small autocorrelation peaks, known as side lobes, which occur
the other 25% of $\tau$. If the sampling frequency is 1,023 samples/ms, the side lobes will equal either $\frac{65}{1,023}$ or $\frac{63}{1,023}$ [Tsui 00].

An example of $R(\tau)$ for $|\tau|$ less than or equal to 25 $\mu$s is shown in Figure 2.1. Note that the width of each autocorrelation peak is equal to 2 times the duration of each C/A code.

![Autocorrelation of a C/A code](image)

**Figure 2.1:** $R(\tau)$ for $|\tau|$ less than or equal to 25 $\mu$s for Satellite Vehicle Identification Number 5
chip. As long as the sampling frequency of the C/A code is at least 1,023 samples/ms, the width of each autocorrelation peak will be the same.

2.3 Noise Power

Noise will accompany the GPS signal into the GPS receiver’s antenna. Typically, a filter and a low noise amplifier will follow the antenna. The filter and amplifier combination will add thermal noise power, \( N_T \), given by the following:

\[
N_T = kTB \tag{2.2}
\]

where \( k \) is Boltzmann’s constant \((1.38 \times 10^{-23} \text{ J/K})\), \( T \) is the temperature in K, and \( B \) is the bandwidth in Hz. Assume that the total noise power in the receiver may be modeled as thermal noise power. The equivalent noise power, \( No \), is equal to the equivalent thermal noise power in the receiver; \( No \) is given by the following:

\[
No = kT_eB \tag{2.3}
\]

where \( T_e \) is the equivalent temperature in K that produces a \( No \) equal to the total noise power in the system. \( C/No \) is given by the following:

\[
C/No = 10 \log_{10} \left( \frac{S}{No}B \right) \tag{2.4}
\]

where \( S \) is the power of the GPS signal.
3 C/No ESTIMATE BASED ON WIDE AND NARROW BAND POWERS

One method used to estimate C/No is to compare signal plus noise power over two different bandwidths. The GPS receiver must process the incoming GPS signal and ambient noise before estimating the C/No. The following is both a discussion of the signal processing performed by the receiver and a derivation for the C/No estimate. This derivation closely follows the derivation provided by [Van Dierendonck 1996]. However, it differs from [Van Dierendonck 1996] in that a complete mathematical derivation for the expression of the C/No estimate and the error in the C/No estimate is provided.

3.1 GPS Receiver Signal Processing

Figure 3.1 shows a block diagram of a typical GPS receiver. The basic functionality described in this diagram is common to most GPS receivers. However, specific details of architecture design will vary from receiver to receiver.

3.1.1 Antenna, Filter, and Amplifier

The GPS signal and noise enters the receiver via a right-hand circularly polarized, RHCP, antenna. The antenna should have a large spatial angle at near uniform gain. Typically, the antenna will attenuate signals that have elevation angles less than 10° to the horizon so that error due to multipath is limited and so that weak signals do not interfere with stronger signals. If the receiver only tracks the C/A code on the L1
frequency, then the antenna should have a bandwidth of at least 2.046 MHz centered at a radio frequency, rf, of 1575.42 MHz [Kaplan 1996].

A filter and an amplifier immediately follow the antenna. The purpose of the filter is to reduce out-of-band interference, and the purpose of the amplifier is to raise the signal plus noise close to the maximum range of the analog-to-digital converter (ADC). The GPS signal, $s(t)$, is given by the following:

$$s(t) = AC(t)D(t)\cos(\omega t + \theta) \quad [3.1]$$
where \( A \) is the amplitude, \( \omega \) is the frequency at the antenna, and \( \theta \) is the Doppler offset.

The total noise is \( n(t)\cos(\omega t) \) where \( n(t) \) is given by the thermal noise produced by the filter and amplifier plus the noise that entered the antenna.

3.1.2 Downconversion to IF and Baseband

One method of signal processing requires downconversion of the rf signal and noise to an intermediate frequency, IF. The IF signal and noise are then downconverted to baseband.

Figure 3.2 shows this downconversion process.

![Figure 3.2: Downconversion Process](image)

The downconversion from rf to IF is achieved by mixing the rf signal with a local oscillator, LO. If the LO is equal to \( LO_1 = 2\cos(\omega_1 t) \) [Van Dierendonck 1996], then the signal at IF, \( s_{IF}(t) \), plus the noise at IF, \( n_{IF}(t) \), and the upper band are given by

\[
s_{IF}(t) + n_{IF}(t) + \text{upper band} = 2\cos(\omega_1 t)[s(t) + n(t)]
\]  

[3.2]
where $s(t)$ was defined in equation [3.1]. Since only $s_{IF}(t)$ and $n_{IF}(t)$ are of interest, bandpass filters are used to eliminate the upper band. Bandpass filters aid the frequency plan in limiting LO feedthrough, harmonics, etc. [Van Dierendonck 1996]. After filtering, $s_{IF}(t)$ plus $n_{IF}(t)$ is given by the following:

$$s_{IF}(t) + n_{IF}(t) = AC(t)D(t)\cos[(\omega - \omega_1)t + \theta] + n(t)\cos[(\omega - \omega_1)t]$$  \[3.3\]

Filters, as well as amplifiers, will alter $A$, but these alterations do not effect the overall results of the receiver. Thus, they will be ignored in the following discussion.

After downconversion to IF, the receiver converts $s_{IF}(t)$ and $n_{IF}(t)$ to baseband. Conversion to baseband is the process of converting the IF signal to that of in-phase and quadraphase components of the signal envelope, but still modulated with residual Doppler [Van Dierendonck 1996]. This conversion is realized by mixing $s_{IF}(t)$ and $n_{IF}(t)$ with two LO’s of frequency $\omega - \omega_1$. These LO’s are 90° phase-shifted with respect to one another.

The in-phase component is realized by mixing both $s_{IF}(t)$ and $n_{IF}(t)$ with an LO given by $LO_{2,l} = \sqrt{2}\cos(\omega_2 t)$ [Van Dierendonck 1996]. The lower band of the in-phase signal plus noise component, $I_{s,l}(t) + I_{n,l}(t)$, plus the upper band of the in-phase signal plus noise component, $I_{s,u}(t) + I_{n,u}(t)$, is

$$I_{s,l}(t) + I_{n,l}(t) + I_{s,u}(t) + I_{n,u}(t) = [s_{IF}(t) + n_{IF}(t)]LO_{2,l}$$  \[3.4\]
Only the lower frequency band of the in-phase signal plus noise component is desired. Low pass filters are used to eliminate the upper band of the signal plus noise. Since

$$\omega - \omega_1 - \omega_2 = 0 \tag{3.5}$$

the in-phase component of the signal is the following:

$$I_s(t) = I_{s,l} = \frac{A}{\sqrt{2}} C(t) D(t) \cos(\theta)$$

The in-phase component of the noise is

$$I_n(t) = I_{n,l} = \frac{1}{\sqrt{2}} n(t) = x_n(t) \tag{3.6}$$

The quadruphase component is realized by mixing both $s_{IF}(t)$ and $n_{IF}(t)$ with an LO equal to $LO_{2,Q} = \sqrt{2} \cos(\omega_2 t + 90^\circ)$ [Van Dierendonck 1996]. The lower band quadruphase signal plus noise component, $Q_{s,l}(t) + Q_{n,l}(t)$, plus the upper band quadruphase signal plus noise component, $Q_{s,u}(t) + Q_{n,u}(t)$, is

$$Q_{s,l}(t) + Q_{n,l}(t) + Q_{s,u}(t) + Q_{n,u}(t) = [s_{IF}(t) + n_{IF}(t)]LO_{2,Q}(t) \tag{3.7}$$

As in the case of the in-phase signal plus noise component, only the lower frequency band of the quadruphase signal plus noise component is desired. Filters are again used to eliminate the upper frequency band of both the signal and, consequently, the noise. Using the fact that $\omega - \omega_1 - \omega_2 = 0$, the quadruphase component of the signal is the following:

$$Q_s(t) = Q_{s,l} = -\frac{A}{\sqrt{2}} C(t) D(t) \sin(\theta) \tag{3.8}$$
The quadraphase component of the noise is

\[ Q_n(t) = Q_{n,t}(t) = -\frac{1}{\sqrt{2}} n(t) = y_n(t) \]  

[3.9]

3.1.3 Analog-to-Digital Conversion

After conversion to baseband, the ADC’s sample and quantize the in-phase and quadraphase components. This means that at time \( t_k \), where \( k = 0, 1, 2, \ldots \), the in-phase signal component is

\[ I_{s,k} = \frac{A}{\sqrt{2}} C_k D_k \cos(\theta_k) \]  

[3.10]

where \( C_k \), \( D_k \), and \( \theta_k \) are the C/A code, data navigation bit, and Doppler offset at time \( t_k \). The in-phase noise component at time \( t_k \) is

\[ I_{n,k} = x_{n,k} \]  

[3.11]

where \( x_{n,k} \) is the in-phase noise component at time \( t_k \). The quadraphase signal component at time \( t_k \) is

\[ Q_{s,k} = -\frac{A}{\sqrt{2}} C_k D_k \sin(\theta_k) \]  

[3.12]

The quadraphase noise component at time \( t_k \) is

\[ Q_{n,k} = y_{n,k} \]  

[3.13]

where \( y_{n,k} \) is the quadraphase noise component at time \( t_k \).
In order to satisfy Nyquist’s theory, the sample frequency of the ADC should be greater than twice the bandwidth of the signal plus noise [Van Dierendonck 1996]. Also, the sampling frequency should not be an integer multiple of the C/A code chip rate [Tsui 2000]. If the sample frequency is an integer multiple of the C/A code chip rate, then the samples are synchronized with the C/A code. This means that the precise time shift of the C/A code is impossible to detect and that signal processing will not provide fine time and distance resolution [Tsui 2000]. Aliasing due to sampling is prevented via a low pass filter. An automatic gain control, AGC, is then used to increase the dynamic range, control the quantization level, and suppress pulse interference [Van Dierendonck 1996].

At this point in the receiver, the in-phase and quadraphase components enter multiple receiver channels. These channels perform the same signal processing, but each channel tracks a different satellite. This means that the number of satellites that can be tracked simultaneously equals the number of receiver channels. The following is a discussion of the signal processing that occurs in each channel.

3.1.4 Doppler Removal

The Doppler shift is removed from the signal via phase rotation of the in-phase and quadraphase components [Van Dierendonck 1996]. This phase rotation is achieved in a carrier tracking loop via a Numerically Controlled Oscillator, NCO. The NCO creates cosine and sine terms that operate at a reference phase offset $\theta_{rd}$. 
Let $\theta_{rD,k}$ equal the reference phase offset at time $t_k$, then the in-phase component of the signal is the following:

$$I_{D,s} = \frac{A}{\sqrt{2}} C_k D_k \cos(\theta_k - \theta_{rD,k})$$  \[3.14\]

Expansion of the cosine term yields

$$I_{D,s} = \frac{A}{\sqrt{2}} C_k D_k \left[ \cos(\theta_k) \cos(\theta_{rD,k}) + \sin(\theta_k) \sin(\theta_{rD,k}) \right]$$  \[3.15\]

Substitution of equations [3.10] and [3.12] into equation [3.15] yields the following:

$$I_{D,s} = I_{s,k} \cos(\theta_{rD,k}) - Q_{s,k} \sin(\theta_{rD,k})$$  \[3.16\]

Similarly, the quadraphase component of the phase-shifted signal is

$$Q_{D,s} = \frac{A}{\sqrt{2}} C_k D_k \sin(\theta_k - \theta_{rD,k})$$  \[3.17\]

Expansion of the sine term yields

$$Q_{D,s} = \frac{A}{\sqrt{2}} C_k D_k \left[ \sin(\theta_k) \cos(\theta_{rD,k}) - \cos(\theta_k) \sin(\theta_{rD,k}) \right]$$  \[3.18\]

Substitution of equations [3.10] and [3.12] into equation [3.18] yields the following:

$$Q_{D,s} = -Q_{s,k} \cos(\theta_{rD,k}) - I_{s,k} \sin(\theta_{rD,k})$$  \[3.19\]

The in-phase noise component and the quadraphase noise component remain random, but their phases are also shifted [Van Dierendonck 1996]. The in-phase noise component may be written as the following:

$$I_{D,n} = x_{D,k}$$  \[3.20\]
where $x_{D,k}$ is the in-phase noise component at time $t_k$ after Doppler removal. The quadraphase noise component may be written as the following:

$$Q_{D,n} = y_{D,k} \quad [3.21]$$

where $y_{D,k}$ is the quadraphase noise component at time $t_k$ after Doppler removal.

### 3.1.5 Correlator

During the correlation process, the in-phase and quadraphase components are multiplied by a reference C/A code. The reference code is created in a second NCO that operates in a code tracking loop. The reference code is shifted by a fraction of a C/A chip to produce early, punctual, and late phases. A common fraction used for this shift is $\frac{1}{2}$ a C/A code chip. Let $p$ denote the phase, then $C_{k, rp}$ denotes the digital reference C/A code at phase $p$ and time $t_k$. Using equation [3.14], the correlated in-phase signal component is the following:

$$I_{c,s} = I_{D,s} C_{k,rp} = \frac{A}{\sqrt{2}} C_k C_{k,rp} D_k \cos(\theta_k - \theta_{rD,k}) \quad [3.22]$$

Using equation [3.17], the correlated quadraphase signal component is the following:

$$Q_{c,s} = Q_{D,s} C_{k,rp} = \frac{A}{\sqrt{2}} C_k C_{k,rp} D_k \sin(\theta_k - \theta_{rD,k}) \quad [3.23]$$

The in-phase noise component is:

$$I_{c,n} = I_{D,n} C_{k,rp} = x_{D,k} C_{k,rp} \quad [3.24]$$
The quadraphase noise component is:

\[ Q_{c,n} = Q_{D,n} C_{k,ry} = y_{D,k} C_{k,ry} \]  \[3.25\]

3.1.6 Accumulator

After correlation, the in-phase and quadraphase samples are summed in an accumulator. The samples are summed over an epoch equal to \( T \); \( S_c \) is the number of samples summed within epoch \( T \). In the following discussion, \( T \) equal to 1 ms will be assumed since it is the length of an entire C/A code sequence and since \( D_k \) is constant during this period [Van Dierendonck 1996]. If the accumulated correlated values are equal to or greater than a selected threshold, then the receiver is tracking the signal. However, if the accumulated correlated values are less than the threshold, then the signal is not being tracked.

The accumulator serves as a time average of the correlated samples [Van Dierendonck 1996]. Thus, the expected value of the accumulated in-phase signal component, \( I_{A,s} \), is the following:

\[
I_{A,s} = \mathbb{E} \left[ \sum_{k=1}^{S_c} I_{c,s} \right] = \frac{A}{\sqrt{2}} D_i \mathbb{E} \left[ \sum_{k=1}^{S_c} C_k C_{k,ry} \cos(\theta_k - \theta_{D,k}) \right] \]  \[3.26\]

where \( D_i \) is equal to the constant \( D_k \) over \( T \) and \( I_{c,s} \) is the correlated in-phase signal component defined in equation [3.22]. The expected value of a sum is equal to the sum of the expected values [Leon-Garcia 1994]. Thus,
The cosine term in the above equation is constant for each $t_k$. Thus,

$$I_{A,s} = \frac{A}{\sqrt{2}} D_i \sum_{k=1}^{S_i} E \left[ C_k C_{k,rp} \cos(\theta_k - \theta_{rD,k}) \right] \quad [3.27]$$

The expected value of $C_k C_{k,rp}$ is equal to the autocorrelation of the C/A code, $R(\tau)$, where $\tau$ is the delay between $C_k$ and $C_{k,rp}$ at time $t_k$ [Van Dierendonck 1996]. $R(\tau)$ is not dependent upon $k$ since $\tau$ is constant for all $k$. Thus, $R(\tau)$ is constant for all $k$.

$I_{A,s}$ may be written as

$$I_{A,s} = \frac{A}{\sqrt{2}} D_i \sum_{k=1}^{S_i} R(\tau) \cos(\theta_k - \theta_{rD,k})$$

$$= \frac{A}{\sqrt{2}} D_i R(\tau) \sum_{k=1}^{S_i} \cos(\theta_k - \theta_{rD,k}) \quad [3.29]$$

Similarly, the expected value of the accumulated quadraphase signal component, $Q_{A,s}$, is given by the following:

$$Q_{A,s} = E \left[ \sum_{k=1}^{S_i} Q_{r,s} \right] = \frac{A}{\sqrt{2}} D_i E \left[ \sum_{k=1}^{S_i} C_k C_{k,rp} \sin(\theta_k - \theta_{rD,k}) \right]$$

$$= \frac{A}{\sqrt{2}} D_i \sum_{k=1}^{S_i} E\left[C_k C_{k,rp} \sin(\theta_k - \theta_{rD,k})\right] \quad [3.30]$$

where $Q_{c,s}$ is the correlated quadraphase signal component defined in equation [3.23].

The sine term in the above equation is constant at each time $t_k$. Thus,
\[ Q_{A,s} = \frac{A}{\sqrt{2}} D_i \sum_{k=1}^{S_i} E[C_{k,r,p}] \sin(\theta_k - \theta_{rD,k}) \]  \[3.31\]

\[ Q_{A,s} \] may be written as

\[ Q_{A,s} = \frac{A}{\sqrt{2}} R(\tau) D_i \sum_{k=1}^{S_i} \sin(\theta_k - \theta_{rD,k}) \]  \[3.32\]

Integration may be used to approximate a discrete sum. Consider a function \( f(x) \) defined on the closed interval \([a, b]\). Consider a partition \( P \) with \( n \) subintervals and points \( a = x_0 < x_1 < x_2 < ... < x_n = b \). The Riemann sum, \( R_P \), is equal to the following:

\[ R_P = \sum_{k=1}^{n} f(\bar{x}_k) \Delta x_k \]  \[3.33\]

where \( \Delta x_k \) is equal to \( x_k - x_{k-1} \) and \( \bar{x}_k \) is a sample point for the \( k \)th subinterval [Varberg 1992]. The sample point \( \bar{x}_k \) is arbitrary and, thus, may be an endpoint on the \( k \)th subinterval [Varberg 1992]. Define the norm of \( P \), \( |P| \), to be the length of the longest of the subintervals of the partition \( P \). The definite integral of \( f(x) \) over the interval \([a, b]\) is equal to the Riemann sum of \( f(x) \) as \( |P| \) approaches 0 [Varberg 1992]:

\[ \int_a^b f(x) \, dx = \lim_{|P| \to 0} R_P = \lim_{|P| \to 0} \sum_{k=1}^{n} f(\bar{x}_k) \Delta x_k \]  \[3.34\]

The summations in equations \[3.29\] and \[3.32\] may be approximated using the above technique. These two summations have \( S_i \) subintervals, of equal length \( t_k - t_{k-1} \), on the
interval \([0, T]\). If the \(k\)th sample point is the endpoint of the \(k\)th subinterval for both summations, then both summations have the form

\[
\sum_{k=1}^{S} f(\theta_k - \theta_{rD,k}) \approx \frac{1}{\Delta t_k} \int_0^T f(\theta_k - \theta_{rD,k}) dt \quad [3.35]
\]

Since the subintervals are of equal length, \(T = S_c (t_k - t_{k-1})\). Recall from chapter 1 that a stationary user was assumed and, thus, the frequency error, \(f_i\), and the phase error, \(\Delta \theta_i\), of the residual carrier on the signal are approximately constant over \(T\) equal to 1 ms [Van Dierendonck 1996]. Thus, substitution of the linear approximation \(2\pi f_i t + \Delta \theta_i\) for \((\theta_k - \theta_{rD,k})\) in the above equation yields

\[
\sum_{k=1}^{S} f(2\pi f_i t_k + \Delta \theta_i) \approx \frac{S_c}{T} \int_0^T f(2\pi f_i t + \Delta \theta_i) dt \quad [3.36]
\]

An approximation of the accumulated in-phase signal component is

\[
I_{A,s} = \frac{A}{\sqrt{2}} R(\tau) D_i \frac{S_c}{T} \int_0^T \cos(2\pi f_i t + \Delta \theta_i) dt \quad [3.37]
\]

or

\[
I_{A,s} = \frac{A}{\sqrt{2}} R(\tau) D_i \frac{S_c}{2\pi f_i T} \left[ \sin(2\pi f_i T + \Delta \theta_i) - \sin(\Delta \theta_i) \right] \quad [3.38]
\]

Expansion of the sine terms yield

\[
\sin(2\pi f_i T + \Delta \theta_i) - \sin(\Delta \theta_i) = 2 \sin(\pi f_i T) \cos(\pi f_i T) \cos(\Delta \theta_i) +
\]

\[
- \sin(\Delta \theta_i) [1 - \cos(2\pi f_i T)] \quad [3.39]
\]

Further simplification yields
\[
\sin(2\pi f_i T + \Delta \theta_i) - \sin(\Delta \theta_i) = 2 \sin(\pi f_i T) \cos(\pi f_i T) \cos(\Delta \theta_i) + \\
- 2 \sin(\Delta \theta_i) \sin^2(\pi f_i T)
\]  
\[3.40\]

Thus, the accumulated in-phase signal component may be written as

\[
I_{A,s} = \frac{A}{\sqrt{2}} R(\tau) D_i \frac{\sin(\pi f_i T)}{\pi f_i T} \cos(\pi f_i T + \Delta \theta_i)
\]  
\[3.41\]

Similarly, an approximation of the accumulated quadrature signal component is

\[
Q_{A,s} = \frac{A}{\sqrt{2}} R(\tau) D_i \frac{S_c}{T} \int_0^\tau \sin(2\pi f_i t + \Delta \theta_i) \, dt
\]  
\[3.42\]

or

\[
Q_{A,s} = -\frac{A}{\sqrt{2}} R(\tau) D_i \frac{S_c}{2\pi f_i T} \left[ \cos(2\pi f_i T + \Delta \theta_i) - \cos(\Delta \theta_i) \right]
\]  
\[3.43\]

Expansion of the cosine terms yield

\[
\cos(2\pi f_i T + \Delta \theta_i) - \cos(\Delta \theta_i) = -\sin(2\pi f_i T) \sin(\Delta \theta_i) + \\
- \cos(\Delta \theta_i)[1 - \cos(2\pi f_i T)]
\]  
\[3.44\]

Simplification yields

\[
\cos(2\pi f_i T + \Delta \theta_i) - \cos(\Delta \theta_i) = -2 \sin(\pi f_i T) \cos(\pi f_i T) \sin(\Delta \theta_i) + \\
- 2 \sin^2(\pi f_i T) \cos(\Delta \theta_i)
\]  
\[3.45\]

The accumulated quadrature signal component may be written as

\[
Q_{A,s} = \frac{A}{\sqrt{2}} R(\tau) D_i \frac{\sin(\pi f_i T)}{\pi f_i T} \sin(\pi f_i T + \Delta \theta_i)
\]  
\[3.46\]
The in-phase and quadraphase noise components are also summed in the accumulator.

Thus, the accumulated in-phase noise component is

\[
I_{A,n} = E\left[ \sum_{k=1}^{S} I_{c,n} \right] \quad [3.47]
\]

and the accumulated quadraphase noise component is

\[
Q_{A,n} = E\left[ \sum_{k=1}^{S} Q_{c,n} \right] \quad [3.48]
\]

Assume that \( I_{c,n} \) and \( Q_{c,n} \) are both white Gaussian noise. This means that

\[
I_{A,n} = \sum_{k=1}^{S} E[I_{c,n}] = 0 \quad [3.49]
\]

and

\[
Q_{A,n} = \sum_{k=1}^{S} E[Q_{c,n}] = 0 \quad [3.50]
\]

since white Gaussian noise has mean equal to 0 [Sklar 1988]. The variance of \( \sum_{k=1}^{S} I_{c,n} \) and

\[
\sum_{k=1}^{S} Q_{c,n} \]

is \( \sigma^2_n \) equal to

\[
\sigma^2_n = E\left[ \left( \sum_{k=1}^{S} I_{c,n} \right)^2 \right] \quad [3.51]
\]

This summation may also be simplified using the definition of Riemann sums. Hence,

\[
\sigma^2_n = E\left[ \left( \frac{1}{\Delta t_k} \int_0^T I_{c,n} \, dt \right)^2 \right] = E\left[ \left( \frac{1}{\Delta t_k} I_{c,n} T \right)^2 \right] \quad [3.52]
\]

Simplification yields the following:
\[ \sigma_n^2 = \left( \frac{T}{\Delta t_k} \right)^2 E[I_{c,n}^2] \] 

[3.53]

The expectation of \( I_{c,n}^2 \) is equal to the integral of the Fourier transform of the function.

The Fourier transform of two-sided bandwidth white Gaussian noise is \( \frac{No}{2} \) [Sklar 1988].

Thus,

\[ E[I_{c,n}^2] = \int_{-\infty}^{\infty} \frac{No}{2} df = \frac{No}{2T} \] 

[3.54]

Using the fact that \( T = S_c \Delta t_k \), the variance of \( \sum_{k=1}^{S_c} I_{c,n} \) and \( \sum_{k=1}^{S_c} Q_{c,n} \) is given by the following:

\[ \sigma_n^2 = \frac{T}{2 \Delta t_k^2} No = \frac{No}{2 \Delta t_k} S_c \] 

[3.55]

In order to normalize the variance of the noise for later calculations, the in-phase and quadrature components are multiplied by a factor of \( \sqrt{\frac{2 \Delta t_k}{S_c No}} \) [Van Dierendonck 1996].

The average power of a GPS signal is equal to \( S = \frac{A^2}{2} \) [Van Dierendonck 1996]. Due to signal processing, \( A \) has been altered from its original value. However, these alterations are small and will be ignored. The \( i \)th accumulated in-phase component at time \( t_k \), \( I_i \), is given by the following:

\[ I_i = \sqrt{\frac{2 \Delta t_k}{S_c No}} (I_{A,s} + I_{A,p}) \]
\[ \sqrt{2 \frac{S}{No}} TR(\tau) D_i \frac{\sin(\pi f_i T)}{\pi f_i T} \cos(\pi f_i T + \Delta \theta_i) + n_{I,i} \]  \hspace{1cm} [3.56]

where \( I_{A,s} \) was defined in equation [3.41] and \( n_{I,i} \) is equal to normalized white Gaussian noise. The \( i \)th accumulated quadrature component at time \( t_k \), \( Q_i \), is equal to the following:

\[ Q_i = \frac{2 \Delta t_k}{S, No} (Q_{A,s} + Q_{A,n}) \]

\[ = \sqrt{2 \frac{S}{No}} TR(\tau) D_i \frac{\sin(\pi f_i T)}{\pi f_i T} \sin(\pi f_i T + \Delta \theta_i) + n_{Q,i} \]  \hspace{1cm} [3.57]

where \( Q_{A,s} \) was defined in equation [3.42] and \( n_{Q,i} \) is normalized white Gaussian noise.

The ratio \( \frac{S}{No} \) is the signal-to-noise ratio.

3.2 Comparison of Wide and Narrow Band Powers

Comparison of signal plus noise power in two different bandwidths provides an estimate for C/No [Van Dierendonck 1996]. If \( g(t) \) is a discrete function of time \( t \), then at time \( t_k \) the total power dissipated in \( M \) samples, \( P_k \), is given by the following:

\[ P_k = \sum_{i=1}^{M} |g^2(t_i)| \]  \hspace{1cm} [3.58]

where \( t_i \) is the time at sample \( i \) and \( t_k \) is the time at sample \( M \).
The bandwidth for wide band power at time $t_k$, $WBP_k$, is $1/T$ where $T$ is the length of
the interval used in the accumulation process [Van Dierendonck 1996]. For example, if
$T$ is equal to 1 ms, then $WBP_k$ has a bandwidth of 1 kHz. At time $t_i$, the GPS signal
plus noise, $s_i$, is given by

$$s_i = \sqrt{I_i^2 + Q_i^2}$$  \hspace{1cm} [3.59]

where $I_i$ is the in-phase component at time $t_i$ defined in equation [3.56], and $Q_i$ is the
quadraphase component at time $t_i$ defined in equation [3.57]. Therefore, $WBP_k$ over $M$
samples is given by the following:

$$WBP_k = \sum_{i=1}^{M} |s_i^2| = \sum_{i=1}^{M} (I_i^2 + Q_i^2)$$  \hspace{1cm} [3.60]

where $t_k$ is the time at sample $M$.

The bandwidth for narrow band power at time $t_k$, $NBP_k$, is $1/MT$ where $M$ is the same
number of samples and $T$ is the same interval length used in the $WBP_k$ calculation [Van
Dierendonck 1996]. For example, if $T$ is equal to 1 ms and $M$ is equal to 20 samples,
then $NBP_k$ has a bandwidth of 50 Hz. The in-phase component accumulated over a
bandwidth of $1/MT$ at $t_k$, $I_k$, is

$$I_k = \sum_{i=1}^{M} I_i$$  \hspace{1cm} [3.61]

The quadraphase component accumulated over a bandwidth of $1/MT$ at time $t_k$, $Q_k$, is
\[ Q_k = \sum_{i=1}^{M} Q_i \]  

[3.62]

The GPS signal plus noise accumulated over a bandwidth of \( \frac{1}{MT} \) at \( t_k \), \( s_k \), is

\[
s_k = \sqrt{I_k^2 + Q_k^2} = \sqrt{\left( \sum_{i=1}^{M} I_i \right)^2 + \left( \sum_{i=1}^{M} Q_i \right)^2} \]

[3.63]

Thus, \( NBP_k \) is given by the following:

\[
NBP_k = |s_k^2| = \left( \sum_{i=1}^{M} I_i \right)^2 + \left( \sum_{i=1}^{M} Q_i \right)^2
\]

[3.64]

where \( t_k \) is the time at sample \( M \).

Normalized power at time \( t_k \), \( NP_k \), is given by the following:

\[
NP_k = \frac{NBP_k}{WBP_k}
\]

[3.65]

[Van Dierendonck 1996]. The expected value and variance of \( NP_k \) provide a monotonic function of C/No [Van Dierendonck 1996]. The expected value of the quotient of two random variables, \( X \) and \( Y \), is approximated by the following:

\[
E\left[ \frac{X}{Y} \right] \approx \frac{E[X]}{E[Y]} - \frac{1}{E^2[Y]} \text{COV}[X,Y] + \frac{E[X]}{E^3[Y]} \text{VAR}[Y]
\]

[3.66]

[Mood 1974]. The variance of the quotient of two random variables, \( X \) and \( Y \), is approximated by the following:

\[
\text{VAR}\left[ \frac{X}{Y} \right] \approx \left( \frac{E[X]}{E[Y]} \right)^2 \left( \frac{\text{VAR}[X]}{E^2[X]} + \frac{\text{VAR}[Y]}{E^2[Y]} - \frac{2\text{COV}[X,Y]}{E[X]E[Y]} \right)
\]

[3.67]
[Mood 1974]. Thus, if $WBP_k$ and $NBP_k$ are treated as random variables, the expected value and variance of $NP_k$ may be approximated. In order to approximate the expected value and variance of $NP_k$, let $M$ equal 20 samples and assume fine tracking. Recall from chapter 1 that fine tracking implies that the frequency and code tracking errors present in the in-phase and quadrature components are approximately equal to 0 [Van Dierendonck 1996]. If the frequency and code tracking errors equal zero, then $R(\tau)$ and $\sin(\pi f t)/\pi f t$ equal 1. Thus, from equation [3.56], $I_i$ is given by

$$I_i = \sqrt{2 \frac{S}{N_0} T D_i + n_{i,j}}$$

[3.68]

and, from equation [3.57] $Q_i$ is given by

$$Q_i = n_{Q,i}$$

[3.69]

Substitution of equations [3.68] and [3.69] into equation [3.60] yields

$$WBP_k = \sum_{i=1}^{M} \left( \left( 2 \frac{S}{N_0} T D_i + n_{i,j} \right)^2 + n_{Q,i}^2 \right)$$

[3.70]

As stated in chapter 2, the navigation data bits are equal to either +1 or -1. Thus, $D_i^2$ is always equal to +1. Hence,

$$WBP_k = \sum_{i=1}^{M} \left( 2 \frac{S}{N_0} T + 2 \left( \frac{S}{N_0} T D_i n_{i,j} + n_{i,j}^2 + n_{Q,i}^2 \right) \right)$$

$$= 2 \frac{S}{N_0} T M + \sum_{i=1}^{M} \left( 2 \left( \frac{S}{N_0} T D_i n_{i,j} + n_{i,j}^2 + n_{Q,i}^2 \right) \right)$$

[3.71]
\(NBP_k\), defined in equation [3.64], may also be written as

\[
NBP_k = \sum_{i=1}^{M} \sum_{j=1}^{M} (I_{i,j} + Q_{i,j})
\]  

[3.72]

Substitution of equation [3.68] and [3.69] into the above equation yields

\[
NBP_k = \sum_{i=1}^{M} \sum_{j=1}^{M} \left( \sqrt{\frac{S}{No}} TD_j + n_{i,j} \right) \left( \sqrt{\frac{S}{No}} TD_j + n_{i,j} \right) + n_{Q_i,n_{Q,j}} \]

[3.73]

As stated in chapter 2, the length of a navigation data bit is 20 ms. Assume that \(i\) equal to one occurs at a data bit change. Then, for \(M\) equal to 20 samples and \(T\) equal to 1 ms, \(D,D_i\) equals +1 and

\[
NBP_k = \sum_{i=1}^{M} \sum_{j=1}^{M} \left( \frac{2}{No} S \right) T + \sqrt{\frac{2}{No}} S \left(D_i n_{i,j} + n_{i,j} T D_j + n_{i,j} n_{i,j} + n_{Q_i,n_{Q,j}} \right)
\]

\[
= 2 \frac{S}{No} TM^2 + \sum_{i=1}^{M} \sum_{j=1}^{M} \left( \sqrt{\frac{2}{No}} S \left(D_i n_{i,j} + n_{i,j} T D_j + n_{i,j} n_{i,j} + n_{Q_i,n_{Q,j}} \right) \right)
\]

[3.74]

3.2.1 Expected Value and Variance of \(WBP_k\)

The expected value of \(WBP_k\) is given by the following:

\[
E[WBP_k] = E \left[ \frac{2}{No} S \right] TM + \sum_{i=1}^{M} \left( \sqrt{\frac{2}{No}} S \right) TD_i n_{i,i} + n_{i,j}^2 + n_{Q_i,j}^2 \right]
\]

\[
= 2 \frac{S}{No} TM + \sum_{i=1}^{M} E \left[ \frac{2}{No} S \right] TD_i n_{i,i} + n_{i,j}^2 + n_{Q,i,j}^2 \]

[3.75]

where \(n_{i,j}\) and \(n_{Q,i,j}\) are normalized white Gaussian noise. Normalized white Gaussian noise has an expected value of 0 and a variance of 1. Any two different samples of white
Gaussian noise will have correlation of 0 [Sklar 1988]. In other words, any sample of white Gaussian noise is independent of another sample of white Gaussian noise. Also, normalized white Gaussian noise is independent of $D_i$. The expected value of the product of two independent random variables, $X_z$ and $X_h$, is given by

$$E[X_z X_h] = E[X_z]E[X_h]$$  \[3.76\]

[Leon-Garcia 1994]. Since the expected value of both $n_{i,j}$ and $D_i$ is 0 and the variance of normalized white Gaussian noise is 1, then

$$E\left[2\sqrt{\frac{S}{No}} T D_i n_{i,j} + n_{i,j}^2 + n_{Q,i}^2\right] = 1 + 1 = 2$$ \[3.77\]

The expected value of $WBP_k$ is

$$E[WBP_k] = 2\frac{S}{No} TM + \sum_{j=1}^{M} 2 = 2M\left(\frac{S}{No} T + 1\right)$$ \[3.78\]

The variance of $WBP_k$ is given by the following:

$$\text{VAR}[WBP_k] = \text{VAR}\left[2\sqrt{\frac{S}{No}} T D_i n_{i,j} + n_{i,j}^2 + n_{Q,i}^2\right]$$ \[3.79\]

The variance of a random variable plus a constant is equal to the variance of the random variable [Freund 1992]. Also, the variance of the sum $\sum_{j=1}^{N} X_j$ where $X_j$ is a random variable is given by the following:

$$\text{VAR}\left[\sum_{j=1}^{N} X_j\right] = \sum_{j=1}^{N} \text{VAR}[X_j] + \sum_{j=1}^{N} \sum_{h=1}^{N} \text{COV}[X_j, X_h]$$ \[3.80\]
The covariance of \( X_j \) and \( X_h \) is given by

\[
\text{COV}[X_j, X_h] = \text{E}[(X_j - \text{E}[X_j])(X_h - \text{E}[X_h])]
\]

\[
= \text{E}[X_j] \text{E}[X_h] - \text{E}[X_j] \text{E}[X_h]
\]

[3.81]

[Leon-Garcia 1994]. Thus,

\[
\text{VAR}[WBP_k] = \sum_{i=1}^{M} \text{VAR} \left[ 2\sqrt{\frac{S}{\text{No}}} TD_i n_{i,j} + n_{i,j}^2 + n_{Q,j}^2 \right] +
\]

\[
+ \sum_{i=1}^{M} \sum_{j=1}^{M, i \neq j} \text{COV} \left[ 2\sqrt{\frac{S}{\text{No}}} TD_i n_{i,j} + n_{i,j}^2 + n_{Q,j}^2, 2\sqrt{\frac{S}{\text{No}}} TD_j n_{i,j} + n_{i,j}^2 + n_{Q,j}^2 \right]
\]

[3.82]

Using equations [3.77] and [3.81] the covariance term in the above equation is given by

\[
\text{COV} \left[ 2\sqrt{\frac{S}{\text{No}}} TD_i n_{i,j} + n_{i,j}^2 + n_{Q,j}^2, 2\sqrt{\frac{S}{\text{No}}} TD_j n_{i,j} + n_{i,j}^2 + n_{Q,j}^2 \right] = \text{E} \left[ \left( 2\sqrt{\frac{S}{\text{No}}} TD_i n_{i,j} + n_{i,j}^2 + n_{Q,j}^2 \right) \left( 2\sqrt{\frac{S}{\text{No}}} TD_j n_{i,j} + n_{i,j}^2 + n_{Q,j}^2 \right) \right] - (2)(2)
\]

\[
= 8 \frac{S}{\text{No}} T \text{E}[n_{i,j} n_{i,j}] + 2 \sqrt{\frac{S}{\text{No}}} T \text{E}[D_i n_{i,j} n_{i,j}^2 + D_i n_{i,j}^2 n_{Q,j}^2, + n_{i,j}^2 D_j n_{i,j} + n_{i,j}^2 n_{Q,j} n_{Q,j}^2] +
\]

\[
+ \text{E}[n_{i,j}^2 n_{i,j}^2 + n_{i,j}^2 n_{Q,j}^2 + n_{Q,j}^2 n_{i,j}^2 + n_{Q,j}^2 n_{Q,j}^2] - 4
\]

[3.83]

Any sample of normalized white Gaussian noise is independent of both the navigation data bit and of another sample of white Gaussian noise. The expected value of two discrete random variables, \( X \) and \( Y \), raised to the \( a \) and \( b \) powers is given by the following:

\[
\text{E}[X^a Y^b] = \sum_z \sum_w x_z^a y_w^b p_{X,Y}(x_z, y_w)
\]

[3.84]
where \( p_{X,Y}(x_z, y_w) \) is the joint probability mass function of \( X \) and \( Y \) [Leon-Garcia 1994]. If \( X \) and \( Y \) are independent, then \( p_{X,Y}(x_z, y_w) \) is equal to the product of the marginal probability mass function of \( X \), \( p_X(x_z) \), and the marginal probability mass function of \( Y \), \( p_Y(y_w) \) [Leon-Garcia 1994]. The above equation may be written as

\[
E[X^a Y^b] = \sum_z \sum_w x_z^a y_w^b p_X(x_z)p_Y(y_w)
\]

\[
= \sum_z x_z^a p_X(x_z) \sum_w y_w^b p_Y(y_w)
\]

\[
= E[X^a]E[Y^b]
\]  \[3.85\]

Since the expected value of both the navigation data bit and normalized white Gaussian noise is 0 and since the variance of normalized white Gaussian noise is 1,

\[
\text{COV}\left[2\sqrt{\frac{S}{N_0}}TD_i n_{i,j} + n_{i,j}^2, 2\sqrt{\frac{S}{N_0}}TD_j n_{i,j} + n_{i,j}^2\right] = 1 + 1 + 1 - 4
\]

\[
= 0 \quad \text{[3.86]}
\]

Thus, the variance of \( WBP_k \), defined in equation [3.82], is given by

\[
\text{VAR}[WBP_k] = \sum_{i=1}^{M} \text{VAR}\left[2\sqrt{\frac{S}{N_0}}TD_i n_{i,j} + n_{i,j}^2\right]
\]  \[3.87\]

The variance of a random variable, \( Y \), is given by the following:

\[
\text{VAR}[Y] = E[(Y - E[Y])^2]
\]

\[
= E[Y^2] - E[Y]^2 \quad \text{[3.88]}
\]

[Leon-Garcia 1994]. The above equation and equation [3.77] indicate that
The square of a normalized Gaussian random variable is equal to a chi-square random variable with $\nu$ equal to 1 degree of freedom [Freund 1992]. The expected value of a chi-square random variable is equal to $\nu$, and the variance of a chi-square random variable is equal to $2\nu$ [Freund 1992]. Thus, both $n_{i,j}^2$ and $n_{Q,j}^2$ are chi-square random variables with expected value equal to 1 and variance equal to 2. The above equation simplifies to

$$\text{VAR} \left[ 2 \sqrt{\frac{S}{No}} TD_i n_{i,j}^2 + n_{i,j}^2 + n_{Q,j}^2 \right] = 8 \frac{S}{No} T + 2 + 2 = 8 \frac{S}{No} T + 4 \quad [3.91]$$
The variance of $WBP_k$ is given by the following:

$$\text{VAR}[WBP_k] = \sum_{i=1}^{M} \left( 8 \frac{S}{N_o} T + 4 \right) = 4M \left( \frac{S}{N_o} T + 1 \right) \quad [3.92]$$

3.2.2 Expected Value and Variance of $NBP_k$

Using equation [3.74], the expected value of $NBP_k$ is given by the following:

$$E[NBP_k] =$$

$$+ E \left[ 2 \frac{S}{N_o} T M^2 + \sum_{i=1}^{M} \sum_{j=1}^{M} \left( \frac{S}{N_o} T \left( D_{i,j} n_{i,j} + n_{i,j} D_{j} \right) + n_{i,j} n_{i,j} + n_{Q,i} n_{Q,j} \right) \right] \quad [3.93]$$

Since white Gaussian noise is independent of the navigation data bit, and it has an expected value equal to 0,

$$E[NBP_k] = 2 \frac{S}{N_o} T M^2 + \sum_{i=1}^{M} \sum_{j=1}^{M} E[n_{i,j} n_{i,j} + n_{Q,i} n_{Q,j}] \quad [3.94]$$

Expansion of the expected value term in the above equation yields

$$\sum_{i=1}^{M} \sum_{j=1}^{M} E[n_{i,j} n_{i,j} + n_{Q,i} n_{Q,j}] = \sum_{i=1}^{M} E[n_{i,j}^2 + n_{Q,i}^2] + \sum_{i=1}^{M} \sum_{j=1}^{M} E[n_{i,j} n_{i,j} + n_{Q,i} n_{Q,j}]$$

$$= 2M \quad [3.95]$$

The expected value of $NBP_k$ is given by the following:

$$E[NBP_k] = 2 \frac{S}{N_o} T M^2 + 2M = 2M \left( \frac{S}{N_o} T M + 1 \right) \quad [3.96]$$
The variance of a constant plus a random variable is equal to the variance of the random variable [Freund 1992]. Thus, the variance of $NBP_k$, defined in equation [3.74], is given by the following:

$$\text{VAR}[NBP_k] =$$

$$\text{VAR} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} \left( \frac{2}{No} T \left(D_i n_{i,j} + n_{i,j} D_j \right) + n_{i,j} n_{i,j} + n_{Q,j} n_{Q,j} \right) \right]$$

[3.97]

The variance of the sum of the random variables $X$ and $Y$ is given by

$$\text{VAR}[X + Y] = \text{VAR}[X] + \text{VAR}[Y] + 2 \text{COV}[X,Y]$$

[3.98] [Leon-Garcia 1994]. Also, the variance of a random variable multiplied by a constant is equal to the constant squared times the variance of the random variable [Leon-Garcia 1994]. Thus,

$$\text{VAR}[NBP_k] = 2 \frac{S}{No} T \text{VAR} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} \left(D_i n_{i,j} + n_{i,j} D_j \right) \right] + \text{VAR} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} \left(n_{i,j} n_{i,j} + n_{Q,j} n_{Q,j} \right) \right] +$$

$$+ 2 \text{COV} \left[ \left( \sum_{i=1}^{M} \sum_{j=1}^{M} \sqrt{\frac{S}{No} T \left(D_i n_{i,j} + n_{i,j} D_j \right)} \right), \left( \sum_{i=1}^{M} \sum_{j=1}^{M} \left(n_{i,j} n_{i,j} + n_{Q,j} n_{Q,j} \right) \right) \right]$$

[3.99]

Using equation [3.89], the first variance term in equation [3.99] is given by the following:

$$\text{VAR} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} \left(D_i n_{i,j} + n_{i,j} D_j \right) \right] = \mathbb{E} \left[ \left( \sum_{i=1}^{M} \sum_{j=1}^{M} \left(D_i n_{i,j} + n_{i,j} D_j \right) \right)^2 \right] +$$

$$- \mathbb{E}^2 \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} \left(D_i n_{i,j} + n_{i,j} D_j \right) \right]$$

[3.100]
As stated above, the navigation data bit is independent of normalized white Gaussian noise, and it has an expected value equal to 0. Hence,

\[
\text{VAR} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} (D_i n_{i,j} + n_{i,j} D_j) \right] = \text{E} \left[ \left( \sum_{i=1}^{M} \sum_{j=1}^{M} (D_i n_{i,j} + n_{i,j} D_j) \right)^2 \right]
\]

\[= 4 \text{E} \left[ \sum_{p=1}^{M} \sum_{u=1}^{M} D_p n_{i,p} \left( \sum_{v=1}^{M} \sum_{z=1}^{M} D_v n_{i,z} \right) \right] \quad [3.101] \]

It was assumed that the data bit transition occurred such that the data bit is constant for the interval \(T\) equal to 1 ms. Thus, \(D_p, D_u\) is always equal to 1. Hence,

\[
\text{VAR} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} (D_i n_{i,j} + n_{i,j} D_j) \right] = 4 \sum_{p=1}^{M} \sum_{u=1}^{M} \sum_{v=1}^{M} \sum_{z=1}^{M} \text{E} \left[ n_{i,p}^2 \right] + 4 \sum_{p=1}^{M} \sum_{u=1}^{M} \sum_{v=1}^{M} \sum_{z=1}^{M} \text{E} \left[ n_{i,s} n_{i,z} \right]
\]

\[= 4M^3 \quad [3.102] \]

Using equation [3.98], the second term of equation [3.99] is equal to

\[
\text{VAR} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} (n_{i,s} n_{i,j} + n_{Q,i,n_{Q,j}}) \right] = \text{VAR} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} n_{i,j} n_{i,j} \right] + \text{VAR} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} n_{Q,i,n_{Q,j}} \right] +
\]

\[+ 2 \text{COV} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} n_{i,j} n_{i,j}, \sum_{i=1}^{M} \sum_{j=1}^{M} n_{Q,i,n_{Q,j}} \right] \quad [3.103] \]

The covariance of two independent random variables is equal to 0 [Leon-Garcia 1994]. Since the variance of \(n_{i,s} n_{i,j}\) is equal to the variance of \(n_{Q,i,n_{Q,j}}\), then equation [3.89] yields

\[
\text{VAR} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} (n_{i,j} n_{i,j} + n_{Q,i,n_{Q,j}}) \right] = 2 \text{VAR} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} n_{i,j} n_{i,j} \right]
\]
\[
= 2 \left( E \left[ \sum_{p=1}^{M} \sum_{u=1}^{M} \sum_{z=1}^{M} n_{i,p} n_{u,z} n_{t,i} n_{t,z} \right] - E^2 \left[ \sum_{p=1}^{M} \sum_{u=1}^{M} n_{t,i} n_{t,j} \right] \right)
\]

[3.104]

Normalized white Gaussian noise has an expected value equal to 0, and any one sample of normalized white Gaussian noise is independent of a second sample of white Gaussian noise. Therefore,

\[
\text{VAR} \left[ \sum_{j=1}^{M} \sum_{j=1}^{M} \left( n_{i,j} n_{t,i,j} + n_{Q,j} n_{Q,j} \right) \right] = 2 \left( \sum_{p=1}^{M} E \left[ n_{i,p}^4 \right] + 3 E \left[ \sum_{p=1}^{M} \sum_{u=1}^{M} n_{t,p}^2 n_{t,u}^2 \right] - M^2 \right)
\]

\[
= 2 \left( \sum_{p=1}^{M} E \left[ n_{i,p}^4 \right] + 3M(M-1) - M^2 \right)
\]

[3.105]

Since the expected value of \( n_{i,p}^2 \) is 1, the above equation may also be written as

\[
\text{VAR} \left[ \sum_{j=1}^{M} \sum_{j=1}^{M} \left( n_{i,j} n_{t,i,j} + n_{Q,j} n_{Q,j} \right) \right] = 2 \left( \sum_{p=1}^{M} \left( E \left[ n_{i,p}^4 \right] - E^2 \left[ n_{i,p}^2 \right] \right) + 2M(M-1) \right)
\]

\[
= 2 \left( \sum_{p=1}^{M} \text{VAR} \left[ n_{i,p}^2 \right] + 2M^2 - 2M \right)
\]

[3.106]

\( n_{i,p}^2 \) is a chi-square random variable with variance equal to 2. Simplification of the above equation yields

\[
\text{VAR} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} \left( n_{i,j} n_{t,i,j} + n_{Q,i} n_{Q,j} \right) \right] = 2(2M + 2M(M-1)) = 4M^2
\]

[3.107]

The covariance term in equation [3.99] is equal to

\[
2 \text{COV} \left[ \left( \sum_{i=1}^{M} \sum_{j=1}^{M} \sqrt{\frac{S}{N_0}} T(D_i n_{t,i,j} + D_j n_{t,i}) \right), \left( \sum_{i=1}^{M} \sum_{j=1}^{M} \left( n_{i,j} n_{t,i,j} + n_{Q,i} n_{Q,j} \right) \right) \right] = 0
\]

[3.108]
since the navigation data bit is independent of normalized white Gaussian noise and since it has an expected value equal to 0. Thus, from equations [3.99], [3.102], [3.107], and [3.108], the variance of $NBP_k$ is equal to the following:

$$\text{VAR}[NBP_k] = 2 \frac{S}{N_o} T (4M^3) + 4M^2 + 0 = 4M^2 \left( 2 \frac{S}{N_o} TM + 1 \right)$$ \hspace{1cm} [3.109]

3.2.3 Covariance of $NBP_k$ and $WBP_k$

Using equation [3.81], the covariance of $NBP_k$ and $WBP_k$ is equal to the following:

$$\text{COV}[NBP_k, WBP_k] = \text{E}((NBP_k - \text{E}[NBP_k])(WBP_k - \text{E}[WBP_k]))$$ \hspace{1cm} [3.110]

Using equations [3.71] and [3.78],

$$WBP_k - \text{E}[WBP_k] = 2 \frac{S}{N_o} TM + \sum_{i=1}^{M} \left( 2 \sqrt{2 \frac{S}{N_o} T D_i, n_{i,j}^2 + n_{Q,j}^2} \right) - 2M \left( \frac{S}{N_o} T + 1 \right)$$

$$= \sum_{i=1}^{M} \left( 2 \sqrt{2 \frac{S}{N_o} T D_i, n_{i,j}^2 + n_{Q,j}^2} \right) - 2M$$ \hspace{1cm} [3.111]

Using equations [3.74] and [3.96],

$$NBP_k - \text{E}[NBP_k] =$$

$$= 2 \frac{S}{N_o} TM^2 + \sum_{i=1}^{M} \sum_{j=1}^{M} \left( \sqrt{2 \frac{S}{N_o} T (D_i, n_{i,j} + n_{Q,j})} \right) - 2M \left( \frac{S}{N_o} TM + 1 \right)$$

$$= \sum_{i=1}^{M} \sum_{j=1}^{M} \left( \sqrt{2 \frac{S}{N_o} T (D_i, n_{i,j} + n_{Q,j})} \right) - 2M$$ \hspace{1cm} [3.112]

Thus,
\[
\text{COV}[\text{NBP}_k, \text{WBP}_k] = 4M^2 - 2M \sum_{i=1}^{M} \sum_{j=1}^{M} E \left[ \sqrt{\frac{S}{N_0} T \left(D_i n_{i,j} + n_{i,j}^2 + n_{Q,i}^2 \right.} \right] + \\
- 2M \sum_{i=1}^{M} E \left[ 2 \sqrt{\frac{S}{N_0} T D_i n_{i,j} + n_{i,j}^2 + n_{Q,i}^2} \right] + 4 \frac{S}{N_0} T \sum_{p=1}^{M} \sum_{u=1}^{M} E \left[D_p n_{i,p} \left(D_u n_{i,v} + n_{i,u} D_v \right)\right]
\]

\[
+ \sqrt{\frac{S}{N_0} T} \sum_{p=1}^{M} \sum_{u=1}^{M} \left(2 E[D_p n_{i,p} \left(n_{i,a} n_{i,v} + n_{Q,a} n_{Q,v} \right)] + E[n_{Q,p}\left(n_{i,a} n_{i,v} + n_{Q,a} n_{Q,v} \right)\right], [3.113]
\]

The navigation data bit is independent of normalized white Gaussian noise, and it has an expected value equal to 0. Also the navigation data bit is constant for all \(M\) samples since \(M\) equals 20, \(T\) equals 1 ms, and \(i\) equal to 1 was a navigation data bit change. Thus, any one sample of the navigation data bit multiplied by itself or a second sample will equal 1. Substitution of equations [3.77] and [3.95] into the above equation yields

\[
\text{COV}[\text{NBP}_k, \text{WBP}_k] = 4M^2 - 4M^2 - 4M^2 + 4 \frac{S}{N_0} T \sum_{p=1}^{M} \sum_{u=1}^{M} E[n_{i,p} \left(n_{i,v} + n_{i,u} \right)] + \\
+ \sum_{p=1}^{M} \sum_{u=1}^{M} \left(2 E[n_{i,p}^2 \left(n_{i,a} n_{i,v} + n_{Q,a} n_{Q,v} \right)] + E[n_{Q,p}\left(n_{i,a} n_{i,v} + n_{Q,a} n_{Q,v} \right)\right], [3.114]
\]

Normalized white Gaussian noise is independent of any second sample of normalized white Gaussian noise, and it has an expected value equal to 0. Thus, expansion of the above equation yields

\[
\text{COV}[\text{NBP}_k, \text{WBP}_k] = \frac{S}{N_0} T \sum_{p=1}^{M} \sum_{u=1}^{M} E[n_{i,p}^2] + \sum_{p=1}^{M} \left(2 E[n_{i,p}^4] + 2 E[n_{Q,p}^2] + n_{Q,p}^2 \right]
\]
\[ + \sum_{p=1}^{M} \sum_{u=1}^{M_w u p} \left( E \left[ n_{i,p}^2 \left( n_{i,u}^2 + n_{Q,u}^2 \right) \right] + E \left[ n_{Q,p}^2 \left( n_{i,u}^2 + n_{Q,u}^2 \right) \right] \right) - 4M^2 \]

\[ = 8 \frac{S}{N_{0}} T M^2 + \sum_{p=1}^{M} \left( E \left[ n_{i,p}^2 \right] - E^2 \left[ n_{i,p} \right] + E \left[ n_{Q,p}^2 \right] - E^2 \left[ n_{Q,p}^2 \right] \right) + 

+ M + 2M + 2M(M - 1) + 2M(M - 1) - 4M^2 \]

\[ = 8 \frac{S}{N_{0}} T M^2 + \sum_{p=1}^{M} \left( \text{VAR}[n_{i,p}^2] + \text{VAR}[n_{Q,p}^2] \right) \quad [3.115] \]

\[ n_{i,p}^2 \text{ and } n_{Q,p}^2 \text{ are chi-square random variables with variance equal to 2. Thus, the} \]

covariance of \( NBP_k \) and \( WBP_k \) is given by the following:

\[ \text{COV}[NBP_k, WBP_k] = 8 \frac{S}{N_{0}} T M^2 + 2M + 2M = 4M \left( 2 \frac{S}{N_{0}} T M + 1 \right) \quad [3.116] \]

3.2.4 Expected Value and Variance of \( NP_k \)

Using equations [3.65] and [3.66], the expected value of \( NP_k \) is given by the following:

\[ E[NP_k] = E \left[ \frac{NBP_k}{WBP_k} \right] \]

\[ \approx \frac{E[NBP_k]}{E[WBP_k]} - \frac{1}{E^2[WBP_k]} \text{COV}[NBP_k, WBP_k] + \frac{E[NBP_k]}{E^3[WBP_k]} \text{VAR}[WBP_k] \quad [3.117] \]

Substitution of equations [3.78], [3.92], [3.96], and [3.116] into the above equation yields

\[ E[NP_k] \approx \frac{2M \left( \frac{S}{N_{0}} T M + 1 \right)}{2M \left( \frac{S}{N_{0}} T + 1 \right)} - \frac{1}{\left( 2M \left( \frac{S}{N_{0}} T + 1 \right) \right)^2} \left( 4M \left( 2 \frac{S}{N_{0}} T M + 1 \right) \right) + \]
\[
\begin{align*}
E[NP_k] &\approx \frac{2M \left( \frac{S_{TM} + 1}{N_o} \right)}{\left( 2M \left( \frac{S}{N_o} T + 1 \right) \right)^3} \left( 4M \left( 2 \frac{S}{N_o} T + 1 \right) \right) \\
\end{align*}
\]

[3.118]

Expansion of the above equation yields
\[
\begin{align*}
E[NP_k] &\approx \frac{8M^3 \left( \frac{S_{TM} + 1}{N_o} \right) \left( \frac{S}{N_o} T + 1 \right)^2}{\left( 2M \left( \frac{S}{N_o} T + 1 \right) \right)^3} - \frac{8M^2 \left( 2 \left( \frac{S}{N_o} T \right)^2 M + \frac{S}{N_o} T + 2 \frac{S}{N_o} T_{TM} + 1 \right)}{\left( 2M \left( \frac{S}{N_o} T + 1 \right) \right)^3} + \\
&+ \frac{8M^3 \left( 2 \left( \frac{S}{N_o} T \right)^2 M + 2 \frac{S}{N_o} T + \frac{S}{N_o} T_{TM} + 1 \right)}{\left( 2M \left( \frac{S}{N_o} T + 1 \right) \right)^3} \\
\end{align*}
\]

[3.119]

Simplification of the above equation yields
\[
\begin{align*}
E[NP_k] &\approx \frac{M \left( \frac{S_{TM} + 1}{N_o} \right) \left( \frac{S}{N_o} T + 1 \right)^2 - \frac{S}{N_o} T(M - 1)}{M \left( \frac{S}{N_o} T + 1 \right)^3} \\
\end{align*}
\]

[3.120]

Using equations [3.65] and [3.67], the variance of \( NP_k \) is given by the following:
\[
\begin{align*}
\text{VAR}[NP_k] &= \text{VAR}\left[ \frac{NBP_k}{WBP_k} \right] \\
&\approx \left( \frac{E[NBP_k]}{E[WBP_k]} \right)^2 \left( \text{VAR}[NBP_k] \right) + \text{VAR}[WBP_k] - 2 \text{COV}[NBP_k, WBP_k] \\
&\approx \left( \frac{E[NBP_k]}{E[WBP_k]} \right)^2 \left( \frac{\text{VAR}[NBP_k]}{E[NBP_k]} \right) + \frac{\text{VAR}[WBP_k]}{E[WBP_k]} - 2 \frac{\text{COV}[NBP_k, WBP_k]}{E[NBP_k]E[WBP_k]} \\
\end{align*}
\]

[3.121]

Substitution of equations [3.78], [3.92], [3.96], [3.109], and [3.117] yields
Simplification of the above equation yields

\[
\frac{\text{VAR}[NBP_k]}{E^2[NBP_k]} + \frac{\text{VAR}[WBP_k]}{E^2[WBP_k]} - \frac{2\text{COV}[NBP_k, WBP_k]}{E[NBP_k]E[WBP_k]} = \frac{4M^2}{\left(2M\left(\frac{S}{No}TM + 1\right)\right)^2} + \frac{4M}{\left(2M\left(\frac{S}{No}T + 1\right)\right)^2} - \frac{2\left(4M\left(\frac{S}{No}TM + 1\right)\right)}{\left(2M\left(\frac{S}{No}TM + 1\right)\right)\left(2M\left(\frac{S}{No}T + 1\right)\right)}
\]

\[
M\left(\frac{2}{\left(\frac{S}{No}TM + 1\right)^2}\left(\frac{2}{\frac{S}{No}T + 1}\right)^2 + \left(\frac{2}{\frac{S}{No}T + 1}\right)^2\right) = \frac{4M\left(\frac{S}{No}TM + 1\right)}{\left(\frac{S}{No}TM + 1\right)^2\left(\frac{S}{No}T + 1\right)^2} + \left[3.122\right]
\]

Thus, using equations [3.78], [3.96], [3.121], and [3.123], the variance of $NP_k$ is given
by the following:

$$\text{VAR}[N_{P_k}] \approx \left( \frac{2M \left( \frac{S}{N_0} TM + 1 \right)}{2M \left( \frac{S}{N_0} T + 1 \right)} \right)^2 \left( \frac{M - 1 \left( \frac{S}{N_0} TM \left( \frac{S}{N_0} T + 2 \right) + 1 \right)}{M \left( \frac{S}{N_0} TM + 1 \right)^2 \left( \frac{S}{N_0} T + 1 \right)^2} \right)$$

$$= \frac{(M - 1 \left( \frac{S}{N_0} TM \left( \frac{S}{N_0} T + 2 \right) + 1 \right)}{M \left( \frac{S}{N_0} T + 1 \right)^4} \tag{3.124}$$

3.2.5 C/No Estimate Using $N_{P_k}$

According to [Van Dierendonck 1996], the following equation provides an estimate for C/No:

$$\text{C/No} \approx 10 \log_{10} \left( \frac{1}{T} \left( \frac{M - \text{E}[N_P]}{M - \text{E}[N_P]} - 1 \right) \right) \tag{3.125}$$

where $N_P$ is $N_{P_k}$ averaged over $K$ samples. Thus, the expected value of $N_P$ is given by the following:

$$\text{E}[N_P] = \frac{1}{K} \sum_{k=1}^{K} \text{E}[N_{P_k}] \tag{3.126}$$

Substitution of equation [3.120] into the above equation yields

$$\text{E}[N_P] \approx \frac{1}{K} \sum_{k=1}^{K} \frac{M \left( \frac{S}{N_0} TM + 1 \right) \left( \frac{S}{N_0} T + 1 \right)^2 - \frac{S}{N_0} T (M - 1)}{M \left( \frac{S}{N_0} T + 1 \right)^3}$$
After some simplification, equation [3.125] is approximated by

\[
C/No \approx 10\log_{10} \left\{ \frac{M\left(\frac{S}{No}TM + 1\right)\left(\frac{S}{No}T + 1\right)^2 - \frac{S}{No}T(M - 1) - M\left(\frac{S}{No}T + 1\right)^3}{TM\left(M\left(\frac{S}{No}T + 1\right)^2 - M\left(\frac{S}{No}TM + 1\right)\left(\frac{S}{No}T + 1\right) + S\frac{T}{No}(M - 1)\right)} \right\}
\]

\[
= 10\log_{10} \left\{ \frac{\frac{S}{No}T(M - 1)\left(M\left(\frac{S}{No}T + 1\right)^2 - 1\right)}{T(M - 1)\left(M\left(\frac{S}{No}T + 1\right)^2 + \frac{S}{No}T\right)} \right\}
\]

Finally, C/No is approximated by the following:

\[
C/No \approx 10\log_{10} \left\{ \frac{\frac{S}{No}\left(M\left(\frac{S}{No}T + 1\right)^2 - 1\right)}{M\left(\frac{S}{No}T + 1\right)^2 + \frac{S}{No}T} \right\}
\]

\[
= 10\log_{10}\left[\frac{\frac{S}{No}}{M}\right] + 10\log_{10} \left\{ \frac{M\left(\frac{S}{No}T + 1\right)^2 - 1}{M\left(\frac{S}{No}T + 1\right)^2 + \frac{S}{No}T} \right\}
\]

The C/No estimate obtained from the above equation is approximately equal to

\[10\log_{10}\left(\frac{S}{No}\right)\]. However, the C/No estimate is biased by the amount given in the second
term in equation [3.129]. For example, let $M$ equal 20 samples, $T$ equal 1 ms, and $\frac{S}{No}$ equal 0.016; the bias in the C/No estimate, $C/No_{bias}$, is given by the following:

$$C/No_{bias} = 10\log_{10} \left[ \frac{20(1.6 \times 10^{-5} + 1)^2 - 1}{20(1.6 \times 10^{-5} + 1)^2 + .001 \frac{S}{No}} \right] = -0.2228 \text{ dB - Hz} \quad [3.130]$$

The magnitude of this bias is consistent with the value provided by [Van Dierendonck 1996]
4 A BLOCK PROCESSING C/No ESTIMATOR

A block processing C/No estimator was developed for the software radio. The software radio is a GPS receiver created by the Avionics Engineering Center at Ohio University. The following is a discussion of the signal processing performed by this receiver. A detailed derivation of the block processing C/No estimation technique will be provided as well.

4.1 Software Radio Signal Processing

Figure 4.1 shows a block diagram of the software radio. The GPS signal enters the receiver via a RHCP antenna with a low noise amplifier. This low noise amplifier sets the noise floor by adding thermal noise to the signal. Immediately following the antenna is a bandpass filter with a center frequency of 1575.42 MHz and a bandwidth of 86 MHz [Gunawardena 2000]. The signal and noise are then amplified and filtered again before

![Figure 4.1: Block Diagram of the Software Radio](image-url)
they are mixed with a frequency of 1554.17 MHz and downconverted to an intermediate frequency of 21.25 MHz [Gunawardena 2000]. The signal and noise are then filtered to remove feed through from the mixer and harmonics. A combination of filters and amplifiers then limits out-of-band noise and amplifies the signal and the in-band noise.

An ADC digitally downconverts the signal and noise at a rate of 5 Megasamples per second, Msps [Gunawardena 2000]. The result is a second intermediate frequency of 1.25 MHz and a bandwidth of 2.5 MHz. Because the signal and noise are filtered before downconversion, the signal bandwidth is 2.2 MHz. However, a bandwidth of 2.5 MHz will be used in the following discussion. At time \( t_k \), where \( k = 0, 1, 2, \ldots \), signal, \( s(k) \), plus noise, \( n(k) \), is given by

\[
s(k) + n(k) = AC_k D_k \cos(2\pi(f_{IF} + \Delta f_k) + \Delta \theta_k) + An_k \tag{4.1}
\]

\( C_k, D_k, \Delta f_k, \Delta \theta_k, \) and \( n_k \) are the C/A code, navigation data bit, frequency offset, phase offset, and noise, respectively at time \( t_k \). \( A \) is the amplitude of the signal; note that the noise is also scaled by the same factor \( A \). \( f_{IF} \) is the intermediate frequency of 1.25 MHz. Assume that \( n_k \) is white Gaussian noise.

At this point in the receiver, the samples are stored for block processing purposes. This means that each sample is stored within a block of memory until a preset number of samples is obtained in that block of memory. When the preset amount of samples is
obtained, the processor may access any sample within the block of memory at any given
time. This differs from the processing technique discussed in section 3.1 where each
sample is not stored and must, therefore, be accessed sequentially. This block processing
technique stores samples for an entire C/A code period of 1 ms. Since the sampling
frequency of the ADC is 5 Msps, \( K \) equal to 5000 samples are stored. The incoming
signal plus noise for block \( i \), \( s_i(k) + n_i(k) \), is given by the following:

\[
s_i(k) + n_i(k) = A_i C_k D_{i,k} \cos(2\pi(f_{IF} + \Delta f_i) t_{i,k} + \Delta \theta_i) + A_i n_{i,k}
\]  

[4.2]

where \( k \), equal to 1,2,\ldots,\( K \), is the number of the sample within block \( i \). \( A_i \), \( \Delta f_i \), and
\( \Delta \theta_i \) are the amplitude, frequency offset, and phase offset, respectively, of the incoming
signal for block \( i \). These three parameters are assumed constant for the 1 ms block of
data and, thus, are independent of \( k \). \( t_{i,k} \) is the time of block \( i \)’s sample \( k \). Note that
the C/A code has length equal to 1 ms and is independent of \( i \). It is possible that the
navigation data bit, \( D_{i,k} \), will have a bit change inside block \( i \). However, assume that
\( D_{i,k} \) is constant for all \( K \) samples and may be written as \( D_i \).

Circular correlation may be computed using circular convolution. Let \( x(k) \) and \( y(k) \) be
a sampled incoming signal and a discrete locally generated signal with \( k = 1,2,\ldots,\( K \) .
Then \( z(c) \), equal to the circular correlation of \( x(k) \) and \( y(k) \), is given by the following:

\[
z(c) = \sum_{k=1}^{K} x(k) y(c+k)
\]  

[4.3]
where \( c \), the sample offset, has an integer value that lies within \([0, K-1]\). The discrete Fourier transform of \( z(c) \) is given by the following:

\[
F[z(c)] = \sum_{c=0}^{K-1} \sum_{k=1}^{K} x(k)y(c + k)e^{(-j2\pi f)c/K} \\
= \sum_{k=1}^{K} x(k)e^{(-j2\pi f)c/K} \sum_{c=0}^{K-1} y(c + k)e^{(-j2\pi((c+k)/f))c/K} \tag{4.4}
\]

where \( F[\cdots] \) denotes the discrete Fourier transform [Tsui 2000]. If both \( x(k) \) and \( y(c + k) \) are periodic and the sum in the above equation includes every possible product, then \( y(c + k) \) is independent of \( k \) [Engineering Productivity Tools Ltd. 1999]. Thus,

\[
F[z(c)] = F[x(k)]\sum_{c=0}^{K-1} y(c + k)e^{(-j2\pi((c+k)/f))c/K} \tag{4.5}
\]

where, if \( y(c + k) \) is real,

\[
\sum_{c=0}^{K-1} y(c + k)e^{(-j2\pi((c+k)/f))c/K} = F^*[y(c + k)] \tag{4.6}
\]

where \( F^*[\cdots] \) denotes the complex conjugate of the discrete Fourier transform [Couch 1997]. Thus, \( z(c) \) is equal to the inverse Fourier transform of equation [4.5], i.e.

\[
z(c) = F^{-1}[F[x(k)]F^*[y(c + k)]] \tag{4.7}
\]

where \( F^{-1}[\cdots] \) denotes the inverse discrete Fourier transform.

At this point in the processor, \( s_i(k) + n_i(k) \) is circularly correlated with a discrete locally generated in-phase signal component, \( I_g(k) \), and a discrete locally generated quadraphase signal component, \( Q_g(k) \). Both \( I_g(k) \) and \( Q_g(k) \) have length equal to one
C/A code period of 1 ms, and they are both sampled at the sample rate of the ADC, 5
Msps. Thus, both \( I_g(k) \) and \( Q_g(k) \) consist of \( K \), or 5000, samples. At time \( t_{ik} \), \( I_g(k) \)
is given by

\[
I_g(k) = C_{g,k} \cos(2\pi(f_{IF} + \Delta f_g)k_{i,k} + \Delta \theta_g) \tag{4.8}
\]

where \( C_{g,k} \) is the generated C/A code. \( \Delta \theta_g \) is the generated phase offset and is assumed
constant over the 1 ms duration. \( \Delta f_g \), the locally generated offset frequency, ranges from
–10 kHz to +10 kHz in steps of 1 kHz. A circular correlation is performed for each
possible value of \( \Delta f_g \); \( \Delta f_g \) is also assumed constant over the 1 ms duration. \( f_{IF} \) is the
second intermediate frequency, equal to 1.25 MHz, of the downconverted signal plus
noise. \( Q_g(k) \) is given by

\[
Q_g(k) = C_{g,k} \sin(2\pi(f_{IF} + \Delta f_g)k_{i,k} + \Delta \theta_g) \tag{4.9}
\]

The circular correlation between \( s_i(k) + n_i(k) \) and \( I_g(k) \), \( I_r(c) \), is given by the
following:

\[
I_r(c) = F^{-1}[F[s_i(k) + n_i(k)]F^*[I_g(c + k)]] \tag{4.10}
\]

where \( c \) is the sample offset, also known as the chip offset, and ranges from 0 to
\( K - 1 \), or 4999. Similarly, the result of the circular correlation between \( s_i(k) + n_i(k) \) and
\( Q_g(k) \), \( Q_r(c) \), is given by the following:

\[
Q_r(c) = F^{-1}[F[s_i(k) + n_i(k)]F^*[Q_g(c + k)]] \tag{4.11}
\]
\( I_r(c) \) and \( Q_r(c) \) are combined to form a total correlated signal plus noise, \( C_T(c) \), given by the following:

\[
C_T(c) = \sqrt{(I_r(c))^2 + (Q_r(c))^2}
\]  \[4.12\]

If there exists a \( c_L \) such that \( C_T(c_L) \) is the largest possible \( C_T(c) \) and \( C_T(c_L) \) is greater than a preset threshold, then the receiver is considered to be tracking a satellite. The \( \Delta f_g \) that produces \( C_T(c_L) \), \( \Delta f_{L} \), is approximately equal to the frequency offset of the incoming signal, \( \Delta f_k \). \( c_L \) and \( \Delta \theta_g \) are approximately equal to the chip offset and phase offset of the incoming signal. Figure 4.2 is a graph of the results for an example of this circular correlation technique. In this example, \( c_L \) is approximately 1000, \( \Delta f_L \) is approximately 3 kHz, \( \Delta \theta_f \) is approximately 0, and the C/No is approximately 50 dB-Hz.

4.2 Average Power

The average power of a random process \( X(t) \) is equal to the expected value of \( X^2(t) \) [Leon-Garcia 1994]. Thus, the average power of \( C_T(c) \), defined in equation [4.12], is equal to the expected value of \( C_T^2(c) \). The expected value of \( C_T^2(c) \) is given by the following:

\[
E[C_T^2(c)] = E[I_r^2(c) + Q_r^2(c)]
\]

\[
= E[I_r^2(c)] + E[Q_r^2(c)]
\]  \[4.13\]
since the expected value of a sum is equal to the sum of the expected values. Thus, the average power of $C_r(c)$ is equal to the average power of $I_r(c)$ plus the average power of $Q_r(c)$

4.2.1 Average Power of the Correlated In-phase Component

Using the definition of correlation given in equation [4.3], $I_r(c)$, defined in equation [4.10], is given by the following:
\[
I_i(c) = \sum_{k=1}^{K} [s_i(k) + n_i(k)] I_g(c + k)
\]

\[
= A_i D_i \sum_{k=1}^{K} C_k C_{g,c+k} \cos(2\pi(f_{IF} + \Delta f_i) t_{i,k} + \Delta \theta_i) \cos(2\pi(f_{IF} + \Delta f_g) t_{i,c+k} + \Delta \theta_g) + \\
+ A_i \sum_{k=1}^{K} n_{i,k} C_{g,c+k} \cos(2\pi(f_{IF} + \Delta f_g) t_{i,c+k} + \Delta \theta_g)
\]

[4.14]

where \( I_g(k) \) was defined in equation [4.8] and \( s_i(k) + n_i(k) \) was defined in equation [4.2]; \( C_{g,c+k} \) is the generated C/A code for sample \( c + k \). The time at sample \( c + k \), \( t_{i,c+k} \), is equal to the following:

\[
t_{i,c+k} = t_{i,k} + t_{i,c}
\]

[4.15]

where \( t_{i,c} \) is the time between sample \( k \) and sample \( c + k \). Equation [4.14] may be written as

\[
I_i(c) = \frac{A_i}{2} D_i \sum_{k=1}^{K} C_k C_{g,c+k} \cos(2\pi(\Delta f_i t_{i,k} - \Delta f_g (t_{i,k} + t_{i,c}) - f_{IF} t_{i,c}) + \Delta \theta_i - \Delta \theta_g) + \\
+ \frac{A_i}{2} D_i \sum_{k=1}^{K} C_k C_{g,c+k} \cos(2\pi(2 f_{IF} t_{i,k} + f_{IF} t_{i,c} + \Delta f_i t_{i,k} + \Delta f_g (t_{i,k} + t_{i,c}))) + \Delta \theta_i + \Delta \theta_g) + \\
+ A_i \sum_{k=1}^{K} n_{i,k} C_{g,c+k} \cos(2\pi(f_{IF} + \Delta f_g) t_{i,c+k} + \Delta \theta_g)
\]

[4.16]

Assume the receiver is fine tracking a signal. Recall from chapter 1 that this implies that \( \Delta f_g \) and \( \Delta \theta_g \) are approximately equal to \( \Delta f_i \) and \( \Delta \theta_i \), respectively; also, \( C_{g,c+k} \) is equal to \( C_{c+k} \). The above equation may be written as
\[ I_r(c) = \frac{A_1}{2} D_i \sum_{k=1}^{K} C_k C_{c+k} \cos(2\pi(f_{IF} + \Delta f_g)_{i,c} + \Delta \theta_g) + \]
\[ + \frac{A_1}{2} D_i \sum_{k=1}^{K} C_k C_{c+k} \cos(4\pi(f_{IF} + \Delta f_g)_{i,k} + 2\pi(f_{IF} + \Delta f_g)_{i,c} + 2\Delta \theta_g) + \]
\[ + A_1 \sum_{k=1}^{K} n_{i,k} C_{c+k} \cos(2\pi(f_{IF} + \Delta f_g)_{i,c+k} + \Delta \theta_g) \quad [4.17] \]

The cosine of the negative of an argument is equal to the cosine of the positive of the argument. Expansion of the above equation yields

\[ I_r(c) = \frac{A_1}{2} D_i \cos(2\pi(f_{IF} + \Delta f_g)_{i,c}) \sum_{k=1}^{K} C_k C_{c+k} \left[1 + \cos(4\pi(f_{IF} + \Delta f_g)_{i,k} + 2\Delta \theta_g)\right] + \]
\[ - \frac{A_1}{2} D_i \sin(2\pi(f_{IF} + \Delta f_g)_{i,c}) \sum_{k=1}^{K} C_k C_{c+k} \sin(4\pi(f_{IF} + \Delta f_g)_{i,k} + 2\Delta \theta_g) + \]
\[ + A_1 \sum_{k=1}^{K} n_{i,k} C_{c+k} \cos(2\pi(f_{IF} + \Delta f_g)_{i,c+k} + \Delta \theta_g) \quad [4.18] \]

The sum of \( C_k C_{c+k} \) from \( k \) equal to 1 to \( K \) is equal to the autocorrelation of the C/A code, \( R(c) \), multiplied by \( K \). Thus, the above equation is given by

\[ I_r(c) = \frac{A_1}{2} D_i \cos(2\pi(f_{IF} + \Delta f_g)_{i,c}) \left[KR(c) + \sum_{k=1}^{K} C_k C_{c+k} \cos(4\pi(f_{IF} + \Delta f_g)_{i,k} + 2\Delta \theta_g)\right] + \]
\[ - \frac{A_1}{2} D_i \sin(2\pi(f_{IF} + \Delta f_g)_{i,c}) \sum_{k=1}^{K} C_k C_{c+k} \sin(4\pi(f_{IF} + \Delta f_g)_{i,k} + 2\Delta \theta_g) + \]
\[ + A_1 \sum_{k=1}^{K} n_{i,k} C_{c+k} \cos(2\pi(f_{IF} + \Delta f_g)_{i,c+k} + \Delta \theta_g) \quad [4.19] \]

Let

\[ 2\pi(f_{IF} + \Delta f_g) = \alpha \quad [4.20] \]
The above equation may be written as

\[ I_r (c) = \frac{A_i}{2} D \cos (\alpha t_{i,c}) \left[ K R(c) + \sum_{k=1}^{K} C_k C_{c+k} \cos (2(\alpha t_{i,k} + \Delta \theta_g)) \right] + \]

\[- \frac{A_i}{2} D \sin (\alpha t_{i,c}) \sum_{k=1}^{K} C_k C_{c+k} \sin (2(\alpha t_{i,k} + \Delta \theta_g)) + \]

\[ + A_i \sum_{k=1}^{K} n_{i,k} C_{c+k} \cos (\alpha t_{i,c+k} + \Delta \theta_g) \quad [4.21] \]

Since \( D_i^2 \) is always equal to 1, \( I_r^2 (c) \) is given by the following:

\[ I_r^2 (c) = \frac{A_i^2}{4} \cos^2 (\alpha t_{i,c}) \left[ K^2 R^2(c) + \left( \sum_{k=1}^{K} C_k C_{c+k} \cos (2(\alpha t_{i,k} + \Delta \theta_g)) \right)^2 \right] + \]

\[ + \frac{A_i^2}{4} \sin^2 (\alpha t_{i,c}) \left( \sum_{k=1}^{K} C_k C_{c+k} \sin (2(\alpha t_{i,k} + \Delta \theta_g)) \right)^2 + A_i^2 \left( \sum_{k=1}^{K} n_{i,k} C_{c+k} \cos (\alpha t_{i,c+k} + \Delta \theta_g) \right)^2 + \]

\[ + \frac{A_i^2}{2} K R(c) \cos^2 (\alpha t_{i,c}) \sum_{k=1}^{K} C_k C_{c+k} \cos (2(\alpha t_{i,k} + \Delta \theta_g)) + \]

\[ + A_i^2 D \cos (\alpha t_{i,c}) \sum_{k=1}^{K} n_{i,k} C_{c+k} \cos (\alpha t_{i,c+k} + \Delta \theta_g) \sum_{k=1}^{K} C_k C_{c+k} \cos (2(\alpha t_{i,k} + \Delta \theta_g)) + \]

\[ - \frac{A_i^2}{2} \cos (\alpha t_{i,c}) \sin (\alpha t_{i,c}) \sum_{k=1}^{K} C_k C_{c+k} \cos (2(\alpha t_{i,k} + \Delta \theta_g)) \left( \sum_{k=1}^{K} C_k C_{c+k} \sin (2(\alpha t_{i,k} + \Delta \theta_g)) \right) + \]

\[ - \frac{A_i^2}{2} K R(c) \cos (\alpha t_{i,c}) \sin (\alpha t_{i,c}) \sum_{k=1}^{K} C_k C_{c+k} \sin (2(\alpha t_{i,k} + \Delta \theta_g)) + \]

\[ - A_i^2 D \sin (\alpha t_{i,c}) \sum_{k=1}^{K} n_{i,k} C_{c+k} \cos (\alpha t_{i,c+k} + \Delta \theta_g) \sum_{k=1}^{K} C_k C_{c+k} \sin (2(\alpha t_{i,k} + \Delta \theta_g)) + \]

\[ + A_i^2 D R(c) \cos (\alpha t_{i,c}) \sum_{k=1}^{K} n_{i,k} C_{c+k} \cos (\alpha t_{i,c+k} + \Delta \theta_g) \quad [4.22] \]
The expected value of a constant multiplied by a random variable is equal to the constant multiplied by the expected value of the random variable. The expected value of a sum is equal to the sum of the expected values. White Gaussian noise is independent of the C/A code chip. The expected value of white Gaussian noise is 0. Equation [3.76] states that the expected value of the product of two independent random variables is equal to the product of the expected value of the independent random variables; the average power of $I_r(c)$ is given by the following:

$$E[I_r^2(c)] = \frac{A_i^2}{4} E\left[\cos^2(\alpha_{i,c}) \left(K^2 R^2(c) + \left(\sum_{k=1}^{K} C_k C_{c+k} \cos(2(\alpha_{i,k} + \Delta \theta_g))\right)^2\right)\right] +$$

$$+ \frac{A_i^2}{4} E\left[\sin^2(\alpha_{i,c}) \left(\sum_{k=1}^{K} C_k C_{c+k} \sin(2(\alpha_{i,k} + \Delta \theta_g))\right)^2\right] +$$

$$+ A_i^2 E\left[\left(\sum_{k=1}^{K} n_{i,k} C_{c+k} \cos(\alpha_{i,c+k} + \Delta \theta_g)\right)^2\right] +$$

$$- \frac{A_i^2}{2} E\left[\cos(\alpha_{i,c}) \sin(\alpha_{i,c}) \sum_{k=1}^{K} C_k C_{c+k} \cos(2(\alpha_{i,k} + \Delta \theta_g))\left(\sum_{k=1}^{K} C_k C_{c+k} \sin(2(\alpha_{i,k} + \Delta \theta_g))\right)\right] +$$

$$+ \frac{A_i^2}{2} K E\left[R(c) \cos^2(\alpha_{i,c}) \sum_{k=1}^{K} C_k C_{c+k} \cos(2(\alpha_{i,k} + \Delta \theta_g))\right] +$$

$$- \frac{A_i^2}{2} K E\left[R(c) \cos(\alpha_{i,c}) \sin(\alpha_{i,c}) \sum_{k=1}^{K} C_k C_{c+k} \sin(2(\alpha_{i,k} + \Delta \theta_g))\right]$$

[4.23]
4.2.2 Average Power of the Correlated Quadrature Component

Similarly, using the definition of correlation given in equation [4.3], \( Q_r(c) \), defined in equation [4.11], is given by the following:

\[
Q_r(c) = \sum_{k=1}^{K} [s_r(k) + n_r(k)] Q_g(c+k)
\]

\[
= A_D \sum_{k=1}^{K} C_k C_{g,c+k} \cos(2\pi(f_{IF} + \Delta f_i) t_{i,k} + \Delta \theta_i) \sin(2\pi(f_{IF} + \Delta f_g) t_{i,c+k} + \Delta \theta_g) +
\]

\[
+ A_D \sum_{k=1}^{K} n_{i,k} C_{g,c+k} \sin(2\pi(f_{IF} + \Delta f_i) t_{i,c+k} + \Delta \theta_i)
\]

\[\text{[4.24]}\]

where \( Q_g(k) \) was defined in equation [4.9] and \( s_r(k) + n_r(k) \) was defined in equation [4.2]. Using equation [4.15], the above equation may also be written as

\[
Q_r(c) = \frac{A_D}{2} \sum_{k=1}^{K} C_k C_{g,c+k} \sin(2\pi(f_{IF} t_{i,k} + f_{IF} t_{i,c} + \Delta f_i t_{i,k} - \Delta f_g (t_{i,k} + t_{i,c})) + \Delta \theta_i + \Delta \theta_g) +
\]

\[
- \frac{A_D}{2} \sum_{k=1}^{K} C_k C_{g,c+k} \sin(2\pi(\Delta f_i t_{i,k} - \Delta f_g (t_{i,k} + t_{i,c}) - f_{IF} t_{i,c}) + \Delta \theta_i - \Delta \theta_g) +
\]

\[
+ A_D \sum_{k=1}^{K} n_{i,k} C_{g,c+k} \sin(2\pi(f_{IF} + \Delta f_g) t_{i,c+k} + \Delta \theta_g)
\]

\[\text{[4.25]}\]

As in the case of \( I_r(c) \), assume that the receiver is fine tracking a signal. This means that \( C_{g,c+k} \) equals \( C_{c+k} \), and that \( \Delta f_i \) and \( \Delta \theta_i \) are equal to \( \Delta f_g \) and \( \Delta \theta_g \), respectively. Thus, \( Q_r(c) \) is given by the following:

\[
Q_r(c) = \frac{A_D}{2} \sum_{k=1}^{K} C_k C_{c+k} \sin(4\pi(f_{IF} + \Delta f_g) t_{i,k} + 2\Delta \theta_g + 2\pi(f_{IF} + \Delta f_g) t_{i,c}) +
\]
\[-\frac{A_i}{2} D_i \sum_{k=1}^{K} C_k C_{c+k} \sin(-2\pi(f_{IF} + \Delta f_g)_{i,c,k}) + \]

\[+ A_i \sum_{k=1}^{K} n_{i,k} C_{c+k} \sin(2\pi(f_{IF} + \Delta f_g)_{i,c,k + \Delta \theta_g}) \quad [4.26]\]

The sine of the negative of an argument is equal to the negative of the sine of the argument. Expansion of the above equation yields

\[Q_r(c) = \frac{A_i}{2} D_i \sin(2\pi(f_{IF} + \Delta f_g)_{i,c}) \sum_{k=1}^{K} C_k C_{c+k} \left[1 + \cos(4\pi(f_{IF} + \Delta f_g)_{i,k} + 2\Delta \theta_g)\right] + \]

\[+ \frac{A_i}{2} D_i \cos(2\pi(f_{IF} + \Delta f_g)_{i,c}) \sum_{k=1}^{K} C_k C_{c+k} \sin(4\pi(f_{IF} + \Delta f_g)_{i,k} + 2\Delta \theta_g) + \]

\[+ A_i \sum_{k=1}^{K} n_{i,k} C_{c+k} \sin(2\pi(f_{IF} + \Delta f_g)_{i,c+k} + \Delta \theta_g) \]

\[= \frac{A_i}{2} D_i \sin(2\pi(f_{IF} + \Delta f_g)_{i,c}) \left[KR(c) + \sum_{k=1}^{K} C_k C_{c+k} \cos(4\pi(f_{IF} + \Delta f_g)_{i,k} + 2\Delta \theta_g)\right] + \]

\[+ \frac{A_i}{2} D_i \cos(2\pi(f_{IF} + \Delta f_g)_{i,c}) \sum_{k=1}^{K} C_k C_{c+k} \sin(4\pi(f_{IF} + \Delta f_g)_{i,k} + 2\Delta \theta_g) + \]

\[+ A_i \sum_{k=1}^{K} n_{i,k} C_{c+k} \sin(2\pi(f_{IF} + \Delta f_g)_{i,c+k} + \Delta \theta_g) \quad [4.27]\]

where \(KR(c)\) is the sum of \(C_k C_{c+k}\) from \(k\) equal to 1 to \(K\). Substitution of equation [4.20] into the above equation yields
\[ Q_r(c) = \frac{A_i}{2} D_i \sin(\alpha_{i,c}) \left[ KR(c) + \sum_{k=1}^{K} C_k C_{c+k} \cos(2(\alpha_{i,k} + \Delta \theta_g)) \right] + \]
\[ + \frac{A_i}{2} D_i \cos(\alpha_{i,c}) \sum_{k=1}^{K} C_k C_{c+k} \sin(2(\alpha_{i,k} + \Delta \theta_g)) + \]
\[ + A_i \sum_{k=1}^{K} n_{i,k} C_{c+k} \sin(\alpha_{i,c+k} + \Delta \theta_g) \quad [4.28] \]

Since \( D_i^2 \) is always equal to 1, \( Q^2_r(c) \) is given by the following:

\[ Q^2_r(c) = \frac{A_i^2}{4} \sin^2(\alpha_{i,c}) \left[ K^2 R^2(c) + \left( \sum_{k=1}^{K} C_k C_{c+k} \cos(2(\alpha_{i,k} + \Delta \theta_g)) \right)^2 \right] + \]
\[ + \frac{A_i^2}{4} \cos^2(\alpha_{i,c}) \left( \sum_{k=1}^{K} C_k C_{c+k} \sin(2(\alpha_{i,k} + \Delta \theta_g)) \right)^2 + A_i^2 \left( \sum_{k=1}^{K} n_{i,k} C_{c+k} \sin(\alpha_{i,c+k} + \Delta \theta_g) \right)^2 + \]
\[ + \frac{A_i^2}{2} KR(c)\sin^2(\alpha_{i,c}) \sum_{k=1}^{K} C_k C_{c+k} \cos(2(\alpha_{i,k} + \Delta \theta_g)) + \]
\[ + A_i^2 D_i \sin(\alpha_{i,c}) \sum_{k=1}^{K} n_{i,k} C_{c+k} \sin(\alpha_{i,c+k} + \Delta \theta_g) \sum_{k=1}^{K} C_k C_{c+k} \cos(2(\alpha_{i,k} + \Delta \theta_g)) + \]
\[ - \frac{A_i^2}{2} \cos(\alpha_{i,c}) \sum_{k=1}^{K} C_k C_{c+k} \cos(2(\alpha_{i,k} + \Delta \theta_g)) \sum_{k=1}^{K} C_k C_{c+k} \sin(2(\alpha_{i,k} + \Delta \theta_g)) + \]
\[ - \frac{A_i^2}{2} KR(c) \cos(\alpha_{i,c}) \sum_{k=1}^{K} C_k C_{c+k} \sin(2(\alpha_{i,k} + \Delta \theta_g)) + \]
\[ - A_i^2 D_i \sum_{k=1}^{K} n_{i,k} C_{c+k} \sin(\alpha_{i,c+k} + \Delta \theta_g) + \sum_{k=1}^{K} C_k C_{c+k} \sin(2(\alpha_{i,k} + \Delta \theta_g)) + \]
\[ + A_i^2 D_i KR(c) \sum_{k=1}^{K} n_{i,k} C_{c+k} \sin(\alpha_{i,c+k} + \Delta \theta_g) \quad [4.29] \]
The expected value of a random variable multiplied by a constant is equal to the constant multiplied by the expected value of the random variable; the expected value of a sum of random variables is equal to the sum of the expected values of the random variables.

White Gaussian noise is independent of the C/A code bit, and it has an expected value equal to 0. Using equation [3.76], the average power of $Q_c(c)$ is given by the following:

$$E[Q^2_c(c)] = \frac{A^2}{4} E\left[ \sin^2(\alpha_{t,c}) \left( K^2 R^2(c) + \left( \sum_{k=1}^{K} C_k C_{c+k} \cos(2(\alpha_{t,k} + \Delta \theta_g)) \right)^2 \right) \right] +$$

$$+ \frac{A^2}{4} E\left[ \cos^2(\alpha_{t,c}) \left( \sum_{k=1}^{K} C_k C_{c+k} \sin(2(\alpha_{t,k} + \Delta \theta_g)) \right)^2 \right] +$$

$$+ A^2 E\left[ \sum_{k=1}^{K} n_{i,k} C_{c+k} \sin(\alpha_{i,c+k} + \Delta \theta_g) \right]^2 +$$

$$+ \frac{A^2}{2} E\left[ \cos(\alpha_{t,c}) \sin(\alpha_{t,c}) \sum_{k=1}^{K} C_k C_{c+k} \cos(2(\alpha_{t,k} + \Delta \theta_g)) \sum_{k=1}^{K} C_k C_{c+k} \sin(2(\alpha_{t,k} + \Delta \theta_g)) \right] +$$

$$+ \frac{A^2}{2} K E\left[ R(c) \sin^2(\alpha_{t,c}) \sum_{k=1}^{K} C_k C_{c+k} \cos(2(\alpha_{t,k} + \Delta \theta_g)) \right] +$$

$$+ \frac{A^2}{2} K E\left[ R(c) \cos(\alpha_{t,c}) \sin(\alpha_{t,c}) \sum_{k=1}^{K} C_k C_{c+k} \sin(2(\alpha_{t,k} + \Delta \theta_g)) \right]$$

[4.30]

4.2.3 Average Power of the Total Correlated Signal Plus Noise

The average power of $C_T(c)$ is given by the following:

$$E[C^2_T(c)] = E[I^2_T(c)] + E[Q^2_T(c)]$$
\[
\frac{A^2}{4} E \left[ \cos^2(\alpha t_{\lambda, i}) + \sin^2(\alpha t_{\lambda, i}) \left[ K^2 R^2(c) + \left( \sum_{k=1}^{K} C_k C_{c+k} \cos(2(\alpha t_{i,k} + \Delta \theta_g)) \right)^2 \right] \right] + \\
\frac{A^2}{4} E \left[ \cos^2(\alpha t_{\lambda, i}) + \sin^2(\alpha t_{\lambda, i}) \left( \sum_{k=1}^{K} C_k C_{c+k} \sin(2(\alpha t_{i,k} + \Delta \theta_g)) \right)^2 \right] + \\
A^2 E \left[ \left( \sum_{k=1}^{K} n_{i,k} C_{c+k} \cos(\alpha t_{i,c+k} + \Delta \theta_g) \right)^2 + \left( \sum_{k=1}^{K} n_{i,k} C_{c+k} \sin(\alpha t_{i,c+k} + \Delta \theta_g) \right)^2 \right] + \\
\frac{A^2}{2} K E \left[ R(c) \right] \left[ \cos^2(\alpha t_{\lambda, i}) + \sin^2(\alpha t_{\lambda, i}) \sum_{k=1}^{K} C_k C_{c+k} \cos(2(\alpha t_{i,k} + \Delta \theta_g)) \right] \tag{4.31}
\]

where the expected value of \( I_{\lambda}^2(c) \) was defined in equation [4.23] and the expected value of \( Q_{\lambda}^2(c) \) was defined in equation [4.30]. The cosine squared of an argument plus the sine squared of the same argument is equal to 1. Simplification of the above equation yields the following:

\[
E[C_{\lambda}^2(c)] = A_i E \left[ \left( \sum_{k=1}^{K} n_{i,k} C_{c+k} \cos(\alpha t_{i,c+k} + \Delta \theta_g) \right)^2 + \left( \sum_{k=1}^{K} n_{i,k} C_{c+k} \sin(\alpha t_{i,c+k} + \Delta \theta_g) \right)^2 \right] + \\
\frac{A^2}{4} E \left[ \left( \sum_{k=1}^{K} C_k C_{c+k} \cos(2(\alpha t_{i,k} + \Delta \theta_g)) \right)^2 + \left( \sum_{k=1}^{K} C_k C_{c+k} \sin(2(\alpha t_{i,k} + \Delta \theta_g)) \right)^2 \right] + \\
\frac{A^2}{4} E \left[ 2K R(c) \sum_{k=1}^{K} C_k C_{c+k} \cos(2(\alpha t_{i,k} + \Delta \theta_g)) + K^2 R^2(c) \right] \tag{4.32}
\]

Recall that white Gaussian noise is independent of all other samples of white Gaussian noise not equal to itself. Also, the expected value of white Gaussian noise is equal to 0.

Thus, the above equation is given by the following:
\[
E[C_i^2(c)] = A_i^2 E \left[ \sum_{k=1}^{K} n_{i,k}^2 C_{c+k} \left[ \cos^2 (\alpha t_{i,c+k} + \Delta \theta_g) + \sin^2 (\alpha t_{i,c+k} + \Delta \theta_g) \right] \right] + \\
+ \frac{A_i^2}{4} \left[ \sum_{k=1}^{K} \sum_{j=1}^{K} C_k C_{c+k} C_{c+j} \cos(2\alpha (t_{i,k} - t_{i,j})) + 2KR(c) \sum_{k=1}^{K} C_k C_{c+k} \cos(2(\alpha t_{i,k} + \Delta \theta_g)) \right] + \\
+ \frac{A_i^2}{4} K^2 E[R^2(c)]
\]

\[
= \frac{A_i^2}{4} E \left[ \sum_{k=1}^{K} C_k C_{c+k} \left[ 2KR(c) \cos(2\alpha (\Delta \theta_g)) + \sum_{j=1}^{K} C_j C_{c+j} \cos(2\alpha (t_{i,k} - t_{i,j})) \right] \right] + \\
+ \frac{A_i^2}{4} K \left[ K E[R^2(c)] + 2No \right] \quad [4.33]
\]

where \( \frac{No}{2} \) is the variance of white Gaussian noise.

4.3 A Block Processing C/No Estimation Technique

As stated in chapter 2, C/No is equal to
\[
10 \log_{10} \left[ \frac{S}{No} B \right]
\]
where \( S \) is the signal power,
\( No \) is the noise power, and \( B \) is the two-sided bandwidth of the \( \frac{S}{No} \) estimate. Thus, for the case where the signal is upsampled to 5 Msps, \( B \) is 2.5 MHz. For the following discussion, assume that the receiver is fine tracking a GPS signal.
4.3.1 Maximum Signal Power and Maximum Noise Power

Recall from chapter 2 that $R(\tau)$ for $\tau$ equal to 0 is equal to the C/A code’s maximum autocorrelation peak, which is equal to 1. $\tau$ equal to 0 corresponds to a sample offset, $c$, equal to 0. Thus, the maximum average signal power of $C_\tau(c)$ may be determined when $c$ is equal to 0. The average power of $C_\tau(0)$ is given by the following:

$$\begin{align*}
E[C_\tau^2(0)] &= \frac{A_i^2}{4} E \left[ \sum_{j=1}^{K} C_j C_j \left[ 2KR(0) \cos(2(\alpha t_i, k + \Delta \theta_g)) + \sum_{j=1}^{K} C_j C_j \cos(2\alpha(t_{i, k} - t_{i, j})) \right] \right] + \\
&\quad + \frac{A_i^2}{4} K \left[ KE[R^2(0)] + 2No \right] \\
&= \frac{A_i^2}{2} E \left[ \sum_{k=1}^{K} \cos(2(\alpha t_i, k + \Delta \theta_g)) \right] + \frac{A_i^2}{4} E \left[ \sum_{k=1}^{K} \sum_{j=1}^{K} \cos(2\alpha(t_{i, k} - t_{i, j})) \right] + \\
&\quad + \frac{A_i^2}{4} K \left[ K + 2No \right] \quad [4.34]
\end{align*}$$

where $E[C_\tau^2(c)]$ was defined in equation [4.33].

Recall from equation [4.20] that $\alpha = 2\pi(f_{IF} + \Delta f_g)$. Also, recall from section 4.1 that $f_{IF}$ equals 1.25 MHz and that $\Delta f_g$ ranges between −10 kHz and +10 kHz. In the above equation, the period of the cosine terms, $T_{cos}$, lies between $\frac{1}{1.26}$ µs and $\frac{1}{1.24}$ µs. Since $T_{cos}$ is much smaller than the 1 ms period over which the summations occur, the expected
values of both cosine terms in the above equation is 0. The above equation may be written as

\[ E[C_T^2(0)] = \frac{A_i^2}{4} K[K + 2No] \quad [4.35] \]

Recall from chapter 2 that as long as the sampling frequency of the signal is at least 1023 samples/ms, the width of each autocorrelation peak for the C/A code is equal to 2 times the duration of a C/A code chip. When the sampling frequency is 5000 samples/ms, there are between 9 and 10 samples within this period. Thus, the width of each autocorrelation peak will be between 9 and 10 samples. In order for \(|\tau|\) to be greater than 1 \(\mu s\), the \(c\) that corresponds to \(\tau\) must lie in the interval between 6 and 4996. The maximum average noise power of \(C_T(c)\) may be determined when \(c\) is in this interval.

4.3.2 C/No Estimates

The average power of \(C_T(c)\) for \(c\) in the interval from 6 to 4996 divided by the average power of \(C_T(0)\) is given by the following:

\[
\frac{E[C_T^2(c)]}{E[C_T^2(0)]} = \frac{\sum_{k=1}^{K} C_k C_{c+k} \left[ 2KR(c)\cos(2(\alpha t_{i,k} + \Delta \theta_g)) + \sum_{j=1}^{K} C_j C_{c+j} \cos(2\alpha(t_{i,k} - t_{i,j})) \right]}{K(K + 2No)} + K \frac{E[R^2(c)]}{K + 2No} + \frac{2No}{K + 2No} \quad [4.36]
\]
There are three approaches that may be used to determine C/No. The first approach is to assume the average noise power dominates the above equation. Then,

\[
\frac{\mathbb{E}[\text{C}_T^2(c)]}{\mathbb{E}[\text{C}_T^2(0)]} \approx \frac{2N_0}{K + 2N_0}
\]

[4.37]

Recall that the average signal power of a GPS signal is \(\frac{A^2}{2}\); the average power of \(A\) times white Gaussian noise is equal to \(\frac{A^2N_0}{2}\). Thus, \(\frac{S}{N_0}\) of the GPS signal plus white Gaussian noise is given by the following:

\[
\frac{S}{N_0} = \frac{2}{A^2N_0} = \frac{1}{N_0}
\]

[4.38]

Using equation [4.37], the first approach yields an estimate of \(\frac{S}{N_0}\), \(\frac{S_1}{N_{01}}\), given by the following:

\[
\frac{S_1}{N_{01}} = \frac{2}{K} \left( \frac{\mathbb{E}[\text{C}_T^2(0)]}{\mathbb{E}[\text{C}_T^2(c)]} - 1 \right) \approx \frac{2}{K} \left( \frac{K + 2N_0 - 2N_0}{2N_0} \right) = \frac{1}{N_0}
\]

[4.39]

The second approach requires that the values of \(c\) that may result in partial or full autocorrelation not be used in equation [4.36]. In order not to use those values of \(c\), the autocorrelation of the C/A code, sampled at a rate of 1023 samples/ms, is determined and then upsampled at a rate of 5000 samples/ms. Since \(R(\tau)\), and not the C/A code, is upsampled, the side lobes and the maximum autocorrelation peak will actually be square
shaped and will have width equal to one C/A code chip duration, 4 or 5 samples. The width of each autocorrelation square is then extended so that the duration of each autocorrelation square is at least equal to 2 times the duration of a C/A code chip. Thus, the width of each autocorrelation square is at least equal to the maximum width of the autocorrelation peaks of the C/A code sampled at a rate of 5000 samples/ms. This means that all values of \( c \) that can result in partial or full autocorrelation peaks are found.

Figure 4.3 is a graph of the autocorrelation squares and the autocorrelation peaks of a C/A code sampled at a rate of 5000 samples/ms for \( |\tau| < 25 \mu s \).

Let \( w \) be an integer sample offset in the interval \([0, K - 1]\) such that \( w \) is not equal to the values of \( c \) that can result in partial or full autocorrelation. \( R(w) \) is equal to \(- \frac{1}{1023}\). The average power of \( C_T(w) \) provides an estimate for the maximum average noise power. Thus, the average power of \( C_T(w) \) divided by the average power of \( C_T(0) \) is given by the following:

\[
\frac{E[C_T^2(w)]}{E[C_T^2(0)]} = \frac{E\left[ \sum_{k=1}^{K} C_k C_{w+k} \left[ \sum_{j=1}^{K} C_j C_{w+j} \cos(2\alpha(t_{i,k} - t_{i,j})) - \frac{2K}{1023} \cos(2(\alpha t_{i,k} + \Delta \theta_j)) \right] \right]}{K(K + 2No)} + \frac{K}{1023^2(K + 2No)} + \frac{2No}{K + 2No}
\]

[4.40]
Assume that \( \frac{K}{1023^2(K + 2No)} \) and the cosine terms in the above equation are dominated by the noise power. Thus,

\[
\frac{E[C_7^2(w)]}{E[C_7^2(0)]} \approx \frac{2No}{K + 2No} \quad [4.41]
\]
The second approach yields an estimate of \( \frac{S}{No}, \frac{S_2}{No_2} \), given by the following:

\[
\frac{S_2}{No_2} = 2 \left( \frac{E[C_r^2(0)]}{E[C_r^2(w)]} - 1 \right) \approx \frac{1}{No} \quad \text{[4.42]}
\]

The third approach requires the subtraction of the cosine terms from equation [4.40]. The cosine terms may be generated since \( \alpha \) and \( \Delta \theta_g \) are known. Let \( G(w) \) be given by the following:

\[
G(w) = E\left[ \sum_{k=1}^{K} C_k C_{w+k} \left[ \sum_{j=1}^{K} C_j C_{w+j} \cos(2\alpha(t_{i,k} - t_{i,j})) - \frac{2K}{1023} \cos(2(\alpha t_{i,k} + \Delta \theta_g)) \right] \right] \quad \text{[4.43]}
\]

\( A_i \), and, therefore, \( K(K + 2No) \) are unknown. In order to eliminate

\[
\frac{G(w)}{K(K + 2No)} + \frac{K}{1023^2(K + 2No)}
\]

from equation [4.40], the average power of \( C_r(0) \) must be approximated using \( K^2 \). Thus,

\[
\frac{E[C_r^2(w)]}{E[C_r^2(0)]} = \frac{1023^2 G(w) + K^2}{1023^2 K^2}
\]

\[
= \frac{1023^2 K G(w) + K^3 + 1023^2 K^2 (2No) - 1023^2 (K + 2No) G(w) - (K + 2No) K^2}{1023^2 K^2 (K + 2No)}
\]

\[
= \frac{(1023^2 K^2 - 1023^2 G(w) - K^2)(2No)}{1023^2 K^2 (K + 2No)} \quad \text{[4.44]}
\]

Let \( C_3 \) be given by the following:
The third approach yields an estimate of \( \frac{S}{No} \), \( \frac{S_3}{No_3} \), given by the following:

\[
\frac{S_3}{No_3} = \frac{2}{K} \left( \frac{1}{C_3} - 1 \right) = \frac{1}{No}
\]

[4.46]

Using a MATLAB™ simulation, the percent error of the C/No estimates obtained using the three approaches described above was determined and compared to actual C/No’s. The results are shown in Figure 4.4. The percent errors for the first approach are similar to the percent errors from the second and third approaches between about 40 and 47 dB-Hz. The percent errors from the second approach are similar to the third approach, but better than the first approach, between 48 and 59 dB-Hz. The third approach is better than the first and second approaches for any C/No value above 60 dB-Hz.

The block processing C/No estimation technique is comprised of a combination of all three approaches. The block processing C/No estimation technique first estimates C/No using the first approach. If the C/No estimate obtained using the first approach is greater than 47 dB-Hz, then the block processing C/No estimation technique utilizes the second approach to estimate C/No. Similarly, if the C/No estimate using the second approach is
greater than 59 dB-Hz, then the block processing C/No estimation technique utilizes the third approach to estimate C/No. The results from the block processing C/No estimation technique are shown in Figure 4.5.

4.3.3 Estimation of C/No for More than One Millisecond of Data

The block processing C/No estimation technique may determine estimates of C/No for more than 1 ms worth of total correlated signal plus noise data. Let \( v \) be an integer
number of milliseconds worth of correlated signal plus noise data available for an estimate of C/No. Let \( E_i[C_i^2(c)] \) equal the average power of \( C_i(c) \) for ms \( i \) where \( i \) equal to 1,2,\ldots,v. Thus, using equation [4.33], \( E_i[C_i^2(c)] \) is given by the following:

\[
E_i[C_i^2(c)] = \frac{A_i^2}{4} \left( E \left[ \sum_{k=1}^{K} C_k C_{c+k} \left[ 2KR(c) \cos(2(\alpha t_{i,k} + \Delta \theta_g)) + \sum_{j=1}^{K} C_j C_{c+j} \cos(2\alpha(t_{i,k} - t_{i,j})) \right] \right] \right)_i + \\
+ \frac{A_i^2}{4} K \left[ \mathcal{K}(E[R^2(c)])_i + 2No_i \right] \tag{4.47}
\]
where $N_{0,i}$ is the average noise power and $A_i$ is the amplitude of the signal and noise for
ms $i$. Assume that $A_i$ is approximately constant for all $v$ ms and is equal to $A_v$. The
expected value of all $v$ estimates of $E_i[C_T^2(c)]$ is given by the following:

$$
E[E_i[C_T^2(c)]] = \frac{1}{v} \sum_{i=1}^{v} E_i[C_T^2(c)]
$$

$$
= \frac{A_v^2}{4v} \sum_{i=1}^{v} \left( \sum_{k=1}^{K} C_k C_{c+k} \left[ 2KR(c) \cos(2(\alpha_{i,k} + \Delta \theta_g)) + \sum_{j=1}^{K} C_j C_{c+j} \cos(2\alpha(t_{i,k} - t_{i,j})) \right] \right) +
$$

$$
+ \frac{A_v^2 K}{4v} \sum_{i=1}^{v} \left[ K(E[R^2(c)]) + 2N_{0,i} \right]
$$

[4.48]

The block processing C/No estimation technique requires the expected value of all $v$
estimates of the average power in the maximum autocorrelation peaks. Recall that the
maximum average signal power of $C_T(c)$ may be determined when $c$ is equal to 0.

Thus, using the above equation, $E[E_i[C_T^2(0)]]$ is given by the following:

$$
E[E_i[C_T^2(0)]] = \frac{1}{v} \sum_{i=1}^{v} E_i[C_T^2(0)]
$$

$$
= \frac{A_v^2 K}{4v} \left( K_v + 2 \sum_{i=1}^{v} N_{0,i} \right)
$$

[4.49]

The block processing C/No estimation technique’s first approach requires the expected
value of all $v$ estimates of the average power of $C_T(c)$ for $c$ not within an absolute value
of 5 of a maximum autocorrelation peak. If the sampling frequency is 5 Mspss, then there
are 4990 possible values of \( c \) in 1 ms. The first approach assumes that the noise power dominates the total correlated signal plus noise power for \( c \) in this interval. Using this assumption for \( c \) in this interval, equation [4.48] is approximated by the following:

\[
E[E_i[C_T^2(c)]] \approx \frac{4^2 K}{2v} \sum_{i=1}^{v} N_0
\]  

[4.50]

\( \frac{S_1}{N_0} \) is given by the following:

\[
\frac{S_1}{N_0} = \frac{2}{K} \left( \frac{E[E_i[C_T^2(0)]]}{E[E_i[C_T^2(c)]]} - 1 \right) \approx \frac{v}{\sum_{i=1}^{v} N_0} = \frac{1}{E[N_0]}
\]  

[4.51]

Each millisecond of data undergoes side lobe subtraction using the technique described in section 4.2. Let \( w \) equal a sample offset that does not result in partial or full correlation. The second approach to estimate \( C/No \) requires the expected value of all \( v \) estimates of the average power in ms \( i \) of the total correlated signal plus noise samples that correspond to a sample offset of \( w \). Recall that \( R(w) \) is equal to \( \frac{1}{1023} \). Thus, using equation [4.48], \( E[E_i[C_T^2(w)]] \) is given by the following:

\[
E[E_i[C_T^2(w)]]
\]

\[
= \frac{A_v^2}{4v} \sum_{i=1}^{v} \left( \sum_{k=1}^{K} C_k C_{c+k} \left[ \sum_{j=1}^{K} C_j \cos(2\alpha(t_{i,k} - t_{j,j})) \right] - \frac{2K}{1023} \cos(2(\alpha t_{i,k} + \Delta \theta)) \right) + \frac{A_v^2 K^2}{4(1023^2)} + \frac{A_v K}{2v} \sum_{i=1}^{v} N_0
\]  

[4.52]
The second approach assumes that the noise power dominates the total correlated signal plus noise power for all \( w \). Using this assumption, \( E_i[C_T^2(w)] \) is approximated by the following:

\[
E_i[C_T^2(w)] \approx \frac{A_i^2K}{2v} \sum_{i=1}^{N_o} No_i \quad [4.53]
\]

\( \frac{S_2}{No_2} \) is given by the following:

\[
\frac{S_2}{No_2} = \frac{2}{K} \left( \frac{E_i[C_T^2(0)]}{E_i[C_T^2(w)]} - 1 \right) \approx \frac{1}{E[No_i]} \quad [4.54]
\]

The third approach to calculate C/No requires the subtraction of the cosine terms from equation [4.52]. The cosine terms for each ms \( i \) must be determined. Let \( G_i(w) \) be given by the following:

\[
G_i(w) = \sum_{i=1}^{v} \left( E \left[ \sum_{k=1}^{K} C_k C_{-k} \left[ \sum_{j=1}^{K} C_j C_{-j} \cos(2\alpha(t_{i,k} - t_{i,j})) - \frac{2K}{1023} \cos(2(\alpha_{i,k} + \Delta \theta)) \right] \right] \right) \quad [4.55]
\]

\( C_3 \) is given by the following:

\[
C_3 = \frac{1023^2K^2v}{1023^2K^2v - 1023^2G_i(w) - K^2} \left( \frac{E_i[C_T^2(w)]}{E_i[C_T^2(0)]} - \frac{1023^2G_i(w) + K^2}{1023^2K^2v} \right)
\]
\[
\frac{S_3}{N_0}\text{ is given by the following:}
\]
\[
\frac{S_3}{N_0} = \frac{2}{K} \left( \frac{1}{C_3} - 1 \right) = \frac{1}{E[N_0]} \tag{4.57}
\]
5 COMPARISON OF C/No ESTIMATORS

This work examined two different types of C/No estimators. One C/No estimator is based on wide and narrow band powers and is ideal for sequential processing. The second C/No estimator is based on the average power of the total correlated signal plus noise and is ideal for block processing. The following is a comparison of the results obtained from these two techniques.

Figure 5.1 shows data, based on [Van Dierendonck 1996], for the magnitude of the expected value and the 95% accuracy of the error, in dB, obtained using the wide and narrow band powers method over a period of 1 s for C/No’s between 40 and 56 dB-Hz. For the period of 1 s, the C/No estimate is obtained using $M$ equal to 20 samples of the

Figure 5.1: Magnitude of the Expected Value and the 95% Accuracy of the Error obtained using the Wide and Narrow Band Powers C/No Estimator over 1 s. Data Based on [Van Dierendonck 1996].
accumulated in-phase and quadraphase components and \( K \) equal to 50 samples of normalized power. Note that both the 95% accuracy and the magnitude of the expected value of the error in the C/No estimate are approximately constant.

A MATLAB™ simulation was used to calculate the 97.5% confidence interval for the expected value and standard deviation of the error in 1000 C/No estimates obtained using the block processing C/No estimation technique over 1-ms time intervals. The results of this MATLAB™ simulation are shown in Figure 5.2 for C/No’s between 40 and 85 dB-Hz. One advantage of the block processing C/No estimation technique is that it only requires 1 ms to produce an accurate estimate of C/No. The C/No estimation technique based upon wide and narrow band powers requires a minimum of 20 ms to provide an accurate estimate of C/No. Note that the expected value of the error steadily increases while the standard deviation of the error steadily decreases after approximately 63 dB-Hz. The MATLAB™ simulation introduces errors due to the Fast Fourier Transform operation used to approximate circular correlation. For actual C/No’s below 63 dB-Hz, these errors are dominated by the noise in the system. However, for actual C/No’s above 63 dB-Hz, these errors begin to dominate the noise in the system.

The block processing C/No estimator may provide one estimate of C/No for 1 s worth of data. A MATLAB™ simulation was used to calculate the 97.5% confidence interval for the expected value and standard deviation of the error in 10 C/No estimates obtained
Figure 5.2: 97.5% Confidence Interval for (a) the Expected Value and (b) the Standard Deviation of the Error in 1000 C/No Estimates obtained using the Block Processing C/No Estimator over 1-ms Time Intervals.

using the block processing C/No estimation technique over 1-s time intervals. The results of this MATLAB™ simulation are shown in Figure 5.3. Since only 1 ms of data is used for each correlation, the magnitude of the expected value of the error is largest between
Figure 5.3: 97.5% Confidence Interval for (a) the Expected Value and (b) the Standard Deviation of the Error in 10 C/No Estimates obtained using the Block Processing C/No Estimator over 1-s Time Intervals.

40 and 43 dB-Hz. More accurate C/No estimates could be obtained for C/No’s below 43 dB-Hz using coherent processing of data blocks over time intervals larger than 1 ms.
Comparisons may be made between Figure 5.3 and Figure 5.1. For example, for actual C/No’s between 40 and 41 dB-Hz the magnitude of the expected value of the error obtained using the block processing C/No estimator is worse than the magnitude of the expected value of the error obtained using the wide and narrow band powers C/No estimator. However, for actual C/No’s between 42 and 56 dB-Hz, the magnitude of the expected value of the error obtained using the block processing C/No estimator is much better than the magnitude of the expected value of the error obtained using the wide and narrow band powers C/No estimator. This means that the block processing C/No estimator can provide more accurate estimates than the wide and narrow band powers C/No estimator for longer time frames and for actual C/No’s between 42 and 56 dB-Hz. Also, from Figure 5.3 (b), note that the 97.5% accuracy between 1 s estimates obtained from the block processing C/No estimator is smaller than 0.09 dB. This indicates that there is a high degree of repeatability between 1 s estimates.
6 CONCLUSIONS AND RECOMMENDATIONS

This work provided a complete mathematical derivation of an existing C/No estimation technique based upon wide and narrow band powers. It was determined that this estimation technique has a bias of approximately –0.22 dB-Hz and an accuracy better than 0.5 dB (95%) for C/No’s between 40 and 56 dB-Hz averaged over 1 s. Also, a block processing C/No estimation technique was developed. The block processing estimation technique is unbiased and has an accuracy of better than 0.09dB (97.5%) for C/No’s between 40 and 56 dB-Hz averaged over 1 s.

Based upon the results of this work, there are future considerations for the block processing C/No estimation technique. The first consideration is implementation of this technique in hardware. For example, currently the Software Radio implements the estimation technique based upon wide and narrow band powers. Thus, although the software radio utilizes block processing, it obtains an estimate for C/No using an estimation technique best suited for sequential processing. The block processing C/No estimation technique may be substituted for the estimation technique based upon wide and narrow band power for this receiver and, thus, C/No estimates may be obtained after just one millisecond and more accurate estimates may be determined for slightly longer intervals. Other future considerations include estimation of C/No using static and kinematic data obtained from a GPS receiver and a more detailed analysis of the error sources for this technique.
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