TESTING THE LOW ENERGY THEOREM FOR SPINLESS
“PROTON-NEUTRON” BREMSSTRAHLUNG

A thesis presented to
the faculty of
the College of Arts and Sciences of Ohio University

In partial fulfillment
of the requirements for the degree
Master of Science

Yurii Pidopryhora
June 2003
This thesis entitled

TESTING THE LOW ENERGY THEOREM FOR SPINLESS

“PROTON-NEUTRON” BREMSSTRAHLUNG

BY

YURIII PIDOPRYHORA

has been approved for

the Department of Physics and Astronomy

and the College of Arts and Sciences by

Daniel R. Phillips

Assistant Professor of Physics and Astronomy

Leslie A. Flemming

Dean, College of Arts and Sciences
Testing the Low Energy Theorem

for Spinless “Proton-Neutron” Bremsstrahlung (193pp.)

Director of Thesis: Daniel R. Phillips

In this thesis the accuracy of using the Low energy theorem (soft-photon approximation) to estimate the amplitude for spinless “proton-neutron” bremsstrahlung is investigated. For this purpose spinless elastic scattering and bremsstrahlung theories are constructed. An exact expression for the bremsstrahlung amplitude is derived and an original proof of the Low energy theorem is given in this case. Computations are performed which compare the exact value of the bremsstrahlung amplitude to the Low energy theorem’s approximation thereof in different scattering configurations. We find that in most of the cases considered the Low energy theorem is a good estimate for the “exact” spinless “proton-neutron” bremsstrahlung amplitude. This result may be used to justify Low energy theorem estimates of cross-sections for neutrinostrahlung processes, relevant in astrophysics. Detailed explanations are given so that this text may be used as a reference by students and teachers of nuclear physics and quantum mechanics.

Approved:

Daniel R. Phillips

Assistant Professor of Physics and Astronomy
To my wife Elena and my son Peter
Acknowledgments

First of all I would like to thank my scientific advisor Dr. Daniel Phillips. Without his help and support this thesis would never appear. I am also very thankful to the other members of my thesis defence committee, Dr. Charlotte Elster and Dr. Joseph Shields for their numerous helpful comments. Next I should say thanks to all the members of Ohio University nuclear physics community for not necessarily direct but nevertheless invaluable support they provided for this project. Furthermore I am grateful to my graduate teacher of Quantum Mechanics Dr. Sergio Ulloa (of Ohio University) and my undergraduate teacher of higher mathematics Dr. Lidia Kovalenko (of Moscow Institute of Physics and Technology) who gave me the skills crucial for successful finishing of this project. Finally, I owe an unrepayable debt of gratitude to my wife Elena, who was not only patiently enduring the last two years of my work on this thesis, but also was helping me in all ways possible. And I must also mention my 1 ½ - year old son Peter, who was often making me a good company during my night home-working sessions and even sometimes encouraging me to work by bringing project materials and saying “Daddy, do!”.
Contents

Abstract 3

Acknowledgments 5

Contents 6

List of Figures 11

1 Introduction 14

1.1 Astrophysical Relevance of Bremsstrahlung 16

1.2 Testing the Low Energy Theorem 20

1.3 Pedagogical Relevance of Bremsstrahlung 23

2 Elastic Proton-Neutron Scattering 25

2.1 The Schrödinger Equation in Coordinate Space 26

2.1.1 The Schrödinger Equation for \( pn \) Elastic Scattering 26

2.1.2 Adjusting the Coupling Constant 29

2.1.3 \( S \)-Wave Phase Shift Computation 31
2.2 Elastic Scattering in Momentum Space:

- the Lippmann-Schwinger Equation .................................. 36
- 2.2.1 The Yukawa Potential in $P$-Space .......................... 36
- 2.2.2 General Theory of Elastic Scattering and the Lippmann-Schwinger Equation .......................... 38
- 2.2.3 Partial-Wave Expansion of the Lippmann-Schwinger Equation .................................. 48
- 2.2.4 Partial Wave Expansion of the Yukawa Potential .......................... 57
- 2.2.5 Numerical Solution of the Lippmann-Schwinger Equation .................................. 61

2.3 The Born Approximation and Discussion of Results .......................... 72

- 2.3.1 The Born Approximation .................................. 72
- 2.3.2 Proton-Neutron Elastic Scattering: the Results .......................... 74

3 The Scattering Amplitude

- of Proton-Neutron Bremsstrahlung .................................. 88

- 3.1 Proton-Neutron Bremsstrahlung Theory .................................. 89
  - 3.1.1 The Time-Independent Green’s Function $\hat{G}(E^+)$ .................................. 90
  - 3.1.2 Radiation Hamiltonian .................................. 95
  - 3.1.3 The Green’s Function for Bremsstrahlung .................................. 103
  - 3.1.4 Building the Scattering State with the Green’s Function .................................. 108
5.2.1 Different Energies .............................................. 148
5.2.2 Different Photon Emission Angles $\theta_\omega$ .................. 151
5.2.3 Different Final Momentum Angles $\theta_\star$ ................. 154
5.2.4 Different Final Momentum Angles $\phi_\star$ .................. 154
5.2.5 Conclusions ..................................................... 159

6 Conclusion .................................................................. 160

A Codes in C for the Schrödinger Equation in Coordinate Space 163
A.1 The Code that Determines Optimal $g^2$ Based
  on the Value of Deuteron Binding Energy ......................... 163
A.2 The Code that Computes the $S$-Wave
  Phase Shifts and Cross-Sections .................................. 166

B The Legendre Functions and Gauss-Legendre Quadratures 169
B.1 The Legendre Functions of 1st and 2nd Kind
  of Orders 6 through 10 ............................................. 169
B.2 The Gauss-Legendre Abscissas and Weights
  of Order 96 at the Interval $[-1, 1]$ ............................... 171

C Useful Theorems of Quantum Mechanics 172
C.1 Definition of a Finite-Range Potential .......................... 172
C.2 The Free Green’s Function $\hat{G}_0(E^+, \vec{K}_0)$
  in the Coordinate Representation ............................... 174
C.3 The Equivalence of \((\vec{P}, \vec{R}, \vec{p}, \vec{r})\) and \((\vec{p}_1, \vec{r}_1, \vec{p}_2, \vec{r}_2)\)

Description of a Free Two-Particle System .......................... 177

C.4 Computation of a \(T\)-Matrix in

a Non-Center-of-Mass Frame of Reference .............................. 178

C.5 The Equivalence of \(\hat{T}(\vec{p}, \vec{q}; E)\) and \(\hat{T}(\vec{q}, \vec{p}; E)\) ......................... 180

C.6 The Cross-Section of Spinless

“Proton-Neutron” Bremsstrahlung ............................... 181

D Algorithm for Computing an On-Shell \(T\)-Matrix

and Corresponding Phase Shifts \(\delta_l\) .............................. 187

List of References .......................................................... 190
List of Figures

1.1 Comparison of neutrino and photon bremsstrahlung .................. 20

2.1 S-wave phase shift from the Schrödinger equation in r-space ...... 34
2.2 Comparing s-wave and ESC96 model $^3S_1$ phase shifts ............... 35
2.3 Integration contour of the Green’s function ............................ 42
2.4 Coinciding Schrödinger and Lippmann-Schwinger $\delta_0$

compared to the Born approximation ................................. 75

2.5 Relative difference of Schödinger and Lippmann-Schwinger $\delta_0$ .... 76
2.6 Relative difference of Born and Lippmann-Schwinger $\delta_0$ ............ 76
2.7 Lippmann-Schwinger $\delta_1$ compared to the Born approximation ...... 78
2.8 Relative difference of Born and Lippmann-Schwinger $\delta_1$ ............ 78
2.9 Lippmann-Schwinger $\delta_2$ compared to the Born approximation ...... 79
2.10 Relative difference of Born and Lippmann-Schwinger $\delta_2$ ............ 79
2.11 Lippmann-Schwinger $\delta_3$ compared to the Born approximation ...... 80
2.12 Relative difference of Born and Lippmann-Schwinger $\delta_3$ ............ 80
2.13 Lippmann-Schwinger $\delta_4$ compared to the Born approximation ...... 81
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.14</td>
<td>Relative difference of Born and Lippmann-Schwinger $\delta_4$</td>
<td>81</td>
</tr>
<tr>
<td>2.15</td>
<td>Lippmann-Schwinger $\delta_5$ compared to the Born approximation</td>
<td>82</td>
</tr>
<tr>
<td>2.16</td>
<td>Relative difference of Born and Lippmann-Schwinger $\delta_5$</td>
<td>82</td>
</tr>
<tr>
<td>2.17</td>
<td>Lippmann-Schwinger $\delta_6$ compared to the Born approximation</td>
<td>83</td>
</tr>
<tr>
<td>2.18</td>
<td>Relative difference of Born and Lippmann-Schwinger $\delta_6$</td>
<td>83</td>
</tr>
<tr>
<td>2.19</td>
<td>Born approximation of $\delta_7$, $\delta_8$, $\delta_9$ and $\delta_{10}$</td>
<td>84</td>
</tr>
<tr>
<td>2.20</td>
<td>Lippmann-Schwinger and Born approximation $NN$ elastic cross-sections</td>
<td>85</td>
</tr>
<tr>
<td>2.21</td>
<td>Relative difference of Born cross-sections of two kinds</td>
<td>86</td>
</tr>
<tr>
<td>2.22</td>
<td>Relative difference of Born and Lippmann-Schwinger cross-sections</td>
<td>87</td>
</tr>
<tr>
<td>3.1</td>
<td>Feynman-like rules for the Green’s function</td>
<td>95</td>
</tr>
<tr>
<td>3.2</td>
<td>Expansion diagram of the full time-independent Green’s function</td>
<td>95</td>
</tr>
<tr>
<td>3.3</td>
<td>Charge exchange bremsstrahlung diagram, excluded from our model</td>
<td>102</td>
</tr>
<tr>
<td>3.4</td>
<td>Integration variable transformation</td>
<td>107</td>
</tr>
<tr>
<td>3.5</td>
<td>Additional Feynman-like rules</td>
<td>119</td>
</tr>
<tr>
<td>3.6</td>
<td>Feynman-like diagrams of $pn$ bremsstrahlung</td>
<td>120</td>
</tr>
<tr>
<td>5.1</td>
<td>Two examples of the Low theorem testing computations</td>
<td>147</td>
</tr>
<tr>
<td>5.2</td>
<td>Low energy theorem testing: different energies, $Re$</td>
<td>149</td>
</tr>
<tr>
<td>5.3</td>
<td>Low energy theorem testing: different energies, $Im$</td>
<td>150</td>
</tr>
<tr>
<td>5.4</td>
<td>Low energy theorem testing: different $\theta_\omega$, $Re$</td>
<td>152</td>
</tr>
<tr>
<td>5.5</td>
<td>Low energy theorem testing: different $\theta_\omega$, $Im$</td>
<td>153</td>
</tr>
<tr>
<td>5.6</td>
<td>Low energy theorem testing: different $\theta_*$, $Re$</td>
<td>155</td>
</tr>
</tbody>
</table>
5.7 Low energy theorem testing: different $\theta_\star$, $Im$  \hspace{1cm} 156
5.8 Low energy theorem testing: different $\phi_\star$, $Re$  \hspace{1cm} 157
5.9 Low energy theorem testing: different $\phi_\star$, $Im$  \hspace{1cm} 158
Chapter 1

Introduction

The non-relativistic quantum-mechanical few-body problem belongs to the very heart of nuclear physics. Much of what we presently know about the forces at work inside the atomic nucleus we have learned through various kinds of scattering experiments. Thus, not surprisingly, this area of physics attracts the constant and passionate attention of scientists all over the world for many decades.

One of the simplest occurrences of a few-body problem in nuclear physics is an elastic scattering of two nucleons. This has been classic text-book material for many years. Yet even in this plain and basic case there are still some unanswered questions, like the exact form of the nucleon-nucleon interaction potential.

If we want to move deeper into the field, the next logical thing to consider is either an inelastic scattering or a third particle produced as a result of the two-body interaction. The latter case is slightly more complicated because of the recoil momen-
tum which makes an analysis of the initial and final states in the same center-of-mass frame of reference impossible. One of the examples of such a process is bremsstrahlung. Originally this term applied only to processes with photon emission (electromagnetic bremsstrahlung), but currently it is often used to designate scattering processes with emission of other particles, e.g. neutrinos (neutrino bremsstrahlung).

Historically the main motivation for studying nucleon-nucleon bremsstrahlung was based on the belief that this process uncovers properties of nucleon-nucleon interaction, which are hidden by a natural symmetry of an elastic scattering. As we are going to see in Chapter 3, the scattering amplitude of bremsstrahlung is determined by half-on-shell $T$-matrix elements, which in turn are connected to non-diagonal matrix elements of the interaction potential. On the other hand, the elastic scattering amplitude is determined by on-shell $T$-matrix elements and thus the full interaction potential matrix cannot be determined from it. However, as a recent study shows [1], the off-shell amplitude is not a physically well defined quantity and as a matter of principle cannot be measured in $NN$ bremsstrahlung.

Another important property of $NN$ bremsstrahlung lies in the fact that the interaction responsible for the third particle radiation (electromagnetic for photon, weak nuclear for neutrino) is much weaker than the strong nuclear interaction governing $NN$ scattering. Thus the ratio of the coupling constants of the two interactions provides us with a natural small parameter, which can be used to simplify the theoretical analysis of the phenomenon. In particular, asymptotically true statements
can be formulated. One such statement is the object of this study: the Low energy theorem.

Finally, electromagnetic bremsstrahlung can be measured experimentally \[2\]. This means that the theoretical issues of the previous two paragraphs are not idle speculations, but actually can be used to address data for the \(pp \rightarrow pp\gamma\) or \(pn \rightarrow pn\gamma\) processes.

Of course, as in the case of elastic scattering, electromagnetic bremsstrahlung is thoroughly studied, see e. g. the classic paper \[3\]. At present a well known Dutch group investigates the phenomenon both experimentally \[2\] and theoretically \[4\], \[5\], and \[6\]. The name of the last theoretical paper, “Soft Electroweak Bremsstrahlung: Theorems and Astrophysical Relevance”, hints at one of the reasons why the interest in bremsstrahlung is still high after all these years of extensive research. The motivation for this project comes from the same source: the astrophysical relevance of bremsstrahlung.

### 1.1 Astrophysical Relevance of Bremsstrahlung

The problem of the core-collapse supernova mechanism \[7\] and the closely related problem of neutron star evolution \[8\] are possibly the most tantalizing and complicated problems of modern day astrophysics. Currently it is believed that cooling by neutrino emission plays an essential role for both of these processes.

The simplest such process is the so-called direct Urca process:
\[
\begin{align*}
n \rightarrow & \ p + e^- + \bar{\nu}_e; \\
p + e^- & \rightarrow \ n + \nu_e. \quad (1.1)
\end{align*}
\]

With the conditions characteristic of neutron stars the protons, neutrons and electrons participating in these reactions, are close to beta equilibrium, where the sum of the electron and proton chemical potentials is equal to the neutron chemical potential. Meanwhile, the temperatures of neutrino-emitting sources — late-stage supernova cores and mantles, proto-neutron and young neutron stars — are thought to be much lower than the Fermi temperatures of their constituents, which are of order 100 MeV or \(10^{12}\) K. Thus the matter is in the degenerate state. If the temperature is \(T\) then the fermions are excited above their Fermi surfaces by energies of order \(\sim k_B T\) \((k_B\) is the Boltzmann constant). Thus the processes (1.1) occurring in a cycle produce neutrinos and antineutrinos each carrying away energy \(\sim k_B T\) [9].

However, if the proton fraction of the matter is low enough, the direct Urca process is kinematically suppressed. This can be proved by a simple argument as follows. Since the neutrino momentum \(\sim k_B T/c\) is much smaller than the momenta of protons, neutrons and electrons — which are close to the corresponding Fermi momenta \(\vec{p}_{F_p}, \vec{p}_{F_n}\) and \(\vec{p}_{F_e}\) in degenerate matter — three-momentum conservation requires the following inequality to be true:

\[
p_{F_p} + p_{F_e} \geq p_{F_n}. \quad (1.2)
\]
In the degenerate state the number density of fermions of each kind is proportional to the Fermi momentum cubed:

\[ n_i = \frac{p_{F_i}^3}{3\pi^2\hbar^3}, \]  
(1.3)

where \( i \) stands for any of the fermions. Thus, assuming that the matter is neutral:

\[ n_e = n_p \Rightarrow p_{F_e} = p_{F_p}. \]  
(1.4)

Hence (1.2) turns into:

\[ p_{F_p} \geq \frac{p_{F_n}}{2}. \]  
(1.5)

Combining it with (1.3) we obtain:

\[ n_p \geq \frac{n_n}{8}, \]  
(1.6)

meaning that for the direct Urca process the proton fraction must be about \( \frac{1}{9} \) or greater.

Of course, models of neutron matter can be built with the proton fraction exceeding this threshold value [9] and our knowledge of neutron star physics is too limited to completely rule them out. However this reasoning is certainly enough to motivate a search for another neutrino emission mechanism.

The most plausible candidate was found in the form of the modified Urca process [9], [10]:

\[ n + n \rightarrow n + p + e^- + \bar{\nu}_e; \]

\[ n + p + e^- \rightarrow n + n + \nu_e. \]  
(1.7)
Another process that may be important is weak neutral current \[^1\]:

\[
N + N \rightarrow N + N + \bar{\nu}_x + \nu_x; \tag{1.8}
\]

\[
e^+ + e^- \rightarrow \bar{\nu}_x + \nu_x. \tag{1.9}
\]

\(N\) here designates either proton or neutron, and \(\nu_x\) stands for any of the neutrino species: \(\nu_e, \nu_\mu\) or \(\nu_\tau\). Various processes involving mesons may also be considered, but they are beyond the scope of our discussion here.

While previous estimates \[^10\] showed that weak neutral current processes are insignificant, new research \[^11\] reinstates their importance, since they are dominant as a production mechanism for \(\mu\)- and \(\tau\)-neutrinos in regions of high nucleon density.

The processes (1.7) and (1.8) are neutrino bremsstrahlung processes (they are also sometimes called “neutrinostrahlung”). Let us remember that the electron-antineutrino pair in (1.7) results from a decay of a virtual \(W^-\) boson and the neutrino-antineutrino pair correspondingly from a decay of a \(Z^0\) boson.\(^1\) Thus we can schematically represent these processes as shown in Fig. 1.1 (a) and (b). It is easy to see the similarity of the diagrams to the usual photon bremsstrahlung process Fig. 1.1 (c). As a result, the computations of cross-sections and energy-loss rates for these processes are similar.

\(^1\)As these are virtual bosons, very high weak boson rest mass is of no consequence kinematically. The energy of the lepton pairs is therefore not limited from below in any way.
1.2 Testing the Low Energy Theorem

Because processes (1.7) and (1.8) are the leading candidates for cooling through the neutrino emission mechanism, knowledge of accurate cross-section values for such “neutrinostrahlung” processes is crucial in realistic astrophysical modelling. It is an important input in evaluating neutron-star neutrino emissivities and hence in constructing the cooling curves of these objects.

One way to estimate bremsstrahlung cross-sections is using the Low energy theorem (see e.g. [12]). We will discuss this theorem in detail in Chapter 4 for the case of spinless “proton-neutron” bremsstrahlung we are investigating in this study. However the basic statement is the same for any bremsstrahlung: the scattering amplitude of the bremsstrahlung process is given by an asymptotic relation:

\[ f(\omega) = \frac{f_{-1}}{\omega} + f_0 + \mathcal{O}(\omega). \]  

(1.10)
In the case of electromagnetic bremsstrahlung the coefficients $f_{-1}$ and $f_0$ are determined by the corresponding non-radiative process\(^2\) — and thus are much easier to compute than the exact bremsstrahlung amplitude. They are also independent of details of the bremsstrahlung model. In the case of $NN$ bremsstrahlung they can be extracted from nucleon-nucleon scattering data thereby providing modelling of better accuracy. Recent work on the $NN \rightarrow NN\nu\bar{\nu}$ process \cite{12} showed that this “soft-radiation” approximation leads to neutrino energy loss rates which are roughly a factor of four smaller than earlier estimates based on a crude one-pion-exchange $NN$ interaction model.

Unfortunately, as for any asymptotically true statement, the deviation of the full bremsstrahlung amplitude from the estimate (1.10) is unknown. Once $\omega$ is large enough that the $O(\omega)$ term becomes leading, the Low energy theorem is worthless.

That is why it is very important to compare the Low energy theorem predictions to more precise calculations, at least for simple processes. Taking into account the similarity of neutrino and photon bremsstrahlung on the one hand and a much larger bulk of knowledge assembled for the photon bremsstrahlung on the other, it is natural to start the theorem testing with the electromagnetic case. Furthermore, as one of the most serious complications of the exact calculation comes from dealing with spins of the particles, it is convenient to start investigating the problem by considering spinless particles. Indeed, historically, the spinless particle case was the one F. E. Low first

\(^2\)For neutrino bremsstrahlung only $f_{-1}$ is determined by the non-radiative process \cite{13}.
proved his theorem for [14]. We will still call the particles “proton” and “neutron” and assume their masses to be characteristic to the nucleons. However, we will always remember that the particles have no spin.

From the physical point of view this situation can be roughly understood as an interaction of particles of certain relative angular momentum configuration, e. g. $^3S_1$ of deuteron formation. With a different choice of particle masses we can also interpret a spinless bremsstrahlung as a pion-pion interaction. Unfortunately, pion-pion bremsstrahlung is not relevant to any real physical problem and would be very difficult to detect experimentally.

The choice of some particular $pn$ interaction potential is not very important. Once the theory is created and codes are written, the potential can easily be replaced by any other reasonable one. Thus a decision was made to use the simplest — Yukawa — potential. The concept of the “nucleons” interacting in the definite angular momentum configuration of deuteron creation allows us to normalize the potential, since we adjust its strength so as to reproduce the value of the deuteron’s binding energy. Of course, we also assume that the particles escape to infinity after the interaction and so the creation of the bound state is no more than a reference point. Mathematically we require the inequality$^3$:

$$\omega < \sqrt{4ME},$$

(1.11)

to be always true and we never investigate the process $NN \rightarrow d\gamma$. Here $\omega$ is the

$^3$We will derive it in Subsection 3.1.5.
radiated photon’s energy, $M$ is nucleon mass and $E$ is the final nucleon energy.

What we are going to accomplish in this study is no more than a first step in testing the accuracy of the Low energy theorem as a mean of computing the neutrino bremsstrahlung cross-section. However even though this step is far away from the ultimate goal of understanding the neutrinostrahlung, it may be very useful from another point of view — the pedagogical one.

1.3 Pedagogical Relevance of Bremsstrahlung

A few-body problem is not only a cornerstone of nuclear physics. It also focuses in itself many basic principles of quantum mechanics. Textbooks (like e. g. [15] and [16]) handle the advanced scattering theory at the very end of the course because the material requires understanding of almost all previous chapters. Operators and states, the coordinate and momentum representation, Schrödinger equation and angular momentum theory, etc. etc. find their application in the theory of two-body scattering, which we construct in Chapter 2. In Chapter 3 we continue with the spinless bremsstrahlung theory and we cannot move forward without comprehension of Green’s functions and time-evolution operators. Not to mention that using the results of both elastic scattering and bremsstrahlung theories requires numerical solution of differential and integral equations and thus provides training also in the field of computational mathematics. Last but not least, studying bremsstrahlung may supply a student with that necessary link between abstract quantum theory and physical reality: brems-
strahlung theory not only involves many fundamental quantum principles, but also offers revealing examples to explain them.

Thus basically the bremsstrahlung theory provides us with a “treasure box” of exercises and “toy” research projects in quantum mechanics, covering almost all essential topics. For this reason we have tried to give a detailed explanation of every aspect of this study, paying special attention to the murkiest pieces, poorly covered in common textbooks. It is the author’s hope that this text is going to be used by both teachers and students of quantum mechanics as a valuable reference. It also was written to be an introduction into the field for students starting their career in theoretical few-nucleon physics.
Chapter 2

Elastic Proton-Neutron Scattering

In this chapter we consider the elastic scattering of a proton and a neutron.\footnote{We use the words “proton” and “neutron” for denotation purposes only. The particles are assumed to be spinless and the electric charge of the proton plays no role in this chapter.} We also perform computations of $\rho n$ phase shifts, assuming that the interaction is described by a Yukawa potential.

Firstly, in Section 2.1 we solve the Schrödinger equation in coordinate space for the $s$-wave (i. e. when the orbital angular momentum quantum number $l = 0$). The main purpose of this is to find the value of the model potential’s coupling constant corresponding to a bound state of energy 2.2246 MeV, i. e. the binding energy of the deuteron. This ensures that the potential describes the strong nuclear interaction of real protons and neutrons as closely as possible in the model employed. We also calculate the phase shifts for a range of energies and compare them to the Nijmegen partial-wave analysis PWA93 [17] results.
Then, in Section 2.2 we switch to momentum space, which is used for the rest of the current study. Using a potential equivalent to the one chosen in Section 2.1 the solution of the Lippmann-Schwinger equation for different partial waves is found. This provides a basis for the \( pn \) bremsstrahlung investigation of Chapter 3.

Finally, in Section 2.3 we analyze the results obtained in Section 2.1 and Section 2.2 by comparing them to each other and to the Born approximation.

### 2.1 The Schrödinger Equation in Coordinate Space

In this section we investigate elastic proton-neutron scattering in the case of \( s \)-waves by solving the Schrödinger equation with the Yukawa potential. The general method to solve the equation and the numerical algorithm are explained in Subsection 2.1.1. Then the value of the coupling constant that gives a bound state with the deuteron binding energy is found in Subsection 2.1.2. Finally, this value of the coupling is used to compute the \( s \)-wave phase shifts in Subsection 2.1.3.

#### 2.1.1 The Schrödinger Equation for \( pn \) Elastic Scattering

Let us assume that the proton-neutron strong nuclear interaction is described by the Yukawa potential:

\[
V(\vec{r}) = -\frac{g^2}{4\pi} \frac{e^{-m_{\pi}r}}{r},
\]  

(2.1)

with \( \vec{r} \) being the separation of the two particles and the pion mass \( m_{\pi} \) equal to 139 MeV. The coupling \( g^2 \) remains to be determined in Subsection 2.1.2.
It is well known that for a local potential a two-body problem can be treated in the center-of-mass frame of reference as a one-body problem for the “particle” with radius-vector \( \vec{r} \) and “reduced” mass:

\[
\mu = \frac{m_1 m_2}{m_1 + m_2}.
\] (2.2)

For simplicity we will neglect the difference in proton’s and neutron’s masses and take

\[
\mu = \frac{1}{2} M_{nucleon} \approx 469.5 \text{ MeV}.
\]

In even more specific case of a central potential \( V(\vec{r}) = V(r) \) such an equivalent one-body Schrödinger equation

\[
\hat{H} \psi = \frac{\hat{p}^2}{2\mu} \psi + V \psi = E \psi,
\] (2.3)

evaluated in spherical coordinates \((r, \theta, \phi)\) leads to the set of eigenstates:

\[
\psi = A_{norm} \frac{u_l(r)}{r} Y_{lm}(\theta, \phi),
\] (2.4)

where \( l \) and \( m \) are, respectively, the orbital and the magnetic angular momentum quantum numbers, and \( A_{norm} \) is an arbitrary normalization factor. Here \( Y_{lm}(\theta, \phi) \) is a spherical harmonic and \( u_l(r) \) is determined by the radial Schrödinger equation:

\[
- \frac{\hbar^2}{2\mu} \frac{d^2 u_l}{dr^2} + \left[ V(r) + \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2} \right] u_l(r) = Eu_l(r).
\] (2.5)

We are mainly interested in the simplest case \( l = 0 \) when the radial equation (2.5) transforms into

\[^2\text{i. e. depending on radius-vectors of particles } \vec{r}_1 \text{ and } \vec{r}_2 \text{ only in the combination } \vec{r} = \vec{r}_2 - \vec{r}_1\]
\[ E \geq 0 : \quad \frac{d^2 u}{dr^2} + [k^2 - U(r)]u(r) = 0, \quad k^2 \overset{\text{def}}{=} \frac{2\mu E}{\hbar^2}; \quad (2.6) \]
\[ E < 0 : \quad \frac{d^2 u}{dr^2} - [\kappa^2 + U(r)]u(r) = 0, \quad \kappa^2 \overset{\text{def}}{=} -\frac{2\mu E}{\hbar^2}; \quad (2.7) \]
\[ U(r) \overset{\text{def}}{=} \frac{2\mu V(r)}{\hbar^2}. \quad (2.8) \]

Almost any standard numerical method is sufficient to solve this equation. We will transform this second-order ordinary differential equation into a set of two first-order ones by introducing a new function \( v \) so that:

\[ v(r) \overset{\text{def}}{=} \frac{du}{dr}. \quad (2.9) \]

We then use the explicit Euler algorithm:

\[
\begin{align*}
    u_{i+1} &= u_i + (r_{i+1} - r_i)v_i, \\
    v_{i+1} &= v_i - (r_{i+1} - r_i)[k^2 - U(r_i)],
\end{align*}
\]

(2.10)

where \( r_{i+1} - r_i \overset{\text{def}}{=} \rho = \text{const} \) if we use a lattice with a constant step. Here we have \( E \geq 0; \) in the case of negative energies \( k^2 \) needs to be replaced by \( -\kappa^2. \)

The boundary condition for \( u \) is naturally determined by the fact that \( r \overset{\text{def}}{=} |\vec{r}| \geq 0. \) Thus all the solutions \( u(r) \neq 0, r < 0 \) are unphysical and must be removed by fixing \( u = 0 \) for all negative \( r \)'s. Then the wavefunction's continuity requires:

\[ (u)_{r=0} = 0. \quad (2.11) \]

To define the Cauchy problem and thus ensure the existence of one and only one solution of the equation we need one more boundary condition, namely the one for
It will fix not only the functional dependence of the wavefunction but also its norm, which is irrelevant for our goals. Thus the second boundary condition is also irrelevant as long as it is not zero (otherwise the system (2.10) has the trivial solution \( u(r) \equiv 0 \)). Arbitrarily we take:

\[
(v)_{r=0} = 1. \tag{2.12}
\]

To make the computations simpler and to avoid dealing with numbers which are too large or too small, it is good to use a convenient system of units. For our purposes the fermi-based system works well: let us measure all lengths in fermis (1 fm = \(10^{-13}\) cm is a typical nuclear scale) and take both \(\hbar = 1\) and the speed of light \(c = 1\), then

\[
\hbar c = 1 = 197 \text{ MeV} \cdot \text{fm} \Rightarrow 1 \text{ fm}^{-1} = 197 \text{ MeV}, \tag{2.13}
\]

and thus energies, momenta and masses (all measured in MeV because of \(c = 1\)) are easily converted into fm\(^{-1}\).

Now we have everything ready to compute the radial wavefunction for known \(E\) and \(g^2\).

### 2.1.2 Adjusting the Coupling Constant

The proton and neutron are known to create a bound state called the deuteron with binding energy 2.2246 MeV. It is also known that the deuteron has no excited states. We can use these facts to choose the “best physical” value of the potential parameter \(g^2\) in our model.
As the wavefunction of a bound state must be localized, we should look for the coupling constant value that leads to the only non-diverging solution $u(r)$ with $E = -2.2246$ MeV of the negative-energy radial Schrödinger equation (2.7).

The model potential (2.1) can be neglected for a sufficiently large radius $r$. In such a case equation (2.7) turns into

$$\frac{d^2u}{dr^2} - \kappa^2 u(r) = 0 \quad (2.14)$$

and has the solution (we assume that it has a finite norm):

$$u(r) = Ae^{\kappa r} + Be^{-\kappa r}. \quad (2.15)$$

If we know $u(r_0)$ and its first derivative $u'(r_0)$ at some large radius $r_0$ we can get the constants $A$ and $B$ from

$$A = \frac{\kappa u(r_0) + u'(r_0)}{2\kappa e^{\kappa r_0}} \quad (2.16)$$

and

$$B = \frac{\kappa u(r_0) - u'(r_0)}{2\kappa e^{-\kappa r_0}}. \quad (2.17)$$

Obviously for a non-diverging solution $A$ must be equal to zero.

Now we can check if a given $g^2$ leads to a diverging or a non-diverging (bound state) solution by applying the numerical algorithm (2.10) with the boundary conditions (2.11–2.12) up to a large radius $r_0$ ($r_0 = 30$ fm was used) and then using (2.16) to see if $A = 0$. 
Of course, numerically we never can get \( A \) to be \textit{exactly} zero. However calculated in the way described above \( A \) is a continuous function of \( g^2 \) and as such the equation

\[
A(g^2) = 0
\]  

(2.18)

can be solved numerically by any of the common methods, such as dividing the interval in half or using the Newton sequence. Taking into account the meager computational power required we can just “scan” over the range of \( g^2 \), calculating the corresponding \( A \)'s and so find the value of \( g^2 \) that satisfies (2.18) with any accuracy we need.

A simple code was written in C to do this (see Appendix A.1). By running it the value of \( g^2 \) is determined to be \( 4.473 \pm 0.001 \). Here the error is determined by the exactness of the initial parameters (\( \mu, m_\pi \) and \( E \)). The numerical error is absolutely irrelevant.

### 2.1.3 \textit{S}-Wave Phase Shift Computation

In this subsection we finally get to proton-neutron scattering, i.e. positive energies. In this case the equation (2.6) applies.

Let us utilize the same approach we used in the previous subsection and assume that the potential (2.1) could be neglected at a sufficiently large distance. Then analogously to (2.14) the radial Schrödinger equation transforms into

\[
\frac{d^2 u}{dr^2} + k^2 u(r) = 0
\]  

(2.19)
and leads to a well known free solution

\[ u_0(r) = A \sin kr + B \cos kr \]

\[ = A(\sin kr + \tan \delta_0 \cos kr), \quad (2.20) \]

\[ \tan \delta_0 \overset{\text{def}}{=} B/A. \quad (2.21) \]

We labelled \( u(r) \) with index 0 to emphasize that this is the \( s \)-wave solution. As the normalization of the wavefunction is arbitrary, we are free to set \( A = \cos \delta_0 / k \). Now our solution is consistent with the common definition of phase shifts \( \delta_l \):

\[ R_l(r) = \cos \delta_l [j_l(kr) - \tan \delta_l n_l(kr)], \quad (2.22) \]

where \( R_l(r) \) is the radial part of the wavefunction (in our case \( R_l(r) = u_l(r)/r \), cf. (2.4)), \( j_l(kr) \) and \( n_l(kr) \) are spherical Bessel and Neumann functions; for \( l = 0 \):

\[ j_0(kr) = \frac{\sin(kr)}{kr}, \quad n_0(kr) = -\frac{\cos(kr)}{kr}. \quad (2.23) \]

Note that in this study we adopt a convention of the negative sign in \( n_0(x) \) (and also in \( n_l(x) \)'s asymptotic at infinity (2.91) etc.) and thus have to \textit{subtract} the second term in the definition of phase shifts (2.22).

A well known formula (2.25) follows from (2.22) if we differentiate \( R_l(r) \) at some large \( r_0 \) and then divide by \( R_l(r_0) \) thus producing the logarithmic derivative:

\[ \gamma_l \overset{\text{def}}{=} \frac{d}{dr} \log R_l(r) \bigg|_{r=r_0} = \frac{R_l'(r_0)}{R_l(r_0)} = k \frac{j_l'(kr_0) - \tan \delta_l n_l'(kr_0)}{j_l(kr_0) - \tan \delta_l n_l(kr_0)}, \quad (2.24) \]

which can be easily solved for \( \tan \delta_l \):

\[ \tan \delta_l = \frac{k j_l'(kr_0) - \gamma_l j_l(kr_0)}{k n_l'(kr_0) - \gamma_l n_l(kr_0)}. \quad (2.25) \]
In the simplest case of \( l = 0 \) a modification of this formula follows from (2.20) if we introduce

\[
\tilde{\gamma}_0 \overset{\text{def}}{=} \frac{d}{dr} \log u_0(r) \bigg|_{r=r_0} = \frac{u'_0(r_0)}{u_0(r_0)}.
\]

(2.26)

It leads to a relation similar to (2.25), but more convenient to use with s-waves:

\[
\tan \delta_0 = \frac{k \cos kr_0 - \tilde{\gamma}_0 \sin kr_0}{k \sin kr_0 + \tilde{\gamma}_0 \cos kr_0}.
\]

(2.27)

We can now solve the radial Schrödinger equation numerically the same way we did in Subsection 2.1.2 (i.e. apply the numerical scheme (2.10) with the boundary conditions (2.11–2.12)) ending at an “infinite” radius \( r_0 \). Then we just need to substitute the calculated values of \( u_0(r_0) \) and \( u'_0(r_0) \) into (2.26–2.27) to get \( \tan \delta_0 \).

For this purpose we slightly modify the code used to find the value of \( g^2 \) (see Appendix A.2). The resulting graph of \( \delta_0 \) versus energy can be seen in Fig. 2.1.

By using a conversion formula that follows from elementary relativistic mechanics we can translate our center-of-mass energy into laboratory energy (\( E_{\text{lab}} \) and \( E_{\text{cm}} \) represent only kinetic energies and thus nucleon masses must be added and subtracted where needed to be consistent with expressions for full relativistic energies):

\[
E_{\text{lab}} = \frac{(E_{\text{cm}} + 2M_{\text{nucleon}})^2}{2M_{\text{nucleon}}} - 2M_{\text{nucleon}},
\]

(2.28)

with \( M_{\text{nucleon}} = 939 \) MeV.

The even simpler non-relativistic relation \( E_{\text{lab}} = 2E_{\text{cm}} \) works well for our non-relativistic energies.
Figure 2.1: S-wave phase shift as a function of center-of-mass energy obtained by solving the radial Schrödinger equation in coordinate space.

Now we are ready to compare the calculated s-wave phase shifts $\delta_0$ to real $^3S_1$ phase shifts ($^3S_1$ state is consistent with the deuteron creation and thus should be the closest to our deuteron-energy-normalized model). The results of comparison with the phase shift from the Nijmegen partial-wave analysis (done by V.G.J. Stoks et al., [17]), PWA93, obtained online at [18] are shown in Fig. 2.2. They provide a rather good illustration of the usefulness of our spinless, single-Yukawa, one-pion approximation: the agreement is satisfactory for $E_{\text{lab}} \lesssim 25$ MeV but deteriorates drastically for higher energies. This can be understood if we remember that our model potential is purely
Figure 2.2: $S$-wave phase shift calculated in this section is compared to PWA93 $^3S_1$ phase shift from NN OnLine website [18].

attractive, which is different from realistic nucleon-nucleon interaction potentials, which usually include both long-range attractive and short-range repulsive parts.

We will analyze these results further in Section 2.3.
2.2 Elastic Scattering in Momentum Space:  

the Lippmann-Schwinger Equation

In this section we again investigate the problem of proton-neutron elastic scattering. However this time we switch to momentum space and solve the integral “twin” of the Schrödinger equation — the Lippmann-Schwinger equation. We start in Subsection 2.2.1 by deriving the $p$-space equivalent of our model potential (2.1). Then in Subsection 2.2.2 we consider the general theory of scattering and derive the Lippmann-Schwinger equation for an arbitrary local finite-range potential. The criterion for a potential to be “finite-range” is also formulated. In Subsection 2.2.3 the partial-wave expansion of this equation is introduced assuming the potential is central. We return to the model potential in Subsection 2.2.4, where we prove that the partial wave expansion of the Yukawa potential is proportional to the Legendre function of the second kind and thus set the stage for solving the Lippmann-Schwinger equation numerically. The last Subsection of this section, 2.2.5, introduces the Gauss-Legendre quadrature and subtraction numerical techniques and explains the details of the numerical solution.

2.2.1 The Yukawa Potential in $P$-Space

Equation (2.1) gives only the coordinate-space representation of the model potential or, in operator notation, it gives the potential matrix element $\langle \vec{r}'|\hat{V}|\vec{r} \rangle$, where $|\vec{r} \rangle$
is a “before interaction” state and $|\vec{r}'\rangle$ is an “after interaction” state. As our model potential is local, the interaction should be possible only if “plain” and “primed” coordinate states coincide (otherwise two spatially separate points would influence each other which violates locality). In other words, the matrix of $\hat{V}$ in the $r$-representation must be diagonal. So it should be more correctly written as:

$$\langle \vec{r}'|\hat{V}|\vec{r}\rangle = -\frac{g^2}{4\pi} \frac{e^{-m\pi r}}{r} \delta^{(3)}(\vec{r}' - \vec{r}),$$

(2.29)

where $\delta^{(3)}(\vec{x})$ is a three-dimensional Dirac delta-function.

To work with our model potential in momentum space we are interested in deriving an expression for its matrix element in the $p$-representation:

$$\langle \vec{p}'|\hat{V}|\vec{p}\rangle = \int d^3\vec{r}'d^3\vec{r} \langle \vec{p}'|\vec{r}'\rangle \langle \vec{r}'|\hat{V}|\vec{r}\rangle \langle \vec{r}|\vec{p}\rangle$$

$$= -\frac{g^2}{4\pi} \int d^3\vec{r}'d^3\vec{r} e^{-i\vec{p}'\cdot\vec{r}'} \frac{e^{-m\pi r}}{r} \delta^{(3)}(\vec{r}' - \vec{r}) e^{i\vec{p}\cdot\vec{r}}$$

$$= -\frac{g^2}{4\pi} \int d^3\vec{r} e^{i(\vec{p}' - \vec{p})\cdot\vec{r}} \frac{e^{-m\pi r}}{r}.$$  

(2.30)

Note that in this study we choose the convention

$$\langle \vec{p}|\vec{r}\rangle = e^{-i\vec{p}\cdot\vec{r}}, \quad \langle \vec{r}|\vec{p}\rangle = e^{i\vec{p}\cdot\vec{r}},$$

(2.31)

and the factor of $1/(2\pi)^3$ will be introduced each time we integrate in momentum space.

Now we can choose $\vec{p} - \vec{p}'$ as the polar axis of a spherical coordinate system and perform the integration:

$$-\frac{g^2}{2} \int_0^{+\infty} \int_{-1}^1 r^2 dr d(\cos \theta) e^{i|\vec{p}' - \vec{p}|r \cos \theta} \frac{e^{-m\pi r}}{r} =$$
\[ = -\frac{g^2}{2i|\vec{p} - \vec{p}'|} \int_0^\infty dr \, e^{-m_\pi r} (e^{i|\vec{p} - \vec{p}'|r} - e^{-i|\vec{p} - \vec{p}'|r}) \]

\[ = -\frac{g^2}{2i|\vec{p} - \vec{p}'|} \left( \frac{1}{m_\pi - i|\vec{p} - \vec{p}'|} - \frac{1}{m_\pi + i|\vec{p} - \vec{p}'|} \right). \tag{2.32} \]

By simplifying this we get what we were seeking:

\[ \langle \vec{p}' | \hat{V} | \vec{p} \rangle = -\frac{g^2}{(\vec{p}' - \vec{p})^2 + m_\pi^2}. \tag{2.33} \]

### 2.2.2 General Theory of Elastic Scattering and the Lippmann-Schwinger Equation

The chapter on scattering theory from the textbook by Iraj R. Afnan \[16\] was extensively used when preparing this and the next subsections. However we keep our conventions and our derivations are not identical to those in the textbook. We also try to pay as much as possible attention to details that could lead to confusion for those readers who encounter the general theory of elastic scattering for the first time.

Let us continue investigating the system of two particles with the Hamiltonian defined by (2.3) and the reduced mass \( \mu \) (2.2). For some time we will work with an arbitrary local finite-range potential \( \hat{V} \). “Finite-range” does not necessarily mean that in the coordinate representation it equals exactly zero for all distances larger than some finite range, it is acceptable if the potential decreases fast enough with distance. For instance, the Yukawa potential we use for most of this study is of finite range even though it never equals zero at a finite distance. The Coulomb potential is an example of an “infinite-range” potential. For the moment we adopt this semi-
intuitive explanation, leaving the strict definition of the finite-range potential concept for later.

The eigenstates of the free Hamiltonian $\hat{H}_0 = \hat{p}^2/2\mu$ are defined by the free Schrödinger equation

$$\hat{H}_0 |\phi_P\rangle = E |\phi_P\rangle = \frac{p^2}{2\mu} |\phi_P\rangle,$$

(2.34)

and coincide with the eigenstates of momentum operator $\hat{p}$:

$$|\phi_P\rangle \equiv |\vec{p}\rangle,$$

(2.35)

so in future we will not distinguish between them.

The Schrödinger equation for the full Hamiltonian defines the scattering state $|\psi\rangle$:

$$(\hat{H}_0 + \hat{V}) |\psi\rangle = E |\psi\rangle \Rightarrow (E - \hat{H}_0) |\psi\rangle = \hat{V} |\psi\rangle.$$

(2.36)

If we assume that the operator $(E - \hat{H}_0)^{-1}$ is not singular,\(^3\) we can easily find the particular solution of this equation as:

$$|\psi\rangle = \frac{1}{E - \hat{H}_0} \hat{V} |\psi\rangle,$$

(2.37)

and then by adding the free solution (2.35) of the same momentum the full solution can be found:

$$|\psi_P\rangle = |\vec{p}\rangle + \frac{1}{E - \hat{H}_0} \hat{V} |\psi_P\rangle.$$

(2.38)

\(^3\)which is actually not true for $E > 0$, as we are going to see later — we will have to redefine this operator by adding an imaginary infinitesimal in order to avoid the singularity
This expression means that the scattering state $|\psi_{\vec{p}}\rangle$ is formed by the superposition of the incident beam (plane wave) $|\vec{p}\rangle$ and $(E - \hat{H}_0)^{-1} \hat{V}|\psi_{\vec{p}}\rangle$ — the scattered wavefunction. We will now show that the second piece actually represents an outgoing spherical wave.

In the coordinate representation (2.38) can be written as:

$$\langle \vec{r} | \psi_{\vec{p}} \rangle = \langle \vec{r} | \vec{p} \rangle + \int d^3\vec{r}' \langle \vec{r} | (E - \hat{H}_0)^{-1} | \vec{r}' \rangle \langle \vec{r}' | \hat{V} | \psi_{\vec{p}} \rangle.$$  (2.39)

To go further we need to find an explicit expression of the Green’s function. Let us start by expanding it over a complete set of momentum eigenstates:

$$\langle \vec{r} | (E - \hat{H}_0)^{-1} | \vec{r}' \rangle = \int \frac{d^3\vec{k}}{(2\pi)^3} \langle \vec{r} | (E - \hat{H}_0)^{-1} | \vec{k} \rangle \langle \vec{k} | \vec{r}' \rangle$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\langle \vec{k} | \vec{r} \rangle \langle \vec{r} | \vec{k} \rangle}{E - \frac{k^2}{2\mu}}.$$  (2.40)

Here we have made use of the fact that momentum eigenstates $|\vec{k}\rangle$ are also the eigenstates of the free Hamiltonian $\hat{H}_0$ with corresponding eigenvalues $k^2/2\mu$.

Remembering that $E$ is the energy of the incident beam of momentum $\vec{p}$ and thus can be written as $E = p^2/2\mu$ and also using the relations (2.31) we can finally transform the Green’s function into a trivial integral:

$$\langle \vec{r} | (E - \hat{H}_0)^{-1} | \vec{r}' \rangle = \frac{2\mu}{(2\pi)^3} \int d^3\vec{k} \frac{e^{i\vec{k} \cdot (\vec{r} - \vec{r}')}}{p^2 - k^2}$$

$$= \frac{2\mu}{(2\pi)^2} \frac{1}{i|\vec{r} - \vec{r}'|} \int_0^\infty \frac{k^2 dk}{p^2 - k^2} \int_{-1}^1 d(cos \theta) e^{ik|\vec{r} - \vec{r}'| \cos \theta}$$

$$= \frac{2\mu}{(2\pi)^2} \frac{1}{i|\vec{r} - \vec{r}'|} \int_{-\infty}^{+\infty} k e^{ik|\vec{r} - \vec{r}'|} dk.$$  (2.41)
Because \( p^2 > 0 \), this integral actually is not defined (there are two poles \( \pm p \) at the integration path) — this is a consequence of the above-mentioned fact that the operator \( (E - \hat{H}_0)^{-1} \) is singular. We are going to deal with this problem by replacing \( E \) with an “infinitesimally complex energy”

\[
E^{\pm} \overset{\text{def}}{=} \lim_{\epsilon \to 0} E \pm i\epsilon. \tag{2.42}
\]

We are not going to explicitly write the limit sign in later discussion but everywhere where \( \epsilon \) is used it is supposed that the limit of \( \epsilon \to 0 \) is taken. Also we will not differentiate in between different kinds of infinitesimals, e. g. in

\[
E^{\pm} = E \pm i\epsilon = \frac{p^2 \pm i\epsilon'}{2\mu} = \frac{(p \pm i\epsilon'')^2}{2\mu}, \tag{2.43}
\]

the primes will be omitted.

Taking into account the introduction of \( E^{\pm} \) the integral in (2.41) is transformed into:

\[
\langle \vec{r}'|(E - \hat{H}_0)^{-1}|\vec{r}'' \rangle = \frac{2\mu}{2\pi^2 i|\vec{r}' - \vec{r}''|} \int_{-\infty}^{+\infty} \frac{ke^{i|\vec{r}' - \vec{r}''|}}{(p \pm i\epsilon)^2 - k^2} dk \tag{2.44}
\]

Again (2.44) has two poles \(^4\):

\[
k_{\pm} = \pm(p \pm i\epsilon), \tag{2.45}
\]

but now they are not at the integration path, which is illustrated in Fig. 2.3.

The figure shows how the poles are situated in the case of \( E^{+} \): the integration contour encloses the positive pole \( k_+ = p + i\epsilon \). In the opposite case the poles are symmetric by the real axis to those shown: the positive pole is then below and the

\(^4\)i. e. two for each choice of the sign in \( E^{\pm} \)
negative one is above the real axis and so the integration contour encloses the negative pole \( k_- = -p + i\epsilon \).

Now we can perform the integral in (2.44) using the Cauchy Residue Theorem:

\[
\langle \mathbf{r} | (E^\pm - \hat{H}_0)^{-1} | \mathbf{r}' \rangle = \frac{2\mu}{(2\pi)^2} \frac{1}{i|\mathbf{r} - \mathbf{r}'|} \oint_C \frac{k e^{ik|\mathbf{r} - \mathbf{r}'|}}{(p \pm i\epsilon - k)(p \pm i\epsilon + k)} \, dk = -\frac{\mu}{2\pi} \frac{e^{\pm ip|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|}.
\]

This gives us an analytic expression for the Green’s function which we can now substitute into (2.39):

\[
\langle \mathbf{r} | \psi^{(\pm)}_{\mathbf{p}} \rangle = \langle \mathbf{r} | \mathbf{p} \rangle - \frac{\mu}{2\pi} \int d^3\mathbf{r}' e^{\pm ip|\mathbf{r} - \mathbf{r}'|} \langle \mathbf{r}' | \hat{V} | \psi^{(\pm)}_{\mathbf{p}} \rangle.
\]

We see that the solution corresponding to \( E^- \) is an ingoing spherical wave and thus is physically irrelevant when scattering is considered. So from now on we will stick
to the $E^+$ case.

The most important and interesting characteristic of a scattering process is the scattering cross-section. In the case of the elastic scattering the cross-section is connected to the scattering amplitude $f(p, \theta)$ by a trivial and well-known relation:

$$d\sigma = |f(p, \theta)|^2 d\Omega.$$  \hfill (2.48)

Here $p$ is the modulus of the incident beam’s momentum, $\theta$ is the observation angle (i.e. the angle between the momentum $\vec{p}_i$ and the radius-vector of observation $\vec{r}$) and $d\Omega$ is the corresponding element of the solid angle.

Thus we will concentrate on the scattering amplitude of the elastic scattering process. Meanwhile $f(p, \theta)$ is defined via:

$$\psi(\vec{r}) \bigg|_{\text{outside the interaction region}} = e^{i\vec{p}_i \cdot \vec{r}} + f(p, \theta) e^{i p_f r} \frac{e^{i p_f r}}{r}.$$  \hfill (2.49)

Here $\vec{p}_i$ is the momentum of the incident beam with $|\vec{p}_i| = p = |\vec{p}_f|$ due to the elasticity of the process, while in turn $\vec{p}_f$ is the local momentum of the outgoing spherical wave: because we measure the scattered wave far from its origin, it can be locally approximated by a flat wave of a single definite momentum $\vec{p}_f$. Basically by assuming (2.49) we define what is a finite-range potential: no matter how complicated is the structure of $V(\vec{r})$ within the potential’s range, at large distances it wears off and leaves only this simple configuration.

Instantly one may notice a striking resemblance of (2.47) to (2.49). To get the expression for $f(p, \theta)$ by comparing them we need first to perform the integration
over $\mathbf{r}'$ in the former. However before doing this we must understand the physical meaning of vectors $\mathbf{r}$ and $\mathbf{r}'$ and, as often happens in quantum mechanics, it is hard to do this unless we think of an actual measurement. Let us imagine we shoot a beam of particles at a target. Somewhere around the target the interaction happens and then a bunch of scattered particles goes in all directions. Of course when we place a detector to measure the flux we make sure that it stands sufficiently far from the target: this way we detect only the particles that went through the interaction and are free again. In real experiments such a detector separation distance is of the meter scale — gigantic when compared to the nuclear interaction range of about a fermi, at least $10^{15}$ times larger! This provides us with a clue of what $\mathbf{r}$ is when we look again at the equation (2.49). The wavefunction we are interested in is measured “outside the interaction region” and is a function of $\mathbf{r}$, so essentially $\mathbf{r}$ can be understood as the radius-vector of the detector. On the other hand when we look at (2.47) we see that the integration is performed over the potential operator acting onto the $\langle \mathbf{r}' \rangle$ state. But we assumed that our potential is a finite-range one, i. e. there is such a finite radius $R_{\text{range}}$ that:

$$\forall |\mathbf{r}'| > R_{\text{range}} : \langle \mathbf{r}' | \hat{V} | \mathbf{r}' \rangle \cong 0.$$  

(2.50)

We used symbol $\cong$ to show that for radii larger than $R_{\text{range}}$ either $\langle \mathbf{r}' | \hat{V} | \mathbf{r}' \rangle \equiv 0$ or the value is so small that it may be neglected. Thus the potential becomes zero outside the interaction region and we may as well exclude all $\mathbf{r}'$’s with absolute values $r'$ larger than radius $R_{\text{range}}$ from the integral in (2.47). But the detector’s radius-vector
\( \vec{r} \) must have its modulus \( r \) to be much larger than \( R_{range} \) and thus:

\[
\forall \ r' \leq R_{range} : \ r' \ll r.
\]  

(2.51)

Taking into account conditions (2.50) and consequently (2.51) requires replacing the integration region in (2.47) by the sphere \( r' \leq R_{range} \) and then taking the limit of \( r \to \infty \):

\[
|\vec{r} - \vec{r}'| = r\sqrt{1 - 2 \frac{\vec{r} \cdot \vec{r}'}{r'^2} + \left( \frac{r'}{r} \right)^2} \approx r - \vec{e}_r \cdot \vec{r}' ,
\]

(2.52)

Now we can write the asymptotic part of the scattering wavefunction (2.47) — i.e. what stands at the left-hand-side of (2.49) — as:

\[
\langle \vec{r} | \psi^{(+)}_{\vec{p}_i} \rangle \approx \left. \frac{e^{i\vec{p}_i \cdot \vec{r}}}{2\pi} \right|_{r \to \infty} \int_{r' \leq R_{range}} d^3 \vec{r}' e^{-i \vec{r}' \cdot \vec{e}_r} \langle \vec{r}' | \hat{V} | \psi^{(+)}_{\vec{p}_i} \rangle .
\]

(2.53)

We added the subscript \( i \) to the momentum \( \vec{p} \) just to emphasize the fact that this is the momentum of the incident beam.

The left-hand-side of the equation (2.53) seems to depend on the quantity \( R_{range} \), however it is not so. In the beginning of this section we assumed \( \hat{V} \) to be a finite-range potential. It can be shown (see Appendix C.1) that if we define a finite-range potential so that for any \( \vec{\alpha} \) the integral

\[
\int d^3 \vec{r}' e^{-i \vec{\alpha} \cdot \vec{r}'} \langle \vec{r}' | \hat{V} \rangle
\]

converges, then the integral in (2.53) may be considered to be performed over the whole coordinate space.
The next thing we need to take care of is the exponent being integrated in (2.53). Referring to the above-mentioned property of the spherical scattered wave far from the interaction region to be approximately a plane wave — note that asymptotically, in the limit \( r \to \infty \), a spherical wave \( \text{is} \) locally a plane wave — we make use of the fact that the plane wave’s momentum \( \vec{p}_f \) is parallel to the observation radius vector \( \vec{r} \) and has modulus equal to \( \| \vec{p}_i \| = \mu \) :

\[
e^{-i \mu \vec{p}_i \cdot \vec{r}} \approx e^{-i \vec{p}_f \cdot \vec{r}} = \langle \vec{p}_f | \vec{r} \rangle.
\]

Finally we are ready to perform the integration in (2.53)

\[
\langle \vec{r} | \psi_{\vec{p}_i}^{(+)} \rangle \approx e^{i \mu \vec{p}_i \cdot \vec{r}} - \frac{\mu}{2\pi} \frac{e^{i \mu r}}{r} \int d^3 \vec{r}' \langle \vec{p}_f | \vec{r}' \rangle \langle \vec{r}' | \hat{V} | \psi_{\vec{p}_i}^{(+)} \rangle = e^{i \mu \vec{p}_i \cdot \vec{r}} - \frac{\mu}{2\pi} \frac{e^{i \mu r}}{r} \langle \vec{p}_f | \hat{V} | \psi_{\vec{p}_i}^{(+)} \rangle,
\]

and deduce from (2.49) that:

\[
f(p, \theta) \equiv -\frac{\mu}{2\pi} \langle \vec{p}_f | \hat{V} | \psi_{\vec{p}_i}^{(+)} \rangle, \quad \cos \theta = \frac{\vec{p}_i \cdot \vec{p}_f}{\mu},
\]

Because the scattering amplitude is directly related to the scattering cross-section through (2.48) (or more complicated but nevertheless direct relations in cases of inelastic scattering), we may consider \( f(p, \theta) \) to be an observable. Thus there must be a corresponding quantum-mechanical operator. Usually it is designated \( \hat{T} \) and defined by the relation:

\[
\hat{T}(E^+) | \vec{p}_i \rangle \overset{\text{def}}{=} \hat{V} | \psi_{\vec{p}_i}^{(+)} \rangle.
\]
With this definition (2.57) is transformed into

\[ f(p, \theta) = -\frac{\mu}{2\pi} \langle \vec{p}_f | \hat{T}(E^+) | \vec{p}_i \rangle, \tag{2.59} \]

and our main task of finding the scattering amplitude has become equivalent to the

task of deriving and solving the equation for the \( T \)-matrix.

We can rewrite (2.38) in our new, more correct, designations as

\[ |\psi_{\vec{p}_i}^{(+)}\rangle = |\vec{p}_i\rangle + \frac{1}{E^+ - H_0} \hat{V} |\psi_{\vec{p}_i}^{(+)}\rangle, \tag{2.60} \]

and then transform it, taking into account the definition of the \( T \)-matrix (2.58):

\[ |\psi_{\vec{p}_i}^{(+)}\rangle = |\vec{p}_i\rangle + \frac{1}{E^+ - H_0} \hat{T}(E^+) |\vec{p}_i\rangle. \tag{2.61} \]

Now let us act on this equation with the operator \( \hat{V} \) and then once again apply the
definition of the \( T \)-matrix:

\[ \hat{V} |\psi_{\vec{p}_i}^{(+)}\rangle = \hat{V} |\vec{p}_i\rangle + \hat{V} \frac{1}{E^+ - H_0} \hat{T}(E^+) |\vec{p}_i\rangle, \tag{2.62} \]

\[ \hat{T}(E^+) |\vec{p}_i\rangle = \hat{V} |\vec{p}_i\rangle + \hat{V} \frac{1}{E^+ - H_0} \hat{T}(E^+) |\vec{p}_i\rangle. \tag{2.63} \]

Since \( \vec{p}_i \) is arbitrary this equation is valid for any momentum eigenstate. And because
all momentum eigenstates together form a complete set of states, the equation (2.63)
is true for absolutely any state. This in turn means that the corresponding operator
relation is true:

\[ \hat{T}(E^+) = \hat{V} + \hat{V} \frac{1}{E^+ - H_0} \hat{T}(E^+). \tag{2.64} \]

Equation (2.64) is called the Lippmann-Schwinger equation. For a defined scat-
tering process (and thus known \( \hat{V} \) and \( H_0 \)) this equation can be solved to yield the
$T$-matrix of the process and hence its scattering amplitude and cross-section. Now it is clear why we had to write $\hat{T}$ as a function of $E^+$ when defining it in (2.58): the right-hand-side of (2.64) depends both on $E$ and the choice of boundary condition (+, i.e. the outgoing wave in our case) and so must its solution.

Note that the results of this subsection apply to any local finite-range potential: we never assumed anything more than that.

### 2.2.3 Partial-Wave Expansion

of the Lippmann-Schwinger Equation

In order to solve the Lippmann-Schwinger equation (2.64) it should be written in a certain representation. An obvious and convenient choice for us is the momentum representation:

$$
\langle \vec{p}' | \hat{T}(E^+) | \vec{p} \rangle = \langle \vec{p}' | \hat{V} | \vec{p} \rangle + \langle \vec{p}' | \hat{V} \frac{1}{E^+ - \hat{H}_0} \hat{T}(E^+) | \vec{p} \rangle,
$$

$$
= \langle \vec{p}' | \hat{V} | \vec{p} \rangle + \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\langle \vec{p}' | \hat{V} | \vec{k} \rangle \langle \vec{k} | \hat{T}(E^+) | \vec{p} \rangle}{E + i\epsilon - \frac{k^2}{2\mu}}.
$$

(2.65)

Here as usual we have made use of the completeness of the momentum eigenstates and their coincidence with the free Hamiltonian eigenstates.

In a relatively simple case of elastic scattering the energy conservation sets natural conditions of the physical relevance of momenta $\vec{p}$ and $\vec{p}'$:

$$
|\vec{p}| = |\vec{p}'| = p, \quad p = \sqrt{2\mu E}.
$$

(2.66)
Momenta satisfying (2.66) for a given energy $E$ are called on-shell momenta and a matrix element $\langle \vec{p}'|\hat{T}(E^+)|\vec{p} \rangle$ calculated with such momenta is called an on-shell $T$-matrix. Ultimately we need only on-shell momenta and $T$-matrices in order to calculate elastic scattering observables.

A three-dimensional integral equation (2.65) is obviously very hard to solve directly [19]. There is a good way of dealing with it though: we may expand all matrix elements over the eigenstates of the system’s total angular momentum and its $z$-axis projection. This method is called the partial-wave expansion. In our approximation of spinless particles the total angular momentum coincides with the orbital angular momentum, so the expansion is easy to perform:

$$\langle \vec{p}'|\hat{V}|\vec{p} \rangle = \langle \vec{e}_{p'}; p'|\hat{V}|p; \vec{e}_p \rangle$$

$$= \sum_{l'm';lm} \langle \vec{e}_{p'}|l'm'; p'|\hat{V}|p; lm \rangle \langle lm|\vec{e}_p \rangle,$$

(2.67)

$$\langle \vec{p}'|\hat{T}(E^+)|\vec{p} \rangle = \langle \vec{e}_{p'}; p'|\hat{T}(E^+)|p; \vec{e}_p \rangle$$

$$= \sum_{l'm';lm} \langle \vec{e}_{p'}|l'm'; p'|\hat{T}(E^+)|p; lm \rangle \langle lm|\vec{e}_p \rangle.$$

(2.68)

While the equation (2.65) was written for an arbitrary local finite-range potential, to go further we need to emphasize that the potential $\hat{V}$ we are working with from now on will always be assumed to have one more specific property: it will be
central, i.e. we assume:

\[
\langle \vec{r}' | \hat{\mathcal{V}} | \vec{r} \rangle = V(r) \delta^{(3)}(\vec{r}' - \vec{r}),
\]  

(2.69)

and hence that \( V \) is spherically symmetric.

There are two important simplifications following from the fact that the model potential is central. Firstly a state of definite angular momentum remains so after an interaction and thus the matrix elements of \( \hat{\mathcal{V}} \) (and consequently of \( \hat{T}(E^+) \)) must be diagonal in the orbital quantum number \( l \). Secondly all the directions of the angular momentum must be equivalent and thus both matrix elements of \( \hat{\mathcal{V}} \) and \( \hat{T}(E^+) \) must be independent of the magnetic quantum number \( m \). In other words because \([\hat{H}, \hat{L}^2] = [\hat{H}, \hat{L}_z] = 0\):

\[
\langle l'm', p'|\hat{\mathcal{V}}|p, lm \rangle = \delta_{ll'}\delta_{mm'}\langle p'|\hat{\mathcal{V}}|p \rangle,
\]

\[
\langle l'm', p'|\hat{T}(E^+)|p, lm \rangle = \delta_{ll'}\delta_{mm'}\langle p'|\hat{T}(E^+)|p \rangle.
\]  

(2.70)

As from now on we are going to work mostly in the momentum representation, we would like to introduce a more compact way of designating matrix elements:

\[
V(\vec{p}', \vec{p}) \overset{\text{def}}{=} \langle \vec{p}'|\hat{\mathcal{V}}|\vec{p} \rangle,
\]

\[
T(\vec{p}', \vec{p}; E^+) \overset{\text{def}}{=} \langle \vec{p}'|\hat{T}(E^+)|\vec{p} \rangle,
\]

\[
V_i(p', p) \overset{\text{def}}{=} \langle p'|\hat{\mathcal{V}}_i|p \rangle,
\]

\[
T_i(p', p; E^+) \overset{\text{def}}{=} \langle p'|\hat{T}_i(E^+)|p \rangle.
\]  

(2.71)

Thus, finally, taking into account both (2.70–2.71) and the fact that \( \langle \vec{e}_{\vec{p}'}, lm \rangle \) and \( \langle lm|\vec{e}_{\vec{p}} \rangle \) are actually just the familiar spherical harmonics \( Y_{lm}(\vec{e}_{\vec{p}'}) \) and \( Y_{lm}^*(\vec{e}_{\vec{p}}) \) we
can rewrite the equations (2.67–2.68) as:

\[ V(\vec{p}', \vec{p}) = \sum_{lm} Y_{lm}(\vec{e}_{\vec{p}'}) V_l(\vec{p}', p) Y_{lm}^*(\vec{e}_p), \]  

\[ T(\vec{p}', \vec{p}'; E^+) = \sum_{lm} Y_{lm}(\vec{e}_{\vec{p}'}) T_l(p', p; E^+) Y_{lm}^*(\vec{e}_p). \]  

Because both \( V_l \) and \( T_l \) are independent of \( m \), we can sum the product of the spherical harmonics over \( m \), making use of the addition theorem:

\[ \sum_m Y_{lm}(\vec{e}_{\vec{p}'}) Y_{lm}^*(\vec{e}_p) = \frac{2l + 1}{4\pi} P_l(\vec{e}_{\vec{p}'}, \vec{e}_p). \]  

Here \( P_l(x) \) is the Legendre polynomial of order \( l \). If we designate \( \vec{e}_{\vec{p}'}, \vec{e}_p \), i. e. the cosine of the angle in between \( \vec{p} \) and \( \vec{p}' \) as \( \eta \):

\[ \eta \equiv \cos(\vec{p} \cdot \vec{p}') \equiv \vec{e}_{\vec{p}'}, \vec{e}_p, \]  

then the expansions (2.72–2.73) are transformed into:

\[ V(\vec{p}', \vec{p}) = \frac{1}{4\pi} \sum_l (2l + 1) P_l(\eta) V_l(\vec{p}', p), \]  

\[ T(\vec{p}', \vec{p}'; E^+) = \frac{1}{4\pi} \sum_l (2l + 1) P_l(\eta) T_l(p', p; E^+). \]  

Thus we have obtained the partial-wave expansion of the matrix elements involved in the Lippmann-Schwinger equation. To proceed we need to expand the equation itself. In the new designations (2.65) turns into:

\[ T(\vec{p}', \vec{p}'; E^+) = V(\vec{p}', \vec{p}) + \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{V(\vec{p}', \vec{k}) T(\vec{k}, \vec{p}; E^+)}{E^+ - \frac{k^2}{2\mu}}. \]  

Though the expansions (2.76–2.77) are readily available for the left-hand-side and the first term of the right-hand-side of the equation, to deal with the integral we have
to return to (2.72–2.73):

$$\int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{V(p', \vec{k}) \ T(k', \vec{p}; E^+)}{E^+ - \frac{k^2}{2\mu}}$$

$$= \int \frac{d^3 \vec{k}}{(2\pi)^3} \sum_{l m} Y_{lm}(\vec{e}_{p'}) V_l(p', k) Y^*_{lm}(\vec{e}_k) \frac{1}{E^+ - \frac{k^2}{2\mu}} \times$$

$$\times \sum_{l' m'} Y_{l' m'}(\vec{e}_{\vec{k}}) T_{l'}(k, p; E^+) Y^*_{l m'}(\vec{e}_{\vec{p}})$$

(2.79)

We now may change the order of the summation and the integration and at the same time separate the radial and the angular parts of the integral, noting that

$$d^3 \vec{k} \equiv k^2 \, dk \, d\Omega_{\vec{k}}$$

and also that only the spherical harmonics contain the angular dependence. Here is what we obtain then:

$$\int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{V(p', \vec{k}) \ T(k', \vec{p}; E^+)}{E^+ - \frac{k^2}{2\mu}}$$

$$= \frac{1}{(2\pi)^3} \sum_{l m} \int_0^\infty \frac{k^2 \, dk}{E^+ - \frac{k^2}{2\mu}} Y_{lm}(\vec{e}_{p'}) V_l(p', k) T_l(k, p; E^+) Y^*_{lm}(\vec{e}_{\vec{p}}) \times$$

$$\times \int d\Omega_{\vec{k}} Y_{l' m'}(\vec{e}_{\vec{k}}) Y^*_{lm}(\vec{e}_{\vec{k}}).$$

(2.80)

However the spherical harmonics are orthogonal to each other and so:

$$\int d\Omega_{\vec{k}} Y_{l' m'}(\vec{e}_{\vec{k}}) Y^*_{lm}(\vec{e}_{\vec{k}}) \equiv \delta_{l l'} \delta_{m m'},$$

(2.81)

thus simplifying (2.80) into:

$$\int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{V(p', \vec{k}) \ T(k', \vec{p}; E^+)}{E^+ - \frac{k^2}{2\mu}}$$

$$= \frac{1}{(2\pi)^3} \sum_l \int_0^\infty \frac{k^2 \, dk}{E^+ - \frac{k^2}{2\mu}} V_l(p', k) T_l(k, p; E^+) \sum_m Y_{lm}(\vec{e}_{p'}) Y^*_{lm}(\vec{e}_{\vec{p}})$$

$$= \frac{1}{4\pi} \sum_l (2l + 1) P_l(\eta) \frac{1}{(2\pi)^3} \int_0^\infty \frac{k^2 \, dk \, V_l(p', k) T_l(k, p; E^+)}{E^+ - \frac{k^2}{2\mu}}. \quad (2.82)$$
While doing the last transformation we again made use of the addition theorem for spherical harmonics.

In \((2.82)\) the integral is expanded in exactly the same form the matrix elements are in \((2.76–2.77)\). Hence the substitution of these three expansions into the three-dimensional Lippmann-Schwinger equation \((2.78)\) splits it into a set of one-dimensional equations:

\[
T_l(p', p; E^+) = V_l(p', p) + \frac{1}{(2\pi)^3} \int_0^\infty \frac{k^2 dk V_l(p', k) T_l(k, p; E^+)}{E^+ - \frac{k^2}{2\mu}}.
\] (2.83)

Solving the equation \((2.83)\) is an easy computational task we are going to discuss in Subsection 2.2.5. It is clear though that numerically it can be solved only for a finite number of \(l\)'s, while the reconstruction of our primary goal \(T(\vec{p}', \vec{p}; E^+)\) with \((2.77)\) requires that we know \(T_l\) for each \(l\) of the infinite natural set. Fortunately in the case of a finite-range potential this contradiction is solved by the nature of the potential itself: as there is a maximum interaction range \(R_{\text{range}}\) the contribution of interactions with the orbital quantum number higher than some critical value \(l_{\text{max}}\) is negligibly small. In terms of classical mechanics, the interaction does not occur when the impact parameter is larger than \(R_{\text{range}}\) and thus the particle of momentum \(p\) cannot be scattered if it has the angular momentum higher than:

\[
l_{\text{max}} = pR_{\text{range}}.
\] (2.84)

Assuming that the potential has a range typical for nuclear interactions, i. e. \(R_{\text{range}} \sim 2\) fm and the maximum energy we are interested in is e. g. 500 MeV\(^5\) we

\(^5\)Actually for a pair of nucleons of this center-of-mass energy relativistic effects play quite a large
can perform a simple calculation to find the relevant $l_{\text{max}}$:

$$p = \sqrt{(E + \mu)^2 - \mu^2} \approx 848 \text{ MeV}/c \approx 4.3 \text{ fm}^{-1} \Rightarrow l_{\text{max}} \sim 10. \quad (2.85)$$

As we know the scattering of a particular partial wave is characterized by the corresponding phase shift $\delta_l$. Because all the information about scattering is contained in the $T$-matrix there must be a connection between $T_l(p', p; E^+)$ and $\delta_l$. Let us find it.

Returning to the general solution of Schrödinger equation (2.4) and taking into account (2.22) and the fact that $Y_{l0}(\theta) \propto P_l(\cos \theta)$ we can write the partial wavefunction as:

$$\psi_l(r, \theta) = \tilde{A}_l [\cos \delta_l J_l(pr) - \sin \delta_l n_l(pr)] P_l(\cos \theta), \quad (2.86)$$

with the normalization constant $\tilde{A}_l$ to be determined later. When writing (2.86) we took $m = 0$ (thus excluding the dependence on $\phi$) by assigning the $z$-axis of our coordinate system along the initial momentum direction (in this case the angular momentum is always perpendicular to the $z$-axis). If we didn’t do this we would have had to sum over a set of wavefunctions with different $m$’s, but finally would still get $\psi_l$ independent of $\phi$: because we are dealing with a central potential the same argument made when deriving (2.70) applies and $m$ is irrelevant.

When we look at the asymptotic scattering wavefunction (2.49) and remember role and so our non-relativistic approach would be a very rough approximation.
the partial wave expansion of a plane wave:

\[ e^{i \vec{p} \cdot \vec{r}} = \sum_l i^l (2l + 1) j_l(pr) P_l(\cos \theta); \]  
\[ \cos \theta \overset{\text{def}}{=} \vec{e}_p \cdot \vec{e}_r, \]  
\[ (2.87) \]

it is natural to look for the scattering wavefunction in the form:

\[ \psi = \sum_l i^l (2l + 1) A_l R_l(pr) P_l(\cos \theta), \]  
\[ (2.89) \]

with \( R_l(pr) \) being the part of \( (2.86) \) depending on \( r \) (or, more exactly, on \( pr \)) and some still unknown normalization factor \( A_l \). The spherical Bessel \( j_l(x) \) and Neumann \( n_l(x) \) functions have the following asymptotic behavior:

\[ j_l(x) \overset{r \to \infty}{\simeq} \frac{1}{x} \sin \left( x - \frac{\pi l}{2} \right); \]  
\[ n_l(x) \overset{r \to \infty}{\simeq} -\frac{1}{x} \cos \left( x - \frac{\pi l}{2} \right). \]  
\[ (2.90) \]

We can thus asymptotically transform \( (2.86) \):

\begin{align*}
R_l(pr) & \overset{r \to \infty}{\simeq} \frac{1}{pr} \sin \left( pr - \frac{\pi l}{2} + \delta_l \right) \\
& = \frac{1}{2ipr} \left[ e^{i(pr-\frac{\pi}{2}+\delta_l)} - e^{-i(pr-\frac{\pi}{2}+\delta_l)} \right] \\
& = \frac{1}{2ipr} e^{-i\delta_l} \left[ e^{2i\delta_l} e^{i(pr-\frac{\pi}{2})} - e^{-i(pr-\frac{\pi}{2})} \right] \\
& = \frac{1}{pr} e^{-i\delta_l} \left[ e^{2i\delta_l} - 1 \frac{e^{i(pr-\frac{\pi}{2})} + \sin \left( pr - \frac{\pi l}{2} \right)}{2i} \right] \\
& \overset{r \to \infty}{\simeq} e^{-i\delta_l} \left[ j_l(pr) + (-i)^l e^{2i\delta_l} - \frac{1}{2i} \frac{e^{ipr}}{pr} \right].
\end{align*}  
\[ (2.92) \]

Substituting this result into \( (2.89) \) and comparing to \( (2.49) \) we get:

\[ \psi \overset{r \to \infty}{\simeq} \sum_l i^l (2l + 1) A_l e^{-i\delta_l} \left[ j_l(pr) + (-i)^l e^{2i\delta_l} - \frac{1}{2i} \frac{e^{ipr}}{pr} \right] P_l(\cos \theta) \\
\equiv e^{i \vec{p} \cdot \vec{r}} + f(p, \theta) \frac{e^{ikr}}{r}. \]  
\[ (2.93) \]
We used the sign $\equiv$ instead of $\simeq_{r \to \infty}$ in the second equality, because there must be only one asymptotic wavefunction at $r \to \infty$ for the physically relevant $\psi$ and hence the upper row in (2.93) must be an exact expansion of the lower row. Furthermore, as Legendre polynomials are a complete set of orthogonal polynomials, such an expansion must be the only possible expansion of the asymptotic scattering wavefunction over the Legendre polynomials. In particular, the expansion coefficients of the first term (plane wave) must be the same as in (2.87) and therefore we see that $A_l \equiv e^{i\delta_l}$. Finally, by analyzing the second term we get the partial wave expansion of the scattering amplitude:

$$f(p, \theta) = \frac{1}{p} \sum_l (2l + 1) \frac{e^{2i\delta_l} - 1}{2i} P_l(\cos \theta).$$

(2.94)

It is convenient to introduce the partial scattering amplitudes:

$$f_l(p) \equiv \frac{e^{2i\delta_l} - 1}{2i} = e^{i\delta_l} \sin \delta_l.$$

(2.95)

Because $\vec{r}$ is parallel to $\vec{p}'$ (cf. the discussions preceding (2.50) and (2.55)) the relations (2.88) and (2.75) actually define the same value $\eta$. Hence we may expand the left-hand-side of (2.59) using (2.94) and the right-hand-side of (2.59) using (2.77):

$$\frac{1}{p} \sum_l (2l + 1) \frac{e^{2i\delta_l} - 1}{2i} P_l(\eta) = -\frac{\mu}{2\pi} \cdot \frac{1}{4\pi} \sum_l (2l + 1) P_l(\eta) T_l(p', p; E^+).$$

(2.96)

Again referring to the completeness of the Legendre polynomials we may equate each particular term on both sides of (2.96). Dropping the common factors and using (2.95) we obtain:

$$f_l(p) = e^{i\delta_l} \sin \delta_l = \frac{\mu p}{8\pi^2} T_l(p', p; E^+).$$

(2.97)
In fact, as we noted before, both $\vec{p}$ and $\vec{p'}$ are on-shell in the case of elastic scattering and thus $T_i(p', p; E^+)$ $\equiv$ $T_i(p, p; E^+) \equiv T_i(p, p)$ with $p = \sqrt{2\mu E}$. Note that from now on we are going to omit the excessive indication of the dependence on $E^+$ in on-shell $T$-matrix elements.

The relation we were seeking, which connects the $T$-matrix and the phase shifts of a given elastic scattering process, can now be found by taking the imaginary and real parts of (2.97) and dividing them by each other:

$$\tan \delta_l = \frac{Im[T_l(p, p)]}{Re[T_l(p, p)]}. \quad (2.98)$$

At last we have all the fundamental scattering theory we need to solve the Lippmann-Schwinger equation numerically with the Yukawa potential and then calculate the first few corresponding phase shifts.

### 2.2.4 Partial Wave Expansion of the Yukawa Potential

To solve the equation (2.83) with a known potential we need to know the functions $V_l(p', p)$ for this potential, i.e. to perform the expansion (2.76). From the theory of polynomial series we know that the expansion coefficients are given by the relation:

$$V_l(p', p) = \text{const} \int_{-1}^{1} d\eta P_l(\eta) V(p', p; \eta). \quad (2.99)$$

As we know from the previous subsection, a central potential (because of its symmetry) depends explicitly only on three scalar variables: the absolute values of the momenta and angular variable $\eta$ — which is in fact what makes the expansion (2.76)
possible. This is why we have written $V(\vec{p}', \vec{p})$ as $V(p', p; \eta)$ in (2.99). The latter variable is defined by (2.75) and thus it is obvious that $-1 \leq \eta \leq 1$, which determines the integration limits.

The constant in (2.99) can easily be determined by substituting it into (2.76) with $V(p', p; \eta) \equiv P_l'(\eta), \forall l'$:

$$V(p', p; \eta) \equiv P_l'(\eta)$$

$$= \frac{1}{4\pi} \sum_l (2l + 1) P_l(\eta) \cdot \text{const} \int_1^{-1} d\eta P_l(\eta) P_l'(\eta)$$

$$= \frac{1}{4\pi} (2l' + 1) P_l'(\eta) \cdot \text{const} \frac{2}{2l' + 1} = \frac{\text{const}}{2\pi} P_l'(\eta)$$

$$\Rightarrow \text{const} = 2\pi.$$  \hspace{1cm} (2.100)

So now we get instead of (2.99):

$$V_l(p', p) = 2\pi \int_1^{-1} d\eta P_l(\eta) V(p', p; \eta).$$  \hspace{1cm} (2.101)

Let us write down this integral for the potential we are interested in: the Yukawa potential, defined in $p$-space by (2.33). First, of course, we need to write (2.33) as an explicit function of $\eta$:

$$V(\vec{p}', \vec{p}) = -\frac{g^2}{(\vec{p}' - \vec{p}) + m_\pi^2} = -\frac{g^2}{\vec{p}'^2 + p^2 - 2pp' \eta + m_\pi^2}$$

$$= -\frac{g^2}{2pp'} \frac{1}{A - \eta},$$  \hspace{1cm} (2.102)
with:

\[ A \overset{\text{def}}{=} \frac{p'^2 + p^2 + m^2_\pi}{2pp'} . \] (2.103)

Hence the transformation integral for the model potential is given by:

\[ V_l(p', p) = -\frac{2\pi g^2}{pp'} \cdot \frac{1}{2} \int_{-1}^{1} \frac{P_l(\eta) d\eta}{A - \eta} = -\frac{2\pi g^2}{pp'} Q_l(A). \] (2.104)

Here we recognized the integral representation of the Legendre function of the second kind \( Q_l(x) \). In the cases of \( l = 0 \) and \( l = 1 \), when the Legendre polynomials are given by:

\[ P_0(x) = 1; \] (2.105)
\[ P_1(x) = x, \] (2.106)

is easy to find the explicit expressions for \( Q_l(x) \)'s as they are related to simple table integrals:

\[ Q_0(x) = \frac{1}{2} \ln \left( \frac{x + 1}{x - 1} \right); \] (2.107)
\[ Q_1(x) = \frac{x}{2} \ln \left( \frac{x + 1}{x - 1} \right) - 1. \] (2.108)

If \( x = A \) then the logarithm is defined for any \( \vec{p} \) and \( \vec{p}' \), because:

\[ \frac{A + 1}{A - 1} = \frac{p'^2 + p^2 + m^2_\pi + 2pp'}{p'^2 + p^2 + m^2_\pi - 2pp'} = \frac{(p' + p)^2 + m^2_\pi}{(p' - p)^2 + m^2_\pi} > 0. \] (2.109)

Note that if we had to deal with \(-1 < x < 1\) then the sign of the denominator in the logarithm would have had to be changed to the opposite.
To obtain expressions for higher values of \( l \) we need to remember that both the Legendre polynomials (the Legendre functions of the first kind) \( P_l(x) \) and the Legendre functions of the second kind obey the same recurrence relations:

\[
(l + 1)P_{l+1}(x) = (2l + 1) x P_l(x) - l P_{l-1}(x);
\]

\[
(l + 1)Q_{l+1}(x) = (2l + 1) x Q_l(x) - l Q_{l-1}(x).
\]

These formulae allow us to get the explicit expressions of both \( P_l(x) \) and \( Q_l(x) \) for \textit{any} \( l \). Let us apply them to get the next four functions of both kinds.

The Legendre polynomials:

\[
P_2(x) = \frac{3 x^2}{2} - \frac{1}{2}; 
\]

\[
P_3(x) = \frac{5 x^3}{2} - 3 x;
\]

\[
P_4(x) = \frac{35 x^4}{8} - \frac{15 x^2}{4} + \frac{3}{8};
\]

\[
P_5(x) = \frac{63 x^5}{8} - \frac{35 x^3}{4} + \frac{15 x}{8}.
\]

The Legendre functions of the second kind:

\[
Q_2(x) = -\frac{3 x}{2} + \frac{1}{4} (3 x^2 - 1) \ln \left( \frac{x + 1}{x - 1} \right); 
\]

\[
Q_3(x) = -\frac{5 x^2}{2} + \frac{2}{3} - \frac{3}{4} x \left( -\frac{5 x^2}{3} + 1 \right) \ln \left( \frac{x + 1}{x - 1} \right); 
\]

\[
Q_4(x) = -\frac{35 x^3}{8} + \frac{55 x}{24} + \frac{3}{16} \left( \frac{35 x^4}{3} - 10 x^2 + 1 \right) \ln \left( \frac{x + 1}{x - 1} \right); 
\]

\[
Q_5(x) = -\frac{63 x^4}{8} + \frac{49 x^2}{8} - \frac{8}{15} + P_5(x) Q_0(x).
\]
Though \((2.85)\) demonstrates that we will hardly need to include \(l > 5\) into our consideration nevertheless let us give in Appendix B.1 and then use in our computations the expressions of the Legendre functions of both kinds for \(6 \leq l \leq 10\), not only because we want to be excessively accurate, but also to make sure that the semi-classical estimate \((2.85)\) works well. Also it is almost impossible to find the expressions for \(l > 5\) in reference literature.

The formula \((2.104)\) combined with the expressions for the Legendre functions clearly provides us with \(V_l\)'s to substitute into \((2.83)\) so now we can proceed with its numerical solution. Note that in this subsection we never assumed any relation in between \(\vec{p}'\) and \(\vec{p}'_0\) so they can be off-shell as well as on-shell.

2.2.5 Numerical Solution

of the Lippmann-Schwinger Equation

There is one last obstacle that must be dealt with before we can write an actual numerical scheme for solving the equation \((2.83)\): the integral in it has a singularity at the on-shell momentum modulus \(k = k_0:\)

\[
k_0 \overset{\text{def}}{=} \sqrt{2\mu E},
\]

\((2.120)\)

and thus cannot be adequately represented by a discrete sum. A common way of overcoming such a problem is the subtraction technique: expressing the divergent integrand as a sum of two functions, one of which is regular and suitable for numerical integration and the other is divergent but simpler so that the integral of it can be
performed analytically. Clearly, the former function must be equal to the initial integrand minus the latter divergent term; this is what gives the technique its name.

In our case as the denominator of the integrand is zero at \( k_0 \) it is natural to regularize the function by adjusting the numerator so that it also has a zero at \( k_0 \):

\[
\frac{k^2 V_l(p', k) T_l(k, p; E^+)}{(k_0^2)^+ - k^2} = \frac{k^2 V_l(p', k) T_l(k, p; E^+) - k_0^2 V_l(p', k_0) T_l(k_0, p; E^+)}{k_0^2 - k^2} + \frac{k_0^2 V_l(p', k_0) T_l(k_0, p; E^+)}{(k_0^2)^+ - k^2}. \tag{2.121}
\]

We removed the \(^+\) superscript from \( k_0^2 \) in the former term because for a regular function there is no need to add an imaginary infinitesimal to the denominator as the integral of this function is defined. In the latter term only the fraction depends upon the integration variable \( k \), so that the integral to be performed analytically is very simple:

\[
\int_0^\infty \frac{dk}{(k_0^2)^+ - k^2} = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dk}{(k_0^2)^+ - k^2} = \frac{1}{2} \oint_C \frac{dk}{(k_0 + i\epsilon - k)(k_0 + i\epsilon + k)} = -\frac{i\pi}{2k_0}. \tag{2.122}
\]

Here we made use of the Cauchy Residue Theorem the same way we did in (2.46), the integration contour is shown in Fig. 2.3 (of course, \( p \) must be replaced by \( k_0 \)).

Thus the one-dimensional Lippmann-Schwinger equation now transforms into:
\[ T_i(p', p; E^+) = V_i(p', p) \]

\[ + \frac{2\mu}{(2\pi)^3} \int_0^{\infty} k^2 V_i(k', k; E^+) - k_0^2 V_i(k_0, k_0) T_i(k_0, p; E^+) \]

\[ - \frac{i\mu}{8\pi^2} k_0 V_i(k', k_0) T_i(k_0, p; E^+), \]  

(2.123)

and is ready to be discretized. For this purpose we are going to apply the method of Gaussian quadratures in its simplest modification, Gauss-Legendre. We will not give the detailed theory behind this method here (interested readers should refer to a textbook on computational mathematics, e. g. [20]). What we need to know for the purpose of utilizing the Gauss-Legendre quadratures method in this study is the following.

The most general possible representation of the integral of a function \( f(x) \) regular at an interval \([a, b]\) in the form of a finite sum is:

\[ \int_a^b f(x)dx \approx \sum_{i=1}^{N} w_i f(x_i). \]

(2.124)

Here \( N \) is the order of the method, the set of \( \{x_i\} \) is called abscissas and the associated \( \{w_i\} \) are weights. Different methods of numerical integration are, in fact, just different in the way they define the set of pairs \( \{x_i, w_i\} \).

To apply the technique of Gauss-Legendre quadratures (which is one of the most advanced and thus the most precise numerical integration methods) let us first change
the limits of integration from \([a, b]\) to \([-1, 1]\):

\[
\int_{a}^{b} f(x)dx = \frac{b - a}{2} \int_{-1}^{1} f\left(\frac{b - a}{2} \xi + \frac{a + b}{2}\right) d\xi = \int_{-1}^{1} g(\xi) d\xi. \tag{2.125}
\]

We can write:

\[
g(\xi) = g_{2N-1}(\xi) + \mathcal{O}(\xi^{2N}), \tag{2.126}
\]

or

\[
g(\xi) \approx g_{2N-1}(\xi), \tag{2.127}
\]

where \(g_{2N-1}(\xi)\) is a polynomial of order no more than \(2N - 1\). Of course, while (2.126) is true for any regular \(g(\xi)\), the approximation (2.127) does not necessarily work well for every \(g(\xi)\). Actually the exactness of relation (2.127) directly determines how well the technique of Gauss-Legendre quadratures works for the given \(f(x)\).

However, in order to find the abscissas let us suppose that equation (2.127) is an exact equality and that in such a case (2.124) is also exactly true. Let us now take such \(g(\xi)\) that \(g(\xi) = g_{N-1}(\xi) P_N(\xi)\), where \(g_{N-1}(\xi)\) is an arbitrary polynomial of order no more than \(N - 1\) and \(P_N(\xi)\) is a Legendre polynomial of order \(N\). The elementary theory of Legendre polynomials states that the integral of such \(g(\xi)\) over the interval of \([-1, 1]\) must be zero, because \(g_{N-1}(\xi)\) can be represented as a linear combination of \(P_0(\xi), P_1(\xi), \ldots, P_{N-1}(\xi)\) and \(P_N(\xi)\) by construction is orthogonal to all of them. Thus, using (2.124) as an exact relation we get:

\[
\int_{-1}^{1} g_{N-1}(\xi) P_N(\xi) d\xi = \sum_{i=1}^{N} w_i^{(\xi)} g_{N-1}(\xi_i) P_N(\xi_i). \tag{2.128}
\]
Here we designated the weights by \( w^{(\xi)}_i \) to distinguish them from the weights \( w_i \) at the interval \([a,b]\). If we take into account that by definition \( \forall i : w^{(\xi)}_i \neq 0 \) and that \( g_{N-1}(\xi) \) is an arbitrary polynomial (and thus for any set of \( N \) points \( \{\xi_i\} \) we can choose \( N \) polynomials \( g^j_{N-1}(\xi) \) such that each of them is zero at all \( \xi_i \)’s but \( \xi_j \)), the equation (2.128) can only be true if:

\[
P_N(\xi_1) = 0, \ P_N(\xi_2) = 0, \ldots, P_N(\xi_N) = 0.
\] (2.129)

or, in other words, the set of abscissas \( \{\xi_i\} \) for the Gauss-Legendre technique of order \( N \) is the set of zeros of the Legendre polynomial of order \( N \).

Now it is not difficult to compute the values of the corresponding weights. The most obvious though not the most convenient way to do this is to solve the system of linear equations:

\[
\begin{bmatrix}
P_0(\xi_1) & \ldots & P_0(\xi_N) \\
P_1(\xi_1) & \ldots & P_1(\xi_N) \\
\vdots & \ddots & \vdots \\
P_{N-1}(\xi_1) & \ldots & P_{N-1}(\xi_N)
\end{bmatrix}
\begin{bmatrix}
w^{(\xi)}_1 \\
w^{(\xi)}_2 \\
\vdots \\
w^{(\xi)}_N
\end{bmatrix}
= \begin{bmatrix}
\int_{-1}^{+1} P_0(\xi) d\xi = 2 \\
0 \\
\vdots \\
0
\end{bmatrix},
\] (2.130)

which is constructed by requiring that the set of abscissas and weights, substituted into (2.124), gives the exact values of the integrals of all the Legendre polynomials of orders from 0 to \( N - 1 \). By construction of the Legendre polynomial set, only the first of these integrals is non-zero.

Here we also provide (without proof) a much more convenient formula for the
Gauss-Legendre weights:

\[ w_i^{(\xi)} = \frac{2}{(1 - \xi_i^2) \left[ \frac{dP_N(\xi)}{d\xi} \right]_{\xi_i}}. \]  

(2.131)

It is usually convenient not to incorporate the routine calculating Gaussian abscissas and weights into the integration code. Instead it is better to use such a routine to create files of \( \{\xi_i, w_i^{(\xi)}\} \) pairs of certain orders and then refer to them from the code. As a reference we provide in Appendix B.2 a table of values of the Gauss-Legendre abscissas and weights of 96th order, which is enough to compute integrals of most regular functions with sufficient accuracy.

Of course, when we need to calculate integrals at intervals other than \([-1, 1]\), we need to rescale our abscissas and weights in accordance with (2.125):

\[ x_i = \frac{b - a}{2} \xi_i + \frac{a + b}{2}; \]  

(2.132)

\[ w_i = \frac{b - a}{2} w_i^{(\xi)}. \]  

(2.133)

Using the above formulae with the set of \( \{\xi_i, w_i^{(\xi)}\} \) pairs of order \( N_1 \) at an interval \([a, c]\) and the analogous set of order \( N_2 \) at a matching interval \([c, b]\), we effectively produce a set of \( N_1 + N_2 \) abscissas and weights for the interval \([a, b]\). Therefore formulae (2.132–2.133) allow us to construct sets of Gaussian abscissas-weights of higher order for a given interval provided we have at least one set of \( \{\xi_i, w_i^{(\xi)}\} \) pairs for the interval \([-1, 1]\) (e. g. if we have a set of order \( N \), we can produce sets of order \( 2N, 3N \) etc.). This method is better than a simple generation of the higher order set over the whole interval, because we can cover some parts of it with more Gaussian
points thus achieving better accuracy; this is useful when there are indications that
the integrand’s behavior is generally simple over most of the integration interval while
there is a fine structure at a small portion of it.

Though the equation (2.123) includes the integral over an infinite interval, we
still can apply the method of Gauss-Legendre quadratures to discretize it. One way
to do this would be to use a mapping function to transform the infinite interval of
integration into a finite one. However for the sake of clarity we are just going to replace
the infinity with a finite “cutoff” parameter. Just by simple logic we can conclude that
as we are dealing with non-relativistic particles most of integrand’s structure should
be contained at low momenta. If we use the pion mass \( m_\pi \) as a measure of momentum
it is quite safe to assume that the main region of integration lies below \( 10 m_\pi \). On the
other hand, the high momentum “tail” needs to be estimated: although the integral is
quite small at momenta higher than \( 10 m_\pi \), it still can contribute significantly when
integrated over a large interval. Hence the overall cutoff (let us designate it \( \Delta_{\text{cut}} \))
should be placed at a sufficiently high momentum (we use \( 1000 m_\pi \) for this purpose)
while most of the computational power should be concentrated at \( k \in [0, 10 m_\pi] \) Using
\( N \) Gaussian points we can distribute half of them at momenta lower than \( 10 m_\pi \) and
another half at the interval \( [10 m_\pi, 1000 m_\pi] \) as is described in the previous paragraph.

Let us assume that we have chosen an optimal set of the Gauss-Legendre abscissas
and weights \( \{k_i, w_i\} \) of order \( N \) at the interval \( [0, \Delta_{\text{cut}}] \). Before we apply it to (2.123)
there is a small correction to be done to the equation itself. When deriving (2.123)
we analytically integrated the divergent part of the integrand (2.121) over $[0, +\infty]$ because we supposed that the regular part is also going to be integrated over this infinite interval. But as now we are actually integrating the regular part over $[0, \Delta_{\text{cut}}]$,

the same interval should be used for analytical integration. In order for our previous derivation to be correct the analytic integral must be restored to the infinite one: 

$$f_{0}^{\Delta_{\text{cut}}} = f_{0}^{\Delta_{\text{cut}}} + f_{\Delta_{\text{cut}}}^{+\infty} - f_{\Delta_{\text{cut}}}^{+\infty} = f_{0}^{+\infty} - f_{\Delta_{\text{cut}}}^{+\infty}.$$ 

Therefore a new term is added to the equation (2.123):

$$- \frac{2\mu}{(2\pi)^3} k_{0}^2 V_{i}(p', k_{0}) T_{i}(k_{0}, p; E^{+}) \int_{\Delta_{\text{cut}}}^{+\infty} \frac{dk}{k_{0}^2 - k^2}$$

$$= - \frac{2\mu}{(2\pi)^3} k_{0}^2 V_{i}(p', k_{0}) T_{i}(k_{0}, p; E^{+}) \left( \frac{1}{2k_{0}} \ln \left( \frac{k + k_{0}}{k - k_{0}} \right) \right)^{+\infty}_{\Delta_{\text{cut}}}$$

$$= \frac{\mu}{(2\pi)^3} k_{0} V_{i}(p', k_{0}) T_{i}(k_{0}, p; E^{+}) \ln \left( \frac{\Delta_{\text{cut}} + k_{0}}{\Delta_{\text{cut}} - k_{0}} \right).$$

(2.134)

It transforms (2.123) into:

$$T_{i}(p', p; E^{+}) \approx V_{i}(p', p)$$

$$+ \frac{2\mu}{(2\pi)^3} \int_{0}^{\Delta_{\text{cut}}} \frac{k^2 V_{i}(p', k) T_{i}(k, p; E^{+}) - k_{0}^2 V_{i}(p', k_{0}) T_{i}(k_{0}, p; E^{+})}{k_{0}^2 - k^2}$$

$$+ \frac{\mu}{(2\pi)^3} k_{0} V_{i}(p', k_{0}) T_{i}(k_{0}, p; E^{+}) \left[ \ln \left( \frac{\Delta_{\text{cut}} + k_{0}}{\Delta_{\text{cut}} - k_{0}} \right) - i\pi \right].$$

(2.135)

Taking on-shell $p = k_{0}$ and applying (2.124) to discretize the integral we get:
\[ T_i(p', k_0 ; E^+) \approx V_i(p', k_0) \]
\[ + \frac{2\mu}{(2\pi)^3} \sum_{i=1}^{N} w_i \frac{k_i^2 V_i(p', k_i) T_i(k_i, k_0 ; E^+) - k_0^2 V_i(p', k_0) T_i(k_0, k_0 ; E^+)}{k_0^2 - k_i^2} \]
\[ + \frac{\mu}{(2\pi)^3} k_0 V_i(p', k_0) T_i(k_0, k_0 ; E^+) \left[ \ln \left( \frac{\Delta_{cut} + k_0}{\Delta_{cut} - k_0} \right) - i\pi \right]. \] (2.136)

This equation contains \( N + 2 \) independent unknown quantities: \( T_i(p', k_0 ; E^+) \), \( \{ T_i(k_i, k_0 ; E^+) \} \) and \( T_i(k_0, k_0 ; E^+) \). However the first one of them still depends upon an arbitrary variable and thus can be made equal to each of the other \( N + 1 \) matrix elements by choosing \( p' = k_0, p' = k_1, \ldots, p' = k_N \). This produces \( N + 1 \) linear equations:

\[ \forall j = 0, 1, \ldots, N : \]
\[ T_i(k_j, k_0 ; E^+) \approx V_i(k_j, k_0) \]
\[ + \frac{2\mu}{(2\pi)^3} \sum_{i=1}^{N} w_i \frac{k_i^2 V_i(k_j, k_i) T_i(k_i, k_0 ; E^+) - k_0^2 V_i(k_j, k_0) T_i(k_0, k_0 ; E^+)}{k_0^2 - k_i^2} \]
\[ + \frac{\mu}{(2\pi)^3} k_0 V_i(k_j, k_0) T_i(k_0, k_0 ; E^+) \left[ \ln \left( \frac{\Delta_{cut} + k_0}{\Delta_{cut} - k_0} \right) - i\pi \right]. \] (2.137)

Let us introduce simpler designations for the unknown \( T \)-matrix elements:

\[ T_i^{(l)} \overset{\text{def}}{=} T_i(k_i, k_0 ; E^+) , \] (2.138)

and the known potential matrix elements:

\[ V_{ij}^{(l)} \overset{\text{def}}{=} V_i(k_i, k_j) . \] (2.139)
We can now rearrange the terms of the system (2.137):

\[
\forall j = 0, 1, \ldots, N : \\
\left\{ \frac{2\mu}{(2\pi)^3} \left[ \frac{1}{2} \ln \left( \frac{\Delta_{\text{cut}} + k_0}{\Delta_{\text{cut}} - k_0} \right) - \frac{i\pi}{2} - k_0 \sum_{i=1}^{N} \frac{w_i}{k_0^2 - k_i^2} \right] k_j V_j^{(l)} - \delta_{0j} \right\} T_0^{(l)} \\
+ \sum_{i=1}^{N} \left[ w_i \frac{2\mu}{(2\pi)^3} \frac{k_i^2 V_j^{(l)}}{k_0^2 - k_i^2} - \delta_{ij} \right] T_i^{(l)} \approx -V_j^{(l)}. \tag{2.140}
\]

and by introducing:

\[
Q \overset{\text{def}}{=} \frac{1}{2} \ln \left( \frac{\Delta_{\text{cut}} + k_0}{\Delta_{\text{cut}} - k_0} \right) - k_0 \sum_{i=1}^{N} \frac{w_i}{k_0^2 - k_i^2} - \frac{i\pi}{2}, \tag{2.141}
\]

rewrite it in a matrix form:

\[
\left( \begin{array}{cccc}
Q k_0 V_{00}^{(l)} & w_1 \frac{k_0^2 k_1^2 V_{01}^{(l)}}{k_0^2 - k_1^2} & \cdots & w_N \frac{k_0^2 k_N^2 V_{0N}^{(l)}}{k_0^2 - k_N^2} \\
Q k_0 V_{10}^{(l)} & w_1 \frac{k_0^2 k_1^2 V_{11}^{(l)}}{k_0^2 - k_1^2} & \cdots & w_N \frac{k_0^2 k_N^2 V_{1N}^{(l)}}{k_0^2 - k_N^2} \\
\vdots & \vdots & \ddots & \vdots \\
Q k_0 V_{N0}^{(l)} & w_1 \frac{k_0^2 k_1^2 V_{N1}^{(l)}}{k_0^2 - k_1^2} & \cdots & w_N \frac{k_0^2 k_N^2 V_{NN}^{(l)}}{k_0^2 - k_N^2} \\
\end{array} \right) \left[ \begin{array}{c}
T_0^{(l)} \\
T_1^{(l)} \\
\vdots \\
T_N^{(l)}
\end{array} \right] \approx \left[ \begin{array}{c}
V_{00}^{(l)} \\
V_{10}^{(l)} \\
\vdots \\
V_{N0}^{(l)}
\end{array} \right]. \tag{2.142}
\]

where \([1]\) stands for the matrix unity:

\[
[1] \overset{\text{def}}{=} \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix}. \tag{2.143}
\]
The matrix equation (2.142) is what we were looking for: both the matrix in
the left-hand-side and the column in the right-hand-side can be calculated by trivial
algebra. Then we can solve for \( \{ T_i^{(l)} \} \) by common means (all modern programming
languages have library functions that solve linear matrix equations). For the purpose
of this chapter it is enough just to know \( T_0^{(l)} = T_i(k_0, k_0) \) — the on-shell \( T \)-matrix
element, related to the phase shifts and thus to the cross-section through the relation
(2.98). But our efforts in getting other — half-on-shell at the Gaussian abscissas —
matrix elements did not go in vain: we do need them (combined with the equation
(2.136) enabling us to compute the half-on-shell \( T \)-matrix for any momentum) for the
calculation of the bremsstrahlung amplitude in the next chapters.

The actual code performing the computations described in this subsection was
written in Java. Unfortunately it is too long and cumbersome to be presented here.
Instead in Appendix D we present the plan of the code which provides much better
illustration than the text of the code itself. The results obtained by running this code
are presented and discussed in the next section of this chapter.
2.3 The Born Approximation

and Discussion of Results

This section is the conclusion to the spinless “proton-neutron”\textsuperscript{6} elastic scattering problem. In Subsection 2.3.1 we introduce the Born approximation, giving us a valuable reference to compare our results with. We use it in Subsection 2.3.2, where we finally present the computed phase shifts and cross-sections and then briefly analyze and discuss them.

2.3.1 The Born Approximation

Before we start applying the analytical and computational methods developed in this chapter to more sophisticated problems in the next parts of this study we need to make sure that these methods produce reasonable results and provide sufficient accuracy. Unfortunately the only independent reference source we have so far is the s-wave phase shift $\delta_0$ computed in Subsection 2.1.3. It is clear that at least some estimates need to be made for other partial waves. The simplest way to do this is to utilize the Born approximation.

As this method is explained in any standard textbook on quantum mechanics, here we just provide the minimal information necessary for our purposes.

The first-order Born approximation treats the potential $\hat{V}$ as a small perturbation

\textsuperscript{6}or, more exactly, “nucleon-nucleon”, because in this problem we were dealing exclusively with the nuclear interaction and thus there is no difference if any of the particles is charged
of the free Hamiltonian $\hat{H}_0 = \hat{p}^2/2\mu$ so that first-order perturbation theory can be applied to solve the Schrödinger equation. This approach obviously works well only in cases of weak potentials and high energies.

The task of calculating the total elastic cross-section for the Yukawa potential (2.1) in the first-order Born approximation can be easily performed analytically (see e. g. the textbook by L. D. Landau and E. M. Lifshits [15]) and gives the following result:

$$\sigma_{\text{Born}} = \frac{16\pi}{m^2} \left( \frac{\alpha_{\pi}\mu}{m_{\pi}} \right)^2 \frac{1}{4p^2/m^2 + 1},$$

(2.144)

$$\alpha_{\pi} \overset{\text{def}}{=} \frac{g^2}{4\pi}.$$  

(2.145)

However by the way it is derived this formula — we will refer to it as the perturbative Born cross-section — obviously includes all partial waves and in our case the Born approximation of each individual partial wave phase shift would be more useful.

Let us return to the operator Lippmann-Schwinger equation (2.64). If we infinitely substitute it into itself, i. e. replace $\hat{T}$ in the right-hand-side by the whole right-hand-side and then repeat this procedure infinitely, then we obtain:

$$\hat{T}(E^+) = \hat{V} - i\hat{V}\hat{G}_0(E^+)\hat{V} - \hat{V}\hat{G}_0(E^+)\hat{V}\hat{G}_0(E^+)\hat{V} + \ldots,$$

(2.146)

$$\hat{G}_0(E^+) \overset{\text{def}}{=} \frac{i}{E^+ - \hat{H}_0}.$$  

(2.147)

Only the first term must be kept in the first order by $\hat{V}$. Hence:

$$\hat{T}(E^+) = \hat{V} \Rightarrow T_l(p, p) = V_l(p, p).$$

(2.148)
As the $T$-matrix is now purely real we cannot directly apply relations (2.97–2.98) to get the phase shift: they simply give $\delta_l \equiv 0$ which is not true. Let us however remember that we directly observe neither the phase shift nor the scattering amplitude but the cross-section, which is given by:

$$\sigma = \sum_{l=0}^{\infty} \sigma_l = \frac{4\pi}{p^2} \sum_{l=0}^{\infty} (2l + 1) |f_l(p)|^2 = \frac{4\pi}{p^2} \sum_{l=0}^{\infty} (2l + 1) \sin^2 \delta_l.$$  \hfill (2.149)

Of course we may envision an experiment involving only a certain partial wave and thus reducing the sum just to one term. A physically reasonable approximation therefore must preserve the relations involving $\sigma_l \propto |f_l(p)|^2$ even if it does not preserve relations for $f_l(p)$. This means that the formula (2.97) is true if squared:

$$|f_l(p)|^2 = \sin^2 \delta_l = \left( \frac{\mu p}{8\pi^2} \right)^2 |T_l(p,p)|^2 = \left( \frac{\mu p}{8\pi^2} \right)^2 V_l^2(p,p). \hfill (2.150)$$

We can now take the square root of it\(^7\):

$$\sin \delta_l = -\frac{\mu p}{8\pi^2} V_l(p,p). \hfill (2.151)$$

This expression is very easy to incorporate into our code because we have there already a routine calculating $V_l(p,p)$. Now we have the reference we were looking for and are ready to present the results of the phase shift computation.

### 2.3.2 Proton-Neutron Elastic Scattering: the Results

Fig. 2.4 presents the s-wave phase-shift computed by our Lippmann-Schwinger code compared to $\delta_0$ obtained by solving the Schrödinger equation and the Born approxi-

\(^7\)We choose the negative sign because attractive potentials are always negative while the sine of the phase shift for such potentials must be positive.
Figure 2.4: $S$-wave phase shifts $\delta_0$ obtained by solving the Lippmann-Schwinger and the Schrödinger equations (they coincide at this scale), compared to their Born approximation.

The relative difference of the Lippmann-Schwinger and Born $\delta_0$'s is shown in Fig. 2.6. When we look at this graph, the natural question emerges: if (and at what energies) the Born approximation gets sufficiently close to the exact solution?
Figure 2.5: Relative difference \( \frac{\delta_0^{(LSch)} - \delta_0^{(Sch)}}{\delta_0^{(Sch)}} \cdot 100\% \) of the s-wave phase shifts from the Lippmann-Schwinger and the Schrödinger equations.

Figure 2.6: Relative difference \( \frac{|\delta_0^{(LSch)} - \delta_0^{(B)}|}{\delta_0^{(LSch)}} \cdot 100\% \) of the s-wave phase shifts from the Lippmann-Schwinger equation and the Born approximation.
Unfortunately with our choice of computation parameters we cannot correctly use the Lippmann-Schwinger code for energies larger than 2 GeV (corresponding to $\Delta_{mid} = 10m_\pi$). However the Schrödinger code can be run with any $E$. The computation for high energies was performed, it shows that the Born approximation constantly gets closer to the exact solution, as expected. The relative difference is lower than 1% at energies larger than 5 GeV.

The next set of figures — Fig. 2.7 through Fig. 2.18 — presents the Lippmann-Schwinger and Born phase shifts for $l = 1$ through 6. There are a few important conclusions that can be made by looking at them.

- For $l \geq 1$ and fixed energy the higher the angular momentum, the lower is the corresponding phase shift. This supports our argument that the contribution of high $l$ partial waves is fairly small.

- Starting with $l = 3$ the phase-shift increases with energy. Thus, again as we argued, the contribution of higher partial waves is larger for larger energies.

- The higher the angular momentum, the closer the Born approximation is to the exact solution.

Also we notice a serious problem: a numerical instability arises due to the insufficient computational accuracy of Java. Starting from $l = 7$ it effectively destroys the Lippmann-Schwinger solution, that is why only the Born approximation is shown for higher angular momenta in Fig. 2.19.
Figure 2.7: $P$-wave phase shift $\delta_1$ obtained by solving the Lippmann-Schwinger equation, compared to its Born approximation.

Figure 2.8: Relative difference $\left| \frac{\delta_1^{(LSch)} - \delta_1^{(B)}}{\delta_1^{(LSch)}} \right| \cdot 100\%$ of the $p$-wave phase shifts from the Lippmann-Schwinger equation and the Born approximation.
Figure 2.9: $D$-wave phase shift $\delta_2$ obtained by solving the Lippmann-Schwinger equation, compared to its Born approximation.

Figure 2.10: Relative difference $\left| \frac{\delta_2^{(LSch)} - \delta_2^{(B)}}{\delta_2^{(LSch)}} \right|^\cdot 100\%$ of the $d$-wave phase shifts from the Lippmann-Schwinger equation and the Born approximation.
Figure 2.11: $F$-wave phase shift $\delta_3$ obtained by solving the Lippmann-Schwinger equation, compared to its Born approximation.

Figure 2.12: Relative difference $|\delta_3^{(LSch)} - \delta_3^{(B)}|$. 100% of the $f$-wave phase shifts from the Lippmann-Schwinger equation and the Born approximation.
Figure 2.13: $G$-wave phase shift $\delta_4$ obtained by solving the Lippmann-Schwinger equation, compared to its Born approximation.

Figure 2.14: Relative difference \( \left| \frac{\delta_4^{(LSch)} - \delta_4^{(B)}}{\delta_4^{(LSch)}} \right| \cdot 100\% \) of the $g$-wave phase shifts from the Lippmann-Schwinger equation and the Born approximation.
Figure 2.15: $H$-wave phase shift $\delta_5$ obtained by solving the Lippmann-Schwinger equation, compared to its Born approximation. The curves almost coincide. Several dots climbing up at low energies are due to a numerical instability.

Figure 2.16: Relative difference $\left| \frac{\delta_5^{(LSch)} - \delta_5^{(B)}}{\delta_5^{(LSch)}} \right| \cdot 100\%$ of the $h$-wave phase shifts from the Lippmann-Schwinger equation and the Born approximation.
Figure 2.17: $i$-wave phase shift $\delta_6$ obtained by solving the Lippmann-Schwinger equation, compared to its Born approximation. The numerical instability is a serious problem for partial waves starting from $l = 6$.

Figure 2.18: Relative difference $\frac{|\delta_6^{(LSch)} - \delta_6^{(B)}|}{\delta_6^{(LSch)}} \cdot 100\%$ of the $i$-wave phase shifts from the Lippmann-Schwinger equation and the Born approximation. Starting from the $i$-wave the relative difference drops below 1%. Thus we can actually use the Born approximation with a very good accuracy for $l = 6$ and higher, especially when we take into account that the contribution of high angular momentum partial waves into the total scattering process is small for $E_{cm} \leq 500$ MeV.
If we wanted to perform a very accurate computation we would have to rewrite the code in Fortran. However taking into account the above conclusions we can replace the \( T \)-matrix by the \( V \)-matrix both avoiding the instability and saving computational time: the small contribution of higher partial waves combined with the fact that the Born approximation works well for large \( l \) ensures sufficient accuracy.

Let us see how this approach works in practice. Fig. 2.20 shows the \( pn \) elastic scattering cross-section\(^8\) calculated by using formula (2.149) with the sum truncated at \( l = 5 \) and the Lippmann-Schwinger phase shifts \( \delta_0 - \delta_5 \),\(^9\) compared to the per-

\(^8\)The unit of measurement we use for cross-sections is the barn, \( 1 \text{ barn} \overset{\text{def}}{=} 10^{-24} \text{ cm}^2 = 100 \text{ fm}^2 \).

\(^9\)shown in figures 2.4, 2.7, 2.9, 2.11, 2.13 and 2.15
Figure 2.20: $NN$ elastic scattering cross-section. The upper curve is constructed from the first six partial waves obtained by solving the Lippmann-Schwinger equation. The lower curve is the Born approximation result (2.144), which includes all partial waves.

First of all, let us estimate how much we are losing due to neglecting higher partial waves. Fig. 2.21 shows the relative difference of the Born approximation result (2.144), which includes an infinite number of partial waves, and the cross-section calculated using (2.149) with the sum truncated at $l = 10$ and the Born approximation phase shifts $\delta_0 - \delta_{10}^{10}$, calculated with (2.151). We see that neglecting partial waves of $l > 10$ works fine up to 300 MeV and then we start to lose accuracy.

---

10shown in figures 2.4, 2.7, 2.9, 2.11, 2.13, 2.15, 2.17 and 2.19
Figure 2.21: The relative difference $\frac{|\sigma_{\text{Born}}^{(10)} - \sigma_{\text{Born}}^{(\text{pert})}|}{\sigma_{\text{Born}}^{(\text{pert})}}$ of the perturbative Born approximation of the cross-section and the cross-section constructed from the first eleven (up to $l = 10$) Born approximations of the phase shifts. The figure shows that the partial wave Born approximation is equivalent to the original perturbative version of it. The rise at high energies is due to the larger contribution of partial waves with $l \geq 11$.

Finally, we compare two different ways to calculate the exact cross-section in Fig. 2.22. It is obvious that adding the higher $l$ Born terms to the results found for $0 \leq l \leq 5$ using the Lippmann-Schwinger equation is a useful thing to do, especially for $E > 200$ MeV. Presumably the agreement for $E \geq 300$ MeV could be improved by adding Born approximation $\delta$’s for $l > 10$ to the Lippmann-Schwinger evaluation.

After going through all these results we must conclude that the methods developed in this chapter are consistent with theory and perform well. We are ready now to consider the main problem of this study: $pn$ bremsstrahlung.
Figure 2.22: The relative difference \[
\frac{\sigma^{(\text{pert})}_{\text{Born}} - \sigma^{(\text{pert})}_{\text{LSch}}}{\sigma^{(\text{pert})}_{\text{Born}}} \]
of the perturbative Born approximation of the cross-section and the cross-section constructed from the first six (up to \( l = 5 \)) phase shifts obtained by solving the Lippmann-Schwinger equation is compared to a similar relative difference where the “original” Lippmann-Schwinger cross-section is improved by adding the Born approximation of the phase shifts of the next five (\( l = 6 \) through 10) partial waves (“improved” curve). We see that for higher energies (where the perturbative Born approximation is closer to the real cross-section than the exact solution limited to the first few partial waves) the improvement is quite dramatic.
Chapter 3

The Scattering Amplitude

of Proton-Neutron Bremsstrahlung

In the previous chapter we performed a detailed investigation of spinless $pm$ elastic scattering. Though we were calling one of the particles “proton”, assuming it is charged, and the other “neutron”, assuming it is not, the presence of an electric charge played no role in our theory so far: the nuclear interaction has a different nature from electromagnetism. To a good approximation it is charge independent.

This is going to change in this chapter, where we are going to consider $pn$ bremsstrahlung: a process involving both nuclear and electromagnetic interactions, namely electromagnetic radiation by a proton being scattered on a neutron. In classical physics a charged particle involved in any interaction which results in its acceleration emits electromagnetic waves. Quantum physics does not require a charged particle
to radiate whenever it interacts with other particles or fields but such an event is still likely to occur. As electromagnetism is much weaker than the strong nuclear interaction, we will work to first order in the electric charge $e$, i.e. consider emission of just one photon during the $NN$ interaction.

In Section 3.1 we find an analytic expression for the $pn$ bremsstrahlung amplitude based on the nucleon-nucleon scattering theory of Chapter 2 and the basic laws of quantum physics.

Then in Section 3.2 this expression is analyzed and a numerical algorithm for its computation is developed.

### 3.1 Proton-Neutron Bremsstrahlung Theory

In this section we build the spinless “proton-neutron” bremsstrahlung theory. Our main task is to find the amplitude of this process. We start in Subsection 3.1.1 by introducing the Green’s function for a time-independent Hamiltonian and developing its basic properties. However, the bremsstrahlung Hamiltonian is time-dependent, consisting of the nucleon-nucleon interaction part we worked with in Chapter 2 and the radiation part due to the photon emission by the proton. The latter contains a time dependence, which we obtain explicitly in Subsection 3.1.2, assuming that a photon of energy $\omega$ is emitted. As the results of Subsection 3.1.1 cannot be applied to this case, the bremsstrahlung Green’s function needs a special derivation which we perform in Subsection 3.1.3. Then we employ it to construct the bremsstrahlung
scattering state in Subsection 3.1.4. Using the coordinate representation of the free Green’s function in an arbitrary inertial frame of reference found in Appendix C.2, we proceed with deducing the bremsstrahlung amplitude from the \( r \)-representation of the scattering state in Subsection 3.1.5.

### 3.1.1 The Time-Independent Green’s Function \( \hat{G}(E^+) \)

At the end of the previous chapter we introduced an operator \( \hat{G}_0(E^+) \) (2.147) for the free Hamiltonian \( \hat{H}_0 \). It, and its generalization for the full Hamiltonian (i.e. the sum of the free Hamiltonian and some time-independent potential), are going to play a crucial role in the derivation of the bremsstrahlung amplitude. So in this subsection we are going to make ourselves familiar with this operator and derive some of its properties.

For the sake of convenience starting from this chapter we are going to replace the reduced mass \( \mu \) by \( M_{\text{nucleon}}/2 \) and omit the subscript “nucleon” in the nucleon mass designation. In an arbitrary frame of reference the free Hamiltonian is thus written as:

\[
\hat{H}_0 = \frac{\hat{P}^2}{4M} + \frac{\hat{\rho}^2}{M},
\]

with \( \hat{P} \) being the total momentum of both particles (clearly by definition \( \hat{P} = 0 \) in the center-of-mass frame) and \( \hat{\rho} \) being the relative momentum of the two particles\(^1\).

---

\(^1\)Note that in the previous chapter, when we mostly considered the case of \( \hat{P} = 0 \), \( \hat{\rho} \) was referred to as “the system momentum”.
A similar operator $\hat{G}(E^+)$:

$$\hat{G}(E^+) = \frac{i}{E^+ - \hat{H}},$$

(3.2)

can be introduced for the full Hamiltonian:

$$\hat{H} = \hat{H}_0 + \hat{V}.$$  

(3.3)

It follows from the discussion in Subsection 2.2.2 that $\hat{G}_0(E^+)$ is the Green's function of the Schrödinger equation (factor $i$ is introduced just to be consistent with the common definition). However even deeper physical meaning of the operators $\hat{G}_0(E^+)$ and $\hat{G}(E^+)$ is uncovered if we redefine them as:

$$\hat{G}_0(E^+) \overset{\text{def}}{=} \int_0^{+\infty} dt e^{iE^+ t} \hat{U}_0(t, 0);$$

(3.4)

$$\hat{G}(E^+) \overset{\text{def}}{=} \int_0^{+\infty} dt e^{iE^+ t} \hat{U}(t, 0),$$

(3.5)

where $\hat{U}_0(t', t)$ and $\hat{U}(t', t)$ are the time-evolution operators correspondingly for the free and the full Hamiltonians:

$$\hat{U}_0(t', t) \overset{\text{def}}{=} e^{-i(t' - t)\hat{H}_0};$$

(3.6)

$$\hat{U}(t', t) \overset{\text{def}}{=} e^{-i(t' - t)\hat{H}},$$

(3.7)

and parameters $t$ and $t'$ can be understood as the moments of time for the initial and the final states of the quantum-mechanical system’s evolution.

By performing the integration it is easy to see that the two definitions are equivalent. But the second definition is obviously more fundamental, it shows that the
Green’s function $\hat{G}(E^+)$ is directly related to the evolution of the system. Note that we always mark the system’s energy $E$ with the superscript $^+$ to emphasize the fact that it is defined with an imaginary infinitesimal $E^+ = E + i\epsilon$. As it has been shown in Subsection 2.2.2 such an infinitesimal is essential for a mathematically correct definition of the Green’s function, otherwise it would diverge.

Let us find the direct operator relation in between the Green’s functions $\hat{G}(E^+)$ and $\hat{G}_0(E^+)$. The difference of the corresponding inverse operators is equal to:

$$\hat{G}_0^{-1}(E^+) - \hat{G}^{-1}(E^+) = -i(E^+ - \hat{H}_0 - E^+ + \hat{H}) = -i\hat{V}. \quad (3.8)$$

By left-multiplying equation (3.8) with $\hat{G}_0(E^+)$ and right-multiplying it with $\hat{G}(E^+)$ we get:

$$\hat{G}_0(E^+)\hat{G}_0^{-1}(E^+)\hat{G}(E^+) - \hat{G}_0(E^+)\hat{G}^{-1}(E^+)\hat{G}(E^+)$$

$$= \hat{G}(E^+) - \hat{G}_0(E^+) = -i\hat{G}_0(E^+)\hat{V}\hat{G}(E^+), \quad (3.9)$$

which can be rewritten as:

$$\hat{G}(E^+) = \hat{G}_0(E^+) + \hat{G}_0(E^+)(-i\hat{V})\hat{G}(E^+). \quad (3.10)$$

This relation can actually be used to derive the Lippmann-Schwinger equation (2.64) as follows. Applying the Schrödinger equation we can express the scattering state $|\psi_{\vec{p}}\rangle$ through the unperturbed plane wave $|\vec{p}\rangle$ (cf. (2.38)):

$$|\psi_{\vec{p}}\rangle = |\vec{p}\rangle - i\hat{G}_0(E^+)^*\hat{V}|\psi_{\vec{p}}\rangle. \quad (3.11)$$
By definition (2.58) \( \hat{T}(E^+)|\vec{p}^\prime \rangle = \hat{V}|\psi_{\vec{p}^\prime} \rangle \) and so (3.11) yields:

\[
\hat{T}(E^+)|\psi_{\vec{p}^\prime} \rangle = \hat{T}(E^+)|\vec{p}^\prime \rangle - i \hat{T}(E^+ \hat{G}_0(E^+) \hat{V}|\psi_{\vec{p}^\prime} \rangle \\
\Rightarrow \hat{T}(E^+)|\psi_{\vec{p}^\prime} \rangle = \hat{V}|\psi_{\vec{p}^\prime} \rangle - i \hat{T}(E^+ \hat{G}_0(E^+) \hat{V}|\psi_{\vec{p}^\prime} \rangle, \tag{3.12}
\]

which is, of course, true for any \(|\psi_{\vec{p}^\prime} \rangle \) and thus:

\[
\hat{T}(E^+) = \hat{V} - i \hat{T}(E^+) \hat{G}_0(E^+) \hat{V} \tag{3.13}
\]

— a variation of the Lippmann-Schwinger equation (2.64). If we multiply it with \( \hat{G}(E^+) \) and use (3.10) we get:

\[
\hat{T}(E^+) \hat{G}(E^+) = \hat{V} \hat{G}(E^+) - i \hat{T}(E^+) \hat{G}_0(E^+) \hat{V} \hat{G}(E^+) = \hat{T}(E^+) \hat{G}_0(E^+) - i \hat{T}(E^+) \hat{G}_0(E^+) \hat{V} \hat{G}(E^+), \tag{3.14}
\]

leading to:

\[
\hat{V} \hat{G}(E^+) = \hat{T}(E^+) \hat{G}_0(E^+). \tag{3.15}
\]

Thus (3.10) is equivalent to:

\[
\hat{G}(E^+) = \hat{G}_0(E^+) - i \hat{G}_0(E^+) \hat{T}(E^+) \hat{G}_0(E^+). \tag{3.16}
\]

The same way we did in (2.146) we can write (3.10) in the form of an infinite expansion:

\[
\hat{G}(E^+) = \hat{G}_0(E^+) - i \hat{G}_0(E^+) \hat{V} \hat{G}_0(E^+) - \hat{G}_0(E^+) \hat{V} \hat{G}_0(E^+) \hat{V} \hat{G}_0(E^+) - \hat{G}_0(E^+) \hat{V} \hat{G}_0(E^+) \hat{V} \hat{G}_0(E^+) \hat{V} \hat{G}_0(E^+) + \ldots \tag{3.17}
\]
For particular initial and final momentum states it turns into:

\[
\langle \vec{p}' | \hat{G}(E^+) | \vec{p} \rangle = \langle \vec{p}' | \hat{G}_0(E^+) | \vec{p} \rangle \\
+ \int \frac{d^3 \vec{q}_1}{(2\pi)^3} \frac{d^3 \vec{q}_2}{(2\pi)^3} \langle \vec{p}' | \hat{G}_0(E^+) | \vec{q}_1 \rangle \langle \vec{q}_1 | (-i \hat{V}) | \vec{q}_2 \rangle \langle \vec{q}_2 | \hat{G}_0(E^+) | \vec{p} \rangle \\
+ \int \frac{d^3 \vec{q}_1}{(2\pi)^3} \frac{d^3 \vec{q}_2}{(2\pi)^3} \frac{d^3 \vec{q}_3}{(2\pi)^3} \frac{d^3 \vec{q}_4}{(2\pi)^3} \langle \vec{p}' | \hat{G}_0(E^+) | \vec{q}_1 \rangle \langle \vec{q}_1 | (-i \hat{V}) \ldots | \vec{p} \rangle \\
+ \int \frac{d^3 \vec{q}_1}{(2\pi)^3} \frac{d^3 \vec{q}_2}{(2\pi)^3} \frac{d^3 \vec{q}_3}{(2\pi)^3} \frac{d^3 \vec{q}_4}{(2\pi)^3} \frac{d^3 \vec{q}_5}{(2\pi)^3} \frac{d^3 \vec{q}_6}{(2\pi)^3} \langle \vec{p}' \ldots | \vec{p} \rangle \\
+ \ldots
\]

(3.18)

Making use of the fact that momentum eigenstates are also free Hamiltonian (and thus free Green’s function) eigenstates and applying the orthonormality relations:

\[
\langle \vec{p} | \vec{q} \rangle = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}),
\]

(3.19)

we can simplify (3.18):

\[
\langle \vec{p}' | \hat{G}(E^+) | \vec{p} \rangle = \frac{(2\pi)^3 i \delta^{(3)}(\vec{p}' - \vec{p})}{E^+ - \frac{p^2}{M}} \\
+ \frac{i}{E - \frac{p^2}{M}} \langle \vec{p}' | (-i \hat{V}) | \vec{p} \rangle \frac{i}{E - \frac{p^2}{M}} \\
+ \frac{i}{E - \frac{p^2}{M}} \left[ \int \frac{d^3 \vec{q}}{(2\pi)^3} \langle \vec{p}' | (-i \hat{V}) | \vec{q} \rangle \frac{i}{E - \frac{p^2}{M}} \langle \vec{q} | (-i \hat{V}) | \vec{p} \rangle \right] \frac{i}{E - \frac{p^2}{M}} \\
+ \ldots
\]

(3.20)

This expansion allows us to introduce a set of Feynman-like rules (see Fig. 3.1) and therefore (3.20) can be represented in a diagrammatic form (see Fig. 3.2).
3.1.2 Radiation Hamiltonian

In Section 2.2 we investigated systems with the Hamiltonian (3.3) and different kinds of potential $\hat{V}$: arbitrary local finite-range, its limitation to a central one, and finally our model Yukawa. But as we were dealing with elastic scattering, in every case by assigning a constant energy $E$ to the system we assumed that the potential is stationary, i. e. time-independent. In contrast, bremsstrahlung involves photon emission, an
electromagnetic process. Its Hamiltonian must be time-dependent: although in this subsection we are going to derive the exact form of this time dependence, the truth of the statement is obvious just from the fact that at the moment of time \( t = -\infty \) we have just two-particle system while at \( t = +\infty \) we have already three-particle system.

Let us consider a free particle, propagating with momentum \( \vec{p}_1 \). If it is charged then the electromagnetic interaction can be included into the particle’s Hamiltonian by performing a usual quantum-mechanical procedure of minimal substitution:

\[
\hat{p}_1 \rightarrow \hat{p}_1 - e\hat{A}.
\]  

(3.21)

Here \( e \) is the charge of the particle, in our case we assume it to be a proton and thus \( e \) is the positive elementary charge. \( \hat{A} \) is the operator of electromagnetic field’s vector potential \( \vec{A} \), for bremsstrahlung it is the potential of the photon — a plane electromagnetic wave — being emitted.

Thus the new Hamiltonian is:

\[
\hat{H} = \frac{(\hat{p}_1 - e\hat{A})^2}{2M} \approx \frac{\hat{p}_1^2}{2M} - \frac{e}{2M}(\hat{p}_1\hat{A} + \hat{A}\hat{p}_1).
\]  

(3.22)

Here and in all the following study we take into account only the first-order perturbation in the electromagnetic interaction, disregarding the terms proportional to \( e^2 \) and higher powers of \( e \). Also one should note that in general \( \hat{A} \) and \( \hat{p}_1 \) do not commute with each other.

As we know from the classical electromagnetic theory, \( \vec{A} \) can be defined in different physically equivalent gauges. Here electrostatic potential — the time-like component
of the electromagnetic 4-vector potential — is assumed to be zero since we choose the Coulomb gauge, \( A_0 \equiv 0 \). In this gauge the photon’s momentum \( \vec{k}_\omega \) is always perpendicular to \( \vec{A} \):

\[
\vec{A} \cdot \vec{k}_\omega = 0.
\]  

(3.23)

The operator \( \hat{A} \) can be expanded over all possible photon momenta (this also is sometimes called “second quantization”):

\[
\hat{A} (\vec{r}_1, t) = \int \frac{d^3 \vec{k}'_\omega}{(2\pi)^3 2\omega'} \left[ \vec{\epsilon}_{\omega'}^* \hat{a} (\vec{k}'_\omega) e^{-i\omega' t + i\vec{k}'_\omega \cdot \vec{r}_1} + \vec{\epsilon}_{\omega'} \hat{a}^\dagger (\vec{k}'_\omega) e^{i\omega' t - i\vec{k}'_\omega \cdot \vec{r}_1} \right].
\]  

(3.24)

Here \( \omega' \overset{\text{def}}{=} |\vec{k}'_\omega| \) is photon’s energy, \( \vec{\epsilon}_{\omega'} \) and \( \vec{\epsilon}_{\omega'}^* \) are complex polarization vectors\(^2\). The radius-vector \( \vec{r}_1 \) is related to the charged particle, thus we are assuming that the proton is in a certain coordinate state \( |\vec{r}_1\rangle \). Operators \( \hat{a} (\vec{k}'_\omega) \) and \( \hat{a}^\dagger (\vec{k}'_\omega) \) are correspondingly the destruction and creation operators of a photon state with momentum \( \vec{k}'_\omega \). They are defined via:

\[
\hat{a} (\vec{k}'_\omega) |\vec{k}'_\omega\rangle = |0\rangle; \tag{3.25}
\]

\[
\hat{a}^\dagger (\vec{k}'_\omega) |0\rangle = |\vec{k}'_\omega\rangle; \tag{3.26}
\]

\[
\hat{a} (\vec{k}'_\omega) |0\rangle = 0, \tag{3.27}
\]

where \( |0\rangle \) is the vacuum state, and 0 in the right-hand-side of the last equation is the number zero (which is not the same as the vacuum state).

\(^2\)At the moment we do not need to know anything about their properties, except that, as follows from (3.23), they are always perpendicular to \( \vec{k}'_\omega \).
Let us consider radiation of a photon of momentum $\vec{k}_\omega$. Before the emission there was the vacuum state $|0\rangle$, which is transformed into $|\vec{k}_\omega\rangle$ after the emission. Hence the matrix element to consider is:

$$
\langle \vec{k}_\omega \left| \hat{A} \right| 0 \rangle = \int \frac{d^3k'}{(2\pi)^3 2\omega'} \epsilon_{\omega'} \langle \vec{k}_\omega \left| \vec{k}'_{\omega'} \right\rangle e^{i\omega't - ik'_\omega \vec{r}_1} = \epsilon_{\omega} e^{i\omega t - ik_\omega \vec{r}_1},
$$

(3.28)

where we made use of the destruction and creation operator properties (3.25–3.26) and of the relativistic norm of photon momentum states:

$$
\langle \vec{k}_\omega \left| \vec{k}'_{\omega'} \right\rangle = (2\pi)^3 2\omega \delta^{(3)}(\vec{k}_\omega - \vec{k}'_{\omega'}).
$$

(3.29)

This equation together with the particular form of “second quantization” expansion (3.24) should be understood as declarations rather than as relations, because both are matters of convention.

Let us designate the time-independent part of the left-hand-side of (3.28) with:

$$
\langle \vec{k}_\omega \left| \hat{A}^{(ti)} \right| 0 \rangle \overset{\text{def}}{=} \epsilon_{\omega} e^{-ik_\omega \vec{r}_1}.
$$

(3.30)

Thus the time dependence of (3.28) can be explicitly expressed as:

$$
\langle \vec{k}_\omega \left| \hat{A} \right| 0 \rangle = \langle \vec{k}_\omega \left| \hat{A}^{(ti)} \right| 0 \rangle e^{i\omega t}.
$$

(3.31)

We can rewrite this expression in a more correct form remembering that from the very beginning we are assuming that the proton is in a certain coordinate state and
this is where the coordinates $\vec{r}_1$ are coming from:

$$\langle \vec{k}_\omega | \hat{A} | 0 \rangle \otimes | \vec{r}_1 \rangle = | \vec{r}_1 \rangle \hat{\epsilon}_\omega e^{-i\vec{k}_\omega \cdot \vec{r}_1} e^{i\omega t}. \quad (3.32)$$

Now we can finally start evaluating the matrix element of the electromagnetic part of the Hamiltonian (3.22).

To do this we should consider the states of both the photon and the charged (radiating) particle, because the operator $\hat{\vec{A}}$ acts only on the photon states while $\hat{\vec{p}}_1$ acts only on the nucleon states.

Since we are assuming one-photon emission, we may write the state of our system before the emission as $| \vec{p}_1 \rangle \otimes | 0 \rangle$, and after it as $| \vec{p}'_1 \rangle \otimes | \vec{k}_\omega \rangle$. Evaluating the products of operators $\hat{\vec{A}}$ and $\hat{\vec{p}}_1$, appearing in (3.22) on the states yields the following.

For $\hat{\vec{p}}_1 \hat{\vec{A}}$:

$$\langle \vec{p}'_1 | \otimes \langle \vec{k}_\omega | \hat{\vec{p}}_1 \hat{\vec{A}} | \vec{p}_1 \rangle \otimes | 0 \rangle$$

$$= \int d^3\vec{r}_1 d^3\vec{r}'_1 \langle \vec{p}_1 | \vec{r}_1 \rangle \langle \vec{r}_1 | \hat{\vec{p}}_1 \rangle \otimes \langle \vec{k}_\omega | \hat{\vec{A}} | 0 \rangle \otimes | \vec{r}_1 \rangle \langle \vec{r}_1 | \vec{p}_1 \rangle$$

$$= \hat{\epsilon}_\omega e^{i\omega t} \int d^3\vec{r}_1 d^3\vec{r}'_1 e^{-i\vec{p}'_1 \cdot \vec{r}_1} \delta^{(3)}(\vec{r}'_1 - \vec{r}_1) e^{-i\vec{k}_\omega \cdot \vec{r}_1} e^{i\vec{p}_1 \cdot \vec{r}_1}$$

$$= \hat{\epsilon}_\omega e^{i\omega t} \int d^3\vec{r}_1 d^3\vec{r}'_1 e^{-i\vec{p}'_1 \cdot \vec{r}_1} \delta^{(3)}(\vec{r}'_1 - \vec{r}_1) e^{i(\vec{k}_\omega - \vec{p}_1) \cdot \vec{r}_1}$$

$$= -\hat{\epsilon}_\omega (\vec{k}_\omega - \vec{p}_1) e^{i\omega t} \int d^3\vec{r}_1 e^{-i(\vec{p}'_1 + \vec{k}_\omega - \vec{p}_1) \cdot \vec{r}_1}$$

$$= (\vec{p}_1 - \vec{p}'_1 - \vec{k}_\omega) \delta^{(3)}(\vec{p}_1 - \vec{p}'_1 - \vec{k}_\omega). \quad (3.33)$$

We replaced $\vec{p}_1 - \vec{k}_\omega$ by $\vec{p}'_1$ using momentum conservation. Note that we have obtained the momentum conservation law for the radiation process without invoking
it externally.

Similarly, for $\hat{A} \hat{p}_1$:

$$\langle \hat{p}_1' | \otimes \langle \vec{k}_\omega | \hat{A} \hat{p}_1 | \vec{p}_1 \rangle \otimes | 0 \rangle = (\vec{e}_\omega \vec{p}_1) e^{i \omega t} \delta^{(3)}(\vec{p}_1 - \vec{p}_1' - \vec{k}_\omega).$$  \hspace{1cm} (3.34)

Finally, substituting (3.33–3.34) into (3.22) we get the radiation Hamiltonian matrix element and can pull out explicitly its time dependence:

$$\langle \vec{k}_\omega | \hat{H}_{EM} | 0 \rangle \overset{\text{def}}{=} \langle \vec{k}_\omega | \hat{H}^{(ti)}_{EM} | 0 \rangle \cdot e^{i \omega t};$$

$$\langle \vec{p}_1' | \otimes \langle \vec{k}_\omega | \hat{H}^{(ti)}_{EM} | \vec{p}_1 \rangle \otimes | 0 \rangle = -\frac{e}{2M} \vec{e}_\omega (\vec{p}_1 + \vec{p}_1') \delta^{(3)}(\vec{p}_1 - \vec{p}_1' - \vec{k}_\omega).$$

Here we have introduced a designation $\hat{H}^{(ti)}_{EM}$ for the time-independent part of the radiation Hamiltonian. As $\vec{p}/M = \vec{v}$ is the velocity of the particle, $\frac{\vec{p}_1 + \vec{p}_1'}{2M} = \vec{v}_1$ is an average velocity of the proton during the photon emission. On the other hand, the proton’s dipole moment is $e \vec{r}_1$ and thus $e \vec{v}_1$ is the actual dipole moment change rate at the moment of emission:

$$\frac{e}{2M} (\vec{p}_1 + \vec{p}_1') \equiv \left( \frac{d \vec{d}}{dt} \right)_{t = \text{moment of emission}} \overset{\text{def}}{=} \left( \frac{d \vec{d}}{dt} \right)_{\text{rad}}.$$  \hspace{1cm} (3.37)

The velocity changes abruptly with the radiation of a photon and so the dipole moment $\vec{d}$ is discontinuous as a function of time $t$. That is why in order for the derivative in (3.37) to be defined we must take the half-sum of the left derivative just before the radiation $\vec{p}/M$ with the right derivative just after the radiation $\vec{p}'/M$ — this is
where the averaging comes from. Now we can rewrite (3.36) as:

\[
\langle \vec{p}'_1 | \otimes \langle \vec{k}_\omega | \hat{H}_{EM}^{(ti)} | \vec{p}_1 \rangle | 0 \rangle = -\vec{\varepsilon}_\omega \left( \frac{d\vec{d}}{dt} \right)_{rad} ^{\delta^{(3)}}(\vec{p}_1 - \vec{p}'_1 - \vec{k}_\omega). \tag{3.38}
\]

The physical meaning of this expression is clear. The left-hand-side of (3.38) is in fact the time-independent transition matrix element. Applying the perturbation theory and using Fermi’s Golden Rule for a single transition in this case we can get an expression for the radiation probability in this case:

\[
W = 2\pi |\langle \vec{p}'_1 | \otimes \langle \vec{k}_\omega | \hat{H}_{EM}^{(ti)} | \vec{p}_1 \rangle | 0 \rangle|^2. \tag{3.39}
\]

If we sum over all possible photon polarizations, and then integrate over all possible directions of the photon momenta (see [21] for details), we get:

\[
W = \frac{\omega}{3\pi} \left| \frac{d\vec{d}}{dt} \right|^{2}_{rad}. \tag{3.40}
\]

Then by integrating over all possible directions of \(d\vec{d}/dt\) and multiplying by the photon energy \(\omega\) we obtain the radiation intensity:

\[
I_\omega = \frac{4\omega^2}{3} \langle \left| \frac{d\vec{d}}{dt} \right|^{2}_{rad} \}. \tag{3.41}
\]

In the classical case in order to emit radiation of frequency \(\omega\) the dipole moment must change with the same frequency. Therefore \(d\vec{d}/dt = \omega \vec{d}\) and so the relation (3.41) agrees with the classical Larmor formula [22]:

\[
I_\omega = \frac{4\omega^4}{3} |\vec{d}|^2. \tag{3.42}
\]

The results of Chapter 2 and this subsection provide us with all the components necessary to start evaluating the bremsstrahlung amplitude, but before we go further an important remark should be made.
Figure 3.3: An example of a bremsstrahlung process being excluded from our model by assuming charge conservation of each of the nucleons and thus not considering charged $NN$ interaction mediators. Here the photon is emitted by a charged virtual mediating particle.

Of course, we know from electrodynamics that only an accelerating particle radiates, so some another external force must be present. In the case of $pn$ bremsstrahlung we are investigating this is the nuclear interaction with the other particle we assume to be non-charged. But it is obvious that neither $\hat{A}$ nor $\hat{p}_1$ act on the state vectors of the second particle, if it is neutral. Thus the states of the second particle are not essential for the electromagnetic Hamiltonian matrix element evaluation and so in this subsection we are not going to include them explicitly into the initial and final states of our system.

In fact, the assumption of one of the particles being neutral both before and after photon emission is not a trivial consequence of considering the proton-neutron interaction. If the nucleon-nucleon interaction is mediated by a charged particle (like $\pi^+$ or $\pi^-$, which are always included in realistic $NN$ interaction models), then bremsstrahlung can occur through a process like the one shown in Fig. 3.3. By
assuming charge conservation for each of the interacting nucleons we exclude such processes from our consideration and therefore constrain our bremsstrahlung model.

### 3.1.3 The Green’s Function for Bremsstrahlung

The full Hamiltonian for bremsstrahlung consists of three parts: the free Hamiltonian \( \hat{H}_0 (3.1) \), the nucleon-nucleon interaction potential \( \hat{V} \) (in our model it is represented by Yukawa potential (2.1) and (2.33), but until the end of this section we just assume it to be local, finite-range and central, and use the results of Section 2.2 based on their conditions) and the radiation Hamiltonian \( \hat{H}_{EM} (3.35-3.36) \), which contains all of the time-dependence. Thus, for a known emitted photon energy \( \omega \) we can write total \( \hat{H}(t) \) as:

\[
\hat{H}(t) = \hat{H}_0 + \hat{V} + \hat{H}_{EM}^{(ti)} e^{i\omega t}. \tag{3.43}
\]

For convenience we will designate the time-independent “NN elastic scattering” part of this Hamiltonian by \( \hat{H}_{NN} \):

\[
\hat{H}_{NN} \overset{\text{def}}{=} \hat{H}_0 + \hat{V}. \tag{3.44}
\]

A time-evolution operator \( \hat{U}(t', t) \) can be constructed for the Hamiltonian (3.43) with a relation analogous to (3.7), but more complicated:

\[
\hat{U}(t', t) = e^{-i \int_{t}^{t'} \hat{H}(\tau) d\tau}. \tag{3.45}
\]

To get this relation we need to consider infinitesimal propagations from time \( t \) to \( t + dt \). The Hamiltonian for each of these propagations may be assumed to be
time-independent with value $\hat{H}(t)$ and thus $\hat{U}(t + dt, t)$ is given by (3.7). Applying these elementary evolutions in consequence we can obtain the finite time-evolution operator:

$$\hat{U}(t', t) = \lim_{\Delta t \to 0} \left[ e^{-i\Delta t \hat{H}(t)} e^{-i\Delta t \hat{H}(t + \Delta t)} e^{-i\Delta t \hat{H}(t + 2\Delta t)} \ldots e^{-i\Delta t \hat{H}(t')} \right].$$

(3.46)

This expression is the same as (3.45) if Hamiltonians for different moments of time commute and we can apply the formula $e^{\hat{A}} e^{\hat{B}} = e^{\hat{A} + \hat{B}}$. Fortunately, this is true for the bremsstrahlung Hamiltonian (3.43): the time dependence is given solely by $e^{i\omega t}$ that commutes with any time-independent operator. For an arbitrary time-dependence of the Hamiltonian the expression (3.46) cannot be simplified in this way.

The task of finding the bremsstrahlung amplitude is solved if we find an explicit expression for the time-evolution operator $\hat{U}(t', t)$. Then we would simply evolve the initial plane wave into the final scattering state. This is essentially what we do, only we express the procedure via the Green’s function.

To start with let us observe that the evolution operator $\hat{U}(t', t)$ itself obeys the Schrödinger equation:

$$i \frac{\partial \hat{U}}{\partial t'} = \hat{H}(t')\hat{U}(t', t).$$

(3.47)

Let us introduce another operator related to $\hat{U}(t', t)$:

$$\tilde{\hat{U}}(t', t) \overset{\text{def}}{=} e^{i\hat{H}_{\text{NN}} t'} \hat{U}(t', t).$$

(3.48)

\[3\text{In fact, this is usually taken as one of the two basic axioms of quantum mechanics.}\]
Then:

\[ \hat{U}(t', t) = e^{-i\hat{H}_{NN}t'} \hat{U}(t', t). \] (3.49)

We can write a Schrödinger-like equation for this operator:

\[
i \frac{\partial \hat{U}}{\partial t'} = -\hat{H}_{NN} e^{i\hat{H}_{NN}t'} \hat{U}(t', t) + e^{i\hat{H}_{NN}t'} \hat{H}(t') \hat{U}(t', t)
= e^{i\hat{H}_{NN}t'} [\hat{H}(t') - \hat{H}_{NN}] \hat{U}(t', t)
= e^{i\hat{H}_{NN}t'} \hat{H}_{EM}^{(ti)} e^{i\omega t'} e^{-i\hat{H}_{NN}t'} \hat{U}(t', t). \] (3.50)

This differential equation can be solved to get the following relation (we take into account \( \hat{\tilde{U}}(t, t) = \hat{U}(t, t) = 1 \)):

\[
\hat{\tilde{U}}(t', t) = e^{-i \int_{t}^{t'} \hat{\tilde{H}}(\tau) d\tau};
\] (3.51)

\[
\hat{\tilde{H}}(\tau) \overset{\text{def}}{=} e^{i\hat{H}_{NN}\tau} \hat{H}_{EM}^{(ti)} e^{-i\hat{H}_{NN}\tau} e^{i\omega \tau}. \] (3.52)

Because, as we can see from (3.36), \( \hat{H}_{EM}^{(ti)} \propto e \), to the order in \( e \) to which we are working we have:

\[
\hat{\tilde{U}}(t', t) \approx 1 - i \int_{t}^{t'} \hat{\tilde{H}}(\tau) d\tau.
\] (3.53)

and hence:

\[
\hat{U}(t', t) \approx e^{-i\hat{H}_{NN}t'} - i e^{-i\hat{H}_{NN}t'} \int_{t}^{t'} \hat{\tilde{H}}(\tau) d\tau.
\] (3.54)

By the Green’s function’s definition (3.5):

\[
\hat{G}(E^+) \overset{\text{def}}{=} \int_{0}^{+\infty} d\tau e^{iE^+\tau} \hat{U}(\tau, 0).
\] (3.55)
Combining it with (3.54) we get:

\[
\hat{G}(E^+) = \int_0^{+\infty} d\tau e^{iE^+\tau} \left[ e^{-i\hat{H}_{NN}\tau} - i e^{-i\hat{H}_{NN}\tau} \int_0^{\tau} \hat{H}(\tau') d\tau' \right]
\]

\[
= \int_0^{+\infty} d\tau e^{i(E^+ - \hat{H}_{NN}\tau)} - i \int_0^{+\infty} d\tau e^{iE^+\tau} e^{-i\hat{H}_{NN}\tau} \int_0^{\tau} \hat{H}(\tau') d\tau'.
\]

(3.56)

The first term of (3.56) is just the Green’s function for the nucleon-nucleon Hamiltonian \(\hat{H}_{NN}\):

\[
\int_0^{+\infty} d\tau e^{i(E^+ - \hat{H}_{NN}\tau)} = \frac{i}{E^+ - \hat{H}_{NN}} \text{def} \hat{G}_{NN}(E^+).
\]

(3.57)

And the second, double integral term of (3.56) can be transformed as follows:

\[
-i \int_0^{+\infty} d\tau e^{iE^+\tau} e^{-i\hat{H}_{NN}\tau} \int_0^{\tau} \hat{H}(\tau') d\tau'
\]

\[
= -i \int_0^{+\infty} d\tau e^{iE^+\tau} e^{-i\hat{H}_{NN}\tau} \int_0^{\tau} e^{i\hat{H}_{NN}\tau'} \hat{H}^{(t)}_{EM} e^{-i\hat{H}_{NN}\tau'} e^{i\omega \tau'} d\tau'
\]

\[
= -i \int_\Omega d\tau d\tau' e^{iE^+\tau} e^{-i\hat{H}_{NN}\tau} e^{i\hat{H}_{NN}\tau'} \hat{H}^{(t)}_{EM} e^{-i\hat{H}_{NN}\tau'} e^{i\omega \tau'}.
\]

(3.58)

The area of integration \(\Omega\) is shown in Fig. 3.4 (a).

Let us introduce new integration variables \(t_1 = \tau - \tau'\) and \(t_2 = \tau\). It is easy to check that the Jacobian of such a variable transformation is to 1. Fig. 3.4 (b) shows that now both integration variables must change from 0 to \(+\infty\) and therefore (3.58) can be rewritten as:

\[
-i \int_\Omega dt_1 dt_2 e^{iE^+(t_1 + t_2)} e^{-i\hat{H}_{NN}(t_1 + t_2)} e^{i\hat{H}_{NN}t_2} \hat{H}^{(t)}_{EM} e^{-i\hat{H}_{NN}t_2} e^{i\omega t_2}
\]

\[
= -i \int_0^{+\infty} dt_1 e^{iE^+t_1} e^{-i\hat{H}_{NN}t_1} \int_0^{+\infty} dt_2 e^{iE^+t_2} \hat{H}^{(t)}_{EM} e^{-i\hat{H}_{NN}t_2} e^{i\omega t_2}
\]
Finally we can combine \((3.57)\) and \((3.59)\) to obtain the expression for the bremsstrahlung Green’s function in terms of time-independent operators:

\[
\hat{G}(E^+) = \hat{G}_{NN}(E^+) + \hat{G}_{NN}(E^+) \left(-i\hat{H}^{(ti)}_{EM}\right) \hat{G}_{NN}(E^+ + \omega). \tag{3.60}
\]

Now we are ready to build the scattering state with it.
3.1.4 Building the Scattering State with the Green’s Function

It is easy to find the scattering state for a time-independent Hamiltonian. We already did it in Subsection 2.2.2. Let us return to the equation (2.38) and rewrite it using the free Green’s function (2.147):

\[
|\psi_{\vec{p}}\rangle = |\vec{p}\rangle + \hat{G}_0(E^+) (-i\hat{V}) |\psi_{\vec{p}}\rangle.
\]

(3.61)

Note that as we are considering a non-radiative process, here \( p = \sqrt{ME} \).

Let us postulate that:

\[
|\Psi\rangle \overset{\text{def}}{=} \hat{G}_{NN}(E^+) \hat{G}_0^{-1}(E^+) |\vec{p}\rangle,
\]

(3.62)

is a solution of (3.61). To check this we substitute it in and obtain:

\[
\hat{G}_{NN}(E^+) \hat{G}_0^{-1}(E^+) |\vec{p}\rangle = |\vec{p}\rangle + \hat{G}_0(E^+) (-i\hat{V}) \hat{G}_{NN}(E^+) \hat{G}_0^{-1}(E^+) |\vec{p}\rangle.
\]

(3.63)

Writing \( |\vec{p}\rangle \) as \( \hat{G}_0(E^+) \hat{G}_0^{-1}(E^+) |\vec{p}\rangle \), we see that (3.63) is actually the basic relation for the Green’s functions of a time-independent Hamiltonian (3.10), acting on the state \( \hat{G}_0^{-1}(E^+) |\vec{p}\rangle \). Thus the equation (3.63) is true and (3.62) is the expression of the scattering state through the Green’s function of the corresponding Hamiltonian.

It may look like there is a problem with this solution: \( \hat{G}_0^{-1}(E^+) |\vec{p}\rangle = 0 \) for on-shell \( p \). However, it is easy to check (e. g. using the relation (3.16)) that in this case \( \hat{G}_{NN}(E^+) |\vec{p}\rangle \) tends to infinity and thus there is no contradiction.
In the case of a time-dependent Hamiltonian we may expect the relation of the scattering state and the Green’s function to be similar to (3.62) indeed: with the time-dependent Hamiltonian in the form (3.43) we must get (3.62) in the limit $\omega \to 0$.

In order to derive the desired relation, let us return to the evolution operator $\hat{U}(t', t)$ (defined correspondingly by (3.45) in the case of the bremsstrahlung Hamiltonian (3.43)). The main property of $\hat{U}(t', t)$ is its ability to transform the “initial” state of a quantum-mechanical system occurring at the moment of time $t$ (let us designate it $|\psi_i(t)\rangle$) into the “final” state, occurring at the moment of time $t' > t$ (we will designate it $|\psi_f(t')\rangle$):

$$|\psi_f(t')\rangle = \hat{U}(t', t) |\psi_i(t)\rangle. \quad (3.64)$$

States $|\psi_i(t)\rangle$ and $|\psi_f(t')\rangle$ are the “full” states including explicit time dependence in them. We know that for a state of energy $E$ this time dependence is expressed with the factor $e^{-iE(t-t_0)}$, if we take $t_0$ as the initial moment of time — we will choose it to be 0 for convenience. Thus to get back to the “space” states (like $|\vec{p}\rangle$, $|\vec{r}\rangle$ etc.) we are usually dealing with, we need to multiply every $|\psi(t)\rangle$ with a factor of $e^{iEt}$ and integrate over all possible time evolutions. In fact, the states of definite time are just Fourier-like images of the states of definite energy, so we can get from the former to the latter by applying the inverse transformation:

$$|\Psi_\omega\rangle = \int_0^{+\infty} dt \, e^{iE+\epsilon t} \hat{U}(t, 0) |\psi_i(0)\rangle = \hat{G}(E^+) |\psi_i(0)\rangle. \quad (3.65)$$

Here we added an imaginary infinitesimal to $E$ as usual, to make the integral well defined. We see that the Green’s function is actually the operator that produces the
scattering state out of the initial state $|\psi_i(0)\rangle$. But what is this initial state $|\psi_i(0)\rangle$?

Applying the same logic we can deduce that if allowed to evolve freely — in which case no photon emission would occur and thus the energy would remain $E + \omega$ — this state would produce just a plane wave $|\vec{p}\rangle$ with $|\vec{p}| = \sqrt{M(E + \omega)}$. Thus:

$$
|\vec{p}\rangle = \int_{t_0}^{+\infty} dt e^{i(E^+ + \omega)(t - t_0)} \hat{U}_0(t, t_0) |\psi_i(t_0)\rangle
= \int_{0}^{+\infty} dt e^{i(E^+ + \omega)t} \hat{U}_0(t, 0) |\psi_i(0)\rangle = \hat{G}_0(E^+ + \omega)|\psi_i(0)\rangle.
$$

(3.66)

This equation can be inverted to obtain $|\psi_i(0)\rangle$. Substituting it into (3.65) we get the relation for the scattering state we were seeking:

$$
|\Psi_\omega\rangle = \hat{G}(E^+)\hat{G}_0^{-1}(E^+ + \omega)|\vec{p}\rangle,
$$

(3.67)

This is indeed analogous to (3.62) and transforms into it in the limit of $\omega \to 0$.

On the other hand, let us recall that Subsection 3.1.3 has resulted in the relation (3.60). Combining it and (3.67) with the equation (3.16), which is true for any Green’s function of a time-independent Hamiltonian, and so applies to $\hat{G}_{NN}(E^+)$, we get:

$$
|\Psi_\omega\rangle = \left[ \hat{G}_{NN}(E^+) + \hat{G}_{NN}(E^+) (-i\hat{H}_{EM}^{(ti)} \hat{G}_{NN}(E^+ + \omega)) \right] \hat{G}_0^{-1}(E^+ + \omega)|\vec{p}\rangle
$$

$$
= \left[ \hat{G}_0(E^+) - i\hat{G}_0(E^+)^\dagger \hat{T}_{NN}(E^+) \hat{G}_0(E^+) + \hat{G}_0(E^+) (-i\hat{H}_{EM}^{(ti)} \hat{G}_{NN}(E^+ + \omega)) \right. \\
- \left. i\hat{G}_0(E^+)\hat{T}_{NN}(E^+)\hat{G}_0(E^+) (-i\hat{H}_{EM}^{(ti)} \hat{G}_{NN}(E^+ + \omega)) \right] \hat{G}_0^{-1}(E^+ + \omega) |\vec{p}\rangle
$$

$$
= \hat{G}_0(E^+) \hat{G}_0^{-1}(E^+ + \omega) |\vec{p}\rangle - i\hat{G}_0(E^+)\hat{T}_{NN}(E^+)\hat{G}_0(E^+) \hat{G}_0^{-1}(E^+ + \omega) |\vec{p}\rangle
$$
\[- \hat{G}_0(E^+) \left[ i \hat{H}_{EM}^{(ti)} + \hat{T}_{NN}(E^+) \hat{G}_0(E^+) \hat{H}_{EM}^{(ti)} \right] \hat{G}_{NN}(E^+ + \omega) \hat{G}_0^{-1}(E^+ + \omega) |\vec{p}\rangle. \]

(3.68)

Here \( \hat{T}_{NN}(E^+) \) is the nucleon-nucleon \( T \)-matrix operator of energy \( E \), as introduced in Section 2.2.

We should explicitly note here that we are solving the bremsstrahlung problem in the center-of-mass frame for the initial nucleon-nucleon system. Thus the initial state is just a plane wave \( |\vec{p}\rangle \) with \( |\vec{p}| = \sqrt{M(E + \omega)} \). Hence:

\[
\hat{G}_0(E^+) \hat{G}_0^{-1}(E^+ + \omega) |\vec{p}\rangle = \frac{1}{E^+ - (E + \omega)} \left[ E^+ + \omega - (E + \omega) \right] = 0. \tag{3.69}
\]

As \( \hat{G}_0(E^+) \hat{T}_{NN}(E^+) |\vec{p}\rangle \) is also finite, equation (3.69) means that both the first and the second terms of (3.68) are zero. These two terms vanish because we postulated the emission of a photon of energy \( \omega \) in our scattering process by assuming the Hamiltonian to be in the form (3.43). The first of them corresponds to an unperturbed plane wave propagation and the second to pure nucleon-nucleon interaction; both are impossible by definition if the radiation event occurs. The same reasoning cannot be applied to the other terms: similar to the non-radiative scattered state (3.62), one can check using (3.16) that \( \hat{G}_{NN}(E^+ + \omega) \) tends to infinity at \( |\vec{p}\rangle \) when \( p = \sqrt{M(E + \omega)} \).
3.1.5 Derivation of the Bremsstrahlung Amplitude
by Computing the Scattering State
in the Coordinate Representation

As we noted before in Subsection 2.2.2, the scattering state configuration \((2.49)\) is common for a two-body elastic scattering with any finite-range potential. It also defines the scattering amplitude \(f(p, \theta)\) but assumes that we are working in the center of mass frame of reference. Due to Galilean invariance, the scattering state should not change when viewed from another inertial frame. Thus we now generalize the scattering amplitude definition can be generalized to an arbitrary frame where the system has total momentum \(\vec{P}\) and the center-of-mass’s radius-vector is \(\vec{R}\). We do this by multiplying by the corresponding bracket \(\langle \vec{R} | \vec{P} \rangle\) (similarly to Appendix C.3):

\[
\langle \vec{r} \vec{R} | \psi_{\text{scat}} \rangle \simeq \langle \vec{r} \vec{R} | \vec{p} \vec{P} \rangle + f(\lambda) \frac{e^{ipr}}{r} e^{i\vec{P} \cdot \vec{R}}.
\] (3.70)

Here \(\lambda\) represents all the parameters the scattering amplitude depends on: the cylindrical symmetry of the elastic scattering is broken, because a photon of momentum \(\vec{k}_\omega\) and polarization \(\vec{e}_\omega\) is radiated and thus the \(f\) is no longer function of \(p\) and \(\theta\) alone. Note that all these parameters \((p, \theta, \vec{k}_\omega, \text{ and } \vec{e}_\omega)\) are still defined in the center-of-mass frame.

Now, in our case, because we assume the emission of a photon of fixed energy, there cannot be a plane wave term in the scattering state \(|\Psi_\omega\rangle\). To produce an equation analogous to (3.70) we would have to integrate over all possible \(\omega\) and also
would have to include the trivial case of \(NN\) elastic scattering. What we are going to obtain in this section is instead a piece of (3.70), correlated to a specific photon energy and thus should be understood in the asymptotic regime as:

\[
\langle \vec{r} \vec{R} \mid \Psi_\omega \rangle \simeq \frac{f_\omega(\lambda)}{r} e^{ipr} e^{i \vec{p} \vec{R}}.
\] (3.71)

Using the result of Appendix C.2, we can write the non-zero terms of (3.68) in the coordinate representation.

\[
\langle \vec{r} \vec{R} \mid \Psi_\omega \rangle = -\langle \vec{r} \vec{R} \mid \hat{G}_0(E^+) \left[ i\hat{H}_{EM}^{(t)} + \hat{T}_{NN}(E^+) \hat{G}_0(E^+) \hat{H}_{EM}^{(t)} \right] \times
\]

\[
\times \hat{G}_{NN}(E^+ + \omega) \hat{G}_0^{-1}(E^+ + \omega) \mid \vec{p} \rangle = -\int d^3\vec{r}' d^3\vec{R}' \langle \vec{r}' \vec{R}' \mid \hat{G}_0(E^+) \mid \vec{r} \vec{R} \rangle \times
\]

\[
\times \langle \vec{r}' \vec{R}' \mid \left[ i\hat{H}_{EM}^{(t)} + \hat{T}_{NN}(E^+) \hat{G}_0(E^+) \hat{H}_{EM}^{(t)} \right] \hat{G}_{NN}(E^+ + \omega) \hat{G}_0^{-1}(E^+ + \omega) \mid \vec{p} \rangle
\]

\[
= \int d^3\vec{r}' d^3\vec{R}' \left( \frac{iM}{4\pi} \frac{e^{i\vec{p} \cdot (\vec{r}' - \vec{r}')}}{|\vec{r}' - \vec{r}'}| e^{-i\vec{k}_\omega \cdot (\vec{R}' - \vec{R})} \right) \langle \vec{r}' \vec{R}' \mid \left[ i\hat{H}_{EM}^{(t)} + \hat{T}_{NN}(E^+) \hat{G}_0(E^+) \hat{H}_{EM}^{(t)} \right] \times
\]

\[
\hat{G}_{NN}(E^+ + \omega) \hat{G}_0^{-1}(E^+ + \omega) \mid \vec{p} \rangle
\] (3.72)

To transform the product \(\hat{G}_{NN}(E^+ + \omega) \hat{G}_0^{-1}(E^+ + \omega)\) we utilized the relation (3.16) at energy \(E + \omega\). We also took into account that the shift from the initial center-of-mass frame of reference is due to the emission of a photon of energy \(\omega\) and thus the total momentum of the after-radiation nucleon-nucleon system must be opposite to the photon’s momentum, since the total momentum of the system is fixed to zero (see Appendix C.2 for definitions of notation):

\[
\vec{K}_0 = -\vec{k}_\omega;
\] (3.73)

\[
|\vec{k}_\omega| \overset{\text{def}}{=} \omega.
\] (3.74)
We have also introduced a new designation in accordance with (C.6 – C.7):

\[ p_\star^2 \overset{\text{def}}{=} ME - \frac{\omega^2}{4}. \]  

(3.75)

A physically reasonable momentum \( \vec{p}_\star \) with absolute value \( p_\star \) can exist only if \( p_\star \) is real, i.e. when:

\[ \omega < \sqrt{4ME}. \]  

(3.76)

This condition can be violated even though within our approach the non-relativity condition is obeyed:

\[ \omega < (E + \omega) \ll M, \]  

(3.77)

if almost all of the initial nucleon energy \( E + \omega \) goes into the radiation. We should be aware that all our future findings (expressions for the bremsstrahlung amplitude etc.) do not apply to this extreme case.

The same way we dealt with a similar expression in (2.53), finally obtaining (2.56), in the limit of \( r \to \infty \) the first exponent in (3.72) asymptotically transforms into (all the accompanying reasoning of Subsection 2.2.2 applies):

\[ \frac{e^{i\vec{p}_\star |\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} \approx \frac{e^{i\vec{p}_\star r}}{r} e^{i\vec{p}_\star \vec{e}_r \vec{r}'} = \frac{e^{i\vec{p}_\star r}}{r} \langle \vec{p}_\star | \vec{r}' \rangle. \]  

(3.78)

We introduced a designation \( \vec{p}_\star \) for the vector of modulus \( p_\star \) in the direction of the observation radius vector \( \vec{r}' \). Here it is going to play the same role the vector \( \vec{p}_f \) played in Subsection 2.2.2.

The second exponent of (3.72) can be written as \( e^{-i\vec{k}_\omega \cdot \vec{R}} e^{i\vec{k}_\omega \cdot \vec{R}'} \), and the factor of \( e^{-i(\vec{k}_\omega \cdot \vec{R})} \vec{R}' \) combined with \( \langle \vec{p}_\star | \vec{r}' \rangle \) (cf. Appendix C.3) produces the bracket
\[ \langle \vec{p}_* (\vec{k}_\omega) | \vec{r}' \vec{R}' \rangle. \] Thus \(3.72\) now can be transformed into:

\[
\langle \vec{r} \vec{R} | \Psi_\omega \rangle \simeq \frac{iM}{4\pi} \frac{e^{ip_*r}}{r} e^{-ik_\omega \vec{R}} \int d^3\vec{r}' d^3\vec{R}' \langle \vec{p}_* (\vec{k}_\omega) | \vec{r}' \vec{R}' \rangle \times
\]

\[
\times \langle \vec{r}' \vec{R}' | \left[ i\hat{H}_{EM}^{(tt)} + \hat{T}_{NN}(E^+) \hat{G}_0(E^+) \hat{H}_{EM}^{(tt)} \right] \cdot \left[ 1 - i \hat{G}_0(E^+ + \omega) \hat{T}_{NN}(E^+ + \omega) \right] | \vec{p} \rangle
\]

\[
= \frac{M}{4\pi} \frac{e^{ip_*r}}{r} e^{-ik_\omega \vec{R}} \times
\]

\[
\times \left[ -\langle \vec{p}_* (\vec{k}_\omega) | \hat{H}_{EM}^{(tt)} | \vec{p} \rangle \right] (3.79)
\]

\[
+ i \langle \vec{p}_* (\vec{k}_\omega) | \hat{T}_{NN}(E^+) \hat{G}_0(E^+) \hat{H}_{EM}^{(tt)} | \vec{p} \rangle (3.80)
\]

\[
+ i \langle \vec{p}_* (\vec{k}_\omega) | \hat{H}_{EM}^{(tt)} \hat{G}_0(E^+ + \omega) \hat{T}_{NN}(E^+ + \omega) | \vec{p} \rangle (3.81)
\]

\[
+ \langle \vec{p}_* (\vec{k}_\omega) | \hat{T}_{NN}(E^+) \hat{G}_0(E^+) \hat{H}_{EM}^{(tt)} \hat{G}_0(E^+ + \omega) \hat{T}_{NN}(E^+ + \omega) | \vec{p} \rangle \right]. (3.82)
\]

Comparing this result to \(3.71\) we can clearly see that the \(\omega\)-dependent bremsstrahlung amplitude \(f_\omega(\lambda)\) is given by the four terms \(3.79–3.82\) multiplied by their mutual factor \(\frac{M}{4\pi}\). Let us take a closer look at each of them in turn.

The first term \(3.79\) is a “one-particle bremsstrahlung” term, containing no \(NN\)-interaction and thus from general physical considerations we expect it to have no contribution. Indeed, the expression for \(\hat{H}_{EM}^{(tt)} (3.36)\) contains a momentum-conserving delta-function \(\delta^{(3)}(\vec{p}_1 - \vec{p}_1' - \vec{k}_\omega)\), stating momentum conservation for this “spontaneous radiation” event:

\[
\vec{p}_1 = \vec{p}_1' + \vec{k}_\omega;
\]

\[
\sqrt{M(E + \omega)} \vec{e}_{\vec{p}_1} = \vec{p}_* + \frac{1}{2} (-\vec{k}_\omega) + \vec{k}_\omega = \vec{e}_{\vec{p}_*} \sqrt{ME - \omega^2} + \frac{1}{2} \vec{k}_\omega. \quad (3.83)
\]
Here we are making use of (3.75) and “translating” the state $|\vec{p}_* (−\vec{k}_ω)\rangle$ into the two-particle state $|\vec{p}_1' \vec{p}_2'\rangle$. Taking the absolute value of this vector relation and squaring it we get:

$$M_ω = ω \sqrt{ME - \frac{ω^2}{4} \cos α}, \quad (3.84)$$

where $α$ is just the angle in between vectors $\vec{p}_*$ and $\vec{k}_ω$.

As $ω$ is not zero, for (3.84) to be true the following condition must be fulfilled:

$$M ≤ E - \frac{ω^2}{4M} \quad (3.85)$$

However this is impossible in the case of non-relativistic energies we are concerned with, which must obey the inequalities (3.77). Hence the momentum cannot be conserved and the $δ$-function turns the term (3.79) into zero.

The next term (3.80) — the first that truly contributes to the bremsstrahlung amplitude — can be transformed into a computable form as follows:

$$f^{(1)}_ω = \frac{iM}{4π} \langle \vec{p}_* (−\vec{k}_ω) | T_{NN}(E^+) | \hat{G}_0(E^+) \hat{H}_{EM}^{(i)} | \vec{p} \rangle$$

$$= \frac{iM}{4π} \int \frac{d\vec{q}^3}{(2π)^3} \langle \vec{p}_* (−\vec{k}_ω) | T_{NN}(E^+) | \vec{q} (−\vec{k}_ω) \rangle i \langle \vec{q} (−\vec{k}_ω) | \hat{H}_{EM}^{(i)} | \vec{p} \rangle \frac{1}{E^+ - \frac{ω^2}{4M} - \frac{q^2}{M}}$$

$$= \frac{e}{8π} \int \frac{d\vec{q}^3}{(2π)^3} \langle \vec{p}_* (−\vec{k}_ω) | T_{NN}(E^+) | \vec{q} (−\vec{k}_ω) \rangle \frac{\vec{e}_ω \cdot (\vec{p} + \vec{q} - \frac{\vec{k}_ω}{2})}{E^+ - \frac{ω^2}{4M} - \frac{q^2}{M}} \frac{δ^{(3)}(\vec{p} - \vec{q} - \frac{\vec{k}_ω}{2})}{E^+ - \frac{ω^2}{4M} - \frac{(\vec{p} - \frac{\vec{k}_ω}{2})^2}{M}} \frac{\vec{e}_ω \cdot (\vec{p} - \frac{\vec{k}_ω}{2})}{(E^+ - \frac{ω^2}{4M}) - \frac{(\vec{p} - \frac{\vec{k}_ω}{2})^2}{M}} \quad (3.86)$$

Using the results of Appendix C.4 and the compact notation of Chapter 2, and also
remembering that in the Coulomb gauge (3.23) $\vec{\epsilon}_\omega \cdot \vec{k}_\omega = 0$, we can rewrite (3.86) as:

$$ f^{(1)}_\omega = \frac{e}{4\pi} T_{NN} \left( \vec{p}_*, \vec{p} - \frac{\vec{k}_\omega \cdot \vec{p}}{2} ; E^+ - \frac{\omega^2}{4M} \right) \frac{\vec{\epsilon}_\omega \cdot \vec{p}}{\sqrt{E^+ - \omega^2} - \frac{\vec{p} - \vec{k}_\omega \cdot \vec{p}}{2M}}. \quad (3.87) $$

The same way (3.81) gives us:

$$ f^{(2)}_\omega = \frac{e}{4\pi} \frac{\vec{\epsilon}_\omega \cdot \vec{p}_*}{(E^+ + \omega) - \frac{(\vec{p}_* + \frac{\vec{k}_\omega}{2})^2}{M}} T_{NN} \left( \vec{p}_* + \frac{\vec{k}_\omega}{2}, \vec{p} ; E^+ + \omega \right). \quad (3.88) $$

Finally, analogous transformation of (3.82) containing two $T$-matrix elements requires double integration, of which one integral remains after momentum conservation is employed:

$$ f^{(3)}_\omega = \frac{e}{4\pi} \int \frac{d^3\vec{q}}{(2\pi)^3} T_{NN} \left( \vec{p}_*, \vec{q} ; E^+ - \frac{\omega^2}{4M} \right) \frac{T_{NN} \left( \vec{q} + \frac{\vec{k}_\omega}{2}, \vec{p} ; E^+ + \omega \right)}{(E^+ - \omega^2) - \frac{\vec{q}^2}{M} \frac{(E^+ + \omega)}{M} - \frac{(\vec{q} + \frac{\vec{k}_\omega}{2})^2}{M}}. \quad (3.89) $$

The expressions (3.87–3.89) added together form the bremsstrahlung amplitude of spinless proton-neutron bremsstrahlung with the emission of a photon of energy $\omega$.

Now the task is to understand the meaning of these three terms and to find a way of computing them numerically.

### 3.2 Analysis and Computation of the Bremsstrahlung Amplitude

In this section we analyze the formulae for the $pn$ bremsstrahlung amplitude obtained in the previous section and then find a way to compute them numerically. First we
employ the method of Feynman-like diagrams to interpret the three terms of the amplitude in Subsection 3.2.1. Then in Subsection 3.2.2 we define the coordinate system and the set of parameters, determining the problem of bremsstrahlung amplitude calculations. There we also discuss the numerical algorithm to compute the first two diagrams. Finally, in Subsection 3.2.3 we utilize the subtraction technique to regularize the expression for the third diagram and thus prepare its integral to be performed using Gauss-Legendre quadratures.

3.2.1 Diagrammatic Representation of the Bremsstrahlung Amplitude

Let us rearrange the expressions (3.87–3.89) for the $pn$ bremsstrahlung amplitude terms:

$$f^{(1)}_{\omega} = -\frac{M}{4\pi} \times -iT_{NN} \left( \tilde{p}_* , \vec{p} - \hat{k}_\omega; \frac{E^+ - \omega^2}{4M} \right) \times$$

$$\times \frac{i}{(E^+ - \omega^2) - \frac{(\vec{p} - \hat{k}_\omega)^2}{M}} \times -\frac{e}{M} \tilde{\epsilon}_\omega \frac{2\vec{p} - \hat{k}_\omega}{2}; \quad (3.90)$$

$$f^{(2)}_{\omega} = -\frac{M}{4\pi} \times \frac{e}{M} \tilde{\epsilon}_\omega \frac{2\vec{p}_* + \hat{k}_\omega}{2} \times$$

$$\times \frac{i}{(E^+ + \omega) - \frac{(\vec{p}_* + \frac{\epsilon_\omega}{2})^2}{M}} \times -iT_{NN} \left( \vec{p}_* + \frac{\hat{k}_\omega}{2}, \vec{p}; E^+ + \omega \right); \quad (3.91)$$
\[
\begin{align*}
    f^{(3)}_\omega &= -\frac{M}{4\pi} \times \int \frac{d^3 \vec{q}}{(2\pi)^3} \left[ -iT_{NN} \left( \vec{p}_*, \vec{q}; E^+ - \frac{\omega^2}{4M} \right) \times \frac{i}{(E^+ - \frac{\omega^2}{4M}) - \frac{q^2}{M}} \times \right. \\
    &\quad \left. \times -\frac{e}{M} \varepsilon_\omega \frac{2\vec{q} + \vec{k}_\omega}{2} \times \frac{i}{(E^+ + \omega) - \frac{(\vec{q} + \frac{\vec{k}_\omega}{2})^2}{M}} \times \right. \\
    &\quad \left. \times -iT_{NN} \left( \vec{q} + \frac{\vec{k}_\omega}{2}, \vec{p}; E^+ + \omega \right) \right].
\end{align*}
\]

(3.92)

In these equations we can easily distinguish some expressions of the previously introduced Feynman-like rules (see Fig. 3.1), namely those for free propagation and integration over loops. If we add two additional rules (see Fig. 3.5), then it is easy to give a diagrammatic interpretation of (3.90 – 3.92), which is given in Fig. 3.6.

---

Figure 3.5: Additional Feynman-like rules.
Figure 3.6: Feynman-like diagrams, representing the three terms of $pn$ bremsstrahlung amplitude: (a) corresponds to (3.87) and (3.90); (b) — to (3.88) and (3.91); (c) — to (3.89) and (3.92).
Thus now we can see the nature of the three terms, the first two representing photon emission before and after the nucleon-nucleon interaction, while the third corresponds to radiation in between two \( NN \) interactions.

### 3.2.2 Computation of the First Two Diagrams

Up to this point we assumed that only the resulting nucleon energy \( E \) and the photon energy \( \omega \) are known. As the expressions (3.87–3.89) also explicitly depend upon the direction of momenta, some angles must be introduced as parameters in order for the task of calculating the bremsstrahlung amplitude to be defined.

Let us choose the direction of initial nucleon momentum \( \vec{e}_p \) to be the \( z \)-axis. Then:

\[
\vec{p} = (0, 0, \sqrt{M(E + \omega)}). \tag{3.93}
\]

Next let us choose the photon momentum to be in the \( xz \)-plane of our coordinate system. This way a single angle \( \theta_\omega \) determines the vector \( \vec{k}_\omega \):

\[
\vec{k}_\omega = (\omega \sin \theta_\omega, 0, \omega \cos \theta_\omega). \tag{3.94}
\]

Finally, the vector \( \vec{p}_* \) needs to be defined. As its modulus is already determined by (3.75), only its direction needs to be fixed by the spherical angles \( \theta_* \) and \( \phi_* \):

\[
\vec{p}_* = (p_* \sin \theta_* \cos \phi_*, p_* \sin \theta_* \sin \phi_*, p_* \cos \theta_*). \tag{3.95}
\]

Thus all the three momenta of (3.87–3.89) are accounted for and hence the problem of computing the \( pn \) bremsstrahlung scattering amplitude is defined with the following
set of five parameters:

\[ E, \omega, \theta, \phi, \phi. \] (3.96)

However fixing these parameters still allows different polarizations of the emitted photon. We will consider positive and negative circular polarizations. If \( \vec{k}_\omega \) were parallel to the \( z \)-axis, then the corresponding polarization vectors would have had components \( \frac{1}{\sqrt{2}}(1, i, 0) \) and \( \frac{1}{\sqrt{2}}(1, -i, 0) \). To get such a coordinate system from the one chosen here we need to rotate it around the \( y \)-axis by an angle \( \theta_\omega \). Thus the transformation matrix is:

\[
\begin{pmatrix}
\cos \theta_\omega & 0 & \sin \theta_\omega \\
0 & 1 & 0 \\
-\sin \theta_\omega & 0 & \cos \theta_\omega
\end{pmatrix},
\] (3.97)

yielding the following polarization vector components in our coordinate system:

\[
\vec{\epsilon}_\omega^{(+)} = \frac{1}{\sqrt{2}} (\cos \theta_\omega, i, -\sin \theta_\omega);
\] (3.98)

\[
\vec{\epsilon}_\omega^{(-)} = \frac{1}{\sqrt{2}} (\cos \theta_\omega, -i, -\sin \theta_\omega).
\] (3.99)

Now that all relevant vector components are defined, the computation of the first two diagrams (3.87 – 3.88) is just trivial algebra — apart from the task of calculating the \( T \)-matrix elements. It is easy to check though that all the \( T \)-matrix elements in expressions (3.87 – 3.89) are half-on-shell. Referring to (2.77) one can see that actually \( T(\vec{p}', \vec{p}; E) \) depends only on the moduli of the momenta \( p' \) and \( p \), and on the cosine of the angle in between them \( \cos(\hat{p}'\hat{p}) \). Taking into account Appendix C.5 where we show that initial and final momenta can be interchanged with each other
without changing the value of the $T$-matrix element, it is logical to introduce a new notation for half-on-shell $T$-matrix elements, with on-shell momentum at the first place and thus defining the energy:

$$T[p, p', \cos(p'p)] \overset{\text{def}}{=} T(p', p; E), \text{with } p = \sqrt{ME}.$$  \hfill (3.100)

Therefore, the $T$-matrix elements of (3.87) and (3.88) now can be written as:

$$T_{NN} \left( \vec{p}_*, \vec{p} - \frac{\vec{k}_\omega}{2}; E^+ - \frac{\omega^2}{4M} \right) = T \left\{ p_*, \left| \vec{p} - \frac{\vec{k}_\omega}{2} \right|, \cos \left[ \vec{p}_* (\vec{p} - \frac{\vec{k}_\omega}{2}) \right] \right\};$$  \hfill (3.101)

$$T_{NN} \left( \vec{p}_* + \frac{\vec{k}_\omega}{2}, \vec{p}; E^+ + \omega \right) = T \left\{ p, \left| \vec{p}_* + \frac{\vec{k}_\omega}{2} \right|, \cos \left[ \vec{p} (\vec{p}_* + \frac{\vec{k}_\omega}{2}) \right] \right\}.$$  \hfill (3.102)

For convenience we have also omitted the $NN$ index in the new notation.

We have previously mentioned that relation (2.136) combined with the elastic scattering $T$-matrix computation algorithm of Subsection 2.2.5 allows us to compute any half-on-shell partial wave $T$-matrix element $T_l(p, p'; E^+)$. After performing this procedure with $l$'s from 0 to some $l_{max}$, we then use (2.77) with the cosine from the third position in the new $T$-matrix notation to reconstruct the full $T$-matrix element.
3.2.3 Computation of the Third ("Loop") Diagram

With the conventions of the previous section the third amplitude component (3.89) can be written as:

\[
I = \frac{e}{4\pi} \epsilon_\omega \int \frac{d^3 \vec{q}}{(2\pi)^3} q^3 T[p_\star, q, \cos(\vec{p_\star} \cdot \vec{q})] T\left\{ p, \left| \vec{q} + \frac{\vec{k_\omega}}{2} \right|, \cos \left[ \vec{p} \left( \vec{q} + \frac{\vec{k_\omega}}{2} \right) \right] \right\} \\
\frac{[(p^+_*)^2 - q^2]}{(p^+_*)^2 - \left| \vec{q} + \frac{\vec{k_\omega}}{2} \right|^2}.
\]

(3.103)

If we introduce the components of the variable vector \( \vec{q} \) as:

\[
\vec{q} = (q \cos \phi \sin \theta, q \sin \phi \sin \theta, q \cos \theta),
\]

(3.104)

then we can integrate (3.103) in spherical coordinates and get:

\[
I = \frac{e}{4\pi} \epsilon_\omega \int \int \sin \theta d\theta d\phi \left( \cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta \right) I(\theta, \phi),
\]

(3.105)

with \( I(\theta, \phi) \) given by:

\[
I(\theta, \phi) = \int_0^{+\infty} dq \, q^3 T[p_\star, q, \cos(\vec{p_\star} \cdot \vec{q})] T\left\{ p, \left| \vec{q} + \frac{\vec{k_\omega}}{2} \right|, \cos \left[ \vec{p} \left( \vec{q} + \frac{\vec{k_\omega}}{2} \right) \right] \right\} \\
\frac{[(p^+_*)^2 - q^2]}{(p^+_*)^2 - \left| \vec{q} + \frac{\vec{k_\omega}}{2} \right|^2}.
\]

(3.106)

Obviously, this expression has two poles, the first at \( |\vec{q}| = q = p_\star = \sqrt{ME - \frac{\omega^2}{4}} \), and the second at \( |\vec{q} + \frac{\vec{k_\omega}}{2}| = p = \sqrt{M(E + \omega)} \). In the latter case, the modulus of \( \vec{q} \) can be found by solving a quadratic equation (one of the roots is always negative, so there is only one solution):

\[
|\vec{q}| = \frac{1}{2} \left\{ \sqrt{4p^2 - \omega^2[1 - \cos^2(\vec{q} \cdot \vec{k_\omega})]} - \omega \cos(\vec{q} \cdot \vec{k_\omega}) \right\} \text{def} q_2.
\]

(3.107)
Because of the poles the integral (3.106) cannot be adequately represented by a discrete sum. We encountered a similar situation in Subsection 2.2.5 and dealt with it using the subtraction technique. The same technique can be applied here, only because of the two poles we have to make two subtractions.

After the first subtraction the part of (3.106) that is regular at the first pole is given by:

\[
R_1(\theta, \phi) = \int_0^{+\infty} dq \frac{q^3}{(p^* - q^2)} \left[ T \left[ p_*, q, \cos(p_* \vec{q}) \right] T \left\{ p, \left| \vec{q} + \frac{k_\omega}{2} \right|, \cos \left[ \vec{p} \left( \vec{q} + \frac{k_\omega}{2} \right) \right] \right\} \right.
\]

\[
- \frac{2(p^2 - q^2)}{(p^* - q^2)} \left[ (p^*)^2 - \left| \vec{q} + \frac{k_\omega}{2} \right|^2 \right].
\]

(3.108)

Here \( \vec{q}_1 \) is defined as \( \vec{q} \) with modulus \( p_* \), i.e. in accordance with (3.104):

\[
\vec{q}_1 \overset{\text{def}}{=} (p_* \cos \phi \sin \theta, p_* \sin \phi \sin \theta, p_* \cos \theta).
\]

(3.109)

The divergent part, which is easily calculated with the Cauchy Residue Theorem (see details in Subsection 2.2.5, equation (2.123)) equals:

\[
D_1(\theta, \phi) = -i\pi p_*^2 \frac{T \left[ p_*, p_*, \cos(p_* \vec{q}) \right] T \left\{ p, \left| \vec{q}_1 + \frac{k_\omega}{2} \right|, \cos \left[ \vec{p} \left( \vec{q}_1 + \frac{k_\omega}{2} \right) \right] \right\}}{2 \left( p^2 - \left| \vec{q}_1 + \frac{k_\omega}{2} \right|^2 \right)}.
\]

(3.110)
To make $R_1(\theta, \phi)$ regular also at the second pole we need to perform the second subtraction. The part of $R_1(\theta, \phi)$ (and hence of (3.106)) that is regular in both poles is then given by:

$$R_2(\theta, \phi) = \int_0^{+\infty} dq \left( \frac{T[p_\star, q, \cos(p_\star q)] T \left\{ p, \left| \tilde{q} + \frac{k_\omega}{2} \right|, \cos \left( \frac{\tilde{p} \cdot \tilde{q} + k_\omega}{2} \right) \right\}}{q^3} \right)$$

$$- \left( p_\star^2 - q^2 \right) \left( p^2 - \left| \tilde{q} + \frac{k_\omega}{2} \right|^2 \right)$$

$$- \left[ T[p_\star, p_\star, \cos(p_\star q)] T \left\{ p, \left| \tilde{q}_1 + \frac{k_\omega}{2} \right|, \cos \left( \frac{\tilde{p} \cdot \tilde{q}_1 + k_\omega}{2} \right) \right\} \right)$$

$$- \left( p_\star^2 - q_2^2 \right) \left( p^2 - \left| \tilde{q}_2 + \frac{k_\omega}{2} \right|^2 \right)$$

$$- q_2^3 \left( p_\star^2 - q^2 \right) \left( p^2 - \left| \tilde{q} + \frac{k_\omega}{2} \right|^2 \right)$$

(3.111)

Vector $\tilde{q}_2$ is defined analogously to $\tilde{q}_1$:

$$\tilde{q}_2 \overset{\text{def}}{=} (q_2 \cos \phi \sin \theta, q_2 \sin \phi \sin \theta, q_2 \cos \theta), \quad (3.112)$$

where $q_2$ is given by (3.107).

To calculate the divergent part of $R_2(\theta, \phi)$ we need to perform the following integral, which is different from the integrals of type (2.44), (2.123) etc. we encountered before:

$$\int_0^{+\infty} dq \left( \frac{d}{q^2 - \left| \tilde{q} + \frac{k_\omega}{2} \right|^2} \right) = - \int_0^{+\infty} dq \left( \frac{d}{(q - q_2^\star)(q - \tilde{q}_2^\star)} \right) = - \frac{i\pi}{q_2 - \tilde{q}_2}. \quad (3.113)$$
We again used the Cauchy Residue theorem to perform the integral. Here \( \bar{q} \) is the negative root of the quadratic equation defining \( q \) and thus is given by:
\[
\bar{q} = -\frac{1}{2} \left\{ \sqrt{4p^2 - \omega^2[1 - \cos^2(\bar{q}\vec{k}\omega)]} + \omega \cos(\bar{q}\vec{k}\omega) \right\}.
\] (3.114)

So finally the divergent part is:
\[
D_2(\theta, \phi) = -i\pi \bar{q}_2 \times T \left[ p_*, q_2, \cos(\vec{p}_* \cdot \vec{q}) \right] T \left\{ \begin{array}{|c|} \bar{q}_2 + \frac{\vec{k}\omega}{2} \end{array} \right\} \left\{ \begin{array}{|c|} p, \end{array} \right\} \cos \left[ \vec{p} \left( \bar{q}_2 + \frac{\vec{k}\omega}{2} \right) \right].
\] (3.115)

Thus the integral \( I(\theta, \phi) \) breaks into the sum:
\[
I(\theta, \phi) = R_2(\theta, \phi) + D_1(\theta, \phi) + D_2(\theta, \phi).
\] (3.116)

Terms \( D_1 \) and \( D_2 \) have already had the integral over \( q \) performed, while the integral \( R_2 \) is ready to be approximated by a discrete sum and computed numerically. To do this we will utilize the method of Gauss-Legendre quadratures of order \( N_q \) the same way we did in Subsection 2.2.5, replacing the infinite integration interval by a finite \([0, \Delta_{q-cut}]\). And similarly to Subsection 2.2.5 this results in the necessity of correcting the subtraction terms \( D_1 \) and \( D_2 \). As in (2.134) we find that the new subtraction terms are:
\[
D_1(\theta, \phi) = \left[ \ln \left( \frac{\Delta_{q-cut} + p_*}{\Delta_{q-cut} - p_*} \right) - i\pi \right] \times
\]
\[
T \left[ p_*, p_*, \cos(\vec{p}_* \cdot \vec{q}) \right] T \left\{ \begin{array}{|c|} \bar{q}_1 + \frac{\vec{k}\omega}{2} \end{array} \right\} \left\{ \begin{array}{|c|} p, \end{array} \right\} \cos \left[ \vec{p} \left( \bar{q}_1 + \frac{\vec{k}\omega}{2} \right) \right].
\] (3.117)
\[ D_2(\theta, \phi) = \left( \ln \left\{ \frac{2\Delta_{\text{cut}} + \omega \cos(\vec{q} \cdot \vec{k}_\omega)}{2\Delta_{\text{cut}} + \omega \cos(\vec{q} \cdot \vec{k}_\omega)} - \frac{\sqrt{4p^2 - \omega^2[1 - \cos^2(\vec{q} \cdot \vec{k}_\omega)]}}{\sqrt{4p^2 - \omega^2[1 - \cos^2(\vec{q} \cdot \vec{k}_\omega)]}} \right\} - i\pi \right) \times \]

\[
\times \frac{T\left[p_*, q_2, \cos(\vec{p} \cdot \vec{q})\right]}{q_2^3} \frac{T\left\{ p, \left| \vec{q}_2 + \frac{\vec{k}_\omega}{2}\right|, \cos \left[ \vec{p} \left( \vec{q}_2 + \frac{\vec{k}_\omega}{2}\right) \right]\right\}}{\left(p_*^2 - q_2^2\right)\sqrt{4p^2 - \omega^2[1 - \cos^2(\vec{q} \cdot \vec{k}_\omega)]}}.
\]

(3.118)

After the numerical integration over \( q \) is accomplished, we are ready to perform the double integral (3.105). This trivial numerical task is carried out with the same techniques of Gauss-Legendre quadratures, only with much smaller orders \( N_\theta \) and \( N_\phi \) correspondingly. The only comment we need to make is that it is more convenient to calculate the scalar product of vectors \( \vec{\epsilon}_\omega \) and \( \vec{q}/q \) first, so that only one integral for each given photon polarization needs to be computed.
Chapter 4

The Low Energy Theorem

for Proton-Neutron

Bremsstrahlung

A very important statement can be formulated for the bremsstrahlung process we have been considering in the previous chapter: the bremsstrahlung amplitude\(^1\) (in our case expressions (3.87 - 3.89) summed together) can be represented as:

\[ f(\omega) = \frac{f_{-1}}{\omega} + f_0 + O(\omega), \]  

(4.1)

where both \( f_{-1} \) and \( f_0 \) are given in terms of the corresponding non-radiative process, i.e. the elastic scattering we were discussing in Chapter 2. In this study we are

\(^1\)A similar statement can also be formulated for the cross-sections, but it is beyond the scope of our discussion here.
mostly calling this statement *the Low energy theorem* due to the fact that it was first proved by F. E. Low in [14]. Another widely accepted name for this theorem is *the soft-photon approximation*.

In a “strict” soft-photon approximation (i.e. when precisely all the terms of orders linear in $\omega$ and higher are omitted) the expression (4.1) means that $\omega f(\omega)$ is a linear function of $\omega$. However, here we will show that the most convenient form of the Low energy theorem (4.46) has $f_{-1}$ and $f_0$ depending on $\omega$ and the direction of photon momentum $\vec{k}_\omega$ (this form, though derived by us in an original way, is not our invention, it exactly agrees with a similar result by Leon Heller [23]). This happens because actually we keep some of the higher order in $\omega$ terms that can be expressed through on-shell $T$-matrix elements. Thus, in addition to the convenience of computation, expression (4.46) provides us with extra accuracy compared to the “strict” soft-photon approximation (see Chapter 5).

In Section 4.1 of this chapter we are going to give a proof of the Low energy theorem with the variables and designations used in this study. Section 4.2 explains how the terms $f_{-1}$ and $f_0$ can be computed.

### 4.1 Proof of the Low Energy Theorem

In this section we are going to prove the Low energy theorem formulated above in the case of spinless $pn$ bremsstrahlung. Historically, in [14] F.E. Low first proved his theorem exactly for such a case, the case of two spin zero bosons. Though his original
proof was given within the quantum relativistic paradigm, we will stay in the bounds of non-relativistic quantum mechanics, so that the results are relevant to the current study and consistent with the results of other chapters.

An outline of such a non-relativistic approach was given in [23]. However this paper doesn’t provide an accurate proof of the theorem. Though it was extensively used when preparing this chapter, and a few key points, such as the set of four parameters (4.4 – 4.7) and the idea of representing the loop matrix element in a derivative form of the type (4.41), were borrowed, most of the chapter contains independent and original derivations.

The section is divided into three subsections, which deal, respectively, with deriving the soft-photon approximation for the first two diagrams (Subsection 4.1.1), deriving it for the loop diagram (Subsection 4.1.2), and combining the former with the latter (Subsection 4.1.3).

4.1.1 First Two Diagrams in the Soft-Photon Approximation

For convenience, let us define vector matrix elements by omitting the factor $\frac{e}{4\pi} \epsilon_\omega$ in (3.87–3.88):

$$\tilde{M}_1 = \frac{\tilde{p}}{E - \frac{\omega^2}{4M} - \frac{(\tilde{p} - \frac{\epsilon_\omega}{2})^2}{M}} \langle \tilde{p}_\star | \hat{T} \left( E - \frac{\omega^2}{4M} \right) | \tilde{p} - \frac{\epsilon_\omega}{2} \rangle; \quad (4.2)$$

$$\tilde{M}_2 = \frac{\tilde{p}_\star}{E + \omega - \frac{(\tilde{p} + \frac{\epsilon_\omega}{2})^2}{M}} \langle \tilde{p}_\star + \frac{\epsilon_\omega}{2} | \hat{T} \left( E + \omega \right) | \tilde{p} \rangle. \quad (4.3)$$

We have omitted the subscripts $_{NN}$ and superscripts $^+$ for the sake of brevity.
Consider an arbitrary off-shell $T$-matrix element $\langle \vec{p}' | \hat{T}(e) | \vec{p} \rangle$. It is defined by four scalar parameters — interaction energy $e$, momentum moduli $p$ and $p'$, and the cosine of the angle between the momenta $\eta$ (see e. g. the relation (2.77)) — or by any four non-degenerate scalar functions of them. For this chapter we will choose those to be:

1. The average center-of-mass energy $\bar{E}_{c.m.}$:

$$\bar{E}_{c.m.} \overset{\text{def}}{=} \frac{1}{2M} (p'^2 + p^2);$$

2. the momentum transfer parameter $t$, analogous to the Mandelstam variable $t$ of relativistic quantum two-body theory:

$$t \overset{\text{def}}{=} (\vec{p}' - \vec{p})^2;$$

3. the initial “off-shellness” $\Delta_i$:

$$\Delta_i \overset{\text{def}}{=} e - \frac{p^2}{M};$$

4. and the final “off-shellness” $\Delta_f$:

$$\Delta_f \overset{\text{def}}{=} e - \frac{p'^2}{M},$$

so that now:

$$\langle \vec{p}' | \hat{T}(e) | \vec{p} \rangle \equiv T(E_{c.m.}, t, \Delta_i, \Delta_f).$$
For the $T$-matrix element in (4.2), the value of $\Delta_f$ is zero, while the values of the other three parameters are:

\[
\bar{E}_{c.m.}^{(1)} = \frac{1}{2M} \left[ \left( \frac{p - \vec{k}_\omega}{2} \right)^2 + p^2 \right] = E + \frac{\omega}{2} - \frac{\vec{p} \cdot \vec{k}_\omega}{2M}; \quad (4.9)
\]

\[
t^{(1)} = \left( \vec{p}_* - \vec{p} + \frac{\vec{k}_\omega}{2} \right)^2 = 2(EM - \vec{p}_* \cdot \vec{p}) + \omega M + (\vec{p}_* - \vec{p}) \cdot \vec{k}_\omega; \quad (4.10)
\]

\[
\Delta \overset{\text{def}}{=} \Delta_f^{(1)} = E - \frac{\omega^2}{4M} - \frac{(\vec{p} - \vec{k}_\omega/2)^2}{M} = -\omega + \frac{\vec{p} \cdot \vec{k}_\omega}{M} - \frac{\omega^2}{2M}. \quad (4.11)
\]

Similarly, for the $T$-matrix in (4.3) $\Delta_i = 0$, and the other parameters are:

\[
\bar{E}_{c.m.}^{(2)} = \frac{1}{2M} \left[ p^2 + \left( \vec{p}_* + \frac{\vec{k}_\omega}{2} \right)^2 \right] = E + \frac{\omega}{2} + \frac{\vec{p}_* \cdot \vec{k}_\omega}{2M}; \quad (4.12)
\]

\[
t^{(2)} = \left( \vec{p}_* + \frac{\vec{k}_\omega}{2} - \vec{p} \right)^2 = 2(EM - \vec{p}_* \cdot \vec{p}) + \omega M + (\vec{p}_* - \vec{p}) \cdot \vec{k}_\omega = t^{(1)}; \quad (4.13)
\]

\[
\Delta' \overset{\text{def}}{=} \Delta_f^{(2)} = E + \omega - \frac{(\vec{p}_* + \frac{\vec{k}_\omega}{2})^2}{M} = \omega - \frac{\vec{p}_* \cdot \vec{k}_\omega}{M}. \quad (4.14)
\]

As $t^{(1)} = t^{(2)}$, we will designate both as $t$.

Finally, the sum of the two matrix elements:

\[
\bar{M}_{1+2} \overset{\text{def}}{=} \bar{M}_1 + \bar{M}_2 = \frac{\vec{p}}{\Delta} T \left( E + \frac{\omega}{2} - \frac{\vec{p} \cdot \vec{k}_\omega}{2M}, t, \Delta, 0 \right) + \frac{\vec{p}_*}{\Delta'} T \left( E + \frac{\omega}{2} + \frac{\vec{p}_* \cdot \vec{k}_\omega}{2M}, t, 0, \Delta' \right) \quad (4.15)
\]

In the limit of $\omega \to 0$ both $T$-matrix elements are equal to:

\[
T \overset{\text{def}}{=} T(E, 2(EM - \vec{p}_* \cdot \vec{p}), 0, 0). \quad (4.16)
\]
Expanding (4.15) into the Taylor series and keeping only the first-order terms in $\omega$, we get:

$$\vec{M}_{1+2} = \left( \frac{\vec{p}}{\Delta} + \frac{\vec{p}_*}{\Delta'} \right) \left\{ T + \frac{\partial T}{\partial \Delta} \left[ \omega M + (\vec{p}_* - \vec{p}) \vec{k}_\omega \right] \right\}$$

$$+ \frac{\partial T}{\partial E_{c.m.}} \frac{1}{2} \left[ \frac{\vec{p}}{\Delta} \left( \omega - \frac{\vec{p} \cdot \vec{k}_\omega}{M} \right) + \frac{\vec{p}_*}{\Delta'} \left( \omega + \frac{\vec{p}_* \cdot \vec{k}_\omega}{M} \right) \right]$$

$$+ \frac{\partial T}{\partial \Delta_i} \vec{p} + \frac{\partial T}{\partial \Delta_f} \vec{p}_* + O(\omega). \quad (4.17)$$

All the partial derivatives are taken at $\bar{E}_{c.m.} = E$, $t = 2(EM - \vec{p}_* \cdot \vec{p})$ and zero off-shellness.

To leading order in $\omega$:

$$\vec{M}_{1+2} \approx \frac{1}{\omega} \left( \frac{\vec{p}_*}{1 - \frac{\vec{p}_* \cdot \vec{k}_\omega}{M\omega}} - \frac{\vec{p}}{1 - \frac{\vec{p} \cdot \vec{k}_\omega}{M\omega} + \frac{\omega}{2M}} \right) T. \quad (4.18)$$

If we also take the extreme non-relativistic limit $E \ll M$, in which $\Delta' \sim -\Delta \sim \omega$, then:

$$\vec{M}_{1+2} \approx \frac{\vec{p}_* - \vec{p}}{\omega} T. \quad (4.19)$$

### 4.1.2 The Loop Diagram in the Soft-Photon Approximation

Analogously, the vector matrix element for the loop diagram, derived from (3.89) equals:

$$\vec{M}_3 = \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{\langle \vec{p}_* | \hat{T} \left( E - \frac{\omega^2}{4M} \right) | \vec{q} \rangle \langle \vec{q} + \frac{\vec{k}_\omega}{2} | \hat{T}(E + \omega) | \vec{p} \rangle}{E + \omega - \frac{\omega^2}{4M} \frac{\vec{q}^2}{M}} \frac{\langle \vec{q} + \frac{\vec{k}_\omega}{2} | \hat{T}(E + \omega) | \vec{p}_* \rangle}{E + \omega - \frac{\omega^2}{4M} \frac{\vec{q}^2}{M}}. \quad (4.20)$$
Both $T$-matrices appearing in this expression can also be written as functions of the four parameters defined in the previous subsection. Then we can show that the first-order Taylor expansion of these matrix elements is equivalent to a similar expansion of simpler matrices. Thus, to the first order in $\omega$:

\[
\langle \vec{p}_* | \hat{T} \left( E - \frac{\omega^2}{4M} \right) | \vec{q} \rangle = T \left[ \frac{p_*^2 + q^2}{2M}, (\vec{p}_* - \vec{q})^2, E - \frac{\omega^2}{4M} - \frac{q^2}{M}, 0 \right] \\
= T \left[ \frac{p_*^2 + q^2}{2M}, (\vec{p}_* - \vec{q})^2, E - \frac{q^2}{M}, 0 \right] + \mathcal{O}(\omega) = \langle \vec{p}_* | T(E) | \vec{q} \rangle + \mathcal{O}(\omega);
\]

(4.21)

\[
\langle \vec{q} + \frac{\vec{k}_\omega}{2} | \hat{T}(E + \omega) | \vec{p} \rangle \\
= T \left[ \frac{p^2 + q^2 + \omega^2 + \vec{q} \cdot \vec{k}_\omega}{2M}, (\vec{q} + \frac{\vec{k}_\omega}{2} - \vec{p})^2, 0, E + \omega - \frac{(\vec{q} + \frac{\vec{k}_\omega}{2})^2}{M} \right] \\
= T \left[ \frac{p_*^2 + q^2}{2M}, (\vec{p}_* - \vec{q})^2, -\omega, E - \frac{q^2}{M} \right] + \mathcal{O}(\omega) = \langle \vec{q} | T(E) | \vec{p} \rangle + \mathcal{O}(\omega).
\]

(4.22)

Thus, omitting terms of $\mathcal{O}(\omega)$ in the numerator and keeping only the terms independent of $\omega$ in the denominators (this is justified because, as we will see in the end of this derivation, the loop diagram does not contribute at $\mathcal{O}(\omega^{-1})$):

\[
\tilde{M}_3 \approx \int \frac{d^3 \vec{q}}{(2\pi)^3} \langle \vec{p}_* | \hat{T}(E) | \vec{q} \rangle \frac{\vec{q}}{(E - \frac{\omega^2}{4M})^2} \langle \vec{q} | \hat{T}(E) | \vec{p} \rangle \\
= \frac{M}{2} \int \frac{d^3 \vec{q}}{(2\pi)^3} \langle \vec{p}_* | T(E) | \vec{q} \rangle \frac{\partial}{\partial \vec{q}} \left[ \frac{E - \vec{q}^2/M}{\vec{q}} \right] \langle \vec{q} | - i\hat{T}(E) | \vec{p} \rangle \\
= \frac{M}{2} \int \frac{d^3 \vec{q}}{(2\pi)^3} \langle \vec{p}_* | \hat{T}(E) \left[ \frac{\partial \hat{G}_0(e, \hat{p})}{\partial \vec{p}} \right] \rangle_{e=E} | \vec{q} \rangle \langle \vec{q} | - i\hat{T}(E) | \vec{p} \rangle \\
= \frac{M}{2} \langle \vec{p}_* | - i\hat{T}(e) \left[ \frac{\partial \hat{G}_0(e, \hat{p})}{\partial \vec{p}} \right] \hat{T}(e) | \vec{p} \rangle_{e=E}.
\]

(4.23)
Though we write that the derivatives are evaluated at \( e = E \), we should take into account the relations \((4.21 - 4.22)\). Similarly, the energy \( e \) can be shifted to any value of \( E + \mathcal{O}(\omega) \).

Let us prove that the last matrix element can be transformed as follows:

\[
\langle \vec{p}^\ast | -i \hat{T}(e) \frac{\partial \hat{G}_0(e, \vec{p})}{\partial \vec{p}} \hat{\bar{T}}(e) | \vec{p} \rangle = \frac{\partial}{\partial \vec{p}^\ast} \langle \vec{p}^\ast | \hat{T}(e) | \vec{p} \rangle + \frac{\partial}{\partial \vec{p}} \langle \vec{p}^\ast | \hat{T}(e) | \vec{p} \rangle. \tag{4.24}
\]

In order to do this two operators \( \hat{R} \) and \( \hat{L} \) need to be defined:

\[
\langle \vec{p}^\ast | \hat{L} | \vec{p} \rangle \overset{\text{def}}{=} \frac{\partial}{\partial \vec{p}^\ast} \langle \vec{p}^\ast | \hat{T}(e) | \vec{p} \rangle; \tag{4.25}
\]

\[
\langle \vec{p}^\ast | \hat{R} | \vec{p} \rangle \overset{\text{def}}{=} \frac{\partial}{\partial \vec{p}} \langle \vec{p}^\ast | \hat{T}(e) | \vec{p} \rangle. \tag{4.26}
\]

Using the Lippmann-Schwinger equation \((2.64)\) we can construct integral equations, which these operators obey:

\[
\langle \vec{p}^\ast | \hat{L} | \vec{p} \rangle = \frac{\partial}{\partial \vec{p}^\ast} \langle \vec{p}^\ast | \hat{V} | \vec{p} \rangle - i \int \frac{d^3 \vec{q}}{(2\pi)^3} \left[ \frac{\partial}{\partial \vec{p}^\ast} \langle \vec{p}^\ast | \hat{V} | \vec{q} \rangle \right] G_0(e, q) \langle \vec{q} | \hat{T}(e) | \vec{p} \rangle; \tag{4.27}
\]

\[
\langle \vec{p}^\ast | \hat{R} | \vec{p} \rangle = \frac{\partial}{\partial \vec{p}} \langle \vec{p}^\ast | \hat{V} | \vec{p} \rangle - i \int \frac{d^3 \vec{q}}{(2\pi)^3} \langle \vec{p}^\ast | \hat{V} | \vec{q} \rangle G_0(e, q) \left[ \frac{\partial}{\partial \vec{p}} \langle \vec{q} | \hat{T}(e) | \vec{p} \rangle \right]. \tag{4.28}
\]

Note, that the derivative under the integral in \((4.28)\) by definition equals \( \langle \vec{q} | \hat{R} | \vec{p} \rangle \).

Assuming, that the potential matrix element depends on the initial and final momenta through the momentum transfer \( \vec{p}' - \vec{p} \) only, which is the case for any local potential, a class which includes the Yukawa we are using in this study, we can prove
that:
\[
\frac{\partial \langle \tilde{p}^\prime | \hat{V} | \tilde{p} \rangle}{\partial \tilde{p}^\prime} + \frac{\partial \langle \tilde{p}^\prime | \hat{V} | \tilde{p} \rangle}{\partial \tilde{p}} = \frac{\partial f(\tilde{p}^\prime - \tilde{p})}{\partial \tilde{p}^\prime} + \frac{\partial f(\tilde{p}^\prime - \tilde{p})}{\partial \tilde{p}} = 0. \tag{4.29}
\]

Using this property we can introduce a new operator \( \hat{Y} \):
\[
\langle \tilde{p}^\prime | \hat{Y} | \tilde{p} \rangle \overset{\text{def}}{=} \frac{\partial}{\partial \tilde{p}^\prime} \langle \tilde{p}^\prime | \hat{V} | \tilde{p} \rangle = -\frac{\partial}{\partial \tilde{p}} \langle \tilde{p}^\prime | \hat{V} | \tilde{p} \rangle. \tag{4.30}
\]

With this operator the equation (4.28) transforms into:
\[
\langle \tilde{p}^\prime | \hat{R} | \tilde{p} \rangle = -\langle \tilde{p}^\prime | \hat{Y} | \tilde{p} \rangle - i \int \frac{d^3 \tilde{q}}{(2\pi)^3} \langle \tilde{p}^\prime | \hat{V} \rangle G_0(e, q) \langle \tilde{q} | \hat{T}(e) | \tilde{p} \rangle, \tag{4.31}
\]

and can be rewritten in the operator form:
\[
\hat{R} = -\hat{Y} - i\hat{V}G_0(e, \tilde{p})\hat{R} \Rightarrow \hat{R} = -[1 + i\hat{V}G_0(e, \tilde{p})]^{-1} \hat{Y}. \tag{4.32}
\]

Before the equation (4.27) can be transformed similarly, the integral in it needs to be performed by parts:
\[
\int \frac{d^3 \tilde{q}}{(2\pi)^3} \left[ \frac{\partial}{\partial \tilde{p}^\prime} \langle \tilde{p}^\prime | \hat{V} | \tilde{q} \rangle \right] G_0(e, q) \langle \tilde{q} | \hat{T}(e) | \tilde{p} \rangle
\]
\[
= -\int \frac{d^3 \tilde{q}}{(2\pi)^3} \left[ \frac{\partial}{\partial \tilde{q}} \langle \tilde{p}^\prime | \hat{V} | \tilde{q} \rangle \right] G_0(e, q) \langle \tilde{q} | \hat{T}(e) | \tilde{p} \rangle
\]
\[
= -\int \frac{d^3 \tilde{q}}{(2\pi)^3} \left[ \frac{\partial}{\partial \tilde{q}} \langle \tilde{p}^\prime | \hat{V} | \tilde{q} \rangle \right] G_0(e, q) \langle \tilde{q} | \hat{T}(e) | \tilde{p} \rangle
\]
\[
+ \int \frac{d^3 \tilde{q}}{(2\pi)^3} \langle \tilde{p}^\prime | \hat{V} | \tilde{q} \rangle \left[ \frac{\partial G_0(e, q)}{\partial \tilde{q}} \right] \langle \tilde{q} | \hat{T}(e) | \tilde{p} \rangle. \tag{4.33}
\]

Of these three integrals the first one is zero, because in the limit of \( |\tilde{q}| \to \infty \) the matrix elements of \( \hat{V} \) and \( \hat{T} \) are still finite, while \( G(e, q) = i/(e - q^2/M) \) tends to zero:
\[
\int \frac{d^3 \tilde{q}}{(2\pi)^3} \left[ \frac{\partial}{\partial \tilde{q}} \langle \tilde{p}^\prime | \hat{V} | \tilde{q} \rangle G_0(e, q) \langle \tilde{q} | \hat{T}(e) | \tilde{p} \rangle \right] =
\]
\[\sum_i \vec{e}_i \int_{-\infty}^{+\infty} dq_j \int_{-\infty}^{+\infty} dq_k \int_{-\infty}^{+\infty} dq_i \left[ \frac{\partial}{\partial q_i} \langle \vec{p}^* | \hat{V} | q^* \rangle G_0(e, q) \langle q^* | \hat{T}(e) | \vec{p}^* \rangle \right] \]

\[= \sum_i \vec{e}_i \int_{-\infty}^{+\infty} dq_j \int_{-\infty}^{+\infty} dq_k \int_{-\infty}^{+\infty} dq_i \left[ \langle \vec{p}^* | \hat{V} | q^* \rangle G_0(e, q) \langle q^* | \hat{T}(e) | \vec{p}^* \rangle \right]_{-\infty}^{+\infty} = 0. \quad (4.34)\]

Here we adopted an arbitrary Cartesian coordinate system with unit vectors \( \vec{e}_i \), so that \( \vec{q} = q_i \vec{e}_i + q_j \vec{e}_j + q_k \vec{e}_k \) with index \( i \) running through values \( x, y \) and \( z \), while \( j \) and \( k \) designate the other two axes.

The remaining two integrals of (4.33) can be written as:

\[\int \frac{d^3 \vec{q}}{(2\pi)^3} \left\{ \langle \vec{p}^* | \hat{V} \hat{G}_0(e, \hat{p}) \hat{L} | \vec{p}^* \rangle + \langle \vec{p}^* | \hat{V} \left[ \frac{\partial \hat{G}_0(e, \hat{p})}{\partial \hat{p}} \right] \hat{T}(e) | \vec{p}^* \rangle \right\}. \quad (4.35)\]

Using this relation the equation (4.27) can be rewritten in an operator form analogous to (4.32):

\[\hat{L} = \hat{Y} - i\hat{V} \hat{G}_0(e, \hat{p}) \hat{L} - i\hat{V} \left[ \frac{\partial \hat{G}_0(e, \hat{p})}{\partial \hat{p}} \right] \hat{T}(e)\]

\[\Rightarrow \hat{L} = [1 + i\hat{V} \hat{G}_0(e, \hat{p})]^{-1} \left\{ \hat{Y} - i\hat{V} \left[ \frac{\partial \hat{G}_0(e, \hat{p})}{\partial \hat{p}} \right] \hat{T}(e) \right\}. \quad (4.36)\]

Now we can add (4.32) and (4.36) to obtain:

\[\hat{L} + \hat{R} = -[1 + i\hat{V} \hat{G}_0(e, \hat{p})]^{-1} i\hat{V} \left[ \frac{\partial \hat{G}_0(e, \hat{p})}{\partial \hat{p}} \right] \hat{T}(e). \quad (4.37)\]

The expression \([1 + i\hat{V} \hat{G}_0(e, \hat{p})]^{-1} \hat{V}\) can be viewed as a result of infinite iteration of the Lippmann-Schwinger equation (2.64), summed up with the geometric progression formula:
\[
\hat{T}(e) = \hat{V} + (-i\hat{V})\hat{G}_0(e, \vec{p})\hat{V} + (-i\hat{V})\hat{G}_0(e, \vec{p})(-i\hat{V})\hat{G}_0(e, \vec{p})\hat{V} \\
+ (-i\hat{V})\hat{G}_0(e, \vec{p})(-i\hat{V})\hat{G}_0(e, \vec{p})(-i\hat{V})\hat{G}_0(e, \vec{p})\hat{V} + \ldots
\]
\[
= \frac{1}{1 + i\hat{V}\hat{G}_0(e, \vec{p})} \hat{V},
\] (4.38)

Thus finally we obtain:

\[
\hat{L} + \hat{R} = -i\hat{T}(e) \left[ \frac{\partial\hat{G}_0(e, \vec{p})}{\partial\vec{p}} \right] \hat{T}(e). \tag{4.39}
\]

This proves the statement (4.24) and we now may write \( \vec{M}_3 \) as:

\[
\vec{M}_3 \approx \frac{M}{2} \left( \frac{\partial}{\partial \vec{p}_*} \langle \vec{p}_* | \hat{T}(e) | \vec{p} \rangle + \frac{\partial}{\partial \vec{p}} \langle \vec{p}_* | \hat{T}(e) | \vec{p} \rangle \right)_{e=E}. \tag{4.40}
\]

If we write the \( T \)-matrix element \( \langle \vec{p}_* | \hat{T}(e) | \vec{p} \rangle \) as a function of parameters (4.4 – 4.7), then we can perform the derivative:

\[
\vec{M}_3 \approx \frac{M}{2} \left[ \left( \frac{\partial}{\partial \vec{p}_*} + \frac{\partial}{\partial \vec{p}} \right) T \left( \frac{\vec{p}_*^2 + \vec{p}^2}{2M}, (\vec{p}_* - \vec{p})^2, e - \frac{\vec{p}_*^2}{M}, e - \frac{\vec{p}^2}{M} \right) \right]_{e=E} = \frac{\partial T}{\partial E_{c.m.}} \frac{\vec{p}_* + \vec{p}}{2} - \frac{\partial T}{\partial \Delta_i} \vec{p} - \frac{\partial T}{\partial \Delta_f} \vec{p}_*, \tag{4.41}
\]

and it is easy to see that the term proportional to \( \frac{\partial T}{\partial \Delta_f} \) has coefficient zero.

If we take into account that the evaluation energy of all \( T \)-matrices is \( e = E + \mathcal{O}(\omega) \), at the same level of approximation we may think that the derivatives of this expression are evaluated at the same energy as those from (4.17), and thus can cancel each other. This will be of use to us in the next subsection.
4.1.3 Combining All the Three Matrix Elements

Adding together (4.17) and (4.41), we get the soft-photon approximation for the total bremsstrahlung amplitude:

\[
\vec{M} = \left( \frac{\vec{p}}{\Delta} + \frac{\vec{p}_*}{\Delta'} \right) \left\{ T + \frac{\partial T}{\partial E_{c.m.}} \frac{\omega}{2} + \frac{\partial T}{\partial t} \left[ \omega M + (\vec{p}_* - \vec{p}) \tilde{k}_\omega \right] \right\} \\
+ \frac{\partial T}{\partial E_{c.m.}} \frac{1}{2} \left[ \vec{p} + \vec{p}_* - \vec{p} \left( \frac{\vec{p} \tilde{k}_\omega}{\Delta M} \right) + \vec{p}_* \left( \frac{\vec{p}_* \tilde{k}_\omega}{\Delta' M} \right) \right] + \mathcal{O}(\omega),
\]

(4.42)

where the derivatives with respect to both off-shellness parameters have cancelled each other. The expression in curly brackets can be viewed as a Taylor expansion up to the first order in \( \omega \) of the \( T \)-matrix in the vicinity of \( E_{c.m.} = E, t = 2(EM - \vec{p}_* \vec{p}) \).

We may “un-expand” this, thereby obtaining:

\[
T + \frac{\partial T}{\partial E_{c.m.}} \frac{\omega}{2} + \frac{\partial T}{\partial t} \left[ \omega M + (\vec{p}_* - \vec{p}) \tilde{k}_\omega \right] = T_{new} + \mathcal{O}(\omega^2)
\]

\[
= T \left( \frac{\omega}{2} \right) E_{c.m.}, 0, 0 + \mathcal{O}(\omega^2),
\]

(4.43)

with:

\[
t_{new} \equiv 2ME + M\omega - 2\vec{p}_* \vec{p} + (\vec{p}_* - \vec{p}) \tilde{k}_\omega =
\]

\[
= p^2 + p_*^2 - 2\vec{p} \vec{p}_* + 2(\vec{p}_* - \vec{p}) \tilde{k}_\omega \frac{\omega^2}{4} + \mathcal{O}(\omega^2)
\]

\[
= \left( \vec{p}_* - \vec{p} + \frac{\tilde{k}_\omega}{2} \right)^2 + \mathcal{O}(\omega^2).
\]

(4.44)

Note that the squared vector equals:

\[
\vec{p}_2 - \left( -\vec{p}_* - \frac{\tilde{k}_\omega}{2} \right) = \vec{p}_2 - \vec{p}'_2,
\]

(4.45)
i.e. the momentum lost (or acquired, as only its absolute value is calculated) by
the neutron. We should also notice that $T_{\text{new}}$ is an on-shell $NN$ $T$-matrix with a
nucleon energy equal to $E_{\text{c.m.}}$ of the bremsstrahlung process. Importantly, the errors
in (4.43) and (4.44) are of order $O(\omega^2)$ and so lead only to errors of order $O(\omega)$ when
substituted into (4.42).

So, finally:

$$\tilde{M} = \left( \frac{\vec{p}}{\Delta} + \frac{\vec{p}_*}{\Delta^2} \right) T \left[ E + \frac{\omega}{2}, \left( \vec{p}_* - \vec{p} + \frac{\vec{k}_\omega}{2} \right)^2, 0, 0 \right]$$

$$+ \frac{\partial T}{\partial E_{\text{c.m.}}} \frac{1}{2} \left[ \vec{p} + \vec{p}_* - \vec{p} \left( \frac{\vec{p} \cdot \vec{k}_\omega}{\Delta M} \right) + \vec{p}_* \left( \frac{\vec{p}_* \cdot \vec{k}_\omega}{\Delta' M} \right) \right] + O(\omega). \quad (4.46)$$

This equation is actually the expression for the Low energy theorem in the case
of spinless “proton-neutron” bremsstrahlung we were seeking.

We have already derived the extreme non-relativistic expression (4.19) for the first
two matrix elements. Combining them with the loop diagram in the same approxi-
mation we obtain:

$$\tilde{M} \approx \frac{\vec{p}_* - \vec{p}}{\omega} T + \frac{\vec{p} + \vec{p}_*}{2} \frac{\partial T}{\partial E_{\text{c.m.}}}. \quad (4.47)$$

As we can see, the loop diagram does not contribute at $O(\omega^{-1})$ and thus we may
expect the first two diagrams to dominate $\tilde{M}$, at least for smaller $\omega$. 
4.2 Using the Low Energy Theorem to Evaluate the Bremsstrahlung Amplitude

As we have already mentioned in Chapter 1, the Low energy theorem is a very convenient — though maybe not always a very accurate — way to evaluate the amplitude or the cross-section of a bremsstrahlung process.

For this purpose we can use formula (4.46). The $T$-matrix there is on-shell and thus can be easily computed with the method developed in Section 1.2. In order to use this method we need to know the absolute values of the initial and final momenta and the cosine of the angle in between them.

As we know the energy of $T_{\text{new}}$, the moduli of its momenta are equal to:

$$p'_{\text{new}}^2 = p_{\text{new}}^2 = M \left( E + \frac{\omega}{2} \right). \quad (4.48)$$

To determine $\cos(\vec{p}_{\text{new}} \vec{p}'_{\text{new}})$ we first need to compute $t_{\text{new}}$. As by definition $t_{\text{new}} = (\vec{p}'_{\text{new}} - \vec{p}_{\text{new}})^2$, the cosine we are looking for is given by the following expression:

$$\cos(\vec{p}_{\text{new}} \vec{p}'_{\text{new}}) = 1 - \frac{t_{\text{new}}}{2M \left( E + \frac{\omega}{2} \right)}. \quad (4.49)$$

The parameters (4.48) and (4.49) define the on-shell $NN$ $T$-matrix that is to be computed in (4.46).

Similarly, an on-shell $T$-matrix for any energy $E_{\text{c.m.}}$ can be computed. Thus to calculate $\frac{\partial T}{\partial E_{\text{c.m.}}}$, we just need to keep $t_{\text{new}}$ constant and vary the energy of the $T$-matrix in the vicinity of $E + \frac{\omega}{2}$. The derivative can then be evaluated numerically using the resulting data.
Chapter 5

Testing the Low Energy Theorem:

Results

It is now time to combine the outcomes of Chapter 3 and Chapter 4 and test how well the Low energy theorem estimate (4.46) reproduces the exact $pn$ bremsstrahlung amplitude ($3.87 – 3.89$).

Because the Low energy theorem is formulated as an asymptotic statement (4.1) in the limit $\omega \to 0$, a natural way to investigate it is to perform a matrix element computation for a fixed initial energy $E_{initial} \equiv E + \omega$ and angle parameters $\theta_\omega$, $\theta_*$ and $\phi_*$, but varying photon energy $\omega$ and then compare the exact results to the Low energy theorem predictions. All the computations presented in this chapter were performed following this pattern.

As one can see from (4.1), such a dependence $M(\omega)$ would be necessarily diverging
in the limit $\omega \to 0$. Thus, to provide a more convenient graphical representation we plot $\omega M(\omega)$ against $\omega$.

We have proved in Chapter 4 that the loop diagram of Fig. 3.6 (c) does not contribute at $O(\omega^{-1})$ to $\vec{M}$ and thus if we compute the “exact” value of the bremsstrahlung amplitude vector $\vec{M}_{1+2}^{(\text{exact})}$ based on the first two diagrams only, in the limit of $\omega \to 0$ it will agree with the soft photon approximation of $\vec{M}$ given by (4.46) and based on all three diagrams, which we designate as $\vec{M}_{1+2+3}^{(\text{SPA})}$. Also, for the same reason, we may expect these two quantities to be close at those finite $\omega$ for which the soft-photon approximation works well. Thus computing $\vec{M}_{1+2}^{(\text{exact})}$ and $\vec{M}_{1+2+3}^{(\text{SPA})}$ provides an accuracy test of the soft-photon approximation, which we are going to perform in this chapter.

Of course, if we computed $\vec{M}$ based on the exact values of all three diagrams, comparing it to $\vec{M}_{1+2+3}^{(\text{SPA})}$ would have given us even better test of the Low energy theorem. However, the computation of the loop diagram requires a 3D numerical integration over an infinite range of momenta and thus needs considerable computer power to be performed. Just a single run (for a point of parameter space (3.96)) of the loop diagram computing code prototype takes approximately 30 minutes using the computational facilities available for this project. On the other hand, computations with this code have shown that the loop diagram value is no more than a few percent of the other two matrix elements. This confirms that the loop diagram is small compared to the other two diagrams not only asymptotically in the limit $\omega \to 0$, but
also for finite $\omega$. This may be because the coupling constant $g^2$ is not large and thus the loop graph — which contains more $NN$ interactions — contributes less.

In fact, we can test how well the assumption of the loop diagram smallness works by examining the slope of $\omega \tilde{M}_{1+2}^{(exact)}(\omega)$ as $\omega \to 0$ compared to the slope of $\omega \tilde{M}_{1+2+3}^{(SPA)}(\omega)$: it would be impossible to reach agreement between them if the loop diagram contributes significantly. Still, computations of the loop graph need to be performed in future in order to confirm the results presented here.

The computations are described in Section 5.1, while their results and an analysis thereof are presented in Section 5.2.

\section{5.1 Description of the Computations Performed}

We have already demonstrated in Section 3.2 and Section 4.2 that both the “exact” bremsstrahlung amplitude and the soft-photon approximation of it can be computed with relatively simple numerical algorithms.

In this chapter we introduce “positive” $M_{1+2}^+$ and “negative” $M_{1+2}^-$ scalar matrix elements with the relation:

\begin{equation}
M_{1+2}^{\pm} \equiv \tilde{c}_\omega^{(\pm)} (\tilde{M}_1^{(exact)} + \tilde{M}_2^{(exact)}),
\end{equation}

where the $\tilde{M}_i^{(exact)}$'s are given by (4.2 – 4.3) and $\tilde{c}_\omega^{(\pm)}$ are the two photon polarization choices (3.98 – 3.99). Note, that we omit the factor $\frac{\epsilon}{4\pi}$ compared to (3.87 – 3.88). The actual code computing $M_{1+2}^{\pm}$ utilizes the algorithm explained in Subsection 3.2.2.
The approach of Section 4.2 combined with the on-shell $T$-matrix computation algorithm of Appendix D provides us with a way to compute the Low energy theorem estimate of the full bremsstrahlung vector matrix element $\tilde{M}_{1+2+3}^{(SPA)}$, also being represented by its “positive” and “negative” polarization projections $M_{1+2+3}^{\pm}$. Again, the factor of $\frac{e^4}{4\pi}$ is omitted similarly to the derivation of formula (4.46).

All the computations are performed with the Yukawa potential (2.33), having the maximum orbital angular momentum $l_{\text{max}} = 5$ and using $N = 96$ Gaussian quadratures.

### 5.2 Results of the Computations

Typical results of computations are presented in Fig. 5.1. Each set of parameters $(E+\omega, \theta, \phi, \phi)$ yields four graphs, showing real and imaginary parts of the positively and negatively polarized matrix elements correspondingly. In the left column of the figure we see these four graphs for $(100 \text{ MeV}, 30^\circ, 45^\circ, 60^\circ)$, while in the right column — for $(100 \text{ MeV}, 60^\circ, 45^\circ, 30^\circ)$.

Note, that due to the condition (3.76) the scattering matrix element is not defined for values of $E$ close to zero, i.e. values of $\omega$ close to the fixed initial energy value. The maximum possible photon energy can be found from (3.76) by solving a quadratic equation:

$$\omega_{\text{max}} = 2M \left( \sqrt{1 + \frac{E_{\text{initial}}}{M}} - 1 \right).$$

(5.2)
Figure 5.1: Two examples of the numerical results, showing $\omega M^\pm$ vs $\omega$ functional dependence. Each column contains four graphs ($Re$ and $Im$ of both polarizations) for one set of parameters. The initial energy $E + \omega$ in both cases is 100 MeV. The black curve shows the “exact” solution while the grey one — the Low energy theorem approximation.
Higher photon energies correspond to forming a bound state, which we are not considering here.

As we have commented in the introduction to Chapter 4, expression (4.46) is not a “strict” soft-photon approximation, it includes higher order terms in $\omega$. Thus the SPA results are curves, not just straight lines and, as we can see, they provide better accuracy than the “strict” SPA linear approximation would have.

Two preliminary observations can be made by looking at Fig. 5.1:

- though at the high-photon-energy region the behavior of the two curves differs, generally the soft-photon approximation gives a satisfactory estimate of the matrix element;

- the agreement of the approximation and the “exact” solution is better for the real than for the imaginary parts of $M^\pm$.

In the following four subsections we investigate how the numerical results depend on each of the four parameters $(E + \omega, \theta_\omega, \theta_*, \phi_*)$: i. e. we change one of them while keeping others fixed to arbitrarily chosen “base” values (100 MeV, 45°, 15°, 45°).

5.2.1 Different Energies

First of all, let us fix the parameter angles $\theta_\omega$, $\theta_*$ and $\phi_*$ and perform the computations for four different energies. The results are presented in figures Fig. 5.2 and Fig. 5.3. This particular choice of energies is dictated by the low-energy instability at
Figure 5.2: Testing the Low energy theorem for fixed angular parameters with four different initial energies $E + \omega$: (from top to bottom) 20, 100, 250 and 500 MeV. Real parts of positively (left) and negatively (right) polarized matrix elements are presented.
Figure 5.3: Same as Fig. 5.2, imaginary parts of the matrix elements.
$E + \omega \lesssim 10 \text{ MeV}$ (cf. Subsection 2.3.2) and by the fact that initial nucleon energies higher than 500 MeV are inconsistent with our non-relativistic approach.

Except for the cases $\omega \gg E$, the agreement in between $ReM_{1+2}^\pm$ and the soft-photon approximation of $ReM_{1+2+3}^\pm$ is remarkable, especially when we consider that the “exact” $ReM_{1+2}^\pm$ does not include the loop graph of Fig. 3.6 (c). The only significant discrepancy we see is at the last third of the 20 MeV graphs. Imaginary parts seem to differ more, but if we take into account the scale difference of $Re$ and $Im$ graphs, we must conclude that in this case the Low energy theorem works well too.

### 5.2.2 Different Photon Emission Angles $\theta_\omega$

Fig. 5.4 and Fig. 5.5 show $\omega M^\pm$ vs. $\omega$ dependencies for four different photon emission angles with other parameters being at their “base” values. We’ve chosen two degenerate cases of emission along the straight forward and straight backward directions and two “ordinary” $\theta_\omega = 60^\circ$ and $90^\circ$. Both the real and imaginary parts of the degenerate emission cases demonstrate an excellent soft-photon — “exact” solution conformity (note special choices of each graph’s ordinate scale - with usual $-0.03 - 0.04 \text{ MeV}^{-1}$ the gray and black curves would coincide). The “ordinary” angle configurations demonstrate no remarkable difference from the corresponding 100 MeV graphs of Fig. 5.2 and Fig. 5.3.
Figure 5.4: Testing the Low energy theorem for fixed $E + \omega$, $\theta_\omega$ and $\phi_\omega$ with four different photon emission angles $\theta_\omega$: (from top to bottom) $0^\circ$, $60^\circ$, $90^\circ$ and $180^\circ$. Real parts of positively (left) and negatively (right) polarized matrix elements are presented.
Figure 5.5: Same as Fig. 5.4, imaginary parts of the matrix elements.
5.2.3 Different Final Momentum Angles $\theta_*$

Let us now vary the angle $\theta_\omega$ — one of the two spherical angles defining $\vec{p}_*$ or, in other words, the final momentum angle in the after-emission center-of-mass frame of reference. As usual, $E + \omega$, $\theta_\omega$ and $\phi_*$ are kept constant.

As we can see in Fig. 5.6 and Fig. 5.7, the configuration of $M^{\pm}(\omega)$ changes significantly for different choices of $\theta_\omega$. Still, the real part of the soft-photon approximation follows the real part of the “exact” solution very closely. On the other hand, the imaginary part graphs this time show more discrepancy in their high-energy parts, in the worst case the absolute value of the “exact” solution is 5-6 times larger than the Low energy theorem prediction.

5.2.4 Different Final Momentum Angles $\phi_*$

Finally, Fig. 5.8 and Fig. 5.9 display graphs having all the same parameters except $\phi_*$. As we can see, they hardly differ from each other (except, naturally, real parts from imaginary ones) — the actual difference is just a few per cent. The angle $\phi_*$ appears to be the “weakest” parameter of all determining a bremsstrahlung process. And in all the graphs we see a very close similarity of the Low energy and “exact” solution curves.
Figure 5.6: Testing the Low energy theorem for fixed $E + \omega$, $\theta_\omega$ and $\phi_\star$ with four different final momentum c.m. angles $\theta_\star$: (from top to bottom) 0°, 60°, 90° and 180°. Real parts of positively (left) and negatively (right) polarized matrix elements are presented.
Figure 5.7: Same as Fig. 5.6, imaginary parts of the matrix elements.
Figure 5.8: Testing the Low energy theorem for fixed $E + \omega$, $\theta_\omega$ and $\theta_\star$ with four different final momentum c.m. angles $\phi_\star$: (from top to bottom) 0°, 60°, 90° and 180°. Real parts of positively (left) and negatively (right) polarized matrix elements are presented.
Figure 5.9: Same as Fig. 5.8, imaginary parts of the matrix elements.
5.2.5 Conclusions

If $\omega$ is not too close to $E_{\text{initial}}$, most of the graphs in Fig. 5.1 through Fig. 5.9 show a remarkable agreement of “exact” $M_{1+2}^{(\text{exact})} \pm$ and the Low energy theorem evaluation of all three diagrams $M_{1+2+3}^{(SPA)} \pm$, which is consistent with the assumption of the loop diagram being only a small correction. As expected, the graphs coincide asymptotically in the limit of $\omega \to 0$.

Intercepts of both curves always agree, as they should since the loop diagram does not contribute at $O(\omega^{-1})$ to $\vec{M}$. This is a good check of accuracy of our calculations.

Slopes for small $\omega$ also agree quite well in most cases and therefore:

\[
\vec{M}_3(\omega) + \omega \frac{\partial \vec{M}_3(\omega)}{\partial \omega} \ll \vec{M}_{1+2}(\omega) + \omega \frac{\partial \vec{M}_{1+2}(\omega)}{\partial \omega}.
\] (5.3)

This directly confirms our assumption of the loop graph’s small contribution into the bremsstrahlung amplitude for those photon energies, for which the soft-photon approximation is valid.

We can see that, as we expected, our expression for the Low energy theorem (4.46) generates $\omega$-dependence beyond linear, which is, sometimes, in perfect agreement with $\vec{M}_{1+2}^{(\text{exact})}$.

Based on these results we conclude that the soft-photon approximation may be considered an excellent estimate of spinless $pn$ bremsstrahlung amplitude for photon energies $\omega \lesssim \frac{2}{3}E_{\text{initial}}$ for the real part of the amplitude and $\omega \lesssim \frac{1}{3}E_{\text{initial}}$ for the imaginary part. However for larger $\omega$, especially for those very close to the initial $NN$ energy, the exact computation is still preferable.
Chapter 6

Conclusion

Let us give a brief summary of results of this work.

A detailed theory of elastic scattering of two spinless particles has been presented in Chapter 2 of this study. It was designed to complement existing textbooks on the subject.

A theory of spinless “proton-neutron” bremsstrahlung has been developed in Chapter 3. The analytical expressions for the scattering amplitude were derived and the algorithms of their numerical computation were developed.

An original proof of the Low energy theorem for this case of bremsstrahlung has been given in Chapter 4 and a soft-photon approximation formula for the scattering amplitude has been derived.

The scattering amplitude has been calculated with a Yukawa potential normalized to produce a bound state of the deuteron energy. Computations with different scat-
tering angular configurations and initial energy have been performed for both exact expressions of the two leading terms of the bremsstrahlung amplitude and the soft-photon approximation. Most of them show that the Low energy theorem provides a good estimate of the exact scattering amplitude values for almost all photon energies. The exception is when photon energies are close to the initial nucleon energy: in this case the Low energy theorem does not provide a good estimate.

As we said in Chapter 1, this project is no more than an introduction into a wide and deep field of the Low energy theorem for bremsstrahlung. Thus there is still much work to do.

There are at least two unaccomplished tasks left for the spinless $pn$ bremsstrahlung theory. First, any statements cannot be fully conclusive before the loop diagram is evaluated, for which a more efficient algorithm should be found. And second, we limited our study only to the scattering amplitude, but if we want to do a comparison with experimental results, the corresponding scattering cross-sections must be computed (see Appendix C.6).

Of course, the bremsstrahlung theory is of little value before we include nucleon spins in it and this is the most important advancement that needs to be done. Only then the theory will achieve a real physical relevance. The next step is to go from the electromagnetic to neutrino bremsstrahlung – and hence to astrophysical applications. It may also be important to include relativistic effects into the theory.

Nevertheless, until more advanced research on this topic is performed, the results
of this study may be used as partial justification for the use of the Low energy theorem in evaluating bremsstrahlung cross-sections.
Appendix A

Codes in C for the Schrödinger Equation in Coordinate Space

A.1 The Code that Determines Optimal $g^2$ Based on the Value of Deuteron Binding Energy

```
#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#define pi 3.141592654
#define fermi 197.0 /* conversion coefficient, 1 fm^-1 is 197 MeV */
#define mu 469.5 /* mu=0.5*m_proton in MeV */
#define mpi 139.0 /* pion mass in MeV */
#define E -2.2246 /* deuteron energy in MeV */
#define amax 1000
#define N 100000.0 /* size of the lattice*/
#define R 30.0 /* end radius for the solution */

/*
 * author: Yurii Pidopryhora
 * date: 06/01/2001
*/
```
The code solves radial Schrödinger equation \( \frac{d^2}{dr^2}u - k^2u + U(r)u = 0 \) with \( k^2 = -2\mu E < 0 \) and \( U(r) = 2\mu V(r) \) for boundary conditions \( u(0) = 0 \) and \( u'(0) = 1 \). It scans over the range of \( \alpha_{\text{mpi}} \) of the potential.

\[
V(r) = -\alpha_{\text{mpi}} \exp(-\text{mpi}r)/r \quad \text{with} \quad \alpha_{\text{mpi}} = g^2/4\pi,
\]

...calculating the positive exponent coefficient \( A \) of the wavefunction

\[
u(R) = A \exp(kR) + B \exp(-kR)
\]

at a large radius \( R = 30 \text{ fm} \) where the wavefunction could be assumed to be free and thus the above formula is true.

It allows to find the value of the coupling constant \( g \) where the bound state is at given energy \( E \).

/* function prototypes */

```c
#include <stdio.h>
#include <math.h>

double UPOT (double x, double alphapi); /* potential energy */

int main ()
{
    /* variable declaration */
    double u, u1; /* the solution */
    double v, v1; /* its first derivative \( v = du/dr \) */
    double ksq; /* in this case of negative energy represents \( \kappa^2 \) */
    double alphapi;
    double rho; /* the step of the lattice */
    double U; /* potential energy */
    int i, j; /* cycle variables */
    rho = R/N;

    FILE *out;
    out = fopen("a_look.txt", "w");

    for (j = 1; j <= amax; j++) {
        alphapi = 0.3559286 + j*0.000000001;
        ksq = -2.*E*mu/(fermi*fermi); /* negative sign because \( E < 0 \) */

        /* main calculations */
```
/* boundary conditions */

u1 = 0.0;
v1 = 1.0;

/* explicit Euler algorithm */

for (i = 1; i <= N; i++)
{
    u = u1;
v = v1;
    U = UPOT (rho*i, alphapi);
u1 = u + rho*v;
v1 = v - rho*u*(-ksq - U); /* note negative sign in front of ksq */
}

double aux1, aux2, A;
aux1 = sqrt(ksq);
aux2 = 2*exp(aux1*R);
A = (u1 + (v1/aux1))/aux2; /* positive exponent coefficient */

fprintf (out, "%2.9f %e\n", alphapi, A);

fclose(out);
return 0;
}

/* function definitions */

/* calculation of potential energy U */
double UPOT (double x, double alphapi)
{
    double U;
    U = -alphapi*2*mu*exp(-mpi*x/fermi)/(x*fermi);
    return U;
}
A.2 The Code that Computes the $S$-Wave Phase Shifts and Cross-Sections

```c
#include <stdio.h>
#include <math.h>
#include <stdlib.h>

#define pi 3.141592654
#define fermi 197.0 /* 1 fm^-1 is 197 MeV */
#define mu 469.5 /* mu=0.5*m_proton in MeV */
#define mpi 139. /* in MeV */
#define Emax 5000
#define Scale 0.1 /* Emax*Scale defines maximum energy in MeV */
#define alphapi 0.3559286055 /* V = -alphapi*exp(-mpi*r)/r */
#define N 1000000.0 /* size of the lattice*/
#define R 30.0 /* end radius for the solution in fermis */

/* author: Yurii Pidopryhora
 * date: 07/07/2001
 * <<<Description>>>
 * The code solves radial Schroedinger equation (d^2)u/dr^2 + (k^2 - U(r))*u = 0
 * with k^2= 2*mu*E (E>0) and U(r)=2*mu*V(r)
 * for boundary conditions u(0)=0 and u'(0)=1
 * When a large radius R=30 fm is reached the wavefunction is assumed to be free
 * and thus the logarithmic derivative of u(r) could be used to get the s-wave
 * phase shifts as a function of energy E. Then the corresponding s-wave
 * cross-section is calculated and compared to the Born approximation.
 * */

/* function prototypes */

double UPOT (double x); /* potential energy */

int main ()
{
    /* variable declaration */

    double u, u1; /* the solution */
    double v, v1; /* its first derivative v=du/dr */
    double ksq;
    double E; /* energy in MeV*/
```
double rho; /* the step of the lattice */
double U; /* potential energy */
int i, j; /* cycle variables */
rho = R/N;

FILE *out;
out = fopen("delta.txt", "w");

for (j = 1; j <= Emax; j++) {
    /* scanning through the range of energies */
    E = j*Scale;
    ksq = 2*E*mu/(fermi*fermi);

    /* main calculations */

    /* boundary conditions */
    u1 = 0.0;
    v1 = 1.0;

    /* explicit Euler algorithm */
    for (i = 1; i <= N; i++) {
        u = u1;
        v = v1;
        U = UPOT (rho*i);
        u1 = u + rho*v;
        v1 = v - rho*u*(ksq - U);
    }

    double k;
    k = sqrt(ksq);

    double gamma;
    gamma = v1/u1; /* logarithmic derivative of u at r = R */
    double tgdelta;
    double kr;
    kr = k*R;
    double n, d;
    n = k*cos(kr) - gamma*sin(kr);
    d = k*sin(kr) + gamma*cos(kr);
    tgdelta = n/d; /* here we use the formula for tan delta_0 */
    double delta;
    delta = atan(tgdelta); /* in radians */
    double sigma;
    sigma = 4*pi*sin(delta)*sin(delta)/(k*k);
        /* the s-wave cross section in fm^2 */
/* Born approximation for comparison */

double aux1, aux2, sigmab;
aux1 = (16.0*pi*fermi*fermi)/(mpi*mpi);
aux2 = (alphapi*mu/mpi)*(alphapi*mu/mpi);
sigmab = aux1*aux2/(((4.0*k*k*fermi*fermi)/(mpi*mpi)) + 1.0);

fprintf (out, "%.3f\t%.3f\t%.3f\t%.3f\n", E, k, tgdelta, delta, sigma, sigmab);

fclose(out);
return 0;

/* function definitions */

/* calculation of potential energy U */

double UPOT (double x)
{
    double U;
    U = -alphapi*2*mu*exp(-mpi*x/fermi)/(x*fermi);
    return U;
}
Appendix B

The Legendre Functions and Gauss-Legendre Quadratures

B.1 The Legendre Functions of 1st and 2nd Kind of Orders 6 through 10

The Legendre polynomials:

\[ P_6(x) = \frac{1}{16} \left( 231 \, x^6 - 315 \, x^4 + 105 \, x^2 - 5 \right); \quad (B.1) \]

\[ P_7(x) = \frac{1}{16} \left( 429 \, x^7 - 693 \, x^5 + 315 \, x^3 - 35 \, x \right); \quad (B.2) \]

\[ P_8(x) = \frac{6435 \, x^8}{128} - \frac{3003 \, x^6}{32} + \frac{3465 \, x^4}{64} - \frac{315 \, x^2}{32} + \frac{35}{128}; \quad (B.3) \]

\[ P_9(x) = \frac{12155 \, x^9}{128} - \frac{6435 \, x^7}{32} + \frac{9009 \, x^5}{64} - \frac{1155 \, x^3}{32} + \frac{315 \, x}{128}; \quad (B.4) \]
\[ P_{10}(x) = \frac{46189 x^{10}}{256} - \frac{109395 x^8}{256} + \frac{45045 x^6}{128} - \frac{15015 x^4}{128} + \frac{3465 x^2}{256} - \frac{63}{256}. \]  

(B.5)

The Legendre functions of the second kind:

\[ Q_6(x) = -\frac{231 x^5}{16} + \frac{119 x^3}{8} - \frac{231 x}{80} + P_6(x) Q_0(x); \]  

(B.6)

\[ Q_7(x) = -\frac{429 x^7}{16} + \frac{275 x^4}{8} - \frac{849 x^2}{80} + \frac{16}{35} + P_7(x) Q_0(x); \]  

(B.7)

\[ Q_8(x) = -\frac{6435 x^7}{128} + \frac{9867 x^5}{128} - \frac{4213 x^3}{128} + \frac{15159 x}{4480} + P_8(x) Q_0(x); \]  

(B.8)

\[ Q_9(x) = -\frac{12155 x^8}{128} + \frac{65065 x^6}{384} - \frac{11869 x^4}{128} + \frac{14179 x^2}{896} \]  

\[-\frac{128}{315} + P_9(x) Q_0(x); \]  

(B.9)

\[ Q_{10}(x) = -\frac{46189 x^9}{256} + \frac{70499 x^7}{192} - \frac{157157 x^5}{640} + \frac{26741 x^3}{448} \]  

\[-\frac{61567 x}{16128} + P_{10}(x) Q_0(x). \]  

(B.10)
## B.2 The Gauss-Legendre Abscissas and Weights

of Order 96 at the Interval \([-1, 1]\)

<table>
<thead>
<tr>
<th>abscissa (\xi_i)</th>
<th>weight (w_i(\xi))</th>
<th>abscissa (\xi_i)</th>
<th>weight (w_i(\xi))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\pm 0.0162767448496030)</td>
<td>0.0325506144923632</td>
<td>(\pm 0.0488129851360497)</td>
<td>0.0325161187138688</td>
</tr>
<tr>
<td>(\pm 0.0812974954644256)</td>
<td>0.0324471637140643</td>
<td>(\pm 0.113695850110666)</td>
<td>0.0323438225685759</td>
</tr>
<tr>
<td>(\pm 0.145973714654897)</td>
<td>0.0322062047940303</td>
<td>(\pm 0.178096882367619)</td>
<td>0.0320344562319927</td>
</tr>
<tr>
<td>(\pm 0.210031310460567)</td>
<td>0.0318287589441110</td>
<td>(\pm 0.241734156163840)</td>
<td>0.0315893307707272</td>
</tr>
<tr>
<td>(\pm 0.273198812591049)</td>
<td>0.0313164255968614</td>
<td>(\pm 0.304364944354496)</td>
<td>0.0310103325863138</td>
</tr>
<tr>
<td>(\pm 0.33520522892625)</td>
<td>0.0306713761236692</td>
<td>(\pm 0.365696861472314)</td>
<td>0.0302999154208276</td>
</tr>
<tr>
<td>(\pm 0.395797649828909)</td>
<td>0.0298963441363284</td>
<td>(\pm 0.425478988407301)</td>
<td>0.0294610899581679</td>
</tr>
<tr>
<td>(\pm 0.454709422167743)</td>
<td>0.0289946141505552</td>
<td>(\pm 0.483457973920596)</td>
<td>0.0284974110650854</td>
</tr>
<tr>
<td>(\pm 0.511694177154668)</td>
<td>0.027970076168483</td>
<td>(\pm 0.539388108324357)</td>
<td>0.027412962760292</td>
</tr>
<tr>
<td>(\pm 0.566510418561397)</td>
<td>0.0268268667255918</td>
<td>(\pm 0.593032364775752)</td>
<td>0.026212340736724</td>
</tr>
<tr>
<td>(\pm 0.618925840125469)</td>
<td>0.025570030053494</td>
<td>(\pm 0.644163403784967)</td>
<td>0.024900633224836</td>
</tr>
<tr>
<td>(\pm 0.668718310043916)</td>
<td>0.0242048147923647</td>
<td>(\pm 0.692564536642172)</td>
<td>0.0234833990859262</td>
</tr>
<tr>
<td>(\pm 0.715676812348968)</td>
<td>0.0227370696583294</td>
<td>(\pm 0.738030643744400)</td>
<td>0.021966444387444</td>
</tr>
<tr>
<td>(\pm 0.759602341176647)</td>
<td>0.0211729398921913</td>
<td>(\pm 0.780369043867433)</td>
<td>0.0203567971543333</td>
</tr>
<tr>
<td>(\pm 0.800308744139141)</td>
<td>0.0195190811401450</td>
<td>(\pm 0.819400310739732)</td>
<td>0.018660796274115</td>
</tr>
<tr>
<td>(\pm 0.837623511228187)</td>
<td>0.0177825023160453</td>
<td>(\pm 0.85495033434601)</td>
<td>0.0168854798642452</td>
</tr>
<tr>
<td>(\pm 0.871388505909297)</td>
<td>0.0159705629025623</td>
<td>(\pm 0.886894517402420)</td>
<td>0.0150387210269949</td>
</tr>
<tr>
<td>(\pm 0.901460635315852)</td>
<td>0.0140909417723149</td>
<td>(\pm 0.915071423120898)</td>
<td>0.0131282925669161</td>
</tr>
<tr>
<td>(\pm 0.927712456722309)</td>
<td>0.0121516046710883</td>
<td>(\pm 0.939370339752755)</td>
<td>0.0111621020998385</td>
</tr>
<tr>
<td>(\pm 0.950032717784438)</td>
<td>0.0101607705350008</td>
<td>(\pm 0.959688291448743)</td>
<td>0.0091486712307839</td>
</tr>
<tr>
<td>(\pm 0.983258286432624)</td>
<td>0.0082687692569876</td>
<td>(\pm 0.9759391745585136)</td>
<td>0.0070964709711387</td>
</tr>
<tr>
<td>(\pm 0.982517263563015)</td>
<td>0.00605855450423596</td>
<td>(\pm 0.988054126329624)</td>
<td>0.0050262074297252</td>
</tr>
<tr>
<td>(\pm 0.992543900323763)</td>
<td>0.00396455433844469</td>
<td>(\pm 0.995981842987209)</td>
<td>0.0029107318179345</td>
</tr>
<tr>
<td>(\pm 0.998364375683182)</td>
<td>0.00185396078894692</td>
<td>(\pm 0.999689503883231)</td>
<td>0.00079679206552013</td>
</tr>
</tbody>
</table>
Appendix C

Useful Theorems of Quantum Mechanics

C.1 Definition of a Finite-Range Potential

Let us return to the introduction of the quantity $R_{\text{range}}$. It is obvious that if there is a valid $R_{\text{range}}$ for which the condition (2.50) is true then any finite number $R' > R_{\text{range}}$ is also a valid range parameter. Hence the integral in (2.53) must be the same for any $R_{\text{range}}$ larger than a certain constant and thus the region of integration may be expanded back to all the real three-dimensional space. In other words, if our assumption about the finite-range nature of the potential is true then:

$$\forall \vec{\alpha} \exists \left| \int d^3\vec{r}' e^{-i\vec{\alpha} \cdot \vec{r}'} \langle \vec{r}' | \hat{V} | \vec{r}' \rangle < \infty \right. \tag{C.1}$$
We introduced an arbitrary $\vec{\alpha}$ to represent the product of an arbitrary momentum modulus $p$ with an arbitrary detector radius-vector direction $\vec{e}_r$. By using $< \infty$ we designate that the integral converges. Though this is a necessary, but not sufficient condition we can now invert this criterion and *define* a finite-range potential as one which satisfies (C.1). This is the exact definition of the finite-range potential we promised before and utilized in (2.54).

Let us assume that the potential is central. If as usual $\vec{\alpha}$ is the polar axis of the spherical coordinate system of choice (cf. (2.32)) it is easy to perform the angular part of the integral in (2.54) and thus obtain the definition of the finite-range central potential:

$$\forall \alpha > 0 \exists \left| \int_0^{\infty} r' dr' \sin(\alpha r') V(r') \right| < \infty.$$  \hspace{1cm} (C.2)

We dropped the constant factor in front of the integral because it is irrelevant for the convergence. The lower integration limit is designated $0^*$ to represent the fact that currently we are investigating only the convergence at infinity. To avoid dealing with a possible divergence at zero which is absolutely irrelevant when distinguishing between finite-range and infinite-range potentials we replace the actual 0 by a very small but finite number $0^*$. The same approach needs to be applied when determining the convergence of the integral in (C.1): a small vicinity around 0 must be excluded from the integration.

Because $|\sin(\alpha r')| \leq 1$ the integral in (C.2) converges if the limiting integral
converges:
\[ \exists \left| \int_0^\infty r'dr' V(r') \right| < \infty. \]  \hfill (C.3)

If \( V(r') \) satisfies (C.3) then it is a finite-range potential. For example, a power law \( \text{const}/r^\gamma \) with any \( \gamma > 2 \) satisfies this condition and thus is a finite-range potential. The same is true for the Yukawa potential (2.1). Note however that if with \( V(r') \) the integral in (C.3) does not converge the potential still could be a finite-range one because this condition is \textit{not equivalent} to (C.2). In fact, it could be proved that (C.2) is satisfied by a power law \( \text{const}/r^\gamma \) with any \( \gamma > 1 \). By substituting \( \text{const}/r' \) into (C.2) we can see that the Coulomb potential is not a finite-range one — just as we expected.

At last we may state that acceptable criteria for what is a finite-range potential were successfully developed.

\section*{C.2 The Free Green’s Function \( \hat{G}_0(E^+, \vec{K}_0) \)}

\textbf{in the Coordinate Representation}

When we were building the theory of elastic \( pn \) scattering in Subsection 2.2.2, one of the important tasks was to find the analytic expression (2.46) for the free Green’s function in the coordinate representation. This expression is invalid in the case of \( pn \) bremsstrahlung because photon emission changes the total momentum of the system of proton and neutron. The description of it in the same center-of-mass reference
frame both before and after the radiation event is no longer possible.

Before this subsection we were using designations like $|\vec{p}\rangle$ and $|\vec{r}\rangle$ assuming that these are momenta in the center-of-mass frame and the separation vectors: thus if the momentum of the first particle is $\vec{p}$, then the momentum of the second one is $-\vec{p}$ and the total momentum of the $pn$ system is always zero. The same is obviously true for the separation vectors: $\vec{r}$ for the proton means $-\vec{r}$ for the neutron. Thus the state is defined just by one vector. Now when there is no universal center-of-mass description we need two vectors to define a state: either momenta/radius-vectors of each particle $(|\vec{p}_1 \vec{p}_2\rangle$ and $|\vec{r}_1 \vec{r}_2\rangle$) or, in addition to the center-of-mass momenta/separations we used before, the total momentum $\vec{P}$ / the center-of-mass’s coordinate vector $\vec{R}$ $(|\vec{p} \vec{P}\rangle$ and $|\vec{r} \vec{R}\rangle$). For example the states at the beginning of this paragraph must now be correctly written as $|\vec{p} 0\rangle$ and $|\vec{r} 0\rangle$. For convenience, we will still use the $|\vec{p}\rangle$-type designations for states defined in a center-of-mass frame.

Thus the Green’s function in the coordinate representation is given by:

$$\langle \vec{r} \vec{R} | \hat{G}_0(E^+, \vec{K}_0) | \vec{r}' \vec{R}' \rangle = \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{d^3 \vec{K}}{(2\pi)^3} \langle \vec{r} \vec{R} | \hat{G}_0(E^+) | \vec{k} \vec{K} \rangle \langle \vec{k} \vec{K} | \vec{r}' \vec{R}' \rangle \cdot \delta(\vec{K} - \vec{K}_0). \quad (C.4)$$

The delta-function appears in the expression because we are expanding over the complete set of physically relevant momentum states, i.e., those with the same total momentum $\vec{K} = \vec{K}_0$. In fact such a delta-function $\delta(\vec{K})$ was implicitly present every time we expanded over the complete set of momentum states $|\vec{k}\rangle \langle \vec{k}|$, only then $\vec{K}_0$ was equal to zero (meaning we work in the center-of-mass reference frame). From the
mathematical point of view we just calculate the Green’s function for a fixed value of 
\(\vec{K}_0\).

Henceforth, if we take into account the relation (2.147) and make use of the fact
that momentum eigenstates are also Hamiltonian eigenstates (the general expression
for the free Hamiltonian (3.1) must be used here), then (C.4) changes into:

\[
\langle \vec{r} \vec{R} | \hat{G}_0(E^+, \vec{K}_0) | \vec{r}' \vec{R}' \rangle = \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{i \langle \vec{r} \vec{R} | \vec{k} \vec{K}_0 \rangle \langle \vec{k} \vec{K}_0 | \vec{r}' \vec{R}' \rangle}{E^+ - \frac{k^2}{2M} - \frac{K_0^2}{4M}}.
\] (C.5)

Let us introduce a new designation for the on-shell (relative to the Green’s function
energy \(E\)) momentum:

\[
k_0 \overset{\text{def}}{=} \sqrt{ME}.
\] (C.6)

Using it in (C.5) we obtain:

\[
\langle \vec{r} \vec{R} | \hat{G}_0(E^+, \vec{K}_0) | \vec{r}' \vec{R}' \rangle = M \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{i \langle \vec{r} \vec{R} | \vec{k} \vec{K}_0 \rangle \langle \vec{k} \vec{K}_0 | \vec{r}' \vec{R}' \rangle}{k_0^2 - \frac{K_0^2}{4M} - k^2}.
\] (C.7)

Of course there could be values of \(k_0\) and \(K_0\) for which \(k_0^2 - \frac{K_0^2}{4M}\) is negative and cannot
be written as the square of a real number, but we will assume that this is not the
case here (cf. condition (3.76)).

In Appendix C.3 we show that brackets of type \(\langle \vec{r} \vec{R} | \vec{p} \vec{P} \rangle\) are calculated the
same way as \(\langle \vec{r}_1 \vec{r}_2 | \vec{p}_1 \vec{p}_2 \rangle\), i.e. are just plane waves.

Thus, closely following the integral computation in (2.40 – 2.46) we get:
\[
\langle \vec{r} \vec{R} | \hat{G}_0(E^+, \vec{K}_0)| \vec{r}' \vec{R}' \rangle = iM \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k} \cdot \vec{r}} e^{i\vec{K}_0 \cdot \vec{R}} e^{i\vec{k} \cdot \vec{r}'} e^{i\vec{K}_0 \cdot \vec{R}'}}{p^* - k^2}
\]
\[
= iM e^{i\vec{K}_0 \cdot (\vec{R} - \vec{R}')} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k} \cdot (\vec{r} - \vec{r}')}}{p^* - k^2} = -\frac{iM}{4\pi} \frac{e^{i\vec{p} \cdot |\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} e^{i\vec{K}_0 \cdot (\vec{R} - \vec{R}')}.
\]

(C.8)

C.3 The Equivalence of \((\vec{P}, \vec{R}, \vec{p}, \vec{r})\) and \((\vec{p}_1, \vec{r}_1, \vec{p}_2, \vec{r}_2)\)

**Description of a Free Two-Particle System**

If we have two non-interacting particles of masses \(m_1\) and \(m_2\) then they could be described either by a set of each particle’s momentum \(\vec{p}_i\) and radius-vector \(\vec{r}_i\) or by a set of their relative and center-of-mass momenta:

\[
\vec{p} = \frac{m_1 \vec{p}_2 - m_2 \vec{p}_1}{m_1 + m_2},
\]

(C.9)

\[
\vec{P} = \vec{p}_1 + \vec{p}_2,
\]

(C.10)

and their relative and center-of-mass radius-vectors:

\[
\vec{r} = \vec{r}_2 - \vec{r}_1,
\]

(C.11)

\[
\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}.
\]

(C.12)

Let us prove that these two descriptions are equivalent in quantum mechanics. If they are, the wavefunction describing the two-particle system must be given both by \(\langle \vec{p}_1 | \vec{r}_1 \rangle \langle \vec{p}_2 | \vec{r}_2 \rangle\) and \(\langle \vec{P} | \vec{R} \rangle \langle \vec{p} | \vec{r} \rangle\). It is easy to show this using (C.9–C.12):
The equivalence has been proved.

C.4 Computation of a $T$-Matrix in a Non-Center-of-Mass Frame of Reference

In Chapter 2 we were calculating the $T$-matrix elements in a center-of-mass frame of reference: $\langle \vec{p}' | \hat{T}(E^+) | \vec{p} \rangle$. But how do we compute a $T$-matrix element in the frame shifted by a momentum $\vec{K}_0$ from the center-of-mass of the system, i.e. $\langle \vec{p}' | \vec{K}_0 | \hat{T}(E^+) | \vec{p} \vec{K}_0 \rangle$? From general considerations of classical mechanics (see e.g. [24]) we may expect the $T$-matrix in the non-center-of-mass frame of reference to be different only in a sense that its energy is less by $K_0^2 / 2M$ compared to the center-of-mass frame ($M$ is the sum of both particle masses, in our case of proton-neutron scattering it is equal $2M_{\text{nucleon}}$). Let us prove this.

It is obvious that the total momentum of the system cannot be changed by an elastic scattering and so both the $T$-matrix and the potential matrix $V$ must be
diagonal in relation to the macroscopic momentum:

\[
\langle \vec{p}' \vec{P}' | \hat{T}(E^+) | \vec{p} \vec{P} \rangle = (2\pi)^3 \delta(3)(\vec{P}' - \vec{P}) \langle \vec{p}' | \hat{T}(E^+, \vec{P}) | \vec{p} \rangle; \quad (C.14)
\]

\[
\langle \vec{p}' \vec{P}' | \hat{V} | \vec{p} \vec{P} \rangle = (2\pi)^3 \delta(3)(\vec{P}' - \vec{P}) \langle \vec{p}' | \hat{V} | \vec{p} \rangle. \quad (C.15)
\]

We have assumed that \( V \) is independent of \( \vec{P} \), which is true for any local potential.

Now let us use the Lippmann-Schwinger equation (2.64) with these matrix elements:

\[
\delta(3)(\vec{P}' - \vec{P}) \langle \vec{p}' | \hat{T}(E^+, \vec{P}) | \vec{p} \rangle = \delta(3)(\vec{P}' - \vec{P}) \langle \vec{p}' | \hat{V} | \vec{p} \rangle \]

\[
+ \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{d^3\vec{Q}}{(2\pi)^3} (2\pi)^3 \delta(3)(\vec{P}' - \vec{Q}) \langle \vec{p}' | \hat{V} | \vec{q} \rangle \delta(3)(\vec{Q} - \vec{P}) \langle \vec{q} | \hat{T}(E^+, \vec{P}) | \vec{p} \rangle \delta(3)(\vec{P}' - \vec{P}) \langle \vec{p}' | \hat{T}(E^+, \vec{P}) | \vec{p} \rangle.
\]

(C.16)

After integration by \( \vec{Q} \) the two delta-functions under the integral transform into \( \delta(3)(\vec{P}' - \vec{P}) \) and thus the whole equation is multiplied with the same delta-function, which may be omitted:

\[
\langle \vec{p}' | \hat{T}(E^+, \vec{P}) | \vec{p} \rangle = \langle \vec{p}' | \hat{V} | \vec{p} \rangle
\]

\[
+ \int \frac{d^3\vec{q}}{(2\pi)^3} \langle \vec{p}' | \hat{V} | \vec{q} \rangle \frac{1}{E^+ - \frac{q^2}{2\mu} - \frac{P^2}{2M}} \langle \vec{q} | \hat{T}(E^+, \vec{P}) | \vec{p} \rangle. \quad (C.17)
\]

We see that the matrix element \( \langle \vec{p}' | \hat{T}(E^+, \vec{P}) | \vec{p} \rangle \) obeys the same integral equation as \( \langle \vec{p}' | \hat{T}(E^+, 0) | \vec{p} \rangle \) does if \( \vec{E} = E - \frac{P^2}{2M} \). Hence, the answer to the question we asked in the beginning is:

\[
\langle \vec{p}' \vec{K}_0 | \hat{T}(E^+) | \vec{p} \vec{K}_0 \rangle = \langle \vec{p}' 0 | \hat{T}(E^+ - \frac{K^2}{2M}) | \vec{p} 0 \rangle. \quad (C.18)
\]
C.5 The Equivalence of $\hat{T}(p^\dagger, q; E)$ and $\hat{T}(q^\dagger, p; E)$

Using the expression (2.146) we can write an arbitrary $T$-matrix element as:

\[
\langle \vec{q} | \hat{T}(E^+) | \vec{p} \rangle = \langle \vec{q} | \hat{V} \rangle + \int \frac{d^3k_1}{(2\pi)^3} \frac{\langle \vec{q} | \hat{V} \rangle \langle \vec{k}_1 | \hat{V} | \vec{p} \rangle}{E^+ - k_1^2/2\mu} \\
+ \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \langle \vec{q} | \hat{V} \rangle \langle \vec{k}_1 \rangle \left( \frac{1}{E^+ - k_2^2/2\mu} \langle \vec{k}_1 | \hat{V} \rangle \langle \vec{k}_2 | \hat{V} \rangle \frac{1}{E^+ - k_1^2/2\mu} \langle \vec{k}_2 | \hat{V} | \vec{p} \rangle \right) \\
+ \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_3}{(2\pi)^3} \langle \vec{q} | \hat{V} \rangle \langle \vec{k}_1 \rangle \langle \vec{k}_2 \rangle \frac{1}{E^+ - k_3^2/2\mu} \langle \vec{k}_1 | \hat{V} \rangle \langle \vec{k}_2 | \hat{V} \rangle \langle \vec{k}_3 | \hat{V} | \vec{p} \rangle + \ldots
\]

(C.19)

As for a local potential $\langle \vec{q} | \hat{V} | \vec{p} \rangle = \langle \vec{p} | \hat{V} | \vec{q} \rangle$, we can regroup terms:

\[
\langle \vec{q} | \hat{T}(E^+) | \vec{p} \rangle = \langle \vec{p} | \hat{V} \rangle + \int \frac{d^3k_1}{(2\pi)^3} \frac{\langle \vec{p} | \hat{V} \rangle \langle \vec{k}_1 | \hat{V} | \vec{q} \rangle}{E^+ - k_1^2/2\mu} \\
+ \int \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_1}{(2\pi)^3} \langle \vec{p} | \hat{V} \rangle \langle \vec{k}_1 \rangle \left( \frac{1}{E^+ - k_2^2/2\mu} \langle \vec{k}_2 | \hat{V} \rangle \langle \vec{k}_1 | \hat{V} \rangle \frac{1}{E^+ - k_1^2/2\mu} \langle \vec{k}_1 | \hat{V} | \vec{q} \rangle \right) \\
+ \int \frac{d^3k_3}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_1}{(2\pi)^3} \langle \vec{p} | \hat{V} \rangle \langle \vec{k}_1 \rangle \langle \vec{k}_2 \rangle \langle \vec{k}_3 \rangle \langle \vec{k}_2 \rangle \langle \vec{k}_3 | \hat{V} | \vec{p} \rangle + \ldots
\]

(C.20)

which is the same as $\langle \vec{p} | \hat{T}(E^+) | \vec{q} \rangle$. 
A general relativistic formula for the bremsstrahlung cross-section (see e.g. [25]) is:

\[
d\sigma = \frac{1}{2\epsilon_1 2\epsilon_2} \frac{1}{|\vec{v}_1 - \vec{v}_2|} \frac{d^3\vec{p}_1'}{(2\pi)^3} \frac{d^3\vec{p}_2'}{(2\pi)^3} \frac{1}{2\epsilon'_1 2\epsilon'_2} \frac{d^3\vec{k}_\omega}{(2\pi)^3 2\omega} \times
\]

\[
\times |f^{(rel)}_\omega|^2 (2\pi)^6 \delta^{(4)}(\bar{\epsilon}_1 \vec{p}_1' + \bar{\epsilon}_2 \vec{p}_2' - \bar{\epsilon}_1' \vec{p}_1' - \bar{\epsilon}_2' \vec{p}_2' - \bar{\epsilon}_\omega \vec{k}_\omega). \tag{C.21}
\]

Here \(\bar{\epsilon}_i = (\epsilon_i, \vec{p}_i)\) are the initial and \(\bar{\epsilon}'_i = (\epsilon'_i, \vec{p}'_i)\) are the final nucleon 4-momenta, \(\bar{\epsilon}'_\omega = (\omega, \vec{k}_\omega)\) is the emitted photon’s 4-momentum, \(\vec{v}_i\) are the initial nucleon 3-velocities and \(f^{(rel)}_\omega\) is the relativistic bremsstrahlung amplitude. When deriving the expressions for the bremsstrahlung amplitude (3.87–3.89) in Chapter 3 we employed the relativistic normalization only for the photon:

\[
\langle \vec{p}_1' | \vec{k}_\omega \rangle = (2\pi)^3 2\omega \delta^{(3)}(\vec{k}_1' - \vec{k}_\omega), \tag{C.22}
\]

while the nucleon moments were normalized non-relativistically:

\[
\langle \vec{p}_i' | \vec{p}_i \rangle = (2\pi)^3 \delta^{(3)}(\vec{p}_1' - \vec{p}_i). \tag{C.23}
\]

Thus our bremsstrahlung amplitude is related to the amplitude of equation (C.21) by the formula:

\[
f^{(rel)}_\omega = \sqrt{4\epsilon'_1 \epsilon'_2} f_\omega \sqrt{4\epsilon_1 \epsilon_2}. \tag{C.24}
\]

To compute \(|\vec{v}_1 - \vec{v}_2|\) we need to use two well-known relations of relativistic mechanics:

\[
\vec{v}_i = \frac{\vec{p}_i}{\epsilon_i}; \tag{C.25}
\]
\[
\epsilon_i^2 = M^2 + p_i^2. \tag{C.26}
\]

Applying them, after performing trivial transformations we get:
\[
|\vec{v}_1 - \vec{v}_2|^2 = \frac{M^2 (p_1^2 + p_2^2) + 2p_1^2 p_2^2 - 2(\vec{p}_1 \cdot \vec{p}_2) \epsilon_1 \epsilon_2}{\epsilon_1^2 \epsilon_2^2}. \tag{C.27}
\]

On the other hand, if we assume that we are working in a “collinear” frame, in which 
\[|\vec{p}_1 \vec{p}_2| = p_1 p_2\] (both center-of-mass and laboratory frames of reference satisfy this condition), then:
\[
(p_1^2 - \epsilon_1 \epsilon_2)^2 - M^4 = 2p_1^2 p_2^2 - 2p_1 p_2 \epsilon_1 \epsilon_2 + M^2 (p_1^2 + p_2^2). \tag{C.28}
\]

Comparing (C.27) and (C.28) to each other we finally obtain:
\[
|\vec{v}_1 - \vec{v}_2| = \sqrt{(p_1 \vec{p}_2 - \epsilon_1 \epsilon_2)^2 - M^4} \frac{1}{\epsilon_1 \epsilon_2}. \tag{C.29}
\]

Now we can substitute the relations (C.24) and (C.29) into (C.21) and get:
\[
d\sigma = \epsilon_1 \epsilon_2 \left( \frac{d^3 \vec{p}_1'}{(2\pi)^3} \frac{d^3 \vec{p}_2'}{(2\pi)^3} \frac{d^3 \vec{k}_\omega}{(2\pi)^3} \frac{1}{2\omega} |f_\omega|^2 \right) \times \\
(2\pi)^4 \delta(\epsilon_1 + \epsilon_2 - \epsilon_1' - \epsilon_2' - \omega) \delta^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_1' - \vec{p}_2' - \vec{k}_\omega). \tag{C.30}
\]

This is not yet the relation we seek because, as we know from Subsection 3.2.2, the
kinematics of a bremsstrahlung process is determined by five scalar variables, while
in the right-hand-side of (C.30) there are differentials of three vector — and thus nine
scalar — variables. Let us choose a different from (3.96) set of kinematic variables,
namely four final scattering angles of the two nucleons \(\theta_1', \phi_1', \theta_2', \phi_2\) and the photon
emission angle \(\theta_\omega\). This way we will get the differential cross-section \(d\sigma/d\Omega_1 d\Omega_2 d\theta_\omega\)
where \( \Omega_i' \) are the elements of nucleon scattering solid angles — so that it is in the form that can be actually measured (e. g. by using particle detectors to measure the detection angles). Now the nine vector components need to be expressed through these five parameters (we will later choose a slightly different parameter set, but at the moment it is convenient to work with these five angles) and then (C.30) should be integrated four times — until only the differentials of the five kinematic parameters left in the right-hand-side.

Assuming that we are in the center-of-mass frame the differentials and the delta-functions are transformed as follows:

\[
\begin{align*}
    d^3 \vec{p}_i' &= p_i'^2 dp_i' d\Omega_i'; \\
    d^3 \vec{k}_\omega &= \omega^2 d\omega \sin \theta_\omega d\theta_\omega d\phi_\omega; \\
    \delta(\epsilon_1 + \epsilon_2 - \epsilon'_1 - \epsilon'_2 - \omega) &= \delta(E - \frac{p_1'^2}{2M} - \frac{p_2'^2}{2M}); \\
    \delta^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_1' - \vec{p}_2' - \vec{k}_\omega) &= \delta^{(3)}(\vec{p}_1' + \vec{p}_2' + \vec{k}_\omega).
\end{align*}
\]

In the last relation we made use of the fact that in the center-of-mass frame \( \vec{p}_1 + \vec{p}_2 = 0 \).

Thus the relation for the differential cross-section is:

\[
\frac{d\sigma}{d\Omega'_1 d\Omega'_2 d\theta_\omega} = \frac{\epsilon_1 \epsilon_2 |f_\omega|^2}{2(2\pi)^5 \sqrt{(\vec{p}_1 \vec{p}_2 - \epsilon_1 \epsilon_2)^2 - M^4}} \times \mathcal{I},
\]

where \( \mathcal{I} \) stands for:

\[
\mathcal{I} = \int dp_1' dp_2' d\omega d\phi_\omega p_1'^2 p_2'^2 \omega \sin \theta_\omega \delta(E - \frac{p_1'^2}{2M} - \frac{p_2'^2}{2M}) \delta^{(3)}(\vec{p}_1' + \vec{p}_2' + \vec{k}_\omega).
\]

With the help of trivial but cumbersome transformations this integral can be per-
formed:

$$\mathcal{I} = \frac{M p_1'^2 p_2'^2}{(p_2' \cos \theta_1' - p_1' \cos \theta_2') \sin \theta_\omega + \frac{p_1' p_2' (\sin^2 \theta_1' - \sin^2 \theta_2') + (p_2'^2 - p_1'^2) \sin \theta_1' \sin \theta_2' \cos (\phi_2' - \phi_1')} {\sqrt{p_1'^2 \sin^2 \theta_1' + p_2'^2 \sin^2 \theta_2' + 2p_1' p_2' \sin \theta_1' \sin \theta_2' \cos (\phi_2' - \phi_1')}} \cos \theta_\omega}. \quad (C.37)$$

One last task that is left is to express all the variables present in \((C.35)\) and \((C.37)\) through the chosen kinematic parameters:

$$\theta_1', \phi_1', \theta_2', \phi_2', \theta_\omega. \quad (C.38)$$

Let us start with \((C.35)\). Timelike components of 4-momenta \(\epsilon_i\) are expressed through \(p_i\) with the relation \((C.26)\). As in the center-of-mass frame of reference \(\vec{p}_1 = -\vec{p}_2\) and \(p_i^2 = M(E + \omega)\), the equation \((C.35)\) is transformed into:

$$\frac{d\sigma}{d\Omega_1' d\Omega_2' d\theta_\omega} = \frac{[M^2 + M(E + \omega)]|f_\omega|^2}{2(2\pi)^5 \sqrt{[M^2 + 2M(E + \omega)]^2 - M^4}} \times \mathcal{I}, \quad (C.39)$$

where due to the momentum conservation (component along the initial proton momentum direction \(\vec{p}_1\), which, as we agreed in Subsection 3.2.2, serves as the polar axis of the spherical coordinate system):

$$\omega = -\frac{p_1' \cos \theta_1' + p_2' \cos \theta_2'} {\cos \theta_\omega}. \quad (C.40)$$

Now both in \((C.39)\) and in \((C.37)\) there are only two “alien” variables: \(p_1'\) and \(p_2'\), and we are still left with the final energy parameter \(E\). We will see that the latter one does not present a serious complication. To express \(p_i'\) through the parameter set \((C.38)\) we need to solve a system of equations, one of which is just the energy
conservation:

\[ p_1'^2 + p_2'^2 = 2ME. \]  \hspace{1cm} (C.41)

The other equation that we need can be derived by combining (C.40) with the conservation of the other momentum components:

\[
p_1'^2 \sin^2 \theta'_1 + p_2'^2 \sin^2 \theta'_2 + 2p_1'p_2' \sin \theta'_1 \sin \theta'_2 \cos(\phi'_1 - \phi'_2) = \omega^2 \sin^2 \theta_\omega. \] \hspace{1cm} (C.42)

Substituting (C.40) into (C.42) we get:

\[
p_1'^2 \sin^2 \theta'_1 + p_2'^2 \sin^2 \theta'_2 + 2p_1'p_2' \sin \theta'_1 \sin \theta'_2 \cos(\phi'_1 - \phi'_2) = (p'_1 \cos \theta'_1 + p'_2 \cos \theta'_2)^2 \tan^2 \theta_\omega. \] \hspace{1cm} (C.43)

Equations (C.41) and (C.43) constitute a system of quadratic equations for the variables \( p'_1 \) and \( p'_2 \). It can be easily solved to obtain:

\[
p_1'^2 = 2ME \frac{S_2(S_2 - S_1) + 2C^2 \pm \sqrt{C^2(C^2 - S_1S_2)}}{(S_1 - S_2)^2 + 4C^2}; \] \hspace{1cm} (C.44)

\[
p_2'^2 = 2ME \frac{S_1(S_1 - S_2) + 2C^2 \mp \sqrt{C^2(C^2 - S_1S_2)}}{(S_1 - S_2)^2 + 4C^2}, \] \hspace{1cm} (C.45)

with auxiliary parameters:

\[
S_1 \overset{\text{def}}{=} \sin(\theta_\omega - \theta'_1) \sin(\theta_\omega + \theta'_1); \] \hspace{1cm} (C.46)

\[
S_2 \overset{\text{def}}{=} \sin(\theta_\omega - \theta'_2) \sin(\theta_\omega + \theta'_2); \] \hspace{1cm} (C.47)

\[
C^2 \overset{\text{def}}{=} (\sin^2 \theta_\omega \cos \theta'_1 \cos \theta'_2 + \cos^2 \theta_\omega \sin \theta'_1 \sin \theta'_2 \cos 2\Phi)^2; \] \hspace{1cm} (C.48)

\[
\Phi \overset{\text{def}}{=} \frac{1}{2}(\pi + \phi'_1 - \phi'_2). \] \hspace{1cm} (C.49)

As we can see, with the chosen kinematic variables two symmetric solutions are
possible: we cannot determine, which momentum is “the first” and which one is “the second”.

Note that the angles $\phi_1$ and $\phi_2$ are present in (C.37) only as $\cos(\phi_2' - \phi_1')$. Thus we can use a simple trigonometric transformation to replace it with:

$$\cos(\phi_2' - \phi_1') = 2\cos^2 \left( \frac{\phi_2' - \phi_1'}{2} \right) = 2\sin^2 \Phi, \quad (C.50)$$

which changes (C.37) into:

$$I = \frac{M p_1'^2 p_2'^2}{\left| (p_2' \cos \theta_1' - p_1' \cos \theta_2') \sin \theta_\omega + \frac{p_1' p_2' (\sin^2 \theta_1' - \sin^2 \theta_2') + 2(p_2'^2 - p_1'^2) \sin \theta_1' \sin \theta_2' \sin^2 \Phi \cos \theta_\omega}{\sqrt{p_1'^2 \sin^2 \theta_1' + p_2'^2 \sin^2 \theta_2' + 4p_1' p_2' \sin \theta_1' \sin \theta_2' \sin^2 \Phi}} \right|. \quad (C.51)$$

Now if we look at the equations (C.39), (C.51), (C.40) and (C.44 – C.48), we see that they solve the problem of $pn$ bremsstrahlung cross-section, but through a slightly different than (C.38) set of parameters for the right-hand-side:

$$E, \theta_1', \theta_2', \Phi, \theta_\omega. \quad (C.52)$$

Using the variable $\Phi$ instead individual $\phi_1'$ and $\phi_2'$ is common in bremsstrahlung studies (see e.g. [3]).

Of course, before we can substitute $f_\omega$ computed using the algorithms described in Section 3.2 into (C.39), we must transform parameters (3.96) into (C.52) (note that angles $\theta_1'$, $\theta_2'$ and $\Phi$ are measured in the initial center-of-mass frame of reference, while $\theta_*$ and $\phi_*$ are measured in the final center-of-mass frame). This is again a trivial but cumbersome task and we will not discuss it here.
Appendix D

Algorithm for Computing an

On-Shell $T$-Matrix and

Corresponding Phase Shifts $\delta_l$

1. Define constants: nucleon reduced mass $2\mu = M_{\text{nucleon}}$, pion mass $m_\pi$, coupling constant $g^2$ and unit transformation coefficient $\hbar c$.

2. Input physical variables, defining the problem: center-of-mass nucleon energy $E$, orbital angular momentum $l$.

3. Input model variables: the order of Gauss-Legendre method $N$, the main cutoff parameter (numerical infinity) $\Delta_{\text{cut}}$ and the low momentum boundary $\Delta_{\text{mid}}$. In our case $N = 192$, $\Delta_{\text{cut}} = 1000m_\pi$ and $\Delta_{\text{mid}} = 10m_\pi$. 
4. Generate or upload from file the Gauss-Legendre abscissas $\xi_i$ and weights $w_i^{(\xi)}$ of order $N/2$ at the interval $[-1,+1]$. Use formulae (2.132–2.133) first with $a = 0$, $b = \Delta_{mid}$, then with $a = \Delta_{mid}$, $b = \Delta_{cut}$ to produce desired set of $N$ Gauss-Legendre abscissas $k_i$ and weights $w_i$, covering the interval $[0, \Delta_{cut}]$.

5. If energy $E$ and masses $\mu$ and $m_\pi$ are defined in units of MeV, use (2.13) to translate them into fm$^{-1}$. Then calculate the on-shell momentum modulus (2.120) $k_0 = \sqrt{2\mu E}$.

6. Create a subroutine computing $V_l(p', p)$ using (2.103–2.104). Corresponding $Q_l$’s should be either calculated with (2.107–2.108, 2.116–2.119, B.6–B.10) or generated by the recurrence relation (2.111) starting from (2.107) and (2.108).

7. For the right-hand-side of the matrix equation (2.142) create an array of dimension $N + 1$, then fill it with $V_j^{(l)} \equiv V_l(k_j, k_0), j = 0, 1, \ldots, N$.

8. Compute $Q$ with (2.141). Then create an array of dimensions $N + 1$ by $N + 1$ and fill it with the correspondent elements of the matrix in the left-hand-side of (2.142).

9. Use a standard library function (or write a routine) to solve linear matrix equation (2.142) for unknown $T_j^{(l)}$’s with the arrays defined in the previous two steps.

10. Apply the relation (2.98) to $T_0^{(l)} \equiv T_l(k_0, k_0)$ in order to obtain the phase shift $\delta_l$. 
11. Output the value of $\delta_l$ vs. the energy $E$. Also the full array of size $N + 1$

$$\{T_j^{(l)} \equiv T_l(k_j, k_0; E^+)\}$$

should be included into the output so that this code can be used as a subroutine with other codes.
List of References


*http://nn-online.sci.kun.nl/NN/nnphs2.shtml.*

The website is maintained by the Theoretical High Energy Physics Group of University of Nijmegen, the Netherlands. It allows user to calculate NN phase shifts with several models in the range of laboratory energies from 0 to 350 MeV.


