1 Introduction

The main focus of this paper will be on two very different areas in which topology is relevant to the study of infinite graphs. The first is the mechanics of compactness proofs, which use a particular group of lemmas to extend results about finite subgraphs to apply to an entire infinite graph. We will explore these results by using them to prove a result of de Bruijn and Erdős, that an infinite graph is $k$-colorable if its finite subgraphs are $k$-colorable, in several different ways. The name “compactness proof,” as we shall see, comes from the fact that the needed machinery can be derived from Tychonoff’s theorem on products of compact spaces.

The second area is a relatively new area of study pioneered by Diestel which redefines certain concepts of graph theory in terms of a topology on a graph. Specifically, we find that certain basic features of the cycle space cannot be extended verbatim to infinite graphs. But if we define the cycle space in terms of homeomorphic images of the circle $S^1$ in a compactified topology on the graph, we can find extensions. This will be motivated in more detail and some of the consequences explored.

For the purposes of this paper, we define a graph in the usual way, except that we do not require $V(G)$ to be finite. As such, all theorems of finite graph theory carry over with the additional requirement that graphs involved be finite.

2 Compactness Proofs

Compactness proofs use one of a number of results to extend results from the finite subgraphs of an infinite graph to the entire graph. We will demonstrate compactness proofs on one theorem using a few different results, some of which are applicable only in graphs with countably many vertices, while others (including Tychonoff’s Theorem, from which the others can be derived) are more general. We will conclude the section with some discussion of how the results we used are interrelated.
2.1 Four Proofs of a Theorem of de Bruijn and Erdös

A prototypical example of a theorem which can be proved using a compactness argument, and for which we will give four such proofs, is the following:

**Theorem 1** (de Bruijn & Erdös, 1951 [1]). Let $G$ be a graph and $k \in \mathbb{N}$. Then if every finite subgraph of $G$ is $k$-colorable, $G$ is $k$-colorable.

Our first proof will be a very simple one from Rado’s Selection Theorem, an important result which we will state shortly. First, we need to distinguish between two uses of the word “coloring” which can be confused in this context.

The $k$-colorings of $G$ referred to in Theorem 1 are the more specific version commonly used in a graph theoretic context: They are functions $f : V(G) \rightarrow \{1, \ldots, k\}$ satisfying the property that $f(x) \neq f(y)$ when $x$ and $y$ are adjacent. Outside the context of Rado’s Selection Theorem, we will always use this meaning of “coloring”.

The $k$-colorings referred to in Rado’s Selection Theorem are more general: they are simply functions from a set (call it $A$) to $\{1, \ldots, k\}$. The theorem involves defining a set $\mathcal{F}$ of $k$-colorings of subsets of $A$ which we call forbidden, and we call a $k$-coloring of a subset of $A$ $\mathcal{F}$-admissible if it does not contain a member of $\mathcal{F}$ as a restriction. The only place where this is likely to be confusing is in Proof 1 of Theorem 1, and we will make it clear which sense of the term we are using. The first meaning of $k$-coloring is essentially a special case of the more general meaning of the term (which makes proving Theorem 1 from Rado’s Theorem very easy, as we shall see).

**Theorem 2** (Rado’s Selection Theorem). Let $A$ be a set and $\mathcal{F}$ a set of forbidden $k$-colorings of finite subsets of $A$ such that every finite subset of $A$ has an $\mathcal{F}$-admissible $k$-coloring. Then there is an $\mathcal{F}$-admissible $k$-coloring of all of $A$.

This formulation is taken from [10], which also gives a proof using Zorn’s lemma and the theory of ultrafilters. We will give our own proof from Tychonoff’s Theorem near the end of this section, on page 6.

**Proof 1 of Theorem 1 (from Rado’s Selection Theorem), based on [10].** Let $\mathcal{F}$ be the set of $k$-colorings on finite subsets of $V(G)$ that send two adjacent vertices to the same element of $\{1, \ldots, k\}$. By assumption, every finite subgraph of $G$ has an $\mathcal{F}$-admissible coloring. Applying Rado’s Selection Theorem, we conclude that $G$ has an $\mathcal{F}$-admissible coloring, which is exactly a $k$-coloring of $G$ in the normal graph theoretic sense.

Our next proof uses a classic result in infinite graph theory, introduced by Dénes König in 1936 [7]. First, we take the opportunity to define some related structures which will be important from here on.

A ray (Fig. 1) is a one-way infinite path; formally a graph $R$ with $V(R) = \{r_n\}_{n \in \mathbb{N}}$ and $E(R) = \{r_nr_{n+1} \mid n \in \mathbb{N}\}$. We may also write $R$ as $r_0r_1 \ldots$ much as we would with a finite path. A sub-ray of a ray $R$ is called a tail of $R$. 

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A double ray (Fig. 2) is a graph $D$ with $V(D) = \{v_i\}_{i \in \mathbb{Z}}$ and $E(D) = \{v_i v_{i+1} \mid i \in \mathbb{Z}\}$, also written $\ldots v_{-1}v_0v_1\ldots$.

Now we can state:

**Lemma 3** (König’s Infinity Lemma). Let $G$ be a graph and $A_n$ be a set of finite, non-empty, disjoint subsets of $V(G)$, such that for $n > 0$ each $x \in A_n$ has a neighbor $f(x)$ in $A_{n-1}$. Then $G$ contains a ray $R = r_0r_1\ldots$ with $r_n \in A_n$ for each $n$.

König’s Lemma can also be formulated in terms of the existence of rays in trees, but this formulation is most easily applicable for our purposes. A simple proof is possible from basic set theoretic principles, but in order to begin showing the connection between methods of giving compactness proofs, we will give a proof from Rado’s Selection Theorem.

**Proof of König’s Infinity Lemma.** We want $\mathcal{F}$ be a set of forbidden 2-colorings on subsets of $\bigcup_{n \in \mathbb{N}} \{A_n\}$ so that to be admissible a coloring must send exactly one vertex of $A_n$ to 1 if its domain contains $A_n$ and if $x, f(x) \in \text{dom}(F)$ and $F(x) = 1$, $F(f(x))$ must also equal 1. Then a 2-coloring of the entire graph that is $\mathcal{F}$-admissible will have to contain a ray with its vertices all sent to 1 by $F$. To achieve this, we can describe $\mathcal{F}$ as containing those colorings $F : W \mapsto \{1, 2\}$, where $W$ is a finite subset of $\bigcup_{n \in \mathbb{N}} \{A_n\}$, that satisfy one of the following:

i. $\exists n \text{ s.t. } A_n \subset W$ and $F(x) = 2$ for all $x \in A_n$,

ii. $\exists x, y, n \text{ s.t. } F(x) = F(y) = 1, x \neq y$, and $x, y \in A_n$, or,

iii. $\exists x \text{ s.t. } F(x) = 1, F(f(x)) = 2$.

Let $V'$ be a finite subset of $\bigcup_{n \in \mathbb{N}} \{A_n\}$. We can find an $\mathcal{F}$-admissible coloring of $V'$ as follows: Let $n$ be maximal such that $A_n \cap V' \neq \emptyset$. Pick a vertex from $A_n \cap V'$, call it $x_n$ and set $F(x_n) = 1, F(y) = 2$ for all $y \in A_n \cap V' \setminus \{x_n\}$. Then we proceed by backwards recursion: Assume $A_m$ intersects $V'$, and we have picked $x_i$ sent to 1 for all $i > m$ with $A_i$ intersecting $V'$ and sent other vertices in those $A_i$ to 2. If $x_{m+1}$ exists and $f(x_{m+1}) \in V'$, call $x_m := f(x_{m+1})$. Otherwise, pick an arbitrary vertex in $A_m \cap V'$ to be $x_m$. Then set $F(x_m) = 1, F(y) = 2$ for all $y \in V' \cap A_m \setminus x_m$. It should be clear that the resulting coloring
of $V'$ is $\mathcal{F}$-admissible. Now, by Rado’s Selection Theorem, there is an $\mathcal{F}$-admissible coloring $F'$ of $\cup \{A_n\}$. The vertices sent by $F'$ to 1 make up the desired ray.  

Proof 2 of Theorem 1, based on [4] (from König’s Infinity Lemma, for countable $G$). Fix an enumeration $\{v_n\}$ of $V(G)$. Then for each $n \in \mathbb{N}$ define $A_n$ to be the set of $k$-colorings on $G[\{v_0, \ldots, v_n\}]$. By assumption, each $A_n$ is non-empty. Also, if we restrict a $k$-coloring of $G[\{v_0, \ldots, v_n\}]$ to $G[\{v_0, \ldots, v_{n-1}\}]$, we have a $k$-coloring of $G[\{v_0, \ldots, v_{n-1}\}]$. Hence if we define a graph $G'$ with $\cup_{n \in \mathbb{N}} \{A_n\}$ as its vertex set, and for each $f \in A_n$ an edge joining it with its restriction in $A_{n-1}$, the graph fulfills the hypothesis of König’s Infinity Lemma. Hence there is a ray $f_0 f_1 \ldots$ with each $f_n$ being a $k$-coloring of $G[\{v_0, \ldots, v_n\}]$ and a restriction of $f_{n+1}$.

Consider $f' := \cup_{n \in \mathbb{N}} f_n$. Every vertex in $G$ can be written as $v_n$, and so is in the domain of $f_n$ and not in the domain of $f_i$ for $i < n$. Furthermore, every $f_m$ with $m > n$ contains $f_n$ as a restriction, so $f_m(v_n) = f_n(v_n)$. So $f' : V(G) \mapsto \{1, \ldots, k\}$ is a well-defined function. Also, since any two vertices of $G$ are in the domain $f_n$ for large enough $n$, they cannot be adjacent and mapped to the same color by $f'$. Hence $f'$ is the $k$-coloring of $G$ that we seek.  

Theorem 1 can also be proven using topological methods. We will give two such proofs. First we state Tychonoff’s famous theorem on the products of compact spaces:

**Theorem 4** (Tychonoff’s Theorem). Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of topological spaces such that $X_\alpha$ is compact for all $\alpha \in J$. Then the product topology on $\prod_{\alpha \in J} X_\alpha$ is compact.

This is a standard theorem of point set topology; a proof (via Zorn’s Lemma) can be found in [8], pp. 234. It can also be used to prove the results used in the two previous proofs, as we shall show.

Proof 3 of Theorem 1, based on [4] (from Tychonoff’s Theorem). Consider the product space

$$X := \prod_{V(G)} \{1, \ldots, k\}$$

where each $\{1, \ldots, k\}$ is endowed with the discrete topology, and $X$ is endowed with the product topology. We can think of points in this space as functions from $V(G)$ to $\{1, \ldots, k\}$. By Tychonoff’s theorem, since $X$ is a product of compact spaces, it is also compact.

The basic open sets of $X$ are of the form:

$$F_{U,h} := \{f : V(G) \mapsto \{1, \ldots, k\} \mid f|_U = h\}$$

where $U$ is a non-empty finite subset of $V(G)$ and $h$ is a function from $U$ to $\{1, \ldots, k\}$. Let $A_U$ be the set of functions in $X$ whose restriction to $U$ is a valid $k$-coloring of $U$. By assumption, $A_U$ is non-empty for every $U$. Also,

$$A_U = \cup \{F_{U,h} \mid h \text{ is a valid } k\text{-coloring of } U\}$$
and
\[ A_U^C = \bigcup \{ F_{U,h} \mid h \text{ is not a valid } k\text{-coloring of } U \} \]
are both open sets, so \( A_U \) is both open and closed. Furthermore, the sets \( A_U \) have the finite intersection property: For any finite \( \mathcal{U} = \{ U_1, \ldots, U_n \} \), \( \bigcap_{U \in \mathcal{U}} A_U = A_{\mathcal{U}} \), which is non-empty by assumption. Hence by the closed set formulation of compactness, \( \bigcap_{U \subseteq V(G)} A_U \) is non-empty. Each element is a \( k\)-coloring of \( G \).

We will now give one more proof of the countable case of the theorem, which proceeds from a form of compactness but is more similar in form to the proof from König’s Lemma. First, we need the following lemma on the space \( X \) defined in the previous proof:

**Lemma 5.** If \( V(G) \) is countable, \( X := \prod_{V(G)} \{ 1, \ldots, k \} \) is sequentially compact.

**Proof.** Let \( \{ x_n \}_{n \in \mathbb{N}} \), where \( x_n = x_{n1}x_{n2} \ldots \) be an infinite sequence of points in \( X \). \( \{ x_n \} \) induces an infinite sequence \( \{ x_{n1} \}_{n \in \mathbb{N}} \) in the first coordinate, which must have a convergent subsequence \( \{ x_{n1}' \}_{n' \in \mathbb{N}} \). In fact, since \( \{ 1, \ldots, k \} \) has the discrete topology, this subsequence must be constant for all \( n' \geq m_1 \) for some \( m_1 \in \mathbb{N} \). Using this fact, we begin to define a subsequence of \( \{ x_n \}_{n \in \mathbb{N}} \) which we will call \( \{ y_n \}_{n \in \mathbb{N}} \). We let \( y_1 = x_{m_1} \). Now we can proceed recursively; \( \{ x_{n'} \}_{n' \in \mathbb{N}} \) induces an infinite sequence \( \{ x_{n2}' \}_{n' \in \mathbb{N}} \) which has a subsequence \( \{ x_{n2''} \}_{n'' \in \mathbb{N}} \) which is constant for all \( n'' > m_2 \) for some \( m_2 > m_1 \). We let \( y_2 = x_{m_2} \), and continue in the same manner.

Now \( \{ y_n \}_{n \in \mathbb{N}} \) is a subsequence of \( \{ x_n \}_{n \in \mathbb{N}} \), which is constant in the first \( m \) coordinates for all \( n \geq m \). This clearly converges to \( (y_{11}, y_{22}, \ldots) \).

**Proof 4 (from sequential compactness, for countable \( G \)).** Consider the product space \( X \) from the previous proof. Since we now have countable \( G \), the space is sequentially compact. We start by fixing an enumeration \( \{ v_n \} \) of \( V(G) \). Then for each \( n \) we pick \( f_n \in X \) such that \( f_n|_{\{ v_0, \ldots, v_n \}} \) is a valid \( k\)-coloring of \( \{ v_0, \ldots, v_n \} \). Since \( X \) is sequentially compact, \( \{ f_n \} \) has a convergent subsequence. Let \( f' \) be the point of \( X \) to which this sequence converges. We want to show that \( f' \) is a \( k\)-coloring of \( G \).

Suppose not: Then there are adjacent vertices of \( G \), \( v_i, v_j \), such that \( f'(v_i) = f'(v_j) \). Assume without loss of generality that \( i < j \). Let \( A_j \) be the neighborhood of \( f' \) given by
\[ A_j = \{ f : V(G) \rightarrow \{ 1, \ldots, k \} \mid f|_{\{ v_0, \ldots, v_j \}} = f'|_{\{ v_0, \ldots, v_j \}} \} \]
Since a subsequence of \( \{ f_n \} \) converges to \( f' \), \( f_n \in A_j \) for infinitely many \( n \). Hence we can pick \( m > j \) such that \( f_m \in A_j \). Then we have \( f_m|_{\{ v_0, \ldots, v_j \}} = f'|_{\{ v_0, \ldots, v_j \}} \), so \( f_m(v_i) = f_m(v_j) \). But \( f_m \) is a valid \( k\)-coloring of \( \{ v_0, \ldots, v_m \} \supseteq v_i, v_j \), so this is a contradiction. We can conclude that \( f' \) is a \( k\)-coloring of \( G \).
2.2 Derivation from Tychonoff’s Theorem

Now we turn to examining the relation between Tychonoff’s Theorem and the other results used in this section.

First, to further suggest the link between König’s Lemma and Tychonoff’s Theorem, it’s instructive to compare the second and fourth proofs. Suppose that in the second proof, rather than citing König’s Lemma, we prove it as we go, based on the principle that if an infinite set is partitioned into finitely many sets, at least one must be infinite. Essentially, we have a set $K_n$ of valid $k$-colorings on the first $n$ vertices in our enumeration. We know that this set is non-empty and finite for every $n$, and we know that every element of $K_n$ contains one element of $K_{n-1}$ as a restriction.

So, for each $n$, $K_n$ contains a restriction of every one of the infinitely many elements of $K_n' = \cup_{m>n} \{K_m\}$. And since there are finitely many elements of $K_n$, one of them, which we’ll call $k_n$ must be a restriction of infinitely many elements of $K_n'$. Then we can look at only those elements of $K_{n+1}$ that contain $k_n$ as a restriction. Again, this set contains restrictions of infinitely many elements of $K_{n+1}'$, and we can proceed recursively, finding a $k_m$ for each $m \geq n$ contained as a restriction in infinitely many other colorings, and $\cup \{k_m\}$ can be shown to be a coloring of the entire graph (For a classical recursion, we would begin with $K_1$, but we can actually pick any $n$ and begin at $K_n$).

In the fourth proof, we do something similar, only we pick $k_n$ from $K_n$ arbitrarily. The guarantee we take from sequential compactness is that a subsequence of $\{k_n\}$, if we pick a high enough number to begin at, is the type of sequence we found in the first proof: any element is a restriction of every higher element.

We could repeat our proof of sequential compactness in this context, but we merely note that it again hinges on the fact that an infinite number of colorings have an infinite subset whose restrictions to a finite subset of the vertices (in this case, one new vertex at each step) are identical. Of course this is only equivalent to a special case of Tychonoff’s Theorem, but it offers a suggestion of the truth of the entire theorem which might be established with transfinite induction.

We now examine our two more powerful tools, Rado’s Selection Theorem and Tychonoff’s Theorem. One is combinatorial, while the other is topological. But we will now formally state and partly prove what we have already mentioned:

**Theorem 6.** Rado’s Selection Theorem and Tychonoff’s Theorem are logically equivalent.

**Proof of Rado’s Selection Theorem from Tychonoff’s Theorem.** We begin by assuming the hypotheses of Rado’s Selection Theorem: We have a set $\mathcal{S}$ and a set $\mathcal{F}$ of forbidden $k$-colorings on $\mathcal{S}$ such that every finite subset of $\mathcal{S}$ has an $\mathcal{F}$-admissible $k$-coloring.

We proceed very similarly to the third proof of Theorem 1 from Tychonoff’s Theorem: Let $X := \prod_{S} \{1, \ldots, k\}$ where each copy of $\{1, \ldots, k\}$ has the discrete topology and $X$ has the product topology. Each copy of $\{1, \ldots, k\}$ is compact because it is finite, so by Tychonoff’s Theorem, $X$ itself is compact.
The basic open sets of $X$ are of the form:

$$\{f : S \mapsto \{1, \ldots, k\} \mid f|_S = h\} = F_{S,h}$$

for some finite $S \subseteq S$, and if we let $A_S$ be the set of $k$-colorings of $S$ whose restrictions to $S$ are $\mathcal{F}$-admissible, then

$$A^C_S = \cup\{F_{S,h} \mid h \text{ is not an } \mathcal{F}\text{-admissible } k\text{-coloring of } S\}$$

is open, and hence $A_S$ is closed.

By assumption, each $A_S$ is non-empty, and so the sets $\{A_S\}$ have the finite intersection property: if we have a collection $\{S_1, S_2, \ldots, S_k\}$ of subsets of $S$, $\cap\{A_{S_1}, A_{S_2}, \ldots, A_{S_k}\} = A_{\cup\{S_1, \ldots, S_k\}}$, which is non-empty by assumption.

Hence since $X$ is compact, $\cap A_S$ is non-empty. But any element of $\cap A_S$ is an $\mathcal{F}$-admissible $k$-coloring of $S$, so Rado’s Selection Theorem holds.

A proof that Rado’s Selection Theorem (stated in terms of choice functions) implies Tychonoff’s Theorem can be found in [11].

3 Circles and the Cycle Space

Now we turn to our other area of inquiry. This is an attempt to use topology to obtain extensions of theorems on the cycle space of finite graphs. To give an example, we examine an attempt to extend a basic theorem and the difficulties we encounter with it. First we need to back up a bit and examine the cycle space of a finite graph.

3.1 The Finite Cycle Space

By defining a vector space based on the cycles of a graph, we open up a number of options. We can use the tools of linear algebra to talk about the graph, and in particular, it gives us a number of tools for deciding whether or not the graph is planar (see sections 4.5 and 4.6 of [4] for an overview).

Our definition is as follows: The edge set of a cycle is a circuit. And for $D, F \subseteq E(G)$, we define the symmetric difference of $D$ and $F$ to be

$$D \triangle F = \{e \in E(G) \mid e \text{ is in exactly one of } D, F\}.$$

Then if we take the circuits of $G$ as a generating set and $\triangle$ as our operation, the resulting subset of the power set of $E(G)$ is the cycle space of $G$, which we write as $\mathcal{C}(G)$. We note that if we think of each subset of $E(G)$ as a $|E(G)|$-vector with values in $\mathbb{F}_2 = \{0, 1\}$, it is clear that $\mathcal{C}(G)$ is a vector space. It will generally be more convenient to consider its elements as collections of edges, however.

One basic property of $\mathcal{C}(G)$ is:
Theorem 7. Let $G$ be a finite graph and $F \subseteq E(G)$. $F \in C(G)$ if and only if $F$ can be written as a disjoint union of circuits.

A proof can be found in [4], pp. 24. This will be a useful guiding principle when we try to extend theorems on the cycle space to infinite graphs.

Before we can give another basic theorem on the finite cycle space, the first that we will try to extend, we need another definition: A subset $D$ of $E(G)$ is a cut if there is a bipartition of $V(G)$ into sets $A,B$ such that

$$D = \{xy \in E(G) \mid x \in A, y \in B\}$$

Then we can state:

Theorem 8. Let $G$ be a finite graph and $F \subseteq E(G)$. Then $F \in C(G)$ if and only if $|F \cap D|$ is even for every cut $D$ in $G$.

Proof, based on [4]. Let $C = x_0x_1 \ldots x_{n-1}x_n = x_0$ be a cycle in $G$ and $D$ a cut induced by the bipartition of $V(G)$ into sets $A, B$. Then $x_ix_{i+1} \in D$ exactly when $x_i \in A$ and $x_{i+1} \in B$, or $x_i \in B$ and $x_{i+1} \in A$. If $|C \cap D|$ were odd, we would have $x_0 = x_n \in A \cap B = \emptyset$, so $|C \cap D|$ must be even. This property is preserved by symmetric differences since

$$|(F_1 \triangle F_2) \cap D| = |F_1 \cap D| + |F_2 \cap D| - 2|\cap \{F_1, F_2, D\}|,$$

and so it extends to the entire cycle space.

Conversely, suppose $F$ meets every cut evenly. We can form a cut including exactly the edges incident with a particular vertex $x$ by partitioning $V(G)$ into the sets $\{x\}$ and $\{y \in V(G) \mid y \neq x\}$. Hence it is easily shown that $F$ contains an even number of edges incident with any given vertex. Now we can easily show by recursion that $F$ is a union of circuits: We can pick an arbitrary point incident with an edge of $F$ and walk along edges until we have traversed a circuit, and remove it. This preserves the property of meeting each vertex evenly, and so we can continue doing it until we have emptied $F$.

3.2 Problems Extending Theorem 8 to an Infinite Graph

![Figure 3: An infinite cut in the double infinite ladder](image)
We immediately run into problems when we try to apply Theorem 8 to a graph with infinitely many vertices. To begin with, it seems reasonable that in an infinite graph, there might be elements of the cycle space containing infinitely many edges, if they can be well-defined. We will actually give a way of defining such edge sets later (on page 20); for now, we simply assume they are a possibility. It is certainly possible to have infinite cuts: for example, let $G$ be the double infinite ladder (Fig. 3) with the cut consisting of all edges connecting the two double rays (pictured as bold lines).

But then we can have a situation where $F$ and a cut have an infinite intersection, which has neither even nor odd cardinality. Hence if we want our extension of Theorem 8 to be generally applicable, the most direct route is to limit ourselves to either finite cuts or finite $F \subseteq E(G)$. The latter approach gives us the following easy result:

**Theorem 9.** Let $F$ be a finite set of edges in $G$. Then $F \in C(G)$ if and only if $|F \cap D|$ is even for every cut $D$ in $G$.

This can be demonstrated by taking a finite subgraph $G'$ containing $F$ and applying Theorem 8; every cut in $G$ contains a (possibly empty) cut in $G'$, and every cut in $G'$ can be extended to a cut in $G$.

Limiting ourselves to finite cuts and taking into account the possibility of infinite elements of the cycle space gives us more trouble. For example, using the infinite double ladder again, let $F$ be the set of edges in the two double rays (the thin lines in Fig. 3).

We claim that every finite cut intersects $F$ in an even number of edges. To see why, let our cut $D$ be given by the bipartition $\{A, B\}$ of $V(G)$. For $D$ to be finite, we must be able to look far enough to the left or right and see vertices only in $A$ or only in $B$. Hence on either double ray, there are a finite number of $A-B$ edges, which are exactly the edges in $D$, and this number must be even on both double rays, or odd on both double rays, giving us an even intersection of $D$ and $F$.

Hence we will have to either severely limit our extension of Theorem 8, or admit $F$ as an element of our cycle space. We will take the latter route. Further, if we want to extend Theorem 7, it seems that $F$ must be a single circuit; it certainly does not appear to be a union of several disjoint circuits.

Let us examine $F$ to see what might make it comparable to the circuits we know. It is the edge set of a 2-regular subgraph, and a cycle is a 2-regular subgraph. However, a cycle is connected and is intuitively seen as describing a single arc or loop in space. Our goal, then, might be to redefine circuits based on a topology in which $F$ is the edge set of a loop.

Intuitively, to make the two double rays look like a loop, we would need to include two points at infinity (Fig. 4). How would we define such points? To show that the way we will eventually define them is natural, suppose we let the edges connecting the two double rays end after a certain point. Then we can give a finite cut that intersects $F$ in an odd number of edges (Fig. 5 is an example).
It seems that the “connectedness” of our rays is due to the fact that they are connected by infinitely many disjoint paths, or not finitely separable. We claim, then, that it is natural to define our “points at infinity” in terms of rays which are not finitely separable, and do exactly that.

Let $R_1, R_2$ be rays. We write $R_1 \sim R_2$ if for every finite $S \subseteq V(G), R_1$ and $R_2$ have tails in the same component of $G - S$. It is easily verified that $\sim$ is an equivalence relation. We define the ends of a graph to be the equivalence classes of its rays under $\sim$. We write the set of ends of $G$ as $\Omega(G) = \Omega$ and denote a generic end by $\omega \in \Omega$.

In our example, we can now imagine that $F$ is the edge set of a “cycle” made up of the two double rays and the two ends of $G$. We will now begin the work of making this notion precise, by defining a suitable topology on $G \cup \Omega(G)$ and defining circuits in terms of homeomorphic images of the circle.

4 The Topological Space $|G|$

Before we define our topology on $G \cup \Omega(G)$, we need to specify its point set. Vertices and ends of $G$ will be points. We also want to include continuum many points on each edge; for each edge $e$, we include a set $\hat{e}$ of continuum many points, disjoint from each other, $V \cup \Omega$, and $\hat{e}'$ for any other edge $e'$. For each edge $e$, we set a homeomorphism between $(0, 1)$ and $\hat{e}$, then extend it to a homeomorphism $h_e : [0, 1] \mapsto u \cup \hat{e} \cup v$ where $u, v$ are the endpoints
of $e$, with $h_e(0) = u, h_e(1) = v$. This gives us a metric on each edge. We will also write $\hat{F}$ where $F \subseteq E(G)$ for the set of interior points of edges in $F$. Since $\hat{F} = \cup_{e \in F} \{\hat{e}\}$, it is also an open set.

4.1 The Topology of $|G|$

Now we can specify the topology of $|G|$. We already have as open the sets $h_e(U)$ where $e$ is any edge and $U$ is any open set in $(0, 1)$. We write the open set $h_e((0, 1))$ as simply $\hat{e}$. As a neighborhood basis for a vertex $u$, we take the “open stars of radius $\epsilon$”:

$$U(u, \epsilon) = u \cup \{x \in \hat{e} \mid e = uv, d(u, x) < \epsilon\}.$$ 

where $0 < \epsilon < 1$.

Finally, we need to have a neighborhood basis for the ends of $G$. For an end $\omega \in \Omega$, any finite $S \subseteq V(G)$ and $0 < \epsilon < 1$, we first define $C(S, \omega)$ as the unique component of $G - S$ containing a tail of each ray in $\omega$. Then we can define:

$$\Omega(S, \omega) = \{\omega' \in \Omega \mid C(S, \omega') = C(S, \omega)\},$$

$$\hat{E}_\epsilon(S, \omega) = \{x \in u\hat{v} \mid u \in C(S, \omega), v \in V(G) \setminus C(S, \omega), d(u, x) < \epsilon\},$$

and

$$\hat{C}_\epsilon(S, \omega) = C(S, \omega) \cup \Omega(S, \omega) \cup \hat{E}_\epsilon(S, \omega).$$

We declare the sets $\hat{C}_\epsilon(S, \omega)$ to be a neighborhood basis for $\omega$ (see Fig. 6).

Figure 6: A basic neighborhood $\hat{C}_{1/2}(S, \omega_1)$ with $S$ containing only the vertex in white

Subsets of $|G|$ will often be noted simply in graph form ($G' \subseteq |G|$), as it should be clear from context that we are examining them topologically. They will always be given the subspace topology. We denote the closure of a set $X$ by $\overline{X}$.

It is worthy of note that for locally finite graphs, $|G|$ is equivalent to the Freudenthal compactification. A general definition can be found in [9]. Essentially, when $G$ is locally finite, $\Omega(G)$ has a one-to-one correspondence with a set of “ends” which can be defined in
purely topological terms, and the union of this set with $G$ gives us a space that is compact and has other desirable properties. In this paper, we limit ourselves to proving that $|G|$ is compact for locally finite $G$ (Theorem 11).

4.2 Topological Properties of $|G|$

We can start our examination of $|G|$ by looking into some of the basic topological properties, specifically, the Hausdorff condition, compactness, and metrizability. We will find that connected locally finite graphs give us very nice topological spaces; $|G|$ has all of these properties when $G$ is connected and locally finite.

4.2.1 The Hausdorff Condition

Theorem 10. For any graph $G$, $|G|$ is Hausdorff.

Proof. For vertices and interior points of edges, this is quite clear. For an end $\omega$ and a vertex $v$, take a finite set of vertices $S \ni v$ and $\epsilon = 1/2$; then $\hat{C}(S, \omega)$ and $U(v, \epsilon)$ are distinct. For an end and an interior point of an edge $e$, include the endpoints of $e$ in $S$. Finally, for two distinct ends $\omega_1, \omega_2$, we can pick $S$ such that $C(S, \omega_1) \neq C(S, \omega_2)$ (otherwise, $\omega_1 = \omega_2$) and hence $\hat{C}(S, \omega_1) \cap \hat{C}(S, \omega_2) = \emptyset$.

4.2.2 Compactness

Theorem 11. Let $G$ be connected. Then $|G|$ is compact if and only if $G$ is locally finite.

The proof of Theorem 11 will take some building up. We begin with a definition: A comb (Fig. 7) is the union of a ray with an infinite set of disjoint finite paths, each of which has an endpoint on the ray. We call the ray the spine of the comb. We call the endpoints of the paths not on the ray its teeth, except when a path consists of a single vertex (which we allow), in which case that vertex is a tooth.

We now prove an important basic lemma on combs in locally finite graphs.

Lemma 12. Let $U$ be an infinite set of vertices in a connected locally finite graph $G$. Then $G$ contains a comb with teeth in $U$.

Proof, based on [4]. Let us select two vertices of $U$. Since $G$ is connected, there is a path between them; this path is a tree each of whose edges lies in a path, contained in the tree, between two vertices in $U$. By Zorn’s Lemma, there is a maximal such tree $T$.

$T$ must contain infinitely many vertices: If not, there is a vertex $u$ of $U$ not in $T$, and since $G$ is connected there is a $T - u$ path $P$. Then $T \cup P$ is a tree each of whose edges lie in a path contained in $T \cup P$ between two vertices in $U$, contradicting the assumption that $T$ is the maximal such tree.

Now we can show that $T$ contains a ray. Pick $x_0 \in V(T)$ arbitrarily and let $X_n$ be the set of vertices in $T$ whose unique path to $x_0$ contains $n$ edges (the vertices at distance $n$
from $x_0$. Since $G$ is locally finite, it can be easily shown by induction that there are only finitely many vertices in each $X_n$, and hence $X_n$ must be non-empty for every $n \in \mathbb{N}$. Then we can apply König’s Lemma to find a ray $R$ in $T$.

Finally, we can recursively construct a set $\{P_n\}$ of countably many disjoint $R-U$ paths: Suppose we have picked $P_i$ for all $i < n$. $\cup_{i<n}\{P_i\}$ contains only finitely many vertices, so we can choose $m$ large enough that $r_mR$ contains no vertices of $\cup_{i<n}\{P_i\}$. We know because $R \subseteq T$ that $r_mr_{m+1}$ is contained in a path $P' \subseteq T$ between two vertices of $U$. Further, one of the endpoints of $P'$ must be outside of $\cup_{i<n}\{P_i\}$: Otherwise, $(\cup_{i<n}\{P_i\} \cup P') \subseteq T$ would contain a cycle. Let $x$ be that endpoint.

Then let $y$ be the point of $R \cap P'$ that is closest to $x$ (in $P'$). Then $P_n = yP'x$ is an $R-U$ path disjoint from all previous $P_i$.

Because of the usefulness of Lemma 12, we will often restrict ourselves to locally finite graphs in the remainder of this paper. This allows us to avoid many graphs that might trip us up. For an example, see Fig. 8. This is an example of a vertex $v$ that dominates an end, meaning that for any ray $R$ in the end, there is an infinite set of $v-R$ paths, each of which intersects any other only in $v$, or equivalently that for any finite $S \subseteq V(G)$ that does not contain $v$, there is a path from $v$ to a tail of $R$ in $G-S$. A graph with a dominated end may be interesting for other purposes, but would be a counterexample for some of ours. By limiting ourselves to locally finite graphs in many places we avoid such problems.

Now we give a lemma that we will need in our proof of Theorem 11.

**Lemma 13.** Let $X$ be the set of teeth of a comb with spine $R \in \omega \in \Omega$. Then $\omega \in \overline{X}$. Conversely, if $\omega \in \overline{X}$ for some $X \subseteq V(G)$ and $\omega \in \Omega$, then there is a comb with some $R \in \omega$ as its spine and teeth in $X$.

**Proof.** Suppose $G$ contains a comb with spine $R$ and teeth in $X$. Let $\mathcal{P}$ be the infinite set of $R-X$ paths in the comb. Note that the paths of $\mathcal{P}$ can be totally ordered by the
order of their endpoints in $R$. Hence we can write them as $\{P_n\}$ and give a corresponding enumeration of their endpoints (teeth) in $X$ as $\{x_n\}$ and their endpoints in $R$ as $\{r'_n\}$.

Let $U$ be a neighborhood of $\omega$. $U$ must contain a basic open set containing $\omega$, which is of the form $\hat{C}_t(S,\omega)$. Since $R\cup\{P_n\}$ contains infinitely many vertices, we can pick $i$ large enough that $(P_i\cup r'R)\cap S = \emptyset$. Then the component of $G - S$ containing $r'_i R$ is $C(S,\omega)$, since $r'_i R$ is a tail of $R \in \omega$, and $P_i$ is a path from $x_i$ to $r'_i R$ in $G - S$. So $x_i \in C(S,\omega) \subseteq \hat{C}_t(S,\omega) \subseteq U$. We can conclude that every neighborhood of $\omega$ contains a point in $X$, and hence $\omega \in X$.

Conversely, suppose that $\omega \in X$. Let $R \in \omega$. Note that no finite $S \subseteq V(G)$ separates $X$ from $R$ (or more precisely, for any finite $S \subseteq V(G)$ there is an $X - R'$ path in $G - S$ where $R'$ is a tail of $R$ in $G - S$). If not, we could pick $S$ such that $\hat{C}_t(S,\omega)$ contains no vertices of $X$, contradicting $\omega \in X$.

This fact allows us to recursively construct the comb that we want, similarly to our proof of Lemma 12. To begin, let $P_0$ be an $X - R$ path in $G$. Then assume that we have identified $P_0,\ldots,P_{n-1}$, distinct $X - R$ paths. To find $P_n$, let $S = \cup_{i<n}\{P_i\}$. We know from the above that $G - S$ contains $R'$, a tail of $R$ and an $X - R'$ path. Call this path $P_n$; it clearly fits our criteria since it cannot share any points with $P_i$ where $i < n$ and $R' \subseteq R$. Then

$$R \cup \bigcup_{n \in \mathbb{N}} \{P_n\}$$

is the desired comb.}

Now we can give our proof of Theorem 11.

Proof of Theorem 11, based in part on [4]. Suppose $G$ has a vertex $v_0$ of infinite degree. Let $\mathcal{O} = \{h_e((1/4,3/4)) \mid e = v_0v\}$. Then

$$\{U(v,1/3) \mid v \in \{v_0\} \cup N(v_0)\} \cup \mathcal{O}$$

$$\cup \{U(v,2/3) \mid v \in V(G) - (\{v_0\} \cup N(v_0))\} \cup \{\hat{C}_t((v_0),\omega) \mid \omega \in \Omega\}$$

is an open cover of $|G|$ with no finite subcover (For each of the infinitely many $e$ adjoining $v_0$, $h_e(1/2)$ is included in only one element of $\mathcal{O}$).
Conversely, suppose \( G \) is locally finite. Pick \( v_0 \in V(G) \). For each \( n \in \mathbb{N} \), let \( D_n \) be the set of vertices at distance \( n \) from \( v_0 \). Since \( G \) is locally finite, it is easy to show by induction that each \( D_n \) is finite. Let \( S_n := \cup_{i \in \{0, \ldots, n\}} D_i \). For any \( v \in D_n \), let \( C_v \) be the component of \( G - S_{n-1} \) containing \( v \), and let \( \hat{C}_v = \overline{C_v} \cup \{ e \mid e = vu, u \in S_{n-1} \} \). By Lemma 13 this set contains exactly those ends containing a ray with a tail in \( C_v \). Note that

\[
G[S_{n-1}] \cup \bigcup_{v \in D_n} \{ \hat{C}_v \} = |G|.
\]

Now let \( \mathcal{O} \) be an open covering of \(|G|\). Note that for any \( n \), \( G[S_n] \) is compact (since it is a finite set of vertices and edges, each of which is compact). Hence if we can find \( n \) large enough that \( \hat{C}_v \) is a subset of some \( O \in \mathcal{O} \) for every \( v \in D_n \), we can conclude that \(|G|\) is compact. We will show that we can always find such an \( n \).

Suppose this is not the case; i.e. for each \( n \) there is a vertex \( v \in D_n \) such that \( \hat{C}_v \) is not contained in any open set in \( \mathcal{O} \). Then for each \( n \), let \( V_n \) be the set of vertices in \( D_n \) having this property. We can apply König’s Infinity Lemma to the sets \( V_n \): we know that each is non-empty, and for any \( v \in V_n \), its neighbor \( v' \) in \( D_{n-1} \) must be in \( V_{n-1} \), since \( \hat{C}_v \subseteq \hat{C}_{v'} \). Hence there is a ray \( R = v_0v_1 \ldots \) with each \( v_i \subseteq V_i \).

Let \( \omega \) be the end of \( R \). \( \omega \) must be contained in some \( O \in \mathcal{O} \), and \( O \) must contain some basic neighborhood of \( \omega \), \( \hat{C}_\epsilon(S, \omega) \). Since \( S \) is finite, we can pick \( n \) large enough that \( S \subseteq S_{n-2} \). Hence \( v_n \) and its neighbors are outside of \( S \), and \( C_{v_n} \) and its neighbors in \( D_{n-1} \) lie in a component of \( G - S \). Since \( v_n \in v_nR \), this must be \( C(S, \omega) \), so \( \hat{C}_{v_n} \) lies in \( C(S, \omega) \cup \Omega(S, \omega) \). Then we have

\[
\hat{C}_{v_n} \subseteq (C(S, \omega) \cup \Omega(S, \omega)) \subseteq \hat{C}_\epsilon(S, \omega) \subseteq O \in \mathcal{O}
\]

contradicting \( v_n \in V_n \).

Note that our inclusion of \( \Omega(S, \omega) \) in \( \hat{C}_\epsilon(S, \omega) \) is critical to this result: If our basic neighborhood of \( \omega \) generally contained no ends other than \( \omega \), the proof would fail, and in fact it is easy to see that we could form an open cover of a graph with infinitely many ends that had no finite subcover by including an open set containing \( \omega \) and no other end for each \( \omega \), then including an open set containing all remaining vertices and edges, but no ends. That a locally finite graph may have infinitely many, or indeed uncountably many ends is illustrated by Fig. 9, page 20.

4.2.3 Metrizability

Looking into the question of metrizability gives us an interesting indicator of the relation between the topology of \(|G|\) and the structure of the graph. We will need some definitions (for clarity’s sake) before giving our major theorem in this area. A tree \( T \) with root \( r \) induces a partial order called the tree order which we will write as \( \leq \) on its vertices: For each \( v \in V(T) \), there is a unique path \( v - r \) path in \( T \). If \( u \in V(T) \) is contained in this
v − r path, we write \( u \leq v \). That this is in fact a partial order should be clear (When we want to exclude \( u = v \), we can write \( u < v \)). We define \([v] = \{ u \in V(T) \mid v \leq u \}\) and \([v] = \{ u \in V(T) \mid u \leq v \}\).

We call a tree in \( G \) normal if for every \( T \)-path not containing any edges of \( T \), the endpoints of the path are comparable in the tree order of \( T \). If \( T \) is a spanning tree this requirement translates to: \( xy \in E(G) \) only if \( x \) and \( y \) are comparable in the tree order of \( T \). A normal spanning tree is an important structural feature because its separation properties are reflected in the graph: Vertices \( x \) and \( y \) are separated in \( G \) by \( [x] \cap [y] \). It also gives us a necessary and sufficient condition for metrizability:

**Theorem 14.** \(|G|\) is metrizable if and only if \( G \) has a normal spanning tree.

We will prove the backward implication; see [3] for a proof of the other direction. We will prove shortly that any locally finite connected graph has a normal spanning tree, and is therefore metrizable.

**Proof that \(|G|\) is metrizable if \( G \) has a normal spanning tree, based on [3].** Let \( T \) be a normal spanning tree of \( G \) with root \( r \). We begin by defining our metric \( d \) on \( T \). We do this by setting the distance between adjacent vertices in \( T \), then using \( h_e \) on individual edges to extend this to individual points. We call the vertices at distance \( n \) from \( r \) the \( n \)-th level of \( T \). The uniqueness of paths in the tree guarantees that vertices in the \( n \)-th level are only adjacent (in \( T \)) to vertices in levels \( n - 1 \) and \( n + 1 \). Hence we can determine the distance between all vertices in \( G \) by saying that if \( x \) and \( y \) are in the \( n - 1 \)-th and \( n \)-th levels respectively and are adjacent, \( d(x, y) = \frac{1}{2^n} \), then extending this to non-adjacent vertices by adding along the edges of the unique tree path connecting them.

Extending \( d \) to inner points of edges in \( T \) is easy enough, we simply scale \( h_e \) by the distance between the two endpoints of the edge to get the distance to another point of \( e \) or to an endpoint, then add along edges of \( T \) if necessary to find the distance to a vertex or perform the same type of scaling on \( h_{e'} \) to find the distance to a point of \( e' \).

To extend \( d \) to \( \Omega(G) \), we note that if we take a ray \( x_0x_1 \ldots \in \omega \) in \( T \) (such a ray must exist by Theorem. 20) with \( x_0 \) in the \( n \)-th level, \( x_1 \) in the \( n + 1 \)-th level and so on, and sum \( d(x_i, x_{i+1}) \) over the natural numbers, we have

\[
\sum_{k=n+1}^{\infty} \left\{ \frac{1}{2^k} \right\} = \frac{1}{2^n}.
\]

Hence we define \( d(x_0, \omega) = \frac{1}{2^n} \). This can be extended to all points in \( T \) in a natural way by the uniqueness of paths; for a vertex \( v \) and an end \( \omega \), the ray in \( \omega \) based at \( v \) must be increasing (i.e. each vertex is in the level below the next vertex) after a particular point \( x \), let \( d(v, \omega) = d(v, x) + d(x, \omega) \). For example, \( d(r, \omega) = 1 \) for all \( \omega \in \Omega \), and \( d(\omega_1, \omega_2) = \frac{1}{2^{n-1}} \) where \( n \) is the highest level at which the normal rays of \( T \) in \( \omega_1 \) and \( \omega_2 \) intersect.
Now we have only to define $d$ on $E(G) - E(T)$. Let $e \in E(G) - E(T)$ with endpoints $x, y$. Then we define $d$ on points of $\hat{e}$ by scaling $h_e$ by $d(x, y)$.

Clearly, $d$ satisfies symmetry and $d(x, x) = 0$ for all points $x$. We will not go through a thorough demonstration that $d$ satisfies the triangle inequality. We will only note that for $x, y, z \in \overline{T}$, there are unique paths $xTy, yTz, xTz$ that determine $d$ by summing on edges or parts of edges contained in those paths. $d(x, y)$ is defined by summing along the edges of $xTy$, and the union of the paths $xTz$ and $zTy$ contains $xTy$, so $d(x, y) \leq d(x, z) + d(z, y)$. The definition of $d$ outside of $T$ and the normality of $T$ guarantee that there are no “shortcuts” to be found by summing along edges not in $T$. \hfill \Box

Note that the theorem that every connected finite graph has a normal spanning tree does not extend to the infinite case. For example, complete graphs on uncountably many vertices do not have them: Any tree with a vertex $v$ of degree greater than 2 will fail to be normal, because the neighbors of $v$ are all adjacent. This leaves a ray or a double ray as the only possible solutions, and both contain only countably many vertices, so they fail to span the graph.

One useful way of characterizing the graphs with normal spanning trees was given by Jung in 1967. To formulate it, we need the following definition: A set $V$ is allowed to contain vertices of $U$ if and only if for each ray $R$ in $G$, there is a finite $S \subseteq V(G)$ separating $U$ from $R$. Note that this is something like saying $U$ does not dominate any ends, except that our finite separating set is allowed to contain vertices of $U$. Now we can write:

**Theorem 15 (Jung 1967).** A connected graph has a normal spanning tree if and only if its vertex set can be written as a union of countably many dispersed sets.

**Proof.** First, suppose $G$ has a normal spanning tree $T$, with root $r$. Let $U_n$ be the set of vertices at distance $n$ from $r$. We will show that each $U_n$ is dispersed, giving us the desired decomposition into countably many dispersed sets.

Let $R$ be a ray in $G$, and $R'$ the unique normal ray sharing the end of $R$. Let $x$ be the vertex of $R'$ in $U_n$. Let $S = \{x\}$. Since $R$ and $R'$ are equivalent, any tail of $R$ in $G - S$ must be in $\lfloor x \rfloor \setminus \{x\}$, and thus is separated from any vertex in $U_n$ by $S$. Hence, $U_n$ is dispersed, and $V(G)$ can be written as a countable union of dispersed sets.

Conversely, let $\{U_n\}_{n \in \mathbb{N}}$ be a set of dispersed subsets of $V(G)$ such that $\cup U_n = V(G)$. We will use this decomposition to well-order the vertices of $G$, at which point we can begin to recursively construct a normal spanning tree of $G$. By the well-ordering theorem, each $U_n$ has a well-ordering $<_n$, and for each vertex $v$ there is a least $n$ such that $v \in U_n$. Then we can order all of $V(G)$ as follows: For $x, y \in V(G), x \neq y$, let $m$ be minimal such that $x \in U_m$, and let $n$ be minimal such that $y \in U_n$. If $m < n$, then $x < y$, and vice versa. If $m = n$, compare $x$ and $y$ in $<_m$. Since they are distinct, they must be comparable; if $x <_m y$, then $x < y$ and vice versa. This is clearly a well-ordering of $V(G)$; if $S \subseteq V(G)$, there is a least $n$ such that $U_n$ contains some vertices of $S$, and the least vertex of them in
$ \langle \nu \rangle$ is the least vertex in $S$. This well-ordering gives us an isomorphism $\phi$ from $V(G)$ to a subset of the ordinals. Call the vertex sent to $\lambda$ $v_\lambda$.

It is worth noting at this point that the union $T$ of a nested set of normal trees $T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$ with the same root is a normal tree. (Any cycle in $T$ has each of its edges contained in some $T_\lambda$; for maximal $\lambda$, $T_\lambda$ contains the entire cycle, which is impossible since it is a tree. Any $T$-path has both its endpoints in some $T_\lambda$; they must be comparable in $T_\lambda$, hence also in $T$).

So, if we can find normal trees $T_\lambda$ with a common root such that for all $\lambda \in \phi(V(G)), v_\lambda \in T_\lambda$ and $\forall \mu < \lambda, T_\mu \subseteq T_\lambda$, then $\bigcap_{\phi(V(G))} T_\lambda$ is a normal spanning tree. All that remains is to specify the recursion for generating the trees.

We begin by letting $T_\emptyset = v_\emptyset = r$. Then for any $\lambda \in \phi(V(G))$, we have the task of adding $v_\lambda$ to a previously existing normal tree $T$ (If $\lambda$ is a successor ordinal with predecessor $\mu$, $T = T_\mu$; if $\lambda$ is a limit ordinal, $T = \bigcup_{\mu < \lambda} T_\mu$).

If $v_\lambda \in T$, then $T_\lambda := T$. Otherwise, let $C$ be the component of $G - T$ containing $v_\lambda$. Then because $T$ is normal, $N(C)$ is a chain in the tree order of $T$. As long as there is a greatest element $x$ in $N(C)$, we can let $P$ be an $x - v_\lambda$ path, contained in $C$ except for $x$, and let $T_\lambda = T \cup P$. The neighbors of any component $C' \subseteq C$ of $G - T_\lambda$ are contained in $N(C) \cup P$, and are still a chain in $T_\lambda$, so this preserves normality. However, unlike in the finite and countable cases, it is not immediately obvious that there is a greatest element of $N(C)$. We will show by contradiction that such an element does exist.

Assume $N(C)$ is infinite. Since $N(C)$ is a chain, it is a subset of a normal ray $R$. Consider the recursive process by which $R$ was generated: Each addition was a path $P_\mu$ connecting $v_\mu$ to a previous normal tree for some $\mu$. Hence each vertex in $R$ is in $P_\mu$ for some $\mu < \lambda$. Further, since each $P_\mu$ only contains finitely many vertices, there is an infinite set $\mathcal{P}$ of (possibly trivial) disjoint paths, each of which links $v_\mu$ to $R$ for some $\mu < \lambda$.

But, for every $\mu < \lambda$, by the definition of our well-ordering, $v_\mu \in U_m$ for some $m \leq n$. So each $v_\mu$ is in one of finitely many $U_m$. Hence $\mathcal{P}$ must contain infinitely many disjoint paths from some $U_m$ to $R$. But this means that $U_m$ cannot possibly be finitely separable from $R$, contradicting our original requirement that it be dispersed.

Hence, at each step in the recursion, $N(C)$ is finite and we can define $T_\lambda$ as needed, Then $\bigcup_{\lambda \in \phi(V(G))} T_\lambda$ is a normal spanning tree of $G$. \hfill $\square$

Note that any finite set of vertices is dispersed, since it separates itself from any ray. Hence we have:

**Corollary 16.** Any connected countable graph has a normal spanning tree.

It is also the case that any connected locally finite graph is countable; if we pick an arbitrary $x_0 \in V(G)$, there is a simple induction argument to show that there are finitely many vertices at a given distance from it, and since the graph is connected, the countable union of finite sets $\bigcup_{n \in \mathbb{N}} \{x \mid x \text{ is at distance } n \text{ from } x_0\} = V(G)$. So we can also say that a connected locally finite graph has a normal spanning tree.
To conclude our examination of metrizability, it is interesting to show a relation between the structure of $G$ and the topology of $|G|$ in the opposite direction. We can restate our theorem on the existence of a normal spanning tree in topological terms:

**Theorem 17** (Topological version of Jung’s Theorem). A connected graph has a normal spanning tree if and only if its vertex set can be written as a union of countably many closed sets of $|G|$.

This restatement is an immediate consequence of the first version of the theorem and the following lemma, which we will prove immediately:

**Lemma 18.** A set of vertices in $G$ is dispersed if and only if it is a closed set in $|G|$.

*Proof.* Let $U \subseteq V(G)$ be dispersed. Then for each $\omega \in \Omega$ there is a finite $S_\omega \subseteq V(G!$ s.t. $C(S,\omega) \cap U = \emptyset$. Then

$$
\bigcup_{\epsilon \in (0,1)} \{ \hat{C}_\epsilon(S_\omega,\omega) \mid \omega \in \Omega \} \cup \{ U(v,\epsilon) \mid v \in V(G) - U \} \cup \{ \hat{e} \mid e \in E \}
$$

is an open set with $U$ as its complement; hence $U$ is closed.

On the other hand, suppose $U \subseteq V(G)$ is not dispersed. Then for some ray $R$, there is no finite set of vertices separating $U$ from $R$. Let $\omega$ be the end containing $R$ and $\hat{C}_\epsilon(S,\omega)$ any of its open neighborhoods. Since $S$ is finite, for some tail $R'$ of $R$ in $G - S$, there is a $U - R'$ path $P$ in $G - S$, with endpoints $x$ in $U$ and $y$ in $R'$. Then $xPyR'$ is a ray in $C(S,\omega)$, and $U$ intersects $\hat{C}_\epsilon(S,\omega)$ for any $S$ and $\epsilon$. Hence, $\omega \in \overline{U}$, so $U$ is not closed. 

5 The Cycle Space of a Locally Finite Graph

We now return to the problem of Section 3.2 that motivated our construction of $|G|$: How can we extend theorems on the cycle space of the finite graph to locally finite graphs?

5.1 Redefining $C(G)$ in terms of $|G|$

We will need some definitions: A *circle* in $|G|$ is any homeomorphic image of the unit circle $S^1$. Note that for an edge $e$, if a circle includes any point of $\hat{e}$, it must contain all of $e$ (Otherwise, it would fail to be either injective or continuous). Hence a circle determines and is determined by a set of edges. We now redefine “circuit”; a circuit is the edge set of a circle. We can see how the edges of the double rays in Fig. 3 are a circuit under this new definition.

In Section 3.2 we mentioned the possibility of elements of the cycle space with infinitely many edges. We now have infinite circuits; we would like to also introduce the mechanics necessary for dealing with infinite collections of circuits algebraically. Let $\mathcal{E}$ be a collection of subsets of $E$. We say that $\mathcal{E}$ is *thin* if there is no edge $e$ such that $e \in E'$ for infinitely
many $E' \in \mathcal{E}$. When $\mathcal{E}$ is thin, we can define the symmetric difference of $\mathcal{E}$, which we write as $\Delta \mathcal{E}$, to consist of those edges which are in an odd number of elements of $\mathcal{E}$. Note that this can only be defined when $\mathcal{E}$ is thin, since we can’t assign “odd” or “even” to infinite cardinalities.

Now we can define $C(G)$ for a locally finite graph $G$:

$$C(G) := \{ \Delta C \mid C \text{ is a thin family of circuits in } G \}$$

Note that for finite $G$ this is equivalent to the usual definition of the cycle space: The edges of a cycle form a circuit, and there are of course no infinite circuits, so the circuits are exactly the edge sets of cycles. Further, there are only finitely many cycles in $G$, so any family of them is thin. If we take the symmetric difference of two circuits, the even degree of all vertices (a necessary and sufficient condition for membership in the cycle space of a finite graph) is preserved, so we have a disjoint union of circuits. Induction can extend this to any finite family of circuits. On the other hand a disjoint union of circuits is exactly the symmetric difference of those same circuits, so we can see that the two definitions really are the same for finite graphs.

We note here that Theorem 7 can be extended to locally finite graphs, but a proof is quite difficult. One can be found in [6].

Figure 9: An example of a circle containing infinitely many double rays
5.2 A Locally Finite Graph with $|\mathbb{R}|$ Ends

We will now give an interesting example first presented by Diestel in [2]. It illustrates that a locally finite graph may have uncountable many ends, as well as showing an example of how strange the circuits in our new version of the cycle space may be.

Let $V(G)$ be the set of finite binary sequences. The edge set consisting of one edge from each vertex other than $\emptyset$ to the sequence obtained by deleting its last term would give us an infinite binary tree. To define $G$ we take these edges and, for each sequence $s$, add another edge between $s01$ and $s10$ (Figure 9).

Let $D$ be the double ray $\ldots(00)(0)\emptyset(1)(1)\ldots$. For each vertex $s$, let $D_s$ be the double ray $\ldots(s011)(s01)(s10)(s100)\ldots$. Let $D' = D \cup \bigcup_{s\in V(G)} D_s$. Then we have

Example 19. $D'$ is a circle, and $D'$ is therefore a circuit.

Exhibiting the homeomorphism $\phi : D' \mapsto S^1$ is somewhat involved, but is illustrative of the type of work necessary to explicitly confirm the topological constructions we want in particular graphs. We use the standard shortcut of identifying $S^1$ with the quotient map on $[0,1]$ that identifies 0 with 1.

Every point in $D'$ is a point of exactly one double ray, or an end. We begin by deciding where to map the double rays. Each double ray in $D'$ is a connected open set, so we must map it to an open interval. Also, for any pair of distinct ends, or any pair of double rays, there is a double ray “between” them, so our map must reflect this structure by mapping “higher” double rays to smaller intervals.

We will decide the image set of each double ray first, and then specify the map of individual points within double rays. We set

$\phi(D) = (0, \frac{1}{2})$
$\phi(D_\emptyset) = (\frac{1}{2}, \frac{3}{4})$
$\phi(D_s) = (\frac{5}{16}, \frac{7}{16})$
$\phi(D_t) = (\frac{13}{16}, \frac{15}{16})$

At this point, we proceed recursively: with every additional binary sequence $s$ of length $n$, we can identify it as being “between” two “lower” double rays $D_r$ and $D_t$. In the first case, we map $D_s$ into an interval in the space between $\phi(D_r)$ and $\phi(D_t)$, half the length of that space and centered in the middle. Thus the portion of the circle’s circumference not in the image set of some double ray decreases by half with each $n$, approaching 0 as $n$ approaches infinity.

To specify $\phi(x)$ for a point $x$ in a double ray $D_\alpha$, we first specify a point $x_0$ at the “bottom” of the ray: $x_0$ is the vertex $\emptyset$ for $D$, for $D_s$ where $s$ is a finite binary sequence or $\emptyset$, $x_0 = h_e(1/2)$ where $e$ is the edge from $s01$ to $s10$. Let $d$ be the distance between $x$ and
\[ \phi(x) = \begin{cases} 
\frac{y_1 + y_2}{2} & \text{if } x = x_0, \\
\frac{y_1 + y_2}{2} + \frac{y_2 - y_1}{2}(1 - \frac{1}{d+1}) & \text{if } x \text{ is to the right of } x_0, \\
\frac{y_1 + y_2}{2} - \frac{y_2 - y_1}{2}(1 - \frac{1}{d+1}) & \text{if } x \text{ is to the left of } x_0. 
\end{cases} \]

Now we need to determine how to map the ends of \(D'\). Note that each \(\omega \in \Omega(D')\) contains one ray of the original binary tree beginning at \(\emptyset\) and that this ray corresponds to an infinite binary sequence \(S\) (hence the conclusion that \(G\) has continuum many ends: Any two such rays that are distinct are separated by the vertices in all the levels of the tree up to two levels above their highest common vertex). For each \(n \in \mathbb{N}\), let \(s_n\) equal \(S\) truncated at the \(n\)th term. Then let \(x_n\) be a point in \(\phi(D_{s_n})\). Note that \(\{x_n\}\) is a convergent sequence; \(x_n\) is always less than \(1/2^n\) away from \(x_{n-1}\). Let \(\phi(\omega)\) be the point to which \(\{x_n\}\) converges.

We will not give the full details to demonstrate that \(\phi\) is a homeomorphism, but only sketch the reasoning. It should be clear enough that it is bijective. Note that a basic open set in \(D'\) is an open interval on an edge or a vertex with two half-open intervals on its adjoining edges, and in either case its image is an open interval on \(S^1\). A basic neighborhood of an end is trickier, since it contains all of the infinitely many other ends that are connected to it in \(G - S\), and infinitely many double rays that lie beneath those ends. The images of such a collection of ends and double rays are infinitely many consecutive arcs on \(S^1\), making up an open interval when joined with the pieces of double rays in the neighborhood that are separated by \(S\).

On the other hand, a basic open set of \(S^1\) is an open interval, and its pre-image consists of either an open set in a double ray, or the union of open sets in two double rays along with an infinite collection of double rays and ends. This can be decomposed into the kind of collections we found from basic neighborhoods of ends above.

### 5.3 Generating \(C(G)\)

Now we will look at a basis for \(C(G)\). In the process, we will set up machinery that we need for our extension of Theorem 8, and extend another theorem on the cycle space of finite graphs.

Before we can move forward, we need to define a particular type of spanning tree in an infinite graph. First note that if \(T\) is a spanning tree in \(G\), each end of \(T\) is a subset of an end of \(G\) (If \(R_1 \sim R_2\) in \(T\), \(R_1\) and \(R_2\) have tails that are identical, so they certainly cannot be finitely separated in \(G\)). This gives us a map \(\phi : \Omega(T) \mapsto \Omega(G)\) where \(\phi(\omega)\) is the unique end of \(G\) containing \(\omega\). If \(\phi\) is a injective, we say that \(T\) is end-faithful.

We make an important note on spanning trees: Every ray \(R\) in a tree \(T\) with root \(r\) is equivalent to exactly one normal ray (that is, one that begins at \(r\)) in \(T\). On the one hand, it is easy to find a normal ray equivalent to \(R\) by finding the \(r - R\) path in \(T\); on
the other, equivalence of two rays means they share a tail, and if two normal rays share a
tail they must be identical. Now we can state:

**Theorem 20.** If \( T \) is a normal spanning tree of \( G \), then every end of \( G \) contains exactly
one normal ray of \( T \), and hence \( T \) is end-faithful.

*Proof, based on [4].* Let \( \omega \in \Omega(G) \), and \( R \in \omega \). We want to find a normal ray \( R' \) in \( T \) such
that \( R' \sim R \) in \( G \). To do this, it will be sufficient to find a comb in \( T \) with teeth in \( R \); we then have infinitely many disjoint paths between \( R \) and the spine of the comb, and the
spine of the comb will share a tail with a normal ray.

Suppose no such comb exists. Then by Lemma 12, \( T \) contains a vertex \( x \) of infinite
degree. Then \([x]\), the finite set of vertices of the path from \( x \) to the root of \( T \), separates
infinitely many vertices of \( R \) in \( T \), and hence also in \( G \), which is impossible since \( R \) is a ray
(for any finite \( S \subseteq V(G) \), \( G - S \) clearly contains a tail of \( R \), so only finitely many vertices
of \( G \) can be separated by \( S \).

Conversely, let \( R_1, R_2 \) be normal rays of \( T \) such that \( R_1 \sim R_2 \) in \( T \). Then \( S := R_1 \cap R_2 \)
is finite and separates \( R_1 \) from \( R_2 \). By the definition of a normal tree, \( S \) also separates \( R_1 \)
from \( R_2 \) in \( G \), so they are in different ends of \( G \).

Let \( G \) be a graph with spanning tree \( T \). Then a **fundamental circuit** \( C_e \) is the edge set
of the unique cycle contained in \( T \cup \{e\} \) where \( e \in G \setminus E(T) \). A basic theorem in finite
graph theory is the following:

**Theorem 21.** Let \( G \) be a finite connected graph with spanning tree \( T \). Then \( C(G) \), the
cycle space of \( G \), is generated by the fundamental circuits \( C_e \) of \( T \).

A proof of this theorem is given in [4].

The reason for introducing end-faithful spanning trees is so that we can use this concept
of fundamental circuits in our redefined cycle space for locally finite graphs. If we try to
write about these fundamental cycles on a spanning tree \( T \) that is not end-faithful, we
run into a major problem: \( T \) may actually contain a circuit of \( G \). For suppose \( T \) has two
rays \( R_1, R_2 \) such that \( R_1 \cap R_2 \neq \emptyset \), with the same end \( \omega \in G \), but different ends in \( T \)
(hence a finite intersection): In other words, \( R_1, R_2 \) are finitely separable in \( T \), but not in
\( G \). Then \( R_1 \cup R_2 \cup \{\omega\} \) contains a circle in \(|G|\), and so \( R_1 \cup R_2 \subseteq T \) contains a circuit. So
in some topological sense, \( T \) is not a tree at all, and fundamental circuits on it will not be
well-defined.

For an example, see Fig. 10: The tree pictured contains three circles of \(|G|\), the union
of the double rays, and the union of the bold edge between the double rays with the
components of the double rays going right or left from it. As we will see, a related problem
is that a collection of fundamental circuits on such a tree is not necessarily thin.

An end-faithful spanning tree avoids this problem (see Fig. 11), so we will only refer
to fundamental cycles in infinite graphs in relation to end-faithful spanning trees. For
this reason, Diestel refers to the closure of an end-faithful spanning tree as a *topological spanning tree*.

We can now state the infinite extension of Theorem 21:

**Theorem 22.** Let $G$ be a locally finite connected graph with end-faithful spanning tree $T$. Let $C$ be a circuit in $G$. Then $C = \Delta\{C_e \mid e \in C \setminus E(T)\}$.

\[\begin{array}{c}
\omega_1 \cdots \\
\end{array}\]

\[\begin{array}{c}
\omega_2
\end{array}\]

Figure 10: This tree is not normal or end-faithful, and an infinite family of fundamental circuits is not thin.

In view of the problems with non-end-faithful spanning trees, this is in some sense the best possible extension. Before we can prove it, we need two more lemmas about fundamental circuits:

**Lemma 23.** Let $G$ be a locally finite connected graph with end-faithful spanning tree $T$. Then the family of fundamental circuits $C_e$ over all $e \in E(G) \setminus E(T)$ is thin.

Figures 10 and 11 illustrate this.

\[\begin{array}{c}
\omega_1 \\
\end{array}\]

\[\begin{array}{c}
\omega_2
\end{array}\]

Figure 11: A normal (and therefore end-faithful) tree $T$ in the infinite ladder, the edges in the double rays are obtained by taking $\Delta\{C_e \mid e \in E(G) \setminus E(T)\}$.

Proof, based on [5]. Suppose to the contrary that for some edge $f$, $f$ is in infinitely many fundamental circuits $C_e$. Then $f$ must be in $T$, otherwise, it would be contained in only one fundamental circuit, $C_f$. Let $E'$ be an infinite set of independent edges (that is, not sharing any endpoints) such that $f \in C_e$ for all $e \in E'$ (We can require that the edges be independent because $G$ is locally finite). Index these edges as $\{e_i\}_{i \in I}$ where $I$ is some infinite set. Let $u_0, v_0$ be the endpoints of $f$. For each $e_i \in E'$, the fact that $f \in C_{e_i}$ means that the unique path in $T$ connecting the endpoints of $e_i$ contains $f$. Let $U = \{u_i \mid u_i v \in E', u_0 \in u_i T v_0\}$, in other words, the set of endpoints of edges of $E'$ that
are in the component of \( T \setminus \{f\} \) containing \( u_0 \). Call this component \( T_U \) and the other component \( T_V \).

Now, apply Lemma 12 to \( \{u_i\} \) in the graph \( T_U \). This yields a ray \( R_U \) and a set \( \{P_{i'}u_i\}_{i' \in I'} \) of disjoint \( U - R_U \) paths, all contained in \( T_U \), for an infinite index set \( I' \subseteq I \). Call the endpoint of \( P_{i'}u_i \) in \( U u_{i'} \), and let \( U' \subseteq U \) be the set \( \{u_{i'}\} \).

Each \( u_{i'} \) adjoins an edge \( e_{i'} \) in \( E' \), let \( v_{i'} \) be the other endpoint of this edge. Apply Lemma 12 to \( \{v_{i'}\} \) in \( T_V \) to generate a ray \( R_V \) and an infinite set of disjoint \( \{v_{i'}\} - R_V \) paths \( P_{i''}v_{i'} \) each of which has an endpoint \( v_{i''} \), with \( i'' \in I'' \subseteq I' \). Now there are infinitely many disjoint \( R_U - R_V \) paths \( P_{i''}u_{i'}e_{i'}v_{i''}P_{i''}v_{i'} \) in \( G \), so \( R_U \sim R_V \). However, \( R_U \) and \( R_V \) are separated in \( T \) by \( u_0 \), so they belong to different ends of \( T \), contradicting our assumption that \( T \) is end-faithful.

\[ \square \]

**Lemma 24.** Let \( T \) be an end-faithful spanning tree in \( G \), and \( e_0 \in T \). \( T \setminus e_0 \) has two components, \( T_1 \) and \( T_2 \). The \( T_1 - T_2 \) edges in \( G \) other than \( e_0 \) are exactly the edges \( \{e \mid e_0 \in C_e\} \).

**Proof.** Let \( E' \) be the set of \( T_1 - T_2 \) edges in \( G \). For any \( e \in E' \), \( (T \setminus \{e_0\}) \cup \{e\} \) contains no circuit because \( T_1 \) and \( T_2 \) are trees sharing no vertices or ends. On the other hand, for \( e \in E(G) \setminus (E(T) \cup E') \), the endpoints of \( e \) are both in \( T_1 \) or both in \( T_2 \), and so \( C_e \) is entirely within one of those two trees.

We also need a lemma on circuits and finite cuts. This lemma is also an important part of our final extension of Theorem 8.

**Lemma 25.** Let \( D \) be a finite cut and \( C \) a circuit. Then \( |C \cap D| \) is even.

**Proof.** Let \( G_1 \) and \( G_2 \) be the induced subgraphs on the partitioning sets of vertices that give us \( D \). Let \( S_1 \) be the set of neighbors of \( G_1 \) in \( G_2 \), and \( S_2 \) the set of neighbors of \( G_2 \) in \( G_1 \), both of which we now know to be finite. Since these two subgraphs of \( G \) are finitely separable, each ray of \( G \) has a tail in exactly only one of them. If \( G_1 \) does not contain the tail of any ray, then the union of open stars on vertices of \( G_1 \),

\[ \bigcup \{U(x, 2/3) \mid x \in G_1\} \]

is open, and if \( G_1 \) does contain a tail of \( R \in \omega \in \Omega(G) \), \( \tilde{C}_e(S_1, \omega) \) is open and contains the end of every ray with a tail in \( G_1 \). In either case, we have an open set containing every ray with a tail in \( G_1 \), and no point of \( G_2 \). We can do the same for \( G_2 \), and if we restrict these open sets to \( |G| \setminus \tilde{D} \), we have \( \overline{G_1} \) and \( \overline{G_2} \) as an open partition.

Now, let \( C' \) be the circle with edge set \( C \). For some homeomorphism \( \phi \), \( C' = \phi(S^1) \). Taking the inverse image \( \phi^{-1}(C' \setminus \tilde{D}) \), we have a finite collection \( \{I_1, \ldots, I_k\} \) of intervals on the circle (since we have removed finitely many disjoint intervals corresponding to edges in \( D \)). Each \( \phi(I_i) \) is connected and entirely in \( |G| \setminus \tilde{D} \), so it is in exactly one of the open sets \( \overline{G_1}, \overline{G_2} \), and does not intersect the other. Furthermore, the gap between two of these
Proof of Theorem 22, based on [5]. Let $C$ be a circuit in $G$. Let $e_0 \in E(G)$. We want to show that $e_0$ is in an odd number of the fundamental cycles $C_e$ where $e \in C \setminus E(T)$ if and only if $e_0 \in C$. If $e_0$ is not in $T$, then it is clearly in exactly one of these cycles, $C_{e_0}$ if it is in $C$, and in none if it is not in $C$. So, we will assume that $e_0 \in T$.

Let $T_1, T_2$ be the two components of $T \setminus \{e_0\}$, and let $E'$ be the set of $T_1 - T_2$ edges in $G$. By Lemma 24, $E' = \{e \mid e_0 \in C_e\} \cup \{e_0\}$.

So for each $e \in E' \setminus \{e_0\}$, $C_e$ contains $e_0$. By Lemma 23, $\{C_e \mid e \in E' \setminus \{e_0\}\} \subseteq \{C_e \mid e \in E(G) \setminus E(T)\}$ is thin, so $E'$ must contain only finitely many edges. Note that $E'$ is also a cut in $G$ induced by the partition of $V(G)$ into $V(T_1)$ and $V(T_2)$. By Lemma 25, $E'$ and $C$ share an even number $k$ of edges.

We have already identified the edges $e \in E' \setminus \{e_0\}$ as those for which $e_0 \in C_e$. If $e_0 \in C$ an odd number $k-1$ of these edges are in $C \setminus \{e_0\} = C \setminus E(T)$, so we have the desired result, that $e_0 \in \triangle \{C_e \mid e \in C \setminus E(T)\}$. And if $e_0 \notin C$, $k$ of these edges are in $C \setminus \{e_0\} = C \setminus E(T)$, so $e_0 \notin \triangle \{C_e \mid e \in C \setminus E(T)\}$.

Corollary 26. The fundamental circuits of an end-faithful spanning tree $T \subseteq G$ are a basis for $\mathcal{C}(G)$.

Proof, based on [4]. By definition, every element of $\mathcal{C}(G)$ is a symmetric difference of circuits. By Theorem 22, each one of these circuits is a symmetric difference of fundamental circuits of $T$, so the fundamental circuits generate $\mathcal{C}(G)$. To demonstrate that the set $\{C_e \mid e \in E(G) \setminus E(T)\}$ of fundamental circuits of $T$ is a minimal generating set, note that for each $f \in E(G) \setminus E(T)$, there is an element of $\mathcal{C}(G)$ containing it ($C_f$), and that $C_f$ is the only element of $\{C_e \mid e \in E(G) \setminus E(T)\}$ that contains $f$.

5.4 The Extension of Theorem 8 to Locally Finite Graphs

We will now, finally, give the other (with Theorem 9) of the possible locally finite extensions of Theorem 8:

Theorem 27. Let $G$ be a connected locally finite graph and $F \subseteq E(G)$. Then $F \in \mathcal{C}(G)$ if and only if $|F \cap D|$ is even for every finite cut $D$ in $G$.

Proof, based on [4]. Let $F \in \mathcal{C}(G)$ and $D$ a finite cut induced by the partition of $V(G)$ into sets $A$ and $B$. By definition, $F = \delta F$ where $F$ is a thin family of circuits. By definition, only a finite number of circuits in $F$ can meet each edge of $D$, so since $D$ is finite, only finitely many circuits of $F$ meet $D$. But we have already shown in Lemma 25 that each of these circuits meets $D$ in an even number of edges. This suffices to show that $F$ has an even intersection with $D$, since we have already shown (in our proof of Theorem 8) that
To summarize: we have looked into two entirely different applications of topological methods to infinite graphs. In Section 2, we demonstrated the topological roots of a particular

6 Conclusion

To summarize: we have looked into two entirely different applications of topological methods to infinite graphs. In Section 2, we demonstrated the topological roots of a particular

\( F = \triangle \{ C_e \mid e \in F \setminus E(T) \} \)

and hence \( F \in \mathcal{C}(G) \). We know by Theorem 23 that \( \{ C_e \mid e \in F \setminus E(T) \} \subseteq \{ C_e \mid e \in E(G) \setminus E(T) \} \) is a thin set. We need to prove that for every \( f \in E(T) \), \( f \in F \) if and only if \( f \) is in \( C_e \) for an odd number of edges \( e \in F \setminus E(T) \).

There are four cases to treat, similarly to our proof of Theorem 22, covering whether \( f \) is or is not in \( F \) and whether \( f \) is or is not in \( E(T) \).

The first is when \( f \) is in neither \( F \) nor \( E(T) \). Clearly, \( f \) is not in \( C_e \) for any \( e \in F \setminus E(T) \), since those cycles contain only edges of \( E(T) \) and \( F \setminus E(T) \).

The second case is when \( f \in F \setminus E(T) \). Then \( f \) is in exactly one fundamental cycle of \( T \): \( C_f \), which is clearly in \( \{ C_e \mid e \in F \setminus E(T) \} \).

For our last two cases, suppose \( f \in E(T) \). Let \( x, y \) be the endpoints of \( f \). By definition of the tree order and the uniqueness of paths between vertices in \( T \), \( x \) and \( y \) are comparable in the tree order; assume that \( x < y \). Consider \( T \setminus f \): It has two components, \( T_x \ni x \) and \( T_y \ni y \), both trees, which together span \( G \). The vertex sets of these components bipartition \( V(G) \) and hence induce a cut \( D_f \). We claim that \( D_f \) is finite: Every edge in \( D_f \) has an endpoint in \( T_x \) and an endpoint in \( T_y \). Note that \( V(T_y) = [y] \), by the fact that every vertex in \( T_y \) is separated from \( r \in T_x \) by \( f \). Hence the endpoint in \( T_x \) of every edge in \( D_f \) must also be in \( [x] \), since it must be comparable to \( y \) and not in \( [y] \). But \( [x] \) is finite, and \( G \) is locally finite, so there can only be finitely many edges with endpoints in \( [x] \), and hence only finitely many edges in \( D_f \).

Since \( D_f \) is a finite cut, by assumption, \( D_f \cap F \) contains an even number of edges. By Lemma 24, \( (D_f \cap F) \setminus f = \{ e \mid f \in C_e \} \).

Now we return to our cases: In the third case, \( f \in F \cap E(T) \), so \( f \in D_f \cap F \). Then \( D_f \cap F \setminus f \) contains an odd number of edges, and \( f \in C_e \) for an odd number of edges \( e \) in \( F \setminus E(T) \).

In the final case, \( f \in E(T) \setminus F \). Then \( D_f \cap F \setminus f = D_f \cap F \), which contains an even number of edges, and \( f \in C_e \) for an even number of edges \( e \) in \( F \setminus E(T) \). \( \Box \)
method of proof. Since Rado's Selection Theorem can be proven using the theory of ultrafilters [10], using topological methods is not necessary for compactness proofs, but does seem to illuminate the problem and could be the easiest method for some other problems.

On the other hand, the work we have surveyed in Sections 4 through 7 is inherently topological in nature, and generates extensions for a variety of theorems from finite graphs (see [2] for a broader survey). Although the topological redefinition of the cycle space seems strange at the outset, we hope that the extensions shown here (Thms. 22 and 27), as well as the links shown between graph theoretic and topological properties of the graph (Thms. 11, 14, and 18) indicate that it is at least useful, and perhaps the most natural and complete conception of the cycle space for a locally finite graph.

References


