An analysis is performed on a open quantum system of driven and damped Rydberg atoms. The interactions between Rydberg states are characterized by dipole-dipole energy shifts that give rise to a Rydberg blockade with on-resonant driving, and collective quantum jumps while driving with an off-resonant nonzero detuning. Under fully independent emission the system admits a bistable steady state exhibiting collective quantum jumps, in which quantum fluctuations drive transitions between two bistable values predicted by a quasiclassical mean-field model. When competition between fully independent and fully collective emission is considered in a two-channel emission model, a collective emission enhancement impacts the presence of the quantum jumps by shifting the predicted region of bistability for the system to different values of detuning. It is shown that jumps can be recovered to an extent, but that the bistability is ultimately destroyed under fully collective emission.
COLLECTIVE QUANTUM JUMPS OF RYDBERG ATOMS
UNDERGOING TWO-CHANNEL SPONTANEOUS EMISSION

A Thesis

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DEDICATION

For my mom and dad.
ACKNOWLEDGEMENTS

I would like to acknowledge the current and former members of the Rydberg blockade research group for their collaboration and teamwork, my fellow physics graduate students for the fond memories we made in and out of the classroom, and the members of the experimental optics groups for the countless conversations had and ideas shared while in the lab or while traveling across the country to attend conferences. I would like to thank the faculty and staff of the Miami University Department of Physics, especially the members of my committee, for your guidance and partnership. I would like to give a special thanks to my advisor Dr. James P. Clemens for taking me under his wing from the ages of 19 to 23, and who with a combination of encouragement, patience, and common sense, has been an extremely positive influence on my life. Finally, to my family and to my small circle of friends, your love and support made this possible.
Chapter 1

Introduction

To fully describe the interactions between atoms, photons, and other submicroscopic constituents of the universe, it is not enough to solely consider classical Newtonian physics. Physicists began to formulate quantum mechanics at the dawn of the 20th century to solve Max Planck’s famous blackbody radiation problem [2] and to explain Albert Einstein’s award winning photoelectric effect [3]. By the 1980s it was understood that a system of $N$ classical particles has a description size which is linear in $N$, but exponential in $N$ if the same system is made quantum. Following this, Richard Feynman connected the description size of the system to the slowdown of the efficiency of a classical computer simulation due to an increase in $N$ by proposing that a classical system would produce polynomial slowdown time but the quantum counterpart produces a significantly longer, exponential slowdown time [4]. Feynman suggested a way to combat this quantum effect by exploiting other quantum effects; that is, by instead using a computer which adheres to the rules of quantum mechanics rather than classical mechanics. The idea for a quantum computer was born.

Since then a solid theoretical framework has been established for quantum computer science [5], but the production of a substantial quantum computer is still long awaited. One could imagine the difficulty in trying to create a large-scale system that still behaves quantum. The stage of development in which physicists currently preside seems to be the stage in which the best quantum bit system is being debated upon. Two popular qubit systems today include ion traps and neutral atoms. Ions as qubits are promising due to their strong atomic interactions but are hindered by their equally strong interactions with their environment which lead to decoherence. Conversely, neutral atoms have weak interactions
with the environment, but have similarly weak interatomic interactions which lead to slow quantum gate operation times. There is a third option however, which combines the benefits of both the former. Rydberg atoms, although neutral, strongly interact with one another like ions, and have become a popular subject of study for their unique and useful properties.

1.1 Rydberg Atoms

Hypothetically any atom can be turned into a Rydberg atom, the requirement being that atom is excited to a Rydberg state. While in the Rydberg state the atom is described as having one or more valence shell electrons with a high principle quantum number, \( n \). The Rydberg atom is known to be hydrogen-like and is able to be modeled using Niels Bohr’s atomic theory. The electron experiences a Coulomb force which is equal to the centripetal force due to its circular orbit about the nucleus given by

\[
\frac{m_e v^2}{r} = \frac{k e^2}{r^2}.
\]  

(1.1)

In this expression \( m_e \) is the electron mass, \( e \) is the electron charge, and \( k \) is the Coulomb constant. This can then be rewritten to obtain an equation for \( v \), the electron’s velocity, in terms of the orbital radius \( r \)

\[
v = \sqrt{\frac{k e^2}{m_e r}}.
\]

(1.2)

In accordance to Bohr’s model, orbital angular momentum is quantized as \( L = m_e v r = n \hbar \). When the quantum rule is applied to the above expression the result is an expression for orbital radius as a function of principle quantum number

\[
r = \frac{n^2 \hbar^2}{m_e k e^2}.
\]

(1.3)

It shows that the orbital radius scales as the square of the principle quantum number. In terms of Rydberg atoms, the result is that the excited electron takes an orbit around the atom’s nucleus which is essentially classical, and it is for this reason that the atom is described
as hydrogen-like. The large radius means a large charge separation, and the consequence is an atom with a massive electric dipole moment and strong dipole-dipole coupling to other like atoms. These dipole interactions are the premier feature of interest for Rydberg atom systems [6], and certainly the reason why they have become the testing ground for various problems in quantum mechanics.

1.2 Quantum Jumps

While the Bohr Model is now known to be an incomplete description of matter, it brought forth ideas which are still relevant in physics today. The quantization of angular momentum means that the allowed orbits for the electron are also quantized. The electron can then make a transition between the allowed orbits by absorbing or emitting the required amount of energy via some scattering event. It was originally thought that the orbits were physically realizable, such as the planetary orbits around the sun, and that the discrete change in the electron’s energy involved the particle literally jumping from one orbit to another. Thus was the birthplace and coining of the term quantum jump. The quantum jump is a prime example of the seemingly non-intuitive predictions of quantum mechanics, as the quantized structure of the atom can be interpreted to suggest that nature is discontinuous in principle. Despite being bizarre, the validity of the theory behind Bohr’s quantization rule, which began as a convenient piece of mathematical formulation, is based on its agreement with experiment.

The first quantum jumps in a single atom were directly observed in 1986 by a group led by Hans Dehmelt at the University of Washington [7], then again shortly after by groups from the University of Hamburg [8] and the National Bureau of Standards in Boulder, Colorado [9]. Not only did the experiments solidify a basic principle of quantum theory, but experimental motivation brought forward the rapid development of new research techniques. The years leading up to the discovery saw advancements in ion trapping [10], laser cooling [11], and precision measurement [12]. Single atom quantum jumps have also sparked interest in research on collective quantum jumps [13, 14], the quantum jump of an ensemble of atoms. In the context of theoretical quantum optics, a notable consequence of the experiments was
the development of quantum trajectory theory by Howard Carmichael [15] as a result of considering the dynamics of individual atoms and photons.

1.3 Superradiance

Pioneered by Dicke [16], the phenomenon of superradiance is of chief interest in theoretical quantum optics and bears significance in the context of this study. Superradiance considers the emission scattering events which occur in an open quantum system of atoms interacting with a radiation field. Excited atoms exhibit ordinary luminescence via spontaneous emission if the atomic interactions with the radiation field are regarded as independent. Increasing the number of atoms can cause the system to exhibit a spontaneous emission which is inherently collective and generates correlation between all atoms in the sample upon decay. Superradiance is achieved when the perpetual generation of correlation in the system...
via collective emission causes the collective mode to emit more rapidly and intensely than the independent mode, reaching a peak intensity which is proportional to the square of the number of emitters. As shown in figure 1.1, luminescent emission occurs over an isotropic exponential decay time, while superradiant emission is anisotropic and occurs over a short emission burst.

While a full description of superradiance has been made available by Haroche [17], a short calculation involving angular momentum states gives an impression of the theory behind the phenomenon. The derivation assumes that the system interactions are the same regardless of how the atoms in the sample are labeled. Thus the system must be described mathematically by a Hilbert space which is fully symmetric with respect to the exchange of any two atoms. In this basis, the system of \( N \) 2-level atoms can be described with a one of a ladder of \( \left| \frac{N}{2}, m \right\rangle \) angular moment state vectors

\[
|eee...e\rangle = |\frac{N}{2}, \frac{N}{2}\rangle, \\
|gee...e\rangle = |\frac{N}{2}, \frac{N}{2} - 1\rangle, \\
|gge...e\rangle = |\frac{N}{2}, \frac{N}{2} - 2\rangle, \\
\vdots \\
|ggg...g\rangle = |\frac{N}{2}, -\frac{N}{2}\rangle. 
\]

(1.4)

Each state vector is represented by its length, \( \frac{N}{2} \), and by its z-component \( m \), which spans \( \pm \frac{N}{2} \) in integer steps. As indicated by 1.4, \( m = \frac{N}{2} \) corresponds to the fully excited state while \( m = -\frac{N}{2} \) sees all the atoms in the ground state. The emission intensity of the collection of atoms is measured as

\[
I = \gamma \langle J_+ J_- \rangle 
\]

(1.5)

where \( \gamma \) is the natural linewidth of the emission and where \( J_+ J_- \) operates on the ladder states as

\[
\langle \frac{N}{2}, m | J_+ J_- | \frac{N}{2}, m \rangle = \left( \frac{N}{2} + m \right) \left( \frac{N}{2} - m + 1 \right).
\]

(1.6)
When the collective emission intensity is calculated for the ladder of states the superradiant emission enhancement comes to fruition. Starting with the fully excited state $|\frac{N}{2}, \frac{N}{2}\rangle$, the emission intensities are

$$
\langle \frac{N}{2}, \frac{N}{2}|J_+J_-|\frac{N}{2}, \frac{N}{2}\rangle \propto N,
$$

$$
\langle \frac{N}{2}, \frac{N}{2} - 1|J_+J_-|\frac{N}{2}, \frac{N}{2} - 1\rangle \propto 2N,
$$

$$
\langle \frac{N}{2}, \frac{N}{2} - 2|J_+J_-|\frac{N}{2}, \frac{N}{2} - 2\rangle \propto 3N,
$$

\vdots

$$
\langle \frac{N}{2}, 0|J_+J_-|\frac{N}{2}, 0\rangle = (\frac{N}{2})(\frac{N}{2} + 1) \propto N^2.
$$

Hence each collective emission event causes the system to radiate with an intensity that is proportionally stronger than expected by an amount equal to the number of atoms that have previously emitted. The emission intensity continues to increase until it reaches the middle rung of the ladder of states with $m = 0$. At this point, the short emission burst has ended and the intensity begins to decrease, yet the system continues to emit superradiantly until $m = -\frac{N}{2}$.

### 1.4 Outline of Thesis

This thesis introduces a theoretical and computational study of an open quantum system featuring laser driven Rydberg atoms damped by both independent and collective spontaneous emission. It is shown that systems of 2-level Rydberg atoms exhibit collective jumps over many atomic lifetimes in which the excited state population fluctuates between two fixed values. It is then shown that there is a collective emission enhancement, related to superradiance, which influences these jumps. The thesis begins by deriving and explaining important theory and techniques from quantum optics used in this study, including master equation formalism, and Monte-Carlo trajectory theory, and semiclassical mean-field theory. The data and results are then presented as to tell the story of the presence or absence of the
collective quantum jumps. Finally, a conclusion will summarize the details of this project while providing insight to future research endeavors.
Chapter 2

Theoretical Frameworks

The physical dynamics of any system are mathematically described by one or more equations of motion. In this description the characteristics of the system are linked to generalized coordinates, such as $x$, $y$, and $z$ for the position of a particle. For a quantum system the description is required to be valid not only for particles but for waves as well. As systems become more complicated so do the equations of motion which model them. However it is important to find parallels between complicated and simple systems’ equations for better intuitive understanding, and also crucial to introduce generic methods to solve equations which may seem too mathematically intricate to manage.

2.1 Lindblad Master Equation

Analogous to classical equations of motions, the Schrodinger equation is a partial differential equation that describes the time evolution of the characteristics of a quantum system

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle. \quad (2.1)$$

In this case $|\Psi(t)\rangle$ is the quantum state, wave function, or state vector, and is the mathematical representation of the characteristics of the system. The act of making an experimental measurement on the system is represented mathematically by solving for the expectation value of an observable which denotes some physical operation associated with that measurement. This process is equivalent to performing the exact same measurement multiple times and recording an average value of the observed quantity. If $\hat{A}$ represents some
observable then the expectation value for observable \( \hat{A} \) in state \( |\Psi(t)\rangle \) is

\[
\langle \hat{A} \rangle = \langle \Psi(t)|\hat{A}|\Psi(t)\rangle.
\] (2.2)

The preceding assumes that the system in question could be represented by just a single state vector \( |\Psi(t)\rangle \). In many cases a system cannot be represented by a single state alone, but rather by some statistical ensemble of quantum states. It then becomes more useful to work with a quantity known as the density matrix, defined as

\[
\rho = \sum_j N P_j |\psi_j\rangle \langle \psi_j|
\] (2.3)

where \( |\psi_j\rangle \) is the \( j \)th quantum state in the ensemble with probability \( P_j \). In place of the Schrodinger equation is now the Von Neumann equation which utilizes the algebraic commutator \([a, b] = ab - ba\)

\[
i\hbar \frac{d}{dt} \rho = [\hat{H}, \rho]
\] (2.4)

while the expectation value for observable \( \hat{A} \) is now calculated with a matrix trace operation as

\[
\langle \hat{A} \rangle = Tr(\rho \hat{A}) = \sum_j N P_j \langle \psi_j|\hat{A}|\psi_j\rangle.
\] (2.5)

While the equation above is only valid for closed quantum systems, it may be updated in order to account for different physical phenomena or initial conditions. In quantum optics there are well known expressions which describe coherently driven systems simultaneously undergoing dissipation. In a sense the system can evolve in time according to the Schrodinger equation, or experience a wave function collapse due to stochastic processes. Let \( C \) be any operator which is responsible for a state collapse. The Lindblad Superoperator is defined as

\[
L(C)\rho = \frac{1}{2}(2C\rho C^\dagger - \rho CC^\dagger - C^\dagger C \rho).
\] (2.6)

It is important to note that the derivation of this expression is based on the Born-Markov
approximation and contingent on switching to the interaction picture of quantum mechanics [18]. Finally, the Von Neumann equation and Lindblad Superoperator when combined give rise to a versatile expression known as the master equation of Lindblad form

\[
\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + \frac{1}{2} \sum_j \Gamma_j (2C^\dagger_j \rho C_j - \rho C^\dagger_j C_j - C^\dagger_j C_j \rho)
\]

(2.7)

in which \(\Gamma_j\) is the strength of the coupling of the system to collapse operator \(C_j\). From this point, expectation values can be found using techniques from ordinary differential equations.

### 2.2 Quantum Trajectories

![Figure 2.1: A diagram of a quantum optical scattering experiment. Different driving inputs, such as a laser or other coherent electromagnetic field, can enter the interaction region. Incoherent driving might represent input thermal photons. Stimulated emission may correspond to coherent forward driving, while the undirected output may be indicative of spontaneous emission.](image)

While the previous approach is effective in calculating ensemble averages there is motivation to introduce a new method to study the complicated processes of an open quantum system. These processes may have a random probability distribution but can still be analyzed statistically. Hence the method known as quantum trajectories was developed and outlined by Carmichael [15] for the purpose of studying these stochastic quantum dynamics.
Consider an ideal open quantum system undergoing stochastic, state collapsing scattering events, and assume ideally that every scattered photon could be detected by some sort of detector or measuring device. In such a case, it becomes convenient to describe the master equation in terms of quantum states which contain information about the measurement record of the system. Hence the master equation is expressed as

$$\rho(t) = \sum_{REC} P_{REC}|\psi_{REC}(t)\rangle\langle\psi_{REC}(t)|.$$  (2.8)

The measurement record is a chronological list of all detected scattering events and is indicated with the subscript REC. Probability is preserved by imposing that the trace over the density matrix sums to unity as

$$\sum_{REC} P_{REC} = 1.$$  (2.9)

Thus $|\psi_{REC}(t)\rangle$ is the conditional state of a single scattering event and $P_{REC}$ is the probability for a specific scattering event to occur. A numerical integration method is needed to calculate the evolution of a state vector, which has dynamics that are deterministic but governed by random probabilities. Quantum trajectories utilize Monte Carlo methods, which are a popular category of algorithms that can be used to solve any problem with a stochastic interpretation. The quantum trajectory algorithm is as follows:

1. Let $\Gamma_j$ be the system’s coupling strength to collapse operator $C_j$. Then the probability $p_s$ of a scattering event to occur in a given timestep $\Delta t$ is

$$p_s = \sum_j \Gamma_j \langle C_j^\dagger C_j \rangle \Delta t.$$  (2.10)

2. A random number $r$ is generated from a uniform distribution with $0 < r \leq 1$.

3. The probability of scattering $p_s$ is compared to the randomly generated value $r$.

   (a) If $p_s > r$ then the system experiences a wave function collapse $|\psi\rangle \rightarrow C_j|\psi\rangle$. 

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The collapse operator which is chosen is based on the weighted probabilities $p_j = \Gamma_j \langle C_j^\dagger C_j \rangle \Delta t$.

(b) If $r \geq p_s$, the system does not experience a wave function collapse but instead evolves according to a non Hermitian Hamiltonian $|\psi\rangle \rightarrow |\psi\rangle - \frac{i}{\hbar} H_B |\psi\rangle \Delta t$. The non Hermitian Hamiltonian corresponds to performing a null measurement and is defined as

$$H_B = H - i\hbar \sum_j \frac{\Gamma_j}{2} C_j^\dagger C_j.$$  \hfill (2.11)

4. The results are aggregated and the algorithm is repeated on the new state of the system until a stopping condition is met.

### 2.3 System Description

This thesis considers an open quantum system of $N$ Rydberg atoms. Each atom is a 2-level system in which a valence shell electron is in a superposition of ground and Rydberg excitation states. The atoms can experience coherent excitation to the Rydberg state due to photon absorption from an external driving field and can return to the ground state via collective or independent spontaneous emission. Accordingly, the emission due to Rabi oscillation between these two states is observed, with the emission intensity indicating the Rydberg population over a given system’s evolution to the steady state. The Hamiltonian of the system is formulated in the interaction picture by approximating the atoms as dipoles in the presence of an external classical electric field and by applying the rotating wave approximation. This Hamiltonian is [13]

$$H = \sum_j^N \left[ -\Delta \sigma_j^+ \sigma_j^- + \frac{\Omega}{2} (\sigma_j^+ + \sigma_j^-) \right] + \frac{V}{N-1} \sum_{j<k}^N (\sigma_j^+ \sigma_{j^-}^+ \sigma_{k^-} + \sigma_{k^-}^+ \sigma_{j^-}^+).$$ \hfill (2.12)

Where $\sigma_j^-$ is the Pauli lowering operator for the $j$th Rydberg atom, $\Delta$ is the field’s detuning from atomic resonance, $\Omega$ is the Rabi oscillation frequency, and $V$ accounts for the long range Rydberg dipole interaction.
Consider a 1-atom system. Since the atom itself is 2-level, the basis states for the system would be

\[ |e\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |g\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

As multiple atoms are included, the basis states available to the system are the tensor products of the individual atomic states. For a two atom system the basis states would be

\[ |e\rangle \otimes |e\rangle = |ee\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |eg\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |ge\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |gg\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \]

In this fashion, the Hilbert space of the system scales as \(2^N\). Independent emission events are considered in the use of the familiar Pauli operators to build independent emission collapse operators. The Pauli lowering operator is defined as

\[ \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \]

The subscript \(j\) represents which atom will emit and thus there are \(N\)-many independent collapse operators for any given system. The independent operators have emission rate \(\gamma_I\) and are tensor products of the Pauli lowering matrix and the 2-dimensional identity matrix \(I_2\). For the 2-atom system these are

\[ \sigma_{1-} = I_2 \otimes \sigma_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \sigma_{2-} = \sigma_- \otimes I_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \]

Independent emission signifies a detector’s ability to distinguish the source of an emitted photon. If the detector clicks and knows exactly which atom it came from, then the system
has experienced a state collapse due to an independent collapse operator. The order of the tensor product makes this distinction clear, as \( \sigma_1 \) sends \( |ee\rangle \rightarrow |eg\rangle \) and \( |ge\rangle \rightarrow |gg\rangle \), while \( \sigma_2 \) sends \( |ee\rangle \rightarrow |ge\rangle \) and \( |eg\rangle \rightarrow |gg\rangle \).

On the other hand, collective emission signifies a detector’s inability to distinguish the source of emission events. In an open quantum system, collective emission may be possible if the wavelength of the emitted light is longer than the interatomic separation. Unlike independent emission there is only one collective emission collapse operator, defined as

\[
J_- = \sum_j \sigma_{j-}. \tag{2.17}
\]

The collective collapse operator has emission rate \( \gamma_C \) and causes entanglement upon state collapse. For example, when the collective collapse operator is applied to the fully excited 2-atom state the result is

\[
J_-|ee\rangle = (\sigma_1- + \sigma_2-)|ee\rangle = |eg\rangle + |ge\rangle. \tag{2.18}
\]

The dynamical evolution of the system is described by a master equation of the familiar Lindblad form, however to simultaneously consider both channels of emission it is necessary to use two Lindblad superoperators [19, 20]. The 2-channel master equation reads

\[
\dot{\rho} = -i[H, \rho] + \frac{\gamma_I}{2} \sum_j (2\sigma_j\rho\sigma_j^+ - \rho\sigma_j^+\sigma_j - \sigma_j\sigma_j^+\rho) + \frac{\gamma_C}{2} (2J_-\rho J_+ - \rho J_+ J_- - J_+ J_-\rho). \tag{2.19}
\]

To keep the total emission rate of the system consistent over many trajectories it is imposed that

\[
\gamma_I + \gamma_C = 1, \quad \gamma_I, \gamma_C \in [0, 1]. \tag{2.20}
\]

Under this condition, the system exhibits purely collective spontaneous emission when \( \gamma_I = 0 \) and \( \gamma_C = 1 \), while \( \gamma_C = 0 \) and \( \gamma_I = 1 \) corresponds to emission which is purely independent, thus mimicking a tunable dial between the two extremes. It is the inbetween case however,
where there is competition between collective and independent emission, that is of utmost interest in the context of this study.

It must be noted that the model containing the 2 types of emissions extends beyond that of an open system with varying interatomic distances [21]. Geometry dependence can be overshadowed in hybrid cavity QED systems if the group of atoms are coupled to the single cavity mode of a leaky cavity or optical fiber [22]. In this case, spontaneous emission into the cavity by the atoms is considered collective even if the atomic spacing is longer than a wavelength. Conversely, independent emission is achieved by emission leaking into outside modes. By this effect a two-channel master equation given by the form of equation 2.19 is naturally realized.

For an adequate value of \( N \) a mean-field theory can be used to study a semiclassical model of the system. In this model, an approximation is applied in which the density matrix is factorized symmetrically by atoms as

\[
\rho = \bigotimes_j \bar{\rho}
\]

with this new density matrix \( \bar{\rho} \) evolving according to

\[
\dot{\bar{\rho}}_{ee} = -\Omega \text{Im} \bar{\rho}_{eg} - \gamma \bar{\rho}_{ee}
\]

\[
\dot{\bar{\rho}}_{eg} = -i(\Delta - V\bar{\rho}_{ee})\bar{\rho}_{eg} - \frac{\gamma}{2} \bar{\rho}_{eg} + i\Omega(\bar{\rho}_{ee} - \frac{1}{2}).
\]

These are the optical Bloch equations for a two-level atom using the local field correction [23]. The effective emission rate is \( \gamma \) and the reformulated effective laser detuning is \( \Delta_{eff} = \Delta - V\bar{\rho}_{ee} \). Note that this factorization is made natural under independent emission, as the independence of the emissions will tend to destroy any correlations between the atoms. It is this lack of correlations that is expressed by a density matrix which factorizes. In other words, it is assumed each atom has the same factorized density matrix because the symmetry of the master equation treats all of the atoms in exactly the same way. Conversely, the mechanism
of collective spontaneous emission inherently generates correlations. Therefore while the predictions of the mean-field theory are important they can only be truly considered under the requirement of fully independent emission.

2.4 Computational Methods

Rarely is there a case in modern theoretical quantum optics that the system dynamics are easily described analytically. Instead, numerical simulations and classical computing techniques are relied upon to generate and evolve the equations of motion. The most important computing tool used in this study is the Quantum Toolbox in Python, stylized as QuTiP.

In short, QuTiP is an open-source software written in the Python programming language specifically designed for use in simulating the dynamics of open quantum systems. Utilizing other popular Python packages, such as Numpy and Scipy for calculation and Matplotlib as a graphics output, the QuTiP framework has found use in multiple universities and institutions around the world due to its absence of licensing fees and its cross-platform capacities. What is most notable about QuTiP however is its ease of use, as the commands mimic common quantum mechanics language. For example the quantum object class of data structures include commands such as \texttt{qutip.basis} to call basis state vectors and \texttt{qutip.sigmam} to utilize the familiar Pauli lowering operator.

The QuTiP library contains built in algorithms which are extremely relevant to the calculations necessary for this study [24, 25]. While the \texttt{qutip.mesolve} function evolves an initial state and obtains expectation values using ordinary differential equation techniques, the valuable \texttt{qutip.mcsolve} function does so using the quantum trajectory techniques discussed in section 2.2. The \texttt{mcsolve} function requires 5 inputs: the system’s Hamiltonian, the system’s initial state in ket vector form, a list of calculation times, a list of collapse operators, and a list of desired expectation values. It is also optional to input \texttt{ntraj}, the desired number of trajectories to be simulated. The output is a data structure of the \texttt{qutip.Odedata} class which can then be unraveled to return the desired expectation values using the \texttt{expect} command, a list times when a collapse operator was applied using the \texttt{col-times} command,
<table>
<thead>
<tr>
<th>Desired quantity or expectation value</th>
<th>Calculated expression</th>
<th>QuTiP function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Collective Emission Intensity</td>
<td>$\langle J_+ J_- \rangle$</td>
<td>mcsolve</td>
</tr>
<tr>
<td>Independent Emission Intensity</td>
<td>$\sum \langle \sigma_j^+ \sigma_j^- \rangle$</td>
<td>mcsolve</td>
</tr>
<tr>
<td>Relative Intensity</td>
<td>$\left( \frac{1}{N} \right) \text{Intensity}$</td>
<td>mcsolve</td>
</tr>
<tr>
<td>Collective Emission Percentage</td>
<td>$\frac{\text{collective collapse operations}}{\text{total collapse operations}}$</td>
<td>mcsolve</td>
</tr>
<tr>
<td>$g^{(2)}$ Correlation</td>
<td>$\frac{\langle \sigma_1^+ \sigma_1^- \sigma_2^+ \sigma_2^- \rangle}{\langle \sigma_1^+ \sigma_1^- \rangle \langle \sigma_2^+ \sigma_2^- \rangle}$</td>
<td>mcsolve</td>
</tr>
</tbody>
</table>

Table 2.1: Table of relevant quantities calculated in this study, accompanied by the QuTiP function used to solve for that value.

and a list of the corresponding applied collapse operators using the command `col-which`.

Table 2.1 describes calculations done in the context of this study. In general the system is evolved to the steady state and relevant quantities reveal the system’s behavior in terms of relative emission intensity and Rydberg population. For the most part, the calculations are done using certain functions from QuTiP. Mean-field theory predictions are generated using Mathematica by solving for the fixed steady state values of equations 2.22 and 2.23. Further statistical and graphical analysis is done in Microsoft Excel if necessary.
Chapter 3

Results

The initial interest in the system described by Lee et al. was due to the quantum jumps seen in the system’s bistable steady state under fully independent spontaneous emission. Preceding work intended to study the bistability solely under the condition of collective emission. Doing so allows for the switch to a truncated, fully symmetric subspace which would drastically increase the computation time for high $N$. In this way, the primary motivation of the project was to study collective quantum jumps for systems of up to $N \sim 100$. However, the focus of the project changed as it was found that collective emission was suppressing the feature of interest.

3.1 2-Atom System

The probe spectrum calculated in figure 3.1 highlights the atomic interactions of a simple system of 2 atoms. The probability of having exactly 0, 1, or 2 atoms excited to the Rydberg state is calculated using $Tr(\rho|gg\rangle\langle gg|)$, $2Tr(\rho|eg\rangle\langle eg|)$, and $Tr(\rho|ee\rangle\langle ee|)$ respectively, while other expectation values are given in table 2.1. If one photon is absorbed by the system then the amount of energy needed to absorb two is increased by $V$ due to the Rydberg dipole-dipole interactions. This means that when the system is coherently driven on resonance the fully excited state $|ee\rangle$ is uncoupled from the other two states and the system exhibits the familiar Rydberg blockade. To absorb 2 photons, the system needs to be driven off atomic resonance. Detuning the driving laser by a value of $\frac{V}{2}$ allows for excitation to the fully excited state with coupling to intermediate states via spontaneous emission.
(a) Energy level diagram

(b) $V = 10, \Omega = 1.5$

Figure 3.1: (a) When $N = 2$ is driven on resonance the 2 atom excited state is shifted by $V$ and the system exhibits a blockade. When $\Delta = \frac{V}{2}$ the double excited state is on resonance via a two-photon transition. (b) When $V = 10$ and $\Omega = 1.5$ we see double excited state population at $\Delta = 5$.

### 3.2 Mean Field Theory

For $N=16$ the system is sufficiently large and the results predicted by quasiclassical mean-field theory are applicable. The optical Bloch equations are valid in the steady state if $\dot{\rho}_{ee}$ and $\dot{\rho}_{eg}$ are set to zero. The resulting expression is

$$\rho_{ee}(1 + \frac{2}{\Omega^2}(\Delta - V\rho_{ee})^2 + \frac{\gamma^2}{2\Omega^2}) - \frac{1}{2} = 0. \tag{3.1}$$

For a given $V$, $\Omega$, and $\gamma$, the value for $\rho_{ee}$ is solved over a range of positive detunings. These solutions represent the relative Rydberg population in the steady state. The cubic form of $\rho_{ee}$ suggests that there are three possible steady state solutions, each identified with a different color in figure 3.2. Two of the solutions are stable, while the third is unstable. It
becomes apparent that not every value of $\Delta$ corresponds to just one steady state solution, as the vertical line included on figure 3.2 shows that some correspond to all three, but more specifically, bifurcation of both stable solutions. Thus the values of detuning which satisfy the condition of bifurcation form a \textit{bistable region} for the steady state. It is interesting to note that the bistable region can shift to different values of $\Delta$ under different chosen parameters, most notably the total emission rate of the system $\gamma$. Figure 3.2b shows that increasing the total emission rate shifts the bistable region to smaller values of detuning.

## 3.3 Collective Quantum Jumps

If a classical system found itself in a state which satisfies the conditions for bistability described in 3.2, the system would simply relax to one of the two metastable states. The Monte Carlo trajectories calculated in figure 3.3 show the interesting behavior a quantum system exhibits under these conditions while undergoing purely independent spontaneous emission. In this case the bistability manifests itself in collective quantum jumps in which the system sporadically fluctuates between intense or dim emission. That is, quantum fluctuations cause the system to jump in time between a dark state with low Rydberg population and a bright
Figure 3.3: Single quantum trajectories of 16 atoms for $\Delta = 3.4, \Omega = 1.5, V = 10$ demonstrate the suppression of the cooperative jumps by collective emission. Blue and orange arrows point to fixed points predicted by the mean-field theory calculation of Figure 2.
state with high Rydberg population. Furthermore these two values correspond to the stable solutions predicted by mean field theory.

If the collective emission is turned on then there is a clear impact on the collective quantum jumps, as demonstrated in trajectories 3.3b and 3.3c. As the value of $\gamma_C$ is increased it seems qualitatively that the upper metastable branch becomes less accessible and less inhabitable. To quantify the impact statistically, the relative emission intensities calculated over long trajectories for various values of $\gamma_C$ are binned and tallied as a bar graph in figure 3.4a. The bistability can be seen in the appearance of local maxima around 0.1 and 0.4, which respectively represent the lower and higher metastable branches. However for $\gamma_C = 0.05$ the maxima representing the upper branch has almost completely disappeared. It is now evident that collective spontaneous emission is suppressing the collective quantum jumps.

What is most notable is that the value of $\gamma_C$ is less than an entire order of magnitude smaller than $\gamma_I$ and yet its presence seems to govern the dynamics. In addition to this it is important to remember that there are 16 $\sigma_j$ independent collapse operators available to the system and only 1 collective collapse operator $J_-$. This suggests that the system is somehow very sensitive to the collective emission. The question then rises, is the suppression due to in-between-jump evolution, or is there an enhancement of the collective emission? Using trajectories simulated for $\gamma_C = 0$ to $\gamma_C = 0.1$, the relative number of state collapses due to $J_-$ are tallied and graphed as function of the input $\gamma_C$ in figure 3.4b. In general the value of $\gamma_C$ is six times lower than the relative fraction of $J_-$ applications over that trajectory. This means that choosing $\gamma_I = 0.95$ and $\gamma_C = 0.05$ does not necessarily imply that 5 percent of the system’s emissions are collective. Indeed there is a collective emission enhancement. As mentioned in section 2.3, the nature of collective emission is such that the emission inherently generates correlations. This fits the description of the collective mode in section 1.3. In this way, the enhancement of collective emission is due to the superradiance phenomenon.

The statistics shown in figure 3.5 provide insight about the bistable region of the system for various values of $\gamma_C$. A series of Monte Carlo trajectories similar to that of figure 3.3 have been calculated and averaged at each given $\Delta$. The result is a curve which bears resemblance
Figure 3.4: Statistics averaged over trajectories of hundreds of atomic lifetimes. (a) For a given trajectory the relative Rydberg excitation values are sorted into bins of width 0.05. For fully independent emission the system frequents two population values but as $\gamma_C$ is increased the upper branch population value is diminished. (b) Number of state collapses due to $J_-$ weighted by total number of state collapses. The collective emission percentage is 6.1265 times greater than the set value of $\gamma_C$. 
Figure 3.5: Average relative emission intensity as a function of detuning for different values of $\gamma_C$, taken over trajectories of hundreds of lifetimes. The range between the maximum and minimum values of relative intensities decreases as the system becomes more collective.

to the mean field theory plots of figure 3.2. In this case there is no bifurcation but rather a sharp drop off from a maximum value to a asymptotic minimum. This drop off must represent the transition from the upper stable branch to the lower which suggests that the system’s bistable region is located somewhere along this range of detunings. While figure 5 cannot explicitly predict bistability it does imply a trend as $\gamma_C$ increases which effects the bistable region of the system under the given conditions. In particular it indicates a trend in which the bistable region widens and shifts to smaller values of detuning as the system becomes more collective.

### 3.4 Photon Correlation

The nature of the system is one that guarantees symmetric emission among atoms after a sufficiently long time average. So to measure a 2-photon correlation in a system of 16 atoms it is enough to measure the correlation between any 2 atoms in the ensemble. By this
Figure 3.6: $g^{(2)}$ photon correlation as a function of detuning for $V = 10$, $N = 16$, $\Omega = 1.5$, show how the bistable region evolves as $\gamma_C$ increases.
effect, the $g^{(2)}$ correlation function defined in table 2.1 is calculated in the steady state and averaged over hundreds of lifetimes, as shown by figure 3.6. By definition the correlation function is a measure of intensity fluctuation. Hence the expectation is for large intensity fluctuations to occur when the value of detuning is set to correspond to the maximum value of the correlation function. In terms of quantum jumps, the bistable region agrees with a region of rapidly increasing fluctuation which occurs for values of detuning just before the maximum. For example, this region is centered about $\Delta = 3.4$ when $\gamma_c = 0$ (Fig. 3.6a), which were the exact parameters that showed bistability in trajectory 3.3a. Accordingly, the second order correlation functions provide further insight to the suppression of jumps in trajectories 3.3b and 3.3c. There is a familiar trend with increasing $\gamma_C$ in which the bistable region again widens and shifts to smaller values of $\Delta$.

The results in figure 3.6 suggest that bistability is still possible in the presence of small amounts of collective emission. Shown in figure 3.7 are quantum trajectories simulated for $\gamma_C = 0.05$, 0.1, and 0.15, but with new values of $\Delta = 3.1$, 2.8, and 2.5 respectively. Because the detuning has been decreased the systems in trajectories 3.7a, 3.7b, and 3.7c, have found themselves in their respective bistable regions, and the collective quantum jumps appear unsuppressed. This shows that the key factor in preserving the bistability while increasing the coupling the the collective emission channel is a decrease in the detuning of the driving field.

While the bistable behavior can be retrieved by decreasing $\Delta$ it does not go without complication. As highlighted by figure 3.1a, a nonzero detuning is necessary in order to excite multiple atoms in the Rydberg cloud. Since the upper metastable limit is now met for smaller detunings, the Rydberg population corresponding to this branch also decreases. For example, the upper branch of the fully independent system emits with an average relative intensity of just under 0.35, while the upper branch of the system with $\gamma_C=0.2$ has an intensity just under 0.15 (Fig. 3.7c). In either case the bottom branch has consistently low emissions but the gap between the two is ever decreasing as $\gamma_C$ increases. By this effect, the system reaches a point in which transitions between the upper and lower metastable
Figure 3.7: Single quantum trajectories of 16 atoms for $\Omega = 1.5, V = 10$ with values of detuning selected such that the collective jumps appear unsuppressed. Jumps can be recovered to an extent until the upper and lower metastable branches become too close in value.
branches are too difficult to discern from random noise and fluctuation in the Monte Carlo simulations.

### 3.5 Nonlinearity

An semiclassical analysis of the equations of motions done on the cases of fully collective and fully independent emission may provide insight on the role of collective emission in the suppression of jumps. The Hamiltonian of the system provided by equation 2.12 is written in terms of independent quantum operators. Using the definition of $J_z$, the Hamiltonian can be written in terms of collective operators as

$$H = -\Delta(J_z + I) + \frac{\Omega}{2} J_x + \frac{V}{N-1} \sum_{j<k} |e\rangle\langle e| \otimes |e\rangle\langle e|.$$  \hspace{1cm} (3.2)

The final sum in the equation, which represents the dipole-dipole coupling, can be approximated using $J_z$ as

$$\frac{V}{N-1} \sum_{j<k} |e\rangle\langle e| \otimes |e\rangle\langle e| \approx \frac{V}{4(N-1)} (J_z^2 + 2J_z + N^2).$$  \hspace{1cm} (3.3)

This approximation is valid within order $N$. The necessary steps in the analysis are taken to arrive at and compare expressions for $\langle \dot{\sigma}_l \rangle$ and $\langle \dot{J}_- \rangle$, which are the time derivatives of the expectation values of the independent and collective collapse operators. The most important part of the expressions however are the terms which materialize due to the dipole-dipole $V$ term. For fully independent emission this is

$$\langle \dot{\sigma}_l \rangle = \cdots - iV(\langle \sigma_l \rangle \rho_{ee} + \rho_{ee} \langle \sigma_l \rangle) \hspace{1cm} (3.4)$$

where the factors of $N$ have been dropped for simplicity. For fully collective emission it is only necessary to consider contributions from the $J_z^2$ portion of the $V$-terms. The expression is
\[ \langle \dot{J}_- \rangle = \cdots + \gamma \langle J_- \rangle \langle J_z \rangle + iV \langle J_- \rangle \langle J_z \rangle. \]  

(3.5)

Both expressions contain terms of the form \( \langle - \rangle \langle z \rangle \), but in the expression for independent emission the terms also contains a factor of \( \rho_{ee} \), which is the key to the nonlinearity that leads to bistability. In the semiclassical analysis, the extra factor arises from making the mean-field theory approximation, which is valid for fully independent emission but not fully collective. The approximation ignores correlations between atoms in the system and allows for the density matrix to be factorized by atom (Eq. 2.21), leading to the factor of \( \rho_{ee} \). Conversely, the correlations between atoms cannot be ignored under fully collective emission, the mean-field assumption is not made, and the nonlinear factor does not appear. This analysis suggests that bistability is not attainable under fully collective emission.
Chapter 4

Outlook

Since the 1970s Rydberg atoms have been a popular subject of study for in theoretical and experimental quantum optics. Rydberg atoms have been shown to exhibit collective jumps [13] and produce Rydberg blockades [6] and are of exemplary use in optical cooling and trapping systems which require a large number of atoms. The strong dipole-dipole interactions of Rydberg atoms give way to uses in the studies of problems in quantum mechanics, such as quantum information processing [26] and many-body physics [27].

This text has described a model for an open quantum system governed by Lindblad master equation dynamics and whose constituent atoms are coupled via Rydberg dipole-dipole interactions. Under fully independent emission the steady state of the system is bistable in the intensity, as quantum fluctuations drive collective quantum jumps between two emission values. When the collective spontaneous emission is made accessible to the atoms, in proportion to the independent emission, there is a collective emission enhancement and the collective quantum jumps appear suppressed. However, second order correlations functions show that the bistable parameter space of the system’s steady state broadens to include smaller values of detuning, and the collective quantum jumps can be made to reappear unsuppressed by making this change to the driving field. The analysis also shows that as the coupling to the collective emission increases beyond a threshold, then the difference between the upper and lower branch of the bistability lessens, making the collective quantum jumps indistinct as the system becomes more collective. Finally, it is shown that the nonlinearity which admits bistability under fully independent emission is nonexistent in the fully collective case. Thus it is believed that bistability is unattainable under fully collective spontaneous
emission.

4.1 Future Development

The 1 and 2-dimensional geometry dependence of the system is currently being investigated [21]. The two-channel emission model is considered by use of the Lehmberg-Agarwal master equation which interpolates between fully collective and fully independent spontaneous emission. In particular, there is motivation to recover collective quantum jumps for certain geometries by changing the parameter space, similar to the results of section 3.4. For the system analyzed within this text, further work will be done to support the claims of section 3.5. Certain regimes of the parameter space will continue to be tested; the behavior of the system in the large $V$ coupling regime is of particular interest. Additionally, it is once again of interest to switch to the fully symmetric Hilbert subspace to analyze a system of hundreds of atoms under fully collective emission.
Appendix A

Mean-Field Theory Code

This Mathematica code can be used to recreate figure 3.2

```
Remove["Global\*"]
```

Equation to Solve

```
Solve[2/Ω (Δ - V pee)2 pee + γ2 / (2Ω2) pee + (pee - 1/2) == 0, pee];
```

Solution Functions

Solution One

```
sol1[V_, γ_, Ω_] = 
2ΩΔ - (12V2γ2 - 16V2Δ2 + 24V2Ω2) / 
(6 2^3/3 V^2 
(-288V3γ2Δ - 128V3Δ3 + 432V4Ω2 - 576V3ΔΩ2 + 
√(4 (12V2γ2 - 16V2Δ2 + 24V2Ω2)^3 + 
(-288V3γ2Δ - 128V3Δ3 + 432V4Ω2 - 576V3ΔΩ2)^2))^{1/3} ) + 
\frac{1}{12 2^{1/3}V^2} 
(-288V3γ2Δ - 128V3Δ3 + 432V4Ω2 - 576V3ΔΩ2 +
```

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\[
\sqrt{4 \left(12V^2\gamma^2 - 16V^2\Delta^2 + 24V^2\Omega^2\right)^3 + (-288V^3\gamma^2\Delta - 128V^3\Delta^3 + 432V^4\Omega^2 - 576V^3\Delta\Omega^2)^2} \right)^{1/3} = \]

Solution Two

\[
sol2[V,\gamma,\Omega] = \frac{2\Delta}{3V} + \left((1 + i\sqrt{3}) \left(12V^2\gamma^2 - 16V^2\Delta^2 + 24V^2\Omega^2\right)/ \left(12 \, 2^{2/3} V^2 \right)
\right)

(-288V^3\gamma^2\Delta - 128V^3\Delta^3 + 432V^4\Omega^2 - 576V^3\Delta\Omega^2 +

\sqrt{4 \left(12V^2\gamma^2 - 16V^2\Delta^2 + 24V^2\Omega^2\right)^3 +
\left(-288V^3\gamma^2\Delta - 128V^3\Delta^3 + 432V^4\Omega^2 - 576V^3\Delta\Omega^2\right)^2) \right)^{1/3} \right) -

\frac{1}{24 \, 2^{1/3} V^3} \left(1 - i\sqrt{3}\right)

(-288V^3\gamma^2\Delta - 128V^3\Delta^3 + 432V^4\Omega^2 - 576V^3\Delta\Omega^2 +

\sqrt{4 \left(12V^2\gamma^2 - 16V^2\Delta^2 + 24V^2\Omega^2\right)^3 + (-288V^3\gamma^2\Delta - 128V^3\Delta^3 + 432V^4\Omega^2 - 576V^3\Delta\Omega^2)^2} \right)^{1/3}

;}

Solution Three

\[
sol3[V,\gamma,\Omega] = \frac{2\Delta}{3V} + \left((1 - i\sqrt{3}) \left(12V^2\gamma^2 - 16V^2\Delta^2 + 24V^2\Omega^2\right)/ \left(12 \, 2^{2/3} V^2 \right)
\right)

(-288V^3\gamma^2\Delta - 128V^3\Delta^3 + 432V^4\Omega^2 - 576V^3\Delta\Omega^2 +

\sqrt{4 \left(12V^2\gamma^2 - 16V^2\Delta^2 + 24V^2\Omega^2\right)^3 +
\left(-288V^3\gamma^2\Delta - 128V^3\Delta^3 + 432V^4\Omega^2 - 576V^3\Delta\Omega^2\right)^2) \right)^{1/3} \right) -

\frac{1}{24 \, 2^{1/3} V^3} \left(1 + i\sqrt{3}\right)

;
\[
(-288V^3\gamma^2\Delta - 128V^3\Delta^3 + 432V^4\Omega^2 - 576V^3\Delta\Omega^2 + \\
\sqrt[3]{4(12V^2\gamma^2 - 16V^2\Delta^2 + 24V^2\Omega^2)^3 + (-288V^3\gamma^2\Delta - 128V^3\Delta^3 + 432V^4\Omega^2 - 576V^3\Delta\Omega^2)^2})^{1/3}
\]

Dynamic Plot

Manipulate[Plot[{sol1[V, 1, \Omega], sol2[V, 1, \Omega], sol3[V, 1, \Omega]}, \{\Delta, 0, 10\}],
\{{V, 10, “Rydberg Energy Shift”}, 0.1, 20\}, \{\{\Omega, 1.5, “Driving”\}, 1, 5\}]

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Appendix B

Quantum Trajectory and Jump Statistics Code

This code in QuTiP can be used to create the quantum trajectories of figures 3.3 and 3.7 and also utilizes the times and which commands which can generate figure 3.4b. The output data can be further analyzed to create figure ??.

```python
from __future__ import division
from qutip import *
import pylab
import math
import matplotlib.pyplot as plt
import numpy as np
import plotly.plotly as py
from plotly.graph_objs import *

#plot.ly information
py.sign_in("cayayald", "zlae2s4d1i")

#linewidth
gamma = 0.85
G = 1.0 - gamma
```

35
#delta is detuning
delta = 2.5

#n is number of atoms
n = 16

#V is energy shift due to dipole-dipole
V = 10

#omega is rabi freq
omega = 1.5
omegastep = 0

sigmamlist = []
sigmaplist = []
coplist = []

# raising and lowering ops
for j in range (n):
    if j==0:
        sigmajm = sigmam()
    else:
        sigmajm = qeye(2)
    for i in range (1,n):
        if j==i:
            sigmajm = tensor(sigmam(),sigmajm)
        else:
            sigmajm = tensor(qeye(2),sigmajm)
sigmamlist.append(sigmajm)
coplist.append(math.sqrt(gamma)*sigmajm)
sigmajp = sigmajm.dag()
sigmaplist.append(sigmajp)

oplist = []
op = sigmaplist[0]*sigmamlist[0]
for i in range(1,n):
    op += sigmaplist[i]*sigmamlist[i]
oplist.append(op/n)

coplist.append(math.sqrt(G)*sum(sigmamlist))

#state vectors
psi0 = basis(2,1)
for i in range(1,n):
    psi0 = tensor(psi0,basis(2,1))

tlist = np.linspace(0,500,5000)

#Hamiltonians
datalist = []
hlist = []
hilist = []

for i in range(n):
    hlist.append(-delta*(sigmaplist[i]*sigmamlist[i]))
\[
+ \left(\frac{\omega}{2}\right)(\sigma_{\omega}\text{list}[i] + \sigma_{\text{m}}\text{list}[i])
\]

for \(i\) in range\(n\):
    for \(j\) in range\(i\):
        hilist.append\((\frac{V}{(n - 1)})(\sigma_{\omega}\text{list}[j] \sigma_{\text{m}}\text{list}[j])
        \ast (\sigma_{\omega}\text{list}[i] \sigma_{\text{m}}\text{list}[i]))\)

\(ntraj = [1]\)
\(H = \text{sum}(\text{hlist}) + \text{sum}(\text{hilist})\)
\(\text{data1} = \text{mcsolve}(H, \psi0, \text{tlist}, \text{coplist}, \text{oplist}, \text{ntraj})\)

\(\text{time} = \text{zip}(\text{data1}.\text{col}\_\text{times})\)
\(\text{which} = \text{zip}(\text{data1}.\text{col}\_\text{which})\)
\(\text{time} = \text{time}[0]\)
\(\text{which} = \text{which}[0]\)

\(\text{time} = \text{np}.\text{transpose}(\text{time})\)
\(\text{which} = \text{np}.\text{transpose}(\text{which})\)

# plotly stuff
\(\text{alpha} = \text{Scatter}(x=\text{tlist}, y=\text{data1}.\text{expect}[0][-1]) \# \text{independent spectrum}\)
\(\text{epsilon} = \text{Scatter}(x=\text{time}, y=\text{which})\)
\(\text{beta} = \text{Data}([\text{alpha}])\)
\(\text{sigma} = \text{Data}([\text{epsilon}])\)

\(\text{unique\_url} = \text{py}.\text{plot}(\text{beta}, \text{filename} = 'Gc=' + \text{str}(G) + 'D=' + \text{str}(\text{delta}) + ',')\)
\(\text{unique\_url} = \text{py}.\text{plot}(\text{sigma}, \text{filename} = 'n.traj for Gamma=' + \text{str}(G) + ',')\)
Appendix C

Photon Correlation Function Code

Code can be used to generate the correlation functions of figure 3.6.

```python
from __future__ import division
from qutip import *
import pylab
import math
import matplotlib.pyplot as plt
import numpy as np
import csv
#import plotly.plotly as py
#from plotly.graph_objs import *

#py.sign_in("cayayald", "zlae2s4d1i")

#linewidth
#gamma = 1 => fully independent
#G = 1 => fully collective
gamma = .25
G = 1.0 - gamma

delta is detuning
delta = -1
```
deltastep = 0.1
dlist = []

#n is number of atoms
n = 16

#V is energy shift due to dipole-dipole
V = 10

#omega is rabi freq
omega = 1.5
omegastep = 0

#lowering, raising, and state collapsing operators
sigmamlist = []
sigmaplist = []
coplist = []
for j in range(n):
    if j==0:
        sigmajm = sigmam()
    else:
        sigmajm = qeye(2)
    for i in range(1,n):
        if j==i:
            sigmajm = tensor(sigmam(),sigmajm)
        else:
            sigmajm = tensor(qeye(2),sigmajm)
sigmamlist.append(sigmajm)
coplist.append(math.sqrt(gamma)*sigmajm) #n-many indep collapse ops

sigmajp = sigmajm.dag()
sigmalist.append(sigmajp)
coplist.append(math.sqrt(G)*sum(sigmamlist)) #one collective collapse op

# we want expectation values of operators in exoplist
exoplist = []

# numerator of cross correlation
exoplist.append(sigmamlist[0]*sigmamlist[0]*sigmaplist[1]*sigmamlist[1])

# denominator of cross correlation
exoplist.append(sigmamlist[0]*sigmamlist[0])
exoplist.append(sigmamlist[1]*sigmamlist[1])

# initial state vector
psi0 = basis(2,0)
for i in range(1,n):
    psi0 = tensor(psi0,basis(2,0))

# build Hamiltonian

g2 = []
tlist = np.linspace(0,200,2000)
ntraj = 1
while delta <= 7:
    glist = []
hlist = []
vlist = []
for i in range(n):
    hlist.append(-delta*(sigmaplist[i]*sigmamlist[i]))
+ (omega / 2)*(sigmaplist[i] + sigmamlist[i])

for i in range(n):
    for j in range(i):
        vlist.append((V / (n - 1))*((sigmaplist[j]*sigmamlist[j])
                        *(sigmaplist[i]*sigmamlist[i])))
H = sum(hlist) + sum(vlist)

# monte carlo trajectory
montecar = mcsolve(H, psi0, tlist, coplist, exoplist, ntraj)
for i in range(len(montecar.expect[0])):
    numerator = montecar.expect[0][i]
    denominator = montecar.expect[1][i]
    denom2 = montecar.expect[2][i]
    glist.append(numerator/(denominator*denom2))
    g2.append(sum(glist)/len(glist))
dlist.append(delta)
    delta = delta + deltastep

x = np.arange(10)
fig = plt.figure()
ax = plt.subplot(111)
ax.plot(dlist, g2, label='g_c=' +str(G)+ ' 

# Shink current axis by 20%
box = ax.get_position()
ax.set_position([box.x0, box.y0, box.width * 0.8, box.height])
plt.grid(True)
ax.yaxis.tick_right()
ax.yaxis.set_label_position("right")
ax.tick_params(axis='x', labelsize=15)
ax.tick_params(axis='y', labelsize=15)

# Put a legend to the right of the current axis
ax.legend(loc=10, prop={'size':12}, bbox_to_anchor=(0.1, 0.5))

#ax.set_yscale('log')
plt.title('g2 with n=' +str(n)+'')
plt.xlabel('Detuning', size='x-large')
plt.ylabel('Intensity', size='x-large')
plt.show()
Appendix D

Average Relative Intensity Code

This code is used to generate the data for figure 3.5.

```python
from __future__ import division
from qutip import *
import pylab
import math
import matplotlib.pyplot as plt
import numpy as np
import csv
import plotly.plotly as py
from plotly.graph_objs import *

py.sign_in("cayayald", "zlae2s4d1i")

#linewidth
#gamma = 1 => fully independent
#G = 1 => fully collective
gamma = 0.95
G = 1.0 - gamma

#delta is detuning
delta = -1
```
deltastep = 0.5
dlist = []

#n is number of atoms
n = 16

#V is energy shift due to dipole-dipole
V = 10

#omega is rabi freq
omega = 1.5
omegastep = 0

#lowering, raising, and state collapsing operators
sigmamlist = []
sigmaplist = []
coplist = []
for j in range(n):
    if j == 0:
        sigmajm = sigmam()
    else:
        sigmajm = qeye(2)
    for i in range(1,n):
        if j == i:
            sigmajm = tensor(sigmam(), sigmajm)
        else:
            sigmajm = tensor(qeye(2), sigmajm)
sigmamlist.append(sigmajm)
coplist.append(math.sqrt(gamma)*sigmajm) # n-many indep collapse ops

sigmajp = sigmajm.dag()
sigmalist.append(sigmajp)
coplist.append(math.sqrt(G)*sum(sigmamlist)) # one collective collapse op

# we want expectation values of operators in exoplist
exoplist = []

op = sigmaplist[0]*sigmamlist[0]
for i in range(1,n):
    op += sigmaplist[i]*sigmamlist[i]
exoplist.append(op/n) # independent emission relative intensity

# initial state vector

psi0 = basis(2,0)
for i in range(1,n):
    psi0 = tensor(psi0,basis(2,0))

# build Hamiltonian

tlist = np.linspace(0,1000,10000)
ntraj = [1]

avgElist = []
while delta <= 7:
    hlist = []
    vlist = []
    for i in range(n):
        hlist.append(-delta*(sigmaplist[i]*sigmamlist[i])
                      + (omega / 2)*(sigmaplist[i] + sigmamlist[i]))
    for i in range(n):
for j in range(i):
vlist.append((V / (n - 1))*(sigmaplist[j]*sigmamlist[j])
*(sigmaplist[i]*sigmamlist[i]))
H = sum(hlist) + sum(vlist)

# monte carlo trajectory
montecar = mcsolve(H, psi0, tlist, coplist, exoplist, ntraj)
avgE = (sum(montecar.expect[0][-1]))/(len(montecar.expect[0][-1]))
avgEList.append(avgE)
dlist.append(delta)
delta = delta + deltastep

# write to csv
fname = 'pseudomft.csv'
with open(fname, 'wb') as f:
    writer = csv.writer(f)
    for i in range(len(dlist)):
        row = []
        row.append(dlist[i])
        row.append(avgEList[i])
        writer.writerow(row)
References


