A PERTURBED MOON: SOLVING NEREID’S MOTION TO MATCH OBSERVED BRIGHTNESS VARIATIONS

by Andrew J. Hesselbrock

Since its discovery in 1949, Nereid’s photometric variations, orbit, and mass have been well established, however knowledge of its spin, orientation, and shape is lacking. We simulate Nereid’s orbital and rotational motion, dependent on these unknown characteristics, in an attempt to match observations. We show how a time-dependent gravitational torque can cause the body to precess on a timespan as small as \( \sim 17 \) years, following a complicated coning nutation. Modeled as a uniformly reflecting body, we find that if the photometric variations are to be solely explained by geometry, Nereid cannot be either prolate or oblate. We have produced large amplitude, intra-night variations similar to those presented in Schaefer et al. (2008), but are unable to fully match their observations. Our study shows our strongest candidate to have an initial obliquity of \( 60^\circ \), a spin rate of 144 hours, and semi-axial ratios of \( \frac{c}{a} \approx 0.5, \frac{b}{a} \approx 0.6 \).
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A Thesis

by

Andrew J. Hesselbrock

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Advisory and Examination Committee:

______________________________
S. G. Alexander (advisor)

______________________________
M. J. Pechan

______________________________
P. K. Urayama
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1 Introduction

First discovered by Gerard Kuiper (Kuiper, 1949), Nereid is a small satellite orbiting Neptune with a semi-major axis of $5.5 \times 10^6$ km (Vashkov’yak, 2010) and an eccentricity of 0.7512 (Dobrovolskis, 1995). It has a radius of $170 \pm 25$ km (Schaefer et al., 2008), a mass of $3.10 \times 10^{19}$ kg (Wu, 2008), and an orbital period of 360 days (Schaefer et al., 2008). It is unknown how fast Nereid rotates about its axis, its rotational orientation, or its shape as our only image (9 x 4 pixels) of the satellite was taken by Voyager II in 1989 from $4.7 \times 10^6$ km away and reveals very little (Schaefer et al., 2008). Much of Nereid remains a mystery, with much waiting to be discovered.

Nereid is most assuredly not a perfectly spherical object (Dobrovolskis, 1995) and will thus experience a torque as it rotates about its axis during its orbit around Neptune. Nereid has the most eccentric orbit of any regular moon in our solar system being almost seven times farther away from its parent planet at apoapsis than at periapsis. It is unknown how Nereid evolved to this orbit, with some theories proposing that it was a captured Kuiper Belt object, or perhaps it was formerly a close moon to Neptune that was perturbed out by Triton (Goldreich, 1989). At apoapsis Nereid is $9.6 \times 10^6$ km away from Neptune, and at periapsis only $1.4 \times 10^6$ km away (Vashkov’yak, 2010). With a torque $\tau \propto \frac{1}{R^2}$, and the nature of a body to move fastest at periapsis, it is likely that the magnitude of the torque Nereid experiences will change by a factor $\sim 320$ over a very short timespan, a “kick”, as it travels through each orbit.

1.1 Brightness Modulations

Other than Nereid’s high eccentricity, Nereid is an interesting subject of study as it exhibits unexplained photometric variations. These variations in the observed brightness of Nereid can be small, to quite large over very short timespans. Schaefer et al. (2008) have observed Nereid over the past twenty years in an attempt to understand its motion, physical characteristics, surface features, etc. They have observed intra-night and night to night variations that could be the result of high and low albedo regions rotating into view (Schaefer et al., 2008). Figure 1 shows their observations of Nereid’s change in brightness over a 20 year period from the average magnitude of their observations. They propose that Nereid experiences a forced precession as a result of its unique orbit and the nature of the rotational torques it may experience.

1.2 The Problem of “Unknowns”

With the vast uncertainty in the geometry and rotation state of Nereid, and the likelihood that the moon experiences very large torques over very short timespans, we set about to model its rotation in an attempt to solve the problem of its unexplained photometric variations. If we could solve the rotational characteristics of an object in Nereid’s orbit and use conservative estimates for Nereid’s physical characteristics, we may shed some light onto the issue.

There are three main physical characteristics which remain completely unknown that are directly relevant to the initial conditions of our simulations with Nereid (Alexander et al., 2011).
Fig. 1: Light curve showing modulations in Nereid’s brightness from 1987 to 2006. All distance and geometry effects have been removed, meaning only variations due to rotational dynamics remain. It is possible that these modulations show Nereid going through a highly active period (large changes in magnitude) between 1986 and 1991 to a slightly less active (small changes in magnitude) period between 1991 and 2002, and then again to a highly active region between 2002 and 2008 (Schaefer et al., 2008). We will compare all of our results to this figure in an attempt to solve the motion of Nereid and will subsequently call it the “Schaefer Plot”.

1.2.1 Spin Rate

Some have proposed that if Nereid’s spin rate was slow enough the body’s rotational motion may be chaotic. Dobrovolskis (1995) has shown that Nereid’s rotation could be chaotic, but only if Nereid’s spin period was greater than about two weeks for one complete rotation. This is an interesting result, however Schaefer et al. (2008) insists that the spin period is much smaller. They claim that the shortest period variations in their observations of Nereid’s brightness demand that Nereid’s spin period cannot be longer than $\sim 3$ days, as any spin period much larger than this would make it unlikely for Nereid to exhibit large amplitude variations in brightness due to a different portion of its surface rotating into view. Thus, if Nereid’s spin period is on the order of 72 hours, its rotation would not be chaotic.

1.2.2 Shape

No conclusive estimates to Nereid’s physical shape can be made from the only image taken by Voyager (Schaefer et al., 2008). To avoid going into the limitless possibilities of such a body, we can simplify the problem by investigating Nereid’s moment of inertia. The ratio between the moments of inertia of the rotating body’s axes has a major effect on a body’s response to gravitational torques. Schaefer et al. (2008) propose that the ratio of Nereid’s short axis to its long axis could be $> 1.9$, as this
could result in a forced precession period $\leq 16$ years, matching the active and inactive modulations in Nereid’s brightness (Schaefer et al., 2008).

### 1.2.3 Obliquity

An object’s obliquity is the angle between its rotation axis and the normal to its orbital plane. If Nereid is axially symmetric and its obliquity is 0°, then no precession results as the symmetry of the body’s orientation means no torque results; likewise if the obliquity is 90° the torque will be zero when the body’s spin axis is pointed towards Neptune (Schaefer et al., 2008). This will occur at two points in the orbit. Thus, the maximum torque that Nereid can experience would be at an obliquity of 45°. In order to retain the precession period of $\sim 16$ years, Nereid’s obliquity is likely to be $\sim 30°$.

These three unknown characteristics (obliquity, spin, and shape) serve as the foundation to our adjustable variables from trial to trial. While using acceptable estimates of these three variables, our goal is to build a computer simulation to calculate the rotational dynamics of such a body and then relate these results to the Nereid case. Our goal is to discover and explain the nature of Nereid’s rotational dynamics, and perhaps ultimately replicate the observations of Schaefer et al. (2008). We agree with the arguments put forth by Schaefer et al. (2008) that Nereid is likely to have an axial tilt (obliquity) of $\sim 30°$, a spin period on the order of $\sim 3$ days, and a moment of inertia ratio (between the long and short axis) of $\sim 2.4$. We define these variables set at these values as “The Schaefer Case”.

### 1.3 Purpose of This Work

The case for Nereid is an interesting one. With so many “unknowns”, an investigation into the cause of Nereid’s photometric variations can both help reveal characteristics this moon may possibly have, and also explore the dynamical relation between a body’s shape, axial tilt, and spin rate on its rotational motion. It is the intent of this thesis to demonstrate that we can accurately build a computer simulation to calculate Nereid’s orbit, explore the dynamics of a body’s rotational motion, and solve how these motions consequently affect brightness measurements conducted on Earth. This simulation, although specific to Nereid, will be quite versatile and could be applied to situations where the rotational dynamics of a body are not fully understood, or where the light curve of a body is called into question. In Chapter 2 we will provide the physical reasoning and computational methods utilized in solving Nereid’s motion. Chapter 3 will be an exploration of “The Schaefer Case” in a two-body, Neptune-Nereid system. This chapter will focus on understanding the dynamics of the time-dependent gravitational torque Nereid is likely to experience. We will then construct a heliocentric simulation in Chapter 4 that will allow us to calculate how rotational modulations can affect measurements in Nereid’s brightness on Earth. Chapter 4 will also provide an extensive investigation into Nereid’s shape, concluding with our best possible candidate for Nereid. This “Best Case” will be our closest, realistic estimate for Nereid that matches the “Schaefer Plot”, given a specific shape, spin rate, and obliquity. Chapter 5 concludes our work and outlines further work that may be completed with our simulation.
2 Computational Methods

2.1 Solving the Orbital Motion

Our computer model is an adaptation from an existing N-body code (Alexander & Agnor 1998; Abel & Alexander, 2001) that utilizes the Hermite Individual Timestep Scheme (HITS) (Makino, 1992; Lecar, 1986; Beaugé, 1990; Kokubo, 1995, 1996; Alexander et al., 2011) method to simultaneously integrate both the orbital and rotational equations of motion for a body. The HITS method allows each body in a simulation to have its own timestep, conserving calculation time. Bodies with more complex intricate motion have smaller timesteps, and bodies with smaller perturbations have longer timesteps. The HITS method uses position, velocity, acceleration, and jerk (time derivative of acceleration) functions to predict a body’s motion after a perturbation. Knowing a body’s position and velocity vectors, we can find their time derivatives, perform a Taylor series expansion, and then correct our functions to account for the perturbation. This scheme is frequently used for N-body simulations, which gives us the freedom to utilize it to model only two bodies, Neptune and Nereid (which greatly simplifies the process) (Alexander & Agnor, 1998; Abel, 2001; Alexander et al., 2011), or later (after some adjustments) the entire solar system. If we choose to only model the Neptune and Nereid system, we can place Neptune at the origin, and thus need only calculate the motion for one body, Nereid (Alexander et al., 2011). (When modeling the entire solar system, we place the sun at the center and much of our motion can be solved using relative coordinates.)

Figure 2 describes the geometry of our orbital frame. The body \( m_j \) represents either Neptune or any other satellites of Neptune. For the orbital motion of a mass \( m_i \) (representing Nereid) with respect to body \( m_j \), HITS requires the acceleration \( \mathbf{a}_i \) and its time derivative, the jerk \( \dot{\mathbf{a}}_i \):

\[
\mathbf{a}_i = - \sum_{j \neq i} \frac{G m_j}{r_{ij}^3} \mathbf{r}_{ij},
\]

\[
\dot{\mathbf{a}}_i = - \sum_{j \neq i} \frac{G m_j}{r_{ij}^3} \left[ \mathbf{v}_{ij} - \frac{3}{r_{ij}^2} (\mathbf{r}_{ij} \cdot \mathbf{v}_{ij}) \mathbf{r}_{ij} \right],
\]

where \( \mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i \) and \( \mathbf{v}_{ij} = \mathbf{v}_j - \mathbf{v}_i \) are the relative position and velocity of body \( m_j \) with respect to \( m_i \) and \( G \) is the gravitation constant. To find the trajectory of body \( m_i \), equations (1) and (2) are summed over all the bodies, \( m_j \), included in the simulation and integrated as described in Makino and Aarseth (1992).

We shall also define the XYZ frame as an inertial frame with the XY-plane coincident with the equatorial plane of Neptune.

2.2 Solving the Rotational Motion

To describe the rotation of Nereid, we use the three Euler angles \( (\psi, \theta, \phi) \) to orient the body–fixed principal axes (xyz) relative to the inertial axes (XYZ). We adopt the 3-2-1 angle convention used in Hughes (1986) where we rotate about the X-axis by angle \( \psi \), then by \( \theta \) about an intermediate axis \( y' \), and finally about the body z-axis by angle \( \phi \). Therefore the body fixed frame is the result of applying these rotations

\[
\mathbf{a}_i = - \sum_{j \neq i} \frac{G m_j}{r_{ij}^3} \mathbf{r}_{ij},
\]

\[
\dot{\mathbf{a}}_i = - \sum_{j \neq i} \frac{G m_j}{r_{ij}^3} \left[ \mathbf{v}_{ij} - \frac{3}{r_{ij}^2} (\mathbf{r}_{ij} \cdot \mathbf{v}_{ij}) \mathbf{r}_{ij} \right],
\]
Fig. 2: Geometry of the N-body rotational dynamics problem. $X,Y,$ and $Z$ refer to the Neptune-centered coordinate system, $x,y,$ and $z$ are the principal axes of the rotating body $m_i$, and $m_j$ is a body which can exert a gravitational force/torque on the rotating body.

to the inertial frame. Figure 3 shows the sequence of these rotations. The resulting matrix, $T$, transforms components of a vector from the inertial frame (XYZ) to the body frame (xyz):

$$T = \begin{pmatrix} 
\cos \theta \cos \phi & \sin \theta \sin \psi \cos \phi + \cos \psi \sin \phi & -\sin \theta \cos \psi \cos \phi + \sin \psi \sin \phi \\
-\cos \theta \sin \phi & -\sin \theta \sin \psi \sin \phi + \cos \psi \cos \phi & \sin \theta \cos \psi \sin \phi + \sin \psi \cos \phi \\
\sin \theta & -\cos \theta \sin \psi & \cos \theta \cos \psi 
\end{pmatrix}. \quad (3)$$
Fig. 3: Orientation of the body’s z-axis as rotated by the Euler angles. The body’s z-axis is rotated about the x-axis by $\psi$, then about an intermediate $y'$-axis by $\theta$, and finally about the z-axis by $\phi$.

The Euler equations take their simplest form in the body–fixed principal axes frame:

$$A \dot{\omega}_x - (B - C)\omega_y\omega_z = \tau_x,$$

$$B \dot{\omega}_y - (C - A)\omega_z\omega_x = \tau_y,$$

$$C \dot{\omega}_z - (A - B)\omega_x\omega_y = \tau_z,$$

where $\omega_x$, $\omega_y$, and $\omega_z$ are the angular velocity components, $A$, $B$, and $C$ are the principal moments of inertia about the $x$, $y$, and $z$ axes respectively, and the components $\tau_x$, $\tau_y$, and $\tau_z$ are the components of the net torque on the body. Murray and Dermott (1999) give the components of the torque about the body principal axes acting on a non-spherical body $m_i$ by point mass $m_j$ as:

$$\tau_x = 3Gm_j(C - B)yz/r^5,$$

$$\tau_y = 3Gm_j(A - C)zx/r^5,$$

$$\tau_z = 3Gm_j(B - A)xy/r^5.$$

6
For a spheroid (identical in x and y) the moment of inertia about the x-axis equals the moment of inertia about the y-axis (A = B). To understand the motion of the body we calculate the Euler moment equations \( \dot{\omega}_{x,y,z} \) by including the now known torque components. Combining these torque equations with our position and velocity formulae we get the following result:

\[
\dot{\omega}_x = I_{bca} \omega_y \omega_z - 3Gm_j I_{bca} yz/r^5, \tag{10}
\]

\[
\dot{\omega}_y = I_{cab} \omega_z \omega_x - 3Gm_j I_{cab} zx/r^5, \tag{11}
\]

\[
\dot{\omega}_z = I_{abc} \omega_x \omega_y - 3Gm_j I_{abc} yx/r^5. \tag{12}
\]

We invoke the convention \( I_{abc} \) to describe the ratio of the moments of inertia where

\[
I_{abc} = \frac{(A - B)}{C}. \tag{Hughes, 1986}
\]

To calculate the body’s orientation to a high precision we calculate the body’s angular velocity, acceleration, and jerk functions (e.g. \( \dot{\psi}, \ddot{\psi}, \dddot{\psi} \)) and then utilize the HITS integrator to solve for our body’s orientation at any moment in time. Knowing \( \dot{\omega}_{x,y,z} \) we are able to calculate \( \dot{\psi}, \dot{\theta}, \dot{\phi} \) with the following equations:

\[
\dot{\psi} = \cos(\phi) \frac{\omega_x}{\cos(\theta)} - \sin(\phi) \frac{\omega_y}{\cos(\theta)}, \tag{13}
\]

\[
\dot{\theta} = \sin(\phi) \omega_x + \cos(\phi) \omega_y, \tag{14}
\]

\[
\dot{\phi} = -\tan(\theta) \cos(\phi) \omega_x + \frac{\sin(\phi) \sin(\theta)}{\cos(\theta)} \omega_y + \omega_z. \tag{15}
\]

Knowing \( \psi, \theta, \phi \) it is possible to calculate many different quantities to track the body’s orientation, however for our simulation we chose to follow the orbiting body’s rotation by calculating its obliquity and precession angles. As stated earlier, an object’s obliquity (\( \beta \)) is the angle its spin axis makes with the normal to its orbital plane and depends on both the orientation of the body axes and the orbit. Using \( \psi, \theta, \phi \) it is given by

\[
\cos \beta = \sin(I \sin(\Omega \sin \theta + \cos \Omega \sin \psi)) + \cos I \cos \theta \cos \psi, \tag{16}
\]

where \( I \) is the orbital inclination and \( \Omega \) is the longitude of the ascending node. If we define obliquity to be the angle between the spin angular momentum (\( \vec{s} \), with unit vector \( \hat{s} \)) and the orbital angular momentum (\( \vec{h} \)), we can simplify this and calculate \( \beta \) via the following equation:

\[
\beta = \arccos(\frac{\vec{h} \cdot \vec{s}}{hs}). \tag{17}
\]

Precession is the change in a rotating axis’s orientation and can be found by projecting the spin axis onto the orbital plane. We define the precession angle (\( \delta \)) as the angle this projection makes relative to the ascending node. We calculate it via
\[ \delta = \arctan \left( \frac{\mathbf{s} \cdot \mathbf{n}_1}{\mathbf{s} \cdot \mathbf{n}_2} \right), \]  \hspace{1cm} (18) 

where \( \mathbf{n}_1 \) is a unit vector along the line of the ascending node, and \( \mathbf{n}_2 \) is a unit vector in the orbital plane perpendicular to \( \mathbf{n}_1 \).

We use these two angles, shown in Figure 4, to track Nereid’s rotational axis through time. In examining Schaefer’s theory, we first examined two possible cases, a prolate body (where the body’s height is greater than its width) and an oblate body (whose height is shorter than its width).

Fig. 4: Geometry of the body’s obliquity (\( \beta \)) and precession (\( \delta \)) angles. \( \beta \) is the angle that the spin axis (\( \mathbf{s} \)) makes with the normal to the orbital plane (\( \mathbf{n} \)). \( \delta \) is the angle that the projection of the spin axis onto the orbital plane makes with the line of the ascending node (\( \Omega \)) of the orbit.
3 Exploring the Schaefer Case

Schaefer et al. (2008) have discussed various possible models that may fit the Nereid case, including both a gravitationally torque dependent forced precession and a free precession, where no torques result. It is theoretically possible that Nereid may be experiencing a free precession, however the circumstances required of this deem it rather unlikely. In a free precession model a body’s precession is not caused by gravitational torques, but rather is a precession of the body’s axis that is allowed to travel freely. This can occur when a body has been captured or been pushed into this motion by a collision with another body. Therefore Nereid would have to be almost completely spherical (such that no torque results), and would have had to have been pushed into this precession by some sort of glancing collision with some other body (Schaefer et al., 2008).

The forced precession model is significantly more likely and is the entire focus of this work. A body experiencing a forced precession could not be perfectly spherical in order to experience gravitational torques. These torques would then perturb the body’s spin axis and could cause it to precess. The “Schaefer Case” is based upon this model, however to achieve a precession period that matches closely with their observations, they propose that Nereid is a prolate body (with an inertial ratio $A/C = 2.4$) spinning about its long axis. It is the goal of this chapter to explore the effects of Nereid’s rotation when each of the unknown variables (obliquity, spin, shape) are modified.

We borrow our initial conditions from Murray & Dermott (1999). They provide us with the physical characteristics of Neptune and Nereid along with the bodies’ orbital elements, from which we can calculate the Cartesian position and velocity vectors for our initial conditions. Our simulation calculates the initial position and velocity vectors of the body Nereid, then uses the HITS to progress the body’s motion forward in time. From there we can calculate the body’s distance from the parent planet, obliquity, precession angle, axis orientation, etc.

In exploring the Schaefer Case we will present our results over 50 year periods, approximately twice the timespan of the “Schaefer Plot”. The initial position and velocity components remain the same for all simulations. First we will set the obliquity of the body at 30°, and its rotational period at 72 hours. We will also set the ratio of the moment of inertia about the x and z axes such that $A/C = 2.4$. For all simulations and results presented in this chapter, the moment of inertia about the x and y axis is the same, thus $A/B = 1$. From this initial setup we will then change each of these variables independently. For example, we may keep the obliquity of the body set at 30°, and the ratio $A/C = 2.4$, but then vary the period of the body’s rotation.

In Chapter 3.1 we will discuss the effects of changing each of these variables, along with a comparison of modeling Nereid as either an oblate or prolate spheroid. In Chapter 3.2 we will describe the body’s rotational dynamics.

3.1 Effects of Unknown Parameters

First, how does the overall shape of the body affect its motion? Nereid is equally likely to be shaped as either an oblate or prolate spheroid, and our first step was to model both using the constraints proposed by the Schaefer Case. Quickly in
Figure 5 and in Figure 6 it can be seen that interesting motion arises in both the body’s obliquity and angle of precession, with a striking difference between the oblate and prolate cases. Both bodies exhibit oscillations in their obliquity and precession angles of varying magnitudes, with “large” jumps occurring when the body is at periapsis. Looking at the obliquity of the body, we see that oscillations occur at a relatively constant magnitude throughout the orbit until the next periapsis pass. At which point, a large spike in the obliquity can be seen, and the oscillations then continue, however at a new amplitude. The overall effect are regions of “high activity” and regions of “low activity” which in theory could match observed photometric variations in the “Schaefer Plot”. Examining the body’s angle of precession we see that oscillations exist at a relatively constant amplitude throughout the orbit, however as the body leaves periapsis it does not have the same value as when it approached periapsis. Here we see that while traveling through periapsis, the body has been pushed into a new orientation.

In Figure 7 it becomes obvious that as the distance from Neptune approaches the minimum value (periapsis), Nereid exhibits a sizable perturbation in its obliquity and precession angles. The large spikes in the obliquity angle confirm our hypothesis that Nereid is experiencing a “kick” while the changes in the angle of precession of the non-spherical body confirm the theory of forced precession.

![Fig. 5: Obliquity over 50 years for oblate (grey) and prolate (black) spheroids. Both bodies have the same spin rate (72 hours), moment of inertia ratio ($\frac{I}{\bar{I}} = 2.4$), and initial obliquity. As the prolate spheroid shows very large variations in obliquity, along with some periodicity that may match the “Schaefer Plot”, it served as the majority of our focus. The large spikes experienced in both cases occur when the body approaches periapsis.](image)
Fig. 6: Precession angle over a full precession period for oblate (grey) and prolate (black) spheroids. Both bodies have the same spin rate (72 hours), moment of inertia ratio \( (\frac{I_C}{I_A} = 2.4) \), and initial obliquity. As the prolate spheroid shows very large variations in the precession angle and a much faster rate of precession (\( \sim 17 \) years for the prolate case vs. \( \sim 70 \) years for the oblate), it served as the majority of our focus.
Fig. 7: Obliquity (left panel) and precession (right panel) angles over a ten year period for the Schaefer Case of a prolate body plotted with the relative distance to Neptune (middle scale). Here we can clearly see the correlation between the perturbations in both angles to the location of periapsis, where the body is “kicked”. As the distance from Neptune approaches the minimum, both angles are perturbed, and oscillations resume. The amplitude of the oscillations then remain relatively constant throughout the rest of the orbit.

Although modulations arise in both cases, the modulations in the prolate case are significantly more pronounced (as a prolate body spinning about its long axis is less dynamically stable). If the variations in Nereid’s brightness are to be attributed to how the body’s rotational motion reflects sunlight towards Earth, it is obviously apparent a prolate body is a much stronger candidate for this theory. Thus, the prolate case is the focus of the rest of this chapter.

Varying the initial obliquity does have a minimal effect on the amplitude of the body’s variations, but more consequently, it alters the period between active and inactive regions, with higher obliquities resulting in shorter periods. This period is \( \sim 18 \) years for an initial obliquity of \( 15^\circ \), and decreases to \( \sim 9 \) years as the obliquity increases to \( 60^\circ \). This is demonstrated in Figure 8. Lastly, the rate of precession increases as the initial obliquity increases.

Figure 9 shows that varying the spin rate of the body strongly affects the amplitude of the body’s oscillations. As the spin period of the body is increased and the body is more easily perturbed, the amplitude of the obliquity oscillations increases greatly. Faster spinning bodies are more rotationally stiff and have significantly smaller oscillations. Eventually if the body is spinning fast enough, the amplitude of the oscillations
is quite small and the modulated oscillations disappear.

Finally, we found that the occurrence of these modulations had a strong dependence on the distribution of mass, as seen in Figure 10. For these oscillations to grow to any notable level, the body must have an $A/C$ ratio greater than $\sim 2.35$, meaning Nereid would be quite non-spherical. Also, larger amplitude variations in obliquity were experienced by more out-of-round bodies.

Fig. 8: Obliquity for a prolate body with a spin period of 72 hours and moment of inertia ratio $(\frac{A}{C} = 2.4)$. As the initial obliquity angle increases, the period between active and inactive regions decreases.
Fig. 9: Variations in obliquity for a prolate body ($A/C = 2.4$) with various spin rates. As the spin period of the body decreases, the amplitude of the modulations decrease as well. The faster the body spins the more rotationally stiff the body becomes.

Fig. 10: Obliquity angle for a prolate body (spin 72 hours) with varying moment of inertia ratios. For each data set both the moment of inertia ratio ($A/C$) and the corresponding axial ratio ($c:a = \sqrt{2(A/C) - 1}$) are labeled. The effect only becomes largely apparent when $A/C > 2.35$, requiring Nereid to be quite out-of-round.
3.2 Dynamics

What is causing these oscillations and what do they mean? If we track the body’s z-axis, and project that onto a unit sphere in X, Y, and Z, we can follow how the vertical axis travels in time. This can be done by simply transforming the coordinates of $\mathbf{s}$ to the inertial frame. In Figure 11 it becomes clear that the body’s z-axis is undergoing a coning motion in some sort of complicated nutation. Sometimes the coning motion is large, sometimes small. These regions match up exactly with the active and inactive regions mentioned in the previous chapter, where large coning motion subsequently means large amplitude oscillations in the obliquity and precession angles, and vice versa. This figure also allows us to easily see how the precession angle changes over time. Figure 12 plots the vertical axis on this same unit sphere as the body travels through periapsis, showing how the body’s orientation changes during the “kick”.

Projecting this motion onto the XY-plane and plotting against time on the vertical axis (as in Figure 13) we clearly see that the body is coning about one specific orientation, passes through periapsis, is pushed to a new orientation, and begins coning about this new direction. Figure 14 shows how the projection of the z-axis on the XY-plane changes during the “kick”.

![Figure 11: Orientation of the body's z-axis on a unit sphere over a full precession period for the prolate Schaefer Case, revealing a complicated nutation. It can clearly be seen that the body exhibits a coning motion, is perturbed to a new orientation, and then resumes coning. A projection onto the XY plane can also be seen (Alexander et al., 2011).](image-url)
Fig. 12: Orientation of the body’s vertical axis on a unit sphere during a periapsis pass for the prolate Schaefer Case (Alexander et al., 2011).

Fig. 13: Orientation of the body’s $z$-axis on a unit circle over a full precession period for the prolate Schaefer Case versus time on the vertical axis. Here we can see that the body is coning about a specific location and then resumes coning about a new orientation after the kick (Alexander, 2011).
To explain why the body is kicked to a new location during periapsis, we need only to look at the torques. Examining the full magnitude of the torque in all directions ($\tau = \sqrt{\tau_x^2 + \tau_y^2 + \tau_z^2}$) over time we see that the torque the body experiences rises and falls, year-to-year, in a pattern that directly matches the obliquity modulations. When the body experiences a very large torque, a large coning motion results, meaning large modulations in obliquity and the angle of precession.

The magnitude of the torque shows how the body’s angular momentum changes with time and depends upon the orientation of the body’s spin axis as it travels through periapsis. The torque depends upon the cross product of the directional vector and the gravitational force the body is experiencing. For example, if the angle between these vectors is close to 0° or 180°, the resulting torque will be small. However if this angle is close to 90° or 270°, the torque will be at a maximum. Therefore after each periapsis pass the body is kicked to a new orientation meaning that when the body approaches periapsis a Nereid Year later, a different overall torque will be felt. This, of course, means different rotational dynamics occur each year. As the body’s orientation is changed by the kick, the positions of the two zero and two maximum torque locations in Nereid’s orbit are changed resulting in a different torque at the next periapsis. Eventually the zero and maximum locations are all translated about the orbit of Nereid with the pattern eventually repeating.

We can see this in figures 15 and 16. Kick 2 is a minimum torque resulting from when the zero location lies directly at periapsis. This means that when the torque is strongest in terms of distance from Neptune, it is effectively diminished by the orientation of Nereid’s spin axis. This means the oscillations in the obliquity and precession angles will be small. When there are no zero torque locations near periapsis, the torque is maximized as seen in Kick 6, which then sets up the largest
amplitude oscillations of the obliquity and precession angles. Figure 16 shows this in great detail, and that the pattern eventually repeats as Kicks 2 and 11 look nearly identical.

Fig. 15: Magnitude of the overall torque experienced by the body over a full active region in obliquity for the prolate Schaefer Case. Here we see that the magnitude of the torque rises and falls over time with a very sharp peak occurring at periapsis. The torques are labeled by number to be referenced later.
Fig. 16: Distance from Neptune (thin trace) and magnitude of the overall torque experienced by the body (bold trace) for select kicks labeled in Figure 15. If the zero torque locations of the body’s orbit are located near periapsis, as in kicks 2 and 11, the overall torque is minimized. The maximum torque (and “kick”) occurs when no zero torque locations lie near periapsis, as in kick 6. This figure also demonstrates how after each orbital pass, the zero torque locations are different for the next year. Progressing from kick-to-kick, year-to-year, we see the zero torque locations move towards the periapsis point, then pass through, with the cycle eventually repeating (note the similarity between kicks 2 and 11).

3.2.1 A Note on the Sun and Triton

Recently, it was shown that Triton has a long term effect on the orbit of Nereid (Vashkov’vak and Teslenko, 2010). The results presented in this chapter are for the rotational motion of Nereid on much shorter timescales; however, the effect of Triton should be examined. Initially, we included Triton in our simulations of the Schaefer case for runs that lasted as long as 100 y. When these were compared to identical runs without Triton, there was no significant difference. This can also be verified by examining the order of magnitude for the maximum torques on Nereid from Neptune and Triton. This would occur when Neptune, Triton, and Nereid line up in their closest configuration when Nereid is at periapsis. The magnitude of the torque caused by a body of mass $M$ varies as $\tau \propto Mr^{-3}$, so the maximum ratio of
the torque on Nereid from Triton to that from Neptune is

$$\frac{\tau_{\text{Triton}}}{\tau_{\text{Neptune}}} = \left( \frac{M_{\text{Triton}}}{M_{\text{Neptune}}} \right) \left( \frac{(1 - e)a_{\text{Nereid}}}{(1 - e)a_{\text{Nereid}} - a_{\text{Triton}}} \right)^3,$$

where $e$ is the eccentricity of Nereid’s orbit, and Triton’s orbit is treated as nearly circular. For values of the masses and semi-major axes in Murray and Dermott (1999), this ratio is $\sim 5 \times 10^{-4}$. In addition to this, even if Triton and Nereid align at periapsis, Triton’s orbit is retrograde, so the duration of its torque would be much less than that of Neptune, and thus, the resulting change in Nereid’s angular momentum would be much smaller. Therefore, on the timescales examined here, we conclude that Triton has a negligible effect on Nereid’s rotation.

In addition to the possible torque on Nereid from Triton, the torque from the Sun may also have an effect on the rotation of Nereid, particularly at apoapsis. Unlike the torque from Neptune, that from the Sun would be relatively constant over short times. Thus, when Nereid is at periapsis, the torques from Neptune and the Sun would be in the ratio (physical values provided by Murray & Dermott, 1999):

$$\frac{\tau_{\text{Neptune}}}{\tau_{\odot}} = \left( \frac{M_{\text{Neptune}}}{M_{\odot}} \right) \left( \frac{a_{\text{Neptune}}}{(1 - e)a_{\text{Nereid}}} \right)^3 \sim 1.8 \times 10^6.$$

When Nereid is at apoapsis, this ratio reduces to:

$$\frac{\tau_{\text{Neptune}}}{\tau_{\odot}} = \left( \frac{M_{\text{Neptune}}}{M_{\odot}} \right) \left( \frac{a_{\text{Neptune}}}{(1 + e)a_{\text{Nereid}}} \right)^3 \sim 5.2 \times 10^3.$$

Thus, the the torque from Neptune is more than six orders of magnitude stronger than the solar torque when Nereid is at periapsis, and more than three orders of magnitude stronger at apoapsis. Thus, it seems reasonable to neglect the torque from the Sun for the timescales of the results presented here; although for longer timescales the role of the Sun may have a cumulative effect on the rotation of Nereid (Alexander et al., 2011).
4 A More Realistic Model

Having a better understanding of the motion Nereid is likely experiencing due to its time-dependent torque, we intend to model how these rotation dynamics, combined with orbital and physical characteristics, affect how Nereid is viewed from Earth. Although there are many surface characteristics that could possibly affect Nereid’s albedo, with this model we will attempt to explain the variations in Nereid’s brightness using only its geometry and orientation.

First we will alter our simulation from a two-body Neptune centered one, to a heliocentric simulation including all the planets (dwarf planet Pluto too), Triton, and Nereid. In Chapter 4.1 we will explain how this is performed and will verify that our initial conditions are valid. In Chapter 4.2 we describe how we calculate Nereid’s brightness as viewed on Earth. Chapter 4.3 explains how we will set up and conduct our simulations. Chapter 4.4 describes our process of exploring various possible shapes and rotational characteristics Nereid may have, and in Chapter 4.5 we present our best estimate for the case of Nereid.

4.1 Heliocentric

The NASA Jet Propulsion Laboratory has developed an on-line system that contains all of NASA’s physical data for any body in the solar system (planets, comets, asteroids, etc.) and an ephemeris computation service that provides orbital data for all of these bodies (Yeomans & Chamberlin, 2012). The system has many features and possibilities, but relevant to us, it is capable of producing position and velocity vectors at any time, for any body, in the solar system relative to a given point, for example, the Sun. In our original version of the code (and what was subsequently used for the entirety of the results in Chapter 3) we utilized orbital elements provided by Murray & Dermott (1999) to calculate the Cartesian coordinates of each body, but this method is less efficient and not as versatile as the initial conditions produced by the HORIZONS System. HORIZONS provides us very precise data for more bodies and already in Cartesian format, skipping the need to calculate those values. The system also allows the user to select a date for when the ephemeris should be generated. Having the system generate position and velocity vectors for all bodies on January 1st, 1980 allows our simulation to produce data during the timespan when Nereid was actually being observed by Schaefer and others, and into the future.

To build our Heliocentric model we only needed to change our initial conditions. We replaced our Neptune centered conditions with a Sun centered model. Since the HORIZONS system allows us to generate an ephemeris relative to the Sun, after adding each body and their physical data, we used the initial Cartesian position and velocity vectors provided by JPL as these were already inherently relative vectors. We use JPL’s ecliptic and mean equinox coordinate system so that in this heliocentric simulation the XY plane corresponds to the plane of the Earth’s orbit with the X–axis pointing in the direction of the Vernal Equinox, as compared to our 2-body simulation from before where the XY plane was defined as the plane made by Neptune’s equator. As before, this eleven body system is integrated forward in time from the initial conditions using the Hermite Individual Timestep Scheme discussed in Chapter 2.1. Our initial system configuration is the Julian Day 2444239.5 (January 1st, 1980 at
It is important that before we simulate the combined orbital and rotational motion for Nereid that we establish that we are obtaining accurate orbits. Veshkov’yak, and Teslenko (2010) present a study of the orbital evolution of Nereid under the influence of Neptune, the Sun, and Triton. They calculate Nereid’s orbit using several techniques over a few timescales.

In the following plots, Nereid’s semi-major axis and eccentricity are determined for a Neptune centered orbit, and the inclination and longitude of ascending node are relative to the ecliptic. Figure 17 shows how our results for the evolution of Nereid’s semi-major axis compare to those presented in Veshkov’yak and Teslenko (2010) over a period of ten years. As we solve the orbital parameters of Nereid differently, the striking similarity between these two plots is extremely reassuring. Figure 18 compares our results of the evolution of Nereid’s eccentricity, inclination, and longitude of ascending node over a period of twenty thousand years to those shown in Veshkov’yak and Teslenko (2010). The results agree quite favorably. Lastly, Figure 19 is a comparison of the same results, but over a timespan of five hundred thousand years. The results agree. Even though our goal is to simulate Nereid’s motion for approximately 200 y, we believe that the excellent agreement with previous calculations of Nereid’s orbit over several different timescales gives us confidence that we are simulating Nereid’s orbit adequately.
Fig. 17: Modulations in Nereid’s semi-major axis over ten years. Vashkov’yak’s results (using multiple methods) are on top, ours on bottom. It should be noted that our initial conditions are likely to be different than presented in Vashkov’yak and Teslenko (2010), as we begin our simulations on January 1st, 1980. This explains the slight discrepancy between our results over a span of ten years.
Fig. 18: Modulations in Nereid’s eccentricity, inclination, and line of ascending node (Ω) over twenty thousand years. Results in Vashkov’yak and Teslenko (2010) are presented on top, ours on bottom.
Fig. 19: Modulations in Nereid’s eccentricity, inclination, and line of ascending node (Ω) over five hundred thousand years. Results in Vashkov’yak and Teslenko (2010) are presented on top, ours on bottom.

Being confident our simulation was correctly calculating the body orbits, we decided to check the rotation calculations. Still using the Schaefer case, we saw that the variations in obliquity and the precession angle remained, thus defending our results from the 2-body simulations in Chapter 3.

4.2 Projected Area

Having a Heliocentric model allows us to calculate the relative positions of any body in the simulation compared with any other. These position vectors will enable us to calculate how much of any body is illuminated by the Sun, and how much of that illumination is viewable at some other reference point. Therefore, if we calculate the projected area of Nereid towards the Sun, we will know the portion of Nereid that is illuminated. But how does the Earth view the reflected rays from the Sun off of
Nereid?

Instead of investigating how Nereid’s motion is dependent upon the distribution of its mass about its axes, its moment of inertia ratios, we move to an investigation of how Nereid’s shape affects observations of the moon’s brightness. Thus we now shall study how the ratio between the moon’s semi-axes a, b, and c, along with the body’s initial obliquity and spin rate affect Nereid’s visual magnitude. Knowing these parameters, Connelly & Ostro (1984) solve how to calculate the illuminated portion of a body (Nereid) viewed from the Earth. The following describes our derivations based upon their results. First, we choose a coordinate system where the origin is located at the location of Nereid. To calculate how much of the sunlit portion of Nereid is viewable from Earth we need to know the relative vectors between Nereid and the Sun, and Nereid and the Earth;  \( \hat{s} \) and  \( \hat{e} \) respectively (both  \( \hat{e} \) and  \( \hat{s} \) are in inertial coordinates):

\[
\hat{e} = \begin{pmatrix}
e_1 \\
e_2 \\
e_3 
\end{pmatrix}
\quad \text{and} \quad 
\hat{s} = \begin{pmatrix}
s_1 \\
s_2 \\
s_3 
\end{pmatrix},
\]

(22)

These vectors are shown in Figure 20 along with the semi-axes (a, b, and c) of the body along the body principal axes (xyz).

Fig. 20: Orientation of the body’s semi-axes relative to the body axes (xyz) inside the orbital frame (XYZ). The directional vector \( \hat{s} \) points from the body to the Sun, and the directional vector \( \hat{e} \) points from the body towards the Earth.

Connelly & Ostro (1984) prove that the projected area \( A_p \) toward the Earth of
the illuminated portion of an ellipsoid is given by:

\[ A_p = \pi abc \left\{ \left( \hat{e}^t Q \hat{e} \right)^{1/2} \left( \hat{s}^t Q \hat{s} \right)^{1/2} + \hat{e}^t Q \hat{s} \right\} \right\} \right\} \right\}. \quad (23) \]

Here \( \hat{e}^t \) and \( \hat{s}^t \) represent the vector transpose and \( Q \) is a diagonal matrix defined as:

\[ Q = \begin{bmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & \frac{1}{c^2} \end{bmatrix}, \quad (24) \]

where \( a, b, \) and \( c \) are the body semi-axes in the \( x, y, \) and \( z \) directions. The following three equations are the solutions to the three transformations:

\[ \hat{e}^t Q \hat{e} = \left( e_1^2 + \frac{e_2^2}{b^2} + \frac{e_3^2}{c^2} \right), \quad (25) \]

\[ \hat{s}^t Q \hat{s} = \left( s_1^2 + \frac{s_2^2}{b^2} + \frac{s_3^2}{c^2} \right), \quad (26) \]

\[ \hat{e}^t Q \hat{s} = \left( \frac{e_1 s_1}{a^2} + \frac{e_2 s_2}{b^2} + \frac{e_3 s_3}{c^2} \right). \quad (27) \]

This leaves us with one large result for \( A_p \), however we can normalize this result with the maximum viewable projection of Nereid, that is the largest cross-section of the body. For an oblate-like ellipsoid \( (c < b \leq a) \), the maximum viewable area is simply \( A_{obl}^{\text{max}} = \pi ab \). For a prolate-like body \( (c > b \geq a) \), \( A_{pro}^{\text{max}} = \pi bc \). Thus, to simplify the result we can divide \( A_p \) by \( A_{\text{max}} \). For an oblate-like body, the fractional projected area \( (A_f)_{obl} \) follows as

\[ (A_f)_{obl} = \frac{1}{2} \left( \frac{\varepsilon_1^2}{(a/c)^2} + \frac{\varepsilon_2^2}{(b/c)^2} + \frac{\varepsilon_3^2}{(c/a)^2} \right)^{1/2} \left( \frac{s_1^2}{(a/c)^2} + \frac{s_2^2}{(b/c)^2} + \frac{s_3^2}{(c/a)^2} \right)^{1/2} \left( \frac{\varepsilon_1 s_1}{(a/c)^2} + \frac{\varepsilon_2 s_2}{(b/c)^2} + \frac{\varepsilon_3 s_3}{(c/a)^2} \right) \right\} \right\}. \quad (28) \]

Since \( A_{pro}^{\text{max}} = \left( \frac{c}{a} \right) A_{obl}^{\text{max}} \), the fractional projected area for a prolate-like body is simply

\[ (A_f)_{pro} = \left( \frac{a}{c} \right) (A_f)_{obl}. \quad (29) \]

Therefore we do not need to know the specific values of the body’s semi-major axes \( a, b, \) and \( c \), but rather only the ratios between them \( (\varepsilon_1, \frac{b}{a}) \).

The brightness of a body is a measurement of the power with which an object can emit or reflect light (its luminosity, \( L \)) over its surface to an observer some distance away. Astronomers frequently work in magnitudes which is the comparison of the brightness of an object to some known object.

\[ m_{\text{subject}} - m_{\text{reference}} = -2.5 \log_{10} \left( \frac{L_{\text{sub}}}{L_{\text{ref}}} \right). \quad (30) \]
Since we are ultimately looking to replicate the Schaefer Plot, which is how the brightness of Nereid changes over time from some average, we shall make our comparison of Nereid’s brightness to Nereid’s average brightness. But if we assume that Nereid reflects light uniformly across its surface, the change in the observed brightness simplifies from a comparison of the luminosity, to how the illuminated portion of the body is viewed on Earth, a combination of the projected area towards the Sun and the Earth ($A_f$). This approach means we don’t need to know Nereid’s albedo as the luminosity is directly proportional to the viewable area. Therefore using $A_{frac}$ we can calculate the apparent magnitude via the following equation:

$$\Delta m = -2.5 \times \log_{10}\left(\frac{A_f}{\langle A_f \rangle}\right).$$

(31)

Our method of calculating Nereid’s observed change in magnitude from some normalized average has no dependency on the distances involved (there is no $r^{-2}$ contribution) and is thus comparable to the Schaefer Plot, where any distance effects were removed (Schaefer et al., 2008).

### 4.3 The Simulation

If Nereid is represented as an ellipsoid, then there are three motions of Nereid that can have an effect on its projected area in the direction of the Earth. One is caused by the orbital motion of Neptune. Even if there are no torques acting on Nereid, and its rotational axis is fixed inertially, its sunlit area will present a changing projected area toward the Earth as it follows Neptune in its orbit about the Sun. This can cause a long-term brightness modulation on a timespan equal to half of Neptune’s orbital period ($\sim 165$ y). There is a similar, yet much smaller, effect caused by the orbital motion of the Earth and Nereid itself. The second motion that can change Nereid’s projected area is caused by the precession of its spin axis due to torques produced by other objects (primarily Neptune). This is the case introduced by Schaefer et al. (2008) and also considered in Chapter 3. The precession period depends on Nereid’s orientation, spin rate and its shape. Drobovolskis (1995) derives an approximate relation for the precession period (which agrees favorably with our calculations):

$$P_{prec} = -\frac{4}{3} \frac{P_{orb}^2}{P_{spin}} \left[\frac{1 + (b/a)^2}{1 + (b/a)^2 - 2(c/a)^2}\right] \frac{(1 - e^2)^{3/2}}{\cos \beta}.$$  

(32)

Here $P_{orb}$ and $P_{spin}$ are Nereid’s orbital and spin periods, respectively, and $e$ is its orbital eccentricity. Thus, depending on Nereid’s spin state and its geometry, the precession period can be as short as 10’s of years, or longer than Neptune’s orbital period. Equation 32 shows the importance of investigating the resulting effects of these three variables. The third type of motion that can affect Nereid’s projected area toward the Earth is caused by its spin. Depending on its geometry and orientation, this can produce brightness modulations with periods as short as a few days, if in fact Nereid is spinning that rapidly. Thus, we expect that these three motions, all acting on somewhat different timescales, can combine to produce the brightness modulations as reported in the Schaefer Plot.
Therefore, because the orbital period of Neptune is $\sim 165$ $y$, for our heliocentric investigation we shall present our results for a timespan of 200 years in order to clearly show the effects on Nereid’s viewable projected area from each of the three motions. As previously discussed, our physical data, which matches Murray & Dermott (1999), and initial position and velocity components for all bodies are produced by the HORIZONS system and then advanced forward in time using the HITS. From here we can calculate Nereid’s obliquity ($\beta$) and precession angle ($\delta$), along with Nereid’s viewable projected area and magnitude. Additionally we shall calculate another useful quantity known as the aspect angle, $\gamma$. The aspect angle is simply the angle between Nereid’s rotation axis and the line-of-sight to Earth, i.e. the angle made between the Nereid’s z-axis and $\hat{e}$, essentially a measure of how Nereid would be “pointing” towards observers on Earth.

While in Chapter 3 we investigated the effect of varying each variable, here we present a much more comprehensive study, altering multiple variables simultaneously in an attempt to reproduce brightness variations that closely match the “Schaefer Plot”.

4.4 Ellipsoids

We will attempt to see if it is possible to explain the variations measured by Schaefer et al. (2008) with nothing but geometry, ignoring all possible surface features of the body. We examine how only the body’s shape, rotational dynamics, and relative alignment affect changes in the measurable brightness on Earth. To simplify our case further we shall restrict our study to only true ellipsoids (identical in the positive and negative directions of the body axes).

4.4.1 The Schaefer Case

In Schaefer et al. (2008), a model was proposed for Nereid that consists of a spin period of 72 $h$, an initial obliquity of $30^\circ$, and moment of inertia given by the ratios $A/C = 2.4$ and $A/B = 1$. This corresponds to a prolate spheroid with semi-axis ratios $c/a = \sqrt{2(A/C) - 1} = 1.95$ and $b/a = 1$. This configuration yields a precession period $\sim 20y$ and was proposed in an attempt to explain the time between the active regions of photometric variability seen in the “Schaefer Plot”. The rotational dynamics of this model were investigated in detail in Chapter 3. Here we will again study the “Schaefer Case” to study whether that model can reproduce the variability seen in observations.

Figure 21 shows the results of a 200 $y$ simulation for the prolate Schaefer case with an initial obliquity angle of $15^\circ$. The top plot is the obliquity angle, $\beta$, and here we see the same type of behavior that was discussed in Chapter 3. Every periapsis, Nereid receives an impulsive torque, the “kick”. Combined with the precession angle, $\delta$, shown in the next panel, the motion can be described as a series of coning motions with modulated amplitude. The lower three panels of the figure relate to how bright Nereid will appear from the Earth. The aspect angle, $\gamma$, is the angle between the body z-axis (in this case the long axis, or c-axis) and the direction to the Earth. Since the obliquity varies by only $\sim 7^\circ$, the aspect angle, also affected by the orientation of the orbital plane, remains within $\sim 15^\circ$ of $90^\circ$. This means that the majority of the time Nereid is viewed very nearly perpendicular to the long axis of the ellipsoid. This is
verified in the next panel which shows a plot of the fractional projected sunlit area in the direction of the Earth, $A_f$. This is seen to vary between a maximum of 1 and a minimum of $\sim 0.94$ which should give fairly small variations in Nereid’s brightness. The lowest panel in Figure 21 shows that this is the case. Here, the magnitude difference from the mean, $\Delta m$, varies between its faintest at $\sim 0.05$ to its brightest at $\sim -0.02$. Therefore when $\beta$ is small, the projected area as viewed on Earth is mainly along the body $x$ or $y$-axis, or a combination of both. Never do we see the body along the $z$-axis, meaning variations are dependent upon differences between the body’s $x$ and $y$ axes, however in the Schaefer Case, these axes are identical.
Fig. 21: Obliquity ($\beta$), precession angle ($\delta$), aspect angle ($\gamma$), normalized sunlit projected area towards Earth ($A_f$), and the difference from average magnitude ($\Delta m$) for the Schaefer Case with an initial obliquity of $15^\circ$. The motion shown in Figure 5 and discussed in Chapter 3 remains in the heliocentric model. The variations in $\Delta m$ are quite small and no large amplitude intra-night variations can be seen.
Fig. 22: Obliquity ($\beta$), precession angle ($\delta$), aspect angle ($\gamma$), normalized sunlit projected area towards Earth ($A_f$), and the difference from average magnitude ($\Delta m$) for the Schaefer Case with an initial obliquity of 45°. The variations in $\Delta m$ are larger than the 15° case seen in Figure 21, yet still no sizable intra-night variations occur.
Fig. 23: Obliquity ($\beta$), precession angle ($\delta$), aspect angle ($\gamma$), normalized sunlit projected area towards Earth ($A_f$), and the difference from average magnitude ($\Delta m$) for the Schaefer Case with an initial obliquity of $75^\circ$. The variations in $\Delta m$ are larger than both the $15^\circ$ and $45^\circ$ cases, however the pattern is extremely sinusoidal, with no “active” or “inactive” regions. No large amplitude intra-night variations appear.

Figures 22 and 23 show the results of simulations for the prolate Schaefer case for initial obliquities of $45^\circ$ and $75^\circ$, respectively on the same vertical scale as in Figure 21. Again, we see the same motion as in Chapter 3. The bottom panel of each figure shows the magnitude difference from the mean, and here the $75^\circ$ case shows the greatest brightness variation from $\sim 0.45$ to $\sim -0.25$. This is expected since for this high of an obliquity, we see Nereid from the Earth more nearly along its long axis as it precesses and follows Neptune in its orbit. A higher obliquity allows us to see all three possible orientations of the body. The projected area towards Earth can transition between a line-of-sight along the body x, y, and z axes. Thus, variations are dependent upon differences between all three of the body’s axes.

While all three of these simulations show that Nereid will appear with a varying brightness as seen from the earth, a flaw becomes apparent in Figures 21, 22, and
23. In none of these cases does Nereid exhibit large brightness variations on the time scale of a few days as evident in the Schaefer Plot, as the x and y axes of the body are identical. The only short term variations seen are due to the coning motion; however, they are much too small to correspond to the short period variations seen in the observations. The variations in brightness that are seen are caused by the change in orientation due to the precession and Neptune’s orbit, and occur on much longer time scales. Neither does the Schaefer Case produce any “active” and “inactive” periods in the calculated difference from average magnitude. The “Schaefer Plot” clearly shows a region of highly active modulations followed by a period of small amplitude oscillations, an effect that is not present in our investigations.

There remains one major flaw to this proposed model. A significant basis of Schaefer’s argument for a prolate model spinning about its long axis was to obtain a precession rate that fit his data. Unfortunately, this type of rotation is inherently unstable as the long axis of a prolate body is not its major moment of inertia axis. A body in such a rotation will eventually dissipate energy (typically due to flexing of the body’s shape), spin out of this rotation, and begin rotating about its axis of maximum inertia (Goldstein, 2002). The timescale for this to occur is questionable. This, combined with our lack of knowledge about how Nereid arrived in its orbit means our previous results cannot be explicitly dismissed. However Schaefer et al. (2008) citing the work of Burns and Safronov (1973) and Harris (1994) estimate that if Nereid is a prolate body that was spinning about its long axis, the timescale for it to begin rotating about its maximum inertia axis is on the order of a half million years. Again, as this value is largely a “best guess” our previous prolate results are still of interest, however it is much more likely Nereid would not be rotating about a minimum axis of inertia. Therefore to build a more complete, realistic model we move to studying bodies rotating about their maximum axis of inertia.

The problem for the prolate and oblate Schaefer cases is that for both, the rotation axis is a symmetry axis so that there is no change in the projected area as Nereid rotates. This, of course, is assuming a uniformly reflecting surface. Therefore, if Nereid’s brightness variations are to be explained using only its shape and rotation state, we are led to look at scenarios where Nereid is modeled as a tri-axial ellipsoid rotating about its maximum axis of inertia (to avoid any problems of instability).

4.4.2 The Pancake-Cigar Model

We now examine the case of a very squished oblate spheroid by setting the ratio between the long axis and one of the short axes at \( \frac{c}{a} = 0.2 \), and restricting the ratio between the two short axes at \( \frac{b}{a} \leq 1 \). If we decrease the value of \( \frac{b}{a} \) while keeping \( \frac{c}{a} \) constant, we will see the affect of shortening one of the short axes such that the body transforms from a very pancake-like shape (\( \frac{b}{a} = 1 \)) to a very cigar-like shape (\( \frac{b}{a} = 0.25 \)). We should note that Nereid’s mass places the moon in a size category that deems these limits, \( \frac{c}{a} = 0.2 \) and \( \frac{b}{a} = 0.25 \), rather unlikely for such a body. These limits however do exaggerate the situation and simplify the understanding of how the body’s shape will affect any photometric variations in the body’s brightness.

Again, in addition to shape, obliquity and spin period will affect the body’s dynamics and the brightness that would be observed from Earth. We therefore have studied the effects of how all three variables simultaneously alter the body’s change
in magnitude. In building thirty-six cases we have examined the resulting change in average magnitude for a body with an initial obliquity of 15, 45, and 75 degrees, a spin period of 36, 72, and 144 hours, and a $\frac{b}{a} = 1$, 0.75, 0.5, and 0.25.

Figure 24 displays the calculated difference from the time-average magnitude in 9 panels. Each panel (labeled A, B, C, etc.) contains plots of $\Delta m$ for each shape for a specific obliquity and spin period. Going across the panels from left to right, the spin period is increased, while moving vertically down the initial obliquity is increased. The correspondence of each panel to its initial obliquity and spin period is shown in the following table:

<table>
<thead>
<tr>
<th>$\beta$/Spin</th>
<th>36h</th>
<th>72h</th>
<th>144h</th>
</tr>
</thead>
<tbody>
<tr>
<td>15°</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>45°</td>
<td>D</td>
<td>E</td>
<td>F</td>
</tr>
<tr>
<td>75°</td>
<td>G</td>
<td>H</td>
<td>I</td>
</tr>
</tbody>
</table>

Within each of the 9 panels are shown the results from changing the ratio between the short axes. Moving vertically down inside any panel, $\frac{b}{a}$ is adjusted from 1.0, to 0.75, 0.5, and finally 0.25. The horizontal scales are the same for all data sets at 200 years. The vertical scales are also identical for all results, set from +2.0 (less bright) to -1.0 (more bright).
Fig. 24: Difference from average magnitude of Nereid as various tri-axial ellipsoids for several orientations and spin rates. $\frac{b}{a}$ is labeled in each plot.
Figure 24 certainly contains an enormous amount of data, yet it also yields a wealth of information and directs us to several conclusions. Firstly, the affect of altering the body from a pancake to a cigar is identical in all panels. When $\frac{b}{a} = 1.0$, there are no large amplitude, short-term variations as the body is axially symmetric and there is no change in the projected area as the body spins about its axis, the same flaw outlined in Chapter 4.4.1. The only variations we do see are a consequence of Nereid’s precession and its changing orbit as Neptune rotates about the Sun. However, as the body is squished to a more cigar-like shape, this symmetry is destroyed and we begin to see intra-night variations whose period is dependent on the spin rate. As the body is moved further from this symmetry, the projected area as viewed along the body x-axis is quite different than when viewed along the body y-axis, and these intra-night variations are maximized. Over time, this in turn produces active regions with large amplitude modulations along with inactive regions in the body’s calculated difference from average magnitude, an important result. In Chapter 3 we saw that the Schaefer Case produces active and inactive regions in the body’s obliquity, but saw in Chapter 4.4.1 that these modulations do not necessarily produce regions of high and low activity in the body’s brightness. Therefore, to match the Schaefer Plot, $\frac{b}{a}$ needs to be small enough to give us large-amplitude, short-period variations, yet still remain within the limits of known semi-axial ratios for other similarly sized moons.

As the spin period of the body is increased the timespan between overall peaks, valleys, dips, etc. in the magnitude of the body from the average, decreases. This is an important variable to consider when trying to match our results to the Schaefer Plot whose photometric variations appear to have a periodicity of $\sim 20$ years. The other direct effect is on the timespan of the short period, intra-night variations. Although not clearly visible in Figure 24, the speed in which the body rotates determines how quickly the differences due to the projected area viewed along each of the body’s axes become largely apparent. The spin of the body needs to be fast enough to match the timespan for the intra-night variations observed by Schaefer et al. (2008), yet slow enough such that the overall periodicity of the active and inactive regions matches that of the Schaefer Plot.

Lastly we examine how the initial obliquity of the body affects its change in magnitude. As discussed in Chapter 4.4.1, when $\beta$ is large, changes from the average magnitude are controlled by the projected area along all three of the body’s axes. Therefore, as the obliquity is increased, the amplitude of these variations from the average are maximized. Again, the precession rate is dependent on the obliquity and we see that as the obliquity increases the period between active and inactive regions is lengthened. Also, in panels G, H, and I we see that even when the body is cigar-like, the amplitude of the oscillations during the “inactive” regions is quite small and constant, while in panels A, B, and C the difference between “active” and “inactive” periods is difficult to pinpoint. In panels D, E, and F, when $\beta$ is set at a more compromising 45°, we still see clear active and inactive variations in brightness, while the inactive periods still display small intra-night variations.

Figure 24 displays our results in a versatile manner allowing us to understand how the obliquity, spin period, and shape of the body determines the projected area viewed towards Earth and the resulting variations in the magnitude of the body’s
brightness. The conclusions we have drawn from studying how these three parameters simultaneously govern our replication of the Schaefer Plot direct us to explore the region set within the boundaries of panels E, F, H, and I.

4.5 Best Estimate

Although Nereid is small compared to Neptune, in terms of our solar system, it is by no means a tiny moon. Its mass is both large enough that it remains theoretically plausible for the moon to have pulled itself into a decently spherical shape, or small enough that it need not be so spherical. The mass required for a body’s self-gravity to even out any existing asymmetry in the body’s shape is not yet clearly defined and Nereid’s mass places it directly in this unknown, debated region. However we can say that is very unlikely that an object of Nereid’s size and mass would be significantly out-of-round on an order such as presented in the investigations of Chapter 4.4.2. We can also say that Nereid’s mass by no means requires it to be quite spherical and that we may hypothesize its shape is similar to better known irregular moons.

Dobrovolskis (1995) discusses the known semi-axial ratios of known moons in a direct discussion on the shape of Nereid. In matching the Schaefer Plot, Figure 24 directs us to more extreme semi-axial ratios where \( c \neq b \neq a \), yet we find that Nereid is unlikely to have either \( a \leq b \leq c \) or \( b \leq a \leq c \). We have demonstrated that a body with \( \frac{a}{b} = 1 \) will have minimal photometric variations as compared to a body with semi-axes such that \( a \leq b \leq c \) or \( b \leq a \leq c \). If Nereid’s photometric variations are to be strictly attributed to the geometry of orbits and the body’s orientation, panels E, F, H, and I of Figure 24 direct us to a body with an obliquity such that \( 45^\circ \leq \beta \leq 75^\circ \) and a spin period somewhere in the range of \( 72 - 144 \) hours.

We have settled upon the following candidate as our best guess for the Nereid case. We are careful to note that this is not what we propose the case for Nereid actually is, rather the closest approximation we can make in attempting to match intra-night variations, periodicity of active and inactive regions, and the amplitude of variations in the observed magnitude from the average over some timespan. The consideration of these three factors along with the study and results presented in Chapter 4.4 have directed us to conclude that if Nereid’s brightness variations are to be attributed to strictly the geometry of the body, we require Nereid to be an ellipsoid shaped body spinning about is z-axis with an initial obliquity of \( 60^\circ \), a spin period of 144 hours, and the following semi-axial ratios: \( \frac{c}{a} = 0.5, \frac{b}{a} = 0.6 \). We choose these ratios as they lie on the bounds of what is known and will maximize the amplitude of our modulations (similarly sized Hyperion has a semi-axial ratio of \( \frac{c}{a} = 0.4 \)) (Harbison et al., 2011).

Figure 25 presents our results for this case in our 5-panel plot format used in Chapter 4.4.1 (however on a different vertical scale).
Fig. 25: Obliquity ($\beta$), precession angle ($\delta$), aspect angle ($\gamma$), normalized sunlit projected area towards Earth ($A_f$), and the difference from average magnitude ($\Delta m$) for our best candidate in matching the observed variations in the “Schaefer Plot”. Our candidate has an initial obliquity of 60°, a spin period of 144 hours, and ratios amongst the semi-axes such that $\frac{c}{a} = 0.5$ and $\frac{b}{a} = 0.6$. We have matched the period of inactive-active regions quite closely, along with the intra-night variations. The discrepancies in amplitude and shape of $\Delta m$ is examined in Figure 26.

In examining this plot we see the precession period of the body is $\sim 75$ years, the aspect angle oscillates with a period of $\sim 50$ years, and the impulsive modulations in the body’s obliquity due to the kick remain present. These three oscillations combine to vary the illuminated projected area of the body as seen on Earth and the variations in magnitude with a period $\sim 25$ years, the closest we conclude we can attain to matching the periodicity seen in the Schaefer Plot. But how well, overall, do the variations in magnitude of this body compare to the results presented in the Schaefer plot? There are a number of things to consider when reproducing and comparing our
results with a format as presented in the Schaefer Plot. Firstly, Schaefer et al. (2008) have presented their results as a difference from average. They have collected data over 362 nights for a total of 774 observations of Nereid’s magnitude (Schaefer et al., 2008). It is this pool of data, collected over inconsistent periods, that has been averaged to develop this plot. As our simulation does not suffer any observational difficulties, we can collect any number of measurements over a continuous timespan. Therefore, our average is significantly more comprehensive and constant than seen in the Schaefer Plot. Secondly, there are large gaps in the data presented by Schaefer et al. (2008). For six months of every year observations of Nereid’s brightness cannot be collected when the orbit of Earth is such that Nereid is in the daytime sky. Also, Schaefer et al. (2008) simply provide no data for the time periods of \( \sim 1989 \) to \( \sim 1993 \) and \( \sim 2000 \) to \( \sim 2003 \). Lastly, they believe their observations show one full period of variations, yet in examining Figure 1, it is impossible to know if the amplitude of the variations are decreasing or still increasing in 2006, the end of their data set. Their argument that Nereid has entirely moved through an entire active period between 2452500 and 2454500 JD are not convincing. Rather, it is possible that Nereid may only be entering an active region at this point. Thus, it is possible the period of variations do not necessarily have to be the \( \sim 17 \) years they present, but need to be over some window matching the 20-year results in the Schaefer Plot. Therefore the period of the variations from active to inactive may be longer than 20 years, which is our conclusion.

Figure 26 displays our results for the difference from average magnitude for our strongest candidate over a timespan of 20 years. In this plot no results are presented during times when Nereid was only viewable in Earth’s daytime sky. Averaging over this collection of data in this timespan, and plotting as a series of points, we present our results in a way most comparable to the Schaefer Plot.
There are two obvious discrepancies between Figure 26 and Figure 1. First, the overall amplitude of the variations is quite small as compared to those presented in Schaefer et al. (2008). The Schaefer Plot covers a data set with a range of approximately \(-1.0\) to \(+1.5\) magnitudes, whereas our strongest candidate only spans a range of approximately \(-0.3\) to \(\sim +0.6\) magnitudes. Secondly, the Schaefer Plot shows Nereid being both brighter and dimmer during active regions than during inactive regions. In Figure 26 we see that Nereid can be much dimmer during active regions, but is only, at most, as bright during inactive regions. In fact, Nereid is slightly more bright during inactive regions than active ones. Thus we conclude that (assuming the observations of Schaefer et al. (2008) are correct) Nereid’s photometric variability cannot be explained by a uniformly reflecting body of this geometry.

4.5.1 A Note on Nereid’s Semi-Axial Ratios

As discussed, Nereid’s semi-axial ratios are unlikely to be \(< 0.5\), however that doesn’t mean it is impossible! Dobrovolskis (1995) points out that some observations of Nereid indicate it may have an axial ratio such that \(\frac{b}{a} \leq 0.25\). Nereid is only slightly larger than the quite out-of-round Hyperion, a moon of Saturn with \(\frac{c}{a} = 0.4\) (Harbison, 2011). However, it is also quite similar in size to another moon of Saturn, Mimas, which is quite spherical with \(\frac{c}{a} = 0.92\) (Thomas, 2010). In spite of this, I present here, for the sake of argument, the results of a scenario that we believe to
be quite unlikely, but could theoretically be redeemed by future observations. Shown below are the results of a case for Nereid with an initial obliquity of $60^\circ$, spin period of 96 hours, $\frac{c}{a} = 0.2$, and $\frac{b}{a} = 0.25$. The precession period of Nereid in this case is $\sim 78$ years. The results of this case, shown in Figure 27, are plotted in the same manner as Figure 26. These results certainly compare much more favorably to the Schaefer Plot, yet still do not cover the full range of amplitude seen in Figure 1.

Fig. 27: Variations in the difference from average magnitude ($\Delta m$) for a hypothetical case ($\beta = 60^\circ$, spin period of 96 hours, $\frac{c}{a} = 0.2$, $\frac{b}{a} = 0.25$) over 20 years as could possibly be observed in the nighttime sky. Some of the same flaws remain as in Figure 26, however the large amplitude variations are a much closer match to the Schaefer Plot. The axes are identical to Figure 1.
5 Conclusion & Further Work

In this work we have made the following five assumptions: The Schaefer Plot is accurate, Nereid’s spin rate is on the order of a few days, Nereid is rotating about its maximum axis of inertia, the ratios between Nereid’s semi-axes fall within the range of known moons, and lastly that Nereid is a uniformly reflecting body. Even though, as discussed in Chapter 4.5, it is possible to interpret the consequences of the Schaefer Plot in various ways, we still assume the display of its information as accurate. We hold that there is little evidence to contradict any of these assumptions and that our results and arguments presented thus forth are well-founded and complete. That is, except for our last assumption.

The results of this investigation are limited in one sense only: in our attempt to match the Schaefer Plot, we assumed Nereid to be a uniformly reflecting, true ellipsoid. This, of course, is easily not the case, however the efforts of undertaking an investigation that includes this consideration would be significantly too great for one thesis. Building (and understanding) a simulation that would include Nereid’s physical reflective characteristics in its calculations would be vastly too complicated if conducted simultaneously with the results and outcomes of this work. For the many estimates we make on Nereid’s shape, we do have at least some known bounds, however in hypothesizing how Nereid’s surface may reflect light from different orientations, there are none. The possibilities are literally infinite from whether the moon’s surface is pocked with craters or smooth, to whether its surface is black and white.

It is thus the conclusion of this work that accepting all five of our assumptions as correct, we cannot reproduce the photometric variations as seen in the Schaefer Plot. The result of our theoretical work show that these assumptions cannot agree with current real-life observations and therefore one of our founding assumptions must be wrong. We propose the most likely inaccuracy of this study is the acceptance that Nereid is a true, uniformly reflecting ellipsoid. We retain that it remains possible for Nereid to have an extreme geometry, one that could better match observations, yet still hold this scenario to be unlikely.

However there is something beautiful about this study. With everything understood as a result of all the discussions in each of the previous sections, it becomes significantly less daunting to model Nereid as a non-uniformly reflecting body. Already knowing how the body’s shape, obliquity, and spin rate will affect the brightness of Nereid as viewed on Earth, adding the body’s reflectance is now the only variable not understood. Additionally, our derivations to solve the difference in the magnitude of Nereid’s brightness depend only on the body’s viewable projected area. When considering a body that does not reflect light uniformly, this projected area would simply change by some factor, some proportionality. The cleverness behind this derivation is that if Nereid’s axes weren’t symmetric (larger in the +x direction than in the -x direction), if the body had some sort of “lump”, the result would be the same as if the body was axis-symmetric, but whose surface had a “bright spot” (more reflective in the +x direction than in the -x direction).

Thus, it remains only to solve the relationship between the body’s surface features and its viewable projected area. To that oh-so-fortunate soul, I say good luck!
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