ABSTRACT

CAVITY QED WITH CENTER OF MASS TUNNELING

by Charles H. Baldwin

We model the evolution of a single atom in a cavity interacting with two lasers, one far off resonance which creates an optical potential lattice and one near resonance, which can interact with the atom. The atom may tunnel between sites in the lattice. We find that the photon counting statistic \( g^{(2)}(\tau) \) depends on the tunneling rate. Additionally we derive methods to work with a Bose-Einstein Condensate in the same situation. It is shown that oscillations in \( g^{(2)}(\tau) \) are directly proportional to the tunneling coefficient given in our models.
CAVITY QED WITH CENTER OF MASS TUNNELING

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1 Introduction

Cavity quantum electrodynamics (cQED) is the study of ultra-cold atoms interacting with laser light in a high quality mirror cavity. The incoming light interacts with the atom causing stimulated emission and stimulated absorption. This makes a cavity a good system to study many quantum effects. Einstein pioneered the work with a statistical treatment of light, which showed a modification of spontaneous emission inside an optical cavity. Interaction between quantum fields and an atom was derived by Edwin Jaynes and Fred Cummings in 1963 [17]. Further development of the laser and higher quality cavities expanded the field, however, most experimental work was focused on modifying emission rates [17]. In 1975 H.J. Carmichael and D.F. Wells predicted photon anti-bunching, a quantum effect, with fluoresced light from a two level atom [20]. The prediction was confirmed by H.J. Kimble et al. in 1976 [20]. Squeezed light, another quantum effect, was shown in 1985 at Bell Labs [20]. Squeezed light is like a coherent state in that it is a minimum uncertainty condition, however, the uncertainty is not equally distributed between field quadratures. More recently cQED has been used to work with entanglement and quantum information.

Quantum information is a fast growing field motivated by the idea of creating a quantum computer. Describing a quantum computer is beyond this paper, however, we will mention that there have been several useful algorithms developed, like Shor’s algorithm, which are much faster than classical algorithms [3]. Quantum computers use quantum bits (qubits) instead of classical bits. A quantum bit is a superposition of two states (|0⟩ and |1⟩). The field of quantum information deals with manipulation of qubits and their interaction. Quantum algorithms are based on entanglement between multiple bits, however, there must be strong correlation between qubits [20]. Dissipation destroys correlation and entanglement but cQED provides a system where dissipation can be modified [20].

In this paper, we model a single atom in an optical lattice, within a cavity, allowed to tunnel between lattice sites. The model provides a base for further work with multiple atoms, which will be able to interact. We make the atoms bosons in a Bose-Einstein condensate which will be discussed later. We establish methods to study the particle’s behavior by observing emission from the cavity.

1.1 Overview of the Model

There have been many variations of the cQED system studied. We will explain the basic model here. We place the atoms inside a high finesse mirror cavity. The atoms are assumed to only have two allowed energy levels, excited (|e⟩) and ground (|g⟩). The
cavity is then pumped with a single mode driving laser of strength $Y$ that interacts with the atoms with coupling rate $g$. The atoms will then oscillate between its two energy levels with the Rabi frequency equal to $g\sqrt{N}$ for $N$ atoms [17].

This alone, however, is not a full description of the system. We have failed to include the loss of photons from the cavity. There are two types of photon loss, fluorescence and transmission. Fluorescence is a result of the atom absorbing a photon from the driving laser and later emitting it out the side of the cavity in a random direction (spontaneous emission.) This is proportional to the Einstein $A$ coefficient, $\gamma = A$, multiplied by the probability an atom is in the excited state. Einstein's coefficient is the probability of spontaneous emission per unit time determined by the type of atom [16]. The second method, transmission, is a result of a photon traveling through the front mirror (cavity loss.) The rate at which this happens is proportional to a constant, $2\kappa$, times the average number of photons in the cavity [9]. Where $\kappa = cT_{\text{mirror}}/(2L)$ and $L$ is the size of the cavity, $c$ is the speed of light, and $T_{\text{mirror}}$ is the transmission probability of the mirror [9]. Both rates are therefore constantly changing as the system evolves.

So far, we have described the Jaynes-Cumming model with dissipative events. This is the basis for most quantum optics investigations. In addition to these interactions we add, in this paper, add an optical lattice and tunneling. The lattice is created by pumping a far off resonant laser into the cavity creating a finite number of sites that the atom can occupy. The atoms can jump between wells determined by a constant $T$. We can also treat the atoms as Bose-Einstein condensate, which requires accounting for additional interactions between atoms.
To solve for the evolution of the system we utilize quantum trajectory theory. A non-Hermitian Hamiltonian accounts for all of these interactions modeling the atoms evolution in the cavity. We first solve Schroedinger’s equation for the state amplitudes. Next, we implement quantum jumps which are random events modeling the fluorescence and transmission discussed earlier. The result is a list of times when photons are fluoresced (spontaneous emission) and transmitted (cavity loss.) Using these times we can graph different photon counting statistics which can be used to show photon bunching or antibunching however for now our purpose is to find the tunneling coefficient.

The model presented in this paper is a simplified version of this system. We will treat a single atom in a two site optical cavity where the atom tunnels between sites. The interaction with the driving laser outlined earlier still applies.

1.2 Outline of Paper

In section two, we will outline the quantum mechanics principles that will be used in our analysis. In section three, we will derive the weak field trajectory model which treats atoms in a cavity interacting with a driving field, however, does not take into account an optical lattice or tunneling. In section four, we will outline the methods to simplify
the system and also gain information from photon emission from the cavity. In section five, we will cover all the necessary background on Bose-Einstein condensates and derive a Hamiltonian to treat their evolution. In section six we will combine the weak field trajectory model and the Bose-Einstein condensate model to give a single Hamiltonian to model \( N \) two level boson atoms in an optical lattice interacting with a driving field. In section seven, we will present the numerical model of a single two level atom in a two site optical lattice and compare it to theoretical predictions. And finally, in section eight we conclude the analysis and give ideas for future studies.

2 Quantum Mechanics Background

In this section, we will discuss certain parts of quantum mechanics which will be used throughout the rest of the paper. We start with an explanation of the second quantization that we will later apply to photons and boson atoms. Secondly, we will explain how we treat the electric field as a quantum operator. Next, we will explain the use of the density operator and how it is important in our system. Finally, we will discuss optical lattices and their effect on the atoms.

2.1 Second Quantization

To move from classical mechanics to a quantum picture the first step is to make the Hamiltonian function into a quantum operator. Sometimes we refer to as the first quantization. The Hamiltonian, however, is still acting on a wave function \([14]\). It works well for the single particle systems but for more advanced systems, we need to look at the second quantization. To make the second quantization we treat the wave function as an operator. The rules for the second quantization vary based on whether we are looking at fermions or bosons. For this paper we only discuss the rules for bosons since our first application will be to photons and later we wish to work with boson atoms.

We replace the wave function with a quantum operator. Likewise, the complex conjugate is replaced with an adjoint operator. The rules for bosons are as follows

\[
\left[ \hat{\psi}(x,t), \hat{\psi}^\dagger(x,t) \right] = \delta(x - x'), \quad (2.1a)
\]

\[
\left[ \hat{\psi}(x,t), \hat{\psi}(x,t) \right] = \left[ \hat{\psi}^\dagger(x,t), \hat{\psi}^\dagger(x,t) \right] = 0 \quad (2.1b)
\]

The square brackets denote the commutator. These relations come from bosons having integer spins \([14]\). The operators can then be decomposed to consist of a classical wave
function part \( (u_i(x)) \) and a time dependent operator \( (\hat{a}(t)) \)

\[
\dot{\psi}(x, t) = \sum_i \hat{a}_i(t)u_i(x) \tag{2.2a}
\]

\[
\dot{\psi}^\dagger(x, t) = \sum_i \hat{a}^\dagger_i(t)u^*_i(x) \tag{2.2b}
\]

Where the \( u_i \) obey the standard operations for wave functions [14]. \( u_i \) also follows Maxwell’s equations (for photons) or Schrödinger’s equation (for atoms.) The \( \hat{a}_i \) and \( \hat{a}^\dagger_i \) operators follow the same behavior as the raising and lowering operators for a quantum simple harmonic oscillator except they apply to the raising and lowering of excitations in a bosonic field. Since they represent the operator part of the wave operator, they follow the boson relations

\[
[\hat{a}_i(t), \hat{a}^\dagger_j(t)] = \delta_{ij} \tag{2.3a}
\]

\[
[\hat{a}_i(t), \hat{a}_j(t)] = [\hat{a}^\dagger_i(t), \hat{a}^\dagger_j(t)] = 0 \tag{2.3b}
\]

The operators are time dependent since the wave function and thus wave operator are time dependent. In this paper we will use the energy basis, shown in the next section, where the eigenvalues of \( \hat{a} \) and \( \hat{a}^\dagger \) are constants. Therefore, we can use Heisenberg’s equation to easily get the time dependent function.

\[
i\hbar \dot{\hat{a}}_i(t) = [\hat{a}_i(t), \hat{H}_0] \rightarrow \hat{a}_i(t) = e^{-iE_i t/\hbar} \hat{a}_i(0) \tag{2.4a}
\]

\[
i\hbar \dot{\hat{a}}^\dagger_i(t) = [\hat{a}^\dagger_i(t), \hat{H}_0] \rightarrow \hat{a}^\dagger_i(t) = e^{iE_i t/\hbar} \hat{a}^\dagger_i(0) \tag{2.4b}
\]

Where \( \hat{H}_0 = \hbar \omega \hat{a}^\dagger \hat{a} \) is the Hamiltonian for a single mode field derived later and \( E_i \) is the energy. This will be useful later on when dealing with the quantum field.

### 2.2 Field Quantization

In order to model quantum effects inside the cavity we must treat the driving laser quantum mechanically. Maxwell’s equations give us a classical field model of electricity and magnetism as well as a wave solution for the electromagnetic field (EM field). Through many experiments it became obvious that an EM field must also exhibit properties of a particle, or what became known as the photon. Therefore, there must be a quantized version of the EM field to explain the experiments. The idea may seem daunting but luckily, we can use Maxwell’s equations to derive a quantized model of the electromagnetic field.

Our approach to this problem will be to first derive an equation for the vector potential
in the Coulomb gauge. Then, using Maxwell’s equations, we can write out the electric and magnetic fields. Using the equation for the total energy of an EM field, we will substitute our equations for the electric and magnetic field in terms of the vector potential. We will then be ready to make an analogy of the final energy equation to that of the Hamiltonian of a quantum simple harmonic oscillator. Ultimately, we will show that an electromagnetic field behaves like a series of simple harmonic oscillators on a quantum level.

We start with the relations that determine the fields from the vector potential in the Coulomb gauge \( \vec{\nabla} \cdot \vec{A} = 0 \) and \( \rho = \vec{J} = 0 \), as shown

\[
\begin{align*}
\vec{B} &= \vec{\nabla} \times \vec{A} \\
\vec{E} &= -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi
\end{align*}
\]

(2.5a)

(2.5b)

Combining these equations with Maxwell’s equations we can easily derive an equation for \( \vec{A} \) which is the typical wave equation

\[
\nabla^2 \vec{A} = \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2}
\]

(2.6)

As with any other wave equation, the solution will contain separable time dependent and position dependent functions. The functions may be complex. Due to the undefined volume, there may also be many possible modes of the wave, dependent on the system. The solution is below.

\[
\vec{A}(\vec{r}, t) = \sum_l \frac{1}{\sqrt{\epsilon_0 V_l}} q_l(t) \vec{u}_l(\vec{r})
\]

(2.7)

Here \( V_l \) is the volume, \( q_l(t) \) contains the time dependent function while \( \vec{u}_l(\vec{r}) \) contains the position dependent mode function [17]. The sum shows that there are many modes \( l \) which contain different time and position functions.

It is easy to relate the vector potential here to the wave function described in the previous section. We have not yet made any quantization here but eventually we will want to make our function into an operator thereby changing the time dependent part into an operator as well. The electric and magnetic fields are determined by equation 2.5

\[
\begin{align*}
\vec{E}(\vec{r}, t) &= -\sum_l \frac{1}{\sqrt{\epsilon_0 V_l}} \dot{q}_l(t) \vec{u}_l(\vec{r}) \\
\vec{H}(\vec{r}, t) &= \sum_l \frac{1}{\mu_0 \sqrt{\epsilon_0 V_l}} q_l(t) \vec{\nabla} \cdot \vec{u}_l(\vec{r})
\end{align*}
\]

(2.8a)

(2.8b)
In addition to quantizing an EM field, we would also like to quantize the electric field. As discussed before the electric field interacts with the atom.

Using equation 2.3 for $\vec{A}(\vec{r}, t)$ the electric and magnetic field is solved in terms of $q_l(t)$ and $\dot{q}_l(t)$. The Hamiltonian of the EM field, $E$, in terms of $q_l(t)$ and $\dot{q}_l(t)$ is

$$H_{EM} = \frac{1}{2} \int d^3r \left[ \epsilon_0 |\vec{E}(\vec{r}, t)|^2 + \mu_0 |\vec{H}(\vec{r}, t)|^2 \right]$$  \hspace{1cm} (2.9)

$$= \frac{1}{2} \sum_l \left[ |\dot{q}_l(t)|^2 + \omega_l^2 |q_l(t)|^2 \right] V_l$$  \hspace{1cm} (2.10)

Where $\omega_l$ is the frequency of field mode $l$ and $V_l$ is the volume [17]. For a full derivation see [17]. Now we move to quantize the field by replacing the time dependent part of the vector potential by quantum operators $q_l(t) \rightarrow \hat{q}_l$. This also changes the vector potential and thus the electric and magnetic field to operators as well.

$$\hat{H}_{EM} = \frac{1}{2} \sum_l \left[ |\dot{\hat{q}}_l(t)|^2 + \omega_l^2 |\hat{q}_l(t)|^2 \right] V_l$$  \hspace{1cm} (2.11)

This equation does not really mean much to us because we have no concept of what $\hat{q}_l$ represents. In order to gain more insight into the physics of a quantized field we will go on to make an analogy to the quantum simple harmonic oscillator (SHO.)

The Hamiltonian has the same form as the Hamiltonian of the SHO with a term dependent on one coordinate and the second term dependent on its derivative. The difference here is that the coordinate is not position but the time dependent portion of the vector potential. We can still apply the same trick as in the quantum SHO by introducing the raising and lowering operators, however, for this case we also call them creation and annihilation operators for reasons that will become clear later. In this analogy $q_l(t)$ is written in terms of the $\hat{a}^\dagger$ and $\hat{a}$, the raising/creation and lowering/annihilation operators, just as $\hat{x}$ and $\hat{p}$ in the SHO.

$$\dot{\hat{q}}_l(t) = \sqrt{\frac{\hbar}{2\omega_l}} \left( \hat{a}_l(t) + \hat{a}^\dagger_l(t) \right)$$  \hspace{1cm} (2.12a)

$$\dot{\hat{q}}_l(t) = i \sqrt{\frac{\hbar \omega_l}{2}} \left( \hat{a}_l(t) - \hat{a}^\dagger_l(t) \right)$$  \hspace{1cm} (2.12b)

The constants are chosen so as to fit equation (2.11) [17]. The creation/annihilation operators obey the relations established in equation (2.3). The time dependencies are derived in the previous section where $\omega_l = E_l/\hbar$. Now we are ready to write a quantum version of the electric field. From our classical field, equation (2.8), we can substitute the
new equation for \( q_l \).

\[
\hat{E}(\vec{r}, t) = i \sum_l \mathcal{E}_l \vec{u}_l(\vec{r}) \left( \hat{a}_l(t) - \hat{a}^\dagger_l(t) \right)
\]  

(2.13)

The constants \( \mathcal{E}_l = \sqrt{\hbar \omega_l / 2 \epsilon_0 V_l} \) and \( \omega_l \) is the oscillation frequency again. For this paper we will assume that the field is constant for the whole volume, therefore we fold \( \vec{u}(\vec{r}) \) into the constant term \( \mathcal{E}_l \). The same substitution can be made for the magnetic field but in our model, we never consider magnetic interactions only electric dipoles so the quantum version of the magnetic field is irrelevant.

As with any quantum system, there are many states that the wave function can occupy. For this paper, we will only look at the Fock states or number states. The Fock states are not the most common occurrence of the field, however, when dealing with a weak field it is applicable [19]. The Fock states are simply the number of photons in the field which is found by the operator \( \hat{N} \) as shown

\[
\hat{N} = \hat{a}^\dagger \hat{a}
\]

(2.14)

This will be important later on when describing the energy of the field. The number of photons in a given system is then the eigenvalues of the number operator. The Fock states also form a complete basis. The creation and annihilation have the following eigenvalues in the Fock states.

\[
\hat{a} |n\rangle = \sqrt{n} |n - 1\rangle \]

(2.15a)

\[
\hat{a}^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle \]

(2.15b)

\[
\hat{N} |n\rangle = n |n\rangle
\]

(2.15c)

Now it is clear why we have called \( \hat{a} \) and \( \hat{a}^\dagger \) the creation and annihilation operators since they correspond to increasing and decreasing the photon number in the system.

### 2.3 Density Operator

The system we wish to investigate is not isolated from the environment. Instead we allow the photons to leave the cavity by transmission and fluorescence. In order to account for this interaction with the environment we must use the density operator, which is the most general description of a quantum state. We first write the Hamiltonian for the entire process, system, environment and interaction between the two

\[
\hat{H} = \hat{H}_S + \hat{H}_R + \hat{H}_I
\]

(2.16)
The subscripts $S$ denotes the system, $R$ a reservoir or environment, and $I$ the interaction between the two. When a system has no classical probability of being in a certain state we call it a pure state. For pure states, we can use Schroedinger’s equation to solve for the evolution

$$i\hbar \dot{\psi}(t) = \hat{H} \psi(t)$$ \hspace{1cm} (2.17)

Also if we want to find the expectation of a given operator we can use the relation

$$\langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle$$ \hspace{1cm} (2.18)

So far what we have done was sufficient for introductory quantum mechanics where the quantum state was pure [2]. That is not true in this work. Eventually, we will allow the photons to leak out of the cavity, which puts the system in a classical mixture of states. The Schroedinger equation and expectation value equation listed before no longer are valid when we are not modeling a pure state [12]. The density operator $\rho$ models mixed states and is defined as

$$\rho = \sum_i P_i |\psi_i\rangle \langle \psi_i|$$ \hspace{1cm} (2.19)

Where now we have allowed the system to be in any state $|\psi_i\rangle$ with probability $P_i$. In general the expectation value for a given operator is

$$\langle \hat{O} \rangle = \sum_j P_j \langle \psi_j | \hat{O} | \psi_j \rangle$$ \hspace{1cm} (2.20)

To solve in terms of the density operator we take the trace

$$Tr(\hat{O}\rho) = \sum_j \langle \psi_j | \hat{O} \rho | \psi_j \rangle$$

$$= \sum_j \langle \psi_j | \hat{O} \left( \sum_i P_i \langle \psi_i | \psi_i \rangle \right) | \psi_j \rangle$$

$$= \sum_j \sum_i \langle \psi_j | \hat{O} | \psi_i \rangle P_i \langle \psi_j | \psi_i \rangle$$

$$= \sum_i P_i \langle \psi_i | \hat{O} | \psi_i \rangle$$ \hspace{1cm} (2.21)
Being able to find the expectation values of an equation is useful but ultimately what we will need is the time evolution of the density operator [12]. Using Schrödinger’s equation

\[ \dot{\rho} = \sum_i P_i \left( |\dot{\psi}_i\rangle \langle \psi_i| + |\psi_i\rangle \langle \dot{\psi}_i| \right) \]

\[ = \sum_i P_i \left( -\frac{i}{\hbar} \hat{H} |\psi_i\rangle \langle \psi_i| + \frac{i}{\hbar} |\psi_i\rangle \langle \dot{\psi}_i| \right) \]

\[ = -\frac{i}{\hbar} \left[ \hat{H}, \rho \right] \quad (2.22) \]

This equation is commonly called the von Neumann, or Master, equation named after John von Neumann who created the density operator treatment [12].

The density operator may also be written out in matrix form and then called the density matrix. As an example, we will consider the density matrix for a single two level atom. With no interactions there are two possible states for the wave function \(|e\rangle\) and \(|g\rangle\). Therefore, the density matrix for this system is

\[ \rho = \begin{pmatrix} \langle e|\rho|e \rangle & \langle e|\rho|g \rangle \\ \langle g|\rho|e \rangle & \langle g|\rho|g \rangle \end{pmatrix} \quad (2.23) \]

The diagonal elements of the density matrix give the populations of each state while the off diagonal elements show the coherence between states [9].

Most systems will be much larger than the two state atom mentioned. We also have not included the reservoir states, which could be infinite. In our case, the reservoir refers to EM modes outside of the cavity which the internal EM field(s) interacts. If we say there are \(N\) states of the system and \(M\) states of the reservoir the density matrix has dimension \(NM(NM+1)/2\) [9]. The matrix created would describe the behavior of the system and reservoir, however, our investigation is only concerned with the system. Therefore, we can eliminate the reservoir description from the density matrix, thereby shrinking the matrix, by taking the partial trace over the reservoir states

\[ \rho_S = Tr_R(\rho) \quad (2.24) \]

Which has dimension \(N(N+1)/2\). There are other properties of the density operator that can be useful, however, later on we will apply a method to return to the wave function analysis so we will not discuss the density operator further.
2.4 Optical Lattices

Eventually our system will include a 1D optical lattice that will restrict the motion of the atoms inside the cavity. Optical lattices represent a periodic potential created by a set of interfering laser beams. The potential is caused by the ac-stark shift on neutral atoms [7]. The result is the atoms are evenly distributed like a crystal [7]. However, unlike crystals they are easy to manipulate since the size and strength of the lattice sites are determined by the laser creating the lattice.

The model we use here is semi-classical where we treat atom quantum mechanically but the field classically. The Hamiltonian then contains two terms, one for the atom and the other for the atom’s dipole interaction with the field

\[ \hat{H} = \frac{\hbar \omega_0}{2} \hat{\sigma}_z - \vec{E}(z, t) \cdot \hat{d} \]  \hspace{1cm} (2.25)

The electric field from a laser is \( \vec{E}(z, t) = E_0(z)\cos(\omega t)\hat{x} \) oscillates in \( \hat{x} \) and polarized in \( \hat{x} \), which makes the Hamiltonian

\[ \hat{H} = \frac{\hbar \Omega(z)}{2} \hat{\sigma}_x \]  \hspace{1cm} (2.26)

where \( \Omega = E_0(z) \langle e|\hat{x} \cdot \hat{d}|g \rangle \) and \( \hat{\sigma}_x = \hat{\sigma}_+ + \hat{\sigma}_- \) which results from the dot product. The Hamiltonian can be rewritten with the rotating wave approximation, which neglects rapidly oscillating terms

\[ \hat{H} \approx -\frac{\hbar \Delta}{2} \hat{\sigma}_z - \frac{\hbar \Omega(z)}{2} \hat{\sigma}_x \]  \hspace{1cm} (2.28)

Where \( \Delta = \omega - \omega_0 \) is the detuning. The eigenvalues for the Hamiltonian are the energy levels of the system

\[ E_{\pm} = \pm \sqrt{\left( \frac{\hbar \Delta}{2} \right)^2 + \left( \frac{\hbar \Omega(z)}{2} \right)^2} \approx \pm \left( \frac{\hbar \Delta}{2} + \frac{\hbar \Omega^2(z)}{4 \Delta} \right) \]  \hspace{1cm} (2.29)

This is the energy of a two level atom either in the exited (+) or ground state (−). The first term is constant so we can shift the zero point energy to that value. The second term varies with position. Since the lattice is formed by a laser we assume it has the
form $\Omega(z) = \Omega_0 \sin(kz)$ which makes the second term proportional to

$$E_{ls} \propto \sin^2(kz)$$

(2.30)

Where the subscript denotes that this is the light shift energy [7]. The light shift is proportional to the potential the atom feels in the optical lattice. We stated before that the lattice is periodic and this derivation verifies that claim. The laser’s wavelength and intensity define the site spacing and depth respectively. For our model, we treat the lattice like an external potential.

$$V(z) = V_0\sin^2(kz)$$

(2.31)

All of the atoms within the cavity feel an external force dependent on their position. Later we will show how this external potential affects the behavior of a BEC.

3 Jaynes-Cumming model with Trajectory Theory

We can now begin to describe the systems behavior in a cavity. Initially, we will assume that the system does not interact with the environment. This is the Jaynes-Cumming Model. Next, we will include the effects of photon loss with quantum trajectory theory which allows us to still use the wave function to describe the system instead of the density operator. This is what we will call the trajectory model. Finally, we will derive the state amplitudes of the weak field trajectory system and truncate the basis with the weak field limit.

3.1 Jaynes-Cumming Model

The Jaynes-Cumming model accounts for a single atom interacting with a single mode EM field. The atom is assumed to have two internal energy levels. This comes from potential applications to quantum computing as well as adding simplicity to the system. For now only the internal energy of the atom, the energy of the field and the interaction between the field and the atom are considered. The atom’s levels referred to are in the energy basis with a ground state $|g\rangle$ and an excited state $|e\rangle$. The wave function in the energy basis is

$$|\psi(t)\rangle = C_g(t) |g\rangle + C_e(t) |e\rangle$$

(3.1)

In the excited state the atom will have energy $E_e = \hbar\omega_0/2$ and in the ground state will have energy $E_g = -\hbar\omega_0/2$. All the time dependencies are contained in the amplitudes so our eventual analysis will be of their evolution. In this section, however, we will focus on
piecing together a Hamiltonian to describe their evolution.

From now on we will work in the rotating frame which eliminates the time dependent phases that appear with the operators, specifically \( \hat{a} \) and \( \hat{a}^\dagger \). To describe the Hamiltonian of two level atoms the Pauli spin matrices are used

\[
\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\sigma}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{\sigma}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

(3.2)

The second two operators represent excitation and dexcitation of the atom respectively. Therefore the Hamiltonian for the internal energy of a two level atom can be written in terms of the Pauli matrix \( \sigma_z \)

\[
\hat{H}_A = \frac{\hbar \omega_0}{2} \hat{\sigma}_z
\]

(3.3)

This Hamiltonian then has the zero point energy set equally between the excited (positive) and ground state (negative.) We treat external energy terms later.

The energy of the field was solved for in the previous section. Substituting the quantum operators, \( \hat{a} \) and \( \hat{a}^\dagger \), into our classical Hamiltonian for \( \hat{q} \) and \( \dot{\hat{q}} \) we get a new quantum Hamiltonian for the field again similar to the Hamiltonian for the SHO.

\[
\hat{H} = \sum_l \hbar \omega_l \left( \hat{a}^\dagger_l \hat{a}_l + \frac{1}{2} \right)
\]

(3.4)

However, from now on we will drop the ground state term \( \hbar \omega_l / 2 \) by shifting the zero point energy. Then, the energy of a single mode is

\[
\hat{H}_F = \hbar \omega_c \hat{a}^\dagger \hat{a}
\]

(3.5)

Here \( \omega_c \) represents the cavity frequency. This equation is also justified by looking back at the number operator \( \hat{N} \), from equation 2.14, since the energy for a single photon is \( \hbar \omega_l \) then for a group of photons of the same frequency, mode, are just the number times the energy.

The final term in the model results from the interaction between the atom and the electric field. The interaction results from the atom being polarized with a dipole moment \( \vec{d} \). We can write a Hamiltonian for the dipole interaction with the quantum electric field derived in the previous section

\[
\hat{E} = i\vec{\mu}(\vec{r})\vec{E}_0 \left( \hat{a} - \hat{a}^\dagger \right)
\]

(3.6)
The interaction Hamiltonian \( (H_{\text{int}}) \) is defined using the dipole approximation as

\[
\hat{H}_{\text{int}} = -\vec{d} \cdot \vec{E} \tag{3.7}
\]

Where \( \vec{d} \) is the dipole moment of the atom defined by

\[
\vec{d} = \mu_{eg} \hat{\sigma}_+ + \mu_{eg}^* \hat{\sigma}_- \tag{3.8}
\]

Here \( \mu_{eg} = \langle e \mid \vec{r} \mid g \rangle \) is the dipole matrix element between the ground and excited states [16]. Then the interaction Hamiltonian is

\[
\hat{H}_{\text{int}} = i\hbar g(\vec{r}) \left( \hat{a}^\dagger \hat{\sigma}_- - \hat{a} \hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_+ e^{i\phi} - \hat{a} \hat{\sigma}_- e^{-i\phi} \right) \tag{3.9}
\]

The constants are combined to be \( g(\vec{r}) = |\vec{d} \cdot \vec{u}(\vec{r})| E_0 \). The interaction is assumed to be on resonance (\( \omega_0 = \omega_c \)). \( \phi = 0 \) is chosen for simplicity

\[
\hat{H}_{\text{int}} = i\hbar g(\vec{r}) \left( \hat{a}^\dagger \hat{\sigma}_- - \hat{a} \hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_+ - \hat{a} \hat{\sigma}_- \right) \tag{3.10}
\]

There are four terms in the interaction Hamiltonian. It is shown with the rotating-wave approximation that two can be ignored by transforming to the interaction picture [17]. However, it is easier to see with intuition. The third term creates a photon and excites the atom. This process is not very common. This is true for the fourth term, which destroys a photon and dexcites the atom. The first two terms are the most probable and will be the only two we will use [16]. This leaves the interaction Hamiltonian as

\[
\hat{H}_{\text{int}} = i\hbar g(\vec{r}) \left( \hat{a}^\dagger \hat{\sigma}_- - \hat{a} \hat{\sigma}_+ \right) \tag{3.11}
\]

The first term represents absorption, a photon is destroyed and the atom is excited. The second term represents stimulated emission, a photon is created and the energy of the atom drops [16].

We now write a composite Hamiltonian, from the three Hamiltonians derived above, for the system of an atom interacting with a single mode of light that makes up the Jaynes-Cummings model.

\[
\hat{H}_{JC} = \hat{H}_A + \hat{H}_F + \hat{H}_{\text{int}} \]

\[
= \frac{\hbar \omega_0}{2} \hat{\sigma}_z + \hbar \omega_c \hat{a}^\dagger \hat{a} + i\hbar g(\vec{r}) \left( \hat{a}^\dagger \hat{\sigma}_- - \hat{a} \hat{\sigma}_+ \right) \tag{3.12}
\]

From now on we will assume that the coupling constant \( g(\vec{r}) \to g \), a constant, throughout
the entire cavity. The eigenstates of this Hamiltonian are the dressed states

$$|n, \pm\rangle = \frac{1}{\sqrt{2}} [ |n-1, e\rangle \pm |n, g\rangle]$$

(3.13)

Again $n$ is the number of photons in the single mode and $e$ and $g$ are the two possible energy states of the atom [11]. The energy levels of the system can be seen in figure (3.1.)

![Figure 3.1: The figure shows the energy levels of the Jaynes-Cumming model with dissipation events $\kappa$ and $\gamma$ included.](image)

Only the ground state and the first excited state are shown. The driving laser excites the system with $Y$ and the $g$ term brings the system from one excited state to another. The $\kappa$ and $\gamma$ terms dissipate the atom and are explained in the next section.

### 3.2 Quantum Trajectory Theory

To model our system we will have to use the density operator, which was introduced in section (2.3). Our system interacts with a reservoir by emitting photons. We ultimately do not care which mode the photons are emitted into. We only care when they are emitted. These processes are random and put the system into a classical mixture of states. While the density matrix provides a method to deal with these events it is more convenient and intuitive to work with the wave function. Quantum trajectory theory
gives a method to convert processes that can only be done with the density operator into the wave function approach. In this section, we will derive these methods and show their application to the Jaynes-Cumming model.

We have already introduced the von Neumann equation, equation (2.22), to model the evolution of the density matrix but now we add another term for the dissipation known as the Liouvillian superoperator $\mathcal{L}$ [9]

$$
\dot{\rho}(t) = \mathcal{L}\rho(t) = -\frac{i}{\hbar} \left[ \hat{H}_{JC}, \rho(t) \right] + \mathcal{L}_{\text{dis}}\rho(t)
$$

This equation is known as the Linblad master equation [9]. The form of the dissipation terms in the superoperator is dependent on the random process studied. In this paper, we look at two random processes, spontaneous emission (fluorescence) and cavity loss (transmission.) The superoperator dissipation portion is

$$
\mathcal{L}_{\text{dis}}\rho = \frac{\gamma}{2} \left[ 2\sigma_-\rho(t)\sigma_+ - \sigma_+\sigma_-\rho(t) - \rho(t)\sigma_+\sigma_- \right] + \kappa \left[ 2a\rho(t)a^\dagger - \rho(t)a^\dagger a - a^\dagger a\rho(t) \right]
$$

There are additional terms which have been ignored due to the physical limits of the reservoir (small thermal photon number $\bar{n} << 1$ at room temperature with the Born and Markov approximations) [9].

We will need a way to rework the superoperator into an operator on a wave function. To do this the superoperator is broken up into two parts, a dissipation term and a jump term

$$
\dot{\rho} = (\mathcal{L} - \mathcal{S})\rho + \mathcal{S}\rho
$$

Where $\mathcal{S}$ is the jump term [9]. We incorporate the dissipation term into the previous Hamiltonian and together they define the evolution of the system. Occasionally, however, the system will undergo a ”jump” that collapses the wave function with a ”jump operator” [16]. We rewrite the dissipation term to be the inverse commutator of a dissipation Hamiltonian and the density matrix

$$
(\mathcal{L} - \mathcal{S})\rho = -\frac{i}{\hbar} \left[ \hat{H}_{JC}, \rho \right] + \left[ \hat{H}_{D}, \rho \right]_+
$$

Which leads to a non-Hermitian Hamiltonian $\hat{H}$ given with dissipation terms

$$
\hat{H} = \hat{H}_{JC} + i\hbar\hat{H}_D
$$

$$
= \frac{\hbar\omega_0}{2} \hat{\sigma}_z + \hbar\omega_c\hat{a}^\dagger \hat{a} + i\hbar g(\vec{r}) \left( \hat{a}^\dagger \hat{\sigma}_- - \hat{a}\hat{\sigma}_+ \right) - i\hbar \left( \frac{\gamma}{2} \hat{\sigma}_+\hat{\sigma}_- + \kappa\hat{a}^\dagger\hat{a} \right)
$$
The final two terms are the non-Hermitian dissipation terms that comes from the quantum trajectories analysis. The corresponding jump operators are

\begin{align*}
S_1 \rho &= \gamma (\sigma_- \rho \sigma_+) \\
S_2 \rho &= 2\kappa (a \rho a^\dagger)
\end{align*}

The first operator corresponds to spontaneous emission and the second to cavity loss. These equations are found by separating equation (3.15). We have now rewritten the superoperator to operate on the wave function and accounted for all of its effects. Moving to the wave function requires us to use random events to define the system’s evolution. Therefore, to create a model that is not dominated by noise we must reiterate the process many times.

The system will evolve with the non-Hermitian Hamiltonian until sometime when a jump will occur with probability

\begin{align*}
P_\gamma(t) &= \gamma \langle \psi(t) | \sigma_+ \sigma_- | \psi(t) \rangle \ dt \\
P_\kappa(t) &= 2\kappa \langle \psi(t) | a a^\dagger | \psi(t) \rangle \ dt
\end{align*}

A random number can be generated to determine if there is a collapse or not [16]. If not, the wave function will evolve based on Schroedinger’s equation. If there is a collapse, then the wave function is hit by the following operators

\begin{align*}
|\psi_c(t_{n+1}) \rangle &= \sigma_- |\psi_c(t_n) \rangle \\
|\psi_c(t_{n+1}) \rangle &= a |\psi_c(t_n) \rangle
\end{align*}

\[ \frac{\sigma_- |\psi_c(t_n) \rangle}{\sqrt{\langle \psi_c(t_n) | \sigma_+ \sigma_- | \psi_c(t_n) \rangle}} \]

\[ \frac{a |\psi_c(t_n) \rangle}{\sqrt{\langle \psi_c(t_n) | a a^\dagger | \psi_c(t_n) \rangle}} \]

The wave function has to be normalized after every jump since the operator is non-Hermitian [16]. As mention earlier, the operation must be repeated many times.

### 3.3 State Amplitudes and Weak Field Limit

In this section, we will combine all of our analysis thus far to find the state amplitudes of the system. We will put a limit to the number of photons in the system, known as the weak field limit, in order to keep the basis a manageable size. The analysis here follows [5] and [4]. Combining the Jaynes-Cumming model, equation (3.12), with the quantum trajectory theory described above our current Hamiltonian which we call the weak field
trajectory model

\[ \hat{H} = i \hbar \Delta_a \hat{J}_z + \hbar \Delta_c \hat{a}^\dagger \hat{a} + i \hbar g (\hat{a}^\dagger \hat{J}_- - \hat{a} \hat{J}_+) + i \hbar Y (\hat{a}^\dagger - \hat{a}) - i \hbar (\frac{\gamma}{2} \hat{J}_+ \hat{J}_- + \kappa \hat{a}^\dagger \hat{a}) \]  (3.23)

Here \( \Delta_a = \omega - \omega_0 \) and \( \Delta_c = \omega - \omega_c \) are the detuning between the atom’s internal frequency (\( \omega_a \)) and the cavity mode (\( \omega_c \)) and the driving frequency (\( \omega \)) respectfully [5]. These terms are, however, not necessary when talking about probability amplitudes since the terms will only give a dependence on the given state. We add the fourth term to represent a driving laser off resonance. The \( \hat{\sigma} \) have also been replaced with \( \hat{J} \) operators which are generalized for multiple atoms.

\[ \hat{J}_\pm = \sum_{i=1}^{N} \hat{\sigma}_i^\pm \]  (3.24)

Using \( \hat{J} \) allows us to look at systems with \( N \) atoms which is part of our ultimate goal since dealing with a BEC requires having multiple atoms. They act on the symmetric states

\[ |1\rangle_S = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} |\downarrow_1 \downarrow_2 \cdots \downarrow_{k-1} \uparrow_{k+1} \cdots \downarrow_N \rangle \]  (3.25a)

\[ |2\rangle_S = \frac{2}{\sqrt{N(N-1)}} \sum_{k=1}^{N} \sum_{l=k+1}^{N} |\downarrow_1 \downarrow_2 \cdots \downarrow_{k-1} \uparrow_k \downarrow_{k+1} \cdots \downarrow_{l-1} \uparrow_l \downarrow_l \cdots \downarrow_N \rangle \]  (3.25b)

These states are necessary since we are not able to determine specifically which atom is in the ground state or the excited state [4]. Applying the \( \hat{J}_\pm \) to the symmetric states gives the following

\[ \hat{J}_+ |0\rangle_S = |1\rangle_S \]  (3.26a)

\[ \hat{J}_- |1\rangle_S = |0\rangle_S \]  (3.26b)

And the same can be done with \( |2\rangle_S \)

The Hamiltonian we will deal with from now on is

\[ \hat{H} = i \hbar g (\hat{a}^\dagger \hat{J}_- - \hat{a} \hat{J}_+) + i \hbar Y (\hat{a}^\dagger - \hat{a}) - i \hbar (\frac{\gamma}{2} \hat{J}_+ \hat{J}_- + \kappa \hat{a}^\dagger \hat{a}) \]  (3.27)

If we were to solve the Schrödinger equation from this Hamiltonian there would be an infinite number of states. In order to truncate the basis we apply the weak field limit where there is less than two excitations (the number of photons plus the number of excited
Here $C_{nm}$ are the amplitudes of the $|nm\rangle$ state where $n$ is the number of photons and $m$ is the number of atoms in the excited state. The excitation of the atom is then referred to as the sum $n + m$.

This is, however, still not correct since we are missing higher order terms, with the truncated basis, that should appear with $\dot{C}_{20}$, $\dot{C}_{11}$, and $\dot{C}_{02}$. In order to fix this problem we go back to the initial assumption of a weak field limit. Besides requiring less than 2 excited states this $Y$ must also be small (driving field is weak.) Therefore some terms multiplied by $Y^s$, where $s$ is some integer dependent on the state, do not affect the amplitude rates. Then we can scale the amplitudes to be on order 1, $Y$, or $Y^2$ grouped by the excitation [5]. We assume the ground state $C_{00} \sim 1$ therefore we drop the $C_{10}$ term in $\dot{C}_{00}$ since its at least of order $Y$. From here it is clear to see that the first excited states ($C_{10}$ and $C_{01}$) are order $Y$ and then we can drop any higher order terms since $Y$ is small) [5]. Similarly the second excited states ($C_{20}$, $C_{11}$ and $C_{02}$) are of order $Y^2$ and so higher order $Y$ terms can be dropped. The final states are

$$\dot{C}_{00} = 0$$

$$\dot{C}_{10} = Y(C_{00} - C_{20}) + g\sqrt{NC_{11}} - \kappa C_{10}$$

$$\dot{C}_{01} = g\sqrt{NC_{00}} - \frac{\gamma}{2} C_{01}$$

$$\dot{C}_{20} = \sqrt{2}YC_{10} + g\sqrt{2}\sqrt{NC_{11}} - 2\kappa C_{20}$$

$$\dot{C}_{11} = YC_{01} - g\sqrt{2}\sqrt{NC_{01}} + g\sqrt{2}\sqrt{N - 1}C_{02} - (\kappa + \frac{\gamma}{2})C_{20}$$

$$\dot{C}_{02} = -g\sqrt{2}\sqrt{N - 1}C_{11} - \gamma C_{02}$$

These are the final amplitudes for the weak field trajectory model [5]. We must remember, though, that in this weak field limit, where the amplitudes are scaled to 1, $Y$, $Y^2$, ..., that all other quantities calculated from them must also be scaled [5]. For example, the jump probability discussed earlier.
4 Methods used in Modeling

To make our model more efficient we will need to make some approximations. First, we will describe adiabatic elimination to further simplify our Hamiltonian. Next, we introduce photon counting statistics which will be our main tool in investigating the system.

4.1 Adiabatic Elimination

Adiabatic elimination is a method to determine the behavior of the system in different time intervals by invoking limits to the parameters involved. We start here from the Hamiltonian used in the previous section, equation (3.27). The limit we will use in this paper is the bad cavity limit meaning κ dominates all other rates. In the steady state photon creation and annihilation operator’s rates must be zero. We will solve for them with Heisenberg’s equations then substitute back into the master equation.

The rates of creation and annihilation are

\[ \dot{a} = \frac{i}{\hbar} [H, a] = -gJ_\downarrow - Y + \kappa a \]  
\[ \dot{a}^\dagger = \frac{i}{\hbar} [H, a^\dagger] = gJ_\downarrow + Y - \kappa a^\dagger \]  

The equations for \( \dot{\sigma}_- \) and \( \dot{\sigma}_+ \) can be solved similarly, however, that is not the simplification we wish to study here [8]. Taking the limit where the rates of \( \dot{a} \to 0 \) and \( \dot{a}^\dagger \to 0 \) we can get the equations for \( \dot{a} \) and \( \dot{a}^\dagger \).

\[ \dot{a} = \frac{g}{\kappa} J_\downarrow + \frac{Y}{\kappa} \]  
\[ \dot{a}^\dagger = \frac{g}{\kappa} J_\downarrow + \frac{Y}{\kappa} \]

The Hamiltonian we derived previously comes from the density operator [5]. As explained earlier the system is interacting with an external reservoir, therefore we cannot simply replace terms in equation (3.27) with our new expressions for \( a \) and \( a^\dagger \). In order to derive the correct Hamiltonian we must start back with the density operator

\[ \dot{\rho} = -\frac{i}{\hbar} [H, \rho] + \frac{\gamma}{2} \left[ 4\dot{J}_-\rho\dot{J}_+ - J_\downarrow \dot{J}_-\rho - \rho \dot{J}_+ J_\downarrow + \kappa [2\dot{a}\rho - a^\dagger \dot{a} \rho - \rho a^\dagger \dot{a}] \right] \]  

Now we can substitute the new \( \dot{a} \) and \( \dot{a}^\dagger \) into the master equation. Instead of substituting in the terms all at once, we will go through each term that is changed starting first with the interaction term in the Hamiltonian.
The first part comes from the interaction term

\[
\frac{i}{\hbar} g \left( \hat{a}^\dagger \hat{J}_- - \hat{a} \hat{J}_+ \right) = \frac{i}{\hbar} g \left[ \left( \frac{g}{\kappa} \hat{j}_+ + \frac{Y}{\kappa} \right) \hat{j}_- - \left( \frac{g}{\kappa} \hat{j}_- + \frac{Y}{\kappa} \right) \hat{j}_+ \right] = \frac{i}{\hbar} g \frac{Y}{\kappa} (\hat{j}_- - \hat{j}_+) + \frac{i}{\hbar} g \frac{g^2}{\kappa}(\hat{j}_+ \hat{j}_- - \hat{j}_- \hat{j}_+) \tag{4.4}
\]

In this part the second term is equivalent to \( \hat{J}_z \) and can be ignored since it only changes the zero point energy.

The next term changed is the weak driving laser

\[
\frac{i}{\hbar} Y (\hat{a}^\dagger - \hat{a}) = \frac{i}{\hbar} g \frac{Y}{\kappa} (\hat{j}_+ - \hat{j}_-) + \frac{i}{\hbar} Y \frac{Y^3}{\kappa^2} \tag{4.5}
\]

Again, since the final term is a constant, and since we are dealing with changes in energy, it can be ignored. The first term is equal but opposite to the first term coming from the interaction term and so, they eliminate each other. This may be disconcerting at first but we must remember that there are more parts from the \( \kappa \) portion of equation for \( \dot{\rho} \).

The \( \kappa \) dissipation term has three parts from the foiling of the equations for \( \hat{a} \) and \( \hat{a}^\dagger \)

\[
\kappa \left[ 2\hat{a}\rho \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \rho - \rho \hat{a}^\dagger \hat{a} \right] = \frac{g^2}{\kappa} \left( 2\hat{j}_- \hat{\rho} \hat{j}_+ - \hat{j}_+ \hat{j}_- \hat{\rho} - \rho \hat{j}_+ \hat{j}_- \right) + \frac{i}{\hbar} \left[ \frac{gY}{\kappa} (\hat{j}_- - \hat{j}_+), \hat{\rho} \right] \tag{4.6}
\]

The first part comes from the \( g \hat{J}/\kappa \) while the second comes from the cross term between the \( \hat{J} \) and the constant term. There was a third term, which was constant, which we again eliminate by shifting the zero point energy. The first term has the same operation on \( \rho \) as the \( \gamma \) term and so we can combine the two to give a new decay term in our Hamiltonian or letting \( \gamma \rightarrow \frac{\gamma}{2} \left( 1 + \frac{2g^2}{\kappa \gamma} \right) \). We move the second part to the evolution Hamiltonian to create the bad cavity Hamiltonian

\[
\hat{H}_{BC} = \frac{i}{\hbar} \frac{gY}{\kappa} \left( \hat{j}_- - \hat{j}_+ \right) - \frac{\gamma}{2} \left( 1 + \frac{2g^2}{\kappa \gamma} \right) \hat{j}_+ \hat{j}_- \tag{4.7}
\]

This Hamiltonian is much simpler to work with since we eliminated the photon number. By using this method, we can solve analytically for some situations. The probability of a jump from transmission is also edited because of the substitution for \( \hat{a} \) and \( \hat{a}^\dagger \) as well as the collapse operator.

\[
P_\kappa = 2\kappa \left( \frac{Y^2}{\kappa^2} + \frac{gY}{\kappa^2} (\langle \hat{j}_+ \rangle + \langle \hat{j}_- \rangle) + \frac{g^2}{\kappa^2} \langle \hat{j}_+ \hat{j}_- \rangle \right) \Delta t \tag{4.8}
\]
Each term corresponds to a different type of emission [5]. The first is from photons passing through the cavity without being absorbed by an atom. The second term is from coherent emission by the atoms into the cavity mode. The third term is from spontaneous emission of the atoms into the cavity mode.

4.2 Photon Statistics

The only information coming from the system are the photon decays outlined in section (3.2). To determine the behavior of the system we can look at the expectation value of the photons. In section (2.3) we derived an expression for the expectation value of an operator. However, it is more useful to find the expectation of an operator at two different times. These are the first order two-time correlation functions,

$$\langle \hat{O}_1(t) \hat{O}_2(t + \tau) \rangle \quad (4.9)$$

where the first measurement is made at time $t$ and the second at a later time $t + \tau$ [12]. A photon traveling from the cavity is either a transmitted (T) or a fluoresced (F) [9]. As shown in section (3.2) a transmission event corresponds to $\hat{a}$ collapse operator and a fluorescence event corresponds to a $\hat{\sigma}_-$ collapse operator. The first order photon correlation for a transmitted photon is

$$G^{(1)}(t, t + \tau) \propto \langle \hat{a}^\dagger(t) \hat{a}(t + \tau) \rangle \quad (4.10)$$

We can also look at the second order two-time correlation function. This is the expectation value of two operators at two times

$$G^{(2)}(t, t + \tau) \propto \langle \hat{a}^\dagger(t) \hat{a}^\dagger(t + \tau) \hat{a}(t + \tau) \hat{a}(t) \rangle \quad (4.11)$$

This equation relates to detecting a photon at time $t + \tau$ after detecting a photon at time $t$. Most studies use the normalized second order two-time photon correlation function [9],

$$g^{(2)}(t, t + \tau) = \frac{\langle a^\dagger(t)a^\dagger(t + \tau)a(t + \tau)a(t) \rangle}{\langle a^\dagger a \rangle^2} \quad (4.12)$$

Which gives the probability of detecting a photon at $t + \tau$ after detecting a photon at time $t$. The equation is normalized by dividing by $G^{(2)}(\infty)$ which is the two-time correlation function for the steady state of the system. For our system, we will treat each photon
detection separately so we can say that \( t = 0 \) for every detection and therefore
\[
g^{(2)}_T(\tau) = \frac{\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0) \rangle}{\langle a^\dagger a \rangle^2} \tag{4.13}
\]

Now turning to the fluoresced photons, we can apply the same operations to the second order photon correlation to get the normalized second order two-time correlation function
\[
g^{(2)}_F(\tau) = \frac{\langle \sigma_+(0)\sigma_+(\tau)\sigma_-(\tau)\sigma_-(0) \rangle}{\langle \sigma_+\sigma_- \rangle^2} \tag{4.14}
\]

We will use both correlations to identify features of the system. Additionally we could look at the correlation between the two types of photon emissions, however, that is beyond this paper.

By using the Schwartz inequality, the classical behavior can be predicted for \( g^{(2)}_T(\tau) \) and \( g^{(2)}_F(\tau) \)
\[
\begin{align*}
g^{(2)}(0) &\geq 1 \quad (4.15a) \\
g^{(2)}(\tau) &\leq g^{(2)}(0) \quad (4.15b) \\
|g^{(2)}(\tau) - 1| &\leq g^{(2)}(0) - 1 \quad (4.15c)
\end{align*}
\]

If the system behaved purely classically we would expect the delay time between counts, \( \tau \) to follow a Poissonian or wider distribution. However, for quantum situations, where we consider a quantum field, we expect the photons to be anti-bunched following sub-Poissonian statistics or a narrower distribution [12]. For our purposes, we can also see the tunneling rate from the counting statistic. Looking ahead to allowing the atoms to move between different sites it is easy to see that when an atom fluoresces a photon in one site there must be a delay proportional to the tunneling rate before it can fluoresce another photon from that same site. Over a single cycle the probability for this happening is likely small however when run over many cycles this process is easy to observe giving us a method for detecting the tunneling rate. This also means there are more counting statistics when we allow multiple lattice sites.

## 5 Bose-Einstein Condensate Background

So far, we have either treated a single atom (Jaynes-Cumming model) or a group of non-interacting atoms (weak field approximation) in a cavity. In this section, we will add to our model by allowing interactions between atoms and incorporating an optical lattice into the cavity. There are two types of particles, fermions and bosons. Fermions follow
The Pauli Exclusion principle meaning they cannot occupy the same state. Bosons are not restricted by this principle and, at very low temperatures, can share the same state. The interaction we model is dependent on which type we choose. For this paper, we have chosen bosons. Boson’s exemption from the Pauli Exclusion principle allows them to form a Bose-Einstein Condensate (BEC.) A BEC is a weakly interacting boson gas which has been cooled below a critical temperature leaving most of the atoms in the same ground state. In this situation quantum noise can be studied easily.

In this section we will outline Bose-Einstein statistics and the critical temperature to form a BEC. Then derive the Gross-Pitaevskii equation that models a weakly interacting boson gas. The next step is to include the effects of the lattice and derive the Bose-Hubbard Hamiltonian, which is based on the second quantization being applied to the Gross-Pitaevskii equation. We will conclude the section with a discussion on quantum states of a BEC in an optical lattice.

5.1 Bose-Einstein statics and the Critical temperature

The behavior of bosons was first derived in 1924 by S.N. Bose and Einstein [1]. Bose wrote to Einstein describing the behavior of a photon gas as a possible explanation for the black body spectrum. Einstein applied the method to particles [1]. It was later shown that particles (bosons) whose quantum states follow the statistics derived by Bose and Einstein have integer spin. Today, the statistics that govern bosons are standard knowledge after college thermodynamics or quantum mechanics. We wish to show that a boson gas will form a BEC below a critical temperature. We start with the energy distribution of a boson gas

$$f_{BE}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} - 1} \quad (5.1)$$

Where $\epsilon$ is the energy, $\beta$ is a constant related to the temperature and $\mu$ is the chemical potential. We can solve for the number of particles by summing over all energy states $\epsilon_k$

$$N = \sum -k \frac{1}{e^{\beta(\epsilon_k-\mu)} - 1} \quad (5.2)$$

For a large volume there are infinite number of states and we can let the sum go to an integral and, for simplification, look at the number density

$$n = \frac{N}{V} = \frac{1}{(2\pi)^3} \int \frac{d^3k}{e^{\beta(\epsilon_k-\mu)} - 1} = \int_0^\infty d\epsilon \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)} - 1} \quad (5.3)$$

Where $g(\epsilon) = m^{3/2} \epsilon^{1/2} / \sqrt{2\pi^2 \hbar^3}$ and is the density of states. The number density is the proportional to $\beta$ and $\mu$. What we really want to see, however, is a function for the
chemical potential since \(n\) is normally easy to control. To do this we can make many substitutions, which are outlined in [1], yielding an equation for a low density limit

\[
\mu = -\frac{3}{2} k_B T \ln \left( \frac{mk_B T}{2\pi \hbar^2 n^{2/3}} \right) \tag{5.4}
\]

\(k_B\) is Boltzmann’s constant and \(T\) is the temperature. Looking closely at this equation it is clear that at some time \(\mu \to 0\). This is the condition for the formation of a BEC. Remembering introductory thermodynamics, the chemical potential is what forces the particles into other states. Therefore, when \(\mu = 0\) there is no reason for the particles to be in an excited state and they all fall into the ground state. The temperature this occurs at can be solved for from equation 5.4

\[
T_C = \frac{2\pi \hbar^2}{k_B m} \left( \frac{n}{2.612} \right)^{2/3} \tag{5.5}
\]

Cooling past the critical temperature still leaves \(\mu = 0\) and it turns out that further passed the critical temperature a system travels the greater fraction of the particles go to the ground state increasing the density [1]. For our models we assume that all particles are in the ground state and \(T = 0\). This is not possible but with advanced cooling techniques the temperatures can reach nano-Kelvin. Even though these statistics have been well understood for decades the first BEC was created only in 1995 [1].

### 5.2 Gross-Pitaevskii Equation

To describe the dynamics of a BEC we use the Gross-Pitaevskii equation (GPE). As shown the previous section, below the critical temperature most bosons occupy the ground state. If the particles are in some type of external position dependent potential well this would mean that they share the same space. This creates a new potential energy for particles overlapping spatially. We can still use the basic Schroedinger equation to describe the system, however, we must now add the additional potential term [18]. Since we are assuming the gas is dilute, we will only include two particle interaction as a delta function. The new Hamiltonian for \(N\) particles is then

\[
\hat{H}_{BEC} = \sum_{i=1}^{N} \left( \frac{\vec{p}_i^2}{2m} + V(\vec{r}_i) \right) + \frac{u_0}{2} \sum_{i \neq j} \delta(\vec{r}_i - \vec{r}_j) \tag{5.6}
\]

Where \(i\) and \(j\) denote specific particles, \(V(\vec{r})\) is the external potential and \(u_0\) represents the energy from the atoms s-wave scattering [18]. Again, for this study we assume \(T = 0\) and therefore all the particles will share the same ground state. We can then expand
the wave function for the system as the product of wave functions for each particle at position \( i \)

\[
|\Psi(r_1^i, ..., r_N^i)\rangle = \prod_{i=1}^{N} |\psi_1(r_i^i)\rangle
\]  

(5.7)

Applying the Hamiltonian, we can solve for the energy of the system. Since the positions are not quantized we must integrate instead of sum over all possible \( \vec{r}_i \)

\[
\hat{H}_{BEC} |\Psi(r_1, ..., r_N)\rangle \equiv E |\Psi(r_1, ..., r_N)\rangle
\]

\[
E = \int \Psi^* \hat{H}_{BEC} \Psi d\vec{r}_1 ... d\vec{r}_N
\]

\[
= \int \prod_{i=1}^{N} \psi_1^*(\vec{r}_i) \left( \sum_{i=1}^{N} \left( -\frac{\hbar^2}{2m} \nabla^2_i + V(\vec{r}_i) \right) + \frac{u_0}{2} \sum_{i \neq j} \delta(\vec{r}_i - \vec{r}_j) \right)
\]

\[
\times \prod_{i=1}^{N} \psi_1(\vec{r}_i) d\vec{r}_1 ... d\vec{r}_N
\]

Since the operators are specific to each particle there is only one term that results from each iteration of the sum for the kinetic and external potential energy. The second sum for \( u_0 \) has \( N \times (N - 1) \) terms since there are \( N \times (N - 1) \) possible pairs of particles for the delta function [18]. Since we integrate over all space, each set of terms is equivalent giving a final energy

\[
E = N \int \psi_1^*(\vec{r}) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}) + \frac{u_0}{2} (N - 1)|\psi_1(\vec{r})|^2 \right] \psi_1(\vec{r}) d\vec{r}
\]  

(5.8)

The wave function for the condensate is normalized to \( N \) so \( \psi(\vec{r}) = \sqrt{N}\psi_1(\vec{r}) \) which leaves the equation as

\[
E = \int \psi^*(\vec{r}) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}) + \frac{u_0}{2} |\psi(\vec{r})|^2 \right] \psi(\vec{r}) d\vec{r}
\]

(5.9)

Also \( (1 - 1/N) \to 1 \) for a large number of particles. This is what is called the GPE [18]. We can undo our solution for \( E \) and extract out a Hamiltonian for the system

\[
\hat{H}_{BEC} = -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}) + \frac{u_0}{2} |\psi(\vec{r})|^2
\]

(5.10)

Where the eigenvalues are \( \mu \), the chemical potential, of the system [18]. The external potential in our model is the 1D optical lattice outlined in section (2.4) and will be applied in the next section. The final term is the Hartree energy that describes the interaction between the atoms in a mean field limit. The Hartree energy also means the Hamiltonian
is no longer linear.

5.3 Bose-Hubbard Model

The final piece that needs to be added to our understanding of a BEC is the optical lattice. The lattice, as described in section (2.4), is the result of off resonant laser light creating a series of potential wells that particles can occupy. The lattice sites can be changed in depth by adjusting the laser intensity [10]. For this work, we look at a one dimensional lattice so \( \vec{r} \rightarrow x \) and the external potential takes the periodic form we derived in section 2.4. We refer to the potential wells within the lattice as sites. By loading a BEC into the lattice, we force the boson particles to fall into the sites. When two or more particles are inside a site, they interact by the Hartree energy. Additionally the particles can move between lattice sites by tunneling.

Loading a BEC into an optical lattice requires that we use the second quantization laid out in the section (2.1). Making the wave functions into operators allows us to create or destroy bosons at any state, which fits in with our model of atoms inhabiting sites of a lattice. To complete the derivation we will assume both that the lattice is deep and the atoms are sufficiently cold so that they only occupy the bottom state of each site. The state operator is then

\[
\hat{\Psi}(\vec{r}) = \sum_i w_0(x - x_i) \hat{b}_i
\]

The \( w_0(x - x_i) \) is the ground state Wannier function, which is approximately a Gaussian, at lattice site \( i \) with position \( x_i \) [18]. The Wannier functions describe particles within a periodic lattice and are orthonormal. The second term \( \hat{b}_i \), is the annihilation operator for boson particles which, like photons, follow equation 2.3. Now we can insert the state operator into the Hamiltonian derived in the previous section from the GPE, equation (5.10) without \( V(\vec{r}) \) to get

\[
H_{BH} = -\hbar \sum_{i,j} J_{ij} \hat{b}_i^\dagger \hat{b}_j + \frac{1}{2} \sum_{i,j,k,l} U_{ijkl} \hat{b}_i^\dagger \hat{b}_j^\dagger \hat{b}_k \hat{b}_l
\]

Do not confuse \( J_{ij} \) with the internal energy operator \( \hat{J}_z \). \( J_{ij} \) and \( U_{ijkl} \) can be found by

\[
J_{ij} = \frac{\hbar}{2m} \int dx \left( w_0(x - x_i) \left( \frac{d^2}{dx^2} + V(x) \right) + w_0(x - x_j) \right)
\]

\[
U_{ijkl} = u_0 \int dx (w_0^* (x - x_i) w_0^* (x - x_j) w_0(x - x_k) w_0(x - x_l))
\]

We can further reduce the equation by eliminating non-nearest neighbor terms. Atoms interact much more strongly with nearby sites than they do with sites across the lattice.
This means only terms of $J$ with consecutive $i$ and $j$ and only the diagonal terms of $U$ where $i = j = k = l$. Therefore $J_{ij}$ and $U_{ijkl}$ are constant throughout the lattice. $J$ is further broken down into diagonal (chemical potential terms) and off diagonal (tunneling terms) $J_{ij} = \epsilon_i \delta_{ij} - J(1 - \delta_{ij})$ \[18\]. Then

$$J = \frac{\hbar}{2m} \int dx \left( w_0(x - x_i) \left( \frac{d^2}{dx^2} + V(x) \right) w_0(x - x_j) \right)$$ \hspace{1cm} (5.14a)

$$U = u_0 \int dx |w_0(\vec{r} - x_i)|^4$$ \hspace{1cm} (5.14b)

$$\epsilon_i = \frac{\hbar}{2m} \int dx \left( w_0(x - x_i) \left( \frac{d^2}{dx^2} + V(x) \right) w_0(x - x_i) \right)$$ \hspace{1cm} (5.14c)

Where $\mu$ is the chemical potential but is assumed to be constant for all sites. While these are clearly parameters dependent on the Wannier site functions, we will treat them as changeable constants for the rest of our analysis. We call $J$ the tunneling constant, higher $J$ implies higher tunneling rate. Experimentally $U$ is difficult to change since it is dependent on the atoms themselves. The tunneling constant however, is easy to change by making the lattice sites deeper, decreases $J$, or shallower, increases $J$ \[10\]. The final simplified Hamiltonian is the Bose-Hubbard Hamiltonian

$$H_{BH} = -\hbar J \sum_{i,j} \hat{\nabla}_i \hat{\nabla}_j + \frac{U}{2} \sum_i \hat{n}_i(\hat{n}_i - 1) + \sum_i \epsilon_i \hat{n}_i$$ \hspace{1cm} (5.15)

Where we have substituted $\hat{n} = \hat{\nabla}_i \hat{\nabla}_j$. The first term is due to tunneling, a boson is created in one site $i$ and destroyed in another $j$ where $i$ and $j$ are consecutive integers. The second term is for interactions within the lattice sites between particles in the same site $i$ \[8\]. The final term is the chemical potential but from now on is dropped since it is constant for all sites and therefore will not determine the systems evolution. We now have a convenient Hamiltonian to describe our boson particle’s dynamics in a lattice based on the same basic operators as the trajectory model.

### 5.4 BEC Quantum States

A large motivation for looking at this specific system, of a BEC in an optical lattice, is the occurrence of quantum phase transitions. Classical phase transitions, such as ice melting, are caused by thermal fluctuations. Likewise, quantum fluctuations cause quantum phase transitions \[10\]. These can only occur at or very near $T = 0$. A BEC in an optical lattice gives a mechanism to study such transitions. There are four possible quantum phases of a BEC. In this section, we will focus on two, superfluid (SF) and Mott insulator (MI). The other two, supersolid and Bose glass, are less probable.
The MI state has zero compressibility meaning each atom is localized in a lattice site. The lattice is assumed to be very deep and there is little tunneling. The SF is more fluid, atoms interact strongly with each other and the tunneling rate is high. The Bose-Hubbard Hamiltonian is sufficient to model the phase transition from SF to MI. A full derivation can be seen [18] by using a mean-field approximation of the Bose-Hubbard Hamiltonian. Here we will simply discuss the results as the phase diagram below.

Figure 5.1: This figure is taken from [13]. The shaded area represents the MI phase while the rest is the SF phase.

The MI lobe’s numbering represents the number of particles per lattice site. The y-axis is the ratio of the chemical potential, $\mu$, to $U$ and the x-axis the ratio of the tunneling rate to $U$. To understand the relation to the Bose-Hubbard Hamiltonian let us look at two limits.

First, let us assume that $J = 0$. Then there is no tunneling and the system is isolated to the y-axis of the diagram meaning it is always in a MI state. We can check that with physical intuition. When the tunneling rate is zero the particles cannot move to other lattice sites since that term in the Bose-Hubbard Hamiltonian goes to zero [6]. This matches our definition of the MI state.

The second limit to look at is if $J$ is large. Then we are restricted to the x-axis of the diagram, which falls into the SF state. Again, we can check with our physical intuition.
High tunneling means that the atoms travel around the lattice constantly which again matches our definition of a SF [6]. These changes can be made by changing the intensity of the lattice laser beam, which deepens the sites.

The other changeable parameter is the chemical potential. If we keep the ratio of tunneling to $U$ constant but increase the ratio of the chemical potential to $U$ we force more and more atoms into lattice sites. Again this matches the diagram if we start close to the y-axis we cross several MI lobes as the chemical potential ratio increases. The difference between the MI lobes is called the energy gap [18].

### 6 Two-component BEC model

We can now combine the weak field trajectory terms, equation (3.23), with the Bose-Hubbard Hamiltonian, equation (5.15), to get a new Hamiltonian describing the evolution of a two level boson atom in an optical lattice. The Hamiltonian for the system is

$$
H = -\hbar J \sum_{i,j} \hat{b}_i^\dagger \hat{b}_j + \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) + i\hbar g \left( \hat{a}_-^\dagger \hat{J}_- - \hat{a}_- \hat{J}_+ \right) + i\hbar Y \left( \hat{a}_-^\dagger - \hat{a}_- \right) - i\hbar \left( \frac{\gamma}{2} \hat{J}_+ \hat{J}_- + \kappa \hat{a}_-^\dagger \hat{a}_- \right)
$$

(6.1)

The Hamiltonian allows boson atoms to travel between lattice sites and also interact with a weak driving laser $Y$.

The number of states for this system is the product of the number of states from the Bose-Hubbard model and the the number of states from the trajectory model. The number of states from the Bose-Hubbard Hamiltonian is proportional to the number of sites, $L$ and the number of particles $N$ [1]

$$
S_{BH} = \frac{(N + L - 1)!}{N!(L - 1)!}
$$

(6.2)

The number of states for the trajectory Hamiltonian is proportional to the number of particles and the number of excitations. In the weak field limit there are six possible states for $N$ atoms and five for a single atom. Then the total number of states is

$$
S_{tot} = 6 \times \frac{(N + L - 1)!}{N!(L - 1)!}
$$

(6.3)

For example, when there are two sites and one particles there are two states from the atom configuration ($|10\rangle$ and $|01\rangle$ where a 1 denotes the one atom in a site) multiplied by the different trajectory states which for the weak filed limit (two excitations) is five.
Figure 6.1: This is the two lattice site setup with the two-component model where $Y$ is the driving laser, $\gamma$ is spontaneous emission, $\kappa$ the cavity loss and $J$ the tunneling coefficient.

This gives ten possible states.

There is also the bad cavity Hamiltonian found by adiabatic elimination of $\hat{a}$ and $\hat{a}^\dagger$ of the trajectory model as shown in section (4.1).

\[
H_{BC} = -\hbar J \sum_{i,j} \hat{b}_i^\dagger \hat{b}_j + \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) + i\hbar \frac{gY}{\kappa} \left( \hat{J}_- - \hat{J}_+ \right) - \frac{\gamma}{2} \left( 1 + 2 \frac{g^2}{\kappa \gamma} \right) \hat{J}_+ \hat{J}_- \tag{6.4}
\]

In this case the number of states based on the configuration of atoms is the same as the previous Hamiltonian, however, the number of states from the bad cavity Hamiltonian is only equal to the number of excited states since $\hat{a}$ and $\hat{a}^\dagger$ were eliminated.

\[
S_{tot} = (e_{max} + 1) \times \frac{(N + L - 1)!}{N!(L - 1)!} \tag{6.5}
\]

Where the "+1" represents the ground state. For the weak field limit $e_{max} = 2$ for $N$ atoms while for a single atom $e_{max} = 1$. Therefore, for the two sites one particle arrangement there are only four possible states. The number of states from the atom configuration are multiplied by two now. For the following sections we will restrict our analysis to this system of a single atom allowed to jump between two lattice sites as shown in figure (6).
6.1 State Amplitudes for the Two Site One Atom model

Now we analyze, in depth, the single atom two-cavity system with the bad cavity Hamiltonian. We use the bad cavity Hamiltonian to solve Schroedinger’s equation to get the rate of change of the amplitudes. The $U$ term is dropped since that term only appears when there are more than one atoms. This also means that we are not considering an actual BEC since there are not multiple atoms to interact.

We use the notation $C_{nm}^i$ where the subscript shows the number of atoms in site one ($n$) and the number of atoms in site two ($m$) and the superscript denotes the internal energy $i = g, e$. Solving Schroedinger’s equation gives

\[
\frac{\dot{C}_{10}^g}{i} = JC_{01}^g + \frac{gY}{\kappa}C_{10}^e 
\]

(6.6a)

\[
\frac{\dot{C}_{01}^g}{i} = JC_{10}^g + \frac{gY}{\kappa}C_{01}^e
\]

(6.6b)

\[
\frac{\dot{C}_{10}^e}{i} = JC_{01}^e - \frac{gY}{\kappa}C_{10}^g - \frac{\gamma}{2}(1 + 2C)C_{10}^e
\]

(6.6c)

\[
\frac{\dot{C}_{01}^e}{i} = JC_{10}^e - \frac{gY}{\kappa}C_{01}^g - \frac{\gamma}{2}(1 + 2C)C_{01}^e
\]

(6.6d)

Where $C = g^2/\kappa \gamma$. From here we can solve the equations numerically or take a weak field limit and solve them analytically. The numerical solution requires a Runge-Kutta method instead of an Euler method.

We can also solve the system analytically in the weak field limit. The amplitude rates are scaled on $Y$ similar to in the simple weak field trajectory model shown in section (3.3.) The ground states are of order 1 and the excited states are of order $Y$. This results in dropping the $Y$ terms in the ground state rates

\[
\frac{\dot{C}_{10}^g}{i} = JC_{01}^g
\]

(6.7a)

\[
\frac{\dot{C}_{01}^g}{i} = JC_{10}^g
\]

(6.7b)

\[
\frac{\dot{C}_{10}^e}{i} = JC_{01}^e - \frac{gY}{\kappa}C_{10}^g - \frac{\gamma}{2}(1 + 2C)C_{10}^e
\]

(6.7c)

\[
\frac{\dot{C}_{01}^e}{i} = JC_{10}^e - \frac{gY}{\kappa}C_{01}^g - \frac{\gamma}{2}(1 + 2C)C_{01}^e
\]

(6.7d)

These equations are coupled so to separate them we use the symmetry states labeled $D_{\pm}^{g,e}$.
which relate to \( C^{g,e} \) by

\[
\begin{align*}
D_+^g &= \frac{1}{\sqrt{2}} (C_{10}^g + C_{01}^g) \\ \\
D_-^g &= \frac{1}{\sqrt{2}} (C_{10}^g - C_{01}^g) \\ \\
D_+^e &= \frac{1}{\sqrt{2}} (C_{10}^e + C_{01}^e) \\ \\
D_-^e &= \frac{1}{\sqrt{2}} (C_{10}^e - C_{01}^e)
\end{align*}
\] (6.8a-d)

Substituting the \( D \) equations into equation (6.7) gives a series of uncoupled differential equations which can be solved analytically. Undoing the substitutions for \( D \) gives the amplitudes for the weak field bad cavity limit as

\[
\begin{align*}
C_{10}^g(t) &= \frac{1}{\sqrt{2}} (D_+^g(0)e^{i\Gamma t} + D_-^g(0)e^{-i\Gamma t}) \\ \\
C_{01}^g(t) &= \frac{1}{\sqrt{2}} (D_+^g(0)e^{i\Gamma t} - D_-^g(0)e^{-i\Gamma t}) \\ \\
C_{10}^e(t) &= \left( \frac{YD_+^g(0)}{\Gamma} (e^{-\Gamma t} - 1) + D_+^e(0)e^{-\Gamma t} \right) e^{i\Gamma t} \\
&\quad + \left( \frac{YD_-^g(0)}{\Gamma} (e^{-\Gamma t} - 1) + D_-^e(0)e^{-\Gamma t} \right) e^{-i\Gamma t} \\ \\
C_{01}^e(t) &= \left( \frac{YD_+^g(0)}{\Gamma} (e^{-\Gamma t} - 1) + D_+^e(0)e^{\Gamma t} \right) e^{i\Gamma t} \\
&\quad - \left( \frac{YD_-^g(0)}{\Gamma} (e^{-\Gamma t} - 1) + D_-^e(0)e^{\Gamma t} \right) e^{-i\Gamma t}
\end{align*}
\] (6.9a-d)

We let \( \gamma(1+2C)/\Gamma = \Gamma \). These amplitudes only apply when \( Y \) is small due to the weak field limit. Therefore, we have two separate models to compare, one numeric and one analytic. To simplify the solution we can restrict the analysis by applying the initial condition that the particle must start in the ground state of the first well. Then, \( D_+^g(0) = D_-^g(0) = 1/\sqrt{2} \) and \( D_+^e(0) = D_-^e(0) = 0 \). Then the amplitudes are

\[
\begin{align*}
C_{10}^g(t) &= \cos(Jt) \\ \\
C_{01}^g(t) &= \sin(Jt) \\ \\
C_{10}^e(t) &= \frac{Y}{\Gamma} (e^{-\Gamma t} - 1) \cos(Jt) \\ \\
C_{01}^e(t) &= \frac{Y}{\Gamma} (e^{-\Gamma t} - 1) \sin(Jt)
\end{align*}
\] (6.10a-d)

Starting the particle in the other well will only shift the phase of each by \( \pi/2 \). This will
be the starting condition used throughout.

### 6.2 The Model

For both methods, numerical and weak field analytical, programs were created to model the evolution of the system. The process for each program is the same and based on the Quantum Trajectory Theory explained in section (3.2). We start the system in the ground state with the atom in the first well. The program then does a series of steps at a set time interval. At each step the probability of a jump is calculated, as derived in section (3.2), for spontaneous emission or cavity loss. For this case (2 sites 1 atom,) there are three jumps. Two for spontaneous emission (one for each sites) and one for cavity loss. We can also look at a jump from either site, which is the sum of the jumps from each site.

\[
P_{\gamma_1} = \gamma |C_{10}^{e}|^2 \Delta t \]
\[
P_{\gamma_2} = \gamma |C_{01}^{e}|^2 \Delta t \]
\[
P_{\gamma} = P_{\gamma_1} + P_{\gamma_2} = \gamma \left( |C_{10}^{e}|^2 + |C_{01}^{e}|^2 \right) \Delta t \]
\[
P_{\kappa} = 2 \left[ \frac{Y^2}{\kappa} + \frac{Y g}{\kappa} \left( C_{10}^{e} C_{10}^{g} + C_{01}^{e} C_{01}^{g} + C_{10}^{e} C_{01}^{e} + C_{01}^{e} C_{10}^{e} \right) \right] \Delta t \]

The third equation was derived in section (4.1). The program then generates a random number and if it falls into a range determined by the probabilities calculated above a jump occurs. The time of the jump is stored and the collapse operators collapse the system

\[
\hat{\sigma}^{1}_- |\psi\rangle = C_{10}^{e} |10g\rangle \\
\hat{\sigma}^{2}_- |\psi\rangle = C_{01}^{e} |01g\rangle \\
\hat{a} |\psi\rangle = \left( \frac{Y}{\kappa} C_{10}^{g} + \frac{g}{\kappa} C_{10}^{e} \right) |10g\rangle + \left( \frac{Y}{\kappa} C_{01}^{g} + \frac{g}{\kappa} C_{01}^{e} \right) |01g\rangle \\
+ \frac{Y}{\kappa} C_{10}^{e} |10e\rangle + \frac{Y}{\kappa} C_{01}^{e} |01e\rangle
\]

If however the random number does not fall into the jump range then the system will evolve. The numeric and weak field analytic methods differ in this evolution. For the numerical evolution, the program uses the Schroedinger equation solutions with a Runge-Kutta method. The analytical model uses the solutions for the amplitudes. With either case the final amplitudes are renormalized since the Hamiltonian’s are non-Hermitian.
We summarize the steps below

1. Calculate jump probability and random number $R$

2. If $R$ is in jump range collapse state and note time otherwise evolve the amplitude functions

3. Renormalized amplitudes

We repeat the process for a set amount of intervals outputting a list of times where a jump occurred.

Ultimately, the data taken will be the times of the jumps but to check the program we can graph the expectation values $\langle \hat{\sigma}_+ \hat{\sigma}_- \rangle$ and $\langle \hat{a}^\dagger \hat{a} \rangle$ which correspond to the probability of each jump.

\begin{align*}
\langle \hat{\sigma}_+ \hat{\sigma}_- \rangle &= |C_{10}|^2 \quad (6.13a) \\
\langle \hat{\sigma}_+ \hat{\sigma}_- \rangle &= |C_{10}|^2 + |C_{01}|^2 \quad (6.13b) \\
\langle \hat{a}^\dagger \hat{a} \rangle &= \frac{Y^2}{\kappa} + \frac{Yg}{\kappa} (C_{10}^e * C_{10}^g + C_{01}^e * C_{01}^g + C_{10}^g * C_{10}^e + C_{01}^g * C_{01}^e) \\
&\quad + \frac{g^2}{\kappa} (|C_{10}|^2 + |C_{01}|^2) \quad (6.13c)
\end{align*}

Which are plotted below with $J = 1$, $g = \sqrt{10}$, $\kappa = 10$, $\gamma = 1$ so $C = 1$. Each type of jump is labeled to show how the collapse affects the expectation value of the function.
Figure 6.2: A solid arrow shows a collapse in the first site, a dashed arrow a collapse in the second site and a finely dashed arrow a collapse from a transmitted photon. (a) shows \( \langle \hat{\sigma}_+ \hat{\sigma}_- \rangle \), (b) shows \( \langle \hat{\sigma}_- \hat{\sigma}_- \rangle \) and (c) shows \( \langle \hat{a}^\dagger \hat{a} \rangle \).
The figure shows how each collapse operator affects the states differently. For instance, a cavity transmission collapse does not bring the excited state to zero like a fluorescence collapse. Since we do not know if the transmitted photon comes from the atom or the driving laser we do not know the state of the atom exactly. Figure (6.2b) looks like the probability amplitude graphs seen in [5]. The amplitude is not dependent on the site and therefore we cannot see any tunneling effects. Figure (6.2a) looks the same but with oscillations overlaid. This is the effect of adding tunneling. However, we cannot measure the wave function to see the tunneling coefficient. To gain information on the system we look at the correlation function $g^{(2)}(\tau)$ as explained in the next section.

7 Data and Analysis

There are three lists of jump times output by each program, one list for each operator, spontaneous emission from each site and emission from the cavity. For most trials the driving field, $Y$, was turned up in order to create many jumps. This resulted in roughly $10^4$ jumps in $10^8$ iterations of the process described above. We vary the tunneling coefficient between trials to show differences in the results. All other constants are kept at unity ($g = \sqrt{10}$, $\kappa = 10$, $\gamma = 1$ so $C = 1$.) Using these times we can determine the time between jumps, $\tau$ and plot the data in a histogram. This relates to the counting statistic $g^{(2)}(\tau)$ except without normalization. Still, from the histogram data it is possible to gain information about the tunneling coefficient $J$.

There are eight possible types of histograms to create depending on what we count as the starting and stopping time for $\tau$. The start and stops are coincide with a jump. For this paper we focus on four of the possible eight; starting and stopping with a jump from the first site, starting with a jump from the first and stopping with a jump from the second, starting and stopping with a count from either site, and starting and stopping from a jump from the side of the cavity.

7.1 Histogram Setup

The times output by the models are the times of jumps to be used to calculated the time between jumps $\Delta t_j$. We not only want to have $\Delta t_j$ be a list of consecutive counts but also include counts from up to four jumps away to include any possible correlations. We then let $\tau = \gamma \Delta t$ which gives $\tau$ dimensionless units. The result is a list of $\tau$ values which can be binned to form a histogram.

What we want to look at is times less than $\tau = 10$ which would describe photon correlations. The number of $\tau$ values depends on what type of jump we are discussing.
In the numerical method, for counts measuring between two sites there are \( \sim 2 \times 10^3 \), measuring in a single site there are \( \sim 10^4 \), and measuring a count from either site or a transmission there are \( \sim 3 \times 10^4 \) values of \( \tau \) in our range. For the weak field analytical method, the counts are of order 10 less than the numerical method. This is because we neglect the \( Y \) term that drives the atoms to the excited state. After this time the correlations are defined by classical probability and go to zero as \( t \to \infty \). We normalize the histogram counts to be independent of the number of jumps by dividing by the ratio of the total number of \( \tau \)'s to the number of bins. This is not exactly the correlation function, however, it is closely related so we will label it as such.

### 7.2 Results of Models

Here we present the primary data of the analysis shown above. Each program was run for \( 2 \times 10^8 \) iterations with \( J = 1, 2, 5, Y = 0.4 \) and \( g = \sqrt{10}, \kappa = 10, \gamma = 1 \) so \( C = 1 \). The data presented below is created by the method described in the previous section where \( \tau < 10 \) for each type of correlation. The results of the analytical model from the weak field analytical solution appear dashed while the results from the numerical Runge-Kutta method appear as solid lines. The graphs go in order of increasing \( J \) and appear with site correlations and one page and non-site dependent, correlations on the next.
Figure 7.1: Site correlation for $J = 1$ (a) shows the correlation between photons fluoresced from a single site (b) shows the correlations between photons fluoresced from site 1 with those emitted from site 2.
Figure 7.2: Site correlation for $J = 1$ (a) shows the correlation between photons fluoresced regardless of site (b) shows the correlations between photons transmitted out the side of the cavity.
Figure 7.3: Site correlation for $J = 2$ (a) shows the correlation between photons fluoresced from a single site (b) shows the correlations between photons fluoresced from site 1 with those emitted from site 2.
Figure 7.4: Site correlation for $J = 2$ (a) shows the correlation between photons fluoresced regardless of site (b) shows the correlations between photons transmitted out the side of the cavity.
Figure 7.5: Site correlation for $J = 5$ (a) shows the correlation between photons fluoresced from a single site (b) shows the correlations between photons fluoresced from site 1 with those emitted from site 2.
Figure 7.6: Site correlation for $J = 5$ (a) shows the correlation between photons fluoresced regardless of site (b) shows the correlations between photons transmitted out the side of the cavity.

The two methods agree well for all correlations except between the two sites. There, the number of $\tau$ values are only $\sim 200$ for the weak field method. The lack of $\tau$ values for the weak field analysis is expected since we dropped the driving term for the ground
state as mentioned earlier. For the other figures the number of values of $\tau$ is large enough to not be drastically affected.

The graphs of $g^{(2)}(\tau)$ for the single site fluorescence correlation, figures (7.1a), (7.3a), and (7.5a), have the same form as the ordinary weak field trajectory model except with added oscillations. As $J$ increases so does the frequency of oscillations. For figure (7.1a) the period is $\sim \pi$, while in figure (7.3a) where $J$ doubles the period is $\sim \pi/2$. Likewise, in figure (7.5a) the period is $\sim \pi/5$. The correlation is $T = \pi/J$ and will be derived in the next section. The oscillations are not as clear for the correlation between site 1 and site 2. Again, due to the low number of $\tau$ values it is difficult to determine beyond $J = 1$ for the weak field method and $J = 2$ for the numerical method. However, we can see oscillations with the same period as the single site correlations.

The other figures, (7.2, 7.4, and 7.6) show the correlations if we ignore the tunneling. These agree with earlier models of a single atom in following the Hamiltonian derived in section (4.1). The results can be compared to [9] and [5].

7.3 Theoretical Comparison

In this section we will use the analytical solutions derived in section (6.1.) Combining the information from sections (4.2) and (6.1) we can solve for the theoretical $g^{(2)}(\tau)$ function. The correlation function $g^{(2)}(\tau)$ can be rewritten in terms of the collapse states, from equation (4.14)

$$g^{(2)}(\tau) = \frac{\langle \psi_c | \hat{\sigma}_+ (\tau) \hat{\sigma}_- (\tau) | \psi_c \rangle}{\langle \hat{\sigma}_+ \hat{\sigma}_- \rangle}$$  \hspace{1cm} (7.1)

Where $|\psi\rangle$ is the collapsed state. We have separated out $\langle \hat{\sigma}_+ \hat{\sigma}_- \rangle$ and canceled out the same term from the bottom. Therefore, if we know which site the photon fluoresces from we know the collapsed state. Let us assume, for now, that we are looking at $g^{(2)}(\tau)$ for the first site starting $\tau$ and stopping $\tau$. When the first jump is observed, at $t = 0$, the wave state is collapsed to the atom being in the ground state and in the first site. Then the quantity $\langle \hat{\sigma}_+ (\tau) \hat{\sigma}_- (\tau) \rangle$ is simply the probability of being in the excited state in the first site with the initial condition of the atom starting in the ground state of the first site.

$$G^{(2)}(\tau) = |C_{10}(\tau)|^2 = \left( \frac{Y}{1} \right)^2 \left( e^{-\Gamma \tau} - 1 \right)^2 \cos^2(J\tau)$$  \hspace{1cm} (7.2)

We normalize to get $g^{(2)}(\tau)$ by dividing by the time average of the excited state probability which is $(Y/\sqrt{2\Gamma})^2$. Again this is not exactly $g^{(2)}(\tau)$ since the normalization is not exact but it is closely related. This gives the final counting statistic

$$g^{(2)}_{F:1 \rightarrow 1}(\tau) = 2 \left( e^{-\Gamma \tau} - 1 \right)^2 \cos^2(J\tau)$$  \hspace{1cm} (7.3)
The same logic can be applied to the situation when a jump from the first site starts the clock measuring $\tau$ and a jump from the second site stops the clock. Again, the collapsed state is the atom in the ground state of the first site. The difference now is that the unnormalized counting statistic is now proportional to the probability of being in the excited state of the second site since that is where we observe the second jump. Then

$$g_{F:1\rightarrow2}(\tau) = 2\left(e^{-\Gamma\tau} - 1\right)^2 \sin^2(J\tau) \quad (7.4)$$

We can also look at the correlation functions when we start the clock at the second site, which would switch the sines and cosines.

$$g_{F:2\rightarrow2}(\tau) = 2\left(e^{-\Gamma\tau} - 1\right)^2 \sin^2(J\tau) \quad (7.5)$$

$$g_{F:2\rightarrow1}(\tau) = 2\left(e^{-\Gamma\tau} - 1\right)^2 \cos^2(J\tau) \quad (7.6)$$

Similarly we can write the theoretical equation for $g^{(2)}(\tau)$ when we do not note which well the particle comes from. This is equal to the probability is the sum of the probability of the atom in the excited state of either site

$$g^{(2)}(\tau) = |C_{e10}\tau| + |C_{e01}\tau| = \left(e^{-\Gamma\tau} - 1\right)^2 \quad (7.7)$$

The first jump collapses the state to $C_{g10} = C_{g01} = 1/\sqrt{2}$ which are the initial conditions for the excited state. We note that in this equation there is no longer an oscillation term dependent on the tunneling constant $J$.

To find the theoretical $g^{(2)}(\tau)$ we turn to [5] which gives a full derivation of the function in a bad cavity, weak field limit where $g/\kappa << 1$

$$g^{(2)}(\tau) = \left(1 - 4C^2e^{-\frac{2}{\kappa}(1+2C)\mu}\right)^2 \quad (7.8)$$

This equation does not apply exactly to our system since here $g/\kappa = 1$. However, the equation is only dependent on $C$ and $\gamma$. While $C = g^2/\kappa\gamma$ there is no limit on $g/\gamma$ and therefore $C$ can have any value. Therefore, the equation should give a relatively close approximation to our numerical results. Again there is no dependence the tunneling constant $J$.

We can now combine our theoretical results with our numerical data. We do not expect exact matches because our models have $Y = 0.4$ which does not correspond to the weak field limit in which the theoretical equations were derived. Also the models include effects from other jumps which we have not treated theoretically. We compare the theoretical graphs with the numerical method since they give the most jumps. For
this sample $J = 1$ and all other variables are the same. We also expand the number of intervals to $2 \times 10^9$. This now gives $\sim 10^5$ values of $\tau$ for the correlation between the sites and about four times more for the correlation between either site and the transmission graphs.

Figure 7.7: Comparison of theoretical and numerical correlations when $J = 1$ for (a) photons fluoresced from a single cavity (b) two cavities
Figure 7.8: Comparison of theoretical and numerical correlations when $J = 1$ for (a) photons fluoresced from either cavity (b) transmitted through the front of the cavity.
The theoretical model matches well in figure (7.7a), however, fails to match the observed trend in (b). For figure (7.7a) we see again the period is $\sim \pi$ and now by comparison to the theoretical function of $g^{(2)}(\tau)$ we can verify that it in fact should be $\pi$ since the constant is $J = 1$ and the oscillation is from the $\cos^2$ term. This matches our observations in the previous section of the numerical data, which showed the period was inversely dependent to the tunneling constant. Now we can state that $\lambda = \pi/J$. The oscillations in the model begin to dissipate for higher $\tau$ due to other collapses inbetween making longer correlation times less likely to oscillate. The same oscillations are seen in figure (7.7b). These do not follow the same trend as the previous graph for reasons we do not fully understand. A likely explanation is related to having less values of $\tau$ meaning that there may be more jumps inbetween the collapse the states.

The other correlations, figure (7.8), also roughly follow our theoretical expectations. The normalization is altered to reflect since for $\tau < 2$ the value of $g^{(2)}(\tau)$ varies from $\tau > 2$ which is the steady state solution. Therefore, we ignore the $\tau$ counts $< 2$ and divide by the counts for $\tau > 2$. For figure (7.8a) the theory matches with some small deviations around $\tau = 1$ due to the new normalization scheme. Figure (7.8b) matches except for very small values of $\tau$ again due to the new normalization. Both match much better because there on the order of 10 more values of $\tau$ than the single site correlations.

### 7.4 Analysis

Theoretical solutions of $g^{(2)}(\tau)$ and the numerical models agree that the correlation between photon fluorescence when measured between sites is proportional to the tunneling rate. Looking at the problem qualitatively this agrees with what we would expect. The wave function will only be collapsed to that site with a jump for a set amount of time. After that, it will collapse to the other site. If we say the atom emits a photon from site $a$ at $t = 0$ then the wave function of the atom is collapsed to the ground state in $a$. The atom, however, will not stay in $a$ much longer since its wave function will now evolve making it more likely to be in the other site $b$. Now, there is a chance the atom will experience a collapse again, at time $t = \tau$, in $b$ but it could also move back to $a$. During this time the probability of detecting a jump from site $a$ is zero or very small. However, once the atom tunnels back to $a$ the probability increases. This is why there is an oscillation in $g^{(2)}(\tau)$. After the first collapse at $t = 0$ the atom will leave $a$ (small values of $\tau$) meaning there is little chance of a second jump. The atom can travel back and forth many times before the second jump. If we are looking at the correlation between photon counts from a single well then when the atom is in site $b$ the probability of a count is very small. Therefore, the oscillation is driven by the tunneling between sites since there are no counts possible at times when the atom is in the other site.
The other plots, of the correlation between photons fluoresced from either site and the correlation of photons transmitted, verify the results of the trajectory model. If we do not note which site the fluoresced photons come from we are ignoring the optical lattice and the tunneling term in the Hamiltonian which leaves us with only the trajectory terms that have been previously investigated in [5] and [4]. The lattice does not affect the transmitted photons since the lattice is created on the same axis that the transmitted photons are emitted from. Therefore, we cannot tell which site the photon comes. Neither of the theoretical or numerical plots show any dependence on the tunneling constant $J$.

These tests show that in fact our theoretical and numerical models do follow the expected trends and verify that $g^{(2)}(\tau)$ can be used to determine the tunneling constant. Since we know that the model follows the behavior predicted earlier we can continue to expand the model by adding more atoms making the system into a BEC.

8 Conclusions and Further Work

We verified the dependence of $g^{(2)}(\tau)$ on $J$ for the single site fluorescence correlations and independence of $J$ for either site and transmission correlations. The numerical models we made match our theoretical values for the correlation function for a single site, however, diverge when looking at the correlation between two sites. They also are not exactly match when looking at fluorescence from either site or transmission. Some of these patterns can be explained with our choice of $g$, $\kappa$, and $\gamma$ especially with the transmission correlation. However, there may need to be a better method developed to normalize the jump counts. This is likely causing the difference in the correlation from either site and likely between two sites. Larger sample sizes may also help since after $\tau = 20$ the counting statistic becomes prone to noise from the random number generator. Larger sample sizes would reduce the fraction of the noise.

We have also established methods to model more complicated multi-site and multi-atom systems numerically. A larger simulation with three sites and two particles was run but since there are more states (12 with the bad cavity limit, 36 without) it takes much longer, see Appendix A. In the plots developed there were not recognizable oscillations dependent on $J$. Further work is needed to speed up the programs to find the jump times. It is also unlikely that a theoretical solution for $g^{(2)}(\tau)$ can be found since the dimensions of multi-site and multi-atom models can grow quickly. Therefore, future work will likely be dependent on numerical results.

There is also the possibility of adding quantized center of mass motion, which is outlined for a single atom in [9] and [12]. This means that the atoms in the lattice site could move from the ground state of each site. This is a numerical method to study
the superfluid to Mott insulator phase transition. Since quantum noise causes the phase transition we expect to see a spike in the correlation functions around that point. There is also theoretical work to be done analyzing the mean field approximation of the Bose-Hubbard Hamiltonian and the effects when the two-component Hamiltonian is included. However, the immediate next step is to run three site two atom programs for large time scales with different values of $g$, $\kappa$ and $\gamma$. It is likely that the correlation functions will oscillate at much more complex frequencies than in the two sites, one atom models. However, as established in this paper, there should be some type of oscillations based on $J$ that will be observable with large enough times.
A Amplitude rates for three site, two atom setup

For the notation, \(^nC_{xyz}^{g,ie}\), \(n\) is the number of photons, \(x, y, z\) are the number of atoms in each site, and \(g, ie\) are the number of atoms in the excited state where \(g = 0\) atoms excited, \(i = 1, 2\) atoms excited.

Ground state

\[
\begin{align*}
0\dot{C}_{101}^g &= iJ(0C_{110}^g + 0C_{011}^g), \quad (A.1) \\
0\dot{C}_{110}^g &= iJ(0C_{101}^g + \sqrt{2}0C_{200}^g + \sqrt{2}0C_{020}^g), \quad (A.2) \\
0\dot{C}_{011}^g &= iJ(0C_{110}^g + \sqrt{2}0C_{002}^g + \sqrt{2}0C_{020}^g), \quad (A.3) \\
0\dot{C}_{200}^g &= iJ0C_{110}^g - ihU0C_{200}^g, \quad (A.4) \\
0\dot{C}_{020}^g &= iJ\sqrt{2}(0C_{110}^g + 0C_{011}^g) - ihU0C_{020}^g, \quad (A.5) \\
0\dot{C}_{002}^g &= iJ\sqrt{2}0C_{011}^g - ihU0C_{002}^g, \quad (A.6)
\end{align*}
\]

1st excited states

\[
\begin{align*}
1\dot{C}_{101}^g &= iJ(1C_{110}^g + 1C_{011}^g) + Y^00C_{101}^g + g\sqrt{2}0C_{110}^{le} - \kappa 1C_{101}^{le}, \quad (A.7) \\
1\dot{C}_{110}^g &= iJ(1C_{101}^g + \sqrt{2}1C_{200}^g + \sqrt{2}1C_{020}^g) + Y^00C_{110}^g + g\sqrt{2}0C_{110}^{le} - \kappa 1C_{110}^{le}, \quad (A.8) \\
1\dot{C}_{011}^g &= iJ(1C_{110}^g + \sqrt{2}1C_{002}^g + \sqrt{2}1C_{020}^g) + Y^00C_{011}^g + g\sqrt{2}0C_{011}^{le} - \kappa 1C_{011}^{le}, \quad (A.9) \\
1\dot{C}_{200}^g &= iJ1C_{110}^g - ihU1C_{200}^g + Y^00C_{200}^g + g\sqrt{2}0C_{200}^{le} - \kappa 1C_{200}^{le}, \quad (A.10) \\
1\dot{C}_{020}^g &= iJ\sqrt{2}(1C_{110}^g + 1C_{011}^g) - ihU1C_{020}^g + Y^00C_{020}^g + g\sqrt{2}0C_{020}^{le} - \kappa 1C_{020}^{le}, \quad (A.11) \\
1\dot{C}_{002}^g &= iJ\sqrt{2}1C_{011}^g - ihU1C_{002}^g + Y^00C_{002}^g + g\sqrt{2}0C_{002}^{le} - \kappa 1C_{002}^{le}, \quad (A.12)
\end{align*}
\]
2nd excited states

\[ \begin{align*}
\dot{\gamma}_{101} & = iJ(2\gamma_{110} + 2\gamma_{011}) + Y\sqrt{3}1\gamma_{101} + 2y^{1}C_{101}^{1e} - 2\gamma^{2}C_{101}^{0e} \\
\dot{\gamma}_{110} & = iJ(2\gamma_{101} + \sqrt{2}2\gamma_{020} + \sqrt{2}2\gamma_{020}) + Y\sqrt{2}1\gamma_{110} + 2y^{1}C_{110}^{1e} - 2\gamma^{2}C_{110}^{0e} \\
\dot{\gamma}_{111} & = iJ(2\gamma_{110} + \sqrt{2}2\gamma_{002} + \sqrt{2}2\gamma_{020}) + Y\sqrt{2}1\gamma_{111} + 2y^{1}C_{111}^{1e} - 2\gamma^{2}C_{111}^{0e} \\
\dot{\gamma}_{200} & = iJ2\gamma_{110} - ihU^{2}2\gamma_{020} + Y\sqrt{2}1\gamma_{200} + 2y^{1}C_{200}^{1e} - 2\gamma^{2}C_{200}^{0e} \\
\dot{\gamma}_{202} & = iJ2\gamma_{110} + 2\gamma_{011} - ihU^{2}2\gamma_{020} + Y\sqrt{2}1\gamma_{202} + 2y^{1}C_{202}^{1e} - 2\gamma^{2}C_{202}^{0e} \\
\dot{\gamma}_{200}^{2} & = iJ2\gamma_{110} - ihU^{2}2\gamma_{020} + Y\sqrt{2}1\gamma_{200}^{2} + 2y^{1}C_{200}^{1e} - 2\gamma^{2}C_{200}^{0e}
\end{align*} \]
References


