TRIPARTITE ENTANGLEMENT IN QUANTUM OPEN SYSTEMS

by Habtom G. Woldekrastos

We investigate entanglement in an open quantum system, specifically in bipartite and tripartite systems of two dimensions for weak and strong fields based on numerical calculations. We address a systematic approach in calculating a tripartite entanglement using concurrence as a measure of entanglement based on direct detection scheme (quantum jump method - with and without knowledge of which qubit jumps). We are also numerically calculating entanglement using 'Adiabatic Elimination' and the 'Three-way tangle' or 'Residual Entanglement' approaches. We present significant differences of the concurrences of the various bipartite splits among the results of two ways of direct detection scheme and when a phase is introduced in the coupling term of the energy.
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1.1 Quantum Entanglement - Background

Quantum entanglement was first identified as a strange quantum phenomenon and a special future of quantum correlations in 1935 in a paper by Einstein, Podolsky and Rosen (EPR) (1). This laid the foundations for entanglement to be theoretically recognized as a resource. Entanglement features the existence of global states of a composite system of more than two objects that can’t be written as a product of states of the individual objects. EPR used entanglement to try to assign values to physical quantities before they are measured. It was then Bell who showed that entanglement makes it impossible to know the values of the quantities prior to measurement.

In 1964, Bell accepted the EPR conclusion - a deterministic description of the world that regarded the quantum description of the physical reality as incomplete and non-working hypothesis. He, then, used the EPR idea to state the 'local hidden variable model' (LHVM) (2) based on assumptions that imposed constraints in the form of Bell inequalities in experimental bipartite systems. Later he showed that some entangled states violated the inequalities which imply that entanglement is the feature of quantum formalism that has no classical counterpart. Although many experimental results early showed violations of Bell inequalities, general theoretical results were obtained late at the beginning of the 1990s (3; 4).

Since Shrodinger had already related entanglement to the notion of ”knowledge”
in the quantum context by stating “the best possible knowledge of a whole system does not include the best possible knowledge of its subsystems” (5), it was made possible to formalize entanglement in terms of entropic inequalities based on the von Neumann entropy in the second half of the 1990s. Quantum states which violate these inequalities were entangled. Nevertheless, the physical interpretation of this signature of entanglement was ambiguous. Further investigation of entanglement in terms of conditional entropy resulted the capability of transmission of quantum information and thus its application in quantum communication between parties separated by macroscopic distances (6; 7).

Entanglement has important applications in many quantum processes that can’t be implemented based on classical formalism. The new fundamental concepts of quantum cryptography(8), quantum teleportation(9) and quantum dense coding(10) in information processing have been developed based on entangled quantum states in recent years. Even though the role of entanglement in quantum computing is not completely understood (11), it has been proposed to increase the speed of a quantum computer, which has a quantum bit (qubit) that can be in any superpositions of the logical values 0 and 1, exponentially faster than the speed of a classical computer, which is based on a classical bit of values 0 and 1.

Because entanglement has very complex structure and it’s fragile with respect to interactions with the environment, it’s not been generally easy to suitably detect, characterize, manipulate and quantify it. In 1996 Peres showed that, if a state is separable (state that’s not entangled), then the partial transpose of the whole system over one subsystem gives a legitimate state (12). Later it was realized that positive maps can be used as good detectors of entanglement. However, they can’t be measured in the laboratory because they are unphysical. But it was possible to go to physical measurable quantities with the application of Hermitian operators (13). This way of characterizing entanglement provides an important basis for the general theory of detection of entanglement.
Entanglement witnesses for operators detecting entanglement were first introduced by Terhal in 2000 that a violation of a Bell inequality can serve as a witness to entanglement (14). These witnesses are mathematical objects of geometric nature which separate entangled states from separable states. The concept of entanglement witness has been applied to different statistical quantum systems (15; 16; 17; 18) involving multipartite entanglement such as quantum cryptography (19), quantum optics (20; 21), condensed-matter nanophysics (22), bound entanglement (23), experimental realization of cluster states (24) and hidden non locality (25). Entanglement witnesses were also helpful in the estimation of entanglement content.

Entanglement has been experimentally realized in many quantum systems (26; 27; 28; 29; 30; 31; 32; 33) in general and in bipartite systems (34; 35; 36) in particular. This exciting field has currently attracted many researchers to investigate its physical realizations and to dig more into its applications.

A detailed review on quantum entanglement can be referred from (37).
1.2 What is Quantum Entanglement?

Entanglement simply put is the quantum correlation or linear dependency among states comprising of independent states of two or more objects prepared in some general state in which an outcome of a measurement on one object affects the outcome of all other objects when they are measured even when they are far apart from each other. This state of $n$ objects is the superposition of all states composing the independent states of each object. The whole system can be described by the state

$$|\Psi\rangle = \sum_{i,j,\ldots,k} C_{i,j,\ldots,k} |\psi^1_{i,j,\ldots,k}\psi^2_{i,j,\ldots,k}\psi^3_{i,j,\ldots,k}\cdots\psi^n_{i,j,\ldots,k}\rangle$$ (1.1)$$

$$\sum_{i,j,\ldots,k} C^2_{i,j,\ldots,k} = 1$$ (1.2)

where each independent state of each object can have $i, j, \cdots, k$ dimension. This general state $|\Psi\rangle$ is said to be an entangled state if more than one term in the sum of equation 1.1 exist and a measurement on one object affects the state of all other objects. To understand more clearly this intricate quantum phenomenon, it is necessary to consider simple quantum systems of two objects. A state $|\psi\rangle$ of two objects A and B, described by $|\phi\rangle$ and $|\varphi\rangle$ respectively, is said to be a product or separable state if $|\psi\rangle$ factorizes, i.e

$$|\psi\rangle = |\phi\rangle \otimes |\varphi\rangle \quad \exists |\phi\rangle, |\varphi\rangle.$$ (1.3)

Otherwise, if $|\psi\rangle$ is unfactorizable, it is said to be an entangled state, i.e

$$|\psi\rangle \neq |\phi\rangle \otimes |\varphi\rangle \quad \forall |\phi\rangle, |\varphi\rangle.$$ (1.4)

A good example is a Bell state of two independent states of two electrons A and B prepared as

$$|\psi\rangle = \alpha|ud\rangle + \beta|du\rangle$$ (1.5)

where $|ud\rangle$ and $|du\rangle$ refer to each electron’s spin up (say along $+z$ axis) and spin down (along $-z$ axis) polarizations and $\alpha^2$ is the probability that the spins of A and
B are up and down respectively. \( \beta^2 \) is the probability that spins of A and B are down and up respectively. The state is normalized so that the total probability \( \alpha^2 + \beta^2 \) is 1. The two electrons can then be taken apart in two casually disconnected different space-time regions. If the spin of A is measured and the outcome is 'up', then you instantly know the spin of B is 'down' without making a measurement on B. If the outcome of A is 'down', then the spin of B is 'up'. And the reverse is true, i.e if B is measured, then A would be known instantly without performing a measurement on it. Therefore, if both \( \alpha, \beta \neq 0 \), then the state is entangled. And for either \( \alpha = 0 \) or \( \beta = 0 \) the state is not entangled. So \( |du\rangle = |d\rangle \otimes |u\rangle \) and \( |ud\rangle = |u\rangle \otimes |d\rangle \) are product states.

### 1.3 Classical and Quantum Correlations

If we consider two classically correlated objects owned by Alice and Bob, then the state of each object is determined prior to separation and Bob’s measurement result is not affected by Alice’s measurement. Whereas if those two objects were like two spin-half particles which are correlated in the quantum formalism, then neither object has a well-defined state before measurement and Bob’s measurement result is affected by Alice’s measurement. The measurement results in a nonlocal collapse of the state vector of the whole system.

### 1.4 Pure and Mixed states

A density matrix \( \rho \) of a quantum system like the state vector contains information related to probability outcomes of measurements on the system. Depending on the amount of information that the density matrix contains, quantum states are generally categorized in to pure and mixed states. If the system is described by the state vector \( |\psi\rangle \) or the density matrix \( \rho = |\psi\rangle \langle \psi| \), then we have complete knowledge about the state of the whole system and thus it’s a pure state. In quantum open systems however, the interaction of the systems with the environment is inevitable and thus we have lack of certainty about knowledge of the state of the system. Such states
can’t be described by the state vector $|\psi\rangle$ and are called mixed states. For both states, the density matrix is the sum of all possible decompositions of $\rho = |\psi\rangle\langle\psi|$ with classical probabilities $p_i$ of being projected onto its eigenstates and is given by

$$\rho = \sum_{i} p_i |\psi_i\rangle\langle\psi_i|.$$  
(1.6)

1.5 Quantification of Entanglement

Some functions have been proposed to quantify the amount of entanglement of a state of a bipartite system and most of them assign the entanglement numbers ranging from 0 to 1, 1 given to a maximally entangled state and 0 to unentangled state. Therefore there is a degree of entanglement. Some states are weakly entangled while others are strongly entangled. These functions are derived in terms of a density matrix.

1.5.1 Entanglement of Formation

The most basic of all entanglement measures is the entanglement of formation $E$ (38). For a bipartite pure state of subsystems A and B of arbitrary dimensions, the entanglement $E$ is defined as the ‘Von Neumann Entropy’ (39; 40) of either of the two subsystems A and B after tracing over A or B

$$E = -Tr(\rho_A \log_2 \rho_A) = -Tr(\rho_B \log_2 \rho_B) = -\sum_{i} \lambda_A i \log_2 \lambda_A i = -\sum_{i} \lambda_B i \log_2 \lambda_B i$$  
(1.7)

where $\rho_A$ is the partial trace of $\rho$ over subsystem $B$, and $\rho_B$ has a similar meaning. $\lambda_A i$ and $\lambda_B i$ are the non-zero eigenvalues of $\rho_A$ and $\rho_B$ respectively. The entanglement of formation of the mixed state $\rho$ is then defined as the average entanglement of the pure states of the decomposition, minimized over all decompositions of $\rho$ (41)

$$E = \min \sum_{i} p_i E(\psi_i).$$  
(1.8)
1.5.2 The Impurity and I-Concurrence

Another measure of entanglement of pure state $|\psi\rangle$ of subsystems A and B of arbitrary dimension is the I-concurrence (42) which is given by

$$C = \sqrt{2(1 - Tr(\rho_A^2))} = \sqrt{2(1 - Tr(\rho_B^2))} \quad (1.9)$$

where $\rho_A = Tr_B(\rho)$ and $\rho_B = Tr_A(\rho)$.

If A is in mixture after tracing over B, then we have lost information about A that is contained in the correlation between A and B.

For a mixed state $\rho$ of two qubits, the concurrence $C(\rho)$ (40) is given by

$$C = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \quad (1.10)$$

where the $\lambda$s are the decreasingly ordered eigenvalues of the matrix $R \equiv \sqrt{\sqrt{\rho \tilde{\rho}} \sqrt{\rho}}$.

In other words, $\lambda$s are the square roots of the eigenvalues of the non-Hermitian matrix $\rho \tilde{\rho}$ and each of them is a non-negative real number. And $\tilde{\rho}$ is as a result of the spin-flip operation of $\rho$:

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y) \quad (1.11)$$

where $\sigma_y$ is the Pauli operator $\left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right)$. A function similar to concurrence used as an entanglement measure is the impurity ($I = \frac{1}{2} C^2$ for a bipartite system) given by

$$I = 1 - Tr(\rho_A^2). \quad (1.12)$$

For pure and mixed states $Tr(\rho) = 1$. But $Tr(\rho^2)$ is one for pure states and less than one for mixed states. A pure state corresponds to a value of $I = 0$ while $0 < I < 1$ is true for mixed states. If tracing over A makes B a mixed state, then some information of B is tied up in A and hence they are entangled.

Let’s now calculate the entanglement of a pure state $|\psi\rangle$ of a bipartite qubits A and B generally given by
\[ |\psi\rangle = C_{00}|00\rangle + C_{01}|01\rangle + C_{10}|10\rangle + C_{11}|11\rangle \quad (1.13) \]

and the density matrix \( \rho = |\psi\rangle \langle \psi | \) from which we can obtain

\[
\rho_B = Tr_A(\rho) = \langle 0_A|\rho|0_A \rangle + \langle 1_A|\rho|1_A \rangle 
= (C_{00}^2 + C_{10}^2)|0\rangle\langle 0| + (C_{00}C_{*01}^* + C_{10}C_{*11}^*)|0\rangle\langle 1| 
+ (C_{01}C_{*00}^* + C_{11}C_{*10}^*)|1\rangle\langle 0| + (C_{01}^2 + C_{11}^2)|1\rangle\langle 1| 
\quad (1.14)
\]

\[
Tr(\rho_B^2) = (C_{00}^2 + C_{10}^2)^2 + (C_{00}C_{*01}^* + C_{10}C_{*11}^*)^2 
+ (C_{01}C_{*00}^* + C_{11}C_{*10}^*)^2 
\quad (1.15)
\]

Therefore, the concurrence of this general state \( |\psi\rangle \) of bipartite qubits A and B is

\[
C = \sqrt{2(1 - Tr(\rho_A^2))} = \sqrt{2(1 - Tr(\rho_B^2))} = 2(C_{00}C_{11} - C_{10}C_{10}) \quad (1.19)
\]

For a product state, the concurrence \( C = 0 \) which implies

\[
C_{00}C_{11} - C_{10}C_{10} = 0 \quad (1.20)
\]

and it’s easy to tell that the state is not entangled from the probability amplitude coefficients.

Another example could be a pure state
\[ |\psi\rangle = C_{11}|11\rangle + C_{10}|10\rangle = |1\rangle(C_{11}|1\rangle + C_{10}|0\rangle \] of a bipartite qubits. The concurrence for this state is

\[
C = \sqrt{2(1 - (C_{10}^2 + C_{11}^2)^2)} \quad (1.21)
\]

and we know

\[
C_{10}^2 + C_{11}^2 = 1 \quad (1.22)
\]

Therefore, from equation 1.21 the concurrence \( C \) is zero and it’s a product state.
As a final example, let’s consider maximally entangled state of equation (1.5) where \( \alpha = C_{10}, \beta = C_{01} \) and \( C_{00} = C_{11} = 0 \) of the general state given by equation (1.13) with labels 1, 0 corresponding to the \( u, d \) respectively. The entanglement of formation for this state is

\[
E = -Tr(\rho_B \log_2 \rho_B) = -\alpha^2 \log_2(\alpha^2) - (1 - \alpha^2) \log_2(1 - \alpha^2). \tag{1.23}
\]

and the concurrence is

\[
C = 2\alpha \sqrt{(1 - \alpha^2)} \tag{1.24}
\]

Figure (1.1) shows the entanglement of formation, the concurrence and the impurity against \( \alpha^2 \) for the state of the spin of the two electrons given by equation (1.5). The entanglement of formation \( E \) is a monotonically increasing function of the concurrence \( C \) and hence of the impurity \( I \).

1.5.3 The Logarithmic Negativity

Another important measure of entanglement is the negativity (43). For a system of two qubits A and B described by the density matrix \( \rho \), the negativity \( N(\rho) \) is defined by

\[
N(\rho) = \frac{||\rho_A^T||_1 - 1}{2} \tag{1.25}
\]

where \( ||\rho_A^T||_1 \) is the trace norm of \( \rho_A \). It’s the sum of the singular values of \( \rho_A \). The singular values are the roots of the eigenvalues of \( \rho_A \rho_A^\dagger \). The negativity measures how negative the eigenvalues of the partial transpose density matrix are. According to the Peres criterion (44) of separability, density matrices with at least one negative eigenvalue of the partial transpose are entangled. This is sufficient for a 2 qubit and qubit - qutrit systems. However, for systems of higher dimensions the Peres Criterion is not fully valid since we can have states with all positive eigenvalues of the partial transpose (43; 44). Therefore, if Peres Criterion is violated, then certainly entanglement is present. But if it’s not violated, there may or may not be entanglement. One
can take the logarithmic negativity which quantifies the entanglement to some upper bound. The logarithmic negativity in simplified form is

\[ E_N(\rho) = \log_2 \|\rho_A^T\|_1 \]  

(1.26)

**Figure 1.1:** The entanglement of formation, concurrence and impurity plotted against \( \alpha^2 \) for the singlet state of two electrons. The state is maximally entangled and the purest when \( \alpha^2 = 0.5 \) and it’s not entangled for \( \alpha^2 = 0, \alpha^2 = 1 \).
1.6 Quantum Trajectory Theory (Direct Detection)

Quantum trajectory theory deals with the dynamical evolution of a quantum system based on stochastic Schrodinger and master equations. It can be easily manifested if we consider an open system of a photoemissive source specifically a single two-state atom with a ground state $|0\rangle$ and an excited state $|1\rangle$ corresponding to energies $E_0$ and $E_1$ respectively, interacting irreversibly with the environment in thermal equilibrium. Such a damped two-level atom is described by the following master equation:

$$\dot{\rho} = -\frac{i}{2}\omega_0 [\sigma_z, \rho] + \frac{\gamma}{2} (\bar{n} + 1) (2\sigma_- \rho \sigma_+ - \sigma_+ \rho \sigma_- - \rho \sigma_+ \sigma_-)$$

(1.27)

$$+ \frac{\gamma}{2} \bar{n} (2\sigma_+ \rho \sigma_- - \sigma_- \rho \sigma_+ - \rho \sigma_- \sigma_+).$$

(1.28)

For simplicity, we assume that the thermal photon number $\bar{n}$ is equal to zero; this is a valid approximation in many systems, particularly for optical transition and will not have any qualitative effect in this example. Therefore, equation 1.28 becomes

$$\dot{\rho} = -\frac{i}{2}\omega_0 [\sigma_z, \rho] + \frac{\gamma}{2} (2\sigma_- \rho \sigma_+ - \rho \sigma_+ \sigma_-).$$

(1.29)

The general wave function may be written as a pure state as

$$|\psi(t)\rangle = C_0(t)|0\rangle + C_1(t)|1\rangle$$

(1.30)

The dynamical evolution of this state $|\psi(t)\rangle$ is governed by the Schrodinger equation

$$\frac{d}{dt} |\psi(t)\rangle = \frac{1}{i\hbar} H |\psi(t)\rangle$$

(1.31)

with the non-Hermitian Hamiltonian

$$H = \frac{1}{2}\hbar \omega_0 \sigma_z - i\hbar \frac{\gamma}{2} \sigma_+ \sigma(-)$$

(1.32)

where the first term is the free atomic Hamiltonian with the inversion operator

$$\sigma_z = |1\rangle \langle 1| - |0\rangle \langle 0|$$

(1.33)
and the transition frequency, $\omega_0 = \frac{2}{\hbar} E_1 = -\frac{2}{\hbar} E_0$ provided that the zero energy is defined halfway between the states $|0\rangle$ and $|1\rangle$. And the second term is the interaction Hamiltonian with the so-called atomic transition (raising and lowering) operators

$$\sigma_+ = |1\rangle \langle 0|, \quad \sigma_- = |0\rangle \langle 1|$$

(1.34)

and $\gamma$ is the Einstein A coefficient which is the rate of spontaneous emission. The evolution generated by equation 1.31 is interrupted by jumps or collapses $|\psi\rangle \rightarrow \hat{C}|\psi\rangle$

(1.35)

(between the two discrete atomic states) first introduced in the Einstein rate equations (45). The collapse operator $\hat{C}$ is given by

$$\hat{C} = \sqrt{\gamma} \sigma_-.$$

(1.36)

Here we assume that the field radiated (photon) as a result of the jump can be detected all around the $4\pi$ solid angle by measuring the observable $\langle \sigma_+ \sigma_- \rangle$. The probability for a collapse to occur in the interval $(t, t + \Delta t)$ is given by

$$p(t) = (\gamma \Delta t) \langle \sigma_+ \sigma_- \rangle$$

(1.37)

provided that the wave function $|\psi(t)\rangle$ is normalized. The trajectory equations 1.31 and 1.32 can be solved analytically. If the initial state of the atom is simultaneously in both states $|0\rangle$ and $|1\rangle$ with probabilities $c_0^2$ and $c_1^2$ respectively, i.e

$$|\psi(0)\rangle = c_0(0)|0\rangle + c_1(0)|1\rangle$$

(1.38)

and from equation 1.30, we have

$$|\psi(t)\rangle = \hat{C}_0(t)|0\rangle + \hat{C}_1(t)|1\rangle.$$

(1.39)
Now substituting equations 1.39 and 1.32 into equation 1.31, and using the initial condition (1.38), we have the following solution of the normalized probability amplitudes.

\[ c_0(t) = \frac{c_0(0)}{\sqrt{|c_0(0)|^2 + |c_1(0)|^2 e^{-\gamma t}} e^{\frac{1}{2}i\omega_0 t}} \]  

(1.40)

\[ c_1(t) = \frac{c_1(0)e^{-\left(\frac{\gamma}{2}\right)t}}{\sqrt{|c_0(0)|^2 + |c_1(0)|^2 e^{-\gamma t}}} e^{\frac{1}{2}i\omega_0 t}. \]  

(1.41)

Substituting equations (1.40 and 1.41) into equation 1.30 yields the following solution of the state of the atom before it emits a photon, that’s before a collapse occurs:

\[ |\psi(t)\rangle = \frac{c_0(t)e^{\frac{1}{2}i\omega_0 t}|0\rangle + c_1(t)e^{-\left(\frac{\gamma}{2}\right)t}e^{\frac{1}{2}i\omega_0 t}|1\rangle}{\sqrt{|c_0(0)|^2 + |c_1(0)|^2 e^{-\gamma t}}} \]  

(1.42)

The probability for a collapse to occur in the time interval \((t, t + \Delta t)\) is given by

\[ p(t) = (\gamma\Delta t) \frac{|c_1(0)|^2 e^{-\gamma t}}{|c_0(0)|^2 + |c_1(0)|^2 e^{-\gamma t}}; \]  

(1.43)

for the atom initially in the excited state \((c_0(0) = 0)\) this probability is independent of time. And if the atom is initially in the ground state \((c_1(0) = 0)\), then the probability is 0. Since this single atom is undriven there can be one and only one collapse of the wave function in each trajectory, which is

\[ |\psi(t)\rangle = C_0(t)|0\rangle + C_1(t)|1\rangle \rightarrow |0\rangle; \]

(1.44)

after this collapse, the non unitary Shrodinger equation keeps the atom in the ground state forever. If \(c_0(0) \neq 0\), then this state approaches the ground state for times much longer than the lifetime \(\gamma^{-1}\). If a photon hasn’t been detected for many lifetimes, then it’s most likely that the atom actually began in the ground state, from which it couldn’t emit.

If the atom is prepared in the excited state \((c_0(0) = 0)\), then after some random time \(t\) \(c_1(t)\) switches to 0 and \(c_0(t) = 1\). The atom emits a quantum of light in a completely random fashion and this exhibition can be simulated in a computer by
generating a random number and compare it with the collapse probability given by equation 1.43. If this random number is less than the collapse probability, then a jump, detection of a photon, is recorded. If the random number is greater than the probability, then the state evolves according to equation 1.42. For a large number of emissions simulated, the number of emissions occurring in the time interval \((t, t + \Delta t)\) decays exponentially with respect to time.

The idea of this method of quantum trajectories using a direct detection scheme can be extended similarly to an optical damped cavity mode prepared in a Fock state. This method is applied in the following chapters for the investigation of entanglement in driven quantum systems.

### 1.7 Previous work on Entanglement in Open Systems

A steady state entanglement in an open cavity QED system has been investigated applying the entanglement of formation and the concurrence as measures of entanglement (46). This is done for a single two-level atom as in figure (1.2) with \(\gamma\) - a decay rate of the atom’s spontaneous emission into free space, a decay rate \(\kappa\) of the cavity field outside the mirror and with a rate of coherent atom-field coupling \(g\). When the system (atom and field mode) is driven weakly by a laser light with a constant field strength \(Y\), it reaches a steady pure state to a good approximation. Then most of the time we get nothing with no photon and the atom just sits in the ground state assuming initially the atom was in its ground state. But sometimes we get one photon or even two photons. Thus, for weak excitations, the atom sits in the ground state and no photon is present in the cavity mode. In other words, the coefficient of the ground state of the system is one. And the state of the system is then

\[
|\psi\rangle = |0g\rangle + F|1g\rangle + A|0e\rangle + E|1e\rangle + G|2g\rangle \tag{1.45}
\]
where $e$ stands for the exited state of the atom and $g$ for the ground state, and $0, 1$ is the photon number of the field mode. But if $|\psi\rangle$ is a product state, then

$$
|\psi\rangle = |\text{atom}\rangle \otimes |\text{field}\rangle 
$$

$$
= (C_e|e\rangle + |g\rangle) (|0\rangle + D_1|1\rangle + D_2|2\rangle) 
$$

$$
= |0g\rangle + C_e|e0\rangle + D_1|1g\rangle + C_eD_1|1e\rangle + D_2|2g\rangle 
$$

Knowledge of only the one excitation amplitudes is not sufficient to yield any information about the entanglement since we can have $F = D_1$ and $A = C_e$. For weak fields $D_2$ is exactly $G$. The entanglement is determined by the value of $E$. If we compare the equations (1.45) and (1.48) and get $E \neq C_eD_1$ or $FA$, then it’s entangled for certain. If they are equal, then we don’t know if it’s entangled because there could be higher terms. One of such an entangled state with higher terms is

$$
|\psi\rangle = |0g\rangle + \alpha|e0\rangle + \beta|1g\rangle + \alpha\beta|1e\rangle + \Gamma|2e\rangle
$$

where $\Gamma \ll \alpha, \beta$. For practical purposes, it’s not entangled since $\Gamma$ is too small, but it’s necessary to look at the whole thing. It may have some small entanglement. The concurrence given by 1.19 for the bipartite system can be used to determine analytically the amount of entanglement in the system. For such a system in a

Figure 1.2: A single two-level atom in optical cavity with weak driving field strength $Y$, spontaneous emission $\gamma$, coupling rate $g$ and decay rate $\kappa$. 

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steady state, an optimal value of the coupling rate can be analytically calculated that maximizes entanglement (concurrence) (46). This analytical investigation of entanglement is consistent with the numerical results of Nha and Carmichael (47).
CHAPTER 2
BIPARTITE ENTANGLEMENT IN ARBITRARY TWO QUBITS

2.1 The Model

For simplicity, we consider a system of two two-level atoms, A and B either of which can be in a ground or excited state, with one driven and the other undriven as shown in figure 2.1. Another system could be a two-state atom and a driven optical cavity mode but the results apply to any system of two arbitrary qubits. The general

Figure 2.1: Two two-state atoms model
state of such a system could be written as

\[ |\psi\rangle = C_{00}|00\rangle + C_{01}|01\rangle + C_{10}|10\rangle + C_{11}|11\rangle \]  \hspace{1cm} (2.1)

where the labels 0 and 1 refer to the ground and excited energy levels for each atom respectively. The Hamiltonian of the system is

\[ H = i\hbar g (\sigma^A_+ \sigma^-_ - \sigma^B_+ \sigma^-_A) + i\hbar Y (\sigma^A_+ - \sigma^A_-) - i\hbar \gamma_A \sigma^A_+ \sigma^-_ - i\hbar \gamma_B \sigma^B_+ \sigma^-_B \]  \hspace{1cm} (2.2)

where \( g \) is the coupling constant between the two qubits A and B, \( Y \) is the driving strength, \( \sigma_- \), \( \sigma_+ \) are the raising and lowering operators and the \( \gamma_s \) are the decay rates for each qubit.

The evolution of the state is governed by the Schrödinger equation

\[ \frac{d}{dt} |\psi(t)\rangle = -\frac{i}{\hbar} H |\psi(t)\rangle \]  \hspace{1cm} (2.3)

Substituting equation (1.13) into equation (2.3) leads to the following solutions of the probability amplitude rates.

\[ \dot{C}_{00} = -Y C_{10} \]  \hspace{1cm} (2.4)

\[ \dot{C}_{01} = -g C_{10} - Y C_{11} - \gamma_B C_{01} \]  \hspace{1cm} (2.5)

\[ \dot{C}_{10} = g C_{01} + Y C_{00} - \gamma_A C_{10} \]  \hspace{1cm} (2.6)

\[ \dot{C}_{11} = Y C_{01} - (\gamma_A + \gamma_B) C_{11}. \]  \hspace{1cm} (2.7)

### 2.2 Direct Detection Scheme

We can now use trajectories to simulate the evolution of the state of the system based on a 'direct detection scheme' (quantum jump method) (48) where we can have jumps (an atom decays to a ground state after spontaneously emitting a photon). When we encounter a jump, we know which atom decays. A jump measurement (detection of a photon), using the jump operators \( \sqrt{\gamma_A} \sigma^A_+ \) and \( \sqrt{\gamma_B} \sigma^B_- \), would collapse
the state $|\psi\rangle$ of the system to either $\sqrt{\gamma_A}(C_{10}|00\rangle + C_{11}|01\rangle)$ or to $\sqrt{\gamma_B}(C_{01}|00\rangle + C_{11}|10\rangle)$ corresponding to a jump of A or B with probabilities $P_A$ and $P_B$ respectively. These jump probabilities are

\[
P_A = \gamma_A \Delta t (C_{10}^2 + C_{11}^2) \tag{2.8}
\]
\[
P_B = \gamma_B \Delta t (C_{01}^2 + C_{11}^2). \tag{2.9}
\]

Initially at time $t = 0$ both qubits are in the ground state, i.e $C_{00} = 1$ and $C_{10} = C_{01} = C_{11} = 0$. The state $|\psi(0)\rangle$ evolves to $|\psi(t)\rangle$ without emission of a photon and both qubits didn’t jump. After a time step of $\Delta t$, either of two things can happen. The detector goes click indicating a jump of qubit A or B is detected or the state continue evolving according to Shrodinger’s equation (2.3). We are using a random number generator to determine whether there occurs a jump or not. When a jump is detected, we run another random number to know which qubit jumps. Another way to apply the direct detection method is where it’s not known which qubit jumps. In this case, only a single random number is generated and compared to the total probability as a result of the sum of the jump operators. In both cases, the state of the system is recorded in each time step, which is assumed to be less than the correlation time of the system, and is used to determine the value of the concurrence. A total time step of 10s ($10\frac{2}{(\gamma_A + \gamma_B)}$) is used. The steps of the simulation for the case with knowledge of which qubit jumps are depicted in figure (2.2).

2.3 Entanglement Measurement

The average value of the concurrence, as derived in equation 1.19, is measured and used as the amount of entanglement in the system.

2.4 Numerical Results and Analysis

Figure 2.3a shows the result of the simulation for the measurement of the average value of the concurrence each time step until it reaches a steady state. A contour plot of the results of the concurrence is illustrated in figure 2.3b for different values of the
Simulation Steps of Evolution of Probability Amplitudes

1. Start at $t = 0$; $C_{00} = 1$, $C_{01} = C_{10} = C_{11} = 0$;
2. At time $t$ new $C_{nm} = C_{nm}(t) + C_{nm}(0)\Delta t$ (assuming no jumps at first)

- Calculate Jump Probabilities
  
  $P_A = (C_{10} + C_{11})\gamma_A \Delta t$
  
  $P_B = (C_{01} + C_{11})\gamma_B \Delta t$

- Throw a random number $r = [0, 1]$
  
  $r < (P_A + P_B)$
  
  $r \geq (P_A + P_B)$

- Throw another random number $s = [0, 1]$
  
  $s < P_A$
  
  $s \geq P_A$

- new $C_{00} = C_{00}$; new $C_{01} = C_{01}$;
  
  new $C_{10} = C_{11}$; new $C_{11} = 0$

- Normalize and Record Probability Amplitudes
  
  $t = t + \Delta t$

**Figure 2.2:** Simulation of evolution of probability amplitude
driving field strength $Y$ and the coupling constant $g$. As one can infer logically, the entanglement is small for very small values of the coupling between the two qubits for both weak and strong fields. For weak driving fields, there is an optimal value of the coupling constant $g$ that always maximizes the entanglement and is consistent with the analytical and numerical results already done before (46; 47). For strong fields, however, the two qubits are strongly entangled and this entanglement stays fairly the same for a wide range of the coupling rate. The results are the same for both cases where we have knowledge of which qubit jumps or we don’t know which qubit jumps. This is because we have only one definition of entanglement and it doesn’t matter whether to know or not which qubit jumps.
Figure 2.3: a) A steady state plot of the concurrence $C$ Vs time $t$ for $g = 0.6$ and $\gamma = 1$. b) The concurrence $C$ for strong and weak driving fields.
CHAPTER 3
TRIPARTITE ENTANGLEMENT IN ARBITRARY THREE QUBITS

3.1 The Model

A system of three arbitrary qubits (which may be two-level atoms) which may be entangled to each other is considered as shown in the figure 3.1 below. A general

![Diagram of three qubits model]

**Figure 3.1:** Three qubits model
pure state for the three-qubit system is given by

\[ |\psi\rangle = C_{000}|000\rangle + C_{001}|001\rangle + C_{010}|010\rangle + C_{011}|011\rangle + C_{100}|100\rangle \]

\[ + C_{101}|101\rangle + C_{110}|110\rangle + C_{111}|111\rangle \]

and the Hamiltonian of the system is

\[ H = \hbar g_{12}(\sigma^1_+ \sigma^2_- + \sigma^2_+ \sigma^-_2) + \hbar g_{13}(\sigma^1_+ \sigma^3_- + \sigma^3_+ \sigma^-_1) + \hbar g_{23}(\sigma^2_+ \sigma^3_- + \sigma^3_+ \sigma^-_2) \]

\[ + \hbar Y(\sigma^1_+ - \sigma^1_-) - \hbar \gamma_1 \sigma^1_+ \sigma^-_1 - \hbar \gamma_2 \sigma^2_+ \sigma^-_2 - \hbar \gamma_3 \sigma^3_+ \sigma^-_3. \]

The state of the system evolves according to the Shrodinger’s equation (2.3) and we get the following equations of the probability amplitude rates:

\[ \dot{C}_{000} = -YC_{100} \]

\[ \dot{C}_{001} = -\imath g_{13}C_{100} - \imath g_{23}C_{010} - YC_{101} - \gamma_3 C_{001} \]

\[ \dot{C}_{010} = -\imath g_{12}C_{100} - \imath g_{23}C_{001} - YC_{110} - \gamma_2 C_{010} \]

\[ \dot{C}_{011} = -\imath g_{12}C_{101} - \imath g_{13}C_{110} - YC_{111} - (\gamma_2 + \gamma_3)C_{011} \]

\[ \dot{C}_{100} = -\imath g_{12}C_{010} - \imath g_{13}C_{001} + YC_{000} - \gamma_1 C_{100} \]

\[ \dot{C}_{101} = -\imath g_{12}C_{011} - \imath g_{23}C_{110} + YC_{001} - (\gamma_1 + \gamma_3)C_{101} \]

\[ \dot{C}_{110} = \imath g_{13}C_{011} - \imath g_{23}C_{101} + YC_{010} - (\gamma_1 + \gamma_2)C_{110} \]

\[ \dot{C}_{111} = YC_{011} - (\gamma_1 + \gamma_2 + \gamma_3)C_{111} \]

### 3.2 Bipartite Splits (Systematic Approach)

These three qubits 1,2 and 3 may be entangled with each other based on the values of the parameters in equation (3.3). We calculated the entanglement between various bipartite splits specifically the concurrence between the state of one qubit and the state of the other two qubits. The bipartite splits are shown in figure (3.2).
Figure 3.2: Bipartite splits between the three qubits Q1, Q2 and Q3.

3.3 Direct Detection Scheme

The state initially prepared is the ground state of the system. That is

\[ |\psi(0)\rangle = |000\rangle \]  

(3.13)

where \( C_{000} = 1 \) and all the other coefficients are zero. The state then starts to evolve by equations (3.5 - 3.12) followed by collapses or jumps in completely random fashion by the jump operators \( \sqrt{\gamma_1}\sigma_1^- \), \( \sqrt{\gamma_2}\sigma_2^- \) and \( \sqrt{\gamma_3}\sigma_3^- \). The three jump probabilities are

\[
p_1(t) = (|C_{100}|^2 + |C_{101}|^2 + |C_{110}|^2 + |C_{111}|^2) \gamma_1 \Delta t
\]  

(3.14)

\[
p_2(t) = (|C_{010}|^2 + |C_{011}|^2 + |C_{110}|^2 + |C_{111}|^2) \gamma_2 \Delta t
\]  

(3.15)

\[
p_3(t) = (|C_{001}|^2 + |C_{011}|^2 + |C_{101}|^2 + |C_{111}|^2) \gamma_3 \Delta t
\]  

(3.16)

Each trajectory is simulated keeping a record of the state of the system over a total time step of 10s \((10\frac{3}{\gamma_1+\gamma_2+\gamma_3}, 10 \text{ times the life time})\) and the state of the system evolves by equations (3.5 - 3.12) or the evolution is interrupted by collapses each random time by one of the three jump operators. The steps followed here are similar to what is done for the two-qubit system. A random number is generated and compared with the total jump probability to know whether a jump is detected or not and another random number is thrown to exactly know which qubit jumps.
3.4 Entanglement Measures

We simultaneously measured the concurrences $C_{1,23}, C_{2,13}$ and $C_{3,12}$ based on the direct detection scheme over many trajectories until they get to a steady state as in figures 3.3, 3.4 and 3.5. For example, the concurrence between qubit 1 and the pair qubits (2,3) is

$$C_{1,23} = \sqrt{2(1 - tr(\rho_1^2))}$$

(3.17)

where $\rho_1 = tr_{23}(\rho_{123})$, i.e tracing over qubit 1 of the density matrix of the system, $\rho_{123}$, gives us the entanglement between that qubit and the pair of qubits 2 and 3. And similarly

$$C_{2,13} = \sqrt{2(1 - tr(\rho_2^2))}$$

(3.18)

$$C_{3,12} = \sqrt{2(1 - tr(\rho_3^2))}.$$  

(3.19)

Using the concurrence as a measure of entanglement, we plotted the results of the calculations for weak and strong fields for a wide range of a weak and strong values of the coupling rates (See figures 3.10, 3.11, 3.12, 3.13 and 3.14).
Figure 3.3: A plot of concurrences, $C_{1,23}, C_{2,13}$ and $C_{3,12}$, calculated with knowledge of which qubit jumps. $Y = 1$. 

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Figure 3.4: A plot of concurrences, $C_{1,23}, C_{2,13}$ and $C_{3,12}$, calculated without knowledge of which qubit jumps. $Y = 1$. 

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Figure 3.5: A plot of concurrences, $C_{1,23}$, $C_{2,13}$ and $C_{3,12}$, calculated with knowledge of which qubit jumps and $g \rightarrow ig$. $Y = 1$. 

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3.5 Residual Entanglement (The Three-Tangle)

When these three qubits are entangled with each other, not any two of them can be fully entangled with one another unlike classical correlations where it can be shared freely. Qubit 1’s entanglement with qubit 2 and its entanglement with qubit 3 is limited and is given by the inequality (49)

$$C_{12}^2 + C_{13}^2 \leq C_{1,23}^2$$ (3.20)

where $C_{12}$ is the concurrence of the mixed state of qubits 1 and 2 given by equation 1.10, and similarly $C_{13}$ is of qubits 1 and 3. The equality is satisfied if we consider the W state

$$|\psi\rangle = \alpha|100\rangle + \beta|010\rangle + \gamma|001\rangle$$ (3.21)

where we can find $C_{12} = 2|\alpha\beta|$, $C_{13} = 2|\alpha\gamma|$, and $C_{1,23} = 2|\alpha\sqrt{|\beta|^2 + |\gamma|^2}|$. The residual entanglement, $\tau_{123}$, is expressed by

$$C_{1,23}^2 - C_{12}^2 - C_{13}^2 = 4|d_1 - 2d_2 + 4d_3| = \tau_{123}$$ (3.22)

where

$$d_1 = C_{000}^2C_{111}^2 + C_{001}^2C_{110}^2 + C_{010}^2C_{101}^2 + C_{100}^2C_{011}^2$$ (3.23)
$$d_2 = C_{000}C_{111}C_{011}C_{100} + C_{000}C_{111}C_{010} + C_{000}C_{111}C_{110}C_{001} + C_{011}C_{100}C_{010} + C_{011}C_{110}C_{001} + C_{101}C_{010}C_{110}C_{001}$$ (3.24)
$$d_3 = C_{000}C_{110}C_{101}C_{011} + C_{111}C_{001}C_{010}C_{100}$$ (3.25)

This residual entanglement is invariant under permutation of the three qubits, i.e $\tau_{123} = \tau_{231} = \tau_{312}$ to mention some. We can consider, for example the Greenberger-Horne-Zeilinger state $(1/\sqrt{2})(|000\rangle + |111\rangle)$ (50). One can easily show that for this state, $\tau_{123} = 1$ and for the W state equation 3.21 $\tau_{123} = 0$.

For the three-qubit state given by equation 3.1, we measured $\tau_{123}$ using the direct detection method over many trajectories as shown in figures 3.6, 3.7 and 3.8. All
results are consistent with the configuration of the qubits. For instance, the residual entanglement is zero for the bipartite configurations and varies for tripartite configurations according to the values of all the other parameters.

Figure 3.6: The three-tangle calculated with knowledge of which qubit jumps. \( Y = 1; \gamma_1 = \gamma_2 = \gamma_3 = 1 \).

3.6 Adiabatic Elimination

This is a different approach to measuring the entanglement of a mixed state of two qubits since there’s no good or convenient way to do it. When the third qubit decays adiabatically, it can be eliminated from the system and one can look at the
Figure 3.7: The three-tangle calculated with knowledge of which qubit jumps and with substitution $g \rightarrow ig$. $Y = 1; \gamma_1 = \gamma_2 = \gamma_3 = 1$. 
Figure 3.8: The three-tangle calculated without knowledge of which qubit jumps. \( Y = 1; \gamma_1 = \gamma_2 = \gamma_3 = 1. \)
entanglement of the other two qubits. As can be inferred from figure 3.9, the entanglement of the third qubit falls off with increasing its decay rate and leaves the other qubits in a mixed state entanglement with some loss of information.

![Graph showing concurrences](image)

**Figure 3.9:** Third qubit decays adiabatically carrying off some information which is absent in the mixed state of the other two qubits. $Y = 1; g_{12} = g_{23} = g_{13} = 1; \gamma_1 = \gamma_2 = 1.$

### 3.7 Numerical Results and Analysis

We calculated the concurrences for the case with and without knowledge of which qubit jumps and the results are shown in figures 3.3 and 3.4. Figure 3.5 also shows the concurrences for a special case where there is a phase difference between the coupling terms in the Hamiltonian equation 3.3 under the substitution of $(g_{12}, g_{23}$ and $g_{13})$ by $(i g_{12}, i g_{23}$ and $i g_{13})$. As we can see as in figures 3.3a and 3.3c, the results for
the entanglement are consistent with that of the bipartite entanglement of two qubits previously measured as in figure 2.3a. The concurrences in figures 3.3 and 3.4 are symmetric about the configuration of the qubits. But important differences can be observed in the cases with and without knowledge of which qubit jumps and in the case when a phase is introduced in the Hamiltonian. The concurrences in figure 3.4 are slightly greater than the corresponding ones in figure 3.3 for the same values of all the parameters. For the case with knowledge of which qubit jumps, the concurrences, \( C_{1,23} \) and \( C_{3,12} \), are distinguishable as in figure 3.3b, but for the case without knowledge of which qubit jumps the corresponding values are indistinguishable as in figure 3.4b. The same differences between such values of the bipartite entanglement are observed in figures 3.3d, 3.3e and in figures 3.4d, 3.4e respectively. Another important difference between two concurrences, \( C_{2,13} \) and \( C_{3,12} \), is noticed in the case with knowledge of which qubit jumps where all the coupling terms are non-zero and with the substitution of \( g \) by \( ig \) (see figure 3.5f). These two concurrences are not symmetric with respect to the configuration of the three qubits. So tracing over qubit 2 and tracing over qubit 3 result different values of the concurrence. In other words, the phase introduced in the coupling terms of the Hamiltonian matters and breaks the symmetry between the concurrences and the configuration of the qubits.

We have additional plots of all concurrences when the system is driven with weak and strong fields for a wide range of coupling values between any two of the qubits (See figures 3.10, 3.11, 3.12 and 3.13). For weak fields (Figures 3.10 and 3.11), there is a general decrease in the entanglement as the coupling increases. And for a strong driving fields (Figures 3.12 and 3.13), there stays strong for quite a wide range of coupling values and dies off as the coupling increases more. The last figure 3.14 depicts the concurrences plotted for strong and weak fields together.
Figure 3.10: A plot of concurrences $C_{1,23}, C_{2,13}$ and $C_{3,12}$ for relatively weak field and weak coupling $g_{12}$. 

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Figure 3.11: A plot of concurrences $C_{1,23}$, $C_{2,13}$ and $C_{3,12}$ for relatively weak field and strong coupling $g_{12}$. 

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Figure 3.12: A plot of concurrences $C_{1,23}$, $C_{2,13}$, and $C_{3,12}$ for strong field and weak coupling $g_{12}$.
Figure 3.13: A plot of concurrences $C_{1,23}, C_{2,13}$ and $C_{3,12}$ for strong field and strong coupling $g_{12}$. 
Figure 3.14: A plot of concurrences $C_{1,23}, C_{2,13}$ and $C_{3,12}$ for weak and strong fields.
CHAPTER 4

ENTANGLEMENT IN QUTRIT-QUDIT-QUDIT SYSTEM
(THREE-LEVEL ATOM AND TWO-MODE CAVITY)

4.1 The Model

For the bipartite system (single atom and one mode cavity) discussed above, practically there is a big problem. Collecting the fluorescent light is very hard and inefficient since we must have many detectors all the way around to collect it.

![Three-level Model](image)

**Figure 4.1:** Three-level Model

Therefore, we considered a three-level atom in a cavity as shown in figure 4.1 with one driven and another undriven mode fields. The cavity collects spontaneous emissions $\kappa_1$ and $\kappa_2$ from the atomic transition coupled to the driven mode and from
the undriven mode respectively. These two emissions then leak outside the mirror through one direction. We can then have two detectors looking at these two different polarizations. Such an experimental setup could be one as in figure 4.1.

![Experimental setup for the Three-level Model](image.png)

**Figure 4.2:** Experimental setup for the Three-level Model

When driven weakly, the state of the system keeping to two excitations is

\[ |\psi\rangle = \sum_{a,b,c} C_{a,b,c} |a, b, c\rangle \]  \hspace{1cm} (4.1)

where

\[
\begin{align*}
    a &= \text{atom level (1,2,3)} \\
    b &= \text{mode 1 photon number} \\
    c &= \text{mode 2 photon number}
\end{align*}
\]

and we used the following Hamiltonian

\[
H = \hbar Y (a - a^\dagger) + \hbar g (a \sum_{21} - a^\dagger \sum_{12}) + \hbar G (b \sum_{23} - b^\dagger \sum_{32}) - \hbar \frac{1}{2} (\gamma + \Gamma) \sum_{22} - \hbar \kappa_1 a^\dagger a - \hbar \kappa_2 b^\dagger b
\]  \hspace{1cm} (4.2)
where $\sum$ is the raising and lowering operators of the atom between the two different transitions with coupling rates $G$ and $g$ to the two modes. $\Gamma$ and $\gamma$ are the spontaneous emissions of the atom for both transitions. And $a^\dagger a$ and $b^\dagger b$ are the creation and annihilation operators for the driven mode and the non-driven mode respectively.

### 4.2 Steady State Entanglement for Bipartite Splits

We now wish to calculate the steady state entanglement for the bipartite splits and the three tangle hoping to understand the tripartite entanglement in physical sense. We start by finding the steady-state solutions for the probability amplitudes.

Taking the time derivative of equation (4.1) and applying the Shrodinger equation (2.3), we get the following general solutions.

\[
\dot{C}_{1,m,n} = Y\sqrt{n+1}C_{1,n+1,m} - Y\sqrt{n}C_{1,n-1,m} - (n\kappa_1 + m\kappa_2)C_{1,n,m} \quad (4.3)
\]
\[
\dot{C}_{2,m,n} = \frac{g}{\sqrt{n+1}}C_{2,n+1,m} + Y\sqrt{n+1}C_{2,n-1,m} - Y\sqrt{n}C_{2,n,m} - Y\sqrt{n}C_{2,n-1,m} - (n\kappa_1 + m\kappa_2 + \frac{\gamma + \Gamma}{2})C_{2,n,m} + G\sqrt{m+1}C_{3,n,m+1} \quad (4.5)
\]
\[
\dot{C}_{3,m,n} = C_{3,n,m-1} + Y\sqrt{n+1}C_{3,n+1,m} - Y\sqrt{n}C_{3,n,m} - (n\kappa_1 + m\kappa_2)C_{3,n,m} \quad (4.7)
\]
These solutions of the probability amplitudes within order $Y$ and $Y^2$ fields are

\begin{align*}
C_{100} &\approx 1, C_{101} = C_{300} = C_{310} = C_{320} = C_{111} = C_{302} = C_{102} = C_{201} \approx 0 \quad (4.9) \\
C_{110} &= -\frac{Y}{B} \quad (4.10) \\
C_{200} &= -\frac{gY}{AB} \quad (4.11) \\
C_{301} &= -\frac{GgY}{\kappa_bAB} \quad (4.12) \\
C_{210} &= \frac{1}{M} \left( \frac{g}{AB} + \frac{g}{\kappa_bB} + \frac{G^2g}{\kappa_bAB(\kappa_a + \gamma_t)} \right) Y^2 \quad (4.13) \\
C_{120} &= \frac{Y^2}{\sqrt{2}\kappa_bB} - \frac{gC_{210}}{\sqrt{2}\kappa_a} \quad (4.14) \\
C_{311} &= -\frac{GC_{210}}{\kappa_a + \kappa_b} - \frac{YC_{301}}{\kappa_a + \kappa_b} \quad (4.15)
\end{align*}

where

\begin{align*}
\gamma_t &= \frac{\Gamma + \gamma}{2} \\
A &= \gamma_t + \frac{G^2}{\kappa_b} \\
B &= \kappa_a + \frac{g^2}{A} \\
M &= \kappa_a + \gamma_t + \frac{g^2}{\kappa_a} + \frac{G^2}{\kappa_a + \kappa_b}
\end{align*}

The density matrix $\rho = |\psi\rangle\langle\psi|$ for this system is

\[
\rho = \begin{pmatrix}
Z^2 & ZC & ZD & ZE & ZF & ZG & ZH \\
ZC & C^2 & CD & CE & CF & CN & CH \\
ZD & CD & D^2 & DE & DF & DN & DH \\
ZE & EC & ED & E^2 & EF & EN & EH \\
ZF & FC & FD & FE & F^2 & FN & FH \\
ZN & NC & ND & NE & NF & N^2 & NH \\
ZH & HC & HD & HE & HF & HN & H^2
\end{pmatrix}
\]
where we labeled the probability amplitudes using constants as $Z = C_{100}, C = C_{110}, D = C_{120}, E = C_{200}, F = C_{210}, N = C_{301}, H = C_{311}$ and from the density matrix, we can calculate the reduced matrices of the atom, mode 1 and mode 2 tracing over the other two. Thus we have

$$\rho_{bc} = Tr_a(\rho) = \begin{pmatrix} Z^2 + E^2 & ZC + EF & ZD & 0 & 0 \\ ZC + EF & C^2 + F^2 & CD & 0 & 0 \\ ZD & CD & D^2 & 0 & 0 \\ 0 & 0 & 0 & N^2 & NH \\ 0 & 0 & 0 & NH & H^2 \end{pmatrix}$$

$$\rho_{ac} = Tr_b(\rho) = \begin{pmatrix} Z^2 + C^2 + D^2 & ZE + CF & ZN + CH \\ ZE + CF & E^2 + F^2 & EN + FH \\ ZN + CH & EN + FH & N^2 \end{pmatrix}$$

$$\rho_{ab} = Tr_c(\rho) = \begin{pmatrix} Z^2 & ZC & ZD & ZE & ZF & 0 & 0 \\ ZC & C^2 & CD & CE & CF & 0 & 0 \\ ZD & CD & D^2 & DE & DF & 0 & 0 \\ ZE & EC & ED & E^2 & EF & 0 & 0 \\ ZF & FC & FD & FE & F^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & N^2 & NH \\ 0 & 0 & 0 & 0 & 0 & HN & H^2 \end{pmatrix}$$

### 4.3 Numerical Results

We are now ready to calculate the entanglement of the bipartite splits using the concurrence. We calculated the concurrences $C_{a, bc}, C_{b, ac}$ and $C_{c, ab}$ as in figure (14).

$$C_{a, bc} = \sqrt{2(1 - Tr(\rho_{bc}^2))}; C_{b, ac} = \sqrt{2(1 - Tr(\rho_{ac}^2))}; C_{c, ab} = \sqrt{2(1 - Tr(\rho_{ab}^2))}$$
Figure 4.3: $Y = 0.05, G = \frac{g}{\gamma}, \gamma = 1, \Gamma = 0.5$. The entanglement goes away in all figures with increasing decay rate of the non driven mode.
Figure 4.4: A plot of concurrences $C_{a,bc}$, $C_{b,ac}$ and $C_{c,ab}$ for $g = 1$ (a,c,e) and $g = 5$ (b,d,f). $Y = 0.01$. 

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CHAPTER 5
CONCLUSIONS

We find that the numerical calculation of the concurrence, used as a measure of entanglement, in a bipartite system of two qubits are consistent with previous results of analytical work for weak fields. There is an optimal value of the coupling $g$ that always maximizes the entanglement. We also calculated the concurrence for strong fields and we find that the entanglement stays strong for a wide range of values of the coupling. We noted important differences of the concurrences of bipartite splits among two ways of direct detection measurement scheme (with knowledge and without knowledge of which qubit jumps) and when $g$ is substituted by $\nu g$ in the three-qubit system. Results of measurements of all concurrences including the residual entanglement (the Three-tangle) are consistent with the configuration of the qubits. Adiabatic Elimination can be used to measure a bipartite mixed-state entanglement with some loss of information. Further work remains to generalize this systematic approach of calculating entanglement using bipartite splits for large number of trajectories and wider range of values of the various system parameters and to investigate entanglement of higher-dimension open systems using a suitable measure of entanglement.
BIBLIOGRAPHY


