ABSTRACT

PHOTON STATISTICS AND FIELD-INTENSITY CORRELATION OF A CAVITY QED SYSTEM WITH EXTERNAL POTENTIALS

by Joseph R. Leach

In this work, we will look at the dynamics of a two-level atom in a damped, driven optical cavity in which atomic center-of-mass motion (CMM) is quantized. Of particular interest is the addition of an external potential (not related to the cavity field) in the cavity that effects the atom. Specifically, we are introducing to the system Hamiltonian a harmonic potential that effectively represents the presence of an optical lattice within the cavity. Thus, we are interested in the interplay of cold atoms and cavity QED. The form of our investigation of the system’s dynamics will be in the study of the photon statistics and field-intensity correlation of the transmitted and fluoresced fields.
PHOTON STATISTICS AND FIELD-INTENSITY CORRELATION OF A
CAVITY QED SYSTEM WITH EXTERNAL POTENTIALS

A Thesis

Submitted to the
Faculty of Miami University
in partial fulfillment of
the requirements for the degree of
Master of Science
Department of Physics

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2003

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I would like to thank my parents, who have provided me the opportunities to do all that I have enjoyed.

I would also like to thank Dr. Perry Rice, my advisor. I was fortunate enough to cross paths with Dr. Rice my sophomore year in college, my academic interests have not quite been the same since. Dr. Rice has introduced me to many things in physics through the years, and has always helped me to learn and to grow. Beyond physics, Perry is my friend, and I am glad to have met him. Thank you Perry.

I would like to give many thanks to Dr. T.W. Houk. Dr. Houk has been an inspiration to my physics career and life in general since meeting him in PHY 491. Dr. Houk has helped me to feel confident as a physicist, and has taught me volumes about all kinds of crazy things.

Lastly, I would like to thank Dr. Samir Bali for all his help during the past year. Dr. Bali has taught me a lot about what a physicist should be and what a physicist should know. Dr. Bali’s EM course was one of my favorites.

I would like to thank all three members of my thesis committee for not sleeping through my defense.
Quantum Optics deals with the quantum mechanical interactions of light and matter on the atomic scale. This field was essentially born in 1960 with the invention of the laser. Since the early 80’s, quantum optics has seen tremendous growth in the areas of laser cooling and trapping, and cavity QED. This ability to control atomic motion, and localize atoms has led to the development of many new types of experiments in atomic optics, from investigating the foundations of quantum mechanics to applications in optical and quantum computing.[1]

Many principle concepts and problems within the framework of quantum optics have to do with the changes in an atom/radiation field system when placed between two “mirrors” very close together, a cavity. For instance, when placed inside a cavity, the spontaneous decay rates for an atom, as well as the levels of its radiative transitions can be modified from their respective free space values. The subset of quantum optics that studies atom/cavity systems is referred to as Cavity Quantum Electrodynamics (cavity QED), and a typical system is shown in figure 1.1. [2]

Cavity QED allows us to explore the quantum mechanical nature of the atom-field interaction through many avenues. Measurements of the second order intensity correlation function of the transmitted field, like those done by Foster, Mielke, and Orozco [3], and Kimble,[4] show that the photon statistics of this field disobey inequalities that are bounds for any classical light source.
Figure 1.1: A typical cavity QED system. $Y$ is the strength of the driving field. $\gamma$ is the atom’s spontaneous emission rate. $g$ is the coupling rate between the atom and the cavity field mode, and $\kappa$ is the loss rate of the cavity. A transmission event is a cavity-photon loss through the cavity’s forward mirror, in the direction of the arrow labelled $\kappa$. A fluorescence is a photon loss from the atom out the side of the cavity in any direction; one possible direction is denoted by the arrow labelled $\gamma$. 
This non-classical behavior finds its root in the atom-field interactions within the cavity, and its magnitude can be explored as a function of detunings from various resonance conditions, atom-field coupling, and other system parameters [3]. Work done by Smith, Reiner et al. has shown that specific quantum states (of the atom and cavity field) can be “frozen” in their evolution, to be unfrozen at a later time, resuming their evolution as if no interruption had occurred[5]. This phenomenon was also realized by measurements of the second order intensity correlation function for the transmitted field. Recently, work has also been done to gain information from a cavity QED system in a new way; a new type of correlation function is being explored. $h_\theta(\tau)$ is the correlation between a photon detection and the conditional fluctuations in the electromagnetic field[6]. Through this new correlation function, the particle and wave nature of light are joined together in a previously unseen way.

Laser cooling slows the motion of an atom significantly, so that Doppler effects are minimal, allowing us to study a quantum optics system where resonance conditions are more easily achieved. Furthermore, an atom which is “cold” enough can be trapped using various methods, permitting the localization of the system under consideration. Cooling and trapping offers promising possibilities for the construction of certain components that will be essential to the field of quantum computation.

Cooled significantly, an atom (or trapped ion) exhibits quantized center of mass motion, dependent upon the characteristics of the trapping potential. Jessen et al. [7] used a ($\text{lin}_{\perp}\text{lin}$) laser configuration to create a 1-D optical molasses that results from the spatially varying polarization of the electric field. The potential wells that an atom sees can be treated harmonically, and allow the atom to populate quantized momentum states provided it hasn’t enough energy to “fly” out of the well. The depth of the wells, and the effective temperature are determined by the light shift produced by the spatially variant polarization. Quantized CMM was seen when spectra sidebands were present in the ($\text{lin}_{\perp}\text{lin}$) system, but not in a $\sigma_+ - \sigma_-$ configuration of similar temperature. The cause of the sidebands were the
quantized “vibronic” transitions of the atoms in the polarization wells.

It is in the following work that we will study the photon statistics and field-intensity correlation of a cavity QED system in which atomic wave-packet motion is treated quantum mechanically, and the atoms are subject to an external trapping potential such as that from a 1-D optical lattice. We will calculate the second order intensity correlation function $g^2(\tau)$ for both the transmitted (TT) and fluoresced fields (FF), as well as the cross-correlations between the two (TF and FT). We will also calculate the field-intensity correlations $h_{\theta TT}^T(\tau)$ and $h_{\theta FF}^F(\tau)$ for the cavity system.

### 1.1 Closed and Open Systems

Problems in elementary quantum mechanics typically involve closed systems consisting of entities that do not interact with the environment outside of themselves. Coupling to the environment involves energy exchange and/or dissipation to an external agent that is generally not of primary interest. In classical mechanics, many dissipative forces can often be ignored because they alter the dynamics of the system only very slightly. Although ignoring dissipation is also not uncommon in quantum mechanics, there are certain systems in which dissipation not only plays a crucial role in determining the system’s evolution, but actually defines the open system. This can be most readily seen in the laser; energy is pumped into the lasing cavity, where it is amplified, to eventually be “leaked” out of the lasing cavity. Were it not for dissipative processes, the laser would not work.

The state of a closed quantum system can be represented by a wavefunction or state vector $|\Psi(t)\rangle$ that resides in a Hilbert space. This state vector contains all the information on the pure state, or superpositions of pure states, of the system. The time evolution of the state vector (and therefore the system) is governed by the Schrödinger Equation [8],

$$i\hbar \dot{\psi}(t) = H \psi(t),$$

where $H$ is the hermitian Hamiltonian representing the total energy of the system.
In classical mechanics, the addition of a velocity dependent force \( \vec{F} = -\gamma \vec{v} = -\gamma \frac{\vec{v}}{m} \) can account for dissipation to the environment, as seen in a ballistics problem that accounts for air resistance. It can be shown [9] that when considering a quantized harmonic oscillator, characterized by the Hamiltonian

\[
H = \hat{p}^2 / 2m + \frac{1}{2} m \omega^2 \hat{q}^2,
\]  

(1.2)

we cannot simply add in such a force. The equations of motion associated with equation 1.2 after a damping force of \(-\gamma p\) is added are

\[
\dot{q} = \frac{p}{m}, \quad \dot{p} = -\gamma p - m \omega^2 q.
\]  

(1.3)

If we are to look at the time evolution of the commutator \([\hat{q}, \hat{p}]\), we see

\[
\frac{d}{dt} [\hat{q}, \hat{p}] = \hat{q} \dot{\hat{p}} + \hat{p} \dot{\hat{q}} - \dot{\hat{q}} \hat{p} - \dot{\hat{p}} \hat{q} = -\gamma [\hat{q}, \hat{p}],
\]  

(1.4)

which has the solution

\[
[\hat{q}(t), \hat{p}(t)] = e^{-\gamma t} [\hat{q}(0), \hat{p}(0)] = e^{-\gamma t} i \hbar.
\]  

(1.5)

This obviously cannot be the correct quantum mechanical treatment of dissipation, as the commutator of \(\hat{q}\) and \(\hat{p}\) would decay with time, making the lower bound of the uncertainty principle equal to 0. This is shown as

\[
\Delta q \Delta p \geq \frac{i}{2} |\langle [q, p] \rangle| \geq \frac{\hbar}{2} e^{-\gamma t}.
\]  

(1.7)

Because of problems like these in the quantum mechanical treatment of dissipation, the description of a typical quantum optics system that incorporates losses must be fundamentally different than the Hermitian Hamiltonian/state vector picture.

A common type of dissipation encountered in a quantum optics system is the coupling of a two level atom and the electromagnetic field surrounding it, which is nominally in the vacuum state. Here, our system is the two level atom, with the
environment being one or many modes of the electromagnetic field. If we are only concerned with the dynamics of the atom, we will need a formalism that allows us to know when an atom emits energy into the field, but we need not know which mode of the field the atom emitted into. When the atom is able to interact with all modes of the electromagnetic field, the system is allowed to be in a mixture of states governed by some probability distribution, rather than in some pure state. Thus, the state of the system must be represented by some “probabilistic state vector”. We call this state representation the reduced density matrix. Chapter 2 will discuss the reduced density matrix, as well as other methods of state representation. We will also discuss the methods by which an open quantum mechanical system can be studied. Chapter 3 will present the modelling of a cavity QED system in external potentials, where the atom’s center-of-mass-motion is quantized. In chapter 4, methods of determining and analyzing the photon statistics of a QED system are discussed. Focus is given to the CMM system, and results are presented. The field-intensity correlation function will be presented in chapter 5, with results for the CMM system. A summary of this work, and a description of possible future work are presented in chapter 6.
Chapter 2

Methods of investigation

2.1 Density matrix and reduced density matrix

Although quantum optics generally is not concerned with the specific dynamics of the full environment (we may not be interested in which field mode the atom has radiated into, we care only that the atom has radiated), it is instructive to first characterize a general density operator that represents the state of both the system we are concerned with, and the environment. The density operator for a mixed state is the sum of density operators for the possible pure states of the full system (atom + field) with a weight factor of $P_i$, the probability of being projected onto state $i$;

$$\chi = \sum_i P_i |\psi_i\rangle\langle\psi_i|.$$  \hspace{1cm} (2.1)

For a pure state:

$$\chi = |\psi\rangle\langle\psi|.$$ \hspace{1cm} (2.2)

This density operator has a matrix representation using the basis vectors of the full system, we call this the density matrix. If our system $S$ (the atom) and the reservoir $R$ (the EM field modes) are coupled together, they form a direct product space $S \otimes R$. Because the number of possible states for the system and the reservoir are different, we designate $S$ as an $N \times N$ space, and $R$ as a $M \times M$ space, with $N \ll M$. The density matrix for $S \otimes R$, $\chi(t)$, takes the form of an
\[ \chi(t) = \begin{pmatrix} R_{11} \rho(t) & R_{12} \rho(t) & R_{13} \rho(t) & \ldots & R_{1M} \rho(t) \\ R_{21} \rho(t) & R_{22} \rho(t) & R_{23} \rho(t) & \ldots & R_{2M} \rho(t) \\ R_{31} \rho(t) & R_{32} \rho(t) & R_{33} \rho(t) & \ldots & R_{3M} \rho(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{M1} \rho(t) & R_{M2} \rho(t) & R_{M3} \rho(t) & \ldots & R_{MM} \rho(t) \end{pmatrix}, \quad (2.3) \]

where \( R_{ii} \) is the probability of the reservoir being in state \( i \), and \( R_{ij} \) is a measure of the coherence between reservoir states \( i \) and \( j \). This form is true at \( t = 0 \), but at later times the reservoir and system can become entangled. The form of density matrix presented in equation 2.3 assumes that we are working in the Born approximation, this condition will be discussed further into the chapter and in Appendix A. \( \rho(t) \) is the reduced density matrix that is a representation of the statistical mixture of states that the system \( S \) is in. Following the general treatment, \( \rho(t) \) takes the form

\[ \rho(t) = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \ldots & \rho_{1N} \\ \rho_{21} & \rho_{22} & \rho_{23} & \ldots & \rho_{2N} \\ \rho_{31} & \rho_{32} & \rho_{33} & \ldots & \rho_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{N1} & \rho_{N2} & \rho_{N3} & \ldots & \rho_{NN} \end{pmatrix}, \quad (2.4) \]

where, similar to \( \chi(t) \), \( \rho_{ii} \) is the probability of the system \( S \) being in state \( i \), and \( \rho_{ij} \) is a measure of the coherence between system states \( i \) and \( j \). From equation 2.3, we see that the reduced density matrix is really just the trace of the density matrix over the reservoir states

\[ \rho(t) = Tr_R[\chi(t)] = \left[ \sum_i R_{ii} \right] \rho(t) = 1\rho(t), \quad (2.5) \]

and that the trace of the reduced density matrix evaluates as

\[ Tr_S[\rho(t)] = \sum_i \rho_{ii}(t) = 1. \quad (2.6) \]

For the system of a two level atom for example, the density operator acts upon
the energy basis states to produce the density matrix
\[ \rho(t) = \begin{pmatrix} \langle e | \rho | e \rangle & \langle e | \rho | g \rangle \\ \langle g | \rho | e \rangle & \langle g | \rho | g \rangle \end{pmatrix}. \] (2.7)

To calculate expectation values for operators within the basis of \( S \), we take the trace of the product of \( \chi(t) \) and the operator \( \hat{O} \):
\[ \langle \hat{O} \rangle = Tr_S \otimes_R [\hat{O} \chi(t)]. \] (2.8)

This reduces to
\[ Tr_S [Tr_R [\hat{O} \chi(t)]] = Tr_S [\hat{O} Tr_R [\chi(t)]] = Tr_S [\hat{O} \rho(t)] = Tr [\hat{O} \rho(t)] \] (2.9)
because \( \hat{O} \) does not act upon the reservoir basis.

Although we know the reduced density matrix \( \rho(t) \) (from here on referred to as the density matrix, as \( \chi(t) \) serves us no computational purpose) describes the state of the system, we do not yet know anything about the time evolution of the density matrix, and hence know nothing about the evolution of the system. Using the definition of the density operator (equation 2.2) and the Schrödinger equation, we see
\[ \dot{\rho} = \sum_i p_i \left[ |\psi_i \rangle \langle \psi_i | + |\dot{\psi}_i \rangle \langle \dot{\psi}_i | \right] \]
\[ = \sum_i p_i \left[ -\frac{i}{\hbar} H_S |\psi_i \rangle \langle \psi_i | + \frac{i}{\hbar} |\dot{\psi}_i \rangle \langle \dot{\psi}_i | H_S \right] \]
\[ = -\frac{i}{\hbar} [H_S, \rho], \] (2.10)

where \( H_S \) is the Hamiltonian for the system only, containing no terms that act on the reservoir, nor any interaction terms. Thus, if the system Hamiltonian is known to us, we can solve for the time evolution of the system at hand. We still have not, however, added dissipation into our formalism; we’ve simply set up a framework into which dissipative terms may possibly be added in a clear, consistent manner.

For the consideration of dissipation, we must generalize the form of equation 2.10.

### 2.2 The Master Equation with Dissipation

In a typical cavity QED system, we are concerned with the dynamics of one or many two-level atoms and the quantized electromagnetic field within a lossy optical
cavity. Energy is being provided to this system via the driving field, a laser treated classically (in this work) coming through the cavity’s input mirror. Dissipation arises in the system through the coupling of the atoms and the cavity field modes to an environment with many degrees of freedom. In general, we are not concerned with the details of this reservoir’s evolution, we only care about the reservoir’s effect on the atom-cavity system, for it is the source of dissipation. To formally treat the system’s evolution, we use the master equation,

\[ \dot{\rho} = \mathcal{L}\rho, \quad (2.11) \]

where \( \mathcal{L} \) is a generalized Liouvillian superoperator (it operates on other operators, not states)\[10\]. The Liouvillian contains within it the same Hamiltonian evolution given by equation 2.10, but also contains the contributions of dissipative processes.

The reservoir that our system couples to is modeled as a collection of quantized harmonic oscillator modes in thermal equilibrium at some temperature. This allows us to sufficiently model many different environments. In this work, the environment is taken to be a reservoir of electromagnetic field modes. To arrive at the final Liouvillian evolution of our system, we essentially do the following:

1.) Couple the system to the reservoir.
2.) Enact two different approximations dealing with the system and reservoir’s time scales and their coupling strengths.
3.) Trace over the reservoir states to extract the system dynamics.

One type of dissipation we will treat in this work is the decay of the cavity field mode into the vacuum radiation field surrounding the optical cavity. This system is characterized by the Hamiltonian

\[ H_S = \hbar \omega_0 a^\dagger a. \quad (2.12) \]

The reservoir Hamiltonian is given by

\[ H_R = \sum_j \hbar \omega_j r_j^\dagger r_j, \quad (2.13) \]

where \( a \) and \( a^\dagger \) (\( r_j \) and \( r_j^\dagger \)) are the annihilation and creation operators for the cavity field mode \((j^{th} \text{ reservoir field mode})\). The interaction Hamiltonian between
the cavity field mode and the reservoir in the rotating wave approximation (RWA) is given by

$$H_{SR} = \sum_j \hbar \left( \kappa_j^* a_j^\dagger + \kappa_j a_j^\dagger r_j \right),$$

(2.14)

where $\kappa_j$ is the coupling strength between the cavity field mode and the $j$th mode of the reservoir field. The first approximation we make in solving for the system’s evolution is the Born approximation. The Born approximation enters the formal derivation as a statement that the full density matrix can be factored as

$$\tilde{\chi}(t) = \tilde{\rho}(t) R_0,$$

(2.15)

where the tilde above $\rho(t)$ and $\chi(t)$ denote that this equation takes place in the interaction picture and not the Schrödinger picture that we have worked in thus far.

Equation 2.15 is equivalent to saying that the reservoir’s properties are independent of time. In this work, this means that due to the reservoir’s many degrees of freedom it is taken that the system couples only very weakly to any one mode of the vacuum field ($\kappa_j << 1$ for all $j$). Thus, the system has very little effect on the reservoir, while the reservoir modes collectively have a large effect on the system. A simple analogy might involve dropping an ice cube into the Atlantic Ocean. The next approximation made is the Markoff approximation; a Markoffian system’s future dynamics only depend on the system’s present state. This approximation enters the formal derivation in the form of a qualitative comparison between the reservoir’s time scale and that of the system. If the reservoir is able to return to its thermal steady-state very quickly as compared with the time scale of the system’s dynamics, then the reservoir will “have no memory” of past interactions. This irreversibility is then in line with our intuition of dissipative processes. After these approximations are made in the formal derivation, one can then trace over the reservoir’s states to obtain the Liouvillian evolution of the system’s density matrix in the form of the master equation. A more formal discussion of the Born-Markoff approximation, and a derivation of the Master equation is presented in Appendix A.
The Liouvillian evolution of a single cavity field mode coupled to the vacuum radiation field given by equation 2.13 is

\[
\dot{\rho}(t) = -i\omega_0 [a^\dagger a, \rho] + \kappa (\bar{n} + 1) \left[ 2a\rho(t)a^\dagger - \rho(t)a^\dagger a - a^\dagger a\rho(t) \right] \\
+ \kappa \bar{n} \left[ 2a\rho(t)a^\dagger - aa^\dagger \rho(t) - \rho(t)aa^\dagger \right],
\]

(2.16)

where \(\kappa\) and \(\bar{n}\) are the cavity field decay rate and the thermal photon number (the mean number of photons in the reservoir at thermal equilibrium), respectively [9].

To see that equation 2.11 gives results that are consistent with our intuition of a decaying field, we calculate the time dependent photon number of the cavity. Using equation 2.8 to calculate the time derivative of the photon number’s expectation value, we find

\[
\langle \dot{n} \rangle = Tr[a^\dagger a\dot{\rho}(t)] \\
= -i\omega_0 Tr(a^\dagger aa^\dagger \rho - a^\dagger a^\dagger \rho a) + \kappa Tr(2a^\dagger a^2 \rho a^\dagger - a^\dagger aa^\dagger \rho a^\dagger) \\
- a^\dagger a^\dagger \rho a + 2\kappa \bar{n} Tr(a^\dagger a^2 \rho a^\dagger + a^\dagger aa^\dagger \rho a - a^\dagger aa^\dagger \rho a - a^\dagger a^\dagger a^\dagger a) \\
= 2\kappa Tr((a^\dagger)^2 a^2 \rho - (a^\dagger a)^2 \rho) \\
+ 2\kappa \bar{n} Tr((a^\dagger)^2 a^2 \rho + (a^\dagger a)^2 \rho - (a^\dagger a)^2 \rho - a(a^\dagger)^2 a) \\
= -2\kappa \langle n \rangle - \bar{n}.
\]

(2.17)

This differential equation has the solution

\[
\langle n(t) \rangle = \langle n(0) \rangle e^{-2\kappa t} + \bar{n}(1 - e^{-2\kappa t}).
\]

(2.18)

From this example, we see that our method provides us with a photon number that starts at some initial value \(\langle n(0) \rangle\) and evolves to its thermal steady state value \(\bar{n}\); thus, our method is in congruence with our intuition.

Another type of dissipation that plays an important role in the evolution of a cavity QED system is the decay of the atomic inversion, that is radiative decay of the two-level atom. Here, our system Hamiltonian is given by

\[
H_S = \frac{1}{2}\hbar \omega_0 \sigma_z,
\]

(2.19)

12
where $\sigma_z$ is the Pauli pseudo-spin operator that describes the atomic inversion.

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.20}$$

The environment that we will couple the system to is a reservoir of cavity field modes, which has the same Hamiltonian given by equation 2.13; the interaction Hamiltonian is

$$H_{SR} = \sum_{\vec{k},p} \hbar (\kappa_{\vec{k},p}^* r_{\vec{k},p}^\dagger \sigma_- + \kappa_{\vec{k},p} r_{\vec{k},p} \sigma_+), \tag{2.21}$$

where $\vec{k}$ denotes the wavevector of the field mode, and $p$ denotes the mode’s polarization. $\sigma_+$ and $\sigma_-$ are the Pauli pseudo-spin operators that take a two-level atom from the excited state to the ground state and the atom from the ground state to the excited state, respectively.

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{2.22}$$

$$\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.23}$$

Following a derivation analogous to that for the damped cavity field, and making the same approximations, we find that the master equation for a radiatively damped two-level atom is given by

$$\dot{\rho} = -\frac{i}{2} \omega_0 [\sigma_z, \rho] + \frac{\gamma}{2} (\bar{n}_a + 1) (2\sigma_- \rho \sigma_+ - \sigma_+ \sigma_- \rho - \rho \sigma_+ \sigma_-) + \frac{\gamma}{2} \bar{n}_a (2\sigma_+ \rho \sigma_- - \sigma_- \sigma_+ \rho - \rho \sigma_- \sigma_+), \tag{2.24}$$

where $\bar{n}_a$ is now the thermal photon number of the reservoir that the atom couples to, and $\gamma$ now denotes the rate of spontaneous emission.

At the optical wavelengths often considered in cavity QED experiments, the thermal photon number $\bar{n}$ is often very small and can be ignored. At $\omega_0 = 3 \times 10^{15}$, $T = 300K$, $\bar{n} \approx 7 \times 10^{-34} \approx 0$. Taking this approximation, we will show the simple calculation of the spontaneous emission dynamics of the two level atom. By the
definition of the reduced density matrix $\rho(t)$, the probability of finding the two-level atom in its excited state at time $t$ is given by the population

$$\rho_{ee}(t) = \langle e | \rho(t) | e \rangle.$$  

(2.25)

Thus, the time derivative of this population is given by

$$\dot{\rho}_{ee} = \langle e | \dot{\rho}_{ee} | e \rangle = -i\frac{\omega_0}{2} \langle e | (\sigma_z \rho - \rho \sigma_z) | e \rangle + \frac{\gamma}{2} \langle e | (2\sigma_- \rho \sigma_+ - \sigma_+ \sigma_- \rho - \rho \sigma_+ \sigma_-) | e \rangle = -\gamma \rho_{ee}.$$  

(2.26)

This is a familiar differential equation, with the familiar solution

$$\rho_{ee}(t) = \rho_{ee}(0) e^{-\gamma t}.$$  

(2.27)

This result matches the classical result, and is consistent with our intuition of radiative decay.

### 2.3 Two-Time Correlation Functions

We have seen, in the preceding section, that the calculation of expectation values for our system is a relatively straightforward process. We simply define the correct form of the master equation (equation 2.11 for a single cavity field mode), and then apply equation 2.9. Another way to access system information from the master equation is to calculate the time dependent values of the density matrix elements in the basis of interest. Information that is of interest to us, however, is not always in the form of a simple expectation value. The spectrum of the radiatively decaying two-level atom, for instance, is a two-time correlation function. This is the expectation value of the product of two operators evaluated at two different times, and takes the form [11]

$$\langle \hat{O}_1(t) \hat{O}_2(t + \tau) \rangle.$$  

(2.28)

As an example of why two-time correlation functions are of interest to us, let us begin the calculation of the spectrum of the transmitted light from a lossy optical
cavity. Here we consider a single cavity mode, but due to the lossy nature of the cavity, this mode will actually have some finite linewidth. A spectrum measurement experiment on the system would be in the form of an intensity measurement, so we calculate the intensity distribution as a function of frequency:

\[ I(\omega) = E^*(\omega)E(\omega), \]  

where the frequency distribution of the field is given by the Fourier transform of the electric field’s time evolution

\[ E(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E(t)e^{i\omega t} dt. \]  

Using this, we see that equation 2.29 can be written as

\[ I(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'E(t)E^*(t')e^{i\omega(t-t')} \]  

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\tau E(t)E^*(t+\tau)e^{-i\omega\tau}, \]  

where \( \tau = t' - t \). Assuming that the field is ergodic, and that the expectation value is equivalent to a time average, we can further simplify equation 2.31 to achieve the Wiener-Khintchine theorem,

\[ I(\omega) = \frac{T}{2\pi} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \langle E(t)E^*(t+\tau) \rangle, \]

where we have used

\[ \langle E(t)E^*(t+\tau) \rangle = \frac{1}{T} \int_{-T/2}^{T/2} dt E(t)E^*(t+\tau). \]

\( T \) is the total time over which the field is time-averaged. In order to correctly model the experiment, we must make certain that our field \( E(t) \) coincides with the experiment’s field signal. In the physical experiment, a detector is used to collect the transmitted photons, and is thus an absorptive device. This absorption is correctly modeled in our calculation by replacing \( E \) with its negative frequency component \( E^{(-)} \propto a \). So, now we can write equation 2.33 in terms of the cavity field’s creation and annihilation operators.

\[ I(\omega) = \frac{T}{2\pi} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \langle a^\dagger(t+\tau)a(t) \rangle. \]
We have used normal ordering (creation operators to the left, annihilation operators to the right) in the above equation, coinciding with our modeling of the detector as an absorptive device. We are now faced with the task of calculating a two-time correlation function for the field operators, a much simpler problem. We thus need to develop a method with which to attack such a problem. The two methods by which we proceed in this work are the Quantum Regression Theorem and the Quantum Trajectories Formalism.

### 2.3.1 Quantum regression theorem

The quantum regression theorem provides a method with which to formally represent and solve two-time correlation functions, that is, an expectation value of the product of two operators evaluated at two different times. In the Heisenberg picture, this expectation value is given by

\[
\langle \hat{O}_1(t)\hat{O}_2(t') \rangle = \text{Tr}_S \left[ \chi(0)\hat{O}_1(t)\hat{O}_2(t') \right].
\]  

(2.36)

Defining \( \tau = t' - t \), using the cyclic property of the trace, and tracing over the reservoir states eventually leads to the result

\[
\langle \hat{O}_1(t)\hat{O}_2(t + \tau) \rangle = \text{Tr}_S[\hat{O}_2(0)e^{\mathcal{L}_\tau}[\rho(t)\hat{O}_1(0)]].
\]  

(2.37)

If we consider time = \( t \) to be the system steady-state, and exploit statistical stationarity, we may write equation 2.37 as

\[
\langle \hat{O}_1(0)\hat{O}_2(\tau) \rangle = \text{Tr}_S[\hat{O}_2(0)e^{\mathcal{L}_\tau}[\rho_{ss}\hat{O}_1(0)]].
\]  

(2.38)

Next, we define an operator

\[
A(\tau) = e^{\mathcal{L}_\tau}\rho_{ss}\hat{O}_1(0),
\]  

(2.39)

where we only see the evolution of \( A \) in \( \tau \), with initial condition

\[
A(0) = \rho_{ss}\hat{O}_1(0).
\]  

(2.40)

Now we can write

\[
A(\tau) = e^{\mathcal{L}_\tau}A(0),
\]  

(2.41)
\[ \dot{A} = \mathcal{L}A. \]  

(2.42)

Equation 2.42 shows that \( A(\tau) \) obeys the same master equation as \( \rho(t) \), with \( A(0) \) having initial conditions dependent upon \( \rho_{ss} \). The familiar result of this is that the two-time correlation function in equation 2.38 satisfies the same equation of motion with respect to \( \tau \) as expectation values do with respect to \( t \).

\[ \langle \hat{O}_1(0)\hat{O}_2(\tau) \rangle = Tr_S[\hat{O}_2(0)A(\tau)]. \]  

(2.43)

For a more complete derivation of this theory, see reference [9].

### 2.3.2 Quantum trajectory theory

The other method by which we extract information from the master equation is the Quantum Trajectories Formalism (QTF) of Carmichael[10]. In the QTF, we abandon the density matrix, mixed-state representation of the system in favor of an ensemble of systems represented by pure state wavefunctions. Each of these pure state systems will undergo continuous, non-Hermitian Hamiltonian evolution that is interrupted at random times by some form of discontinuous state change. These state changes are the realizations of system decay processes, including spontaneous emission and cavity decay. The pure state representation for each member of the ensemble will take the form governed by its history of discontinuous state changes, and is thus called the system’s *conditioned wavefunction*. To mathematically formalize this approach, we first *unravel* the master equation by writing equation 2.11 as

\[ \dot{\rho} = (\mathcal{L} - S)\rho + S\rho, \]  

(2.44)

where \( S\rho \) contains all terms of form \( \hat{O}^\dagger \rho \hat{O} \). These \( \hat{O} \) operators are those that model collapse events for the system. We express the remaining terms as

\[ (L - S)\rho = -\frac{i}{\hbar}[H_S,\rho] + [H_D,\rho]_+, \]  

(2.45)

where \([\hat{O},\rho]_+ = \hat{O}\rho + \rho\hat{O} \), and \( H_D \) contains the dissipative terms not included in \( S\rho \). Equation 2.45 can only be utilized if the master equation unravelling in modeling a direct detection process. The Hamiltonian that determines the continuous
evolution of the pure states is not Hermitian, and is given by

\[ \hat{H} = H_S + i\hbar H_D, \]  

(2.46)

where \( H_S \) is the Hermitian Hamiltonian for the system, not accounting for dissipation processes. To illustrate these preliminary steps of the QTF method, let us consider the example of the damped two-level atom, for which the master equation is

\[ \dot{\rho} = -\frac{i}{2} \omega_A [\sigma_z, \rho] + \frac{\gamma}{2} (\bar{n} + 1) (2\sigma_- \rho \sigma_+ - \sigma_+ \sigma_- \rho - \rho \sigma_+ \sigma_-) \]

\[ + \frac{\gamma}{2} \bar{n} (2\sigma_+ \rho \sigma_- - \sigma_- \sigma_+ \rho - \rho \sigma_- \sigma_+), \]

(2.47)

where \( \gamma \) is the rate of spontaneous emission. To simplify our example, let us assume that the thermal photon number \( \bar{n} \) is equal to zero; this is a valid approximation in many experimental regimes, and will not qualitatively change the form of our example. So, the master equation for our damped two-level atom is now written as

\[ \dot{\rho} = -\frac{i}{2} \omega_A [\sigma_z, \rho] + \frac{\gamma}{2} (2\sigma_- \rho \sigma_+ - \sigma_+ \sigma_- \rho - \rho \sigma_+ \sigma_-). \]

(2.48)

We can immediately recognize \( \frac{\gamma}{2} (2\sigma_- \rho \sigma_+ \sigma_-) \) as a collapse term of the form \( \hat{O}^\dagger \rho \hat{O} \), and thus we define \( S \rho \) to be

\[ S \rho = \gamma \sigma_- \rho \sigma_+ \]

\[ = \gamma \sigma_- |\Psi\rangle_T \langle \Psi|_T \sigma_+. \]

(2.49)

Thus,

\[ (L - S) \rho = -\frac{i}{2} \omega_A [\sigma_z, \rho] - \frac{\gamma}{2} [\sigma_+ \sigma_-, \rho]_+. \]

(2.50)

Comparing this equation to equation 2.45 shows that

\[ H_S = \frac{\hbar \omega_A}{2} \sigma_z \]

\[ H_D = -\frac{\gamma}{2} \sigma_+ \sigma_-. \]

(2.51)

These terms, added together as prescribed in the non-Hermitian Hamiltonian (equation 2.46) clearly form a direct link, and equivalency, between equations 2.50 and 2.10.
Equation 2.49 identifies $\sqrt{\gamma} \sigma_-$ as the collapse operator for the system; this follows from our model of radiative damping for the two-level atom, as a transition from the excited state to the ground state will radiate a photon into the environment. With the identification of the appropriate collapse operator, we can now evaluate the probability of a collapse event occurring $p_c(t)$ during any random time interval $t \, dt$.

\[
p_c(t) = \text{tr}[S \rho_c(t)] dt = \gamma \frac{\langle \psi_c(t) | \sigma_+ \sigma_- | \psi_c(t) \rangle}{\langle \psi_c(t) | \psi_c(t) \rangle} dt,
\]

where the subscript $c$ denotes the use of the conditional wavefunction and density matrix for the system. The denominator of equation 2.52 is needed because the non-Hermitian Hamiltonian does not generally preserve the normalization of the conditional wavefunction. With these fundamental procedures and details explained, we can now work through the typical operation of a QTF simulation carried out on the computer. First, we consider an ensemble of systems each in some common initial state; for the purpose of this explanation let’s follow one such system, and its conditional wavefunction $|\psi_c(t)\rangle$, through the simulation. The systems undergo evolution as governed by the Schrödinger equation:

\[
\hat{H} |\psi_c(t)\rangle = H |\psi_c(t)\rangle,
\]

where $H$ is the Hamiltonian of equation 2.46. At each time-step $(t_n)$ of the program, a random number $(r_n)$ in the interval $[0, 1]$ is generated and compared to the collapse probability as evaluated in equation 2.52. One of two conditions can be satisfied at this point, determining the evolution of the system for the time-step. If $r_n \geq p_c(t_n)$, then the system will experience a collapse and the evolved conditional wavefunction at time $t_{n+1}$ will be given by

\[
|\psi_c(t_{n+1})\rangle = \frac{\sigma_- |\psi_c(t_n)\rangle}{\sqrt{\langle \psi_c(t_n) | \sigma_+ \sigma_- | \psi_c(t_n) \rangle}}.
\]  

If $r_n < p_c(t_n)$, then the system will undergo continuous Hamiltonian evolution for the duration of the time-step, giving the conditional wavefunction to be

\[
|\psi_c(t_{n+1})\rangle = \frac{e^{-i/\hbar}H \Delta t |\psi_c(t_n)\rangle}{\sqrt{\langle \psi_c(t_n) | e^{i/\hbar}(H^\dagger - H) \Delta t | \psi_c(t_n) \rangle}}.
\]
Notice that the evolved conditional wavefunctions are to be normalized at each
time-step.

It can be shown that the ensemble average of conditional density matrices is
equivalent to the system’s real density matrix:

\[ \rho(t) = \langle \psi_c(t) | \psi_c(t) \rangle. \] (2.56)

From this ensemble-average/system equivalency, it is seen that expectation values
for the system are calculated as usual, only this is done for each member of the
ensemble and the results are averaged to give the system’s expectation value. The
calculation of two-time correlation functions is a potentially complicated proce-
dure. One could essentially allow the ensemble members to evolve to the time in
question, and then examine all possible correlations between the members. When
extracting certain information with the QTF methods, however, this complication
can be avoided by picking an appropriate *unravelling* of the master equation. If
we, for example, were interested in the photon statistics of a damped cavity mode,
the associated master equation would be (with thermal photon number \( \bar{n} \) equal to
zero again)

\[ \dot{\rho} = -i\omega_0 [a^\dagger a, \rho] + \kappa \left( 2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a \right). \]

If we unravel this equation as

\[ S\rho = 2\kappa \rho a^\dagger, \] (2.57)

\[ (L - S)\rho = -i\omega_0 [a^\dagger a, \rho] - \kappa [a^\dagger a, \rho]_+, \] (2.58)

we will have adopted a formalism based upon a simulated photon counting experi-
ment, where the detector is modeled as an absorptive device. Thus, to calculate the
system’s photon statistics, we simple employ the same QTF simulation presented
earlier and keep a record of collapse event times. By making a histogram of pho-
ton absorption delay times, we can characterize the transmitted field’s dynamics
in precisely the same manner as is done experimentally.
Chapter 3

Cavity QED with quantized CMM

3.1 Derivation of the Hamiltonian with external potential

It is in this section that we will derive the form of our full Hamiltonian for the system of a “cooled/trapped atom” coupled to a single cavity field mode; the model of our trapping mechanism is a one-dimensional optical lattice along the cavity’s axis. First, we will derive the form for the Hermitian portion of the Hamiltonian as described in section 2.3.2, equation 2.46.

Following the treatment of Vernooy and Kimble \[12\], we will start with a general form of the Hamiltonian that describes an atom interacting with the quantized cavity field, while “sitting” within an (as of yet, unspecified) external potential.

\[
\hat{H}(\vec{r}) = \frac{\vec{p}^2}{2m} + V_{\text{ext}}(\vec{r}) + \hbar g(\vec{r}) \hat{H}_{IF},
\]

(3.1)

where \(\hat{H}_{IF}\) is the Jaynes-Cummings interaction in a rotating frame, with the atom and field on resonance\[13\].

\[
\hat{H}_{IF} = (\sigma_- a^\dagger + a \sigma_+).
\]

(3.2)

We will be using the “well-dressed state” picture of the system state-function,
which is a simple extension of the “dressed state” eigenstates of $\hat{H}_{IF}$. The well-dressed states are a direct product of the “vibrational state” $|k\rangle$ and the atom-field dressed states $|n,\pm\rangle$.

\[|n,\pm\rangle \equiv \frac{1}{\sqrt{2}}\left[|n-1,e\rangle \pm |n,g\rangle\right], \quad (3.3)\]

\[|k\rangle \otimes |n,\pm\rangle \equiv |n,k,\pm\rangle, \quad (3.4)\]

where $n$ denotes the field’s photon number, $e$ and $g$ specify the excited and ground states of the atom, and $k$ denotes the vibrational state associated with $V_{ext}(\vec{r})$.

Without yet specifying a form for the atom-field coupling factor $g(\vec{r})$, let us follow Vernooy and Kimble’s approach by looking at it’s behavior around the extremum $g(\vec{r}_0)$.

\[g(z) = g_m + \frac{dg(z)}{dz}\bigg|_{z_0} (z - z_0) + \frac{1}{2} \frac{d^2 g(z)}{dz^2}\bigg|_{z_0} (z - z_0)^2 + \ldots, \quad (3.5)\]

where $\vec{r} \rightarrow z$, $\vec{r}_0 \rightarrow z_0$ and $g(\vec{r}_0) \rightarrow g_m$, as we are concerned with only the position of the atom along the 1-D lattice ($z$-axis). We are, in this work, ignoring the transverse variation of $g$ due to the mode’s Gaussian form. Because we’ve chosen to expand $g(z)$ about an extremum, the first derivative in $z$ is by definition equal to zero. If we consider only small displacements from $z_0$, we may neglect terms of higher order, and therefore are left with a harmonic coupling factor:

\[g(z) = g_m \left[1 + \frac{(z - z_0)^2}{2g_m} \frac{d^2 g(z)}{dz^2}\bigg|_{z_0}\right]. \quad (3.6)\]

Defining a “curvature factor”

\[\xi \equiv \left(\left(\frac{1}{g_m}\right)^2 \left(\frac{d^2 g(z)}{dz^2}\bigg|_{z_0}\right)^{-1/2}\right), \quad (3.7)\]

allows us to express equation 3.1 as

\[\hat{H}(\vec{r}) = \frac{\hat{p}^2}{2m} + \hbar g_m \left[1 + \frac{(z - z_0)^2}{2\xi^2}\right] (\sigma_+ a + a^\dagger \sigma_-), \quad (3.8)\]

where we have temporarily set $V_{ext}(z) = 0$.

As we will eventually be working with the well dressed states, let us now examine the form of equation 3.8 acting on the simpler dressed states $|n,\pm\rangle$.

\[\hat{H}|n,\pm\rangle = \left[\frac{\hat{p}^2}{2m} + \hbar g_m \left[1 + \frac{(z - z_0)^2}{2\xi^2}\right] (\sigma_+ a + a^\dagger \sigma_-)\right] \frac{1}{\sqrt{2}}\left[|n-1,e\rangle \pm |n,g\rangle\right]. \quad (3.9)\]
Because the dressed states are eigenstates of the Jaynes-Cummings interaction with $\hat{H}_{IF}|n,\pm\rangle = \pm \sqrt{n}|n,\pm\rangle$, we may rewrite equation 3.9 as

$$\hat{H}|n,\pm\rangle = \left[ \frac{p^2}{2m} \pm \hbar g_m \sqrt{n} \pm \hbar g_m \frac{\sqrt{n}}{\xi^2} (z-z_0)^2 \right]|n,\pm\rangle. \tag{3.10}$$

For now, let us ignore the center term in brackets of equation 3.10 and make the declaration

$$\hat{H}_v \equiv \left[ \frac{p^2}{2m} \pm \hbar g_m \frac{\sqrt{n}}{\xi^2} (z-z_0)^2 \right]. \tag{3.11}$$

If we, for now, only consider the additive form of equation 3.11, we realize that it is just the form of a quantum mechanical harmonic oscillator; we'll further characterize this oscillator by making the following substitutions (the second term in equation 3.11 being negative in sign would represent a “flipped” harmonic potential, a repulsive potential with an unstable equilibrium about $z_0$):

$$\hbar g_m \frac{\sqrt{n}}{\xi^2} (z-z_0)^2 = \frac{1}{2} m \omega^2 x^2, \tag{3.12}$$

$$(z-z_0)^2 = x^2, \tag{3.13}$$

$$\omega^2 = \frac{\hbar g_m \sqrt{n}}{m \xi^2}, \tag{3.14}$$

$$\omega = \sqrt{\frac{\hbar g_m \sqrt{n}}{m \xi^2}}. \tag{3.15}$$

Now, equation 3.11 can be written as

$$\hat{H}_v \equiv \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2, \tag{3.16}$$

which is the common form for the Hamiltonian of a particle in a harmonic well. It is familiar that the energy of the $k$th eigenstate of equation 3.16 is

$$E_k = \left( k + \frac{1}{2} \right) \hbar \omega, \tag{3.17}$$

where

$$\hbar \omega = \sqrt{\frac{(2 \hbar g_m \sqrt{n})}{2 m \xi^2} \frac{\hbar^2}{2m \xi^2}}. \tag{3.18}$$

Keeping the $(z-z_0)^2 \rightarrow x^2$ substitution, and setting $V_{ext}(x) = \frac{1}{2} \hbar \alpha x^2$, we may now write down the full Hamiltonian of equation 3.1 as

$$\hat{H} = \left( \frac{\hbar \omega}{2} + \hbar \omega a^\dagger a \right) + \frac{p^2}{2m} + \frac{1}{2} \hbar \alpha x^2 + h g_m \left( 1 + \frac{x^2}{2 \xi^2} \right) (\sigma_+ a + a^\dagger \sigma_-) + i \hbar E (e^{i \omega t} a - e^{-i \omega t} a^\dagger), \tag{3.19}$$
where the last term denotes the interaction between the (classical) driving field and the cavity field mode. We may further simplify this expression by grouping the terms quadratic in \( \hat{p} \) and \( x \). We must also exploit the fact that this Hamiltonian will be acting upon the well dressed states, where \( (\sigma_+ a + a^\dagger \sigma_-)|n, k, \pm\rangle = \pm \sqrt{n}|n, k, \pm\rangle \).

So,

\[
\hbar g_m (\sigma_+ a + a^\dagger \sigma_-) \longrightarrow \pm \hbar g_m \sqrt{n}.
\] (3.20)

Grouping the quadratic terms, we find that \( H_v \) becomes

\[
\hat{H}_v = \frac{p^2}{2m} + \frac{1}{2} \left[ \hbar \alpha \pm \frac{\hbar g_m \sqrt{n}}{\xi^2} \right] x^2
\]

\[
= \frac{p^2}{2m} + \frac{1}{2} m \Omega_{n,\pm}^2 x^2,
\] (3.21)

where

\[
\Omega_{n,\pm} = \left[ \frac{\hbar}{m} \left[ \alpha \pm \frac{g_m \sqrt{n}}{\xi^2} \right] \right]^{\frac{1}{2}}.
\] (3.22)

This treatment assumes no specific relationship between \( \alpha \) and \( \frac{g_m \sqrt{n}}{\xi^2} \), but in the numerical simulations presented in this work we have assumed a deep trapping limit where \( \alpha \geq \frac{g_m \sqrt{n}}{\xi^2} \). Because we have defined \( V_{\text{ext}} \) to be a harmonic potential, our system is now as shown in figure 3.1.
Figure 3.1: A typical cavity QED system with the inclusion of a harmonic trapping potential.
We recognize that $H_v$ will be the only portion of the full Hamiltonian that addresses the motion/position of the atom, and thus will be the only portion to act upon the CMM states $|k\rangle$. This harmonic oscillator Hamiltonian, when acting upon a CMM state, will simply return the product of the state and its energy ($\hbar\Omega_{n,\pm}|k\rangle$); we may therefore make the substitution

$$\frac{p^2}{2m} + \frac{1}{2}m\Omega_n^2 a^2 \rightarrow \hbar\Omega_{n,\pm} b\dagger b, \quad (3.23)$$

where $b$ and $b\dagger$ are the ladder operators for the CMM states, analogous to $a$ and $a\dagger$ acting upon $|n\rangle$. With this simplification, we may now state the final form of our system’s Hermitian Hamiltonian:

$$\hat{H}_s = \left[\frac{\hbar\omega_\sigma_z}{2} + \hbar\omega a\dagger a\right] + \hbar\Omega_{n,\pm} b\dagger b + \hbar g_m (\sigma_+ a + a\dagger \sigma_-) + \imag \hbar E(e^{i\omega t} a - e^{-i\omega t} a\dagger). \quad (3.24)$$

This is the Hermitian portion of the full Hamiltonian that will determine the time evolution of our “well dressed states”, $|n, k, \pm\rangle$.

To derive the form of the non-Hermitian portion of the full Hamiltonian, we must first write down the dissipative Liouvillian operator that describes the dissipations we account for, as discussed in equations 2.44 and 2.45. The dissipative Liouvillian operator is given by

$$L_{\text{diss}} \rho = \frac{\gamma}{2}[2\sigma_- \rho \sigma_+ - \sigma_+ \sigma_- \rho - \rho \sigma_+ \sigma_-] + \kappa[2a\rho a\dagger - a\dagger a\rho - \rho a\dagger a] + (n_{Th} + 1)\Gamma[2b\rho b\dagger - b\dagger b\rho - \rho b\dagger b] + n_{Th}\Gamma[2b\dagger b\rho - b b\dagger \rho - \rho b b\dagger], \quad (3.25)$$

where $\gamma$ is the spontaneous emission rate, $2\kappa$ is the cavity field decay rate, and $\Gamma$ is the decay rate for a vibrational excitation. Dissecting this operator as prescribed by equations 2.45 and 2.46 allows us to make the following declarations for the collapse terms and the non-Hermitian portion of the full Hamiltonian, respectively:

$$S_1 \rho = \gamma(\sigma_- \rho \sigma_+),$$
$$S_2 \rho = 2\kappa(a\rho a\dagger),$$
$$S_3 \rho = 2(n_{Th} + 1)\Gamma(b\rho b\dagger),$$
$$S_4 \rho = 2n_{Th}\Gamma(b\dagger \rho b), \quad (3.26)$$
\[ \hat{H}_D = -\left[ \frac{\gamma}{2} \sigma_+ \sigma_- + \kappa a^\dagger a + (n_{Th} + 1) \Gamma(b^\dagger b) + n_{Th} \Gamma(bb^\dagger) \right], \]  

(3.27)

where the collapses have probabilities of occurring determined as in equation 2.52 (simply replace \( \sigma_+ \sigma_- \) by \( a^\dagger a \) and \( \gamma \) by \( 2\kappa \), for instance, to calculate the probability for the collapse \( S_2 \)).

We may now present the full, non-Hermitian, Hamiltonian that governs the time evolution of our system’s state function; this is done by combining the terms in equations 3.24 and 3.27 as done in equation 2.45.

\[
\hat{H} = \left[ \hbar \omega \sigma_z + \hbar \omega a^\dagger a \right] + \hbar \Omega_{n_g} b^\dagger b + \hbar g_m (\sigma_+ a + a^\dagger \sigma_-) + i \hbar E' (e^{i\omega t} a - e^{-i\omega t} a^\dagger) \\
- i \hbar \left[ \frac{\gamma}{2} \sigma_+ \sigma_- + \kappa a^\dagger a + (n_{Th} + 1) \Gamma(b^\dagger b) + n_{Th} \Gamma(bb^\dagger) \right].
\]

(3.28)

3.2 Equations for Probability Amplitudes

Solving for the dynamics of the atom-field system in the weak field limit reduces, essentially, to the problem of solving for the time evolution of “state population amplitudes”. We therefore must define a formal representation of the system’s state at any time \( t \). The most general representation for our system’s state is given by the wavefunction

\[
|\Psi\rangle = \sum_{n \neq 0, k} \left[ C_{n, k, +} e^{\frac{-iE_n}{\hbar} t} |n, k, +\rangle + C_{n, k, -} e^{\frac{-iE_n}{\hbar} t} |n, k, -\rangle \right] \\
+ \sum_k C_{0, k, g} e^{\frac{-iE_0}{\hbar} t} |0, k, g\rangle,
\]

(3.29)

where \( C_{0, k, g} \) and \( C_{n, k, \pm} \) are time dependent population amplitudes that are generally complex. The population of a given state \( |n, k, \pm\rangle \) is thus

\[ P_{n, k, \pm} = |C_{n, k, \pm}|^2. \]

(3.30)

We use the Schrödinger equation (eq. 1.1) to give us a generalized equation of motion for the system’s state function \( |\Psi\rangle \). Using the Hamiltonian given by equation 3.28, we can calculate the right hand side of equation 1.1 as

\[
\hat{H}|\Psi\rangle = \sum_k \left[ C_{0, k, g} e^{\frac{-E_0}{\hbar} t} \left( -\frac{\hbar \omega}{2} + \hbar \Omega_{n_g} k - i \hbar (n_{Th} + 1) \Gamma k - i \hbar n_{Th} \Gamma(k + 1) \right) \\
+ \frac{i \hbar E'}{\sqrt{2}} e^{i\omega t} \left[ C_{1, k, +} e^{\frac{-E_n}{\hbar} t} - C_{1, k, -} e^{\frac{-E_n}{\hbar} t} \right] \right]|0, k, g\rangle
\]


\[
\begin{align*}
- \sum_k \left[ C_{0,k,g} e^{-\frac{E_{0,k} + i \hbar E \sqrt{2}}{2}} \right] |1, k, +\rangle e^{-i\omega t} \\
+ \sum_k \left[ C_{0,k,g} e^{-\frac{E_{0,k} + i \hbar E' \sqrt{2}}{2}} \right] |1, k, -\rangle e^{-i\omega t} \\
- \sum_k \left[ C_{1,k,+} e^{-\frac{E_{1,k} + i \hbar E'}{2}} e^{-i\omega t} \left( \frac{1}{\sqrt{2}} + \frac{1}{2} \right) \right] |2, k, +\rangle \\
- \sum_k \left[ C_{1,k,-} e^{-\frac{E_{1,k} + i \hbar E'}{2}} e^{-i\omega t} \left( \frac{1}{\sqrt{2}} - \frac{1}{2} \right) \right] |2, k, -\rangle \\
+ \sum_{n \neq 0,k} \left[ C_{n,k,+} e^{-\frac{E_{n,k} + i \hbar \Omega}{2}} \left( \hbar \omega (n - 1/2) + \hbar \Omega_{n,k} + \hbar \sqrt{n} \right) \\
- \hbar \frac{\gamma}{4} + \kappa (n - 1/2) + (n \Gamma + \Gamma k + \Gamma (k + 1)) \right] |n, k, +\rangle \\
- \left( C_{n,k,-} e^{-\frac{E_{n,k} + i \hbar \Omega}{2}} \hbar \frac{\gamma}{4} + \kappa \frac{n}{2} \right) |n, k, -\rangle \\
+ \sum_{n \neq 0,k} \left[ C_{n,k,-} e^{-\frac{E_{n,k} + i \hbar \Omega}{2}} \left( \hbar \omega (n - 1/2) - \hbar \Omega_{n,k} - \hbar \sqrt{n} \right) \\
- \hbar \frac{\gamma}{4} + \kappa (n - 1/2) + (n \Gamma + \Gamma k + \Gamma (k + 1)) \right] |n, k, -\rangle \\
+ \sum_{n \neq 0,k} \delta_n \left[ C_{n,k,+} e^{-\frac{E_{n,k} + i \hbar E'}{2}} e^{i\omega t} \left( \sqrt{n - 1} + \sqrt{n} \right) |n - 1, k, +\rangle \\
+ \sum_{n \neq 0,k} \delta_n \left[ C_{n,k,+} e^{-\frac{E_{n,k} + i \hbar E'}{2}} e^{i\omega t} \left( \sqrt{n - 1} - \sqrt{n} \right) |n - 1, k, -\rangle \\
+ \sum_{n \neq 0,k} \delta_n \left[ C_{n,k,-} e^{-\frac{E_{n,k} + i \hbar E'}{2}} e^{-i\omega t} \left( \sqrt{n + 1} + \sqrt{n} \right) |n + 1, k, +\rangle \\
+ \sum_{n \neq 0,k} \delta_n \left[ C_{n,k,-} e^{-\frac{E_{n,k} + i \hbar E'}{2}} e^{-i\omega t} \left( \sqrt{n + 1} - \sqrt{n} \right) |n + 1, k, -\rangle \\
- \left. C_{n,k,+} e^{-\frac{E_{n,k} + i \hbar E'}{2}} e^{-i\omega t} \left( \sqrt{n + 1} + \sqrt{n} \right) \right| n + 1, k, + \rangle \\
- \left. C_{n,k,-} e^{-\frac{E_{n,k} + i \hbar E'}{2}} e^{-i\omega t} \left( \sqrt{n + 1} - \sqrt{n} \right) \right| n + 1, k, - \rangle, \right. \\
\end{align*}
\]
where
\[
\delta_n = \begin{cases} 
0 & : n = 1 \\
1 & : n \neq 1
\end{cases}.
\] (3.32)

To see the full procedure of this evaluation refer to Appendix A, where the Non-Hermitian Hamiltonian has been separated into small segments, whose actions on \(|\Psi\rangle\) have been separately and neatly calculated. The left side of equation 1.1 can be calculated from equation 3.29, and is
\[
\frac{i\hbar}{\hbar} |\Psi\rangle = i\hbar \sum_{n \neq 0, k} \left[ \left( \hat{C}_{n,k,+} - \frac{i}{\hbar} E_{n,k,+} \hat{C}_{n,k,+} \right) e^{-iE_{n,k,+} t} |n,k,\rangle \right]
+ \left( \hat{C}_{n,k,-} - \frac{i}{\hbar} E_{n,k,-} \hat{C}_{n,k,-} \right) e^{-iE_{n,k,-} t} |n,k,-\rangle 
+ i\hbar \sum_k \left[ \left( \hat{C}_{0,k,g} - \frac{i}{\hbar} E_{0,k,g} \hat{C}_{0,k,g} \right) e^{-iE_{0,k,g} t} |0,k,g\rangle \right].
\] (3.33)

The method for attaining a differential equation in \(C_{m,l,\pm}\) (or \(C_{0,l,g}\)) is to project the state \(\langle m, l, \pm |\) (or \(\langle 0, l, g |\)) onto both \(H|\Psi\rangle\) and \(i\hbar |\dot{\Psi}\rangle\), and to solve for \(\dot{C}_{m,l,\pm}\) (or \(\dot{C}_{0,l,g}\)). Such a procedure leads us to the following equations of motion for the state population amplitudes. We have explicitly written out the forms of \(\dot{C}_{0,l,g}\) and \(\dot{C}_{m,l,\pm}\) for \(n = 1,2\), as these will be the starting point for any calculations in the weak field limit (at most, two quanta of energy in the system).

\[
\dot{C}_{0,l,g} = - \left[ (n_{Th} + 1) \Gamma l + n_{Th} \Gamma (l + 1) \right] C_{0,l,g}
+ \frac{E'}{\sqrt{2}} e^{it(\Omega_0 l - \Omega_1 l - g)} C_{1,l,+}
- \frac{E'}{\sqrt{2}} e^{it(\Omega_0 l - \Omega_1 l + g)} C_{1,l,-},
\] (3.34)

\[
\dot{C}_{1,l,+} = - \left[ \frac{\gamma}{4} + \frac{\kappa}{2} + (n_{Th} + 1) \Gamma l + n_{Th} \Gamma (l + 1) \right] C_{1,l,+}
- \frac{E'}{\sqrt{2}} e^{it(\Omega_1 l - \Omega_0 l + g)} C_{0,l,g}
- \left[ \frac{\gamma}{4} - \frac{\kappa}{2} \right] e^{it(\Omega_1 l - \Omega_1 l + 2g)} C_{1,l,-}
+ \frac{E'}{2} (1 + \sqrt{2}) e^{it(\Omega_1 l - \Omega_2 l + g(1 - \sqrt{2}))} C_{2,l,+}
+ \frac{E'}{2} (1 - \sqrt{2}) e^{it(\Omega_1 l - \Omega_2 l + g(1 + \sqrt{2}))} C_{2,l,-},
\] (3.35)
\[ \dot{C}_{1,l,-} = -\frac{\gamma}{4} + \frac{\kappa}{2} + (n_{Th} + 1)\Gamma l + n_{Th}\Gamma(l + 1)\left[C_{1,l,-} \right. \\
+ \frac{E'}{\sqrt{2}}e^{i(\Omega_1 - l - \Omega_0 - g)}C_{0,l,g} \\
+ \frac{E'}{2}(1 - \sqrt{2})e^{i(\Omega_1 - l - \Omega_1 + l + g(1 + \sqrt{2}))}C_{1,l,+} \\
+ \frac{E'}{2}(1 + \sqrt{2})e^{i(\Omega_1 - l - \Omega_1 - l + g(1 - \sqrt{2}))}C_{2,l,+} \\
\left. - \frac{\gamma}{4} - \frac{\kappa}{2} e^{i(\Omega_1 - l - \Omega_2 - l - g(2\sqrt{2}))}C_{2,l,-} \right] \]

(3.36)

\[ \dot{C}_{2,l,+} = -\frac{\gamma}{4} + \frac{3\kappa}{2} + (n_{Th} + 1)\Gamma l + n_{Th}\Gamma(l + 1)\left[C_{2,l,+} \right. \\
+ \frac{1}{\sqrt{2}}e^{i(\Omega_2 + l - \Omega_1 + l + g(\sqrt{2} - 1))}C_{1,l,+} \\
+ \frac{1}{\sqrt{2}}e^{i(\Omega_2 + l - \Omega_1 - l + g(\sqrt{2} + 1))}C_{1,l,-} \\
- \frac{\gamma}{4} - \frac{\kappa}{2} e^{i(\Omega_2 - l - \Omega_2 - l + g(2\sqrt{2}))}C_{2,l,-} \right] \]

(3.37)

\[ \dot{C}_{2,l,-} = -\frac{\gamma}{4} + \frac{3\kappa}{2} + (n_{Th} + 1)\Gamma l + n_{Th}\Gamma(l + 1)\left[C_{2,l,-} \right. \\
+ \frac{1}{\sqrt{2}}e^{i(\Omega_2 - l - \Omega_1 + l + g(\sqrt{2} + 1))}C_{1,l,+} \\
- \frac{1}{\sqrt{2}}e^{i(\Omega_2 + l - \Omega_1 - l + g(\sqrt{2} - 1))}C_{1,l,-} \\
- \frac{\gamma}{4} - \frac{\kappa}{2} e^{i(\Omega_2 - l - \Omega_2 - l + g(\sqrt{2} - 1))}C_{2,l,+} \right] \]

(3.38)

\[ \dot{C}_{m,l,+} = -\frac{\gamma}{4} + \kappa(m - 1/2) + (n_{Th} + 1)\Gamma l + n_{Th}\Gamma(l + 1)\left[C_{m,l,+} \right. \\
- \frac{E'}{2}\left[\sqrt{m} + \sqrt{m - 1}\right]e^{i(\Omega_0 + l - \Omega_{m - 1} + l + g(\sqrt{m} - \sqrt{m - 1}))}C_{m-1,l,+} \\
+ \frac{E'}{2}\left[\sqrt{m} - \sqrt{m - 1}\right]e^{i(\Omega_0 + l - \Omega_{m - 1} - l + g(\sqrt{m} + \sqrt{m - 1}))}C_{m-1,l,-} \right] \]
\[
\dot{C}_{m,l,-} = - \left[ \frac{\gamma}{4} - \frac{\kappa}{2} \right] e^{it(\Omega_{m,-l} - \Omega_{m,+l} + g(2\sqrt{2}))} C_{m,l,-} - \frac{E'}{2} \left[ \sqrt{m} + \sqrt{m+1} \right] e^{it(\Omega_{m,-l} - \Omega_{m,+l} + l + g(\sqrt{m} + \sqrt{m+1}))} C_{m+1,l,+} + \frac{E'}{2} \left[ \sqrt{m} - \sqrt{m+1} \right] e^{it(\Omega_{m,-l} - \Omega_{m,+l} - l + g(\sqrt{m} - \sqrt{m+1}))} C_{m+1,l,-},
\]

(3.39)

\[
\dot{C}_{m,l,=} = - \left[ \frac{\gamma}{4} + \kappa(m - 1/2) + (n_{Th} + 1)\Gamma l + n_{Th} \Gamma (l + 1) \right] C_{m,l,-} + \frac{E'}{2} \left[ \sqrt{m} - \sqrt{m-1} \right] e^{it(\Omega_{m,-l} - \Omega_{m,-l-1} - l - g(\sqrt{m} - \sqrt{m-1}))} C_{m-1,l,+} - \frac{E'}{2} \left[ \sqrt{m} + \sqrt{m-1} \right] e^{it(\Omega_{m,-l} - \Omega_{m,+l} - l - g(\sqrt{m} + \sqrt{m-1}))} C_{m-1,l,-} - \left[ \frac{\gamma}{4} - \frac{\kappa}{2} \right] e^{it(\Omega_{m,-l} - \Omega_{m,+l} - l + g(\sqrt{m}))} C_{m,l,+} + \frac{E'}{2} \left[ \sqrt{m} - \sqrt{m+1} \right] e^{it(\Omega_{m,-l} - \Omega_{m,+l+1} - l - g(\sqrt{m} - \sqrt{m+1}))} C_{m+1,l,+} + \frac{E'}{2} \left[ \sqrt{m} + \sqrt{m+1} \right] e^{it(\Omega_{m,-l} - \Omega_{m,+l+1} - l - g(\sqrt{m} - \sqrt{m+1}))} C_{m+1,l,-}.
\]

(3.40)

### 3.3 Solving the “C dot” Equations

In the weak field limit, only states with 2 or fewer quanta of energy are left within the basis (we must keep states with at least two photons, as we wish to calculate photon statistics). Our task is therefore to solve the coupled sets of differential equations 3.34 - 3.38. The limit we are considering is one in which \( E' \to 0; \) this allows us to scale our equations in orders of \( E' \). If no driving field were applied, the atom would certainly be in the ground state, so we make the approximation that for weak fields \( C_{0,l,g} \sim 1 \). The driving field is responsible for populating the atom’s excited states, and thus \( C_{1,l,\pm} \sim E C_{0,l,g} \sim E' \). This reasoning can be continued to reveal that our scaling to two excitation states should be

\[ C_{0,l,g} \sim 1 \]
\[
C_{1,l,\pm} \sim E' \\
C_{2,l,\pm} \sim E'^2.
\] (3.41)

With equation 3.41 as our scaling, we may omit terms from the right sides of equations 3.34 - 3.38 of higher order in \(E'\) than the left sides, as in the limit that \(E' \to 0\) the lower order terms greatly dominate in contribution to the dynamics. This approximation dictates that our results are only valid in the \(E' \to 0\) limit.

As a first step in solving the “C dot” equations, we note that we can solve the equations of motion for the slowly varying population amplitudes \(\dot{D}_{n,k,\pm,}\), defined as

\[
\begin{align*}
D_{0,l,g} &= C_{0,l,g}, \\
D_{1,l,\pm} &= C_{1,l,\pm} e^{-it(\Omega_{1,\pm l - \Omega_0 l})}, \\
D_{2,l,\pm} &= C_{2,l,\pm} e^{-it(\Omega_{2,\pm l - \Omega_0 l} \sqrt{2} g)}.
\end{align*}
\] (3.42)

With the “E scaling” approximation, and the use of the slowly varying population amplitudes, we find that our \(\dot{C}_{n,l,\pm}\) equations become (notice that the term proportional to \(C_{2,l,+}\) has been eliminated from equation 3.35 because it’s multiplier is \(E'\), this would result in a term \(\sim E'^3\))

\[
\begin{align*}
\dot{D}_{0,l,g} &= \dot{C}_{0,l,g} = 0, \\
\dot{D}_{1,l,+} &= -\left(\frac{\gamma}{4} + \frac{\kappa}{2} + i(\Omega_{1,+ l - \Omega_0 l} + g)\right)D_{1,l,+} \\
&\quad - \frac{E'}{\sqrt{2}} D_{0,l,g} \\
&\quad - \left[\frac{\gamma}{4} - \frac{\kappa}{2}\right]D_{1,l,-},
\end{align*}
\] (3.44)

\[
\begin{align*}
\dot{D}_{1,l,-} &= -\left(\frac{\gamma}{4} + \frac{\kappa}{2} + i(\Omega_{1,- l - \Omega_0 l} - g)\right)D_{1,l,-} \\
&\quad + \frac{E'}{\sqrt{2}} D_{0,l,g} \\
&\quad - \left[\frac{\gamma}{4} - \frac{\kappa}{2}\right]D_{1,l,+},
\end{align*}
\] (3.45)
\[
\dot{D}_{2,l,+} = - \left( \frac{\gamma}{4} + \frac{3\kappa}{2} + (i(\Omega_{2,l} + l - \Omega_0 l + \sqrt{2}g)) \right) D_{2,l,+} \\
+ E' \left( \frac{1}{\sqrt{2}} + \frac{1}{2} \right) D_{1,l,+} \\
+ E' \left( \frac{1}{\sqrt{2}} - \frac{1}{2} \right) D_{1,l,-} \\
- \left( \frac{\gamma}{4} - \frac{\kappa}{2} \right) D_{2,l,-},
\]

(3.46)

\[
\dot{D}_{2,l,-} = - \left( \frac{\gamma}{4} + \frac{3\kappa}{2} + (i(\Omega_{2,l} - l - \Omega_0 l - \sqrt{2}g)) \right) D_{2,l,-} \\
+ E' \left( \frac{1}{\sqrt{2}} - \frac{1}{2} \right) D_{1,l,+} \\
- E' \left( \frac{1}{\sqrt{2}} + \frac{1}{2} \right) D_{1,l,-} \\
- \left( \frac{\gamma}{4} - \frac{\kappa}{2} \right) D_{2,l,+}.
\]

(3.47)

In equations 3.43 through 3.47 we have taken \( \Gamma \) to be 0; this condition removes the built-in decay of an atom’s vibrational state. Physically, we have chosen \( \Gamma \ll \gamma, \kappa \) and the atom will take on a vibrational population amplitude distribution on a time scale much shorter than the system’s evolution. We are thus required to account for the vibrational level populations on our own, imposing some distribution for \( l \) upon our system. In our simulations of the system, we will explore the effects of several distinct vibrational level distributions, whose forms will be specified in further chapters.

As we have declared \( D_{0,l,g} \) to be a constant \( \sim 1 \), we note that equations 3.44 and 3.45 form a set of coupled, first-order, inhomogeneous differential equations.

As discussed in section 1.4, the Quantum Trajectories method follows the discontinuous evolution of a \textit{conditioned} wavefunction \( |\Psi_c(t)\rangle \); but, we still have not specified how the nature of the system’s conditioning. We may begin looking at the system at any time, at the occurrence of any event. Only a few events, however, make any physical sense to the problem at hand. We will condition our system by letting it evolve for many lifetimes of the slowest process (either cavity decay or atomic lifetime), eventually reaching the steady state \( |\Psi_{ss}\rangle \). When the sys-
tem has reached the steady state, we assume either a transmission collapse event or a fluorescence collapse event occurs; we say collapse because the steady state wavefunction experiences a discontinuous decay event and defines \( |\Psi_c\rangle \) by

\[
|\Psi_c\rangle = \begin{cases} a|\Psi_{ss}\rangle = \frac{|\Psi_{CT}(0)\rangle}{||\Psi_{CT}(0)\rangle||^2} : \text{Transmission} \\ \sigma_-|\Psi_{ss}\rangle = \frac{|\Psi_{CF}(0)\rangle}{||\Psi_{CF}(0)\rangle||^2} : \text{Flourescence} \end{cases}
\]  

(3.48)

Equation 3.48 requires that we know the form of the steady state wavefunction \( |\Psi_{ss}\rangle \) before we can solve for any other parameter of the system at any other time, we therefore calculate \( |\Psi_{ss}\rangle \).

In the steady state, all population amplitudes \( D_{n,k,\pm} \) are constant, and we may set all \( \dot{D}_{n,k,\pm} = 0 \). Equations 3.44 and 3.45 then become (in a matrix form that makes the coupling obvious)

\[
\left( \begin{array}{cc}
\frac{3}{4} + \frac{\gamma}{2} + i(\Omega_1 - l - \Omega_0 l + g) \\
\frac{3}{4} - \frac{\gamma}{2} + i(\Omega_1 - l - \Omega_0 l - g)
\end{array} \right)
\left( \begin{array}{c}
D_{1,l,+}^{ss} \\
D_{1,l,-}^{ss}
\end{array} \right)
= \left( \begin{array}{c}
-\frac{E'}{\sqrt{2}} E' \\
\frac{E'}{\sqrt{2}}
\end{array} \right) D_{0,l,g}^{ss}.
\]

(3.49)

Solving for \( D_{1,l,+}^{ss} \) and \( D_{1,l,-}^{ss} \), we find

\[
D_{1,l,+}^{ss} = \frac{A_l(G + H_l)}{F_l H_l - G^2},
\]

\[
D_{1,l,-}^{ss} = \frac{-A_l(G + F_l)}{F_l H_l - G^2},
\]

(3.50)

where

\[
A_l = \frac{E'}{\sqrt{2}} D_{0,l,g},
\]

\[
G = -\frac{\gamma}{4} - \frac{\kappa}{2},
\]

\[
H_l = -\frac{\gamma}{4} + \frac{\kappa}{2} + i(\Omega_{1,1} - \Omega_0 l - g),
\]

\[
F_l = -\frac{\gamma}{4} + \frac{\kappa}{2} + i(\Omega_{1,1} - \Omega_0 l + g).
\]

(3.51)

Using the same procedure, we may use our results for \( D_{1,l,+}^{ss} \) and \( D_{1,l,-}^{ss} \) and solve equations 3.46 and 3.47 for \( D_{2,l,+}^{ss} \) and \( D_{2,l,-}^{ss} \), finding

\[
D_{2,l,+}^{ss} = \frac{-G \beta_{2,l} - Z_l \beta_{1,l}}{G^2 - Y_l Z_l},
\]

\[
D_{2,l,-}^{ss} = \frac{-G \beta_{1,l} - Y_l \beta_{2,l}}{G^2 - Y_l Z_l}.
\]

(3.52)
where

\[ Y_l = \frac{\gamma}{4} + \frac{3\kappa}{2} + i(\Omega_2, l - \Omega_0 l + \sqrt{2} g), \]
\[ Z_l = \frac{\gamma}{4} + \frac{3\kappa}{2} + i(\Omega_2, l - \Omega_0 l - \sqrt{2} g), \]
\[ \beta_{1,l} = -E' \left[ \frac{1}{\sqrt{2}} + \frac{1}{2} \right] D_{SS}^{1,l,+} + E' \left[ \frac{1}{\sqrt{2}} - \frac{1}{2} \right] D_{SS}^{1,l,-}, \]
\[ \beta_{2,l} = +E' \left[ \frac{1}{\sqrt{2}} - \frac{1}{2} \right] D_{SS}^{1,l,+} - E' \left[ \frac{1}{\sqrt{2}} + \frac{1}{2} \right] D_{SS}^{1,l,-}. \] (3.53)

Now that the steady state values for the population amplitudes have been calculated, our task is to solve for the time evolution of \( D_{1,l,+}(t) \) and \( D_{1,l,-}(t) \).

As before, it helps to see the coupling of equations 3.44 and 3.45 if we write them in matrix form as

\[ \dot{\vec{A}}(t) = M \vec{A}(t) + \vec{\Delta}, \] (3.54)

where

\[ \vec{A}(t) = \begin{pmatrix} \dot{D}_{1,l,+}(t) \\ \dot{D}_{1,l,-}(t) \end{pmatrix}, \]
\[ M = \begin{pmatrix} -\left[ \frac{\gamma}{4} + \frac{\kappa}{2} + i(\Omega_{1,+} l - \Omega_0 l + g) \right] & -\left[ \frac{\gamma}{4} - \frac{\kappa}{2} \right] \\ -\left[ \frac{\gamma}{4} - \frac{\kappa}{2} \right] & -\left[ \frac{\gamma}{4} + \frac{\kappa}{2} + i(\Omega_{1,-} l - \Omega_0 l - g) \right] \end{pmatrix}, \]
\[ \vec{\Delta} = E' \begin{pmatrix} -1 \\ 1 \end{pmatrix} D_{0,l,g}. \] (3.55)

In solving equation 3.54, we choose a full matrix approach that is rather elegant and compact. First, we make the definition

\[ \vec{B} = \vec{A} + M^{-1} \vec{\Delta}, \] (3.56)

such that (following our assumption that \( D_{0,l,g} \) is constant)

\[ \dot{\vec{B}} = \vec{A} = M(\vec{B} - M^{-1} \vec{\Delta}) + \vec{\Delta} = M \vec{B}. \] (3.57)
Now, we state that
\[ \vec{C} = S^{-1} \vec{B}, \]  
where \( S^{-1} \) is time-independent. Thus, we get the following equation of motion for \( \vec{C} \).
\[ \dot{\vec{C}} = S^{-1} MS\vec{C}. \]  
(3.59)
Now, as we have not yet specified the form of matrix \( S \), we note that a wise choice would be such that \( S^{-1} MS \) is a diagonal matrix. This can be accomplished by defining the columns of \( S \) to be the eigenvectors of matrix \( M \). Now, we’re left with the very simple equation of motion
\[ \dot{\vec{C}} = S^{-1} MS\vec{C} = \Lambda \vec{C}, \]  
(3.60)
where \( \Lambda = S^{-1} MS \) is a diagonal matrix, and \( \vec{C}(t) \) is simply
\[ \vec{C}(t) = e^{\Lambda t} \vec{C}(0). \]  
(3.61)
By stepping back through our previous substitution, we arrive at
\[ \vec{B}(t) = Se^{\Lambda t}S^{-1} \vec{B}(0), \]  
(3.62)
and ultimately, we attain the form of the time evolution of \( D_{1,l,+}(t) \) and \( D_{1,l,-}(t) \):
\[ \vec{A}(t) = \left( Se^{\Lambda t}S^{-1} \right) \vec{A}(0) + \left( Se^{\Lambda t}S^{-1} - 1 \right) M^{-1} \vec{A}. \]  
(3.63)
So, all that’s left to do to attain the solutions for \( D_{1,l,+}(t) \) and \( D_{1,l,-}(t) \) is calculate the appropriate eigenvectors/eigenvalues of \( M \), and find the inverses of matrices \( M \) and \( S \). Without showing the details of such calculations, we arrive at
\[
D_{1,l,+}(t) = \left[ \frac{\beta_2'}{2\chi_2} D_{1,l,+}(0) + \frac{G}{2\chi_2} D_{1,l,-}(0) + \frac{E' D_{0,l,g}}{2\chi_2 \phi \sqrt{2}} \left[ -H_i \beta_2^2 + G^2 - G\beta_2 + GF_i \right] \right] e^{\lambda_1 t} \\
+ \left[ \frac{-\beta_1'}{2\chi_2} D_{1,l,+}(0) - \frac{G}{2\chi_2} D_{1,l,-}(0) + \frac{E' D_{0,l,g}}{2\chi_2 \phi \sqrt{2}} \left[ H_i \beta_1^2 - G^2 + G\beta_1 - GF_i \right] \right] e^{\lambda_2 t} \\
+ \left[ \frac{H_i + G E'}{\phi \sqrt{2}} \right] D_{0,l,g},
\]  
(3.64)
\[
D_{1,l,-}(t) = \left[ \frac{-\beta_1'}{2\chi_2} D_{1,l,-}(0) - \frac{\beta_1' \beta_2'}{2G\chi_2} D_{1,l,+}(0) + \frac{E' D_{0,l,g}}{2\chi_2 \phi \sqrt{2}} \left[ \frac{H_i}{G} \beta_1 \beta_2' - G\beta_1^2 + G\beta_1' \beta_2 - \beta_1' F_i \right] \right] e^{\lambda_1 t} \\
+ \left[ \frac{\beta_2'}{2\chi_2} D_{1,l,-}(0) + \frac{\beta_1' \beta_2'}{2G\chi_2} D_{1,l,+}(0) + \frac{E' D_{0,l,g}}{2\chi_2 \phi \sqrt{2}} \left[ -\frac{H_i}{G} \beta_1 \beta_2' + G\beta_2^2 - \beta_1' \beta_2 + \beta_2' F_i \right] \right] e^{\lambda_2 t} \\
- \left[ \frac{F_i + G E'}{\phi \sqrt{2}} \right] D_{0,l,g},
\]  
(3.65)
where

\begin{align*}
\chi_1 &= \frac{F_l + H_l}{2}, \\
\chi_2 &= \sqrt{(F_l - H_l)^2 + 4G^2} \\
\lambda_1 &= \chi_1 + \chi_2, \\
\lambda_2 &= \chi_1 - \chi_2, \\
\beta'_1 &= F_l - \lambda_1, \\
\beta'_2 &= F_l - \lambda_2, \\
\phi &= F_l H_l - G^2.
\end{align*} (3.66)

We are now equipped with all the necessary information to solve for the dynamics of our system in the weak field limit. We will investigate the various photon statistics of the system in chapter 4, and the field-intensity correlation function in chapter 5.
Chapter 4

Photon Statistics

4.1 Second order intensity correlation function

A property of the system that is of interest to us is the photon statistics of the transmitted and fluoresced fields. What we are looking for are patterns or certain characteristics of the intensity fluctuations for the fields. This can be calculated by making use of another two-time correlation function, the second-order intensity-intensity correlation function. The conditional probability that we detect a second photon after a delay time $\tau$, having detected a primary photon at time $t$, is given by

\[ G^{(2)}(\tau) = \langle : I(t + \tau) I(t) : \rangle \]

\[ = \langle a^\dagger(t) a^\dagger(t + \tau) a(t + \tau) a(t) \rangle, \]

(4.1)

where we have used $I(t) = E^*(t) E(t)$, modeled the detector to be an absorptive device, and $\langle : : \rangle$ denotes the normal ordering seen earlier. It is customary to use a normalized version of equation 4.1, called the normalized second-order correlation function. To normalize equation 4.1, we divide by the probability of detecting two
uncorrelated photons at an infinite delay.

\[
G^{(2)}(\infty) = \langle a^\dagger(0)a(0)\rangle \langle a^\dagger(\infty)a(\infty)\rangle = \langle a^\dagger a\rangle^2. \tag{4.2}
\]

This is equivalent to normalizing equation 4.1 to the conditional probability of detecting a photon at time \(\tau\) given a detection at time 0 for a classical field of the same mean intensity and well-defined phase. Such a field will be a coherent state that obeys Poisson statistics, and the conditional probability will factor as

\[
\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\rangle = \langle a^\dagger(0)a(0)\rangle^2 = \langle a^\dagger a\rangle^2. \tag{4.3}
\]

The normalized second order correlation function is thus given by

\[
g^{(2)}(t, \tau) = \frac{G^{(2)}(t, \tau)}{G^{(2)}(\infty)} = \frac{\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle}{\langle a^\dagger a\rangle^2}, \tag{4.4}
\]

where \(t\) denotes the starting time for the correlation measurement. As the fields we are dealing with are assumed to be statistically stationary, equation 4.4 only depends upon the delay time \(\tau\), and we may declare \(t \to 0\) such that \(t + \tau \to \tau\). We may now refer to equation 4.4 as \(g^{(2)}(\tau)\). The quantum regression and trajectory theories may be applied to evaluate the second order temporal correlation function \(g^{(2)}(\tau)\) for both the transmitted (TT) and fluoresced fields (FF), as well as cross-correlations between the two fields (TF and FT correlation). The behavior of \(g^{(2)}(\tau)\) can help to characterize the field(s), and determine if it demonstrates non-classical behavior.

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4.2 The Schwartz Inequality and $g^{(2)}(\tau)$

The Schwartz inequality must be obeyed by any well-behaved, classical functions, and the inequality may take different forms depending on the functions considered. In the form we are concerned with, the Schwartz inequality is

$$\left( \int dx \int dy |f(x,y)g(x,y)| P(x,y) \right)^2 \leq \left( \int dx \int dy f(x,y)^2 P(x,y) \right) \times \left( \int dx \int dy g(x,y)^2 P(x,y) \right).$$

(4.5)

By absorbing the joint probability function $P(x,y)$ into $f(x,y)$ and $g(x,y)$ as

$$\bar{f}(x,y) = f(x,y)\sqrt{P(x,y)},$$
$$\bar{g}(x,y) = g(x,y)\sqrt{P(x,y)},$$

(4.6)

we may write the Schwartz inequality in its more familiar form: [14].

$$\left( \int dx \int dy |\bar{f}(x,y)\bar{g}(x,y)| \right)^2 \leq \left( \int dx \int dy \bar{f}(x,y)^2 \right) \times \left( \int dx \int dy \bar{g}(x,y)^2 \right).$$

(4.7)

By choosing $x = \bar{I}, y = \bar{I}_0, f(x,y) = \bar{I}, g(x,y) = 1$, and $P(x,y) = P(\bar{I},t+\tau; \bar{I}_0,t)$ where $P(\bar{I},t+\tau; \bar{I}_0,t)$ is the joint probability function that there is field intensity $\bar{I}$ at time $t + \tau$ and intensity $\bar{I}_0$ at time $t$, equation 4.5 becomes

$$\left( \int d\bar{I} \, \bar{I} P(\bar{I},t+\tau) \right)^2 \leq \left( \int d\bar{I} \, \bar{I}^2 P(\bar{I},t+\tau) \right) \times 1$$
$$\langle \bar{I} \rangle^2 \leq \langle \bar{I}^2 \rangle$$
$$\frac{\langle \bar{I}^2 \rangle}{\langle \bar{I} \rangle^2} \geq 1.$$

(4.8)

Next, we write equation 4.8 in terms of the field amplitude, and ultimately in terms of the quantum mechanical field operators ($a$ and $a^\dagger$):
Exploiting the stationarity of the field, we may write this as

\[
\frac{\langle E^*(t)^2 E(t)^2 \rangle}{E^*(t) E(t))^2} \geq 1
\]

\[
\frac{\langle a^\dagger(t)^2 a(t)^2 \rangle}{\langle a^\dagger(t) a(t)\rangle^2} \geq 1.
\]  \hspace{1cm} (4.9)

The left side of equation 4.10 is now just equation 4.4 with zero delay time \((\tau = 0)\), and we see that the Schwartz inequality demands

\[
\frac{\langle a^\dagger(t)^2 a(t)^2 \rangle}{\langle a^\dagger a\rangle^2} \geq 1.
\]  \hspace{1cm} (4.10)

for any field obeying a classical probability distribution. If \(g^{(2)}(0) < 1\), then there is no probability distribution \(P(x, y)\) that is strictly non-negative describing the field.

The Schwartz inequality also demands for classical fields

\[
g^{(2)}(\tau) \leq g^{(2)}(0),
\]

\[
|g^{(2)}(\tau) - 1| \leq g^{(2)}(0) - 1.
\]  \hspace{1cm} (4.12)

The proofs of these inequalities are similar to that above, and are not included. Breaking one of the Schwartz inequalities reveals that the field cannot be described by a classical probability distribution, and is therefore exhibiting quantum mechanical behavior. \(g^{(2)}(0) < g^{(2)}(\tau)\) denotes photon anti-bunching, where pairs of photons are less likely to be very close together; thus, very short and very long delays between successive photons are avoided in favor of a more uniform delay
pattern. $g^{(2)}(0) < 1$ denotes Sub-Poissonian statistics that are indicative of a non-classical field, as a classical field would have a zero-delay correlation function value greater than or equal to 1. There are also violations of the Schwartz inequality known as “overshoot” and “undershoot”, where $|g^{(2)}(\tau) - 1| > |g^{(2)}(0) - 1|$ and $|g^{(2)}(\tau) - 1| < |g^{(2)}(0) - 1|$, respectively.

4.3 $g^{(2)}(\tau)$ in the Weak Field Limit

The second order intensity correlation function is defined as in equation 4.4, it is the autocorrelation of the intensity at some time $t$ with the intensity at another time $t + \tau$. This form assumes that the experimental intensity measurement is a photon-absorptive process, and employs the quantum mechanical field operators. This form also utilizes normal and time ordering of the operators. Because $g^{(2)}(\tau)$ is only dependent upon the delay time $\tau$, and the fields are stationary, that is

$$\langle a^\dagger(0)a(0) \rangle = \langle a^\dagger(\tau)a(\tau) \rangle \equiv \langle a^\dagger a \rangle_{SS}, \quad (4.13)$$

we may write $g^{(2)}(\tau)$ in it’s more conventional form, with $t = 0$ denoting the time the first photon is detected.

$$g^{(2)}(\tau) = \frac{\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0) \rangle}{\langle a^\dagger a \rangle_{SS}^2}, \quad (4.14)$$

where the subscript $SS$ denotes the expectation of $a^\dagger a$ taken in the system steady state.

In an experiment, the system steady state is achieved by waiting a few lifetimes (cavity or atom, whichever is longer) after turning the driving laser on; this allows sufficient time for the transients of the atom-cavity system to decay away.
To calculate $g^{(2)}(\tau)$, we can use quantum trajectory simulations as discussed in section 2.3.2. One waits 10 lifetimes say, and then stores the average photon number as a function of time. This would be calculated as an expectation value of the quantum trajectory wave function,

$$|\psi(t)\rangle = \sum_{n,\{m\}} C_{n,\{m\}}(t)e^{-iE_{n,\{m\}}t/\hbar} |n,\{m\}\rangle,$$

where we have taken $n$ to be the photon number, and $\{m\}$ to be the set of all other quantum numbers like the internal state of the atom, or its vibrational state number if there is an external trapping potential applied. Using this wave function in the usual way, one finds

$$\langle a^\dagger a \rangle = \langle \psi | a^\dagger a | \psi \rangle = \sum_{n,\{m\}=0}^{\infty} n |C_{n,\{m\}}|^2. \quad (4.16)$$

In the weak field limit, where the driving field occasionally deposits a photon into the cavity, we can truncate the expansion at some level, and to first order we have

$$\langle a^\dagger a \rangle = \sum_{\{m\}=0}^{\infty} |C_{1,\{m\}}|^2. \quad (4.17)$$

In the trajectory calculation, one obtains $\langle n \rangle = \langle a^\dagger a \rangle$ as a function of time, for one realization of the trajectory. To obtain $\langle n \rangle$ one must perform a number of simulations, and average over the ensemble of trajectories. Carmichael has shown that this yields the same result as the usual density matrix approach [10]. The advantage of the trajectory method is that the trajectories are specified by $N$ complex amplitudes, where $N$ is the number of degrees of freedom (or equivalently the dimension of the Hilbert space) instead of $N(N-1)/2$ density matrix elements; so,
one must solve fewer equations. But, as discussed in section 2.3.1, those equations must be solved over and over again to achieve a good ensemble, typically $5 - 50$ thousand separate runs. For an ergodic process, a time average for one system is assumed to be equivalent to the ensemble average. If we work under the assumption that our system is ergodic, one can choose to perform a time average over only one trajectory. Choosing either the ensemble or ergodic approach usually is a decision based upon available computational speed and memory. Although $g^{(2)}(\tau)$ can be calculated using the quantum regression theorem, and the density matrix representation, the trajectories method is well suited to the physicist, as his/her intuition of the underlying physics relies more typically upon wavefunctions than density matrices.

In the weak field limit, one only considers states with two photons, or more generally states with an energy equal to or less than $2\hbar \omega$. For the atom cavity system (with no quantized center of mass motion) those states would be

\[ |0,g\rangle, \]
\[ |0,e\rangle, \]
\[ |1,g\rangle, \]
\[ |1,e\rangle, \]
\[ |2,g\rangle, \]

where the number indicates the number of photons in the field and $e(g)$ denotes the atom in the excited (ground) state. In general the state vector will take the
The procedure is to develop the equations of motion for the various state coefficients (the “C-dot” equations), then solve them in the steady state ($\dot{C}_{n,e/g} = 0$). The steady state coefficients will determine the initial conditions for the collapsed wave function which is then propagated in time, and $\langle n(\tau) \rangle_c$ can be calculated. The steady state coefficients can be used to calculate $\langle n \rangle_{SS}$.

In the weak field limit, it is possible to obtain analytic results for $g^{(2)}(\tau)$ utilizing the trajectory theory. For weak driving fields, the system will settle into a steady state $|\psi_{SS}\rangle$ with $\langle n \rangle \ll 1$.

\begin{equation}
|\psi_{SS}\rangle = \sum_{n,\{m\}} C_{n,\{m\}}^{SS} e^{-iE_{n,\{m\}}t/\hbar} |n, \{m\}\rangle.
\end{equation}

The steady state is achieved as the non-Hermitian parts of the trajectory Hamiltonian result in decay terms in the equations for the probability amplitudes, while the driving terms result in growth. After some number of lifetimes, the growth and decay terms will reach a balance, defining a steady state. At some point a collapse happens, and the system is in the state

\begin{equation}
|\psi_c\rangle = \frac{\hat{a} |\psi_{SS}\rangle}{|\hat{a}|\psi_{SS}\rangle|}.
\end{equation}

where the factor in the denominator is for normalization (which is not preserved by the collapse, or the non-Unitary evolution steps). The system will evolve from
this collapsed state back to the steady state. The probability of a cavity collapse in a given time step is given by \( \kappa \langle n \rangle dt \), where \( \kappa \) is the cavity field decay rate. For a small enough time step this probability is quite low, and the system can stay in this steady state for several cavity lifetimes before another collapse.

We will find it useful to represent the collapsed state as

\[
| \psi_c(t) \rangle = \sum_{n, \{m\}} C_{n,\{m\}}^C(t) e^{-iE_{n,\{m\}}t/\hbar} | n, \{m\} \rangle.
\] (4.21)

The probability of a collapse event at a later time \( t = \tau \) is then given by

\[
\kappa \langle \psi_c(\tau) | n | \psi_c(\tau) \rangle dt \equiv \kappa \langle n(\tau) \rangle_c dt,
\] (4.22)

where \( | \psi_c(\tau) \rangle \) is the conditioned wave function, evolved (non-Unitarily) from the collapsed state for a time \( \tau \), and \( \langle n(\tau) \rangle_c \) the conditioned photon number. Thus the ratio \( \langle n(\tau) \rangle_c / \langle n \rangle_{SS} \) gives the probability of the next jump (corresponding to a detection event assuming perfect efficiency) happening at time \( t = \tau \) given a jump at \( t = 0 \) relative to that of a Poissonian field of the same mean intensity. This is not quite \( g^{(2)}(\tau) \). The ensemble of second jump times would give a distribution of waiting times between emissions; \( g^{(2)}(\tau) \) is the probability of a jump at \( t = \tau \) given a jump at \( t = 0 \), not the probability of the next jump happening at time \( \tau \).

In the weak field limit however, the photon flux is so small that the waiting time distribution and \( g^{(2)}(\tau) \) are equivalent; if there is a jump at \( t = 0 \) and another at \( t = \tau \) it is highly unlikely there was another jump in between. The probability of detection of a photon at \( t = 0 \) and another at \( t = \tau \) (the numerator of \( g^{(2)}(\tau) \)) can be written as

\[
\text{Prob}(0, \tau) = \text{Prob}(0) \times \text{Prob}(\tau|0),
\] (4.23)
where \( (\tau|0) \) denotes that we are interested in a probability at time \( \tau \) conditional upon an event having already happened at time \( t = 0 \). The probability of the first detection is proportional to the steady state photon number \( \langle n \rangle_{SS} \), while the conditional probability is given by \( \langle n(\tau) \rangle_c \). So for weak fields we conclude that

\[
g^{(2)}(\tau) = \frac{\langle n \rangle_{SS} \langle n(\tau) \rangle_c}{\langle n \rangle_{SS}^2} = \frac{\langle n(\tau) \rangle_c}{\langle n \rangle_{SS}}. \\
(4.24)
\]

This will be given by

\[
g^{(2)}(\tau) = \sum_{\{m\}=0}^{\infty} \frac{|C_{1,\{m\}}^C(\tau)|^2}{\sum_{\{m\}=0}^{\infty} |C_{1,\{m\}}^{SS}|^2}, \\
(4.25)
\]

where \( C_{1,\{m\}}^C(\tau) \) is the amplitude of the one photon states \textit{conditioned} on the state collapse at \( t = 0 \), and again \( C_{1,\{m\}}^{SS} \) is the set of steady state one photon amplitudes. Experimentally, it is common to measure the waiting time distribution.

### 4.4 Correlations between transmitted and fluorescent fields

So far we have looked at the detection of transmitted photons after first obtaining a transmitted photon. We are now in a position to discuss the detection of fluorescent photons after a previous detection of a fluorescent photon out the side of the cavity.

The fluorescent field is proportional to the dipole moment of the atom,

\[
\hat{E}_{fl} \propto \sigma_- e^{-i\omega t} + \sigma_+ e^{i\omega t}. \\
(4.26)
\]
The intensity of the fluorescent field is given by

\[ I_{fl} \propto \langle \sigma_+ \sigma_- \rangle, \tag{4.27} \]

and we can define dipole quadrature operators

\[ \hat{X}^{fl} = \sigma_- + \sigma_+, \tag{4.28} \]
\[ \hat{Y}^{fl} = \sigma_- - \sigma_+, \tag{4.29} \]
\[ \sigma_\theta = \sigma_- e^{-i\theta} + \sigma_+ e^{i\theta}. \tag{4.30} \]

With these we can define four second-order intensity correlation functions that are the probability of detecting a photon at \( t = \tau \) in transmission (T) or fluorescence (F), given that one was obtained in T or F at \( t = 0 \)

\[ g^{(2)}_{TT} = \frac{\langle a_\dagger(0)a_\dagger(\tau/a(\tau)a(0) \rangle}{\langle a\dagger a \rangle^2}, \tag{4.31} \]
\[ g^{(2)}_{TF} = \frac{\langle a_\dagger(0)\sigma_+(\tau)\sigma_-(\tau)a(0) \rangle}{\langle a\dagger a \rangle \langle \sigma_+ \sigma_- \rangle}, \tag{4.32} \]
\[ g^{(2)}_{FT} = \frac{\langle \sigma_+(0)a_\dagger(\tau)a(\tau)\sigma_-(0) \rangle}{\langle a\dagger a \rangle \langle \sigma_+ \sigma_- \rangle}, \tag{4.33} \]
\[ g^{(2)}_{FF} = \frac{\langle \sigma_+(0)\sigma_+(\tau)\sigma_-(\tau)\sigma_-(0) \rangle}{\langle \sigma_+ \sigma_- \rangle^2}. \tag{4.34} \]

In terms of quantum trajectories, the appropriate collapse operator for a fluorescence decay is \( \sqrt{\gamma} \sigma_- \), thus the collapsed state of the system after the emission of a fluorescent photon is

\[ |\psi_c^{F} \rangle = \frac{\sigma_- |\psi_{SS} \rangle}{| \sigma_- |\psi_{SS} \rangle |}. \tag{4.35} \]

The collapsed state after emission of a transmitted photon is, again

\[ |\psi_c^{T} \rangle = \frac{a |\psi_{SS} \rangle}{| a |\psi_{SS} \rangle |}. \tag{4.36} \]
In the weak field limit, we have four interesting second-order correlations

\[ g_{TT}^{(2)} = \frac{\langle a^\dagger(\tau) a(\tau) \rangle_T^T_c}{\langle a^\dagger a \rangle_{SS}}, \]

\[ g_{FT}^{(2)} = \frac{\langle a^\dagger(\tau) a(\tau) \rangle_F^T_c}{\langle a^\dagger a \rangle_{SS}}, \]

\[ g_{TF}^{(2)} = \frac{\langle \sigma_+(\tau) \sigma_-(\tau) \rangle_T^T_c}{\langle \sigma_+ \sigma_- \rangle_{SS}}, \]

\[ g_{FF}^{(2)} = \frac{\langle \sigma_+(\tau) \sigma_-(\tau) \rangle_F^F_c}{\langle \sigma_+ \sigma_- \rangle_{SS}}, \]

where \( \langle i \rangle_c \) denotes conditioning upon a detection event from mode \( i \), which can either be the transmitted field \( T \), or the fluorescent field \( F \). We next need to formulate the needed averages of dipole operators in terms of probability amplitudes, as in the case of the field operators. To do this, we need to specify the internal degrees of freedom in the state vector. We must explicitly state whether the atom is in the excited or ground state.

\[ | \psi(t) \rangle = \sum_{n,\{m\}} C_{n,e,\{m\}}(t) e^{-iE_{n,e,\{m\}}t/\hbar} | n, e, \{m\} \rangle + C_{n,g,\{m\}}(t) e^{-iE_{n,g,\{m\}}t/\hbar} | n, g, \{m\} \rangle, \]

where again \( \{m\} \) represents the remaining quantum numbers

\[ \langle \sigma_+ \rangle = \sum_{n,\{m\}} C^{*}_{n,e,\{m\}} C_{n,g,\{m\}}, \]

\[ \langle \sigma_+ \sigma_- \rangle = \sum_{n,\{m\}} |C_{n,e,\{m\}}|^2, \]

\[ \langle \sigma_\theta \rangle = \sum_{n,\{m\}} (C^{*}_{n,e,\{m\}} C_{n,g,\{m\}} e^{-i\theta} + C^{*}_{n,g,\{m\}} C_{n,e,\{m\}} e^{i\theta}). \]

So, for weak fields

\[ g_{TT}^{(2)} = \frac{\sum_{\{m\}=0}^\infty |C_{1,g,\{m\}}^{CT}(\tau)|^2}{\sum_{\{m\}=0}^\infty |C_{1,g,\{m\}}^{SS}|^2}. \]
The superscript $CT$ indicates a probability amplitude evolving from a collapse due to a transmitted photon. Here we have ignored terms involving $C_{1,e,\{m\}}$, and $C_{2,g,\{m\}}$ as they involve states with two excitations; we have also ignored higher order terms as well. The other intensity-intensity correlations are given by

\[ g_{FF}^{(2)} = \frac{\sum_{\{m\}=0}^{\infty} |C_{0,e,\{m\}}^{CF}(\tau)|^2}{\sum_{\{m\}=0}^{\infty} |C_{0,e,\{m\}}^{SS}|^2}, \quad (4.46) \]

\[ g_{TF}^{(2)} = \frac{\sum_{\{m\}=0}^{\infty} |C_{0,e,\{m\}}^{CT}(\tau)|^2}{\sum_{\{m\}=0}^{\infty} |C_{0,e,\{m\}}^{SS}|^2}, \quad (4.47) \]

\[ g_{FF}^{(2)} = \frac{\sum_{\{m\}=0}^{\infty} |C_{0,e,\{m\}}^{CF}(\tau)|^2}{\sum_{\{m\}=0}^{\infty} |C_{0,e,\{m\}}^{SS}|^2}. \quad (4.48) \]

The Schwartz inequality provides a condition pertaining to the intensity cross-correlations $g_{FT}^{(2)}(\tau)$ and $g_{TF}^{(2)}(\tau)$. To derive the condition that must be met by a classically behaving system we consider the Schwartz inequality in the form of equation 4.5 and make the following substitutions

\[ x = \bar{I}_i, \]

\[ y = \bar{I}_j, \]

\[ f(x,y) = \bar{I}_i, \]

\[ g(x,y) = \bar{I}_j, \]

\[ P(x,y) = P(\bar{I}_j, \tau; \bar{I}_i, \tau). \quad (4.49) \]

This gives

\[ \left( \int d\bar{I}_i \int d\bar{I}_j \bar{I}_i \bar{I}_j P(\bar{I}_j, \tau; \bar{I}_i, \tau) \right)^2 \leq \left( \int d\bar{I}_i \int d\bar{I}_j \bar{I}_i^2 P(\bar{I}_j, \tau; \bar{I}_i, \tau) \right) \times \left( \int d\bar{I}_i \int d\bar{I}_j \bar{I}_j^2 P(\bar{I}_j, \tau; \bar{I}_i, \tau) \right), \quad (4.50) \]
which, after the integrals are evaluated becomes

\[
\langle \bar{I}_i(\tau)\bar{I}_j(\tau) \rangle^2 \leq \langle \bar{I}_i^2(\tau) \rangle \langle \bar{I}_j^2(\tau) \rangle. \tag{4.51}
\]

Dividing both right and left sides by \(\langle \bar{I}_i(\tau) \rangle^2 \langle \bar{I}_j(\tau) \rangle^2\) and then taking the square root of each side leaves us with

\[
\frac{\langle \bar{I}_i(\tau)\bar{I}_j(\tau) \rangle}{\langle \bar{I}_i(\tau) \rangle \langle \bar{I}_j(\tau) \rangle} \leq \sqrt{\frac{\langle \bar{I}_i^2(\tau) \rangle \langle \bar{I}_j^2(\tau) \rangle}{\langle \bar{I}_i(\tau) \rangle^2 \langle \bar{I}_j(\tau) \rangle^2}}. \tag{4.52}
\]

Comparison to equations 4.1 and 4.4 shows that this is just a statement that

\[
g_{ij}^{(2)} \leq \sqrt{g_{ii}^{(2)} g_{jj}^{(2)}}. \tag{4.53}
\]

The equal-time cross-correlations must obey this inequality if the system is to be described using a classical probability distribution. Along with equations 4.11 and 4.12, violation of equation 4.53 provides us with evidence of a system’s non-classical behavior.

### 4.5 Time asymmetry and detailed balance

This section is by no means an attempt to provide a rigorous derivation of the origins of time asymmetry in cross-correlations. These origins lie deeply within the realm of statistical mechanics and a full treatment can be found in references [15] [16]. The intent here is to briefly present the physics behind the asymmetry that can be seen in the \(g_{ij}^{(2)}(\tau)\) data as has also been seen by A. Denisov.

Autocorrelations, like \(g_{TT}^{(2)}(\tau)\) and \(g_{FF}^{(2)}(\tau)\), are necessarily time-symmetric due the stationarity of the electric field (and hence it’s intensity). A formal statement
of this symmetry is

\[
\frac{\langle a^\dagger(t)a^\dagger(t + \tau)a(t + \tau)a(t) \rangle}{\langle a^\dagger a \rangle^2} = \frac{\langle a^\dagger(t - \tau)a^\dagger(t)a(t)a(t - \tau) \rangle}{\langle a^\dagger a \rangle^2}. \quad (4.54)
\]

A cross-correlation of fluctuations \( \langle B(t+\tau)A(t) \rangle \) about equilibrium (like \( g^{(2)}_{ij}(\tau) \)) however, will only exhibit time symmetry if the equilibrium is held with *detailed balance*. The definition of detailed balance can take many forms, depending on the nature of the system(s) and its(their) possible representations in a generalized configuration space. For the present purposes, it will suffice to give an example of detailed balance using elementary graph theory. If we choose to label the (quantized) states \(|i\rangle\) of our system, we can assign to each state a node in a 2-D plane. Let us now “connect” the states with edges. The existence of an edge between two states \(|i\rangle\) and \(|j\rangle\) means that if the system is in either state, there is some finite possibility for the system to be in the other state at some later time. We are presently ignoring state superpositions, they will be accounted for later to bring our example fully into a quantum mechanical representation. So far, our system is represented by the graph in figure 4.1.
Figure 4.1: A graph of the system, with states $|i\rangle$, $|j\rangle$ and $|k\rangle$ specifically labelled. The system may “move” directly from $|i\rangle$ to $|j\rangle$, but not from $|k\rangle$ to $|j\rangle$ without first going into some intermediary state.
We see from figure 4.1 that the system can “move” directly from $|i\rangle$ to $|j\rangle$, but not from $|k\rangle$ to $|j\rangle$ without first going into some intermediary state. Let us now account for directionality in our system, we replace the line edges with arrows denoting possible directions of “state travel” in configuration space. A double-sided arrow from $|i\rangle$ to $|j\rangle$ will denote that the system can move between states $|i\rangle$ and $|j\rangle$ in either direction. If we allow our system to move from any state to any other state, we have a complete directed graph that looks like figure 4.2 (one node has been omitted to reduce clutter)
Figure 4.2: A complete directed graph of the system, the system may move between any states in any direction.
Now we move our example into the quantum world by saying that our nodes are actually the probability amplitudes ($C_i$) for the states $|i\rangle$, $|j\rangle$, $|k\rangle$, etc. We associate with each edge (and in each direction) a maximum current flow $J_{i,j}$ that denotes the maximum amount of “probability current” that can flow from node $C_i$ to node $C_j$. A statement of balance for the system would be that for every probability amplitude $C_m$ (assuming current always flows maximally)

$$\sum_i J_{i,m} = \sum_j J_{m,j}, \quad (4.55)$$

which states that probability amplitude distribution remains constant and that the net probability current is 0. A statement of detailed balance would be

$$J_{i,m} = J_{m,i} \quad \forall \ m, i, \quad (4.56)$$

which states that for any pair of states, the net probability current between the two states is 0 and thus the overall net current is also 0.

Because the intensity cross-correlations will only display time-symmetry when the condition of detailed balance holds (the balance we consider here is between the atom and the cavity field mode), time-asymmetry indicates the lack of detailed balance. In order to look for the symmetry (or lack thereof), we therefore follow convention in our plotting of the cross-correlation data. For the following plots of $g_{TF}^{(2)}(\tau)$, the following plotting scheme is employed:

$$g_{TF}^{(2)}(\tau) = \begin{cases} 
  g_{TF}^{(2)}(\tau) : \tau \geq 0 \\
  g_{FT}^{(2)}(-\tau) : \tau \leq 0 
\end{cases} \quad (4.57)$$
4.6 Results and discussion

4.6.1 Transmitted and fluoresced field autocorrelations

Figures 4.3 through 4.26 present $g_{ii}^{(2)}(\tau)$ for $i \equiv T$ as well as $i \equiv F$. For all data presented, the vibrational state spacing is defined by

$$\Delta_{1,+} = \frac{\Omega_{1,+} - \Omega_0}{\gamma} = 0.1. \tag{4.58}$$

As previously stated, we are assuming the deep trapping limit where $\alpha \geq \frac{g_m\sqrt{n}}{\xi^2}$ in equation 3.22. Because of this severe imbalance between trapping potential and atom-cavity coupling, we may expand equation 3.22 in the binomial approximation, and define

$$\Omega_{n,\pm} \approx \sqrt{\frac{\hbar \alpha}{m}} \left[ 1 \pm \frac{g_m\sqrt{n}}{2\xi^2\alpha} \right], \tag{4.59}$$

therefore defining $\Delta n, \pm = \Omega n, \pm - \Omega_0$ for $n = 1, 2$ to be

$$\Delta_{1,+} = \frac{g_m\sqrt{n}}{2\xi^2\alpha},$$

$$\Delta_{1,-} = -\Delta_{1,+},$$

$$\Delta_{2,+} = \sqrt{2}\Delta_{1,+},$$

$$\Delta_{1,-} = -\sqrt{2}\Delta_{1,+}. \tag{4.60}$$

The energy level structure of the whole system is shown in figure 4.3.
Figure 4.3: Energy level structure for the quantized CMM cavity-QED system. Ω terms denote generalized vibrational state spacings as derived in chapter 3, $\hbar \omega$ denotes the transition energy of the atom, and $hg$ denotes the splitting of the first excited state due to the a.c. Stark shift.
Figures 4.3 through 4.26 present data first in frame (a) for the case that only the ground vibrational state $|l = 0\rangle$ is populated. Frame (b) shows the autocorrelations when the atom is in an even superposition of $|l = 0\rangle$ and $|l = 5\rangle$. Frame (c) presents data for when the atom’s vibrational states are “Boltzmann populated” using the distribution,

$$P_l = \left[\frac{e^{-l}}{\sum_l e^{-l}}\right]^{1/2},$$

(4.61)

for the population amplitudes. Frame (d) of the figures presents the case where each of the atom’s vibrational states is equally populated.

Figure 4.4 presents the photon statistics of the transmitted field when the system is mildly in the bad-cavity limit. In this regime, $\kappa$ is the fastest rate in the system, and the cavity leaks photons quite easily. We see that varying the vibrational distribution doesn’t change the qualitative shape of the curves, but effects the magnitude of bunching at $\tau = 0$. In frames (a) and (c), we see that the field exhibits an undershoot violation, whereas frames (b) and (d) show no non-classicalities. In frame (a), the atom is in a vibrational pure state, and the quantum mechanical nature of the statistics is shown. The Boltzmann-type distribution (with only 20 possible states) governs that the ratio of ground state population and first excited state population $P_0/P_1$ is about 2.718, a situation more analogous to a vibrational pure state then to the even superposition in frame (b), where no non-classicalities are observed. Even in the Boltzmann-type distribution, the second most populated state ($|l\rangle$) experiences only a relatively small detuning of $\Delta_{1,\pm}$, and thus the most populated vibrational levels are also the least detuned. Figure 4.5 shows the statistics for the fluorescent field for the same parameters,
and we see that the vibrational distribution has no strong effect on the shape of the curves, save for the relative “sharpness” of the curve in frame (d). We see the necessary perfect antibunching at $\tau = 0$, as one atom cannot emit two photons at zero delay.
Figure 4.4: Plots are $g^{(2)}_{TT}(\tau)$ for $g/\gamma = 2$, $\kappa/\gamma = 5$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.5: Plots are $g_{FF}^{(2)}(\tau)$ for $g/\gamma = 2$, $\kappa/\gamma = 5$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}}[|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.6 again shows a similarity between the ground state and Boltzmann distribution cases, however both exhibit strong antibunching. This is no doubt a function of the increase in atom-cavity mode coupling. 4.6(d) exhibits no non-classicalities, where 4.6(b) shows a slight undershoot violation, breaking the similarity shown in figure 4.4. Like figure 4.5, 4.7 shows no real effect from changing the vibrational distribution, in fact there is no strong effect from changing the atom-cavity parameters as $\gamma$ is still the smallest rate considered.
Figure 4.6: Plots are $g^{(2)}_{TT}(\tau)$ for $g/\gamma = 2.2$, $\kappa/\gamma = 10$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.7: Plots are $g_{FF}^{(2)}(\tau)$ for $g/\gamma = 2.2$, $\kappa/\gamma = 10$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
When we increase the effect of atomic decay by making $\gamma$ bigger than either $g$, we see that, while still in the bad-cavity regime ($g/\gamma = .866$, $\kappa/\gamma = 10$), the magnitude of antibunching is severely decreased from the previous plots when $g = 2.2$. This verifies that the strong antibunching in figure 4.6 as compared to figure 4.4 was due to the increased atom-cavity mode coupling. Figure 4.8(a), like 4.6(a) still shows sub-Poissonian statistics and photon antibunching, while 4.8(b) demonstrates sub-Poissonian statistics and an undershoot violation near $\gamma \tau = 0.15$. Figures 4.8(c) and 4.8(d) maintain the sub-Poissonian statistics and undershoot violations. Like figures 4.5 and 4.7, figure 4.9 shows no large changes in behavior for different vibrational-state distributions.

Figure 4.10 shows the photon statistics of the transmitted field in the good-cavity limit ($g/\gamma = .3$, $\kappa/\gamma = .1$). Again, we see a similarity between frames (a) and (c), and (b) and (d). Non-classical behavior is noted in frames (a) and (c), with sub-Poissonian statistics and antibunching, as well as an undershoot violation in frame (c). No non-classicalities are present in frames (b) and (d). Figure 4.11 shows that while the form of non-classicalities present in the fluorescent field’s autocorrelation do not change with vibrational distribution, the exact shape of the curves varies significantly; again, similarity is noted between frames (a) and (c), and (b) and (d). It is suspected that the energy level structure for the system will wash away non-classical behavior for vibrational states that are sufficiently “non-pure”. While the equations of motion presented in chapter 3 are of the same form for different vibrational levels, the rates of oscillation are different. So, while the effects of the distributions are similar to those of detunings as presented in [3],
they may be more properly described as “dephasing” effects. While the Boltzmann-type distribution is obviously not a pure vibrational state, the contribution of the vibrational ground state is highly dominant. Also, because this state evolves more slowly, it may be viewed as a more d.c. term while the more rapidly varying terms “interfere” with one another. Still in the good-cavity limit, figure 4.12 shows the transmitted field’s statistics for $g = 0.05$ and $\kappa = 0.1$. The only pure vibrational state is in frame (a), where we see very slightly sub-Poisonian characteristics, and equally slight antibunching. While it at first appears that frame (c) is more similar to (b) and (d) than (a) (none of the three show non-classical behavior), the scales of the plots reveal that the behavior of $g^{(2)}(\tau)$ in frame (c) is confined about 1.0 so tightly that it’s shape is not of much concern.

Figure 4.13 shows some initially curious behavior for $g^{(2)}_{FF}(\tau)$ in the good cavity limit, where one would expect that the fluorescent field to be robust even for small $\tau$. But with an atom-cavity mode coupling of $g = 0.05$, it is seen that the atom takes a while to re-excite, and then another time $\sim 1/\gamma$ to fluoresce, causing a long relaxation time back to steady-state. The less pure the vibrational distribution in figure 4.13, the more sharp the peak is at about $\gamma \tau = 5.0$. 
Figure 4.8: Plots are $g^{(2)}(\tau)$ for $g/\gamma = .866$, $\kappa/\gamma = 10$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.9: Plots are $g_{FF}^{(2)}(\tau)$ for $g/\gamma = 0.866$, $\kappa/\gamma = 10$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only.

(b) $\frac{1}{\sqrt{2}} (|0\rangle + |5\rangle)$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.10: Plots are $g^{(2)}_{TT} (\tau)$ for $g/\gamma = .3$, $\kappa/\gamma = .1$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} (|0\rangle + |5\rangle)$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.11: Plots are $g_{FF}^{(2)}(\tau)$ for $g/\gamma = .3$, $\kappa/\gamma = .1$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [ |0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.12. Plots are $g_{TT}^{(2)}(\tau)$ for $g/\gamma = .05$, $\kappa/\gamma = .1$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} (|0\rangle + |5\rangle)$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.13: Plots are $g_{FF}^{(2)}(\tau)$ for $g/\gamma = .05$, $\kappa/\gamma = .1$, $\Delta_{1+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [ |0\rangle + |5\rangle ]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figures 4.14 and 4.15 show the autocorrelation data for the system in the atomic-loss limit, specifically $g/\gamma = 20.0$ and $\kappa/\gamma = 10.0$. With such a high atom-cavity mode coupling, energy may be exchanged many times between the atom and the cavity field mode before being leaked out of the cavity system. The rate of this exchange is the Rabi frequency $\Omega$ and is given by

$$\Omega = \left(g^2 - \left[\frac{\kappa}{2} - \frac{\gamma}{2}\right]^2\right)^{1/2}.$$  

(4.62)

If expanded in terms of the operational variables of chapter 3, $\chi_2$ of equation 3.66 is seen to be $\Omega/\gamma$, our “$\gamma$-scaled” Rabi frequency. No non-classical behavior is seen in figure 4.14, but frame (d) does show a marked increase in the magnitude of $g^{(2)}(\tau)$. All vibrational distributions of figure 4.15 demonstrate an overshoot violation at $\gamma\tau \approx 0.2$, and the Rabi oscillations are again quite pronounced.

Figures 4.16 and 4.17 present cavity-loss limit data when $g/\gamma = 3.0$ and $\kappa/\gamma = 0.1$. The general behavior is the same as that in figures 4.14 and 4.15, but we notice a “muting” of the oscillations in frames (b) and (d) at about $\gamma\tau = 3.0$ and $\gamma\tau = 6.0$ in 4.16 and 4.17, respectively. The parameters of the system show relatively strong coupling between the atom and the cavity field mode, and a much smaller cavity field mode loss rate. This means that the atom and field are exchanging energy many times before we witness a transmission. Frames (b) and (d), as previously stated, represent the systems with the largest effective detunings and therefore will show the largest deviations from the on-resonance case due to many lifetimes of off-resonance exchange in the strong coupling regime. This behavior is also noted in figures 4.18 and 4.19. Both sets of plots for the fluorescent field demonstrate
overshoot violations that survive the mentioned detunings.

Both figures 4.20 and 4.22 show no non-classical behavior for the transmitted field’s autocorrelation, and figures 4.21 and 4.23 show rather expected behavior for the fluorescent field. Both 4.20 and 4.21 (where $g/\gamma = 4.0$ and $\kappa/\gamma = 3.0$) show pronounced Rabi oscillations where 4.22 4.23 (for $g/\gamma = 4.0$ and $\kappa/\gamma = 7.9$) do not. This is due to the latter’s dominance of the cavity field-mode decay rate.
Figure 4.14: Plots are $g^{(2)}_{TT}(\tau)$ for $g/\gamma = 20$, $\kappa/\gamma = 10$, $\Delta_{1,\pm}/\gamma = 0.1$. (a) $|0\rangle$ only.
(b) $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.15: Plots are $g_{FF}^{(2)}(\tau)$ for $g/\gamma = 20$, $\kappa/\gamma = 10$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} (|0\rangle + |5\rangle)$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.16: Plots are $g^{(2)}_{TT}(\tau)$ for $g/\gamma = 3$, $\kappa/\gamma = .1$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.17: Plots are $g^{(2)}_{FF}(\tau)$ for $g/\gamma = 3$, $\kappa/\gamma = .1$, $\Delta_{1+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [ |0\rangle + |5\rangle ]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.18: Plots are $g^{(2)}_{TT}(\tau)$ for $g/\gamma = 3$, $\kappa/\gamma = .4999$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}}[|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.19: Plots are $g_{FP}^{(2)}(\tau)$ for $g/\gamma = 3, \kappa/\gamma = .4999, \Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [ |0\rangle + |5\rangle ]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.20: Plots are $g_{TT}^{(2)}(\tau)$ for $g/\gamma = 4$, $\kappa/\gamma = 3$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.21: Plots are \( g_{FE}^{(2)}(\tau) \) for \( g/\gamma = 4, \kappa/\gamma = 3, \Delta_{1,+}/\gamma = 0.1 \).

(a) \( |0\rangle \) only.

(b) \( \frac{1}{\sqrt{2}} (|0\rangle + |5\rangle) \).

c) Pseudo-Boltzmann.

d) All states equal population.
Figure 4.22: Plots are $g^{(2)}(\tau)$ for $g/\gamma = 4$, $\kappa/\gamma = 7.9$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only.
(b) $\frac{1}{\sqrt{2}} (|0\rangle + |5\rangle)$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.23: Plots are $g_{FF}^{(2)}(\tau)$ for $g/\gamma = 4$, $\kappa/\gamma = 7.9$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}}|0\rangle + |5\rangle$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figures 4.24 through 4.27 present data for the system when all rates involved are nearly equal. We thus expect to see a fragile balance of behaviors that are quite sensitive to detunings. In 4.24 we see that when only the ground vibrational level is populated, the system has an overshoot violation as well as an undershoot violation. When the system is in the Boltzmann-type distribution, the undershoot remains, and the overshoot is just barely noticed. When the higher vibrational states become more populated, and the resulting detunings are larger as in frames (b) and (d), the non-classicalities are completely washed out and the field becomes more bunched at $\tau = 0$. When $\kappa/\gamma$ is increased from 0.77 to 1.6 as in figure 4.26, we see near-perfect antibunching at $\tau = 0$ for the ground-state-only case. This antibunching persists in 4.26(c), becomes much less pronounced in 4.26(b), and is wholly absent from 4.26(d). The cases of intermediate detuning (frames (b) and (c)) also show an undershoot violation in addition to their sub-Poissonian statistics. The autocorrelation data for the fluorescent field in figures 4.25 and 4.27 for $g/\gamma = 1.0$, $\kappa/\gamma = 0.77$ and $g/\gamma = 1.0$, $\kappa/\gamma = 1.6$ respectively show the expected antibunching and sub-Poissonian statistics. Figure 4.25 shows a more pronounced remnant of Rabi oscillation than 4.27 due to the lesser role of $\kappa$ in that system.
Figure 4.24: Plots are $g^{(2)}_{TT}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = .77$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} (|0\rangle + |5\rangle)$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.25: Plots are $g_{FF}^{(2)}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = 0.77$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} |0\rangle + |5\rangle$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.26: Plots are $g_{TT}^{(2)}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = 1.6$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only.
(b) $\frac{1}{\sqrt{2}} [ |0\rangle + |5\rangle ]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.27: Plots are $g_{FF}^{(2)}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = 1.6$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. 
(b) $1/\sqrt{2} [ |0\rangle + |5\rangle ]$. (c) Pseudo-Boltzmann. (d) All states equal population.
4.6.2 Effective temperature in Boltzmann distribution

In the Boltzmann-type distribution used in this work so far, there has been no real system parameter dependence; we have simply assigned population amplitudes to the vibrational states assuming factors of $\Delta E = kT = 1.0$ in a uniform state-energy spacing Boltzmann distribution. We want now to look at the behavior of the system when the atom’s vibrational level populations are governed by a more physical distribution. The distribution used for the states’ population amplitudes is now given by

$$P_l = \left[ \frac{e^{-l(\hbar \omega_v)/k_B T}}{\sum_l e^{-l(\hbar \omega_v)/k_B T}} \right]^{1/2}, \quad (4.63)$$

with the vibrational level spacing $\omega_v = 100 \text{Khz}$, and a variable “intercavity temperature” $T$, $k_B$ is the Stephan-Boltzmann constant. We call $T$ the effective temperature because we are not really describing a distribution of entities in the energy basis, but rather assigning probability amplitudes to different states of the same entity (the atom). Thus, by varying the effective temperature we are not really modeling an experimental cavity system as much as we are altering the distribution in a methodical and describable way.

For figures 4.28 through 4.33, frame (a) shows data for $T = 3 \times 10^{-3} K$, (b) for $T = 3 \times 10^{-4} K$, (c) for $T = 3 \times 10^{-6} K$, and (d) shows data for $T = 3 \times 10^{-5} K$.

We see in figure 4.28(c) that the system experiences both undershoot and overshoot violations at a very low effective temperature. As the effective temperature increases, even by just a factor of 10, the non-classicalities vanish without recur-
rence. This is comparable to other systems which behave with a quantum mechanical coherence that is lost upon thermalization. Here, we are simply increasing the occurrence and amount of detuning from resonance. Figure 4.29 shows no appreciable effects from varying the effective temperature, save the “sharpening” of the peak in $g_F^{(2)} F(\tau)$ between $\tau = 2.0$ and $\tau = 3.0$. Figure 4.30 shows drastically different behavior for systems at different effective temperatures. At $T = 3 \times 10^{-6}$ the system behaves essentially as it did when only the ground vibrational state was populated in figure 4.24(a). Indeed, at this low temperature the distribution gives the ground state a population amplitude $P_0 \approx 0.96$, and the first excited state has just $P_1 = 0.2688$. Upon squaring these amplitudes to find the populations, we see that this situation is very close to a pure ground state. Figure 4.28(a) shows $g^{(2)}(\tau)$ when $T = 3 \times 10^{-3} K$, a situation where all states are equally populated to within a few thousandths. Naturally we obtain results very similar to figure 4.24(d). As seen in figure 4.30(d), when the effective temperature is as high as $3 \times 10^{-5} K$ the severe antibunching at $\tau = 0$ disappears and an undershoot violation occurs. This temperature sees the vibrational states $|0\rangle$ and $|1\rangle$ very nearly equally populated with $P_1^2/P_0^2 \approx 0.79$. When $T$ is increased by another factor of 10 in figure 4.30(b), all nonclassicalities cease. Similar trends are noted in figure 4.32. Figures 4.31 and 4.33 show results similar to figures 4.29 and 4.7, respectively.
Figure 4.28: Plots are $g^{(2)}_{TT}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = .77$, $\Delta_{1,+}/\gamma = 0.1$. All trials use Boltzmann distribution for vibrational states. (a) $T = 3 \times 10^{-3}K$. (b) $T = 3 \times 10^{-4}K$. (c) $T = 3 \times 10^{-6}K$. (d) $T = 3 \times 10^{-5}K$.
Figure 4.29: Plots are $g^{(2)}_{FF}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = .77$, $\Delta_{1,+}/\gamma = 0.1$. All trials use Boltzmann distribution for vibrational states. (a) $T = 3 \times 10^{-3}K$. (b) $T = 3 \times 10^{-4}K$. (c) $T = 3 \times 10^{-6}K$. (d) $T = 3 \times 10^{-5}K$.
Figure 4.30: Plots are $g^{(2)}_{TT}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = 1.6$, $\Delta_{1,+}/\gamma = 0.1$. All trials use Boltzmann distribution for vibrational states. (a) $T = 3 \times 10^{-3}K$. (b) $T = 3 \times 10^{-4}K$. (c) $T = 3 \times 10^{-6}K$. (d) $T = 3 \times 10^{-5}K$
Figure 4.31: Plots are $g^{(2)}_{FF}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = 1.6$, $\Delta_{1,+}/\gamma = 0.1$. All trials use Boltzmann distribution for vibrational states. (a) $T = 3 \times 10^{-3} K$. (b) $T = 3 \times 10^{-4} K$. (c) $T = 3 \times 10^{-6} K$. (d) $T = 3 \times 10^{-5} K$. 
Figure 4.32: Plots are $g^{(2)}(\tau)$ for $g/\gamma = 2.2$, $\kappa/\gamma = 10$, $\Delta_{\pm}/\gamma = 0.1$. All trials use Boltzmann distribution for vibrational states. (a) $T = 3 \times 10^{-3}K$. (b) $T = 3 \times 10^{-4}K$. (c) $T = 3 \times 10^{-6}K$. (d) $T = 3 \times 10^{-5}K$. 

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Figure 4.33: Plots are $g_{FF}^{(2)}(\tau)$ for $g/\gamma = 2.2$, $\kappa/\gamma = 10$, $\Delta_{1,+}/\gamma = 0.1$. All trials use Boltzmann distribution for vibrational states. (a) $T = 3 \times 10^{-3} K$. (b) $T = 3 \times 10^{-4} K$. (c) $T = 3 \times 10^{-6} K$. (d) $T = 3 \times 10^{-5} K$. 
4.6.3 Vibrational level spacing and two-state superpositions

Figures 4.34 through 4.39 present the autocorrelations for different vibrational two-state superpositions. As these states are simple superpositions of the ground vibrational state and either $|l = 5\rangle$ or $|l = 1\rangle$, there really is no physical difference between the cases. The dynamics of the state are dependent upon the product of the vibrational state number and the vibrational state spacing, and thus

$$\text{We therefore have only labelled the superpositions differently, and we can say either the state is the same and spacing different, or the spacing is the same and the state different. This equivalence would not necessarily hold for a three-state superposition. Figure 4.34 shows data for an even superposition of } |l = 0\rangle \text{ and } |l = 5\rangle. \text{ Here, we see that the most extreme case of vibrational spacing } (\Delta_{1,+} = 2.0, \text{ frame (c)}) \text{ demonstrates an overshoot violation, while the less extreme cases show no violations. We see in figure 4.36 that the } \Delta_{1,+} = 2.0 \text{ case experiences severe antibunching, sub-Poissonian statistics, and an undershoot violation, while the cases with smaller spacing only demonstrate undershoot violations. The increase in vibrational spacing effects no severe changes in behavior for the data presented in figures 4.35 and 4.37; increased small-amplitude oscillations are, however, seen with increased spacing. Where increased vibrational-state spacing seemed to enhance nonclassical behavior for systems without an obviously dominant rate, the data for a bad-cavity limit system shows the opposite. Figure 4.38 shows that as the spacing}$$
in increased, nonclassical effects diminish. This can be seen with the smallest “detuning” in figure 4.38(b), where the effective detuning is \(5 \times 0.3 = 1.5\). The next highest detuning is in figure 4.44(c) where the detuning is \(1 \times 2.0 = 2.0\); after this level of detuning, non-classical effects do not return. This may be understood, again, in terms of the number of off-resonance exchanges between the cavity field mode and the atom before a transmission occurs. Figure 4.39 shows that \(g_{FF}^{(2)}(\tau)\) has little sensitivity to changes in vibrational-state spacing, all frames showing the same sub-Poissonian statistics and antibunching. The systems with vibrational state distributions that are an even mixture of \(|l = 0\rangle\) and \(|l = 1\rangle\) are at the lower end of the detuning range as compared to the states with even mixtures of \(|l = 0\rangle\) and \(|l = 5\rangle\), and this data is presented in figures 4.40 through 4.45.

Figure 4.40(a) presents an undershoot and an overshoot violation when \(g\gamma = 1.0, \kappa/\gamma = 0.77, \text{and } \Delta_{1,+} = 0.1\). When \(\Delta_{1,+}\) is increased in frames (b) through (d), we see that both of these violations are no longer present. When \(\kappa/\gamma\) is increased to 1.6 in figure 4.42 we see that the non-classicalities are more robust, with all four frames showing undershoot violations and three frames showing sub-Poissonian statistics. It is in figure 4.42(c) that the sub-Poissonian statistics disappear. Figures 4.41 and 4.43 show no sensitivity to changes in vibrational state-spacing with regard to classical/non-classical behavior, although frames with higher detuning show increased small-amplitude fluctuations. Figure 4.44 shows, as previously implied, that below \(\Delta_{1,+} = 2.0\), the system exhibits nonclassicalities in the form of undershoot violations, sub-Poissonian statistics, and antibunching. Figure 4.45 shows practically no effects from variation of vibrational state-spacing.
Figure 4.34: Plots are $g_{TT}^{(2)}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = .77$. All trials use $\frac{1}{\sqrt{2}} \left| 0 \right> + \left| 5 \right>$ as the vibrational state distribution. (a) $\Delta_{1,+}/\gamma = 0.1$. (b) $\Delta_{1,+}/\gamma = 0.3$. (c) $\Delta_{1,+}/\gamma = 2$. (d) $\Delta_{1,+}/\gamma = 0.5$
Figure 4.35: Plots are $g_{FF}^{(2)}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = .77$. All trials use $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$ as the vibrational state distribution. (a) $\Delta_{l,+}/\gamma = 0.1$. (b) $\Delta_{l,+}/\gamma = 0.3$. (c) $\Delta_{l,+}/\gamma = 2$. (d) $\Delta_{l,+}/\gamma = 0.5$
Figure 4.36: Plots are $g^{(2)}_{TT}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = 1.6$. All trials use $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$ as the vibrational state distribution. (a) $\Delta_{1,+}/\gamma = 0.1$. (b) $\Delta_{1,+}/\gamma = 0.3$. (c) $\Delta_{1,+}/\gamma = 2$. (d) $\Delta_{1,+}/\gamma = 0.5$
Figure 4.37: Plots are $g_{FF}^{(2)}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = 1.6$. All trials use $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$ as the vibrational state distribution. (a) $\Delta_{1,+}/\gamma = 0.1$. (b) $\Delta_{1,+}/\gamma = 0.3$. (c) $\Delta_{1,+}/\gamma = 2$. (d) $\Delta_{1,+}/\gamma = 0.5$
Figure 4.38: Plots are $g^{(2)}_{TT}(\tau)$ for $g/\gamma = 2.2$, $\kappa/\gamma = 10$. All trials use $\frac{1}{\sqrt{2}}[|0\rangle + |5\rangle]$ as the vibrational state distribution. (a) $\Delta_{1,+}/\gamma = 0.1$. (b) $\Delta_{1,+}/\gamma = 0.3$. (c) $\Delta_{1,+}/\gamma = 2$. (d) $\Delta_{1,+}/\gamma = 0.5$
Figure 4.39: Plots are $g_{FF}(\tau)$ for $g/\gamma = 2.2$, $\kappa/\gamma = 10$. All trials use $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$ as the vibrational state distribution. (a) $\Delta_{1+}/\gamma = 0.1$. (b) $\Delta_{1+}/\gamma = 0.3$. (c) $\Delta_{1+}/\gamma = 2$. (d) $\Delta_{1+}/\gamma = 0.5$
Figure 4.40: Plots are $g^{(2)}_{TT}(\tau)$ for $g/\gamma = 1, \kappa/\gamma = .77$. All trials use $\frac{1}{\sqrt{2}} |0\rangle + |1\rangle$ as the vibrational state distribution. (a) $\Delta_{1+}/\gamma = 0.1$. (b) $\Delta_{1+}/\gamma = 0.3$. (c) $\Delta_{1+}/\gamma = 2$. (d) $\Delta_{1+}/\gamma = 0.5$
Figure 4.41: Plots are $g_{FF}^{(2)}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = .77$. All trials use $\frac{1}{\sqrt{2}} [|0\rangle + |1\rangle]$ as the vibrational state distribution. (a) $\Delta_{1,+}/\gamma = 0.1$. (b) $\Delta_{1,+}/\gamma = 0.3$. (c) $\Delta_{1,+}/\gamma = 2$. (d) $\Delta_{1,+}/\gamma = 0.5$
Figure 4.42: Plots are $g^{(2)}_{TT}(\tau)$ for $g/\gamma = 1, \kappa/\gamma = 1.6$. All trials use $\frac{1}{\sqrt{2}} \left[ |0\rangle + |1\rangle \right]$ as the vibrational state distribution. (a) $\Delta_{1,+}/\gamma = 0.1$. (b) $\Delta_{1,+}/\gamma = 0.3$. (c) $\Delta_{1,+}/\gamma = 2$. (d) $\Delta_{1,+}/\gamma = 0.5$
Figure 4.43: Plots are $g^{(2)}_{F,F'}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = 1.6$. All trials use $\frac{1}{\sqrt{2}} [|0\rangle + |1\rangle]$ as the vibrational state distribution. (a) $\Delta_{1,+}/\gamma = 0.1$. (b) $\Delta_{1,+}/\gamma = 0.3$. (c) $\Delta_{1,+}/\gamma = 2$. (d) $\Delta_{1,+}/\gamma = 0.5$.
Figure 4.44: Plots are $g_{TT}^{(2)}(\tau)$ for $g/\gamma = 2.2$, $\kappa/\gamma = 10$. All trials use $\frac{1}{\sqrt{2}} |0\rangle + |1\rangle$ as the vibrational state distribution. (a) $\Delta_{1,+}/\gamma = 0.1$. (b) $\Delta_{1,+}/\gamma = 0.3$. (c) $\Delta_{1,+}/\gamma = 2$. (d) $\Delta_{1,+}/\gamma = 0.5$. 
Figure 4.45: Plots are $g_{FF}^{(2)}(\tau)$ for $g/\gamma = 2.2$, $\kappa/\gamma = 10$. All trials use $\frac{1}{\sqrt{2}} |0\rangle + |1\rangle$ as the vibrational state distribution. (a) $\Delta_{1,+}/\gamma = 0.1$. (b) $\Delta_{1,+}/\gamma = 0.3$. (c) $\Delta_{1,+}/\gamma = 2$. (d) $\Delta_{1,+}/\gamma = 0.5$.
4.6.4 Cross correlations

Figures 4.46 through 4.57 present $g^{(2)}_{TT}(\tau)$ and $\sqrt{g^{(2)}_{TT}(\tau)g^{(2)}_{FF}(\tau)}$ data as described in sections 4.4 and 4.5. In all the data presented, inequality 4.53 is broken, indicating that the correlation functions for the cavity QED system cannot be described by classical probability functions. We expect to see a violation of inequality 4.53 at $\tau = 0$, as the autocorrelation for the fluorescent field must be 0 here. We in fact have no data for which the inequality holds at $\tau = 0$, although the data presented in figure 4.52 very nearly has $g^{(2)}_{TF}(0) \leq \sqrt{g^{(2)}_{TT}(0)g^{(2)}_{FF}(0)}$. As inequality 4.53 only applies to equal time correlation functions, $\tau = 0$ is the only time at which the inequality is of any interest to us.

Figure 4.46 shows that the cross correlation is quite symmetric when $|\gamma \tau| \geq 1$, and inequality 4.53 never holds for any vibrational distribution. In fact, the curves in frames (a), (b), and (c) look very similar, differing only slightly in height and width about $\tau = 0$. Figure 4.46(d) shows a marked increase in the magnitude of $g^{(2)}_{TF}(\tau)$, and a significant narrowing about $\tau = 0$. Figure 4.47 shows data for the system when $g/\gamma = 2.2$, $\kappa/\gamma = 10$, a step further into the bad-cavity regime from figure 4.46. Because of $\kappa$’s stronger dominance, we expect that a correlation involving a transmission event would reach steady-state quicker. This would allow less time for the system (atom and cavity field mode) to fall out of balance, and hence we expect an improved time symmetry. We indeed see that the cross-correlation exhibits a greater degree of symmetry about $\tau = 0$; the shapes of the curves are obviously similar and the heights of the peaks of the curve on either side of $\tau = 0$ are in an approximate ratio of 0.886 (a) through (c), and
about 0.731 in frame (d). This is to be compared to an average height ratio of
about 0.82 for figures 4.46(a) through 4.46(c), and about 0.799 for 4.46(d). Like
figure 4.46, we see that inequality 4.53 never holds for any considered vibrational
state distribution. Figure 4.48 is in concert with our explanation of figure 4.47;
as the system goes further into the bad cavity regime (by virtue now of a weaker
atom-cavity field mode coupling), the degree of time symmetry increases for the
cross-correlation. We see that the peaks in figure 4.48 are in a ratio of about
0.95 for frames (a) through (c) and about 0.90 for frame (d). Time asymmetry
persists strongly, however, if we focus our attention tightly around \( \tau = 0 \), where we
note that the cross correlation’s central minimum is actually to the left at about
\( \gamma \tau = -0.08 \). We see in figure 4.49 that the cross-correlation inequality does not
hold for the system in the good-cavity limit (here, \( g/\gamma = 0.3 \) and \( \kappa/\gamma = 0.1 \)). In
this figure, the cross-correlation also shows an obvious lack of symmetry. Moving
further into the good-cavity limit in figure 4.50, with \( g/\gamma = 0.05 \) and \( \kappa/\gamma = 0.1 \),
we see that time-asymmetry becomes very pronounced.

Figures 4.51 through 4.54 show cross-correlation data for when the system
parameters are in the oscillatory regime. Figure 4.51 shows data for \( g/\gamma = 20, \)
\( \kappa/\gamma = 10, \) and \( \Delta_{1,+} = 0.1; \) these parameters also set the system in the bad-cavity
limit, as \( \kappa \) is much larger than \( \gamma \). Figures 4.52 and 4.53 show data for when
\( g/\gamma = 3.0, \kappa/\gamma = 0.1 \) and \( \kappa/\gamma = 0.4999 \) respectively, and \( \Delta_{1,+} = 0.1; \) this choice
of parameters overlaps a bit with the good-cavity regime, in which \( \kappa \) is the smallest
system rate. Like figure 4.51, figure 4.54 shows data for a region of parameter space
that overlaps with the bad-cavity limit, with \( g/\gamma = 4.0, \kappa/\gamma = 3.0, \) and \( \Delta_{1,+} = 0.1. \)
As $g$ is obviously the dominant rate in all of these system configurations, we expect to see Rabi oscillations in the correlations that denote the repeated exchange of energy between the atom and cavity field mode. The Rabi oscillations are quite pronounced in all of the figures.
Figure 4.46: Plots are $g_{TF}^{(2)}(\tau)$ (bold) and $\sqrt{g_{TT}^{(2)}(\tau)g_{FF}^{(2)}(\tau)}$ (dotted) for $g/\gamma = 2$, $\kappa/\gamma = 5$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [ |0\rangle + |5\rangle ]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.47: Plots are $g_{TT}^{(2)}(\tau)$ (bold) and $\sqrt{g_{TT}^{(2)}(\tau)g_{FF}^{(2)}(\tau)}$ (dotted) for $g/\gamma = 2$, $\kappa/\gamma = 10$, $\Delta_{1+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.48: Plots are $g_{TT}^{(2)}(\tau)$ (bold) and $\sqrt{g_{TT}^{(2)}(\tau)g_{FF}^{(2)}(\tau)}$ (dotted) for $g/\gamma = 0.866$, $\kappa/\gamma = 10$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}}[|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.49: Plots are $g_{TF}^{(2)}(\tau)$ (bold) and $\sqrt{g_{TT}^{(2)}(\tau)g_{FF}^{(2)}(\tau)}$ (dotted) for $g/\gamma = .3$, $\kappa/\gamma = .1$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.50: Plots are $g^{(2)}_{TF}(\tau)$ (bold) and $\sqrt{g^{(2)}_{TT}(\tau)g^{(2)}_{FF}(\tau)}$ (dotted) for $g/\gamma = .05$, $\kappa/\gamma = .1$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}}[|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.51: Plots are $g_{TF}^{(2)}(\tau)$ (bold) and $\sqrt{g_{TT}^{(2)}(\tau)g_{FF}^{(2)}(\tau)}$ (dotted) for $g/\gamma = 20$, $\kappa/\gamma = 10$, $\Delta_{1+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [ |0\rangle + |5\rangle ]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.52: Plots are $g_{TF}^{(2)}(\tau)$ (bold) and $\sqrt{g_{TT}^{(2)}(\tau)g_{FF}^{(2)}(\tau)}$ (dotted) for $g/\gamma = 3$, $\kappa/\gamma = .1$, $\Delta_{1,+}/\gamma = 0.1$. All trials use $\frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$ as the vibrational state distribution. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}}[|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.53: Plots are $g_{TF}^{(2)}(\tau)$ (bold) and $\sqrt{g_{TT}^{(2)}(\tau)g_{FF}^{(2)}(\tau)}$ (dotted) for $g/\gamma = 3$, $\kappa/\gamma = .4999$, $\Delta_{1,+}/\gamma = 0.1$. All trials use $\frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$ as the vibrational state distribution. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}}[|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.54: Plots are \( g_{TF}^{(2)}(\tau) \) (bold) and \( \sqrt{g_{TT}^{(2)}(\tau)g_{FF}^{(2)}(\tau)} \) (dotted) for \( g/\gamma = 4 \), \( \kappa/\gamma = 3 \), \( \Delta_{1,+}/\gamma = 0.1 \). All trials use \( \frac{1}{\sqrt{2}} \left[ |0\rangle + |1\rangle \right] \) as the vibrational state distribution. (a) \( |0\rangle \) only. (b) \( \frac{1}{\sqrt{2}} \left[ |0\rangle + |5\rangle \right] \). (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.55: Plots are $g_{TF}^{(2)}(\tau)$(bold) and $\sqrt{g_{TT}^{(2)}(\tau)g_{FF}^{(2)}(\tau)}$(dotted) for $g/\gamma = 4$, $\kappa/\gamma = 7.9$, $\Delta_{1,+}/\gamma = 0.1$. All trials use $\frac{1}{\sqrt{2}} \left[ |0\rangle + |1\rangle \right]$ as the vibrational state distribution. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} \left[ |0\rangle + |5\rangle \right]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.56: Plots are $g_{TF}^{(2)}(\tau)$(bold) and $\sqrt{g_{TF}^{(2)}(\tau)g_{FF}^{(2)}(\tau)}$(dotted) for $g/\gamma = 1$, $\kappa/\gamma = .77$, $\Delta_{1,+}/\gamma = 0.1$. All trials use $\frac{1}{\sqrt{2}} [\left|0\right> + \left|1\right> ]$ as the vibrational state distribution. (a) $\left|0\right>$ only. (b) $\frac{1}{\sqrt{2}} [\left|0\right> + \left|5\right> ]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 4.57: Plots are \( g_{TF}^{(2)}(\tau) \) (bold) and \( \sqrt{g_{TF}^{(2)}(\tau)g_{FF}^{(2)}(\tau)} \) (dotted) for \( g/\gamma = 1, \ \kappa/\gamma = 1.6, \ \Delta_{1,+}/\gamma = 0.1 \). All trials use \( \frac{1}{\sqrt{2}} \left[ |0\rangle + |1\rangle \right] \) as the vibrational state distribution. (a) \( |0\rangle \) only. (b) \( \frac{1}{\sqrt{2}} \left[ |0\rangle + |5\rangle \right] \). (c) Pseudo-Boltzmann. (d) All states equal population.
Chapter 5

Field-Intensity Correlation

5.1 \( h_{\theta}(\tau) \)

We have seen the important roles two-time correlation functions play in quantum optics. In section 2.3 the first order field-field correlation function revealed the spectrum of a quantum mechanical field, and in chapter 4 the second order intensity correlation function \( g^{(2)}(\tau) \) was shown to be a measurement of the photon statistics of our cavity QED fields. Recently, Carmichael and his co-workers have introduced a new intensity-field correlation function \( h_{\theta}(\tau) \) that is of great interest [6]. Because \( h_{\theta}(\tau) \) is an intensity-field correlation function, it takes the general form

\[
h_{\theta}(\tau) = \frac{\langle I(0)E(\tau) \rangle}{\langle I \rangle \langle E \rangle}, \tag{5.1}
\]

and for a quantized field, this becomes

\[
h_{\theta}(\tau) = \frac{\langle a^\dagger(0)a_{\theta}(\tau)a(0) \rangle}{\langle a^\dagger a \rangle \langle a_0 \rangle}, \tag{5.2}
\]
where we have, like for $g^{(2)}(\tau)$, exploited normal and time ordering, and we have used the quantum mechanical field quadrature operator (our calculation is based upon a balanced homodyne detection scheme):

$$a_\theta = ae^{-i\theta} + a^\dagger e^{i\theta}. \quad (5.3)$$

In equation 5.3, $\theta$ is the phase of the local oscillator (LO) with respect to the average signal field. To fully understand our use of the quadrature operator and $h_\theta(\tau)$ itself, it is instructive to first discuss some properties of a quantized electric field, and balanced homodyne detection.

A classical single-mode electric field can be described by an amplitude and a phase, both of which may have well defined values. For the single-mode quantized field however, there is no well defined phase operator, and the amplitude and phase may not be measured simultaneously. We can write the classical field as

$$\hat{E} = A \cos(\omega t - kz + \phi) = X \cos(\omega t - kz) + Y \sin(\omega t - kz), \quad (5.4)$$

where $X = A \cos(\phi)$ and $Y = A \sin(\phi)$ are the two field quadratures, and may simultaneously have well defined values. For the single-mode quantized field, we write the field as

$$\hat{E} = \sqrt{\frac{\hbar \omega}{2\varepsilon_0 V}} \left( a e^{-i(\omega t + \phi)} + a^\dagger e^{i(\omega t + \phi)} \right) = \hat{X} \cos(\omega t) + \hat{Y} \sin(\omega t), \quad (5.5)$$

where $V$ is the mode volume, and $\omega$ is the frequency of the mode. In this representation of the quantized field, the two field quadratures are (with $\phi = 0$)

$$\hat{X} = \sqrt{\frac{\hbar \omega}{2\varepsilon_0 V}} \left( a + a^\dagger \right), \quad (5.6)$$

$$\hat{Y} = \sqrt{\frac{\hbar \omega}{2\varepsilon_0 V}} \left( a - a^\dagger \right). \quad (5.7)$$
These two quadrature operators do not commute,

\[
[\hat{X}, \hat{Y}] = \left( \frac{\hbar \omega}{2\varepsilon_0 V} \right) [(a + a^\dagger), (a - a^\dagger)] \\
= \left( \frac{\hbar \omega}{2\varepsilon_0 V} \right) ([a^\dagger, a] - [a, a^\dagger]) \\
= 2 \left( \frac{\hbar \omega}{2\varepsilon_0 V} \right), \tag{5.8}
\]

and are therefore not simultaneously measurable. By using a balanced homodyne detector (BHD, shown in figure 5.1), where the LO with a variable phase \( \theta \) is combined with the signal field on a beam splitter, and the difference in two photodetectors is measured, one can measure either quadrature, or the more general quadrature,

\[
\hat{X}_\theta = \sqrt{\left( \frac{\hbar \omega}{2\varepsilon_0 V} \right)} (ae^{-i\theta} + a^\dagger e^{i\theta}) = \sqrt{\left( \frac{\hbar \omega}{2\varepsilon_0 V} \right)} a_\theta, \tag{5.9}
\]

simply by varying the phase of the LO. The LO is assumed to be a very strong field, in a coherent state, with a large average photon number, and it’s field operator can be written as

\[a_{LO} \equiv Ae^{i\theta}. \tag{5.10}\]

In this case, the LO can be treated as a classical field, and in fact the noise in the LO cancels out in the BHD measurement. To see how this is done, let us ”walk” through the measurement system as shown in figure 5.1. The two output modes of the beam splitter are

\[
b_1 = \frac{1}{\sqrt{2}} \left( ia + Ae^{i\theta} \right), \tag{5.11}
\]
\[
b_2 = \frac{1}{\sqrt{2}} \left( a + iAe^{i\theta} \right), \tag{5.12}
\]

where we have accounted for the \( \frac{\pi}{2} \) phase shift upon reflection given by the Stokes’
relations. The current from the two photodetectors (which much be balanced in terms of quantum efficiencies) is proportional to the photon fluxes,

\[
\langle b_1^{\dagger} b_1 \rangle = \frac{1}{2} (a^{\dagger} a - i a^{\dagger} A e^{i\theta} + i a A e^{-i\theta} + A^2),
\]

\[
\langle b_2^{\dagger} b_2 \rangle = \frac{1}{2} (a^{\dagger} a + i a^{\dagger} A e^{i\theta} - i a A e^{-i\theta} + A^2),
\]

(5.13)

and the difference current is then given by

\[
\Delta I = \eta \langle b_1^{\dagger} b_1 - b_2^{\dagger} b_2 \rangle
= -i\eta A \langle a^{\dagger} e^{i\theta} - a e^{-i\theta} \rangle
\propto i\eta A \langle a_{\theta} \rangle,
\]

(5.14)

where \( \eta \) is the conversion factor between photon flux and current, which depends on detector and collection efficiencies.

In equation 5.14, we see that balanced homodyne detection allows us to measure the average of the electric field quadrature we desire by varying the phase of the local oscillator. Finally then, we are ready to discuss the measurement of \( h_{\theta}(\tau) \). Simply put, one first detects a photon, waits a time \( \tau \), and then measures \( \langle a_{\theta} \rangle \). A practical way do that is shown in figure 5.1.

To understand this we examine the structure of

\[
h_{\theta}(\tau) = \frac{\langle a^{\dagger}(0) a_{\theta}(\tau) a(0) \rangle}{\langle a^{\dagger} a \rangle \langle a_0 \rangle}.
\]

(5.15)

We see that with the \( a \) acting to the right, and the \( a^{\dagger} \) acting to the left at \( t = 0 \), a collapsed state is prepared, the collapse being a photon loss from the field, corresponding to a detection event. Then at \( t = \tau \) one measures \( \langle a_{\theta} \rangle \) conditioned on the previous detection. This differs from a direct measurement of \( \langle a_{\theta} \rangle \) with
Figure 5.1: This is a common experimental setup for measuring $h_\theta(\tau)$. In this figure, the source would be either the transmitted or fluoresced portion of the field. LO denotes Local Oscillator, a
no conditioning. An ensemble average of the latter measurements (necessary to get a good signal to noise ratio) would yield zero due to phase fluctuations. The conditioned BHD measurement essentially looks at members of the ensemble with the same phase, a phase that is set by the photodetection.

5.2 The Schwartz inequality and \( h_\theta(\tau) \)

As with other correlation functions, like the second-order intensity correlation function \( g^{(2)}(\tau) \), restrictions can be placed on \( h_\theta(\tau) \) if there is an underlying positive definite probability distribution function for amplitude and phase of the electric field, i.e. that the field is classical albeit stochastic. By ignoring third-order moments (a Gaussian fluctuation assumption that is valid for weak fields), one finds

\[
 h_\theta(\tau) = 1 + 2 \frac{\langle \Delta a_\theta(0) \Delta a_\theta(\tau) \rangle}{\langle \Delta a^\dagger \Delta a \rangle},
\]

and we see that the intensity-field correlation function is connected to the spectrum of squeezing [6]

\[
 S_\theta(\omega) \propto \int_0^\infty d\tau \cos(\omega \tau) [h_\theta(\tau) - 1].
\]

From this, it has been shown that the Schwartz inequality would yield

\[
 0 \leq h_\theta(0) - 1 \leq 1,
\]

and more generally

\[
 | h_\theta(\tau) - 1 | \leq | h_\theta(0) - 1 | \leq 1.
\]

Whenever there is squeezing, these inequalities do not hold for \( h_\theta(\tau) \). Giant violations of these inequalities have been predicted for an optical parametric oscillator,
and a group of $N$ atoms in a driven optical cavity, and have been recently observed in the cavity QED system [6].

5.3 $h_\theta(\tau)$ in the weak field limit

As with $g^{(2)}(\tau)$, one can calculate the denominator of $h_\theta(\tau)$ using steady state density matrix elements, and the numerator using the quantum regression theorem. Here we choose to only discuss an analytic solution using the quantum trajectory method, and again we look at weak driving fields. We find

$$\langle a^\dagger(0)a_\theta(\tau)a(0)\rangle = \langle \psi_c \mid a_\theta \mid \psi_c \rangle, \quad (5.20)$$

where $| \psi_c \rangle$ is the collapsed state produced by the photodetection event, as in the case of $g^{(2)}(\tau)$. Once again we need only keep the states with two or less excitations (total in the cavity mode or internal energy) for weak driving fields. The result is that

$$h_\theta(\tau) = \frac{\langle n \rangle_{SS} \langle a_\theta(\tau) \rangle_c}{\langle n \rangle_{SS} \langle a_\theta(\tau) \rangle_{SS}} = \frac{\langle a_\theta(\tau) \rangle_c}{\langle a_0(\tau) \rangle_{SS}}. \quad (5.21)$$

The expectation value of the field quadrature operator is given by

$$\langle a_\theta \rangle = \sum_{n,\{m\}} \left( \sqrt{n} C^*_{n,\{m\}} C_{n-1,\{m\}} e^{-i\theta} + \sqrt{n+1} C^*_{n,\{m\}} C_{n+1,\{m\}} e^{i\theta} \right). \quad (5.22)$$

In the weak field limit we have

$$\langle \hat{a}_\theta \rangle = \sum_{\{m\}} \left( C^*_{1,\{m\}} C_{0,\{m\}} e^{-i\theta} + C^*_{0,\{m\}} C_{1,\{m\}} e^{i\theta} \right). \quad (5.23)$$
So finally then, for weak fields we have

\[
    h_\theta(\tau) = \frac{\sum\{m\} \left( C_{1,\{m\}}^{C_0} C_0^{C_1} e^{-i\theta} + C_{0,\{m\}}^{C_1} C_1^{C_0} e^{i\theta} \right)}{\sum\{m\} \left( C_{SS}^{SS_1} C_{SS_0}^{SS} + C_{SS_1}^{SS_0} C_{SS}^{SS_1} \right)}.
\]  

(5.24)

### 5.4 Cross Correlations in \( h_\theta(\tau) \)

As discussed in section 4.4, cross correlations in the photon statistics are of interest. In these correlations, we look at the photon statistics of one field (either transmitted T, or fluoresced F) after a conditioning photon detection in either T or F. We were able to write the forms of these cross correlations in equations 4.45 and 4.46. Similarly, we have four intensity-field correlations of interest

\[
    h_{TT}^T(\tau) = \frac{\langle a_\theta(\tau) \rangle_{CT}}{\langle a_0(\tau) \rangle_{SS}},
\]  

(5.25)

\[
    h_{TF}^T(\tau) = \frac{\langle a_\theta(\tau) \rangle_{CF}}{\langle a_0(\tau) \rangle_{SS}},
\]  

(5.26)

\[
    h_{TF}^T(\tau) = \frac{\langle \sigma_\theta(\tau) \rangle_{CT}}{\langle \sigma_0(\tau) \rangle_{SS}},
\]  

(5.27)

\[
    h_{FF}^T(\tau) = \frac{\langle \sigma_\theta(\tau) \rangle_{CF}}{\langle \sigma_0(\tau) \rangle_{SS}}.
\]  

(5.28)

In terms of probability amplitudes, we have

\[
    h_{TT}^T(\tau) = \frac{\sum\{n,\{m\}\} \left( C_{n,e,\{m\}}^{CT} C_{n,g,\{m\}}^{CT} (\tau) e^{-i\theta} + C_{n,g,\{m\}}^{CT} C_{n,e,\{m\}}^{CT} (\tau) e^{i\theta} \right)}{\sum\{n,\{m\}\} \left( C_{SS}^{SS_n} C_{SS_{n,g,\{m\}}}^{SS_n} + C_{SS_n}^{SS_g} C_{SS_{n,e,\{m\}}}^{SS_n} \right)}.
\]  

(5.29)

This could be calculated for individual trajectories, and then an ensemble average taken. For weak fields, on resonance, we have

\[
    h_{TT}^T(\tau) = \frac{\sum\{m\} \left( C_{1,g,\{m\}}^{CT} C_{0,g,\{m\}}^{CT} (\tau) \right)}{\sum\{m\} \left( C_{SS}^{SS_1} C_{SS_0}^{SS} \right)} \cos \theta.
\]  

(5.30)

For the other three field-intensity correlation functions we have

\[
    h_{TT}^F(\tau) = \frac{\sum\{m\} \left( C_{1,g,\{m\}}^{CF} C_{0,g,\{m\}}^{CF} (\tau) \right)}{\sum\{m\} \left( C_{SS}^{SS_1} C_{SS_0}^{SS} \right)} \cos \theta,
\]  

(5.31)
\[ h_{\theta}^{TT}(\tau) = \sum_{\{m\}} \frac{C_{0,e,\{m\}}^{CT}(\tau) C_{0,g,\{m\}}^{CT}(\tau)}{\sum_{\{m\}} C_{0,e,\{m\}}^{SS} C_{0,g,\{m\}}^{SS}} \cos \theta, \]  
\[ h_{\theta}^{FF}(\tau) = \sum_{\{m\}} \frac{C_{0,e,\{m\}}^{CF}(\tau) C_{0,g,\{m\}}^{CF}(\tau)}{\sum_{\{m\}} C_{0,e,\{m\}}^{SS} C_{0,g,\{m\}}^{SS}} \cos \theta. \]  

### 5.5 Results for \( h_{\theta}^{TT}(\tau) \) and \( h_{\theta}^{FF}(\tau) \)

#### 5.5.1 \( h_{\theta}^{TT}(\tau) \) and \( h_{\theta}^{FF}(\tau) \)

Although we have spoken about \( h_{\theta}(\tau) \) as measuring the field quadrature at phase \( \theta \) at some time \( \tau \) having detected conditioning event at \( t = 0 \), equation 5.16 shows a different interpretation. For weak fields, where fluctuations are essentially Gaussian, \( h_{\theta}(\tau) \) can be written strictly in terms of the quadrature variance and the fluctuation intensity for the source field as shown in equation 5.16. What the following curves then present are the fluctuations of the source field’s quadrature (we look at the in-phase quadrature for TT, and the out-of-phase for FF) about the mean source field of zero. The envelope of the curve (for oscillatory fluctuations) is therefore the field amplitude in the following plots.

Equation 5.19 shows that in order for the field quadrature to be behaving classically, \( h_{\theta}(0) \) must lie between 0 and 2. For discussion purposes, I will call an upper bound violation of this rule a “high violation”, and a lower bound violation will be called a “low violation”. Equation 5.19 also says that a classical field quadrature must behave such that \( h_{\theta}(\tau) \) deviates from unity no more than does \( h_{\theta}(0) \). I will call an upper limit violation of this type an “overshoot”, and a lower limit violation an “undershoot”; the meanings of these terms are analogous to
Figures 5.2 through 5.25 present $h_{\theta}^{TT}(\tau)$ and $h_{\theta}^{FF}(\tau)$ for the same selection of system parameters presented in section 4.6.1. Figure 5.2 shows the field-intensity correlation for the transmitted field for when $g/\gamma = 2.0$, $\kappa/\gamma = 5.0$, and $\Delta_{1,-} = 0.1$; we are thus in the bad-cavity regime, where $\kappa$ is the largest system rate. We see that for all considered vibrational distributions, the fluctuations exhibit a low violation at $\tau = 0$; for less pure vibrational states there is a greater field at $\tau = 0$. Figure 5.3 shows that the fluorescent field behaves classically for this set of system parameters, in all vibrational distributions considered.

Figure 5.4 shows that for the less detuned vibrational distributions (the $|l = 0\rangle$ only, and the Boltzmann-type) no non-classicalities are present, but for the more detuned cases there is a low violation. Figure 5.5, for the same system parameters of $g/\gamma = 2.2$, $\kappa/\gamma = 10$, $\Delta_{1,-} = 0.1$ shows that there are no non-classicalities in the fluorescent field fluctuations. No non-classical behavior is exhibited in either the transmitted or fluorescent field in figures 5.6 and 5.7 when $g/\gamma = 0.866$, $\kappa/\gamma = 10$, and $\Delta_{1,+} = 0.1$, a system deeply into the bad-cavity regime.
Figure 5.2: Plots are $h^{TT}_\theta(\tau)$ for $g/\gamma = 2$, $\kappa/\gamma = 5$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.3: Plots are \( h_{\theta}^{FF}(\tau) \) for \( g/\gamma = 2, \kappa/\gamma = 5, \Delta_{1,+}/\gamma = 0.1 \). (a) \( |0\rangle \) only. (b) \( \frac{1}{\sqrt{2}} \left( |0\rangle + |5\rangle \right) \). (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.4: Plots are $h_0^{TT}(\tau)$ for $g/\gamma = 2.2$, $\kappa/\gamma = 10$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} (|0\rangle + |5\rangle)$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.5: Plots are $h^{FF}_{\theta}(\tau)$ for $g/\gamma = 2.2$, $\kappa/\gamma = 10$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.6: Plots are $h_{\theta}^{TT}(\tau)$ for $g/\gamma = .866$, $\kappa/\gamma = 10$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [ |0\rangle + |5\rangle ]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.7: Plots are $h_\theta^{FF}(\tau)$ for $g/\gamma = .866$, $\kappa/\gamma = 10$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}}[|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.8 shows $h_\theta(\tau)$ for when $g/\gamma = 0.3$, $\kappa/\gamma = 0.1$, and $\Delta_{1,+} = 0.1$. Because $g$ is bigger than $\kappa$ by a factor of three we see some mild oscillations in the curves. Frames (a) and (c) are similar, as are frames (b) and (d); none of the cases demonstrate non-classicalities of any sort. Like figure 5.8, figure 5.9 shows frames (b) and (d) having mild oscillations. This is expected, because these are the cases with the greatest vibrational detuning effects which add extra oscillations to the data. The curves of $h_\theta(\tau)$ behave in a classical manner in figure 5.9.

Figure 5.10 shows data when the system is in the good-cavity limit, with $g/\gamma = 0.05$, $\kappa/\gamma = 0.1$, and $\Delta_{1,+} = 0.1$. Because the cavity leaks photons slowly, conditioned upon a transmission we expect the field to show little fluctuation. Indeed, the field shows no fluctuations for the case where only the ground vibrational state is populated. Figure 5.10(c) shows very little fluctuation of the field for the Boltzmann-type distribution. Both frames (a) and (c) show no non-classical behavior, while frames (b) and (d) both exhibit overshoot violations of inequality 5.19. Figure 5.11 shows no violations of inequality 5.19 for any of the considered vibrational distributions.
Figure 5.8: Plots are $h^{TT}_0(\tau)$ for $g/\gamma = .3$, $\kappa/\gamma = .1$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. 
(b) $\frac{1}{\sqrt{2}} (|0\rangle + |5\rangle)$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.9: Plots are $h^F_\theta(\tau)$ for $g/\gamma = .3$, $\kappa/\gamma = .1$, $\Delta_{1+}/\gamma = 0.1$. (a) $|0\rangle$ only. 
(b) $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.10: Plots are $h^{TT}_{\theta}(\tau)$ for $g/\gamma = .05, \kappa/\gamma = .1, \Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only.
(b) $\frac{1}{\sqrt{2}}[|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.11: Plots are $h^F_0(\tau)$ for $g/\gamma = .05$, $\kappa/\gamma = .1$, $\Delta_1,+/\gamma = 0.1$. (a) $|0\rangle$ only. 
(b) $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figures 5.12 through 5.19 show $h_{\theta}(\tau)$ for systems in the strong-coupling regime where $g$ dominates both $\gamma$ and $\kappa$, and we see the expected Rabi oscillations in the data. For all of the plots of $h_{\theta}^{TT}(\tau)$ we see low violations no matter what the vibrational distribution is, with the distribution simply changing the magnitude of the field’s fluctuations. All of the $h_{\theta}^{FF}(\tau)$ plots in this section are completely void of non-classical behavior as inequality 5.19 is always obeyed. Figures 5.14 through 5.17 show that when the vibrational detunings are largest (when the distribution is either that in frame (b) or frame(d)), the Rabi oscillations are disturbed and damped. This can be understood through the explanation presented in chapter 4 for the same phenomenon.
Figure 5.12: Plots are $h_{0}^{TT}(\tau)$ for $g/\gamma = 20$, $\kappa/\gamma = 10$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.13: Plots are $h_g^{FF}(\tau)$ for $g/\gamma = 20$, $\kappa/\gamma = 10$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [ |0\rangle + |5\rangle ]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.14: Plots are $h^T_\theta(\tau)$ for $g/\gamma = 3$, $\kappa/\gamma = .1$, $\Delta_{1+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [ |0\rangle + |5\rangle ]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.15: Plots are $h^{EF}_\theta(\tau)$ for $g/\gamma = 3$, $\kappa/\gamma = .1$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} (|0\rangle + |5\rangle)$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.16: Plots are $h_{\theta}^{TT}(\tau)$ for $g/\gamma = 3, \kappa/\gamma = .49, \Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}}[|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.17: Plots are $h_{\theta}^{FF}(\tau)$ for $g/\gamma = 3, \kappa/\gamma = .49, \Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only.
(b) $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.18: Plots are $h^T_0(\tau)$ for $g/\gamma = 4$, $\kappa/\gamma = 3$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [\langle 0 | + |5 \rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.19: Plots are $h_0^{FF}(\tau)$ for $g/\gamma = 4$, $\kappa/\gamma = 3$, $\Delta_{1+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figures 5.20 and 5.21 present the field-intensity correlations for the transmitted and fluoresced field, respectively, when $g/\gamma = 4.0$, $\kappa/\gamma = 7.9$, and $\Delta_{1,+}/\gamma = 0.1$. These parameters define a system in the bad-cavity regime, and thus we expect to see large fluctuations as the cavity leaks energy through the forward mirror. Figure 5.20 shows that all vibrational distributions considered experience a large non-classical fluctuation at $\tau = 0$, this low violation is enhanced by vibrational distributions of greater detuning. $\hat{h}^{FF}$ in figure 5.21 shows no non-classical behavior. When $g/\gamma = 1.0$, $\kappa/\gamma = 0.77$, and $\Delta_{1,+}/\gamma = 0.1$, as in figure 5.22, we again see low violations at $\tau = 0$ that are enhanced by distributions of greater detuning. Like all of the previous plots of $\hat{h}^{FF}$, figure 5.23 reveals no behavior in violation of inequality 5.19. Figure 5.24 shows the field intensity correlation for the transmitted field when all the system rates are of comparable magnitude: $g/\gamma = 1.0$, $\kappa/\gamma = 1.6$, and $\Delta_{1,+}/\gamma = 0.1$. When only the ground vibrational state is populated in frame (a), we see no non-classical behavior. For all other considered vibrational distributions we see low violations. Figure 5.25 presents $\hat{h}^{FF}$ for the same system parameters, and exhibits classical behavior.
Figure 5.20: Plots are $h^{TT}_{\theta}(\tau)$ for $g/\gamma = 4$, $\kappa/\gamma = 7.9$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [ |0\rangle + |5\rangle ]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.21: Plots are $h^F_F(\tau)$ for $g/\gamma = 4$, $\kappa/\gamma = 7.9$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. 
(b) $\frac{1}{\sqrt{2}} (|0\rangle + |5\rangle)$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.22: Plots are $h_{\theta}^{TT}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = .77$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}}[|0\rangle + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.23: Plots are $h_\theta^{FF}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = .77$, $\Delta_{1,+/}\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [(0) + |5\rangle]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.24: Plots are $h_0^{TT}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = 1.6$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $1/\sqrt{2} [ |0\rangle + |5\rangle ]$. (c) Pseudo-Boltzmann. (d) All states equal population.
Figure 5.25: Plots are $h_0^{FF}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = 1.6$, $\Delta_{1,+}/\gamma = 0.1$. (a) $|0\rangle$ only. (b) $\frac{1}{\sqrt{2}} [ |0\rangle + |5\rangle ]$. (c) Pseudo-Boltzmann. (d) All states equal population.
5.5.2 Effective temperature in Boltzmann distribution

Figures 5.26 through 5.31 present field-intensity correlation data for systems in which the atom’s vibrational levels are Boltzmann populated according to equation 4.63. As stated in chapter 4, the cases where \( T = 3 \times 10^{-3} \) and \( T = 3 \times 10^{-6} \) are very similar in structure to the “all states equal” and “vibrational ground state only” distributions. All the figures showing \( h_{\theta}^{EF}(\tau) \) data demonstrate that the system never behaves non-classically, no matter what the vibrational distribution is. This is in line with our previous results.

Figures 5.26 and 5.28 show that the correlation has a low violation, except for frame (c) of figure 5.28. This case is almost identical to that of a strict “vibrational ground state only” distribution, and the data matches figure 5.24(a) well. We see in figure 5.30, where \( g/\gamma = 2.2, \kappa/\gamma = 10, \text{ and } \Delta_{1,+}/\gamma = 0.1 \), that the correlation maintains its low violation until the effective temperature is again at \( T = 3 \times 10^{-6} \). Thus, we conclude that the non-classicalities are diminished in the bad-cavity regime when the detunings are reduced in occurrence and magnitude.
Figure 5.26: Plots are $h_{\theta}^{TT}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = .77$, $\Delta_{1,+}/\gamma = 0.1$. All trials use Boltzmann distribution for vibrational states. (a) $T = 3 \times 10^{-3}K$. (b) $T = 3 \times 10^{-4}K$. (c) $T = 3 \times 10^{-6}K$ (d) $T = 3 \times 10^{-5}K$
Figure 5.27: Plots are $h_{\theta}^{FF}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = .77$, $\Delta_{1,+}/\gamma = 0.1$. All trials use Boltzmann distribution for vibrational states. (a) $T = 3 \times 10^{-3}K$. (b) $T = 3 \times 10^{-4}K$. (c) $T = 3 \times 10^{-6}K$ (d) $T = 3 \times 10^{-5}K$
Figure 5.28: Plots are $h^T_\theta (\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = 1.6$, $\Delta_{1,+}/\gamma = 0.1$. All trials use Boltzmann distribution for vibrational states. (a) $T = 3 \times 10^{-3} K$. (b) $T = 3 \times 10^{-4} K$. (c) $T = 3 \times 10^{-6} K$ (d) $T = 3 \times 10^{-5} K$
Figure 5.29: Plots are $h_{\theta}^{FF}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = 1.6$, $\Delta_{1,+}/\gamma = 0.1$. All trials use Boltzmann distribution for vibrational states. (a) $T = 3 \times 10^{-3}K$. (b) $T = 3 \times 10^{-4}K$. (c) $T = 3 \times 10^{-6}K$ (d) $T = 3 \times 10^{-5}K$
Figure 5.30: Plots are $h^{TT}_o(\tau)$ for $g/\gamma = 2.2$, $\kappa/\gamma = 10$, $\Delta_{1,+}/\gamma = 0.1$. All trials use Boltzmann distribution for vibrational states. (a) $T = 3 \times 10^{-3}K$. (b) $T = 3 \times 10^{-4}K$. (c) $T = 3 \times 10^{-6}K$ (d) $T = 3 \times 10^{-5}K$. 
Figure 5.31: Plots are $h^{FF}_0(\tau)$ for $g/\gamma = 2.2$, $\kappa/\gamma = 10$, $\Delta_{1,+}/\gamma = 0.1$. All trials use Boltzmann distribution for vibrational states. (a) $T = 3 \times 10^{-3}K$. (b) $T = 3 \times 10^{-4}K$. (c) $T = 3 \times 10^{-6}K$ (d) $T = 3 \times 10^{-5}K$
5.5.3 Vibrational level spacing and two-state superpositions

Figures 5.32 through 5.43 show data for the field-intensity correlation when the vibrational state distribution is an even, two-state superposition. We consider distributions of the form

$$|\text{vibrational state}\rangle = \frac{1}{\sqrt{2}} [|l = 0\rangle + |l = m\rangle], \quad (5.34)$$

where $m = 1, 5$. All of the fluorescent field data shows that this field behaves classically after the conditioning event. Figure 5.32 shows that the transmitted field-intensity correlation maintains a low violation of similar magnitude for all strengths of detunings used. The higher detunings ($\Delta_{1,+}/\gamma = 0.5$ and $\Delta_{1,+}/\gamma = 2.0$) are also seen to add small amplitude fluctuations to the data. Figure 5.34 shows that the low violation increases in severity with increasing detuning, but at $\Delta_{1,+}/\gamma = 2.0$ the low violation is replaced by an undershoot. Figure 5.36 shows $h^T_T(\tau)$ when $g/\gamma = 2.2$ and $\kappa/\gamma = 10$. We see that the strength of the low violation increases with increasing detuning, and that high levels of detuning begin to cause secondary oscillations in the data.

The forms of all the curves in figure 5.38 are all generally the same, and again we see the secondary oscillations for high $\Delta_{1,+}$. This figure is showing data for a system in the superposition given by equation 5.34 when $m = 1$. All frames of this figure exhibit a low violation, the magnitudes of which increase with increasing detuning. This fits well with the earlier results as expected; the data for the superposition with $m = 1$ are just more special cases of the data for the system in
the two-state superposition with $m = 5$ (this explanation is also given in chapter 4). The data presented in figure 5.40 also show similar trends, the low violation is of greater magnitude with greater detuning. The data in figure 5.42 fall right into place with the data in figure 5.36, and with the smallest detuning of $\Delta_{1,+}/\gamma = 0.1$ we even see the low violation vanish. This data exhibits that with a small enough effective detuning (detuning as achieved through the vibrational distribution), the non-classicalities are absent from the field fluctuations, except for cases where the atom-cavity field mode coupling is greater than $\gamma$. 
Figure 5.32: Plots are $h_0^{TT}(\tau)$ for $g/\gamma = 1, \kappa/\gamma = .77$. All trials use $\frac{1}{\sqrt{2}} |0\rangle + |5\rangle$ as the vibrational state distribution. (a) $\Delta_{1,+}/\gamma = 0.1$. (b) $\Delta_{1,+}/\gamma = 0.3$. (c) $\Delta_{1,+}/\gamma = 2$ (d) $\Delta_{1,+}/\gamma = 0.5$
Figure 5.33: Plots are $h_0^{EF}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = .77$. All trials use $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$ as the vibrational state distribution. (a) $\Delta_{1,+/\gamma} = 0.1$. (b) $\Delta_{1,+/\gamma} = 0.3$. (c) $\Delta_{1,+/\gamma} = 2$ (d) $\Delta_{1,+/\gamma} = 0.5$
Figure 5.34: Plots are $h^T_{\theta}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = 1.6$. All trials use $\sqrt{2} \left[ |0\rangle + |5\rangle \right]$ as the vibrational state distribution. (a) $\Delta_{1,+}/\gamma = 0.1$. (b) $\Delta_{1,+}/\gamma = 0.3$. (c) $\Delta_{1,+}/\gamma = 2$ (d) $\Delta_{1,+}/\gamma = 0.5$
Figure 5.35: Plots are $h_0^{EF}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = 1.6$. All trials use $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$ as the vibrational state distribution. (a) $\Delta_{1,+}/\gamma = 0.1$. (b) $\Delta_{1,+}/\gamma = 0.3$. (c) $\Delta_{1,+}/\gamma = 2$ (d) $\Delta_{1,+}/\gamma = 0.5$
Figure 5.36: Plots are $h_0^{TT}(\tau)$ for $g/\gamma = 2.2$, $\kappa/\gamma = 10$. All trials use $\frac{1}{\sqrt{2}} [|0\rangle + |5\rangle]$ as the vibrational state distribution. (a) $\Delta_{\gamma} = 0.1$. (b) $\Delta_{\gamma} = 0.3$. (c) $\Delta_{\gamma} = 2$ (d) $\Delta_{\gamma} = 0.5$
Figure 5.37: Plots are $h^{FF}_0(\tau)$ for $g/\gamma = 2.2$, $\kappa/\gamma = 10$. All trials use $\frac{1}{\sqrt{2}} |0\rangle + |5\rangle$ as the vibrational state distribution. (a) $\Delta_{1,+}/\gamma = 0.1$. (b) $\Delta_{1,+}/\gamma = 0.3$. (c) $\Delta_{1,+}/\gamma = 2$. (d) $\Delta_{1,+}/\gamma = 0.5$
Figure 5.38: Plots are $h_g^{TT}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = .77$. All trials use $\frac{1}{\sqrt{2}} [\ket{0} + \ket{1}]$ as the vibrational state distribution. (a) $\Delta_{1,+}/\gamma = 0.1$. (b) $\Delta_{1,+}/\gamma = 0.3$. (c) $\Delta_{1,+}/\gamma = 2$ (d) $\Delta_{1,+}/\gamma = 0.5$
Figure 5.39: Plots are $h_g^{EF}(\tau)$ for $g/\gamma = 1$, $\kappa/\gamma = .77$. All trials use $\frac{1}{\sqrt{2}} [\ket{0} + \ket{1}]$ as the vibrational state distribution. (a) $\Delta_{1+}/\gamma = 0.1$. (b) $\Delta_{1+}/\gamma = 0.3$. (c) $\Delta_{1+}/\gamma = 2$ (d) $\Delta_{1+}/\gamma = 0.5$
Figure 5.40: Plots are $h_0^{TT}(\tau)$ for $g/\gamma = 1, \kappa/\gamma = 1.6$. All trials use $\frac{1}{\sqrt{2}} [|0\rangle + |1\rangle]$ as the vibrational state distribution. (a) $\Delta_{1,+}/\gamma = 0.1$. (b) $\Delta_{1,+}/\gamma = 0.3$. (c) $\Delta_{1,+}/\gamma = 2$ (d) $\Delta_{1,+}/\gamma = 0.5$
Figure 5.41: Plots are \( h_\theta^{EF}(\tau) \) for \( g/\gamma = 1, \kappa/\gamma = 1.6 \). All trials use \( \frac{1}{\sqrt{2}} \langle 0 | + | 1 \rangle \) as the vibrational state distribution. (a) \( \Delta_{1,+}/\gamma = 0.1 \). (b) \( \Delta_{1,+}/\gamma = 0.3 \). (c) \( \Delta_{1,+}/\gamma = 2 \). (d) \( \Delta_{1,+}/\gamma = 0.5 \).
Figure 5.42: Plots are $h_{0}^{TT}(\tau)$ for $g/\gamma = 2.2$, $\kappa/\gamma = 10$. All trials use $\frac{1}{\sqrt{2}} \left[ |0\rangle + |1\rangle \right]$ as the vibrational state distribution. (a) $\Delta_{1,+}/\gamma = 0.1$. (b) $\Delta_{1,+}/\gamma = 0.3$. (c) $\Delta_{1,+}/\gamma = 2$. (d) $\Delta_{1,+}/\gamma = 0.5$
Figure 5.43: Plots are $h^F_\theta(\tau)$ for $g/\gamma = 2.2$, $\kappa/\gamma = 10$. All trials use $\frac{1}{\sqrt{2}} [|0\rangle + |1\rangle]$ as the vibrational state distribution. (a) $\Delta_{1,+}/\gamma = 0.1$. (b) $\Delta_{1,+}/\gamma = 0.3$. (c) $\Delta_{1,+}/\gamma = 2$ (d) $\Delta_{1,+}/\gamma = 0.5$
Chapter 6

Conclusions and future work

As we have seen in the previous chapters, the effects of a non-pure vibrational state vary depending on the parameters of the cavity QED system. It was seen for $g^{(2)}(\tau)$ that increasing the number of populated states in the vibrational distribution usually washed out non-classical effects, but sometimes just changed the form of the non-classical behavior. When looking at systems where the atom is in a strict vibrational two-state superposition, increasing the spacing of the vibrational levels (and therefore increasing the effective detuning) sometimes brought about non-classicalities absent from the vibrational pure-state dynamics.

$g^{(2)}_{FF}(\tau)$ was not nearly as sensitive to adjustments in vibrational distribution as $g^{(2)}_{TT}(\tau)$ was. The intensity cross-correlations were shown to exhibit time-asymmetry to varying degrees depending on the cavity system parameters. Time asymmetry was reduced as the cavity system moved further into the bad-cavity limit. We also saw the violation of inequality 4.53 not to be effected by the vibrational distribution, as it was always present for $\tau = 0$.

We have seen in the field-intensity correlation data that non-classicalities in
the transmitted field are generally enhanced by the “de-purification” of the atom’s vibrational state. Similar to the intensity correlation data for the fluorescent field, the $h_\theta^{FF}(\tau)$ data suggests that the fluorescent field is not very sensitive to the atom’s vibrational distribution. In fact, non-classical behavior was never witnessed for the fluoresced field.

Future work should be done on this system to more realistically model an optical trapping potential, possibly in higher dimensions. Also, the inclusion of a higher number of both field number-states and vibrational states may provide deeper insight to this system’s behavior. This would allow for a driving field of arbitrary strength and would provide a more accurate model of experimental conditions.
Bibliography


Appendix A

Derivation of the Master Equation in the Born-Markoff Approximation

To begin the formal derivation of the Master equation in the Born-Markoff approximation, we start with a generalized form of equation 2.10 that is applicable to the full density operator, \( \chi(t) \)

\[
\dot{\tilde{\chi}}(t) = i\frac{\bar{h}}{\hbar} [\tilde{\chi}(t), H_i], \quad (A.1)
\]

where, as before, tilde denotes that we’re working in the interaction picture and \( H_i \) is the interaction Hamiltonian for the system and the reservoir(s). We now formally integrate equation A.1 and solve for \( \tilde{\chi}(t) \)

\[
\tilde{\chi}(t) = \tilde{\chi}(0) + i\frac{\bar{h}}{\hbar} \int_0^t dt' [\tilde{\chi}(t'), H_i(t')] \quad (A.2)
\]

We now substitute this back into the right hand side of equation A.1 to get

\[
\dot{\tilde{\chi}}(t) = i\frac{\bar{h}}{\hbar} [\tilde{\chi}(0), H_i] - \frac{1}{\hbar^2} \int_0^t dt' [[\tilde{\chi}(t'), H_i(t')], H_i(t)]. \quad (A.3)
\]
We now make the assumption that at $t = 0$ the full density operator factorizes as

$$\tilde{\chi}(0) = \tilde{\rho}(0)R_0(r), \quad (A.4)$$

where $\tilde{\rho}$ is the reduced density operator in the interaction picture, and $R_0(r)$ is the density operator of the reservoir(s) at $t = 0$. This assumption says that at $t = 0$ we have chosen that there is no interaction between the system and the reservoir(s).

We now trace over the reservoir(s) states to obtain an equation of motion for $\tilde{\rho}$ alone.

$$\dot{\tilde{\rho}} = \frac{i}{\hbar} Tr_R [\tilde{\rho}(0)R_0(r), H_i(t)] - \frac{1}{\hbar^2} Tr_R \int_0^t dt' \{ [\tilde{\chi}(t'), H_i(t')], H_i(t) \} . \quad (A.5)$$

For simplicity, we assume $Tr_R [R_0(r), H_i(t)] = 0$. This leaves us with

$$\dot{\tilde{\rho}} = -\frac{1}{\hbar^2} Tr_R \int_0^t dt' \{ [\tilde{\chi}(t'), H_i(t')], H_i(t) \} . \quad (A.6)$$

We now state the Born approximation, that to lowest order in $H_i$ the dynamics of the full density operator only depend upon the dynamics of the system’s reduced density operator.

$$\tilde{\chi}(t) = \tilde{\chi}(t)R_0(r) + O(H_i). \quad (A.7)$$

We may make this assumption because, as discussed in chapter 2, we take the coupling between the system and the reservoir to be small. With this weak coupling, the reservoir is taken to be unaffected by the system due to it’s vast size. Our system however, because of its small size as compared to the reservoir, is greatly affected by the weak coupling to many reservoir modes. Using the Born approximation, to lowest order in $H_i$, equation A.6 becomes

$$\dot{\tilde{\rho}} = -\frac{1}{\hbar^2} Tr_R \int_0^t dt' \{ [\tilde{\rho}(t')R_0(r), H_i(t')], H_i(t) \} . \quad (A.8)$$
It is instructive to complete the derivation with an example such that the form of the Liouvillian superoperator is justified in the equation \( \dot{\rho} = L \rho \). In order to do this, we must first further specify the nature of the interaction Hamiltonian \( H_i \) to be a collection of products between system and reservoir operators.

\[
H_i = \hbar \sum_{j,k} S_k \Gamma_{j,k}, \tag{A.9}
\]

where \( S_k \) are the system operators, and \( \Gamma_{j,k} \) are the operators of the reservoir only.

Equation A.8 now becomes

\[
\dot{\tilde{\rho}} = -\frac{1}{\hbar^2} Tr_R \int_0^t dt' \left[ \tilde{\rho}(t') R_0(r), \hbar \sum_{j,k} S_k(t') \Gamma_{j,k}(t') \right], \tag{A.10}
\]

Expanding equation A.10, and accounting for the sum indexing, we have

\[
\dot{\tilde{\rho}} = -\sum_{j,k,l,m} \int_0^t dt' \left[ Tr_R \left[ \tilde{\rho}(t') R_0(r) S_k(t') \Gamma_{j,k}(t') S_m(t) \Gamma_{l,m}(t) \right] \\
+ Tr_R \left[ S_k(t') \Gamma_{j,k}(t') \tilde{\rho}(t') R_0(r) S_m(t) \Gamma_{l,m}(t) \right] \\
+ Tr_R \left[ S_m(t) \Gamma_{l,m}(t) \tilde{\rho}(t') R_0(r) S_k(t') \Gamma_{j,k}(t') \right] \\
+ Tr_R \left[ S_m(t) \Gamma_{l,m}(t) S_k(t') \Gamma_{j,k}(t') \tilde{\rho}(t') R_0(r) \right] \right]. \tag{A.11}
\]

After performing the trace over the reservoir states, equation A.11 simplifies to

\[
\dot{\tilde{\rho}} = -\sum_{j,k,l,m} \int_0^t dt' \left[ [S_k(t') \tilde{\rho}(t') S_m(t) - \tilde{\rho}(t') S_k(t') S_m(t)] \langle \Gamma_{j,k}(t') \Gamma_{l,m}(t) \rangle_R \\
+ [S_m(t) \tilde{\rho}(t') S_k(t') - S_m(t) S_k(t') \tilde{\rho}(t')] \langle \Gamma_{l,m}(t) \Gamma_{j,k}(t') \rangle_R \right]. \tag{A.12}
\]

We are now in a position to specify the system and reservoir operators in equation A.9 so as to work the example of a single cavity field mode coupled to a reservoir of harmonic oscillator modes. We define the creation and annihilation operators for the cavity field mode in the interaction picture

\[
\tilde{S}_1(t) = e^{i\omega_0 a^\dagger a} e^{-i\omega_0 a^\dagger a} = a^\dagger e^{-i\omega_0 t}, \\
\tilde{S}_2(t) = e^{i\omega_0 a^\dagger a} a e^{-i\omega_0 a^\dagger a} = a e^{+i\omega_0 t}. \tag{A.13}
\]
And we define the reservoir oscillator mode operators to be

\[
\begin{align*}
\hat{\Gamma}_{1,j}(t) & = e^{i\omega_j c_j^\dagger t} \kappa_j c_j e^{-i\omega_j c_j^\dagger t} = \kappa_j c_j e^{-i\omega_j t}, \\
\hat{\Gamma}_{2,j}(t) & = e^{i\omega_j c_j^\dagger t} \kappa_j^\dagger c_j^\dagger e^{-i\omega_j c_j^\dagger t} = \kappa_j^\dagger c_j^\dagger e^{i\omega_j t}.
\end{align*}
\]

Using these operator definitions, we may now bring the sum through equation A.12 to arrive at

\[
\hat{\rho} = \int_0^t dt' \left( [a\hat{\rho}(t')a - a^\dagger a\hat{\rho}(t')] e^{-i\omega_0(t+t')} \sum_{j,l} (\Gamma_{2,j}(t')\Gamma_{2,l}(t))_R + \text{ h.c.} \\
+ [a^\dagger \hat{\rho}(t')a - a^\dagger a^\dagger \hat{\rho}(t')] e^{+i\omega_0(t+t')} \sum_{j,l} (\Gamma_{1,j}(t')\Gamma_{1,l}(t))_R + \text{ h.c.} \\
+ [a\hat{\rho}(t')a^\dagger - a a^\dagger \hat{\rho}(t')] e^{-i\omega_0(t-t')} \sum_{j,l} (\Gamma_{2,j}(t)\Gamma_{1,l}(t'))_R + \text{ h.c.} \\
+ [a^\dagger \hat{\rho}(t')a^\dagger - a a^\dagger \hat{\rho}(t')] e^{+i\omega_0(t+t')} \sum_{j,l} (\Gamma_{1,j}(t)\Gamma_{2,l}(t'))_R + \text{ h.c.} \right),
\]

(A.15)

where \text{h.c.} denotes the Hermitian conjugate of the term immediately preceding.

The specific forms of the reservoir correlations needed to evaluate equation A.15 are

\[
\begin{align*}
\sum_{j,l} (\hat{\Gamma}_{2,j}(t')\hat{\Gamma}_{2,l}(t)) & = \sum_{j,l} \kappa_j^\dagger \kappa_l^\dagger e^{i\omega_j t} e^{-i\omega_l t} Tr(R_0 c_j^\dagger c_l) \\
& = 0, \\
\sum_{j,l} (\hat{\Gamma}_{1,j}(t)\hat{\Gamma}_{1,l}(t')) & = \sum_{j,l} \kappa_j \kappa_l e^{-i\omega_j t'} e^{-i\omega_l t} Tr(R_0 c_j c_l) \\
& = 0, \\
\sum_{j,l} (\hat{\Gamma}_{2,j}(t)\hat{\Gamma}_{1,l}(t')) & = \sum_{j,l} \kappa_j^\dagger \kappa_l e^{i\omega_j t} e^{-i\omega_l t'} Tr(R_0 c_j^\dagger c_l) \\
& = \sum_j |\kappa_j|^2 e^{+i\omega_j (t-t')} \bar{n}(\omega_j, T), \\
\sum_{j,l} (\hat{\Gamma}_{1,j}(t)\hat{\Gamma}_{2,l}(t')) & = \sum_{j,l} \kappa_j \kappa_l^\dagger e^{-i\omega_j t'} e^{+i\omega_l t} Tr(R_0 c_j c_l^\dagger) \\
& = \sum_j |\kappa_j|^2 e^{-i\omega_j (t-t')} (\bar{n}(\omega_j, T) + 1),
\end{align*}
\]

(A.16)

where \(\bar{n}(\omega_j, T)\) is the average thermal photon number for reservoir mode \(j\) in thermal equilibrium at temperature \(T\) as discussed in chapter 2. More specifically,

\[
\bar{n}(\omega_j, T) = Tr_R \left( R_0 c_j^\dagger c_j \right) = \frac{e^{-\hbar\omega_j / k_B T}}{1 - e^{-\hbar\omega_j / k_B T}}.
\]

(A.17)

To continue, it is helpful if we make a change of variables and an approximation about the nature of the reservoir’s mode spacing. First we define the delay time \(\tau = t - t'\). Next, we state that the reservoir’s modes are very tightly spaced, warranting the conversion of the sum in equations A.16 to integrals. Defining the \textit{mode density} \(g(\omega)\) such that \(g(\omega)d\omega\) is the number of reservoir modes of frequencies
between $\omega$ and $\omega + d\omega$ and using $\tau$ in place of $t - t'$, we have
\[
\sum_j |\kappa_j|^2 e^{+i\omega_j(t-t')} \bar{n}(\omega_j, T) \Rightarrow \int_0^\infty d\omega e^{+i\omega \tau} g(\omega) |\kappa(\omega)|^2 \bar{n}(\omega, T),
\]
\[
\sum_j |\kappa_j|^2 e^{-i\omega_j(t-t')} (\bar{n}(\omega_j, T) + 1) \Rightarrow \int_0^\infty d\omega e^{-i\omega \tau} g(\omega) |\kappa(\omega)|^2 (\bar{n}(\omega, T) + 1). \tag{A.18}
\]
Equation A.15 may now be written as
\[
\dot{\tilde{\rho}} = \int_0^t d\tau \left( [a^\dagger \tilde{\rho}(t-\tau)a - aa^\dagger \tilde{\rho}(t-\tau)] \int_0^\infty d\omega e^{+i(\omega-\omega_0)\tau} g(\omega) |\kappa(\omega)|^2 \bar{n}(\omega, T) + \text{h.c.} + [a \tilde{\rho}(t-\tau)a^\dagger - a^\dagger a \tilde{\rho}(t-\tau)] \int_0^\infty d\omega e^{-i(\omega-\omega_0)\tau} g(\omega) |\kappa(\omega)|^2 (\bar{n}(\omega, T) + 1) + \text{h.c.} \right). \tag{A.19}
\]
Now we are able to make the formal statement of the Markoff approximation. The integrals of equations A.18 are essentially delta functions, as any sizable $\tau$ will average the “slower varying” functions $g(\omega)$, $|\kappa(\omega)|^2$ and $\bar{n}(\omega, T)$ to zero. This means that the reservoir correlations “die off” on a time scale much shorter than the natural time scale of the cavity field mode system (and hence $\tilde{\rho}$). We may therefore replace $\tilde{\rho}(t-\tau)$ with $\tilde{\rho}(t)$ and extend the upper limit on the time integral of equation A.19 to infinity. We may now write
\[
\lim_{t \to \infty} \int_0^t d\tau e^{-i(\omega-\omega_0)\tau} \approx \pi \delta(\omega - \omega_0), \tag{A.20}
\]
where we have ignored a small, imaginary offset (this offset is equivalent to the Lamb shift for our cavity field mode “oscillator”). This delta function reduces the remaining integral over $\omega$, and we may now write A.19 as
\[
\dot{\tilde{\rho}} = \frac{\kappa'}{2} \left( 2a \tilde{\rho} a^\dagger - a^\dagger a \tilde{\rho} - \tilde{\rho} a^\dagger a \right) + \kappa' \bar{n} \left( a \tilde{\rho} a^\dagger + a^\dagger a \tilde{\rho} - a^\dagger a \tilde{\rho} a \right), \tag{A.21}
\]
where
\[
\kappa' \equiv 2\pi g(\omega_0) |\kappa(\omega_0)|^2, \\
\bar{n} \equiv \bar{n}(\omega_0, T). \tag{A.22}
\]
We may transform this equation back into the Schrödinger picture by the usual prescription,
\[
\dot{\rho} = -\frac{i}{\hbar} [H_S, \rho] + e^{-i(H_S t)} \tilde{\rho} e^{i(H_S t)} - \frac{1}{\hbar} [H_S, \rho] + e^{-i(H_S t)} \dot{\tilde{\rho}} e^{i(H_S t)}, \tag{A.23}
\]
\[
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\]
to obtain the familiar result, as presented in chapter 2 (using the Linblad form)

\[ \dot{\rho} = -i\omega_0 [a^\dagger a, \rho] + \frac{\kappa'}{2} (\bar{n} + 1) \left( 2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a \right) + \frac{\kappa'}{2} \bar{n} \left( 2a^\dagger \rho a - aa^\dagger \rho - \rho aa^\dagger \right) \]  
(A.24)
Appendix B

Construction of the Hamiltonian

In this appendix, we present the evaluation of $\hat{H}|\Psi\rangle$, where $\hat{H}$ and $|\Psi\rangle$ are given by equations 3.28 and 3.29, respectively. The process is more tractable if we first break $\hat{H}$ into 8 parts that separately address the energetics and/or interactions of the system’s various degrees of freedom. Doing this, we arrive at

$$\hat{H} = [\hat{A} + \hat{B} + \hat{C} + \hat{D} + \hat{H}_1 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4], \quad (B.1)$$

where

$$\begin{align*}
\hat{H}_A &= \frac{\hbar \omega \sigma_z}{2} + \hbar \omega a^\dagger a, \\
\hat{H}_B &= \hbar \Omega_n b^\dagger b, \\
\hat{H}_C &= \hbar g(\sigma_+ a + a^\dagger \sigma_-), \\
\hat{H}_D &= i\hbar E'(e^{i\omega t}a - e^{-i\omega t}a^\dagger), \\
\hat{H}_1 &= -\frac{i\hbar \gamma}{2} \sigma_+ \sigma_-, \\
\hat{H}_2 &= -i\hbar \kappa a^\dagger a, \\
\hat{H}_3 &= -i\hbar (n_{Th} + 1) \Gamma(b^\dagger b), \\
\hat{H}_4 &= -i\hbar n_{Th} \Gamma(b b^\dagger). \quad (B.2)
\end{align*}$$

Now, we examine the operation of each of these smaller Hamiltonians on our wavefunction. Because we chose to describe our system using the “well dressed” states, we will simply show the Hamiltonians’ operation on $|n, k, \pm\rangle$ and $|0, k, g\rangle$.

$$\hat{H}_A |n, k, \pm\rangle = \hbar \omega (n - \frac{1}{2}) |n, k, \pm\rangle,$$
\[
\hat{H}_A |0, k, g\rangle = -\frac{\hbar \omega}{2} |0, k, g\rangle, \\
\hat{H}_B |n, k, \pm\rangle = \hbar \Omega_{n, \pm} |n, k, \pm\rangle, \\
\hat{H}_B |0, k, g\rangle = \hbar \Omega_0 |0, k, g\rangle, \\
\hat{H}_C |n, k, \pm\rangle = \pm \hbar g \sqrt{n} |n, k, \pm\rangle, \\
\hat{H}_C |0, k, g\rangle = 0, \\
\hat{H}_D |n, k, \pm\rangle = \frac{i \hbar E'}{2} \sqrt{n-1} e^{i \omega t} \left[ |n-1, k, +\rangle + |n-1, k, -\rangle \right] \\
\pm \frac{i \hbar E'}{2} \left[ e^{i \omega t} \sqrt{n} \left[ |n-1, k, +\rangle - |n-1, k, -\rangle \right] \\
- e^{-i \omega t} \sqrt{n+1} \left[ |n+1, k, +\rangle - |n+1, k, -\rangle \right] \\
- \frac{i \hbar E'}{2} \sqrt{n} e^{-i \omega t} \left[ |n+1, k, +\rangle + |n+1, k, -\rangle \right], \\
\hat{H}_D |1, k, \pm\rangle = \pm \frac{i \hbar E'}{2} \left[ e^{i \omega t} \frac{2}{\sqrt{2}} |0, k, g\rangle - e^{-i \omega t} \sqrt{2} \left[ |2, k, +\rangle - |2, k, -\rangle \right] \right] \\
- \frac{i \hbar E'}{2} e^{-i \omega t} \left[ |2, k, +\rangle + |2, k, -\rangle \right], \\
\hat{H}_D |0, k, g\rangle = \frac{-i \hbar E' \sqrt{2}}{2} e^{-i \omega t} \left[ |1, k, +\rangle - |1, k, -\rangle \right], \\
\hat{H}_1 |n, k, \pm\rangle = \frac{-i \hbar \gamma}{4} \left[ |n, k, +\rangle + |n, k, -\rangle \right], \\
\hat{H}_1 |0, k, g\rangle = 0, \\
\hat{H}_2 |n, k, \pm\rangle = -i \hbar \kappa n |n, k, \pm\rangle + \frac{i \hbar \kappa}{2} \left[ |n, k, +\rangle + |n, k, -\rangle \right], \\
\hat{H}_2 |0, k, g\rangle = 0, \\
\hat{H}_3 |n, k, \pm\rangle = -i \hbar (n_{Th} + 1) \Gamma k |n, k, \pm\rangle, \\
\hat{H}_3 |0, k, g\rangle = -i \hbar (n_{Th} + 1) \Gamma k |0, k, g\rangle, \\
\hat{H}_4 |n, k, \pm\rangle = -i \hbar n_{Th} \Gamma (k+1) |n, k, \pm\rangle, \\
\hat{H}_4 |0, k, g\rangle = -i \hbar n_{Th} \Gamma (k+1) |0, k, g\rangle.
\]

(B.3)

Combining all of these operations, we can now write down the full version of \(\hat{H}|\Psi\rangle\) as

\[
\hat{H}|\Psi\rangle = \sum_k \left[ (\hat{H}_A + \hat{H}_B + \hat{H}_C + \hat{H}_D + \hat{H}_1 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4)|0, k, g\rangle \left[ C_{0,k,g} e^{-\frac{E_{0,k,g}}{\kappa}} \right] + \hat{H}_D |1, k, +\rangle \left[ C_{1,k,+} e^{-\frac{E_{1,k,+}}{\kappa}} \right] \right]
\]

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\[ + \hat{H}_D |1, k, - \rangle \left[ C_{1,k,-} e^{-\frac{E_{1,k,-} t}{\hbar}} \right] \]
\[ + \sum_{n \neq 0,k} C_{n,k,+} e^{-\frac{E_{n,k,+} t}{\hbar}} (\hat{H}_A + \hat{H}_B + \hat{H}_C + \delta_n \hat{H}_D + \hat{H}_1 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4) |n,k,+ \rangle \]
\[ + \sum_{n \neq 0,k} C_{n,k,-} e^{-\frac{E_{n,k,-} t}{\hbar}} (\hat{H}_A + \hat{H}_B + \hat{H}_C + \delta_n \hat{H}_D + \hat{H}_1 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4) |n,k,- \rangle, \]

where
\[ \delta_n = \begin{cases} 
0 & : \ n = 1 \\
1 & : \ n \neq 1 
\end{cases} \]

(B.4)

Equation 3.31 is now finally achieved if we evaluate equation B.4 as governed by equations B.3.