SYSTEMATIC SYMMETRIES:
AN INQUIRY INTO THE INFINITE VIA THE WORKS OF M.C. ESCHER

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CHAPTER 1

Introduction

"A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas."

-G. H. Hardy, A Mathematician’s Apology

G.H. Hardy’s notion is one that defies the stereotype of mathematicians as mechanical, cold individuals who, as often depicted in the media, may have social issues or mental illness pumping through their symbol obsessed veins. He dares to put the mathematically-minded on the same plane as artists, the historically forward thinking rebels of society. Objection in relation to this equivalence is understandable but unfounded, as the intersection of the lives of mathematicians and artists is non-empty. In the junction, beyond the theoretical urge to abstract and structure a deeper understanding of our small reality, there also exists a (once) breathing example of an artist who dared to traverse the unfamiliar paths that govern the minds of mathematicians and, in the process, created work that will remain both beautiful and precise for generations. This artist goes by a moniker that is familiar to most – M.C. Escher.

Doris Schattschneider’s The Mathematical Side of M.C. Escher describes Escher as a youth, noting that he experienced little pull toward mathematics, describing himself as “an extremely poor pupil in arithmetic and algebra... I was slightly better at solid geometry because it appealed to my imagination, but even in that subject I never excelled at school” [12, p. 706]. The bits of inspiration that struck Escher when confronted with problems of shape would be the small epiphanies that would finally push him toward the appropriate vernacular for working with shape and symmetry - the language of mathematics. The connection may seem clear to those who have had the pleasure of viewing one of Escher’s tiling works- an understanding of transformations underlies every
decision the artist made. Yet, the man denied any ability in the field and never turned to rigorous mathematics in his work. Instead, he took definitions from mathematicians and devised his own method of studying “tessellations,” using only visual tools to answer his own, surprisingly well—posed questions. His son George reflected on his father’s view of mathematics in his work:

“Unfortunately, the specialized language of mathematics hid from [Father] the fact that mathematicians were struggling with the same concepts as he was. Scientists, mathematicians, and M.C. Escher approach some of their work in similar fashion. They select by intuition and experience a likely looking set of rules which defines permissible events inside an abstract world. Then they proceed to explore in detail the consequences of applying these rules. If well—chosen, the rules lead to exciting discoveries, theoretical developments and much rewarding work.” [12, p. 706] An analysis of the biographical sketch provided by Bruno Ernst in his tell–all book *The Magic Mirror of M. C. Escher* reveals that Escher’s initial intuitive inspiration behind the life—long obsession with tilings, and, in the search for understanding, mathematics came from two coming of age visits to the Alhambra in Granada, Spain. There, Escher meticulously sketched the majolica tiles and, after the second trip, traded the classical artistic traditions he rigorously absorbed while studying at the Haarlem School for Architecture and Decorative Arts for a completely different notion. Doris Schattschneider describes Escher’s evolution as the decision of the man to let go of “landscapes” and embrace the concept of “mindscapes” [12, p. 706]. Most students of mathematics would find this similar to letting go of the naive constraints of realizable application that is so prevalent in the promotion of the subject and allowing their algorithmically trained minds to wander into the abstract beauty that is the world of higher mathematics. As a student standing alarmingly near the edge of madness that is the decision to dedicate their life to mathematics, it is our aim to explore the world of someone who did just that
but managed to do so through a medium, giving life to the abstractions in a meaningful and beautiful way. Beyond this existential desire, the main aim of this work is to present an accessible tutorial for anyone, mathematically minded or not, who is curious to understand the delicate theory of symmetry that underlies Escher’s work and its deep roots in mathematics. These roots, I believe, are what will make Escher’s work timeless. Afterall, despite his draw to the aesthetic, he created beauty out of symmetry – a concept far more permanent in the earth’s history than humankind itself.
Although it is often said that one should keep the artist and the work separated, M.C. Escher’s incredible journey is one that requires cataloguing for several reasons, chief among them the fact that his run-ins with inspiration were so randomized that had one event played out in a different format, there is no telling as to what direction his art would have flowed. One of the most concise biographical presentations of Escher’s life is the aforementioned work of Ernst, which graciously provides us with an unbounded pool of information about the artist’s life.

The artistic and wildly mathematical life of Mauritius Cornelis Escher, born in Leeuwarden, Netherlands, in 1898, began, as most great things do, with an ending; Escher dropped out of art school in 1922, after spending two years at the School of Architecture and Decorative Arts. Having mastered the techniques of technical drawing and woodcut printmaking, he emerged wielding a chisel and immediately took off for Italy, as any young art school drop out should, seeking the scenery that would inspire hundreds of drawings and prints. After an initial two week adventure through the country, Escher was determined to return to the stimulating landscape. Although his official college transcript read that he was “... too tight, too literary–philosophical, a young man too lacking in feeling or caprice, too little of an artist,” the young man exhibited his ability to be taken to whimsy as he hopped from country to country on cargo boats, working his way from city to city before arriving in Siena in the spring of 1923 [3, p. 8].

In this year-long journey, Escher’s work showed little of the stylistic sophistication that most associate with the artist. He sketched landscapes, buildings, plants, and his future wife Jetta, whom he met in Italy that year. By 1926, the couple wed, moved from Rome to the outskirts of the country, and welcomed
their first child, George, into the world. Finally, Escher could get down to work. However, he did not fall into the mundane routine of a working husband and father. Instead, he continued to travel, create, and seek out every possible contingency. Each spring, Escher continued to pursue the same notions of youthful exuberance that initially drove him to Italy by setting off on a two month long adventure. Mischief loves company, hence the artist had no issue finding a band of brothers, consisting of fellow artists, who were more than willing to take off and scour the scenery of Abruzzi, Campania, Sicily, Corsica, and Malta. In true bohemian fashion, the men preferred to walk to the lengths of the country side, absorbing the culture and political air of the time and drinking it all down with goat’s milk. It was on one these journeys that Escher sketched for his first critically acclaimed work, *Castrovalva* (1930). Of course, a work of great praise comes with a price, and Escher’s was a small passport tussle with the Italian police, over the suspicion that the artist has attempted to assassinate the king of Italy.

![Figure 1: Castrovalva](image)

An utter disdain for authority and politics motivated Escher to relocate his family from Italy to the neutral environment of Switzerland, where he could peacefully envelop himself in his own ideas and freely play with pouring his thoughts through the sieve of a medium. Although the family’s stay in this
particular country was a brief two years, the brutal winters forced Escher to reexamine his life. The artist felt trapped within the sharp edges of the icy architecture of Chateau d’Oex. Amidst the “horrible white misery of snow,” Escher yearned for the warmth of Southern Italy and, quite literally, dreamt of the sea [3, p.11]. This urge inspired the artist, once accused of being devoid of quirk, to make the arrangement that would take him back on the road and back to the place where his obsession with “periodic space-filling” would begin: Granada, Spain. Although Escher had passed through the city in his initial two-week voyage to Italy and visited the Alhambra, the home of some of the most geometrically intricate decoration one can find, his mind was not altered by the short stay. He attempted a few prints, inadvertently playing with mathematical ideas in several sketches and prints. One could theorize that beneath the superficial current of urging for the beauty and calm of the sea, Escher’s mind yearned to return to the equally breathtaking organization of the Moorish tiles, for he had vaguely recognized, but did not fully understand the beauty of such systematic compositions. Escher himself best describes this cognitive dissonance:

“Long before I discovered a relationship with regular space division through the Moorish artists of the Alhambra, I had already recognized it in myself. In the beginning... I knew no rules of the game and tried, without knowing what I was about, to fit together congruent surfaces... this has remained a very strenuous occupation, a real mania to which I became enslaved...” [3, p. 41]

Pre-enslavement and in hot pursuit of any sort of mania, M.C. and Jetta were able to travel by cargo boat through much of Southern Europe due to a serendipitous deal made by the artist. The arrangement consisted of payment by prints, 48 to be exact, for which he sketched throughout the trip. The company, completely unaware of Escher’s artistic ability, accepted the deal and, inadvertently, provided Escher with a ticket to his future as an artist. This ticket led
Escher and his wife back to the Alhambra in Granada, Spain, where the couple had the pleasure of spending quality time amongst the elegance of the Moorish style. Man and wife sketched their surroundings, getting lost in the “great complexity and geometric artistry” of the tiles [Weirda p. 41]. At the time, Escher had little concern for the mathematics underlying the beauty of the tiles. In the mathematical community, these tiles, along with much of Egyptian and Moorish art, would go on to be thoroughly studied from the perspective of symmetry groups, which are later discussed in this work. Lacking the mathematical background allowed Escher to explore the motifs on his own. In *M.C. Escher: His Life and Complete Graphic Work*, he writes that he spent a large part of his time puzzling with animal shapes, attempting to make connections from his sketches with the idea of making his own, less design orientated tilings. A year later, his half-brother, a geologist, introduced Escher to the field of crystallography, which finally brought mathematics into the artist’s world. He read articles of Haag, which revealed to him the mathematical definition of regular plane division, an idea he immediately began to explore carefully and thoughtfully, as demonstrated in the sketch of tiling lizards (featured in *Reptiles*, a later print). These types of explorations were common in Escher’s meticulous approach to making art. The unbounded work ethic and the ability to fall head first, without fear, into the rabbit holes of mathematical abstraction allowed Escher to carry on a correspondence with several mathematicians throughout his lifetime. The list is impressive, including Roger Penrose, H.S.M. Coxeter and, most notably, George Pólya, with whom he developed a repartee that would support his exploration of symmetry and allow him to develop fully as an amateur researcher of mathematics, an unofficial title he would hold for the rest of his life as he continued to attach aesthetic to pure abstraction [4].

According to Doris Schattschneider’s article *The Mathematical Side of M.C. Escher*, despite Escher’s lack of training or “talent” for mathematics, the relationships he developed with mathematicians were hardly one-sided. Both
parties often experienced the delight of inspiration derived from one another’s work. Escher’s interaction with Pólya was so indepth that it deserves its own portion of this work. Penrose and Coxeter encountered Escher’s work at the same exhibit and walked away independently in awe of various aspects of the other man’s work. For Penrose, Escher’s *Relativity*, the lithograph of a dizzying array of staircases that defy gravity as a whole but whose faceless occupants travel according to the local system established by the orientation of their particular location, stimulated the urge to seek out a simplified and more concise version of such an ‘impossible’ situation. The result, of course, is the infamous Penrose tribar, a physically unrealizable shape whose three perpendicular parts appear to “fit” when considered locally. Penrose returned the favor of inspiration by forwarding all such ‘impossible’ sketches to the artist who, in turn, took the forms and translated them into a representation of perpetual motion, as seen in several of Escher’s most famous works – including *Waterfall* and *Ascending Staircase*. Both works are viewed as ‘optical illusions’, where the eye does not get lost till the global view is taken.

Coxeter, on the other hand, was initially interested in utilizing two of Escher’s
symmetry drawings in an article about hyperbolic geometry. The article where the drawings appeared featured a mathematical image of a hyperbolic tiling—essentially, a tiling trapped within a circle. In order to envision such an object, one must consider the concept of infinite pieces in a finite setting—a topic Escher, naturally, found to be engrossing. The artist analyzed the image in his usual, meticulous way, and was able to produce another one of his more iconic works—Circle Limit 1. Coxeter and Escher continued their correspondence, despite the artist’s disappointment at the mathematician’s inability to provide explanations of certain phenomena that Escher found to be intriguing in a way the artist could grasp. However, Escher knew from experience to not be discouraged. He wrote that his

“... great enthusiasm for this sort of picture and... tenacity in pursuing the study will, perhaps, lead to a satisfactory solution in the end... no matter how difficult it is, I feel great satisfaction from solving a problem like this in my own bumbling fashion”[12, p. 713]. Escher’s fashion prevailed and he was able to produce three more “Coxeters.” His vigilance and beautifully constructed work was even more appreciated by the mathematician. In a Mathematical Review, Coxeter wrote that “Escher’s work, made on his intuition, without any computation, is perfect”[p.713].

Escher’s accomplishments as an amateur mathematician were ultimately exhibited by the work of a Ph.D. student. Heinrich Heesch was interested in the same type of tiling characterizations that Escher had spent much of his artistic life exploring. Twenty years after Escher concluded his studies of the topic, Heesch concluded that there were exactly 28 types of the specific tilings of the type both men found to be intriguing. The clincher lies in the fact that Escher, with his ‘bumbling fashion’ and lack of any formal training, discovered 27 of these tilings on his own.
CHAPTER 3

Escher’s Tessellations

Escher’s aforementioned Reptiles was based upon his first pokes at the idea of a tessellation, or regular division of the plane, described by Haag as “congruent convex polygons joined together; the arrangement by which the polygons adjoin each other is the same throughout” when translated from his text Die regelmäßigen Planteilungen und Punktsysteme. The later print based upon the sketches that attempted to depict such symmetry is the representation of a lizard breaking out of a simple two dimensional tiling contained in a sketch book and transforming into a living, breathing three dimensional version of itself. The lizard scales a book, a geometer’s tool, and, finally, exhibits its prowess as the descendant of a dragon when arriving upon the dodecahedron, a polyhedron consisting of 12 sides, before returning to its flat realm. Despite the cyclic nature of the image, those who believe in seeing the artist in their work could take Reptiles to be a metaphor for the changes that took place in Escher’s aesthetic upon discovering the connection betwixt his intuitive understanding of tiling and the rigorous methods of mathematics. If one suspends one’s belief and allows the lizard to personify the artist himself, the journey from the sketch pad and back can be viewed as a representation of Escher’s journey to the understanding of the tilings, where the climax is represented by the show of smoke upon the dodecahedron, a purely mathematical object, and the journey up the book and set square represent the slow separation of Escher from the purely aesthetic to the more abstract. Upon reaching understanding, or the dodecahedron, the lizard returns to the sketchpad, just as Escher did after learning a new concept.
Before any of the aforementioned interaction with the mathematically minded, Escher was able to reach a true understanding of tilings due to a serendipitous run–in with a paper of George Pólya, the mathematician who lived by many teaching axioms; the most crucial amongst them for Escher’s fate was simple and concrete: “make a picture”[10, p. 293]. According to Schattschneider’s article *The Pólya–Escher Connection*, the Pólya effect on the direction and execution of Escher’s work is undeniable. In 1924, the year that Escher returned to Granada, Pólya’s paper on the 17 symmetries of the plane was published in German. Unlike others who had made the same classifications, Pólya included illustrations with his work. It was this action that earned him the honor of having his name neatly printed across the front of one of Escher’s many (17, to be exact) notebooks devoted to the topic of regular plane division [p.293]. Upon discovery of the illustrations, Escher painstakingly copied the entire paper, even the discussion of classification of groups, which the mathematician was most concerned with. Colors were absent from all of Pólya’s illustrations but one, yet this did not deter the artist from creating a new puzzle for himself. He wanted his coloring to be perfect – to stick to the aesthetic harmony of the tilings themselves. According to Schattschneider, there is a mathematical interpretation of
a ‘perfect coloring’. By her definition, it is a tiling where the symmetry group which acts on the uncolored tiling induces a permutation of the set of colors of the tiles. Do not be deterred if these words mean nothing, they meant nothing to Escher as well. Yet, his intuition and effort led him in the direction of perfection, just as it did with his “Coxeters.” Pólya’s paper provided a set of visual lectures for the very serious student and led him to create a classification system which he utilized in the generation of his tessellations.

Although Escher’s system is rooted deeply in the careful study of geometry and symmetry, he developed it for the purpose of adding an essence of life to the calculated ornamentation or the Moors. The tiles that had tickled his fancy and stimulated an interest in the process of plane division also lacked the element Escher deemed essential, as demonstrated by its presence in every bit of work he created: recognizability. Escher wrote in disbelief about the lack of this quality in the Moorish tiles, noting with astonishment the lack of the artist’s desire to imitate nature in their work. “This is hardly believable,” he wrote. “Recognizability is so important to me that I never could do without it” [2, p. 42].

In his attempt to recognize the underlying method of his desires, Escher constructed two classes of systems that, essentially, yielded vague algorithms for his innately endless tessellations, a concept he based on the belief that “a plane, which one must imagine as extending without boundries in all directions, can be filled or divided into infinity, according to a limited number of systems, with similar geometric figures that are contiguous on all sides without leaving ‘empty spaces’ ” [2, p.93]. We now present The Quadrilateral systems, which are the basis for most of Escher’s work with tessellations, and the Triangle systems, both presented in Schattschneider’s book, *Visions of Symmetry: Notebooks, Periodic Drawings, and Related Work of M.C. Escher*. 
CHAPTER 4

The Quadrilateral Systems

The Quadrilateral systems depend on the selection of one of five polygons and one of the ten specified symmetries desired in the tessellation. Each polygon determines a grid which Escher used as a base for a “motif,” which he defines as “a certain polygonal form that repeats itself in congruent shapes in the plane” [11, p.70]. We will take “congruent” in its colloquial form here, as shapes that are exactly the same in every way except orientation. Escher formed his motifs per system, starting with a grid cell and then trimming sides in a way that preserves symmetry. The Quadrilateral systems boast the most intricate characteristics in regards to these motifs, and Escher stayed organized by assigning notation to each of the systems. The notation begins where each motif does – with the polygonal grid. He assigned letters to these forms, which determine the underlying grid:

A – Parallelogram
B – Rhombus
C – Rectangle
D – Square
E – Isosceles Right Triangle

and numbered the symmetries with Roman numerals [11, p. 58]. So, a system labeled I^A denotes that we will be working with a parallelogram grid and allow translations in two directions (see Table 1). The rules within each system are straightforward and depend on the “movement” of the original shape to the tiles that lie adjacent to it. Since four of the tiles end up sharing an edge and four others share a vertex with the original shape, each tile has eight surrounding tiles. These constraints allowed Escher to not only create tessellations but also color them to his liking, with a minimum of 2 colors, due to the ‘checkerboard’
nature of the arrangement. The symmetries of each system came in some combination of three possible generation varieties: translations, rotations, and glide reflections. As an artist, Escher did not bother with the mathematical definitions of these words (fear not, we shall worry about such things very soon). Instead, Escher drew an example of every system, and so we shall utilize loose definitions of these transformations [p.34].

A translation in this polygonal world is a transformation which slides a motif to a different point and is redrawn there.

Figure 4: A Translation of a Llama

A reflection of a motif is a “flip” of the shape directly to the left or right, top or bottom, or at an angle.

Figure 5: A Reflected Gnome

A glide reflection is a combination of translations and reflection, leading to combinations that boast no reflectional or rotational symmetry. In turn, a rotation implies a spinning action of a polygon about a central point.
Escher also defines and characterizes the movement of his motifs in specific directions. A transformation in a transversal direction is a shift of a motif to a motif of contrasting color that shares a boundary with the original polygon. This direction is parallel to a side of the underlying quadrilateral cell on which the motif is sketched [11, p. 61].

A transformation in a diagonal direction moves a motif to a motif of the same color that touches only a corner of the shape. The direction in this case is parallel to a diagonal of a cell in the underlying grid.
Escher’s constraints led him to ten systems of tessellations with one motif and two colors. Of course, this is not where the story ends. The quadrilaterals that he used for grids share the following, aforementioned common property: at each of the four vertices of each motif, exactly four motifs intersect. It is this property that allowed Escher to 2-color these systems. However, by simply moving one of the lines that collide at a vertex, we end up in a situation where three sides meet at a single point, thus requiring three distinguishing colors. Escher referred to this phenomenon as a “transitional system,” as we can return to one of the ten 2-color systems by continuing the movement of a shifted line on its trajectory until it reaches a vertex where an even number of motifs meet [11, p. 64–67]. The notation for such a system is merely a dash between the notation for the systems in transition (see Figure (b)).

The following visual discussion presents a recreation of a table Escher presents in his 1941–1942 notebooks (Table 1), as presented by Schattschneider on page 61 of *Visions of Symmetry*, along with an example of one such transition given by Escher in the same source. The double lines of division in the table represent groups of systems between that one can comfortably transition while playing by Escher’s rules. Systems that do not belong to the same group cannot be transitioned between. The example which appears in Figures 9, 10, and 11 is a basic transition within the System I constraints. Note that in Figure 10, three colors are required for the tessellation process. This is due to the fact that three congruent polygons meet at a point (this is a subtlety that requires a close examination of point C). The entire transition process, in this case, is based around
the idea of finding a shape within the given motif and ‘cutting’ it off or moving it around [11, p. 63]. Escher gives such examples for each of the five groups of systems, which can be found in Schattschneider’s book.

<table>
<thead>
<tr>
<th>System</th>
<th>Translations</th>
<th>Rotational Axes</th>
<th>Glide Reflections</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Both Transversal</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td>Both Diagonal</td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>One Transversal</td>
<td>2-fold on the vertices, 2-fold on the centers of the parallel sides</td>
<td>None</td>
</tr>
<tr>
<td>III</td>
<td>Both Diagonal</td>
<td>2-fold on the centers of all sides</td>
<td>None</td>
</tr>
<tr>
<td>IV</td>
<td>Both Diagonal</td>
<td>None</td>
<td>Both Transversal</td>
</tr>
<tr>
<td>V</td>
<td>One Transversal</td>
<td>None</td>
<td>One Transversal</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Both Diagonal</td>
</tr>
<tr>
<td>VI</td>
<td>One Diagonal</td>
<td>2-fold on the centers of two adjacent sides</td>
<td>One Diagonal, Both Transversal only in the direction of sides without rotation point</td>
</tr>
<tr>
<td>VII</td>
<td>None</td>
<td>2-fold on the centers of the parallel sides</td>
<td>One Transversal</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Both Diagonal</td>
</tr>
<tr>
<td>VIII</td>
<td>None</td>
<td>2-fold on the four vertices</td>
<td>Both Transversal</td>
</tr>
<tr>
<td>IX</td>
<td>None</td>
<td>4-fold on diagonal vertices</td>
<td>None</td>
</tr>
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<td></td>
<td></td>
<td>2-fold on diagonal vertices</td>
<td></td>
</tr>
<tr>
<td>X</td>
<td>None</td>
<td>4-fold on three vertices</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2-fold on the center of the hypotenuse</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: A copy of Escher’s summary chart as it is presented on page 61 of *Visions of Symmetry.*
(a) An example of an $I^A$ system. Note the parallel grid and the way in which it was clearly deformed here to create a more interesting form.

(b) An example of the previous system in transition from $I^A$ to $I^A$. Note that the grid remains the same although the tessellating form varies.

(c) The new $I^A$ system. Note the change in the parallelogram.

Figure 9: System Transition
Escher also explored the preservation of split single motif symmetries. His approach was to divide the “congruent polygons of the systems with one motif, or the transitional systems, into two unequal parts, in such a way that at each vertex of the newly formed polygons again 4, 6, or 8 polygons meet” [11, p. 70]. Escher utilized this technique in order to highlight duality in his work. The 2–color tilings that are constructed in this manner have been fully classified by Andreas W.M. Dress, who appropriately named them “regular Heaven and Hell patterns,” and are referred to in mathematical literature as 2–icosahedral tilings.

Figure 10: Escher utilized a 2–icosahedral tiling in the construction of Sky and Water I.

Although Escher’s exploration of polygonal tilings seems rather dull, his artistic output was lively, to say the least. It cannot be forgotten that his initial intent was to tessellate realistic forms, most specifically animals. In his essay, The Regular Division of the plane used in surface decoration, he highlights the aesthetic value of the systems that allow rotations and those that do not. He writes,

“When no rotation takes place (so only translation or (and) reflection), then it is logically acceptable to use animal motifs whose most characteristic silhouette is the side view (e.g., horse), or the front–view (e.g. person), and also those viewed from above (e.g. lizard). When a rotation does take place, then the
only animal motifs which are logically acceptable are those which show their most characteristic image when seen from above. From this it follows that the systems without rotation are more important for decoration of vertical surfaces than those with rotation.” [11, p.78]

This description of transformation in the plane sheds light on the clear, fundamental differences between the mind of a mathematician and the mind of M.C. Escher– the former must exploit all possibilities within the domain of their exploration, while the latter was seeking a specific outcome, setting aside the gems that sparkled along the way. Simultaneously, Escher’s statement can be viewed from a unifying perspective – for him, the discovery of Regular Division led to a theory that enabled him to finally solve the problem of the tiling lizard. For mathematicians, the creative utilization of various areas (such as geometry, topology, or analysis) in combination has often led to seemingly unrelated, but brilliant, results.
CHAPTER 5

Triangle Systems

The only type of rotational symmetry present in Escher’s quadrilateral systems is 2–fold or 4–fold. The artist also explored a set of systems which displayed 6–, 3–, and 2– fold rotational axes. The 6– and 3– fold axes never occur for quadrilaterals. However, they always occur for the set of systems he called Triangle Systems, which gained their name from the embedded equilateral triangle that is present in all of the motifs. These systems are in many ways “simpler” than the quadrilateral cases. Although Escher never formally categorized these Regular Divisions in the form of a table, he did provide visual examples and notation for all of the possibilities he considered. He classified these systems according to their rotational symmetry and divided them into two groups: A and B. In group A, he allowed for 3– fold axes only, determined in a specific way.

Figure 11: An example of 3–fold symmetry.

System B allows for 6– and 3– fold axes, arranged in their own specific way.

Figure 12: An example of 6–fold symmetry.

In his usual manner, Escher assigned notation for this set of systems based on
their characteristics, but only described the rotational symmetry in his writings. He denoted 3–fold axis \( \triangle \), 6–fold axis \( \bigcirc \), and a 2–fold axis \( \bigcirc \). Schattschneider decoded the notation he used to label his sketches as follows: number of motifs (I or II), system (A or B), number of colors applied to the system (2, 3, or 4), and type (1, 2, 3, 4) [11, p. 80–81]. The “types” are never clearly defined by either Escher nor Schattschneider, but based on a close observation of Escher’s given examples, it appears as though they are merely there to number his output for a certain sequence of the motif, system, and number of colors applied notation [11, p. 80–81]. At the heart of these systems is the notion of central symmetry, with which Escher began each set of examples. Although central symmetry was not of interest to him aesthetically, he noted that “these type C systems... are the primal types from which all other ones spring” [11, p.79]. We provide the following visual example with \( I A_3 \) type \( C_1 \), \( I A_3 \) type 1, and \( II A_3 \) type 2.
Figure 13: Triangle System Transitions
In this example, we jump from $I A_3$ type $C_1$ to $I A_3$ type 1 by taking the central point of symmetry and connecting to it “three points lying on the boundary of the motif.” In order to make the leap to $A_3$ type 2, we proceed in a manner similar to his polygonal formations – he simply trims off the same shape in each of the adjoining polygons, resulting in a new tiling.
CHAPTER 6

Wallpaper Groups

As mentioned previously, Escher was deeply intrigued by Pólya’s characterization of the 17 symmetries of the plane. Albeit, the stimulation was mostly due to the visual aids as opposed to the mathematics. Yet, Escher’s work and characterizations perfectly reflect many of the notions found in the theory of Wallpaper Groups – algebraic objects that deal with plane symmetry.

In order to draw direct parallels between the pure beauty of Escher’s tessellations and the even more pure topic of Abstract Algebra, one must begin where most mathematicians begin – with definitions. The algebraic study of symmetry requires one to rigorously define the notions Escher so intuitively sketched in his notebooks.

We begin the discussion of the aptly named Wallpaper Groups by clearly defining the intuitive notions of transformation and symmetry, building on these definitions to create a rigorous classification system, and then drawing parallels to Escher’s “sketchy” definitions. The careless reader might be blind to the beauty of the rigor, so we urge such a reader to keep in mind that it is this style of logic and proof that underlies many of Escher’s thought–provoking works. We will utilize Wallace and West’s definitions from their text Roads to Geometry unless otherwise indicated.

We begin our discussion of Wallpaper Groups with another rather flat object: a map.

**Definition 1** Let $A$ and $B$ be sets. A mapping or function $f$ from $A$ to $B$ is a rule that assigns to each element $x$ in $A$ exactly one element $y$ in $B$ [denoted $y = f(x)$]. The set $A$ is called the domain of $f$, $y$ is called the image of $x$ under $f$, and $x$ is called a preimage of $y$ under $f$. The set of all images of elements of $A$ under $f$ (which is some subset of $B$) is called the range of $f$. [p.256]
Similar to their earthly analogs, maps vary according to the lands they describe. Mathematical maps are also numerous and full of variety. In order to attempt to grasp the very localized landscape that consists of wallpaper groups, we must select maps that will be of use in our journey.

**Definition 2** A mapping \( f \) from \( A \) to \( B \) is onto \( B \) if for any \( y \) in \( B \) there is at least one \( x \) in \( A \) for which \( f(x) = y \). We call \( y \) the image of \( x \) and \( x \) the preimage of \( y \). [p. 260]

**Definition 3** A mapping \( f \) from \( A \) to \( B \) is one to one if each element of the range has exactly one preimage. Symbolically, if \( a \) and \( b \) are elements of \( A \) such that \( f(a) = f(b) \), then \( a = b \). [p. 260]

Given these three definitions, we are armed and able to proceed into the world of points and lines that Escher’s work describes so beautifully. Now, the careful reader will be aghast at this carelessness. How can one proceed, in a world as mechanical as mathematics, with forming a theory if one has not defined all of the terms? Fortunately, as Escher’s polygonal systems exist in a Euclidean world, we are allowed to assume the axioms of Euclidean Geometry—where the undefined terms are point, line, and plane. (It is even more fortunate that we do not live in the time of Euclid’s Geometry, where the very existence of points is left to the imagination). Hence, we may proceed to define, with great care, this idea of a transformation under a mathematical scope.

**Definition 4** A transformation \( t \) of the plane is a one to one mapping of points of the plane onto points in the plane. [p. 261]

This definition yields a correspondence between the plane and itself, thus shedding the light of clarity on what it means, exactly, to shift “motifs” around.

It can be shown that the composite of two transformations of the plane (i.e., applying a transformation to a transformation of a point) is in itself a transformation. However, the reader should proceed with caution. Although we
have specified certain distinct characteristics for these maps, this does not imply that the set of transformed points will relate to one another in the same way after each shift. If such is the case, then the points are said to be invariant under said transformation. Fortunately, this is precisely what we are interested in – a way of shifting around polygonal forms in a way that preserves such characteristics as symmetry. This, given its necessary nature, is a defined term.

**Definition 5**  
A *transformation of the plane for which distance between points is invariant* is called an **isometry**. [p. 261]

Isometries come in a variety of shapes and sizes, as well as with a handy, transitive toolbox that comes stocked with a set of invariant properties. At its essence, the definition of an isometry generalizes the well-known (but often incorrectly grasped) notion of congruence.

**Definition 6**  
Two figures are said to be **congruent** if and only if one is the image of the other under some isometry. [p. 263]

This notion of congruence required 5 definitions and the acknowledgement of a few results to grasp in this highly logical world! Escher, on the other hand, felt free to toss the word, happy with the knowledge that “you know what he is talking about.” Although the approach taken here might seem to be overkill to the untrained eye, it is precisely what we need in order to fully understand the underlying theory of Escher’s work.

For example, with our given set of definitions and a well-known theorem from high school geometry, we can now prove that the image of a triangle under an isometry is a congruent triangle, which is exactly the property that Escher utilized in both his quadrilateral and triangle systems. The proof of this fact can be derived directly from definitions 5 and 6.

Now that we have an understanding of basic movement around the Euclidean plane, we can begin to discuss the relevant varieties of these particular mappings.
Definition 7 If $\mathbf{PQ}$ is a vector, then a translation through vector $\mathbf{PQ}$, denoted by $T_{\mathbf{PQ}}$, is a transformation of the plane such that if $A$ is any point in the plane and $T_{\mathbf{PQ}}(A) = A'$, then $\mathbf{AA'}$ and $\mathbf{PQ}$ are equal vectors. [p. 264]

Figure 14: A translation.
Definition 8 A rotation about a point $P$ through an angle $\theta$, $R_{P\theta}$, is a transformation of the plane where $R_{P\theta}(P) = A$, and if $A \neq P$, $R_{P\theta}(A) = A'$ such that the line segments $PA$ and $PA'$ are congruent and $m\angle APA' = \theta$. [p. 265]
Definition 9  A reflection $R_l$ in a line $l$ is a transformation of the plane where, if $A$ is not on $l$, then $R_l(A) = A'$ such that $l$ is the perpendicular bisector of line segment $AA'$, and if $P$ is on $l$, then $R_l(P) = P$. [p. 267]

Figure 16: A reflection.
**Definition 10** A glide reflection $G_{EH,l}$ is the composition of $R_l$ and $T_{EH}$, where $EH$ is parallel to $l$.

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The definitions and examples hopefully give the reader some intuition for the so-called rigid motions of polygons in the plane. Armed with these notions, we are prepared to wade into the world of Wallpaper Groups. Yes, these mathematical objects are named after the archaic decorations that adorn many a grandparent’s wall. Yet, the name is apt, describing a collection of patterns governed by an underlying structure—a clear similarity to Escher’s theory of Regular Divisions of the Plane. Despite the seemingly simple characteristics of these structures, finding a resource that provides a complete and rigorous description of the tilings that initiated Escher’s fixation is difficult. Wallpaper Groups appear in several realms of mathematical literature and are thus described differently according to the context. More over, many definitions carry the type of weight that Escher himself could not bear—the need for in-depth background knowledge in order to describe what seems so “obvious” (we utilize this word with care, as it is one of the most dangerous in mathematics). One
of the goals of this work is to create a concise but accessible resource on the mathematics in Escher’s work. Hence, although the work of Doris Schattschneider, the main authority on all things relating Escher and mathematics, presents the most coherent descriptions of the objects we wish to study, in the spirit of Escher, we proceed with the simplest definitions, hence the terms presented here are the combination of the words of a variety of mathematicians. We begin with the notion of “mathematical wallpaper.”

**Definition 11** Any pattern on an (infinite) plane surface that is periodic in two directions is called a wallpaper pattern. To be periodic in a certain direction describes the case where after a translation of a design a certain nonzero amount in that direction, the pattern exactly matches up with itself. [14]

**Definition 12** The symmetry group of a pattern is the set of all the isometries of the plane that take the wallpaper pattern to itself. A symmetry group of a wallpaper pattern is called a Wallpaper Group. [14]

**Definition 13** The fundamental unit of a wallpaper pattern in the smallest area of the plane that contains all of the information needed such that the entire plane can be filled using compositions of the isometries of the symmetry group. [14]

**Definition 14** The translation group of a wallpaper pattern is the set of all translations that map the pattern onto itself. The smallest region of the plane having the property that the set of its images under this translation group covers the plane is called a unit of the pattern. [9]

For the purpose of this work, we consider only symmetry and translation groups. However, the careful reader whom we keep evoking would ask the obvious question – what is a “group,” exactly? Although we can easily present a symbolic definition, we will present it in conjunction with an easy to follow example adapted from Frederick M. Goodman’s text.
We begin with a concept most do not spend much time pondering. What is, exactly, a symmetry? This word is heavily sprinkled throughout this work, yet never defined. The question is nearly existential – how do we quantify the organized beauty that surrounds us and is, in many ways, a part of us? Goodman described symmetry as “an undetectable motion” and an object with this characteristic as a symmetric object. Observe that if we apply any number of symmetries to an object, it remains the same [5, p. 4]. Hence, if we choose to think of applying different symmetries to an object as an operation – say, in the manner in which we do with addition and integers – it quickly becomes clear that the application of symmetries is associative in nature. Associativity allows us to group our compositions of symmetries (note that order does matter) and maintain consistency. Now, consider the action of leaving the object alone – no motion. This is in itself a symmetry, and thus belongs to the set of all symmetries of the object. We call this element the identity element of the set of symmetries of the object. The identity element is crucial, as it is a starting point which, in turn, allows us to define the inverse of a symmetric motion of an object. Note that every symmetry has its own inverse that is unique to the elements of the object which are being permuted. For example, one could consider the symmetries of a square. Giving the edges an identification, such as a label, forces us to construct the inverse symmetries as the exact opposite movement of the original symmetry. We also note that we can apply either the original symmetry to the inverse one and vice versa. The combination of these characteristics, summarized below, is precisely the definition of a group that will be most applicable in this setting.

**Definition 15** *In our case, a group is a set of isometries \( G \) that, under the operation of functional composition, flaunts the following properties:

- **Closure:** If \( a \) and \( b \) are in \( G \), then \( ab \) is in \( G \).

- **Associativity:** for all \( a, b, c \) in \( G \), one has \((ab)c = a(bc)\)."
• There exists an identity element \( e \) in \( G \) such that \( ea = ae = a \) for all \( a \) in \( G \).

• For each element \( a \) in \( G \) there is an element \( a^{-1} \) in \( G \) satisfying \( aa^{-1} = a^{-1}a = e \)

Each instance of mathematical rigor gives an air of exponentially increasing distance between the geometrical treatment and the ideas of Escher for these plane symmetries. However, the underlying structure is literally the same. Consider a point in a given wallpaper pattern and the set of all images that could occur from applying the elements of the translation group. We can use these translations in composition with one another (and their inverses) to find all of the translational symmetries of our pattern. If we place a dot at every point of translational symmetry, we will achieve a grid that we call a lattice. This idea mirrors Escher’s quadrilateral systems almost exactly. In her article The Plane Symmetry Groups: Their Recognition and Notation, Schattschneider gives several classifications which correspond to Escher’s five groups of transitional systems. First, she defines the lattice unit as the parallelogram which is a fundamental unit that can generate the rest of the pattern via translations only [9, p. 441]. There are five such units and the lattices generated by them – the square, the hexagonal, the rhombic, the rectangular, and the arbitrary parallelogrammatic lattices – and they are reflected in Escher’s choices of grids. One might try to note a difference in his choice of an isosceles triangle for a grid, but by simply imagining joining six such triangles at a central vertex reveals that he was dealing exactly with the hexagonal lattice structure.

**Theorem 1** There exist exactly 17 Wallpaper Groups.

The fact that there exist exactly 17 Wallpaper Groups is proven in a variety of contexts, from Group Theory to elementary geometry, and relies on cases and classification. The more interesting fact is that M.C. Escher was able to uncover many of the possibilities in his work with his meticulous, systematic approach.
In Appendix II, we include a flowchart for the identification of Wallpaper Groups (adapted from table 5.1 in Washburn and Crowe’s *Symmetries of Culture: Theory and Practice of Plane Pattern Analysis*) as well as a selected list of groups along with an Escher example. We encourage the reader, as careless or careful as they might be, to make sure that the patterns are fitting, as it will simply highlight Escher’s knack for design and mathematics to an even higher extent.
The rigid actions of the Wallpaper Groups leaves one thirsting for curves, and no one has a better grasp of this concept than mathematicians. For Escher, it was the aforementioned H.S.M. Coxeter, a researcher whose interests spanned a wide variety of mathematical arenas, that revealed to the artist’s mostly Euclidean eyes the beauty of hyperbolic geometry. According to Wallace and West, a geometry, abstractly, is an independent, consistent, and complete axiomatic system in which theorems are logically deduced from a set of statements consisting of both definitions and undefined terms. Consistency, in this case, refers to the impossibility of a contradiction being derived from the set of axioms; the independence property refers to the ability of an axiom to stand alone—logical deduction from one to another is impossible. Finally, completeness adds a bound on the set of axioms—a system is complete if the addition of any other axiom creates a new contribution to the set of undefined terms. In the Euclidean geometry, where Wallpaper Groups exist, the underdefined terms are point, line, and plane and the axioms are the well known defining characteristics of that world. For example, the fifth postulate that most students encounter in high school geometry states:

a. *Every plane contains at least three non-collinear points*

b. *Space contains at least four non-coplanar points*

Surely this is a triviality! Of course there are at least three points in any plane—thinking otherwise would seem ludicrous! Yet, it is this statement that yields the actual existence of points in the Euclidean geometry in the first place—
a crucial factor for a non–vacuous system.

Euclidean geometry is characterized by the Euclidean Parallel Postulate which states that through a given external point, there is at most one line parallel to a given line. Again, this might strike a lazy reader as a trivial statement. Yet, a careful negation of the statement reveals two other parallel possibilities. A proper negation would read that there exist a line $l$ and a point $P$ not on $l$ such that either there is no line $m$ parallel to $l$ through $P$ or there are two distinct lines $m$ and $n$ parallel to $l$ through $P$ [15, p.70]. It is this second idea—the existence of two distinct lines parallel to a given line through a given point, that characterizes hyperbolic geometry. This negation naturally eliminates the uniqueness of the lines. Hence, we can inductively extend the last portion of the statement to include the possibility of an infinite number of such parallels. It is this concept that defines hyperbolic geometry and gives the models of this geometry their distinct, curvy flavor. Although we will explore this world in an indepth manner, Escher’s knowledge of the concept was, to say the least, lack–luster.

Escher was aware of the fact that hyperbolic geometry can be modeled via the Poincaré Disk Model, the image of which is precisely what Coxeter shared with the artist after their fateful meeting in 1954. In 1985, Escher sent the mathematician a print of Circle Limit 1, his first of many such hyperbolic tessellations (see Figure 2, (a)). According to Goodman–Strauss, the author of the main work we use for the construction of hyperbolic objects, the ideas that were initially presented to Escher were primitive. Yet, he deduced his own construction methods and was able to produce not only beautiful but surprisingly accurate portrayals of deeply mathematical notions. The aim here is to not only describe and define the Poincaré model of hyperbolic geometry, but to construct this model in an Escher–esque manner – simply yet meticulously. Geometric constructions were some of the earliest of forms of mathematical inquiry and discovery. Hence, we are once again presenting the methodology and theory behind the beauty that
Escher grasped with his artistic intuition.

We begin with a description of the Poincaré Disk Model. To be clear, a model is an interpretation of an axiomatic system that does not fail to validate all of the axioms [15, p. 11]. An example so common that most people regard it as an absolute is the Euclidean plane for Euclidean geometry. The Poincaré Disk is in no way the only way to study hyperbolic geometry, but it is the representation that sparked Escher’s imagination and is thus of greatest interest to our study. In the half–plane model of hyperbolic geometry, for example, the notion of infinity remains mysterious as we still have half of the plane to play with. The Poincaré Disk model, on the other hand, creates a boundary to which we can get infinitely close, which allows us, in a way, to put a bound on infinity.

To begin, consider a circle $C$ fixed in $\mathbb{R}^2$, that denotes the Euclidean plane. We kick off with an elementary characterization of intersecting circles.

**Definition 16** Two circles are said to be **orthogonal** if the radius drawn from one of the circles to a point of intersection is perpendicular, at that point, to the radius drawn from the other circle. [15, p. 179]

![Figure 18: Orthogonal Circles](image)

**Definition 17** A point in the Poincaré Disk is a Euclidean point that lies on the interior of $C$. [13, p. 97]

It is important to note the distinction between the interior and boundary of $C$. The points which lie on the boundary of $C$ are not actually members of the
model, but will prove to be somewhat useful in the future.

**Definition 18** A Poincaré line of the first kind is a set of all the points on the diameter of $C$ that lie inside of $C$. [13, p. 97]

**Definition 19** A Poincaré line of the second kind consists of all the points of a Euclidean circle orthogonal to $C$ that lie inside of $C$. These lines are also referred to as geodesics. [13, p.97]

![Figure 19: Poincaré lines of the first and second kind.](image)

Due to these curvy definitions, it is fairly clear that the “usual” notion of a distance between two points (generally taken to be the absolute value of the difference of the real numbers to which the points correspond) is not going to be quite as tidy as in the Euclidean domain. In order to properly define the distance between two points within the Poincaré model, we need to consider the points that lie on the boundary of circle $C$, outside of the model, which we call ideal points.

**Definition 20** The Poincaré distance between two points $A$ and $B$ in the Poincaré disk is defined by $d(A,B) = |\ln\left(\frac{AQ}{BP}\cdot\frac{BQ}{AP}\right)|$, where $P$ and $Q$ are the points at which the Poincaré line containing $A$ and $B$ intersects $C$, and $AQ$, $BQ$, $BP$, $AP$ are standard Euclidean distances between the given points. [13, p. 97]
Figure 20: The points A and B, their corresponding ideal points P and Q, and the line segments necessary for the cross ratio which determines distance

As Escher proceeded to construct his works by hand, we present a rather archaic, geometric construction of the disk. In Euclidean geometry, construction of lines and circles can be done with a compass and a straight edge. In order to construct in hyperbolic geometry, we need to develop an analogous theory in this mostly curvy universe. What is, exactly, a triangle in a space where the angle sum of a triangle has degree less than 180? How do we measure these angles? The definitions and constructions we present here are not new. Chaim Goodman–Strauss, the author of Compass and Straightedge in the Poincaré Disk—the work from where the presented constructions are given—writes that “surely this was all well–known at the end of nineteenth century just as it has long been forgotten at the dawn of the twenty–first” [6, p. 38]. However, despite the dawn of computer software and programming abilities that could quickly produce a tiling of the Poincaré Disk, there is a crucial need to understand the simple geometric constructions that underlie Escher’s work. The man was not a machine but rather an artist with a keen eye for natural mathematics. Patient geometric construction, although often regarded as trivial and tedious, is at the heart of both Escher’s artwork and the mathematical research of many early mathematicians and logicians.
Definition 21 If two lines intersect, then the angles between them are the Euclidean angles contained in the lines on which they lie. [13, p. 97]

This definition of an angle yields two perspectives in the hyperbolic world. If we are considering Poincaré lines of the first kind, the angle at the intersection of these two lines is simply the Euclidean measure of the intersection. If one or more of the lines is a Poincaré line of the second kind, then we measure the angle or angles formed by the intersections of the tangent lines of one or both lines at the intersection point.

![A hyperbolic triangle in the Poincaré disk.](image1)

Definition 22 Given 3 distinct noncollinear points $A', B', C'$ in the Poincaré Disk, the hyperbolic triangle $A'B'C'$ is the set of all points that lie on the line segments $\overline{A'B'}, \overline{B'C'}, \overline{C'A'}$, which are determined by the intersections of geodesics and/or diameters.

Surprisingly, we are now prepared to construct a hyperbolic tessellation! The reader, whom we assumed to be observant (albeit occasionally careless), might note that we are lacking actual construction proofs. The rigorous constructions are known and available for our use. However, a large set of tessellations of triangles is constructible with a Euclidean compass and straight–edge, including those that Escher produced. Before we begin the construction, it must be highlighted that although we are constructing with Euclidean tools, we must not view the outcome from a Euclidean perspective. The notion of congruence, which is so
clear in Euclidean geometry (two triangles, for example, are congruent if their angles and sides are, essentially, the same,) does not reveal itself in the same manner in hyperbolic geometry. As an example, we can cite the Angle, Angle, Angle [AAA] Congruence Condition that holds in hyperbolic geometry

**Theorem 2** AAA Congruence Condition for Hyperbolic Triangles

Two triangles are congruent if all corresponding angles have the same measure.

As we know, AAA is not a theorem in Euclidean geometry – it is easily contradicted by a quick consideration of two non-congruent similar triangles. Hence, tessellations in the Poincaré disk take a different and rather curious form, as demonstrated in Escher’s *Circle Limit* series.

**Theorem 3** Given an unordered triple of whole numbers $p$, $q$, $r$ with $p^{-1} + q^{-1} + r^{-1} < 1$, there is a tiling of the hyperbolic plane by triangles with interior angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, and $\frac{\pi}{r}$. [6, p. 45]

Many of Escher’s works with circles fit into this criterion. We now present a simplistic construction of the $\frac{\pi}{5}$, $\frac{\pi}{7}$, $\frac{\pi}{7}$ triangular tessellation, based on Goodman–Strauss’ instructive paper on the construction of hyperbolic triangle tessellations on the Poincaré Disk for $r = 2$.

**The Construction**

Even for our simplistic construction, we must understand and be able to perform some essential Euclidean constructions with an unlabeled straight-edge and compass. This will, of course, require some very basic Euclidean definitions. For this section, we advise our attentive reader to sketch the constructions as they go, to get a more intuitive feel for the plethora of definition and constructions.

**Definition 23** Given three distinct points $A$, $B$, and $C$ in $\mathbb{R}^2$, point $B$ is said to lie between $A$ and $C$ if the distance from $A$ to $C$ is equal to the sum of the distance from $A$ to $B$ and the distance from $B$ to $C$. 
This idea of distance might be intuitively clear to any reader that has had any type of experience with Euclidean geometry, no matter how horrific their high school geometry experience might have been. To avoid any confusion, we cite the School Mathematics Study Group’s [SMSG] postulates, where the Postulates 2, 3, and 4 yield the idea of a correspondence between points on a given line and the real numbers, define distance between two points, and allows us to assign a coordinate system. These postulates are included in Appendix I.

**Definition 24** Given two distinct points A and B in the plane, the line segment \( \overline{AB} \) is the set of all points \( P \) that lie between A and B.

**Definition 25** Given two distinct points A and B in the plane, the ray \( \overrightarrow{AB} \) is the set consting of \( \overline{AB} \) and the set of all points \( P \) such that B lies between A and \( P \).

**Definition 26** Given three distinct, non–collinear points A, B, C, we define the angle \( \angle ABC \) as the union of \( \overrightarrow{BA} \) and \( \overrightarrow{BC} \).

**Definition 27** We define the interior of \( \angle ABC \) as the set of all points \( P \) that lie between any two points \( Q_1 \) and \( Q_2 \) such that \( Q_1 \) lies on \( \overline{BA} \) and \( Q_2 \) lies on \( \overline{BC} \).

Given these conceptually unchallenging definitions, we are finally prepared to construct points and angles with a compass and straight–edge. As we are keeping things as simple and Escher–esque as possible, we will only explore the constructions of the angles necessary for our goal – the tiling of the Poincaré Disk with the hyperbolic triangle that boasts the angles of \( \frac{\pi}{5} \), \( \frac{\pi}{4} \), and \( \frac{\pi}{2} \). For this purpose, we only need to worry about the constructions of \( \frac{3\pi}{5} \), \( \frac{2\pi}{5} \), \( \frac{\pi}{4} \), and \( \frac{\pi}{20} \). We begin, this time, in the middle.

**Construction 1** To construct the midpoint of \( \overline{AB} \), construct the circle of radius \( AB \) centered at A and the circle of radius \( AB \) centered at B. The intersection of the circles determines two points, \( P_1 \) and \( P_2 \), such that \( P_1P_2 \) intersects \( \overline{AB} \).
call the point of intersection the midpoint of $\overline{AB}$. We call $P_1P_2$ the perpendicular bisector of $\overline{AB}$.

![Figure 22: A perpendicular bisector to a given line segment.](image)

The angles necessary for Escher’s hyperbolic tessellations are easily constructable. We note that $\frac{1}{4} \cdot \frac{2\pi}{5} = \frac{\pi}{10}$, and $\frac{1}{2} \cdot \frac{2\pi}{5} = \frac{\pi}{5}$ so all that is required of us is the construction of $\frac{2\pi}{5}$, $\frac{\pi}{4}$ and the bisector of an angle, which is what allows one to halve angles.

It may seem strange to the geometrically inexperienced reader to consider the “construction” of an angle. Not all angles are constructable with the bare-bones tools of a compass and straightedge. However, the geometric formation of an angle of measure $\frac{2\pi}{5}$ is intimately linked with the formation of a regular pentagon, a five–sided figure with 5 congruence sides.

**Construction 2** Given a line segment $\overline{AB}$, construct the circle $C$, centered at $A$ of radius $AB$. Next, extend the line segment $\overline{AB}$ to intersect $C$. Construct the perpendicular bisector of that line. Label a point of intersection with the circle $C, D$. Next, construct $E$, the midpoint of $\overline{AB}$. Construct the circle centered at $E$ that passes through $A$ and $B$. Next, construct $\overline{DE}$. Mark the intersection of the previously formed circle and the ray as $F$. Create an arc centered at $D$ and
of radius $DF$. Mark the points of intersection of the ray with $C$ as $H$ and $G$.

The angle $\angle HCG$ has measure $\frac{2\pi}{5}$. [1, p.9]

![Figure 23: Construction 2](image)

The construction of $\frac{\pi}{4}$ relies on Construction 1, where we showed how to construct the perpendicular bisector of a line segment as well as the definition of perpendicular lines, which guarantees that the lines will intersect at right, or 90 degree, angles.

**Construction 3** Given line segment $\overline{AB}$, construct the perpendicular bisector of $\overline{AB}$. Construct an arc centered at the midpoint, $P$, of $\overline{AB}$ such that it intersects the perpendicular with the width of the compass set to $d(A, P)$. Call this intersection $Q$. Construct the line segment $\overline{AQ}$. The measure of $\angle QAP$ is $\frac{\pi}{4}$.

In order to construct the final, necessary angle for our hyperbolic tessellation, we need to construct the angles $\frac{\pi}{5}$ and $\frac{\pi}{20}$. We are going to utilize the previously discussed note of the fact that $\frac{\pi}{5} = \frac{1}{2} \frac{2\pi}{5}$. As we already have the tools to construct $\frac{2\pi}{5}$, we only need to understand how to halve, or bisect, this angle. Similarly, as $\frac{\pi}{20} = \frac{1}{4} \frac{\pi}{5}$, the understanding of bisection will allow us to create
this angle measure. The realizability of this idea lies in the fact that such a bisection essentially constructs two congruent triangles, which allows us to make conclusions about the constructed angles.

**Construction 4** Given $\angle ABC$, construct an arc of any radius such that it intersects $\overline{AB}$ and $\overline{BC}$. Call these intersection points $P$ and $Q$, respectively. For convenience, we keep the width of the compass the same and construct intersecting arcs centered at the two new points. Call this intersection $I$. Construct the ray $\overrightarrow{BI}$. The measure of $\angle IAC$ is $\frac{1}{2}$ the measure of $\angle ABC$.

Figure 24: A standard bisection.
Now, the constructions of $\frac{\pi}{5}$ and $\frac{\pi}{20}$ are nearly trivial.

**Construction 5** Given a line segment $\overline{AB}$, construct $\angle CAB$ such that the measure of the angle is $\frac{2\pi}{5}$. Next, construct the bisector of $\angle CAB$. The measure of the resulting angle is $\frac{\pi}{5}$. In order to construct an angle of measure $\frac{\pi}{20}$, simply bisect the resulting angle two more times.

With a deep, cleansing breath and a good grasp of the constructions above, we can proceed to construct our hyperbolic tessellation. As we set off on this journey, we begin where Escher believed to be the only plausible starting point – the largest portion of the tessellation lying central in our circular boundary. In his work *Escher on Escher*, he notes that “the only way to escape the fragmentary character of composition and contain an ‘infinity’ in its entirety within a logical border line is to work at it in reverse order from [previously]” [4, p. 125–126]. Thus instead of beginning with a lattice, we now begin with the central portion of our work – the triangle itself. The construction we give is based on a description given in Goodman–Strauss’ *Straight Edge and Compass in the Poincaré Disk*, which presents the first step of the construction of any hyperbolic tessellation. Interestingly enough, we begin with two points in the Euclidean plane. First, we will create the Poincaré Disk and a hyperbolic triangle with internal angles $\frac{\pi}{5}$, $\frac{\pi}{4}$, and $\frac{\pi}{2}$. Note that we fulfill the hypotheses of the previous theorem and thus such a hyperbolic triangle truly exists.
Construction of the Poincaré Disk and The Hyperbolic Triangle with Internal Angles $\frac{\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}$

1. Begin with two points, $A$ and $O$, in the Euclidean Plane

2. Rotate $A$ about $O$ by $\frac{2\pi}{5}$ to produce $A^*$. 

3. Find the midpoint, $M$, of $AA^*$. 

4. Rotate the line that contains $AA^*$ by $\frac{\pi}{20}$ in the clockwise direction about point $A$ to produce line $k$. 

5. Line $k$ intersects the line that contains $OM$. Call this intersection point $I$. 

6. Construct the circle centered at $I$ passing through $A, A^*$. This is a Poincaré line of the second kind! Call it $\gamma$. 

\[ \begin{array}{c}
\begin{array}{c}
\text{O} \\
\text{M} \\
\text{A} \\
\text{A^*}
\end{array}
\end{array} \] 

\[ \begin{array}{c}
\begin{array}{c}
\gamma \\
\text{I} \\
\text{M} \\
\text{A} \\
\text{A^*}
\end{array}
\end{array} \]
7. Find the midpoint of \( \overline{OI} \). Construct the circle \( C \) such that \( C \) passes through \( I \) and \( O \) and is centered at the found midpoint. Mark one of the intersections of \( \gamma \) and \( C \). Label this intersection \( D \).

8. Construct circle \( P \), centered at \( O \) and intersecting \( D \). This is “Disk” of the Poincaré Disk Model in this tessellation.

9. Let \( Q' \) be the intersection of \( \gamma \) with the line that contains \( \overline{IO} \), and let \( P' \) be the intersection of \( \overline{AO} \) and \( \gamma \).
10. The hyperbolic triangle $P'OQ'$ has internal angles $\frac{\pi}{5}, \frac{\pi}{4}, \frac{\pi}{2}$. 
The Tessellating Process

We are now prepared to create a tiling of the constructed disk with the constructed triangle. In the words of Goodman–Strauss, who presents this construction, we will attempt to keep things “elementary and synthetic.” The most general approach to tiling a triangle of the $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{2}$ sort requires quite a bit of knowledge centered around Euclidean results for circles. To keep things neat and accessible, we present a tiling construction that is specifically focused on tiling the hyperbolic triangle constructed here. Further generalizations are, of course, possible.

We begin our tiling by noting the symmetry of the situation. As it often happens in mathematics, we can greatly simplify the process by considering the symmetry of the tiling we want to create. Hence, we return to the pentagon.

1. Construct the line parallel to $\overline{AA^*}$ through the point $I$. Label the intersection of the geodesic forming circle and the parallel line as $I_1, I_2$. This will allow us to both keep track of the centers of our geodesic yielding circles and construct new circles orthogonal to the Disk.

2. Reflect $\overline{OA}, \overline{OA^*}, \overline{OI}$, the line that passes through $I, I_1, I_2$, and the circle centered at $I$ about $\overline{OA}$. Carry on in the same fashion until the disk is full as
3. Mark the intersection points of all circles and note the sudden appearance of a pentagon in the sketch. Mark the vertices of this pentagon. We will utilize them to create the next set of geodesics.

4. Construct the circles centered at the vertices of the pentagon through the points of intersection of the circles. Below, we demonstrate one such circle as well as the entire set.
6. Our next task is to further flesh out the tiling. We shortcut by rotating the original Poincaré line of the second kind, $\gamma$, by $\frac{\pi k}{5}$ about a point which lies on it, for $k = 1$ and $k = 2$. The figure below yields an example of this shortcut, where the original line is pink and its rotations are blue.
Finally, the reader can continue to flesh out their tiling to their hearts desire by constructing geodesics through points determined by the lines currently present. We present Goodman–Stauss’ constructions necessary for such feat.

**Construction 6** Construct a circle through three given non–collinear points $A, B, C$. Construct segments $\overline{AB}, \overline{BC}$. Construct the perpendicular bisectors $l_1, l_2$ of these segments. Let $O$ be the point of intersection of $l_1, l_2$; the desired circle has center $O$ and passes through, say $A$.

**Construction 7** Invert a point through a circle $C$ with center $O$. If we wish to invert a point $A$ lying inside the circle (other than $O$!), take the perpendicular $l$ through $A$ to $\overline{OA}$. Let $P$ be the point of intersection of $l$ and $C$. Let $l'$ be the perpendicular to $\overline{OP}$ passing through $P$. Then the point $B$ of intersection of $\overline{OA}$ and $l'$ is the desired inverse of $A$ through $C$.

**Construction 8** Given points $A, B$ within the Poincaré Disk, construct the hyperbolic geodesic $\overline{AB}$. Equivalently, given two points $A, B$ and a circle $C$ with center $O$, construct the unique circle through $A, B$ that is orthogonal to $C$. This
is wonderfully simple: Invert $A$ through $C$ to construct $A^1$ and the desired circle is the circle $\gamma$ through $A, B, \text{and } A^1$.

By repeatedly applying the above construction, one successfully produces a hyperbolic tiling of the Poincaré Disk!

The methodology presented in this section is fairly simple – all of the moves are constructible with a straight edge and compass. However, the process is tedious, requiring a painstaking amount of attention to detail as well as some amount of crafting ability. M.C. Escher, of course, had no issue with either of those options and utilized what must have been a construction similar to the one we presented to construct his Circle Limit series. If the reader elects to follow the presented straight–edge and compass construction, he or she will most likely find him or herself frustrated and perhaps a bit confused. Escher himself, despite his methodical ways, was unhappy with his first Circle Limit piece, noting in Escher on Escher: Exploring the Infinite the lack of ‘flow’ in his work. However,
once better acquainted with the process via the Coxeter correspondence, Escher corrected his work and finally, in *Circle Limit III*, captured the totality of infinity, which was his initial goal. He wrote that “without [Coxeter’s] kind assistance, I probably would have never found a satisfactory solution,” highlighting the essential nature of mathematics in his work. [4, p.42]
CHAPTER 8

Conclusion

“\textit{I have come to the end of my lecture. The attention you have so kindly given to my fantasies proves, I hope, that science and art sometimes can touch one another, like two pieces of the jigsaw puzzle which is our human life, and that contact may be made across the borderline between our two respective domains.}”

– M. C. Escher, \textit{Visions of Symmetry}

Previously published work about M.C. Escher, his art, and the mathematics that directs the currents that make his work such a treat to even the untrained eye is plentiful. However, his own methodology is often set aside for the more abstract generalizations that govern mathematical research. The aim of this work is to shed light on how Escher progressed as an artist and an amateur mathematician, thus presenting his so-called “layman’s theory” and drawing the appropriate parallels to the mathematical realm while keeping the constructions accessible and the beauty of the work in the forefront. Doris Schattschneider, the expert authority on all things that lie in the intersection of Escher and Mathematics, published an article entitled \textit{The Mathematical Side of M.C. Escher}, which is frequently cited in this work. Yet she, like many who have written on the topic, creates a divide between the art and the mathematics. Researching Escher’s own approach to creation leads one to believe that the artist had more than a mathematical side– he had mathematical intuition and an acceptance of the subject’s laws which govern his process. M.C. Escher stands as an example of true understanding of symmetry and deductive reasoning, despite the fact that his motivation was completely aesthetic. Many mathematicians publish work drenched in esoteric jargon. An exploration of Escher’s method sheds light on the unnecessary nature of this approach. The point, of both art and mathematics, is to create an understanding of what is right in front of our eyes, not to
alienate the audience. Of mathematics, Escher wrote that the laws of mathematics are not merely human inventions or creations. They simply “are; they exist quite independently of the human intellect. The most that any man with a keen intellect can do is to find out that they are there and to take cognizance of them [4, p.35]. The same, we believe, can be said of the beauty and symmetry that floats right below the surfaces of both nature and art. Hence, we hope that this work reveals to the reader what is ‘just there’ in Escher’s work from both the mathematical and the aesthetically derived perspectives.

We elected to work with tessellations, both Euclidean and hyperbolic, because of the urge to flesh out the abstract notion of infinity that so many college students, including those studying mathematics, take for granted. When on paper, we often believe that we understand what it means for a function to approach infinity or to grow infinitely small or large. Yet, the beauty and human nature of this concept often eludes us and representation rarely even occurs to us until we glimpse something as powerful as an Escher print. The man had more than an appreciation of the concept – he believed that its existence required the tessellating reaction.

“When one dives into endlessness, in both time and space, farther and farther without stopping, one needs fixed points or milestones past which one speeds... There must be stars along which one shoots, beacons from which one can measure the road covered. He must divide his universe in distances of a specific length, in compartments that repeat themselves in endless series. At every border crossing between one compartment and the next, his clock ticks. Whoever wants to create for himself a universe in a two–dimensional plane will notice that time goes by while he is working on his creation. When he is finished, however, and looks at what he has done, then he sees something that is static and timeless. In his representation, no clock ticks... The dynamic, regular ticking of the clock at every border crossing of our trip through space is silenced. However, we can replace it, statically, with the periodic repetition of similar figures in our drawing
plane.” [4, p.124]

We hope that this work allows for greater understanding of this belief.
CHAPTER 9

Bibliography
Works Cited


Figure References

The following figures were created by M.C. Escher and are copywritten by the M.C. Escher Company B.V. (Copyright Pending).

Figure 1:
*Castrovalva*. Lithograph, 1930. 530x421mm

Figure 2 (a):
*Circle Limit I*. Woodcut, 1958. 418mm diameter

Figure 2 (b):
*Waterfall*. Lithograph, 1961. 380 x 300mm

Figure 3:
*Reptiles*. Lithograph, 1943. 334 x 385mm

Figure 9:
*Sky and Water I*. Woodcut, 1938. 435 x 439mm

Figure 23:
*Circle Limit I* (see Figure 2 (a))
*Circle Limit II*. Woodcut, 1959. 417mm diameter
*Circle Limit III*. Woodcut, 1959. 417mm diameter
*Circle Limit IV (Heaven and Hell)*. Woodcut, 1960. 460 diameter

Escher 85: *Lizard/Fish/Bat*. Ink, pencil, watercolor, 1952. 270 x 212mm

Escher 46: *Two Fish*. India ink, colored pencil, watercolor, 1942. 289 x 255mm

Escher 72: *Fish/Boat*. Colored pencil, ink, 1948. 267x 200mm

Escher 91: *Beetle*. India ink, watercolor, 1953. 189 x 194mm

The following figures were recreated by Anna Levina based on Wallace and West's Illustrations in the cited text *Roads to Geometry, 3rd Ed.*

Figure 12: p. 264
Figure 13: p. 265
Figure 14: p. 267

All other images were created by Anna Levina for the purpose of this work.
CHAPTER 10

Appendix I
The SMSG Postulates

Undefined Terms: Point, Line, Plane

Postulates

Postulate 1.
Given any two distinct points there is exactly one line that contains them.

Postulate 2.
Distance Postulate. To every pair of distinct points there corresponds a unique positive number. This number is called the distance between the two points.

Postulate 3.
Ruler Postulate. The points of a line can be placed in a correspondence with the real numbers such that:
1. To every point of the line there corresponds exactly one real number.
2. To every real number there corresponds exactly one point of the line.
3. The distance between two distinct points is the absolute value of the difference of the corresponding real numbers.

Postulate 4.
Ruler Placement Postulate. Given two points $P$ and $Q$ of a line, the coordinate system can be chosen in such a way that the coordinate of $P$ is zero and the coordinate of $Q$ is positive.

Postulate 5.

a. Every plane contains at least three non-collinear points.
b. Space contains at least four non-coplanar points.

Postulate 6.
If two points lie in a plane, then the line containing these points lies in the same plane.

Postulate 7.
Any three points lie in at least one plane, and any three non-collinear points lie in exactly one plane.

Postulate 8.
If two planes intersect, then that intersection is a line.

Postulate 9.
Plane Separation Postulate. Given a line and a plane containing it, the points of the plane that do not lie on the line form two sets such that:
1. each of the sets is convex
2. if $P$ is in one set and $Q$ is in the other, then segment $PQ$ intersects the line.

Postulate 10.
Space Separation Postulate. The points of space that do not lie in a given plane form two sets such that:
1. Each of the sets is convex.
2. If $P$ is in one set and $Q$ is in the other, then segment $PQ$ intersects the plane.
Postulate 11.  
*Angle Measurement Postulate.* To every angle there corresponds a real number between 0° and 180°.

Postulate 12.  
*Angle Construction Postulate.* Let $AB$ be a ray on the edge of the half-plane $H$. For every $r$ between 0 and 180 there is exactly one ray $AP$, with $P$ in $H$ such that $m\angle PAB = r$.

Postulate 13.  
*Angle Addition Postulate.* If $D$ is a point in the interior of $\angle BAC$, then $m\angle BAC = m\angle BAD + m\angle DAC$.

Postulate 14.  
*Supplement Postulate.* If two angles form a linear pair, then they are supplementary.

Postulate 15.  
*SAS Postulate.* Given a one-to-one correspondence between two triangles (or between a triangle and itself). If two sides nd the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Postulate 16.  
*Parallel Postulate.* Through a given external point there is at most one line parallel to a given line.

Postulate 17.  
To every polygonal region there corresponds a unique positive real number called its area.

Postulate 18.  
If two triangles are congruent, then the triangular regions have the same area.

Postulate 19.  
Suppose that the region $R$ is the union of two regions $R_1$ and $R_2$. If $R_1$ and $R_2$ intersect at most in a finite number of segments and points, then the area of $R$ is the sum of the areas of $R_1$ and $R_2$.

Postulate 20.  
The area of a rectangle is the product of the length of its and the length of its altitude.

Postulate 21.  
The volume of a rectangular parallelepiped is equal to the product of the length of its altitude and the area of its base.

Postulate 22.  
*Cavalieri's Principle.* Given two solids and a plane. If for every plane that intersects the solids and is parallel to the given plane the two intersections determine regions that have the same area, then the two solids have the same volume.

Source:

CHAPTER 11

Appendix II
What is the smallest rotation?

- 180
- 120
- 90
- 60

Does the pattern rotate?

- Yes
- No

Is there a glide reflection which doesn't lie on a mirror line?

- Yes
- No

Does the pattern reflect in at least one direction?

- Yes
- No

Are there reflections in two directions?

- Yes
- No

Are all the centers of rotation on mirror lines?

- Yes
- No

Are there mirror lines at 45 degree angles?

- Yes
- No

Is there a glide reflection?

- Yes
- No

What is the smallest rotation?

- 180
- 120
- 90
- 60