QUASIPARTICLE ANISOTROPIC HYDRODYNAMICS IN ULTRA-RELATIVISTIC HEAVY-ION COLLISIONS

A dissertation submitted to
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by
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List of Publications

1. Mubarak Alqahtani, Mohammad Nopoush, and Michael Strickland. Quasiparticle

2. Mubarak Alqahtani, Mohammad Nopoush, and Michael Strickland. Quasiparticle

3. Mubarak Alqahtani and Michael Strickland. Quasiparticle anisotropic hydrodynamics,

4. Mubarak Alqahtani, Mohammad Nopoush, Radoslaw Ryblewski, and Michael Strick-
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Chapter 1

Introduction

In this chapter, I will give a general introduction about high-energy physics. I will start by giving a basic review of elementary particle physics, and the fundamental theories that attempt to explain the basic blocks of the universe as suggested by the standard model. Then I will talk more about the aim of this study: quark-gluon plasma (QGP) and heavy-ion collisions. Finally, I will review the basics of fluid dynamics in both classical and relativistic systems.

Before starting the discussions of this work, I would like to define some conventions and notations used in this dissertation.

1.1 Units and notation

Throughout this study, I will use the natural units, $\hbar = c = k_B = 1$ where $\hbar \equiv h/2\pi$ with $h$ is Planck’s constant, $c$ is the speed of light, and $k_B$ is the Boltzmann’s constant. This means that masses, temperatures, energies, and momenta have the same “energy-like” units such as MeV or GeV. On the other hand in natural units, spatial and temporal separations have units of inverse energy units like GeV$^{-1}$. There is an important relation which is used frequently to transform between GeV and fm ($10^{-15}$ m) units which is worth listing

$$\hbar c = 0.197326938 \text{ GeV fm}, \quad (1.1)$$
Now to give a sense of how these units are compared to the SI units, I list some definitions below:

\[
1 \text{ MeV} = 1.60218 \times 10^{-25} \text{ J},
\]

\[
1 \text{ MeV}/c^2 = 1.78266 \times 10^{-30} \text{ kg},
\]

\[
1 \text{ MeV} = 1.16045 \times 10^{10} \text{ K}.
\]

To make it more clear, let us consider some examples.

- The critical temperature at which quark-gluon plasma can be created is \( T_0 \sim 200 \text{ MeV} \) which is \( \sim 2 \times 10^{12} \text{ K} \).

- The proton's mass is \( 938.272 \text{ MeV} \sim 1 \text{ GeV} = 1.67262 \times 10^{-27} \text{ kg} \).

- The proper time that quark-gluon plasma lasts before hadronization is \( \sim 10 \text{ fm}/c \sim 3 \times 10^{-23} \text{ sec} \).

Following the convention usually used in particle physics field, the metric signature is taken in this study to be “mostly negative”, \((+,-,-,-)\) where the line element in this case is given by

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2,
\]

where \( x^\mu = (t, x, y, z) \). To make the formalism easier usually Milne coordinates is used \( x^\mu = (\tau, r, \phi, \varsigma) \) with the following line element \( ds^2 = d\tau^2 - dr^2 - r^2 d\phi^2 - \tau^2 d\varsigma^2 \). Here, \( \tau \) is the longitudinal proper time defined as

\[
\tau = \sqrt{t^2 - z^2},
\]

\( r \) is the transverse radius which is defined as

\[
r = \sqrt{x^2 + y^2},
\]
φ is the azimuthal angle defined as

$$\phi = \tan^{-1} \left( \frac{y}{x} \right),$$  \hspace{1cm} (1.5)

and ζ is the spacetime rapidity defined as

$$\zeta = \tanh^{-1} \left( \frac{z}{t} \right).$$  \hspace{1cm} (1.6)

### 1.2 The standard model

There are four fundamental forces in nature: gravitational, electromagnetic, weak, and strong forces. The last three are the ones which the standard model of particle physics is based on. Gravity is excluded since it is not yet quantized which means it can not be formulated as a quantum field theory like the other three forces. For each force, there is a corresponding quantum theory which explains the force and its origin. We will give a brief introduction to each one of them [32].

Quantum electrodynamics (QED) is the quantum theory of electrodynamics which describes the electromagnetic force. The force carriers in this case are photons which are neutral massless spin-1 particles. QED can be basically reduced to an interaction between two charged particles like two electrons through the exchange of a photon.

The electroweak theory is the quantum theory of electroweak forces which is responsible for many different decays like beta decay (emission of an electron). As an example of beta decay, a neutron can decay to a proton, an electron, and an antineutrino. This decay was originally discovered when a radioactive nucleus emitted an electron to transform into a lighter nucleus, for example potassium goes to calcium (\(^{40}\text{K} \rightarrow ^{40}\text{Ca}\)). There are two kinds of weak interaction force carriers: charged interactions where carriers are \(W^+\) and \(W^-\) and neutral interactions where the force carrier is \(Z\).

And finally, Quantum Chromodynamics (QCD) is the quantum theory which describes the strong force. The strong force is based on color charges where we have three colors
Figure 1.1: This figure shows the forces, particles, forces carriers that the standard model is based on. Figure is taken from Ref. [1]

Corresponding to green, red, and blue charges. The carriers of the strong force are gluons which play a similar rule to the photon in QED. One fundamental process in this case is quark → quark+gluon (quark emits gluon). The strong force is responsible for holding hadrons, which are composed of quarks together, for example, the proton is made of three quarks (uud). It is also responsible for holding the nucleus together by providing an attractive force that holds protons and neutrons together.

In Fig. 1.1, we show the building blocks of the standard model: leptons, quarks, and the force carriers. There are six spin one-half leptons: the electron, the muon, and the tau electron with a corresponding neutrino for each generation: electron neutrino, muon
neutrino, and tau neutrino. Also, there are six spin one-half quarks: down, up, strange, charm, bottom, and top. For all forces, the mediators are spin-one bosons: eight gluons, the photon, the $W^+$, the $W^-$, and the $Z$.

Since this study will be mainly about QCD, below, I will give a detailed description of QCD. But before that, I will give a very brief introduction to QED since it is similar to QCD and much easier to explain some basic concepts.

1.3 Quantum Electrodynamics (QED)

As explained in the last section, QED describes electromagnetic processes where charged particles interact through the exchange of photons. To write down the Lagrangian density (I will call it simply the Lagrangian for abbreviation), it is helpful to separate this process into two parts. First, let’s consider a free relativistic fermion (like an electron or positron), the Lagrangian is given by

$$L_{\text{Dirac}} = \bar{\psi} \left( i \frac{\partial}{\partial \tau} - m \right) \psi, \quad (1.7)$$

where $\psi$ is a Dirac spinor, and $\bar{\psi}$ is a “conjugate” Dirac spinor, $\bar{\psi} \equiv \psi^\dagger \gamma^0$.

Next, let’s consider the electromagnetic Lagrangian which is given by

$$L_{\text{Maxwell}} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - A_{\mu} J_{\mu}, \quad (1.8)$$

where $A_{\mu}$ is the 4-component vector potential, $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor, and $J^\mu$ is the 4-current.

To get the full QED Lagrangian in quantum electrodynamics (QED) we combine both parts with $J_{\mu} = -e \bar{\psi} \gamma^\mu \psi$

$$L_{\text{QED}} = L_{\text{Dirac}} + L_{\text{Maxwell}}, \quad (1.9)$$

$$= \bar{\psi} \left( i \frac{\partial}{\partial \tau} - m \right) \psi - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + e \bar{\psi} A_{\mu} \psi. \quad (1.10)$$
To make the formalism appear less complicated, the covariant derivative can be introduced

\[ D_\mu \equiv \partial_\mu - ieA_\mu , \]  

Then, the QED Lagrangian can be written as

\[ \mathcal{L}_{\text{QED}} = \bar{\psi} (i \not{D} - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} . \]  

We should note here that the Lagrangian is not uniquely defined since it is invariant under local gauge transformation

\[ A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x) , \]  

where \( \lambda(x) \) is an arbitrary functions vanishing fast as \( x \to \infty \). This symmetry of the Lagrangian is called gauge symmetry.

### 1.4 Quantum chromodynamics (QCD)

The formulation of the QCD Lagrangian is similar to the formulation of the QED Lagrangian. However, in QCD we have six flavors (6 different quarks) with three colors for each one: red, green, and blue. This means that the quarks fields are represented by 3-dimensional column vectors in color space. Moreover, the force carriers, gluons here, are color-charged; they can also have different color charges. Unlike the photon in QED, the gluons are represented by \( 3 \times 3 \) non-commuting matrices in the adjoint representation of \( SU(3) \).

The QCD Lagrangian is

\[ \mathcal{L}_{\text{QCD}} = \sum_{i=1}^{N_f} \bar{\psi}_i (i \not{D} - m_i) \psi_i - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} . \]  

where \( N_f \) is the number of flavors of quarks with masses \( m_i \). The strength field \( F^a_{\mu\nu} \), in this case, is defined as

\[ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc} A^b_\mu A^c_\nu . \]
where $f^{abc}$ is the structure constants related to Gell-Mann matrices in the fundamental representation of the theory.

As we discussed before, gluons carry color charges which make them interact with each other. This can be seen by expanding the second term of the Lagrangian in Eq. (1.14) \[33\]

\[
F^a_{\mu\nu}F^{a\mu\nu} = \left[ \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc} A^b_\mu A^c_\nu \right] \left[ \partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} + g f^{amn} A^{\mu m} A^{\nu n} \right] \\
= \left[ \text{quadratic terms} + 2gf^{abc} A^b_\mu A^c_\nu (\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) \right] \\
+ \left[ g^2 f^{amn} A^b_\mu A^c_\nu A^{\mu m} A^{\nu n} \right] \quad (1.16)
\]

As we can see from this expansion, unlike photons in QCD we have a three-gluon vertex ($A^a_\nu \partial^\mu A^{\nu a}$) and a four-gluon vertex ($A^b_\mu A^c_\nu A^{\mu m} A^{\nu n}$). In Fig. 1.2, we show the fundamental vertices of the QCD theory: quark-gluon vertex, three-gluon vertex, and four-gluon vertex.

### 1.5 The coupling constant in QED and QCD

In this section, I will present the running coupling constants for both QED and QCD derived by one-loop calculations [34, 35]. The one-loop running coupling constant of QED is given by

\[
\alpha(Q) = \frac{\alpha(Q_0)}{1 - \frac{\alpha(Q_0)}{\pi} \log \left( \frac{Q^2}{Q_0^2} \right)}.
\quad (1.17)
\]
On the other hand, the one-loop running coupling constant of QCD is given by

\[ \alpha_s(Q) = \frac{\alpha_s(Q_0)}{1 + \frac{\alpha_s(Q_0)}{4\pi} \left(11 - \frac{2N_f}{3}\right) \log \left(\frac{Q^2}{Q_0^2}\right)}, \tag{1.18} \]

where \( Q \) is the renormalization scale (reference momentum) and \( Q_0 \) is an initial known momentum scale by experiment like in QED when one can use low-energy scattering of electrons or by some theoretical calculations as in QCD when one can use lattice QCD.

In Fig. 1.3, we plot the running coupling constants to see the difference between QED and QCD. In panel (a), we show the one-running coupling constant for QED (Eq. (1.17)) as a function of the scale \( (Q) \) multiplied by \( \alpha_0 = 1/137.036 \). As an initial condition, we used the fine structure value at \( Q_0 = 10^{-9} \) GeV which gives \( \alpha_0(Q_0 = 10^{-9} \text{ GeV}) = 1/137.036 \). As can be seen from this figure, the coupling constant is varying with the scale but with small magnitude over the relevant energy scales. Usually, using the one-loop calculations suffices, unless in some high-energy experiments where the corrections are important. On the other hand, in panel (b), we show the one-running coupling constant for QCD as a function of the scale \( (Q) \) in Eq. (1.18). Here we used as an initial condition, \( \alpha_s(Q_0 = 1.5 \text{ GeV}) \simeq 0.236 \) which is determined by QCD lattice results \([36]\). We also assumed for this plot \( N_f = 3 \). As we can see from panel (b), the QCD running coupling constant increases when decreasing the scale to the ordinary scales of nuclear matter. In Fig. 1.4, we show some experimental measurements of QCD coupling constant \( (\alpha_s) \) as a function of the energy scale \( Q \) taken from the particle data group (PDG) which shows a similar behavior predicted by one-loop running calculations.

This behavior results in different QCD behaviors depending on the energy. First, at high energies, quarks and gluons interact weakly since the coupling constant becomes small. This limit is called “asymptotic freedom” \([37, 38]\). Second, at lower energies, quarks and gluons interact strongly since the coupling constant becomes quite large \( \alpha_s \sim 0.5 \). This limit is called “confinement” where quarks and gluons interact strongly and can not observed as free
Figure 1.3: Panel (a) shows the QED one-loop running coupling constant ($\alpha$) scaled by ($\alpha_0$) as a function of the scale $Q$. In panel (b), we show the QCD one-loop running coupling constant ($\alpha_s$) as a function of the scale $Q$. The plots are adapted from Ref. [2].

Particles but are, instead, confined inside hadrons.

As a result, perturbative methods can be used for QED calculations where one-loop calculations are sufficient for most practical purposes. On the other hand, for QCD, one can rely on perturbative methods only at higher energies, but for the relevant low energies one should calculate the needed quantities using some non-perturbative methods such as lattice QCD. This is what makes QCD a very interesting theory which has a variety of different phenomena, but at the same time hard to solve from first principles.

1.6 Phase diagram of QCD

As we discussed before, QCD is a rich theory with very different properties compared to other known physical theories. This richness has many reflections and one of them is the QCD phase diagram. In Fig. 1.5, we show the QCD phase diagram in the temperature and baryon chemical potential plane. As can be seen from this figure, there are many different regions in the QCD phase diagram depending on the temperature and baryon-chemical potential.
Figure 1.4: Experimental measurements of the QCD coupling constant ($\alpha_s$) as a function of the energy scale $Q$. Figure is taken from Ref. [3] where more details and a recent review can be found about all listed measurements in the plot.

Ordinary nuclear matter such as nucleons (protons or neutrons) are called cold nuclear matter at $T = 0$ and $\mu_B \sim m_p \sim 1$ GeV. If the temperature and/or chemical potential increases, we start to see different forms of QCD matter. If temperature is increased, ordinary nuclear matter start to be excited to produce a hadron resonance gas. At much higher temperatures, with smaller chemical potential ($\mu_B \sim 0$), QCD matter undergoes a transition from hadronic degrees of freedom to the fundamental constituents, quarks and gluons, where the critical temperature is $T_c \sim 150–170$ MeV as indicated by the lattice results [39, 40]. LHC heavy-ion collision experiments are designed to study this region which is similar to the early universe as suggested by the big-bang theory. At lower energies, RHIC experiments are designed to
Figure 1.5: This figure shows the QCD phase diagram in the temperature (T) and baryon chemical potential ($\mu_B$) plane. Figure is taken from Ref. [4].

probe different regions of the QCD phase diagram using gold-gold collisions. The highest energy that RHIC can reach is $\sqrt{s_{NN}} = 200 \text{ GeV}$ but it can cover a wide range of energies which helps probing different regions of QCD phase diagram which is not possible at LHC energies. This program is called the beam-energy scan at RHIC which has the potential of understanding the middle part of the phase diagram at moderate temperatures and baryon chemical potential at different energies ($\sqrt{s_{NN}} < 200 \text{ GeV}$). The main goal of this program is to look for the existence of the critical point since lattice QCD confirmed that at low chemical potential there is a crossover instead of a finite-order phase transition from QGP to hadron resonance gas. By going right at the bottom of the phase diagram, cold but very dense systems, one starts to see other nuclear matter phases such as neutron stars which is important to astrophysics.
1.6.1 Quark gluon plasma (QGP)

The QGP is one of the nuclear matter phases which can only be created in very extreme conditions. In this phase, the QGP, quarks and gluons are de-confined in a thermalized state where they are the fundamental degrees of freedom. The QGP as a state can not be observed directly which means that we have to look for indirect ways to confirm the existence of QGP. One piece of evidence for the formation of QGP is the collective flow seen in heavy-ion collision experiments. The origin of the collective flow can be understood by recognizing that QGP is an approximately thermalized system of quarks and gluons. Since the system is thermal it will have thermal pressure components (pressure gradients) which force the system to expand anisotropically. If there is no collective behavior (there is no pressure) that should be an evidence that QGP was not formed as an approximately equilibrated hot plasma [9].

1.6.2 The heavy-ion collision program

The QGP can be created and studied using ultrarelativistic heavy-ion collision experiments (URHICs) at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC). Before going to further details, let’s define explicitly what we mean by ultra-relativistic heavy-ion collisions. We call any system “ultra-relativistic” if its kinetic energy is much higher than its rest energy which happens only at projectile velocities close to the speed of light. On the other hand, we mean by “heavy-ions” heavy atomic nuclei such as lead (Pb) and gold (Au). \(^1\)So these experiments are designed to use high energy nuclear collisions using heavy ions to create QGP and to study its properties and stages. As a physical field, ultra-relativistic heavy-ion collisions, connects nuclear physics with elementary particles physics. One thing here that we want to note is why do we need heavy-ions rather than some simple systems like proton-proton collisions. This can be answered in two

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\(^1\)In practice people call any ions that is heavier than deuteron a heavy ion, for example helium, calcium ...etc.
different ways [41]. First, using large nuclei will lead to large volume of QCD matter at high energy densities which will help to create strongly interacting QCD matter. In this case, a large number of hadrons will be emitted from the macroscopic volume of QGP itself. Second, the source of final emitted particles in small systems will not be the formation of QGP but rather other sources like fragmentation. ²

People usually ask the question: why billions of dollars should be spent on such experiments? Some may even argue that there is no direct benefits of such works. So, let’s list some potential benefits of heavy-ion collision programs. As we know from the standard model, the fundamental constituents of matter are quarks besides leptons and force carriers. The primary reason for the interest of heavy-ion collision program is to understand the fundamental matter under extreme conditions (at high energies). Using such experiments, we were able to create, in the laboratory, and study a new form of matter, the quark-gluon plasma. This program will help studying the QCD and some of its predictions such as the confinement and deconfinement. Using URHIC we can explore the phase structure of fundamental matter and the macroscopic properties of QCD matter by understanding its constituents and interactions. Moreover, QCD is a non-abelian theory, so by studying QGP one can get insight on how non-abelian theories are different from abelian theories. Finally, as heavy-ion collisions called the little bangs, understanding the properties of QGP will help us understand more the early universe and the big-bang theory.

1.6.3 Heavy-ion collisions stages

In this section, I will describe the main space-time stages of the QGP evolution generated in heavy-ion collisions. In Fig. 1.6, we show a cartoon of the main stages that the QGP goes through until the produced particles reach the detector.

²Of course, in some very rare cases, QGP can be produced from proton-proton collisions due to quantum fluctuations but with much less final particles. The talk here is about the majority of events where QGP is not produced.
Figure 1.6: Stages of relativistic heavy-ion collisions from QGP formation until hadronization where particles reach the detector. Figure is made by Chun Shen Ref. [5].

**Before the initial nuclear impact**  Looking to Fig. 1.6, we see that before the nuclear impact both nuclei look like “pancakes”. This is due to Lorentz contraction where both of them are highly Lorentz contracted in the lab frame. Lorentz contraction can be characterized by the \( \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \) factor which is related to the object velocity with respect to the speed of light. For classical systems \((v \ll c)\), \( \gamma \rightarrow 1 \) while for ultrarelativistic systems \((v \text{ is very close from } c)\), \( \gamma \rightarrow \infty \). For example, at RHIC energies with \( \sqrt{s} = 200 \text{ GeV} \), \( \gamma \approx \sqrt{s}/2 = 100 \) whereas at lowest LHC energies, \( \sqrt{s} = 2.76 \text{ TeV} \), \( \gamma \approx 1000 \). The duration of the collision also depends on the \( \gamma \) factor. It is inversely proportional to \( \gamma \), i.e., the duration of the collision \( \sim R/\gamma \) where \( R \) is the nucleus radius. Roughly at LHC, the duration of the collision \( \sim 10/1000 \approx 0.01 \text{ fm/c} \). By decreasing the collision energy, the duration of the collision increases.
The pre-equilibrium stage  This stage comes after the nuclear impact immediately. In this stage, the system possesses strong gradients and strong fields resulting in a negative pressure ratio $P_L/P_T$. The usual theoretical framework to study the hard particle production is color-glass-condensate/glasma frameworks [42, 43, 44, 45, 46, 47].

The viscous hydrodynamics stage  In this stage the system is close to equilibrium and locally being thermalized. In this case, one can use viscous hydrodynamics since the gradients are small enough to describe the system using this framework. 3

The chemical freeze-out  This stage happens when the inelastic collisions between produced particles stop. Then the dominant processes become the elastic collisions and the strong decays of heavier resonances such as $\Delta$ and $\rho$ to stable particles.

The thermal freeze-out  This transition happens when the hadronic gas transforms from a strongly interacting gas to a weakly interacting gas streaming freely to the detector. This can be understood by noticing that the system is expanding which means that the number density is decreasing which results to the increase of the timescale between collisions, i.e., $\tau_{\text{coll}} \sim 1/\sigma n$. At one point, the timescale between collisions will become much bigger than the expansion time scale $\tau_{\text{coll}} \geq \tau_{\text{exp}}$ where the expansion time scale is inversely proportional to the divergence of the 4-vector velocity $\tau_{\text{exp}} \sim 1/\partial_{\mu} u^\mu$. Due to the lack of interactions, the momenta of particles don’t change anymore (significantly at least), they become frozen and by this reason we call the stage thermal freeze-out.

Eventually, the final particles are reaching the detector freely where their momentum-distribution can be measured and then many observables can be studied.

3We note here that in many recently works, the equilibration and small gradients are not requirements for hydrodynamics to be applicable, for more details I refer the reader to [48, 49, 50, 51].
1.7 Some definitions

In this section, we will introduce some useful quantities that will be used later in the text.

1.7.1 Rapidity

The rapidity is a quantity usually used in particle physics accelerators. It is defined as

\[ y = \frac{1}{2} \ln \left( \frac{E + p_\parallel}{E - p_\parallel} \right), \] (1.19)

where \( p_\parallel \) is the momentum component parallel to the beam line. If the particle is moving perpendicular to the beam line, i.e., \( p_\parallel = 0 \), then \( y = 0 \). On the other hand, if the particle is moving parallel to the beam line i.e., \( p_\perp = 0 \), then \( y \) will have its maximum value depending on the particle mass. For example, for massless particles \( y \to \infty \) when \( p_\perp = 0 \). The real advantage of using the rapidity over the speeds is the fact that it is a Lorentz boost invariant quantity along the beam line. Let’s consider two frames relative to each other by a boost invariant along the beam axis, if one measures the difference between two particles rapidities with respect to each frame, the difference will be the same i.e.,

\[ y_1 - y_2 = y'_1 - y'_2. \] (1.20)

1.7.2 Pseudorapidity

Usually rapidity is hard to measure since one needs to know the energy and the full momentum of each particle, which is hard particularly in the forward or backward directions and at high velocities. Instead of rapidity, pseudorapidity can be used which is a Lorentz boost invariant quantity along the beam line similar to rapidity, but it is much easier to measure. The pseudorapidity is defined as

\[ \eta = \frac{1}{2} \ln \left( \frac{|p| + p_\parallel}{|p| - p_\parallel} \right), \] (1.21)
After some algebra, the pseudorapidity can be written as

$$\eta = \ln \left( \cot \frac{\theta}{2} \right), \quad (1.22)$$

where $\theta$ is the polar angle measured from the beam axis, i.e., $\theta = \pi/2$ when the particle velocity is perpendicular to the beam axis. Similar to rapidity, $\eta = 0$, when the particle direction is perpendicular to the beam axis and $\eta = \infty$ when the particle direction is parallel to the beam axis. In Fig. 1.7, we show a sketch showing the pseudorapidity dependence on the polar angles measured with respect to the beam line.

One should note that at high relativistic velocities where $E = |p|$ or for very small mass particles $\eta \approx y$.

### 1.7.3 Collisions centrality

In heavy-ion collisions, two nuclei collide with each other and it matters how they collide since the produced volume in each case is different, which leads to clear differences in the spectra. They can collide head on, barely overlap, or in between with infinite possibilities. In
Figure 1.8: The left side of the illustration shows the geometrical definition of the impact parameter while the right side shows the spectators and participants nucleons after the collision. Figure is taken from Ref. [6].

Fig. 1.8, we define the impact parameter \( b \) which is the distance between the centers of both nuclei. The overlap region shape determines the impact parameter value. For most central collisions, the impact parameter is the smallest, \( b = 0 \), whereas for peripheral collisions, \( b \) is as big as allowed by the system size. As we can see from this figure, nucleons inside the overlap region are called *participants* where nucleons outside are called *spectators*. Experimentally, it is not possible to directly determine the impact parameter, centrality, and the number of participant nucleons. However, one can use the multiplicity of produced particles (detected by the detector) in a specific range of rapidity to infer the impact parameter of the collision. As one may guess, the multiplicity increases with decreasing the impact parameter. For the most central collisions, the multiplicity will be highest since the initial volume at the impact is as big as possible. In contrast to that, the multiplicity is smallest for peripheral collisions since the initial volume is very small compared with central collisions. As an example, the most central collisions 0 – 5% are the top 5% that have the highest multiplicity.

In Fig. 1.9, panel (a), we show the charged particle distribution as a function of centrality.
Figure 1.9: In panel (a) charged particle distribution is shown as a function of the multiplicity where the centralities classes are shown as well. In panel (b) the number of participating nucleons and binary nucleons, $N_{\text{part}}$ and $N_{\text{bin}}$ respectively as a function of the impact parameter for both Pb-Pb and Au-Au at $\sqrt{s_{NN}} = 2.76$ TeV, and 200 GeV respectively. Figure is taken from Ref. [7] where panel (a) is adapted from Ref. [8] for Pb-Pb collisions with the corresponding centrality percentiles. In panel (b), we show the number of participants and the number of binary collisions as a function of the impact parameter for Pb-Pb and Au-Au at $\sqrt{s_{NN}} = 2.76$ TeV, and 200 GeV respectively. Note that the number of binary collisions for Pb-Pb is much more than binary collisions in Au-Au because of the increase of the inelastic cross section due to the increase of the collision energy.

1.7.4 The Glauber model

As we have seen from the last section, one needs to build a model to find the relation between the “geometric” quantities such as the impact parameter (centrality), number of participants and the observed multiplicity. For this purpose, one needs to find the total inelastic $AA$ cross section which will be used later to find all required quantities. The standard model to do that is Glauber model which helps to find the impact parameter for
each collision depending on the multiplicity observed [52, 53, 54, 55]. In this section, I will review the basics of the Glauber model, and for more details I refer the reader to Ref. [56].

The starting point is to determine the nuclear charge densities since they are not uniformly distributed. The nuclear charge density is modeled by the Woods-Saxon profile with some experimental input parameters

\[ n_A(r) = \frac{n_0}{1 + e^{(r-R)/d}}, \]  

where \( n_0 \) is the central density, \( d \) is the “skin depth”, and \( R \) is the nucleus radius where all of them can be determined by low-energy nuclear physics. The probability per unit transverse area of finding a nucleon located at \( r_\perp \) is called the nuclear thickness function. For a nucleus \( A \) the nuclear thickness function can be determined using

\[ T_A(r_\perp) = \int_{-\infty}^{\infty} dz n_A \left( \sqrt{r_\perp^2 + z^2} \right), \]

where \( r_\perp = \sqrt{x^2 + y^2} \).

The joint probability per unit area of a nucleon being located in the overlapping region between the two colliding nuclei is given by

\[ T_A(x_\perp + b/2) T_B(x_\perp - b/2), \]

Integrating over \( x_\perp \) gives the nuclear overlap function between nuclei \( A \) and \( B \) is

\[ T_{AB}(b) = \int d^2x_\perp T_A(x_\perp + b/2) T_B(x_\perp - b/2), \]

Using that, one may find that the averaged probability for a nucleon-nucleon collision to happen in a nucleus-nucleus collision is given by \( T_{AB}(b)\sigma_{nn} \) where \( \sigma_{nn} \) is the total inelastic nucleon-nucleon cross section. After some work, one can show that the total inelastic \( AA \) cross section is given by

\[ \sigma_{in}^{AB} = 2\pi \int_0^\infty db [1 - \exp(-T_{AB}(b)\sigma_{nn})]. \]
After defining the nuclear overlap function and the total inelastic scattering cross section, one can show that the centrality class can be determined using

\[ c_i = \frac{1}{\sigma_{\text{tot}}} \int_{b_i,\text{min}}^{b_i,\text{max}} d^2b \left[ 1 - \exp(-T_{AB}(b)\sigma_{nn}) \right], \tag{1.28} \]

In a similar way, the average impact parameter in each centrality class can be calculated using

\[ \bar{b}_i = \frac{1}{\sigma_{c_i}^{AA}} \int_{b_i,\text{min}}^{b_i,\text{max}} d^2b \left[ 1 - \exp(-T_{AB}(b)\sigma_{nn}) \right]. \tag{1.29} \]

Finally, the average number of participants in each class can be found

\[ \bar{N}_{\text{part}}^i = \frac{1}{\sigma_{c_i}^{AA}} \int_{b_i,\text{min}}^{b_i,\text{max}} d^2b n_{\text{part}}^{AA}(b) \left[ 1 - \exp(-T_{AB}(b)\sigma_{nn}) \right]. \tag{1.30} \]

In Table 1.1, in Pb-Pb collisions, we list the minimum centrality \( c_{\text{min}} \), and the maximum centrality \( c_{\text{max}} \) for each centrality class. We also show the corresponding minimum and maximum impact parameter in that class \( (b_{\text{min}}, b_{\text{max}}) \). In simulations, usually one takes the average impact parameter \( \langle b \rangle \) corresponding to each class.

In this section, results are derived using optical Glauber model where the number of nucleons for each nucleus is assumed to be very large beside some other simplifications like assuming that the nucleon density distribution is continuous. In practice, one can use Monte Carlo Glauber model which enables specifying the spatial coordinates for nucleons rather than using a continuous distribution [56].

### 1.8 Collective flow

One of the most important heavy-ion collision observables is the collective flow of the produced matter after the impact. This collective behavior is caused by interactions of the fireball.
<table>
<thead>
<tr>
<th>$c_{\text{min}}$</th>
<th>$c_{\text{max}}$</th>
<th>$b_{\text{min}}$</th>
<th>$b_{\text{max}}$</th>
<th>$\langle b \rangle$</th>
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<tr>
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<td>$b_{\text{min}}$</td>
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<td>$\langle b \rangle$</td>
</tr>
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<td>20</td>
<td>15.608</td>
</tr>
</tbody>
</table>

Table 1.1: The relationship between centrality class ($c_{\text{min}}, c_{\text{max}}$) and the minimum and maximum impact parameter in that class ($b_{\text{min}}, b_{\text{max}}$) with the average impact parameter $\langle b \rangle$ derived using Glauber model in Pb-Pb collisions. Note that $b_{\text{min}}, b_{\text{max}},$ and $\langle b \rangle$ have units of fm. This table is taken from Ref. [31].

### 1.8.1 Radial flow

In the case of very central collisions, in the optical Glauber model, the overlap region is circular. So the only possible flow pattern is radial flow since there is no initial spatial anisotropy. As shown in Fig. 1.10, the system is azimuthally symmetric when both nuclei are aligned ($b = 0$). One should note that this simple picture does not hold most of the times because of the quantum fluctuations of the distributions of nucleons and their constituents quarks and gluons. As a result, even perfectly central collisions may develop anisotropic flow and hence fluctuations will dominate over the geometric effects for this class of collisions.

### 1.8.2 Anisotropic flow

The momentum distribution of the produced particles detected by the detector can be written as a Fourier series

$$E \frac{dN}{d^3p} = \frac{1}{2\pi} \frac{dN}{p_T dp_T dy} \left( 1 + 2 \sum_{n=1}^{\infty} v_n \cos [n(\phi - \Psi_R)] \right),$$  \hspace{1cm} (1.31)
where $E$ is the energy, $p$ is the momentum, $p_T$ is the transverse momentum, $\Psi_R$ is the reaction plane angle, and $\phi$ is the measured angle for each particle with respect to the lab coordinates.

The coefficients $v_n$ are called the flow harmonic coefficients where $v_1$ is called the directed flow, $v_2$ is called the elliptic flow, $v_3$ is called the triangular flow, and in general $v_n$ is called the n-th harmonic differential flow. $v_n$ is given by

$$v_n(p_T, y, b) = \langle \cos[n(\phi - \Psi_R)] \rangle,$$  \hspace{1cm} (1.32)

In Fig. 1.11, we show a sketch indicating the definition of the angle $\Psi_R$ which is the angle between the lab coordinates and the impact parameter coordinates in each event.

**The directed flow $v_1$:** At early times, some dense matter is formed which forces some particles to deflect. Particles deflect depending on their rapidity, if the particle is moving with $+y$, it will deflect towards $+x$ and towards $-x$ if moving with $-y$. The magnitude of $v_1$ tells us about the compressibility of matter, for more details, see Ref. [11]. In Fig. 1.11, we show a schematic diagram showing the directed flow showing the direction of spectators’
The elliptic flow $v_2$: The elliptic flow represents the elliptical azimuthal asymmetry of the momentum distribution of the produced particles.

$$v_2(y, p_T, b) = \langle \cos[2(\phi - \Psi_R)] \rangle = \frac{\int d\phi \cos 2(\phi - \Psi_R) \frac{dN}{dy p_T dp_T d\phi}}{\int d\phi \frac{dN}{dy p_T dp_T d\phi}}, \quad (1.33)$$

The triangular flow $v_3$: The triangular flow is defined as

$$v_3(y, p_T, b) = \langle \cos[3(\phi - \Psi_R)] \rangle, \quad (1.34)$$

In Fig. 1.11, we show a schematic diagram showing the shape of the elliptic flow and the triangular flow.
Figure 1.12: A schematic diagram for the directed flow, viewed in the $x - z$ plane, showing the deflection of the spectators along $\pm x$ for spectators moving with $\pm y$. Figure is taken from Ref. [11].

Figure 1.13: The left sketch shows the shape of the overlap region that generates elliptic flow $v_2$ while the right sketch shows the shape of the overlap region that generates triangular flow $v_3$. Figure is taken from Ref. [12] where left, and right sketches are adapted originally from Ref. [13] and Ref. [14] respectively.
1.9 Fluid dynamics

In this section, I will give a very brief introduction to fluid dynamics (the motion of the fluids: liquids and gases) both in non-relativistic and relativistic systems.

1.10 Non-relativistic fluid dynamics

In ideal fluids, dissipative effects such as the viscosity (internal friction) and the thermal conductivity (heat exchange) can be neglected. In this classical ideal limit Euler’s Equation can be used to describe the fluid

\[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p, \]  

where \( \mathbf{v}(t, x) \) is the fluid velocity, \( \rho(t, x) \) is the fluid mass density, and \( p(t, x) \) is the pressure. Clearly from Eq. (1.35), we see that we need two additional equations to close the system. The first equation is the “continuity equation”

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \]

where \( \mathbf{J} \equiv \rho \mathbf{v} \) is the current density. The last needed equation is the equation of state which, in this case, relates the pressure and the density, i.e., \( p = p(\rho) \).

In the case where energy dissipation can’t be neglected, one can use Navier-Stokes’ Equation which is suitable in this case

\[ \rho \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla p + \eta \nabla^2 \mathbf{v} + (\zeta + \frac{2}{3} \eta) \nabla (\nabla \cdot \mathbf{v}), \]

where \( \eta \) is the shear viscosity and \( \zeta \) is the bulk viscosity. Here, the bulk viscosity corresponds to the fluid’s resistance to isotropic volume change, whereas shear viscosity corresponds to the fluid’s resistance to deformation. We should note that in Eq. (1.37) it was assumed that both \( \eta \) and \( \zeta \) are constants. Regarding the details of the derivations and the general form of Eq. (1.37) for general \( \eta \) and \( \zeta \), I refer the reader to Ref. [57].
1.10.1 Relativistic fluid dynamics

1.10.2 The kinetic theory

Consider a homogeneous fluid where we have $N$ particles in a volume $V$, the probability of finding the particles in a unit phase space is given by $f(x,p)\,d^3x\,d^3p$. So, the number of particles per unit phase space which is related to the distribution function is given by

$$dN \propto f(x,p)\,d^3x\,d^3p,$$

(1.38)

Once the distribution function is determined, many thermodynamic quantities can be found, for example, the particle four-current can be defined using the distribution function

$$J^\mu = \int dP \, p^\mu f(x,p),$$

(1.39)

where

$$dP \equiv \frac{N_{\text{dof}}}{(2\pi)^3} \frac{d^3p}{p^0}$$

(1.40)

where $p^0$ is the energy density in the local rest frame and $N_{\text{dof}}$ is the number of degrees of freedom.

This gives the number density and the three-current

$$n = \int dP \, E \, f(x,p),$$

(1.41)

$$\mathbf{J} = \int dP \, \mathbf{p} \, f(x,p),$$

(1.42)

The energy-momentum tensor can be defined as

$$T^{\mu\nu} = \int dP p^\mu p^\nu f(x,p),$$

(1.43)

where, in the local rest frame, the energy density is given by the $T^{00}$ component, and the pressure is given by the $T^{ii}$ component. Energy-momentum conservation can be written compactly as

$$\partial_\mu T^{\mu\nu} = 0.$$

(1.44)
Now, let’s turn to the kinetic equations after defining the distribution function. If there are no external sources, the probability density has to be conserved

$$\frac{df}{dt} = 0.$$  \hfill (1.45)

This is called the Liouville’s theorem. In the presence of collisions, Eq. (1.45) can be written as

$$\frac{df}{dt} = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}}.$$  \hfill (1.46)

Consider the total derivative of the distribution function

$$df = \frac{\partial f}{\partial t} dt + \nabla f \cdot dr + \frac{\partial f}{\partial p} \cdot dp.$$  \hfill (1.47)

Using $dr = (p/m)dt$ and $dp = Fdt$, Eq. (1.46) can be written as

$$\frac{\partial f}{\partial t} + \frac{p}{m} \cdot \nabla f + F \cdot \frac{\partial f}{\partial p} = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}}.$$  \hfill (1.48)

In a compact way, Eq. (1.48) can be written when neglecting the force term as

$$p^\mu \partial_\mu f(x,p) = -\mathcal{C}[f],$$  \hfill (1.49)

where $\mathcal{C}[f]$ represents the collisional kernel which contains all of the microscopic interactions involved in the dynamics. In the absence of collisions $\mathcal{C}[f] = 0$. Eq. (1.49) is known as Boltzmann equation which has a lot of applications in many different physics fields.

### 1.10.3 Relaxation time approximation

The form of $\mathcal{C}[f]$ depends on the nature of the interactions which is complicated in general but can be simplified by some approximations. One of these approximations which is used in many heavy ion collision studies is the relaxation time approximation (RTA) first proposed by Anderson and Witting [58]. It takes the following form

$$\mathcal{C}[f] = \frac{u^\mu p_\mu}{\tau_{eq}} [f(x,p) - f_{eq}(x,p)],$$  \hfill (1.50)
where $u^\mu$ is the four-vector velocity, $f_{eq}$ is the local equilibrium distribution function, and $\tau_{eq}$ is a parameter setting how fast the system approaches equilibrium, i.e., how $f$ approaches $f_{eq}$ as $t \to \infty$. To see how this parameter is related to the relaxation time, let’s take a homogeneous system (0+1d) for simplicity,

$$p^0 \partial_\mu f(t) = -\frac{u^0 p_0}{\tau_{eq}} [f(t) - f_{eq}(t)], \quad (1.51)$$

giving

$$\partial_t f(t) = -\frac{1}{\tau_{eq}} [f(t) - f_{eq}(t)]. \quad (1.52)$$

Integrating, one obtains

$$f(t) = f_{eq}(t) + [f(t = 0) - f_{eq}(t)] e^{-t/\tau_{eq}},$$

If $\tau_{eq} \to 0$, then $f(t) = f_{eq}(t)$ which means the system stays in equilibrium. If $\tau_{eq} \to \infty$, then $f(t) = f(t = 0)$, and the system in principle never goes back to equilibrium. Hence, from this example, we see that $\tau_{eq}$ plays the role of the thermal relaxation time [59].
Chapter 2

Anisotropic hydrodynamics

From the early days after establishing the existence of the QGP, relativistic hydrodynamics was a powerful tool to study the collective behavior seen in heavy-ion collision experiments at RHIC and LHC. In the beginning, ideal hydrodynamics was used to describe heavy-ion collision physics [60, 61, 62]. However, later on it was recognized that the QGP is not an ideal fluid but rather has some dissipative effects. As a result, relativistic viscous hydrodynamics was used and was quite successful describing experiments [63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 23]. The canonical way to derive viscous hydrodynamics from kinetic theory is based on linearization around the isotropic equilibrium distribution function.

\[
f(x, p) = f_{eq}\left(\frac{p^\mu u_\mu}{T}\right)\left[1 + \delta f(x, p)\right], \tag{2.1}\]

As we can see from Eq. (2.1), the expansion is around the isotropic equilibrium state (the leading term), whereas the viscous effects are included as correction terms. For example, in the case of non-zero shear and bulk viscosities, the distribution function used to derive the dynamical equations includes beside the leading term, one term corresponds to the shear viscosity effects and another term corresponds to the bulk corrections

\[
\delta f = \delta f_{\text{shear}} + \delta f_{\text{bulk}}. \tag{2.2}\]

One usually justifies using linearization around the isotropic equilibrium state by the assumption that the QGP is very close from equilibrium. However, as many studies suggest, QGP is a highly anisotropic plasma in the local rest frame. As a concrete example, consider
the pressure anisotropy, $P_L/P_T$ predicted by different methods which shows that the QGP is highly momentum-space anisotropic at early times and relaxes slowly towards isotropy at late times. In Fig. 2.1, the pressure anisotropy, $P_L/P_T$ is shown as a function of proper time $\tau$. As we can see from this cartoon, different methods like color-glass condensate (CGC), Glasma, Boltzmann-Vlasov transport, and relativistic viscous hydrodynamics all show that $P_L/P_T \ll 1$ at early times but relaxes slowly towards equilibrium at late times. One should note that by increasing the shear viscosity to entropy density ratio $(\eta/s)$ the system becomes more anisotropic. It also shows that even if the system stars from isotropy, there is a kind of attractor resulting in $P_L/P_T \ll 1$, as can be seen from the cartoon, at times on the order of a few tenths of fm/c. This simple example shows that in principle QGP can be far from equilibrium [15].

As we can see, one needs to find another way which does not require being close to isotropy but works in principle even for far from equilibrium systems. As an alternative method, anisotropic hydrodynamics was introduced which is based on including the anisotropies in the leading term of the distribution function unlike viscous hydrodynamics where one expands around the equilibrium distribution function[94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 16, 105, 106, 107, 108, 109, 108, 110, 19, 111, 112, 113]. Then viscous corrections are included in the corrections to the leading order. I refer the reader, for more details, to this recent review, see Ref. [15].

### 2.1 Anisotropic hydrodynamics

The one particle distribution function in anisotropic hydrodynamics can be constructed as

$$f(x, p) = f_{iso}\left(\frac{1}{\lambda} \sqrt{p_\mu \Xi^{\mu\nu} p_\nu}\right) + \delta f,$$  \hspace{1cm} (2.3)
where $\lambda$ is a temperature-like parameter which can be identified with the temperature only in the isotropic equilibrium limit and $\Xi^{\mu \nu}$ is a second-order tensor which accounts for momentum-space anisotropy defined as

$$
\Xi^{\mu \nu} = u^{\mu} u^{\nu} + \xi^{\mu \nu} - \Delta^{\mu \nu} \Phi, \tag{2.4}
$$

and

$$
\Delta^{\mu \nu} = g^{\mu \nu} - u^{\mu} u^{\nu}, \tag{2.5}
$$

$$
\Delta^{\mu}_{\mu} = 3, \tag{2.6}
$$

where $u^{\mu}$ is the four-velocity associated with the local rest frame, $\xi^{\mu \nu}$ is a symmetric and traceless tensor, and $\Phi$ is the bulk degree of freedom. The quantities $\Xi^{\mu \nu}$ and $\lambda$ are functions of space and time. There are some constraints that the anisotropy tensor’s components obey
as follows

\[ u^\mu u_\mu = 1, \quad (2.7) \]

\[ \xi^\mu_{\;\mu} = 0, \quad (2.8) \]

\[ u_\mu \xi^{\mu\nu} = u_\mu \Delta^{\mu\nu} = 0. \quad (2.9) \]

Using the above relations, one can easily find

\[ \Xi^\mu_{\;\mu} = 1 - 3\Phi. \quad (2.10) \]

One may wonder, why the anisotropy tensor is decomposed in the way shown in Eq. (2.4). This decomposition is similar to the convention used when defining the energy-momentum tensor \( T^{\mu\nu} \) in relativistic viscous hydrodynamics

\[ T^{\mu\nu} = T_0^{\mu\nu} + \pi^{\mu\nu} + \Pi \Delta^{\mu\nu}, \quad (2.11) \]

where \( \pi^{\mu\nu} \) is a traceless tensor and \( \Pi \) is assigned to the bulk degree of freedom.

Next, we return to the exact form of the distribution function that will be used throughout most of this study. First, the corrections to the leading term are ignored in this study due to its complexity; however, a well-defined method for including them has been presented in Refs \([16, 19]\). Second, \( \xi^{\mu\nu} \) is a \( 4 \times 4 \) matrix where space-like components, in general, are not vanishing; however, at leading order we can assume that \( \xi^{\mu\nu} = \text{diag}(0, \xi_x, \xi_y, \xi_z) \). The distribution function in this case can be written in the local rest frame as \([108]\)

\[ f(x, p) = f_{\text{iso}} \left( \frac{1}{\lambda} \sqrt{p_\mu \Xi^{\mu\nu} p_\nu} \right) = f_{\text{iso}} \left( \frac{1}{\lambda} \sqrt{\sum_i \frac{p_i^2}{\alpha_i^2} + m^2} \right), \quad (2.12) \]

where the sum is over \( i \in \{x, y, z\} \) with the anisotropy parameters

\[ \alpha_i \equiv (1 + \xi_i + \Phi)^{-1/2}. \quad (2.13) \]

This will be the case except for the last chapter, Chap. 6, where we present one method for including the off-diagonal terms in the anisotropy tensor in an approximate way.
The isotropic limit  To go back to the isotropic limit, both \( \xi_i \) and \( \Phi \) vanish giving \( \alpha_i = 1 \). If one wants the isotropic equilibrium limit then beside vanishing \( \xi_i \) and \( \Phi \), one must take \( \lambda \to T \). In the case of isotropic equilibrium limit, \( p_\mu \Xi^{\mu\nu} p_\nu = (p \cdot u)^2 = E^2 \). Using Boltzmann distribution function gives \( f_{eq}(x) = e^{-E/T} \).

The viscous hydrodynamics limit  By assuming small anisotropy parameter in the Romatschke-Strickland form one easily can show that anisotropic hydrodynamics reproduces the second-order viscous hydrodynamics dynamical equations [94]. The equivalence to second order viscous hydrodynamics also can be shown in general 3+1d anisotropic hydrodynamics in the near-equilibrium limit as shown in Ref. [114].

2.2 Why anisotropic hydrodynamics?

Figure 2.2: Particle production as a function of \( 4\pi\eta/s \) predicted by Boltzmann equation exact solution compared with 2\(^{nd}\)-order viscous hydro, 3\(^{rd}\)-order viscous hydro, LO aHydro, and NLO aHydro (vaHydro). Figure taken from Ref. [16]

In Sec. 1.9, I showed the Navier-Stokes equation which was a first-order theory of viscous
hydrodynamics. The Navier-Stokes equation is not a causal theory which is a fundamental problem which means it can not be used in heavy-ion phenomenological studies. Later on, in order to fix the causality problem, second order viscous hydrodynamics was introduced by Müller-Israel-Stewart [115, 116, 117]. The Müller-Israel-Stewart theory is a causal theory which has been used in many phenomenological studies. Recently, second order viscous hydrodynamics was revisited to include, in a systematic way, some neglected terms in the original formalism [88]. This new derivation has been called the Denicol-Niemi-Molnár-Rischke (DNMR) method while the original second order viscous hydro is called IS or MIS theory. Third-order viscous hydrodynamics also was derived recently in Ref. [118].

One should note that hydrodynamic gradient-expansion series does not converge, it is a divergent series. This can be shown by calculating the gradient series coefficients as shown in many works, see for example, Ref. [48, 119, 120, 49].

The IS and DNMR frameworks have been used in phenomenological studies for a long time and show a quite good agreement with experimental data. People usually wonder why do we need to introduce a new method on top of the viscous hydrodynamics methods. I want to show some situations where aHydro clearly does better. In this section, I will consider some simple examples which show how aHydro does better. The exact solutions of Boltzmann equation give us a powerful tool for couple of reasons. They can provide us semi-analytic expressions for thermodynamic quantities like temperature, pressure components, and energy density. They can also help to compare different hydrodynamic models to see which ones best agree with the exact solutions.

I should note a few things before presenting our comparisons between different hydrodynamic methods and exact solutions. In this section, by LO aHydro we mean, aHydro with spheroidally deformed local momentum distribution function, i.e only one anisotropy parameter which is known as Romatschke-Strickland form

\[ f(x, p) = f(\sqrt{p_{\perp}^2 + (1 + \xi)p_z^2}/\lambda) \]
On the other hand, we mean by NLO aHydro or vaHydro anisotropic hydrodynamics where the corrections are included \((\delta f \neq 0)\) while keeping the leading term in Romatschke-Strickland form (spheroidally deformed) \([16, 19]\). In vaHydro, the leading term is treated non-perturbatively while the corrections, \(\delta f\), are treated perturbatively. Note that this is different from what we call LO aHydro in the bulk of this dissertation which is aHydro with an ellipsoidal form.

Second, in some cases presented in this section, the conditions are taken to be extremely far away from real-life physics. For example, experimental data results suggest low \(4\pi\eta/s\), very close from what is known as the lower bound, i.e, \(4\pi\eta/s \approx 1 - 3\) \([23, 122]\), however the comparisons here in some cases are presented at higher \(4\pi\eta/s\) like \(4\pi\eta/s = 10\) or even much higher \(4\pi\eta/s = 100\). This should be understood as test cases to see the applicability and validity of these methods subject to extreme conditions.

### 2.2.1 Conformal 0+1d systems

In this section, I will present comparisons of the exact solution of Boltzmann equation for conformal 0+1d systems for a few thermodynamic quantities. For details concerning the exact solutions for conformal systems in 0+1d I refer the reader to Ref. \([123]\).

#### Entropy generation

In Fig. 2.2, we show the entropy generation predicted by a few dissipative hydrodynamics methods as a function of shear viscosity to entropy density ratio \([16]\). Since in a conformal system, the number density is proportional to the entropy density, we plot the final particle number \(n(\tau_f)/\tau_f\) divided by the initial particle number \(n(\tau_0)/\tau_0\), where \(n\) is the number density, \(\tau_f\) the final proper time, and \(\tau_i\) the initial proper time, as a function of the shear viscosity to entropy density ratio \((\eta/s)\). The number production should vanish in two limits for two different reasons. First, in the ideal hydrodynamics limit when there is no viscosity \((\eta/s = 0)\),
Figure 2.3: Pressure anisotropy as a function of proper time predicted by the exact solution of Boltzmann equation in conformal 0+1d systems versus IS, DNMR, and aHydro methods. In the left panel, $\xi_0 = 0$ where in the right panel $\xi_0 = 10$. In both panels, the top, middle, bottom rows corresponds to $4\pi\eta/s = 1$, $4\pi\eta/s = 3$, and $4\pi\eta/s = 10$ respectively. Figure is taken from Ref. [15] which is adapted from the original Ref. [17].

It should vanish because of entropy conservation. Second, in the free streaming limit when $\eta/s \to \infty$ which results to $\tau_{eq} \to \infty$ because of the absence of interactions. As can be seen from Fig. 2.2, viscous hydrodynamics methods (IS, DNMR, and Jaiswal-3rd order) fail to reproduce the correct asymptotic behavior in the free streaming limit. On the other hand, both aHydro models are able to reproduce the correct asymptotic behavior at large $\eta/s$ with NLO aHydro reproducing the exact solution quite well. LO aHydro captures the behavior at large $\eta/s$ qualitatively, but quantitatively differs from the exact solution.
Pressure anisotropy

In Fig. 2.3, we show comparisons of the pressure ratio as a function of proper time predicted by the exact solution of Boltzmann equation and compared against different hydro methods: IS, DNMR, and aHydro [17, 15]. In the left panel, $\xi_0$ is taken to be 0 (initially the system is taken to be isotropic) where in the right panel, $\xi_0$ is taken to be 10 (initially the system is taken to be anisotropic). In both panels, the top, middle, bottom rows corresponds to $4\pi \eta/s = 1, 3, 10$ respectively. As we can see from these comparisons aHydro shows the best agreement when compared to other viscous methods (IS, and DNMR) even for quite large $\eta/s$ as shown in the bottom row where $4\pi \eta/s = 10$. We note that DNMR is not quite off from the exact solution and does quite better than the IS method.

In Fig. 2.4, we present other results where the authors found the agreement between aHydro and exact solutions of Boltzmann equation by a suitable choice of moments [18]. In Fig. 2.4, the temperature evolution and the pressure anisotropy predicted by aHydro versus the exact solution for different shear viscosity to entropy density ratios are shown for two different sets of initial conditions as shown in the figure. In each figure, the comparisons are presented for different $4\pi \eta/s = 1, 10, 100$ (i.e., different $\tau_{eq}$ since in the conformal limit the relaxation time is given by $\tau_{eq} = 5\eta/(Ts)$) and one extra case where $\tau_{eq} = 1$. As we can see from these comparisons aHydro was able to reproduce the exact solution for each case very well even for large $\eta/s$, e.g. $4\pi \eta/s = 100$.

2.2.2 Non-conformal 0+1d systems

In this section, I will present some comparisons for non-conformal (massive) 0+1d systems where exact solutions are taken from Ref. [124]. In Fig. 2.5, we show comparisons of the pressure anisotropy (left column) and the bulk pressure (right column) as a function of proper time where in the top row $\xi_0 = 0$ and in the bottom row $\xi_0 = 100$ [19]. In this figure, vHydro, LO aHydro, and vaHydro are tested against the exact solution in the non-conformal
Figure 2.4: Comparisons of temperature evolution (top row) and pressure anisotropy (bottom row) as a function of proper time predicted by the exact solution of Boltzmann equation in conformal 0+1d systems versus aHydro at different $\tau_{eq}$ or equivalently at different $\eta/s$. In the left panel, $\xi_0 = 0$, and in the right panel $\xi_0 = 100$. Figure taken from Ref. [18]

0+1d systems. As we can see from these comparisons NLO aHydro (vaHydro) was able to reproduce the exact solution quite well where LO aHydro and vHydro in this case have some differences with the exact solutions.

### 2.2.3 Gubser flow in 1+1d

In the last section we considered the simplest possible transversely and longitudinally expanding system allowed by symmetries. So, let’s look for another system which has different symmetries (1+1d), a system undergoing Gubser flow[125, 126], boost invariant and cylindrically symmetric expansion. In Fig. 2.6, the de Sitter-space effective temperature ($\bar{T}$) and the scaled shear ($\bar{\pi}_\zeta$) are plotted as a function of de Sitter-space time-like $\rho$. The de Sitter-space time-like $\rho$ is related to proper time in a complicated way, however when
Figure 2.5: Comparisons of pressure anisotropy (left column) and bulk pressure (right column) as a function of proper time. In the top, bottom rows $\xi_0$ is taken to be $0$ and $100$ respectively. The predicted results by exact solution is compared versus vHydro, LO aHydro, and vaHydro for non-conformal $0+1$d systems. Figure taken from Ref. [19].

$\rho \rightarrow -\infty (\tau \rightarrow 0)$ and when $\rho \rightarrow \infty (\tau \rightarrow \infty)$. For more details I refer the reader to Ref. [20]. As we can see from Fig. 2.6, we can not tell much about the differences between hydro methods, I-S, DNMR, and aHydro when compared to the exact solution [127, 128] by looking only to the effective temperature. However, by looking at the scaled shear tensor component, we see that aHydro was able to reproduce the exact solution much better than other viscous methods.
Figure 2.6: Comparisons of de Sitter effect temperature (top row) and the scaled shear (bottom row) as a function of de Sitter “time” $\rho$ where in the left, middle, right columns $4\pi\eta/s = 1, 3, \text{ and } 10$ respectively. The IS, DNMR, aHydro methods are tested against the exact solution of Boltzmann equation. Figure taken from Ref. [20]

2.3 Quasiparticle anisotropic hydrodynamics

This dissertation is mainly about quasiparticle anisotropic hydrodynamics which I will cover in the coming three chapters. In the remaining of this chapter I will introduce quasiparticle anisotropic hydrodynamics. In Chap. 3, I will introduce quasiparticle anisotropic hydrodynamics for boost invariant transversely homogeneous systems $(0+1)d$ and compare results with standard anisotropic hydrodynamics (aHydro or standard aHydro). In Chap. 4, I will continue our comparisons of these two aHydro approaches, but this time in $1+1d$. We will compare this time both of aHydro approaches with second order viscous hydrodynamics. Finally, in Chap. 5, I will present results of $3+1d$ quasiparticle anisotropic hydrodynamics only and compare this time with experimental results with ALICE collaboration. For the
Table 2.1: Parameters of the function defined in Eq. (2.15).

<table>
<thead>
<tr>
<th>$h_0$</th>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$f_0$</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$g_1$</th>
<th>$g_2$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1396</td>
<td>-0.1800</td>
<td>0.0350</td>
<td>2.76</td>
<td>6.79</td>
<td>-5.29</td>
<td>-0.47</td>
<td>1.04</td>
<td>0.01</td>
</tr>
</tbody>
</table>

next section, I will introduce the quasiparticle and standard anisotropic hydrodynamics formalism.

2.3.1 Equation of state

In general, besides the hydrodynamic equations, one needs an extra equation to close the system (number of equations equal number of degrees of freedom). This equation comes from what is known as the equation of state, $P = P(\mathcal{E})$, which relates the equilibrium pressure to the energy density. As an example, let’s consider the pressure and energy density of quarks and gluons at high temperatures (the Stefan-Boltzmann (SB) limit) is given by

$$P_{SB} = \frac{E_{SB}}{3} = \frac{3}{\pi^{2}} \left[ T_{4}^{4} \right] = \frac{\pi^{2}T_{4}^{4}}{45} \left( N_{c}^{2} - 1 + \frac{7}{4}N_{c}N_{f} \right), \quad (2.14)$$

here, the number of flavors ($N_{f}$) and the number of colors ($N_{c}$) are taken to be $N_{c} = N_{f} = 3$. We can’t use only these relations since QGP cools down and the degrees of freedom change from quarks and gluons to hadrons. The way to determine the full equation of state (EoS) is lattice QCD where the thermodynamic functions are determined in a non-perturbative way from the first principles. In this study, we use an analytic parameterization provided by lattice data from the Wuppertal-Budapest collaboration for the trace anomaly defined by

$I_{eq} = \mathcal{E}_{eq} - 3P_{eq} [39]$  

$$\frac{I_{eq}(T)}{T^{4}} = \left[ \frac{h_0}{1 + h_3t^2} + \frac{f_0}{1 + g_1t + g_2t^2} \left[ \tanh(f_1t + f_2) + 1 \right] \right] \exp \left( -\frac{h_1}{t} - \frac{h_2}{t^2} \right), \quad (2.15)$$

where $t \equiv T/(0.2 \text{ GeV})$ and the remaining parameters are defined in Table. 2.1.
Using this parameterization of the interaction measure, both the pressure and energy density can be obtained as

\[
\frac{P_{\text{eq}}(T)}{T^4} = \int_0^T \frac{dT}{T} \frac{I_{\text{eq}}(T)}{T^4},
\]

(2.16)

\[
\varepsilon_{\text{eq}}(T) = 3P_{\text{eq}} + I_{\text{eq}}(T),
\]

(2.17)

with \( P_{\text{eq}}(T = 0) = 0 \) since the vacuum pressure vanishes at \( T = 0 \). In Fig. 2.7, we show the temperature dependence of the energy density, pressure, and the speed of sound squared \( c_s^2 \). As expected, at high temperatures, \( \varepsilon, P, \) and \( c_s^2 \) all approach the ideal limit.

### Quasiparticle equation of state

As we know, QCD is a non-conformal theory with a running-coupling constant that depends on the temperature. For high temperatures, \( T \gg \Lambda_{\text{QCD}} \), one can rely on perturbative methods to get some insight into the properties of the QGP. As some perturbative methods suggested like hard thermal loop (HTL) resummation, QGP can be described as a system of massive quasiparticles with temperature dependent mass \( m(T) \)[129, 130, 131, 132, 133, 134].
In this section, we will introduce a consistent way of implementing the equation of state assuming full non-conformality of the system. Then we will use this formalism for the equation of state in deriving the dynamical equations of anisotropic hydrodynamics, which we will refer to as quasiparticle anisotropic hydrodynamics. To implement the equation of state in a consistent way while ensuring the thermodynamic consistency, one must introduce a background field $B(T)$

$$T_{\mu\nu}^{\text{eq}} = T_{\mu\nu}^{\text{kinetic,eq}} + g^{\mu\nu} B_{\text{eq}}, \quad (2.18)$$

with $B_{\text{eq}} \equiv B_{\text{eq}}(T)$ being the additional background contribution as we will define later on. The first part of Eq. (2.18) represents the kinetic contribution to the energy momentum tensor

$$T_{\mu\nu}^{\text{kinetic,eq}} = \int dP \, p^\mu p^\nu f_{\text{eq}}(x,p). \quad (2.19)$$

where as defined before

$$dP \equiv \frac{N_{\text{dof}}}{(2\pi)^3} \frac{d^3 p}{p^0} \quad (2.20)$$

Next, let’s see the effects of introducing the background field $B_{\text{eq}}$ on the bulk thermodynamic variables for an equilibrium massive Boltzmann gas

$$n_{\text{eq}}(T,m) = 4\pi \tilde{N} T^3 \dot{m}_{\text{eq}} K_2 (\dot{m}_{\text{eq}}), \quad (2.21)$$

$$S_{\text{eq}}(T,m) = 4\pi \tilde{N} T^3 \dot{m}_{\text{eq}} [4K_2 (\dot{m}_{\text{eq}}) + \dot{m}_{\text{eq}} K_1 (\dot{m}_{\text{eq}})], \quad (2.22)$$

$$E_{\text{eq}}(T,m) = 4\pi \tilde{N} T^4 \dot{m}_{\text{eq}} [3K_2 (\dot{m}_{\text{eq}}) + \dot{m}_{\text{eq}} K_1 (\dot{m}_{\text{eq}})] + B_{\text{eq}}, \quad (2.23)$$

$$P_{\text{eq}}(T,m) = 4\pi \tilde{N} T^4 \dot{m}_{\text{eq}} K_2 (\dot{m}_{\text{eq}}) - B_{\text{eq}}, \quad (2.24)$$

where $\dot{m}_{\text{eq}} = m/T$. As we can see from Eqs. (2.21) and (2.22), both the number density and entropy density are unchanged due to introducing the background field. However, the energy density is modified by $E_{\text{kinetic}} + B_{\text{eq}}$, and the pressure is modified as well by $P_{\text{kinetic}} - B_{\text{eq}}$. This should hold even in the non-equilibrium case later on when we generalize the method.
Figure 2.8: Panel (a) shows the temperature dependence of the quasiparticle mass scaled by the temperature. Panel (b) shows the temperature dependence of the background field $B_{eq}$ scaled by $T^4$.

Next, to determine $B_{eq}$, one can require the thermodynamic consistency using for example

$$TS_{eq} = \mathcal{E}_{eq} + P_{eq} = T \frac{\partial P_{eq}}{\partial T}, \quad (2.25)$$

For this identity to hold, using Eqs. (2.23), (2.24), and (2.25) one obtains

$$\frac{dB_{eq}}{dT} = - \frac{1}{2} \frac{dm^2}{dT} \int dP f_{eq}(x, p)$$

$$= - 4\pi N \tilde{m}^2 TK_1(\hat{m}_{eq}) \frac{dm}{dT}. \quad (2.26)$$

Now, we are left with finding $m(T)$ which is easily obtained from lattice results by knowing the entropy density using

$$\mathcal{E}_{eq} + P_{eq} = TS_{eq} = 4\pi N T^4 \tilde{m}_{eq}^3 K_3(\hat{m}_{eq}). \quad (2.27)$$

By solving the differential equation Eq. (2.26), one can get $B_{eq}$ and that is all what we need beside the thermal mass to have a full quasiparticle description. In Fig. 2.8-a, we show the temperature dependence of the quasiparticle mass scaled by the temperature. As we can see, at high temperatures the mass depends linearly on $T$, as suggested by perturbative
calculations such as hard thermal loop resummation[129, 130, 131, 132, 133, 134]. In Fig. 2.8-b, we show the temperature dependence of the background field $B_{\text{eq}}$ scaled by $T^4$ which is easily obtained by integrating Eq. (2.26).

### 2.3.2 Moments of Boltzmann Equation

#### Quasiparticle aHydro

The Boltzmann equation for thermal quasiparticles is

$$p^\mu \partial_\mu f(x,p) + \frac{1}{2} \partial_\mu m^2 \partial_\mu f(x,p) = -C[f(x,p)],$$

(2.28)

where $f(x,p)$ is the anisotropic distribution function specified in Eq. (2.12).

To find the bulk properties of the system, one takes the zeroth, first, and second moments of Boltzmann equation

$$\partial_\mu J^\mu = -\int dP C[f],$$

(2.29)

$$\partial_\mu T^{\mu\nu} = -\int dP p^\nu C[f],$$

(2.30)

$$\partial_\mu I^{\mu\nu\lambda} - J^{(\nu} \partial^{\lambda)} m^2 = -\int dP p^{\nu} p^{\lambda} C[f],$$

(2.31)

where in this case the particle four-current $J^\mu$, the energy-momentum tensor $T^{\mu\nu}$, and the rank-three tensor $I^{\mu\nu\lambda}$ are defined as

$$J^\mu \equiv \int dP p^\mu f(x,p),$$

(2.32)

$$T^{\mu\nu} \equiv \int dP p^\mu p^\nu f(x,p) + B(\alpha, \lambda) g^{\mu\nu},$$

(2.33)

$$I^{\mu\nu\lambda} \equiv \int dP p^\mu p^\nu p^\lambda f(x,p).$$

(2.34)

In general we take the background field, $B(\alpha, \lambda)$, to be a non-equilibrium field. As explained in the case of the equilibrium background field, in order to conserve the energy-momentum tensor, there should be a relation between the background field and the thermal mass to be satisfied, namely

$$\partial_\mu B = -\frac{1}{2} \partial_\mu m^2 \int dP f(x,p).$$

(2.35)
Standard aHydro

In the case of a relativistic massless gas, we take the massless limit of the Boltzmann equation which gives

\[ p^\mu \partial_\mu f(x, p) = -C[f(x, p)], \tag{2.36} \]

where \( f(x, p) \) is the anisotropic distribution function specified in Eq. (2.12) with \( m = 0 \)

\[ f(x, p) = f_{\text{iso}} \left( \frac{1}{\lambda} \sqrt{\sum_i \frac{p_i^2}{\alpha_i^2}} \right), \tag{2.37} \]

All the equations in the quasiparticle model can be used in this case with the appropriate distribution function defined in Eq. (2.37). Beside that, one should take \( m \) and \( B \) to be zero when they appear explicitly in the equations. First, by taking the limit \( m \to 0 \) and \( B = 0 \) of Eq. (2.33), the energy-momentum tensor \( T^{\mu\nu} \) is defined as

\[ T^{\mu\nu} \equiv \int dP p^\mu p^\nu f(x, p). \tag{2.38} \]

Second, the second moment defined in Eq. (2.31) is

\[ \partial_\mu T^{\mu\nu\lambda} = -\int dP p^\nu p^\lambda C[f], \tag{2.39} \]

2.3.3 Bulk variables

Quasiparticle method

To find the number density, energy density, and the pressures, we expand \( J^\mu \) and \( T^{\mu\nu} \) over the basis vectors. The particle four-current \( J^\mu \equiv (n, J) \) in this case is

\[ J^\mu = nu^\mu + J_x X^\mu + J_y Y^\mu + J_z Z^\mu, \tag{2.40} \]

where \( u^\mu \) is the four-vector velocity, \( n \) is the number density, and \( J_i \) is the \( i \)-th component of the three-vector current. By appropriate projections, one finds

\[ J_x = J_y = J_z = 0, \tag{2.41} \]
and the only non-vanishing term is

\[ J^\mu = n u^\mu , \]  \hfill (2.42)

where \( n = \alpha n_{eq}(\lambda, m) \) and \( \alpha \equiv \alpha_x \alpha_y \alpha_z \).

The energy-momentum tensor \( T^{\mu\nu} \) can be expanded using the basis vectors as

\[ T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}_x X^\mu X^\nu + \mathcal{P}_y Y^\mu Y^\nu + \mathcal{P}_z Z^\mu Z^\nu . \]  \hfill (2.43)

Using Eqs. (2.12), (2.33), and (2.43) and taking projections of \( T^{\mu\nu} \) one can obtain the energy density and the components of pressure

\[ \mathcal{E} = \mathcal{H}_3(\alpha, \hat{m}) \lambda^4 + B , \]
\[ \mathcal{P}_x = \mathcal{H}_{3x}(\alpha, \hat{m}) \lambda^4 - B , \]
\[ \mathcal{P}_y = \mathcal{H}_{3y}(\alpha, \hat{m}) \lambda^4 - B , \]
\[ \mathcal{P}_z = \mathcal{H}_{3z}(\alpha, \hat{m}) \lambda^4 - B , \]  \hfill (2.44)

where \( \hat{m} \equiv m/\lambda \) and the \( \mathcal{H} \)-functions appearing above defined in App. 8.4.1.

**Standard method**

The only changes in this case are the definitions of the energy density and pressure components due to the vanishing of the background field \( B \) and \( m \)

\[ \mathcal{E} = \mathcal{E}_{eq}(\lambda) \tilde{\mathcal{H}}_3(\alpha) \]  \hfill (2.45)
\[ \mathcal{P}_i = \mathcal{P}_{eq}(\lambda) \tilde{\mathcal{H}}_{3i}(\alpha) . \]  \hfill (2.46)

with \( i \in \{ x, y, z \} \) and the \( \mathcal{H} \)-functions appearing above defined in App. 8.4.3.

**2.3.4 Dynamical equations**

**Quasiparticle method**

The zeroth moment gives the evolution equation for the particle four-current

\[ \partial_\mu J^\mu = \frac{1}{\tau_{eq}} (n_{eq} - n) . \]  \hfill (2.47)
Using Eq. (2.42) one has

\[ D_u n + n \theta_u = \frac{1}{\tau_{eq}} (n_{eq} - n) . \]  

(2.48)

The first moment gives the energy and momentum conservation which by using Eq. (2.43) and taking \( U^- \), \( X^- \), \( Y^- \), and \( Z^- \)-projections, one obtains four independent equations

\begin{align*}
D_u \mathcal{E} + \mathcal{E} \theta_u + \mathcal{P}_x u_{\mu} D_x X^\mu + \mathcal{P}_y u_{\mu} D_y Y^\mu + \mathcal{P}_z u_{\mu} D_z Z^\mu &= 0 , \\
D_x \mathcal{P}_x + \mathcal{P}_x \theta_x - \mathcal{E} X_{\mu} D_{\mu} u^\mu - \mathcal{P}_y X_{\mu} D_y Y^\mu - \mathcal{P}_z X_{\mu} D_z Z^\mu &= 0 , \\
D_y \mathcal{P}_y + \mathcal{P}_y \theta_y - \mathcal{E} Y_{\mu} D_{\mu} u^\mu - \mathcal{P}_x Y_{\mu} D_x X^\mu - \mathcal{P}_z Y_{\mu} D_z Z^\mu &= 0 , \\
D_z \mathcal{P}_z + \mathcal{P}_z \theta_z - \mathcal{E} Z_{\mu} D_{\mu} u^\mu - \mathcal{P}_x Z_{\mu} D_x X^\mu - \mathcal{P}_y Z_{\mu} D_y Y^\mu &= 0 .
\end{align*}

(2.49)

The second moment of Boltzmann equation gives

\[ \partial_{\mu} \mathcal{I}^{\mu \nu \lambda} - J^{(\nu} \partial^{\lambda)} m^2 = \frac{1}{\tau_{eq}} (u_\mu \mathcal{I}^{\mu \nu \lambda}_{eq} - u_\mu \mathcal{I}^{\mu \nu \lambda}) , \]

(2.50)

where \( \mathcal{I}^{\mu \nu \lambda}_{eq} \) is \( \mathcal{I}^{\mu \nu \lambda} \) with \( f \rightarrow f_{eq} \). \( \mathcal{I}^{\mu \nu \lambda} \) can be expanded over the basis vectors and the non-vanishing terms in Eq. (2.34) are those with an even number of the same spatial index

\[ \mathcal{I} = \mathcal{I}_u [u \otimes u \otimes u] + \mathcal{I}_x [u \otimes X \otimes X + X \otimes u \otimes X + X \otimes X \otimes u] + \mathcal{I}_y [u \otimes Y \otimes Y + Y \otimes u \otimes Y + Y \otimes Y \otimes u] + \mathcal{I}_z [u \otimes Z \otimes Z + Z \otimes u \otimes Z + Z \otimes Z \otimes u] . \]

(2.51)

By integration using the distribution function specified in (2.12), one finds

\begin{align*}
\mathcal{I}_u &= \left( \sum_i \alpha_i^2 \right) \alpha \mathcal{I}_{eq}(\lambda, m) + \alpha m^2 n_{eq}(\lambda, m) , \\
\mathcal{I}_i &= \alpha \alpha_i^2 \mathcal{I}_{eq}(\lambda, m) ,
\end{align*}

(2.52, 2.53)

where

\[ \mathcal{I}_{eq}(\lambda, m) = 4\pi \tilde{N} \lambda^5 \tilde{m}^3 K_3(\tilde{m}) . \]

(2.54)
By expanding Eq. (2.50) and taking its \(uu\), \(XX\), \(YY\), and \(ZZ\)-projections one finds

\[
D_u \mathcal{I}_u + I_u \theta_u + 2I_x u_\mu D_x X^\mu + 2I_y u_\mu D_y Y^\mu + 2I_z u_\mu D_z Z^\mu \\
- n D_u m^2 = \frac{1}{\tau_{eq}} (I_{u,eq} - I_u), \quad (2.55)
\]

\[
D_u \mathcal{I}_x + I_x (\theta_u + 2u_\mu D_x X^\mu) = \frac{1}{\tau_{eq}} (I_{eq} - I_x), \quad (2.56)
\]

\[
D_u \mathcal{I}_y + I_y (\theta_u + 2u_\mu D_y Y^\mu) = \frac{1}{\tau_{eq}} (I_{eq} - I_y), \quad (2.57)
\]

\[
D_u \mathcal{I}_z + I_z (\theta_u + 2u_\mu D_z Z^\mu) = \frac{1}{\tau_{eq}} (I_{eq} - I_z). \quad (2.58)
\]

Then, taking \(uX\), \(uY\), and \(uZ\)-projections gives

\[
D_x \mathcal{I}_x + I_x \theta_x + (I_x + I_u) u_\mu D_x X^\mu - I_x X_\mu D_y Y^\mu - I_x X_\mu D_z Z^\mu - \frac{1}{2} n D_x m^2 = 0, \quad (2.59)
\]

\[
D_y \mathcal{I}_y + I_y \theta_y + (I_y + I_u) u_\mu D_y Y^\mu - I_y Y_\mu D_x X^\mu - I_y Y_\mu D_z Z^\mu - \frac{1}{2} n D_y m^2 = 0, \quad (2.60)
\]

\[
D_z \mathcal{I}_z + I_z \theta_z + (I_z + I_u) u_\mu D_z Z^\mu - I_z Z_\mu D_x X^\mu - I_z Z_\mu D_y Y^\mu - \frac{1}{2} n D_z m^2 = 0, \quad (2.61)
\]

finally projecting with \(XY\), \(XZ\), and \(YZ\) one finds

\[
\mathcal{I}_x (Y_\mu D_x X^\mu + Y_\mu D_x u^\mu) + \mathcal{I}_y (X_\mu D_y Y^\mu + X_\mu D_y u^\mu) = 0, \quad (2.62)
\]

\[
\mathcal{I}_z (Z_\mu D_z X^\mu + Z_\mu D_z u^\mu) + \mathcal{I}_z (X_\mu D_z Z^\mu + X_\mu D_z u^\mu) = 0, \quad (2.63)
\]

\[
\mathcal{I}_y (Z_\mu D_y Y^\mu + Z_\mu D_y u^\mu) + \mathcal{I}_z (Y_\mu D_z Z^\mu + Y_\mu D_z u^\mu) = 0. \quad (2.64)
\]

We should note here that Eq. (2.55) is not independent which means that we will not use it as a dynamical equation. This can be shown by subtracting the sum of Eqs. (2.56)-(2.58) from it to obtain

\[
m^2 (D_u n + n \theta_u) = m^2 \left( \frac{\tau_{eq}}{\tau_{eq}} - n \right). \quad (2.65)
\]

which is the same as Eq. (2.48) for non-vanishing mass.

**Standard method**

In the standard method, the dynamical equations are the same as in the quasiparticle method defined above but with the limit \(m \to 0\). For example, in Eqs. (2.59)-(2.61) the last
term in each equation vanishes because it has \( m \) explicitly. Also, the definitions of \( \mathcal{I}_u, \mathcal{I}_i \) change to
\[
\begin{align*}
\mathcal{I}_u &= \left( \sum_i \alpha_i^2 \right) \alpha \mathcal{I}_{eq}(\lambda), \\
\mathcal{I}_i &= \alpha \alpha_i^2 \mathcal{I}_{eq}(\lambda); \quad (i = x, y, z),
\end{align*}
\]
with \( \mathcal{I}_{eq}(\lambda) = 32\pi \tilde{N} \lambda^5 \).

\section*{2.3.5 The relaxation time \( \tau_{eq} \)}

As can be seen from the last section, the collisional kernel in the relaxation-time approximation depends on the relaxation time. In this section, we will introduce a realistic model for the relaxation time which will be used through the study for both aHydro methods. In the quasiparticle method, we use a realistic model for \( \tau_{eq} \) which is related to the shear viscosity to entropy density ratio for massive gases \([135, 136]\). In this case \( \tau_{eq} \) is given by
\[
\eta(T) = \tau_{eq}(T) \mathcal{P}_{eq}(T) \frac{\kappa(\hat{m}_{eq})}{15},
\]
where \( \kappa(x) \) here is defined as
\[
\kappa(x) \equiv x^3 \left[ \frac{3}{x^2 K_2(x)} - \frac{1}{x} + \frac{K_1(x)}{K_2(x)} \right]
- \frac{\pi}{2} \frac{1 - x K_0(x) L_{-1}(x) - x K_1(x) L_0(x)}{K_2(x)}.
\]
where \( K_n(x) \) are modified Bessel functions of second kind and \( L_n(x) \) are modified Struve functions. In this dissertation, we assume that the ratio of the shear viscosity to entropy density \( \eta/\mathcal{S}_{eq} \equiv \bar{\eta} \) is constant, then one can write the \( \tau_{eq} \) from Eq. (2.68) as
\[
\tau_{eq}(T) = \frac{15 \bar{\eta}}{\kappa(\hat{m}_{eq}) T} \left( 1 + \frac{\mathcal{E}_{eq}(T)}{\mathcal{P}_{eq}(T)} \right).
\]
where we used the thermodynamic relation \( \mathcal{S}_{eq} = (\mathcal{E}_{eq} + \mathcal{P}_{eq})/T \).

In the standard method, where we can use \( m \to 0 \) and \( \kappa(\hat{m}_{eq}) \to 12 \), the \( \tau_{eq} \) is given by
\[
\tau_{eq}(T) = \frac{5 \eta}{4 \mathcal{P}_{eq}(T)}.
\]
which is the massless limit of Eq. (2.70)
2.4 Anisotropic Freeze-out

As explained in the introduction, QGP undergoes a couple of stages before the produced particles can be detected by the detector. At low temperatures, the QGP undergoes a transition between quarks and gluon degrees of freedom to hadrons. In the hydrodynamic stage the relevant quantities that one deals with are the energy density, pressure components, transport coefficients while on the experimental side the measured quantities are the particles momenta and energies. As a result, one should look for a procedure to connect these two different stages and the usual procedure is the Cooper-Frye freeze-out [137]. The Cooper-Frye freeze-out is based on the equivalence of the energy momentum-tensor in both stages [138]

\[ T^\mu_\text{kinetic} = \int dP p^\mu p^\nu f(x,p) = T^\mu_\text{hydro}. \]  

In the reminder of this section, I will follow closely the discussions presented in [59, 111]. To see how Cooper-Frye freeze-out works, let’s write the number of particles in a small volume \( dV \) [59]

\[ dN = dV \int d^3p f(x,p). \]  

In a covariant form, this can be written using \( dV = dV_\mu p^\mu / E_p \) as

\[ dN = dV_\mu \int \frac{d^3p}{E_p} p^\mu f(x,p). \]  

Now, in general we will not write \( dV_\mu \) in terms of the 4-vector flow velocity as \( dV_\mu = u_\mu dV \) since many particles can be emitted at early times from the edge of the system. Keeping this in mind, the total number of particles emitted from all fluid elements can be written as

\[ N = \int dV_\mu(x) \int \frac{d^3p}{E_p} p^\mu f(x,p). \]  

Since \( dV_\mu(x) \) can’t be written as \( dV_\mu(x) = dV u^\mu \), one defines, in general, a 3-d hypersurface \( \Sigma_\mu \) which defines the 4-d volume occupied by the fluid at the freeze-out temperature with
a surface normal vector $d^\mu \Sigma_\mu$. The number of particles which decouple at the freeze-out hypersurface is

$$N = \int \frac{d^3p}{E_p} \int d^3\Sigma_\mu p^\mu f(x, p),$$

(2.76)

This formula gives what is called Cooper-Frye formula which counts the number of particles emitted from the fluid through the hypersurface [59, 137]. To do the calculations, one needs to specify an appropriate parameterization to define the hypersurface and its normal. In this study we will use the parametrizations presented in Refs. [74, 75, 139, 140]. This is not the only parameterization in the literature, but it has the advantage that the function $d(\zeta, \phi, \theta)$ is single-valued for most freeze-out surfaces. Let’s present this parametrization in Cartesian-coordinates $(t, x, y, z)$

$$
\begin{align*}
t &= (\tau_0 + d(\zeta, \phi, \theta) \sin \theta \sin \zeta) \cosh \left[ \frac{d(\zeta, \phi, \theta) \cos \theta}{\Lambda} \right], \\
x &= d(\zeta, \phi, \theta) \sin \theta \cos \zeta \cos \phi, \\
y &= d(\zeta, \phi, \theta) \sin \theta \cos \zeta \sin \phi, \\
z &= (\tau_0 + d(\zeta, \phi, \theta) \sin \theta \sin \zeta) \sinh \left[ \frac{d(\zeta, \phi, \theta) \cos \theta}{\Lambda} \right]. \\
\end{align*}
$$

(2.77)

Since we prefer usually to work with Milne coordinates, we rewrite Eq. (2.77) in Milne coordinates,

$$
\begin{align*}
\tau &= \tau_0 + d(\zeta, \phi, \theta) \sin \theta \sin \zeta, \\
r &= d(\zeta, \phi, \theta) \sin \theta \cos \zeta, \\
\phi &= \phi, \\
\zeta &= \frac{d(\zeta, \phi, \theta) \cos \theta}{\Lambda}.
\end{align*}
$$

(2.78)

In Fig. 2.9, we show a diagram to show this parametrization in Milne coordinates.

Now, let’s define what we mean by the above introduced variables. The function $d(\zeta, \phi, \theta)$ is the distance between any point on freeze-out hypersurface to the point $(\tau_0, 0, 0, 0)$. $\theta$ and $\zeta$
Figure 2.9: The system of coordinates used to parametrize the freeze-out hypersurface defined in Eq. (2.78)

are the polar and azimuthal coordinates in this coordinate system. For dimensional reasons the length scale $\Lambda$ is introduced where final results are not affected by such a choice. To construct the normal vector to the hypersurface one takes the derivatives of the orthogonal basis coordinates (2.77) with respect to $\zeta$, $\phi$, and $\theta$

$$d^3\Sigma_\mu = \epsilon_{\mu\alpha\beta\gamma} \frac{\partial x^\alpha}{\partial \zeta} \frac{\partial x^\beta}{\partial \phi} \frac{\partial x^\gamma}{\partial \theta} d\zeta d\phi d\theta,$$

(2.79)

where $\epsilon_{\mu\alpha\beta\gamma}$ is the four-dimensional Levi-Civita symbol.

Now, we can write the total number of particles as a sum over all particle species

$$N = \sum_i \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{m_i^2 + p^2}} (2s_i + 1)(2g_i + 1) \int f_i(x, p) p^\mu d^3\Sigma_\mu,$$

(2.80)

with

$$N = \sum_i \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{m_i^2 + p^2}} (2s_i + 1)(2g_i + 1) \int f_i(x, p) p^\mu d^3\Sigma_\mu,$$

(2.80)

with

where $s_i$ and $g_i$ are the spin and isospin degeneracies of the hadron, $m_i$ is the hadron mass.
mass, and \( f_i \) is the distribution function for the particle species \( i \). Since we have different produced particles with spin-half and spin-one, one needs to consider the appropriate quantum statistics for each particle.

Since we have \( p^\mu \Xi_{\mu\nu} p^\nu \) in the argument of the distribution function, we need to come up with a convenient parameterization for the particle momentum in the lab frame as follows

\[
p^\mu \equiv (m_\perp \cosh y, p_\perp \cos \varphi, p_\perp \sin \varphi, m_\perp \sinh y),
\]

where \( m_\perp = \sqrt{p_\perp^2 + m^2} \), \( y = \tanh^{-1}(p^z/p^0) \) is the particle’s rapidity, and \( \varphi \) is the particle’s azimuthal angle. Using that one finds

\[
\begin{align*}
  p \cdot u &= m_\perp \cosh(\theta_\perp) \cosh(y - \varsigma) - p_\perp \sinh(\theta_\perp) \cos(\phi - \varphi), \\
  p \cdot X &= m_\perp \sinh(\theta_\perp) \cosh(y - \varsigma) - p_\perp \cosh(\theta_\perp) \cos(\phi - \varphi), \\
  p \cdot Y &= p_\perp \sin(\phi - \varphi), \\
  p \cdot Z &= -m_\perp \sinh(y - \varsigma),
\end{align*}
\]

(2.82)

Using that will enable us to calculate the argument of distribution function \( p^\mu \Xi_{\mu\nu} p^\nu \) \[141\]

\[
p^\mu \Xi_{\mu\nu} p^\nu = (1 + \Phi) \left[ m_\perp \cosh \theta_\perp \cosh(y - \varsigma) - p_\perp \sinh \theta_\perp \cos(\phi - \varphi) \right]^2 \\
+ \xi_x \left[ m_\perp \sinh \theta_\perp \cosh(y - \varsigma) - p_\perp \cosh \theta_\perp \cos(\phi - \varphi) \right]^2 \\
+ \xi_y \left[ m_\perp \cosh \theta_\perp \cosh(y - \varsigma) - p_\perp \sinh \theta_\perp \cos(\phi - \varphi) \right]^2 \\
+ \xi_z \left( m_\perp \cosh(y - \varsigma) - p_\perp \sinh \theta_\perp \cos(\phi - \varphi) \right)^2.
\]

(2.83)

Expanding Eq. (2.79) for \( x^\mu = (t, x, y, z) \) and contracting with \( p^\mu \), one obtains \[111\]

\[
\begin{align*}
p^\mu d^3 \Sigma \mu &= \frac{\tau}{\Lambda} \sin \theta d^2 \left[ p_\perp \sin(\phi - \varphi) \frac{\partial d}{\partial \phi} \\
+ \frac{\Lambda}{\tau} m_\perp \cos \zeta \sin \theta \sinh(y - \varsigma) \left( d \cos \theta + \frac{\partial d}{\partial \theta} \sin \theta \right) \\
+ \cos \zeta \sin \theta \left( p_\perp \cos \zeta \cos(\phi - \varphi) + m_\perp \sin \zeta \cosh(y - \varsigma) \right) \left( d \sin \theta - \frac{\partial d}{\partial \theta} \cos \theta \right) \\
+ \cos \zeta \frac{\partial d}{\partial \zeta} \left( p_\perp \sin \zeta \cos(\phi - \varphi) - m_\perp \cos \zeta \cosh(y - \varsigma) \right) \right] d\zeta d\phi d\theta,
\end{align*}
\]

(2.84)

where \( d = d(\zeta, \phi, \theta) \).
2.4.1 Anisotropic Freeze-out between $a_{\text{Hydro}}$ and $v_{\text{Hydro}}$

As can be seen from the last section, by using $a_{\text{Hydro}}$, the distribution function is always positive definite at all space-time points. This is not guaranteed in viscous hydrodynamics where one includes the viscous corrections as correction terms. For example, by including shear and bulk viscous correction, the distribution function becomes $f = f_{\text{eq}} + f_{\text{shear}} + f_{\text{bulk}}$. In this case, it is not guaranteed that the corrections will not be bigger than the leading term. In this study, we show that including the bulk correction can give negative pion numbers in some cases as will be shown later in Chap. 4.

For now, to explain the point, let’s consider the simplest possible form of the viscous hydrodynamics distribution function. Following the discussion in Ref. [15], the first order distribution function (Navier-Stokes theory) in 0+1d systems can be written as

$$f_{\text{NS}}(x, p) = f_{\text{eq}} \left( \frac{E}{T} \right) \left[ 1 + \frac{\eta}{s} \left( \frac{p_x^2 + p_y^2 - 2p_z^2}{3\tau T^3} \right) + \ldots \right] , \quad (2.85)$$

As can be seen from Eq. (2.85), there are some regions of phase space where the distribution function in viscous hydrodynamics can be negative. To see this, in Fig. 2.10-a, we
Figure 2.11: Contour plot of the distribution function specified in Eq. (2.85) times the phase factor \( p^2 f(p) \). Figure is adapted from Ref. [15].

show the distribution function dependence of \( p_L \) with \( p_T = 0 \). In Fig. 2.10-b, we plot the distribution function times the space factor \( p^2 f(p) \) (the integrand of the number density) as a function of \( p_L \) with \( p_T = 0 \). As we can see from both panels, that the distribution function can be negative in some phase space regions. In Fig. 2.11, we show a contour plot of the distribution function \( p^2 f(p) \) as a function of both \( p_L \) and \( p_T \) which shows that possibility of the distribution function being negative in viscous hydrodynamics. In both Fig. 2.10, and Fig. 2.11, we take \( T = 230 \) MeV, \( \tau = 0.6 \) fm/c, and \( \eta/s = 3/(4\pi) \).
Chapter 3

0+1d quasiparticle anisotropic hydrodynamics

In Chap. 2, we presented the formalism and the dynamical equations of both aHydroQP and standard aHydro in 3+1d systems. However, solving for 3+1d is very hard numerically to be done in one step beside one needs more comparisons with other methods before going to the full 3+1d case. Hence, the simplest system that one may consider is the boost-invariant and cylindrically symmetric system (0+1d). In this chapter, I will simplify the dynamical equations presented in the last chapter to the case of 0+1d systems for both aHydro approaches, aHydroQP and standard aHydro presented in details in the last chapter. We will solve the dynamical equations and compare some first order quantities like the temperature and the energy density. We will also compare the pressure anisotropy and the bulk pressure between these two aHydro approaches to see how they are compared to each other. This chapter is mainly based on Ref. [142].

3.1 Dynamical equations in 0+1d systems

In this section, we will present the dynamical equations for both standard aHydro and quasiparticle aHydro in 0+1d systems.

3.1.1 Dynamical equations in aHydroQP

In 0+1d systems, the bulk variables and many dynamical equations can be simplified. First, in a transversally-symmetric system one has \( P_T \equiv P_x = P_y \) and \( P_L \equiv P_z \) and the energy-momentum tensor can be simplified to

\[
T^{\mu \nu} = (E + P_T) u^\mu u^\nu - P_T g^{\mu \nu} + (P_L - P_T) Z^\mu Z^\nu ,
\]

(3.1)
where

\[ E = \tilde{\mathcal{H}}_3(\alpha, \hat{m}) \lambda^4 + B, \]
\[ \mathcal{P}_T = \tilde{\mathcal{H}}_{3T}(\alpha, \hat{m}) \lambda^4 - B, \]
\[ \mathcal{P}_L = \tilde{\mathcal{H}}_{3L}(\alpha, \hat{m}) \lambda^4 - B. \] (3.2)

The various \( \mathcal{H} \)-functions appearing above are defined in App. 8.4.2. There is one quantity that we want to define, which is the bulk pressure

\[ \Pi = \frac{1}{3} (\mathcal{P}_L + 2\mathcal{P}_T) - \mathcal{P}_{eq}. \] (3.3)

The zeroth moment can be written as

\[ \partial_\tau n + \frac{n}{\tau} = \frac{1}{\tau_{eq}} (n_{eq} - n), \] (3.4)

where we used derivatives defined in Apps. 8.3.2.

The first moment can be simplified to

\[ \partial_\tau E = -\frac{E + \mathcal{P}_L}{\tau}, \] (3.5)

with three other trivial equations resulting from the boost invariance and transverse homogeneity of the 0+1d system

\[ \partial_\tau \mathcal{P}_T = \partial_\phi \mathcal{P}_T = \partial_\varsigma \mathcal{P}_L = 0. \] (3.6)

In this case, \( I_x = I_y \) and the second moment equations listed in Eqs. (2.56)-(2.58) simplify to

\[ \partial_\tau \log I_x + \frac{1}{\tau} = \frac{1}{\tau_{eq}} \left( \frac{I_{eq}}{I_x} - 1 \right), \] (3.7)
\[ \partial_\tau \log I_z + \frac{3}{\tau} = \frac{1}{\tau_{eq}} \left( \frac{I_{eq}}{I_z} - 1 \right). \] (3.8)

where the equations in Eqs. (2.59)-(2.64) are trivially satisfied.
In this case, we need four equations to solve for the four unknown variables that we have, \( \alpha_x, \alpha_z, \lambda, \) and \( T. \) Three equation will come from the first and second moments and the fourth equation will be the matching condition. Using Eqs. (3.2), (2.53), and (2.54) one can expand (3.5), (3.7), and (3.8) to obtain

\[
4 \tilde{\mathcal{H}}_3 \partial_\tau \log \lambda + \tilde{\Omega}_m \partial_\tau \log \hat{m} + \tilde{\Omega}_L \partial_\tau \log \alpha_x + \tilde{\Omega}_T \partial_\tau \log \alpha_z^2 + \frac{\partial_\tau B}{\lambda^4} + \frac{\tilde{\Omega}_L}{\tau} = 0, \tag{3.9}
\]

\[
4 \partial_\tau \log \alpha_x + \partial_\tau \log \alpha_z + 5 \partial_\tau \log \lambda + \partial_\tau \log (\hat{m}^3 K_3(\hat{m})) + \frac{1}{\tau} \]

\[
= \frac{1}{\tau_{eq}} \left[ \frac{1}{\alpha_x^2 \alpha_z} \left( \frac{T}{\lambda} \right)^2 K_3(\hat{m}_{eq}) - 1 \right], \tag{3.10}
\]

\[
2 \partial_\tau \log \alpha_x + 3 \partial_\tau \log \alpha_z + 5 \partial_\tau \log \lambda + \partial_\tau \log (\hat{m}^3 K_3(\hat{m})) + \frac{3}{\tau} \]

\[
= \frac{1}{\tau_{eq}} \left[ \frac{1}{\alpha_x^2 \alpha_z^3} \left( \frac{T}{\lambda} \right)^2 K_3(\hat{m}_{eq}) - 1 \right], \tag{3.11}
\]

where the functions \( \tilde{\Omega}_T, \tilde{\Omega}_L, \) and \( \tilde{\Omega}_m \) are defined in App. 8.4.2.

The last equation we need is the matching condition which is a result of the energy-momentum conservation \( \partial_\mu T^{\mu\nu} = 0. \) This means that both sides of Eq. (2.30) vanish and \( T \) can then be written in terms of \( \alpha \) and \( \lambda. \) Using (2.12) in the RTA approximation one obtains

\[
\mathcal{E}_{\text{kinetic}} = \mathcal{E}_{\text{kinetic, eq}} \text{,} \tag{3.12}
\]

which gives

\[
\tilde{\mathcal{H}}_3 \lambda^4 = \tilde{\mathcal{H}}_{3, eq} T^4. \tag{3.13}
\]

One can use root solver to solve Eq. (3.13), but instead it is much easier to transform it as a differential equation and then solve for a system of coupled differential equation. Taking the proper time derivative of Eq. (3.13) and using Eq. (3.5), one obtains

\[
4 \tilde{\mathcal{H}}_{3, eq} \partial_\tau \log T + \tilde{\Omega}_{m, eq} \partial_\tau \log \hat{m}_{eq} + \frac{\tilde{\Omega}_L}{\tau} \left( \frac{\lambda}{T} \right)^4 + \frac{\partial_\tau B}{T^4} = 0. \tag{3.14}
\]

There is one problem that one faces when using thermal masses in the first moment equations since they involved the background contribution \( B \) and it’s derivatives, for example in
0+1d beside $B$, there is its proper-time derivative. In practice, the $B$ proper-time derivative can be written in terms of derivatives of $m$ using Eq. (2.35).

$$\partial_\tau B = -\frac{\lambda^2}{2} \tilde{H}_3 B(\alpha, \hat{m}) \partial_\tau m^2.$$ \hspace{1cm} (3.15)

However, to get the full energy density, and pressure components one needs to know $B$ itself as shown in Eq. (2.44). Since the system reaches equilibrium at late times, we can integrate the dynamical equations from $\tau_f$ to the initial time $\tau_0$. The advantage of this procedure is to use the boundary condition $B(\tau_f) = B_{eq}(T(\tau_f))$ where $B_{eq}$ is provided from the realistic EoS.

### 3.1.2 Dynamical equations of Standard aHydro

In this section, we present the details of the “standard EoS” method where the particles are taken to be massless which yields to $B \to 0$. For the massless transversally-symmetric case (0+1)d, Eqs. (3.2) become

$$\mathcal{E} = \tilde{H}_3(\alpha) \lambda^4,$$

$$\mathcal{P}_T = \tilde{H}_{3T}(\alpha) \lambda^4,$$

$$\mathcal{P}_L = \tilde{H}_{3L}(\alpha) \lambda^4.$$ \hspace{1cm} (3.16)

where all $\tilde{H}$-functions are defined in App. 8.4.3.

By looking at the above equations in the standard aHydro, we see that there is a multiplicative factorization of the energy density and pressures into a function that only depends on the anisotropy parameters $\alpha$ and another function which only depends on the temperature-like scale $\lambda$. The idea that the standard method of the EoS relies on in aHydro is to relate the isotropic part to the energy density and pressures of a massless conformal
Figure 3.1: The three panels show: (a) the temperature-like parameter \( \lambda \) scaled by the initial temperature-like \( \lambda_0 \) as a function of the proper time, (b) the anisotropy parameter \( \alpha_x \) as a function of the proper time, and finally (c) the anisotropy parameter \( \alpha_z \) as a function of the proper time. For this figure we compared predictions of aHydroQP and aHydro and took \( 4\pi \eta/s = 1 \).

gas where

\[
\mathcal{E}_{eq}(T) = 24\pi \tilde{N}T^4, \\
\mathcal{P}_{eq}(T) = 8\pi \tilde{N}T^4,
\]

(3.17)

Using the above relations, one can use the multiplicative factorization to rewrite Eqs. (3.16) in terms of the equilibrium thermodynamic functions, then one can impose the realistic EoS by replacing \( \mathcal{E}_{eq}(\lambda) \) and \( \mathcal{P}_{eq}(\lambda) \) by lattice QCD calculations.

\[
\mathcal{E} = \frac{\mathcal{E}_{eq}(\lambda)}{2} \alpha_x^4 \bar{H}_2 \left( \frac{\alpha_z}{\alpha_x} \right), \\
\mathcal{P}_T = \frac{3\mathcal{P}_{eq}(\lambda)}{4} \alpha_x^4 \bar{H}_{2T} \left( \frac{\alpha_z}{\alpha_x} \right), \\
\mathcal{P}_L = \frac{3\mathcal{P}_{eq}(\lambda)}{2} \alpha_x^4 \bar{H}_{2L} \left( \frac{\alpha_z}{\alpha_x} \right).
\]

(3.18)

To obtain the dynamical equations in this case, one needs to take the massless limit and \( B \to 0 \) of the equations obtained earlier in the quasiparticle case and then substitute \( \mathcal{E} \) and \( \mathcal{P}_{T,L} \) from Eq. (3.18). Starting from Eq. (3.5) and using Eq. (3.18) one obtains the first
moment equation,

\[ \partial_\tau \log \mathcal{E}_{eq}(\lambda) + (1 + \chi) \partial_\tau \log \alpha_z + (3 - \chi) \partial_\tau \log \alpha_x = -\frac{1}{\tau} - \frac{3P_{eq}(\lambda)}{\tau \mathcal{E}_{eq}(\lambda)} \chi, \quad (3.19) \]

with \( \chi \equiv \tilde{H}_2 \tilde{L} / \tilde{H}_2 \). The second-moment equations, which are obtained by taking the massless limit of (3.10) and (3.11), are

\[ 4 \partial_\tau \log \alpha_x + \partial_\tau \log \alpha_z + 5 \partial_\tau \log \lambda + \frac{1}{\tau} = \frac{1}{\tau_{eq}} \left[ \left( \frac{T}{\lambda} \right)^5 \frac{1}{\alpha_x^4 \alpha_z} - 1 \right], \quad (3.20) \]

\[ 2 \partial_\tau \log \alpha_x + 3 \partial_\tau \log \alpha_z + 5 \partial_\tau \log \lambda + \frac{3}{\tau} = \frac{1}{\tau_{eq}} \left[ \left( \frac{T}{\lambda} \right)^5 \frac{1}{\alpha_x^2 \alpha_z^3} - 1 \right]. \quad (3.21) \]

Finally, the matching condition in this case is

\[ \partial_\tau \log \mathcal{E}_{eq}(T) = -\frac{1}{\tau} - \frac{3P_{eq}(\lambda)}{\tau \mathcal{E}_{eq}(\lambda)} \chi. \quad (3.22) \]

where \( \tau_{eq} \) is defined earlier in Eq. 2.71.

### 3.2 Numerical results

Here, we present our numerical results of solving the dynamical equations specified in the last section for 0+1d case. We compare the results of the “standard EoS” and the “quasiparticle EoS” methods. We take the initial proper time to be \( \tau_0 = 0.25 \text{ fm/c} \), the final time to be \( \tau_f = 500 \text{ fm/c} \), and the initial temperature to be \( T_0 = 600 \text{ MeV} \). This final time...
is really very long but we went to late times to assure that the system is reaching equilibrium to get the full energy density and pressures by finding $B(\tau)$ using $B(\tau_f) = B_{eq}(T(\tau_f))$. When comparing both the "standard EoS" and the "quasiparticle EoS" methods, we match the physical quantities rather than the model parameters like $\alpha$’s. The physical quantities we match at the initial temperature $T_0$ are $P_L(T_0)/P_T(T_0)$, and the initial bulk correction $\Pi(\tau_0)$.

Now, let’s turn to our numerical results where, in Fig. 3.1-Fig. 3.3, we show in panel (a) the temperature-like parameter $\lambda$ scaled by the initial temperature-like $\lambda_0$ as a function of the proper time, (b) the anisotropy parameter $\alpha_x$ as a function of the proper time, and in panel (c) the anisotropy parameter $\alpha_z$ as a function of the proper time. In each figure we assumed a different value of the $\eta/s = 1, 3,$ and 10 where we see that the differences between methods increase by increasing $\eta/s$. We should note here that they are not initially the same because we fixed at the initial time the physical observables as discussed before.

In Figs. 3.4 - 3.7, there are four panels in each figure which corresponds to: (a) the effective temperature scaled by the initial temperature $T/T_0$ as a function of proper time $\tau$, (b) $(\tau/\tau_0)^{4/3}$ times the energy density scaled by the initial energy density $E_0$, (c) the pressure anisotropy $P_L/P_T$, and finally in panel (d) the bulk correction scaled by the equilibrium pressure, $\Pi/P_{eq}$. 

Figure 3.3: Same as Fig. 3.1 except here we take $4\pi\eta/s = 10$. 

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Figure 3.4: The four panels show: (a) the effective temperature scaled by $T_0$, (b) $(\tau/\tau_0)^{4/3}$ times the energy density scaled by the initial energy density, $\mathcal{E}_0$, (c) the pressure anisotropy, and (d) the bulk correction to the pressure scaled by $P_{\text{eq}}$ where in all panels we took $4\pi\eta/s = 1$.

**Isotropic initial conditions:** First, we consider the case when the initial conditions are isotropic, i.e., $\alpha_x = \alpha_z = 1$, which gives $P_L(T_0)/P_T(T_0) = 1$ and $\Pi(\tau_0) = 0$ in Figs. 3.4 - 3.6. In panel (a) of all three figures, we see that both aHydro approaches have a good agreement describing the effective temperature where the maximum difference between them is on the order of 1%. Next, in panel (b) of all three figures, we present the scaled energy density multiplied by a factor of $(\tau/\tau_0)^{4/3}$. By multiplying the energy density by this factor, it should
Figure 3.5: Same as Fig. 3.4 except here we take $4\pi\eta/s = 3$.

approach unity at late times when the system approach ideal behavior. From this figure, we see that both approaches are in quite good agreement with the maximum difference being approximately 4%. In panel (c) of all three figures, we compare results for the pressure anisotropy predicted by both approaches where we see some quantitative differences, but with the same overall qualitative behavior. The maximum differences seen in this figure were less than or in the order of 20%. Finally, in panel (d) of Figs. 3.4 - 3.6 we show the bulk pressure scaled by the equilibrium pressure $\Pi/P_{\text{eq}}$ where we see clear differences qualitatively between the two methods. However, one should note that they approach the same limits asymptotically as expected since at late times they should be approaching equilibrium.
Anisotropic initial conditions: The second case we consider is taking the initial conditions to be anisotropic, i.e., $\alpha_x = \alpha_z \neq 1$ and investigate how the two methods compare to each other in Fig. 3.7. We see similar results as in the isotropic initial conditions where both the effective temperature and the energy density are in quite good agreement between both methods. On the other hand, the pressure anisotropy shows some differences and the bulk corrections shows some large quantitative differences.
Figure 3.7: Same as Fig. 3.5 except here we take anisotropic initial conditions.
Chapter 4

1+1d quasiparticle anisotropic hydrodynamics

In Chap. 3, we presented our results of comparing aHydroQP and standard aHydro in a 0+1d system. We noticed some quantitative differences in the pressure anisotropy ($P_L/P_T$), but more importantly clear differences in the evolution of the bulk pressure scaled by the equilibrium pressure ($\Pi/P_{eq}$). Given such discrepancies, we need to investigate more the differences between these two methods. The next step should be studying 1+1d systems which are boost invariant and azimuthally-symmetric to compare particle spectra predicted by these two methods. If they turn out to be similar or with some negligible differences, this will give us an advantage for using the standard aHydro which is straightforward to implement and numerically much cheaper than aHydroQP. This chapter is mainly based on Ref. [141].

4.1 Dynamical equations in 1+1d systems

We need to revisit the dynamical equations since they are the only difference between the previous case (0+1d) and this new case. By going back to Sec. 2.3.4 and specifying the dynamical equations in the 1+1d case one ends up with two equations from the first moment, three from the second moment, together with the matching condition (which ensures energy-momentum conservation). As a result, we end up with six equations for six independent
variables $\alpha$, $\lambda$, $T$, and $\theta_\perp$ as

\begin{align}
D_u\mathcal{E} + \mathcal{E}\theta_u + \mathcal{P}_x D_x\theta_\perp + \frac{\mathcal{P}_y}{r}\sinh\theta_\perp + \frac{\mathcal{P}_z}{\tau}\cosh\theta_\perp &= 0, \\
D_x\mathcal{P}_x + \mathcal{P}_x\theta_x + \mathcal{E} D_u\theta_\perp - \frac{\mathcal{P}_y}{r}\cosh\theta_\perp - \frac{\mathcal{P}_z}{\tau}\sinh\theta_\perp &= 0,
\end{align}

\begin{align}
\frac{D_u\mathcal{I}_x}{\mathcal{I}_x} + \theta_u + 2 D_x\theta_\perp &= \frac{1}{\tau_{\text{eq}}}\left(\frac{\mathcal{I}_{\text{eq}}}{\mathcal{I}_x} - 1\right), \\
\frac{D_u\mathcal{I}_y}{\mathcal{I}_y} + \theta_u + 2 \frac{r}{\tau}\sinh\theta_\perp &= \frac{1}{\tau_{\text{eq}}}\left(\frac{\mathcal{I}_{\text{eq}}}{\mathcal{I}_y} - 1\right), \\
\frac{D_u\mathcal{I}_z}{\mathcal{I}_z} + \theta_u + 2 \frac{r}{\tau}\cosh\theta_\perp &= \frac{1}{\tau_{\text{eq}}}\left(\frac{\mathcal{I}_{\text{eq}}}{\mathcal{I}_z} - 1\right),
\end{align}

where the derivatives $D_\alpha$ and divergences $\theta_\alpha$, with $\alpha \in \{u, x, y, z\}$, are defined in App.~8.3.1.

The matching condition equation gives

\begin{align}
\mathcal{E}_{\text{kinetic}} &= \mathcal{E}_{\text{kinetic, eq}}, \\
\mathcal{H}_3(\alpha, \hat{m})\lambda^4 &= \mathcal{H}_{3, \text{eq}}(\hat{m}_{\text{eq}})T^4,
\end{align}

where $\hat{m}_{\text{eq}} \equiv m/T$ and $\mathcal{H}_{3, \text{eq}}(\hat{m}_{\text{eq}}) \equiv \mathcal{H}_3(\mathbf{1}, \hat{m}_{\text{eq}})$.

The dynamical equations in the standard aHydro are the same but with appropriate definitions of the bulk variables ($\mathcal{E}$, $\mathcal{P}_x$, $\mathcal{P}_y$, and $\mathcal{P}_z$), $\mathcal{I}$ components ($\mathcal{I}_x$, $\mathcal{I}_y$, $\mathcal{I}_z$, and $\mathcal{I}_{\text{eq}}$), and the matching condition, $\mathcal{E}_{\text{eq}}(\lambda)\hat{\mathcal{H}}_3(\alpha) = \mathcal{E}_{\text{eq}}(T)$.

### 4.2 Freeze-out in 1+1d

The freeze-out procedure is needed, as explained before, to produce predictions for experimental observables such as the spectra and the number of produced particles. In this section, we consider a boost-invariant cylindrically-symmetric system (1+1)d following the discussion in Ref.~[111]. As we will see this will simplify the formalism presented in Sec.~2.4.

First, let’s take the system to be cylindrically symmetric (no $\phi$ dependence), then one has $d(\zeta, \phi, \theta) \to d(\zeta, \theta)$. Next, we consider the system to be also boost invariant which means the hypersurface has no longitudinal rapidity $\varsigma$ dependence. This also means that the projection
of \(d(\zeta, \theta)\) normal to the \(\zeta\)-axis is constant, i.e., \(d(\zeta, \theta) \sin \theta = \text{constant}\). As a result, we are free to choose any values, so we can pick \(\zeta = 0\) \((\theta = \pi/2)\), then one has

\[
\begin{align*}
    d(\zeta, \theta) &= \frac{d(\zeta)}{\sin \theta}, \\
    \frac{\partial d(\zeta, \theta)}{\partial \theta} &= -d(\zeta) \frac{\cot \theta}{\sin \theta},
\end{align*}
\]

(4.8)

where \(d(\zeta) \equiv d(\zeta, \theta = \pi/2)\). Using the above simplifications, we can determine the appropriate form of \(p^\mu d^3 \Sigma_\mu\) needed to do the freeze-out in 1+1d

\[
p^\mu d^3 \Sigma_\mu = \frac{\tau}{\Lambda} d(\zeta) \cos \zeta \csc^2 \theta \left[ d'(\zeta) \left( p_{\perp} \sin \zeta \cos(\phi - \varphi) - m_{\perp} \cos \zeta \cosh(y - \zeta) \right) \\
+ d(\zeta) \left( p_{\perp} \cos \zeta \cos(\phi - \varphi) + m_{\perp} \sin \zeta \cosh(y - \zeta) \right) \right] d\zeta d\phi d\theta.
\]

(4.9)

And this should be the only difference from what presented in Sec. 2.4.

### 4.3 Results

We now present our numerical results where we compare our new method (aHydroQP) with some standard methods presented previously in the literature. The first method we compare with is the standard aHydro and the second method is second order viscous hydrodynamics [88, 92]. In App. 8.2 we provide more details about the vHydro equations and the transport coefficients used herein. In this section, we present the comparisons for both Pb-Pb collisions and p-Pb collisions. In both cases we assume the system to be initially isotropic which means in both aHydro methods \(\alpha_x(\tau_0) = \alpha_y(\tau_0) = \alpha_z(\tau_0) = 1\) and in vHydro \(\pi^{\mu\nu}(\tau_0) = \Pi(\tau_0) = 0\) where we choose the initial proper time to be \(\tau_0 = 0.25\) fm/c. In all cases, for more accuracy, we use fourth-order Runge-Kutta integration for the temporal updates and fourth-order centered differences for the evaluation of all spatial derivatives. We take the freeze-out temperature to be \(T_{\text{eff}} = T_{\text{FO}} = 150\) MeV. To do the freeze-out we sum over 371 hadronic resonances \((M_{\text{hadron}} \leq 2.6\) GeV), taken from the SHARE table of hadronic resonances [143, 144, 145]. We note that we only consider the primordial spectra
Figure 4.1: Comparisons of (a) the effective temperature, (b) the transverse flow rapidity ($\theta_\perp$) in all three models: aHydroQP, aHydro, and vHydro in 1+1d numerical solutions at $\tau = 5.25$ fm/c. The shear viscosity to entropy density ratio was taken to be $4\pi\eta/s = 1$.

Figure 4.2: Same as Fig. 4.1 except here we take $4\pi\eta/s = 3$.

(no resonance feed down) which limits us from any direct comparisons with experimental data.

**Pb-Pb collisions:** First, we consider Pb-Pb collisions with $\sqrt{s_{NN}} = 2.76$ TeV. To set the initial energy density profile we use smooth Glauber wounded-nucleon overlap where the inelastic nucleon-nucleon scattering cross-section is $\sigma_{NN} = 62$ mb. For both aHydro models,
we use 200 points in the radial direction with a lattice spacing of $\Delta r = 0.15$ fm and temporal step size of $\Delta \tau = 0.01$ fm/c. On the other hand, for the vHydro results, we use 600 points in the radial direction with a lattice spacing of $\Delta r = 0.05$ fm and temporal step size of $\Delta \tau = 0.001$ fm/c since the code is more sensitive to the spatial lattice spacing and required a smaller temporal step size for stability. The initial central temperature in this case is taken to be $T_0 = 600$ MeV at the initial time $\tau_0$.

In Figs. 4.1 and 4.2 we present comparisons of the numerical 1+1d solutions to all three
Figure 4.5: Comparisons of the freeze-out hypersurfaces in all three models: aHydroQP, aHydro, and vHydro in 1+1d solutions. We consider the shear viscosity to entropy density ratio to be $4\pi\eta/s = 1$, and $4\pi\eta/s = 3$ in panel (a) and panel (b) respectively.

methods, aHydroQP, aHydro, and vHydro for $4\pi\eta/s = 1$, and 3 respectively. In each figure, we show in panel (a) the effective temperature, and in panel (b) the transverse rapidity profiles at $\tau = 5.25$ fm/c. At this time, we see a quite good agreement between all three methods, however, the differences become more by increasing $4\pi\eta/s$ as can be seen from Figs. 4.2 when compared with results in Fig. 4.1. To see how these solutions compare at different times, we kept $4\pi\eta/s = 1$ and vary only the time in Figs. 4.3 and 4.4 at 2.25 fm/c, and 10.25 fm/c respectively. From panel (a) of both figures we see that the differences in the effective temperature profile grow with the proper time. In the central region, we notice that aHydro is different from both aHydroQP and vHydro at late times, as can be seen from Fig. 4.4. Looking at panel (b) of Figs. 4.1 - 4.4, the transverse rapidity profiles show somewhat larger differences at very early times ($\tau = 2.25$ fm/c), but at late times ($\tau = 10.25$ fm/c), the agreement between all models becomes quite good.

Next, we turn to Fig. 4.5. In this figure, we present comparisons of the freeze-out hypersurfaces for $4\pi\eta/s = 1$ in (Fig. 4.5a), and $4\pi\eta/s = 3$ in (Fig. 4.5b) for all three methods.
Figure 4.6: Comparison of the $\pi^0$, $K^+$, and $p$ spectra as a function of transverse momentum $p_T$ obtained using aHydroQP, aHydro and vHydro. The shear viscosity to entropy density ratio is $4\pi\eta/s = 1$, 3, and 10 in the left, middle, right panels respectively.

As panel (a) demonstrates one finds that the freeze-out hypersurfaces are quite similar for $4\pi\eta/s = 1$, however, by increasing $4\pi\eta/s$ as panel (b) demonstrates, there are larger differences between the three methods. Clearly, standard aHydro shows a clear difference when compared to other methods. This difference seems to be very important, as we will see in Chap. 5, where standard aHydro fails to describe the spectra at low $p_T$.

In Fig. 4.6 we show our comparisons of all three methods for the primordial pion, kaon, and proton spectra produced in left, middle, right panels for $4\pi\eta/s = 1$, 3, and 10, respectively. As can be seen from the left panel, for $4\pi\eta/s = 1$, both aHydro approaches are in
Figure 4.7: In the left panel, we show comparisons of the total number of pions ($\pi^0$), kaons ($K^0$), and protons as a function of $4\pi\eta/s$. In the right panel, we show comparisons of the neutral pions, kaons ($K^+$), and protons $\langle p_T \rangle$ as a function of $4\pi\eta/s$. In both panels, we compare results obtained using aHydroQP, aHydro, and vHydro.

...good agreement with the largest differences occurring at low momentum. On the other hand, the vHydro results show a clear downward curvature in the pion spectrum. This results in
many fewer high-$p_T$ pions which will have an impact on other observables as we will see later on. The same thing can be seen in the kaon and proton spectra with less downturn for larger masses. If one extends the plots to high $p_T$, the pions spectra in vHydro will turn negative eventually. Although this turn to negative spectra didn’t appear in the left panel at $p_T < 3$ GeV, it clearly happens at larger $\eta/s$ as can be seen from both middle and right panels. From the middle panel, for $4\pi\eta/s = 3$ the differential pion spectrum goes negative at $p_T \sim 1.6$ GeV while from the right panel, for $4\pi\eta/s = 10$ it goes negative at $p_T \sim 0.9$ GeV.

This problem does not occur in either aHydro approach since the one-particle distribution function we are using is positive-definite by construction. In vHydro, we found that the bulk-viscous correction to the distribution function is responsible for the negative spectra. If we only include the shear viscous correction to the distribution function the resulting spectra is positive definite in the $p_T$ range shown in fig. 4.6. This behavior is a known issue in second order viscous hydrodynamic and has already been seen by other authors [146].

To further study the differences between aHydroQP and the other two standard methods, we present more comparisons in Fig. 4.7. In the left panel, we show the total number of $\pi^0$’s (top), $K^0$’s (middle), and $p$’s (bottom) as a function of $4\pi\eta/s$ which is obtained by integrating the differential yields over $p_T$. As can be seen from this comparison, all approaches are in agreement at small $\eta/s$, however, at large $\eta/s$ they give totally different results. One thing to note which is related to the problem we addressed above in vHydro, we note that the number of pions drops much quicker than the two aHydro approaches (went negative at $4\pi\eta/s \sim 22$). This can be explained by the fact that at high $p_T$ there is a negative number of pions. If we compare both aHydro approaches we see that there is a reasonable agreement in the total number of pions, whereas there are big differences in kaons and proton yields. This means that there is a strong dependence on the way the EoS is implemented in aHydro.
especially in the total number of primordial protons.

In the right panel of Fig. 4.7, we show the average $p_T$ for $\pi^0$, $K^+$, and $p$ in the top, middle, and bottom panels respectively. We see that in both aHydro approaches, $\langle p_T \rangle$ of the pions and kaons shows a weak dependence on the assumed value of $\eta/s$. On the other hand, vHydro predicts that $\langle p_T \rangle$ for the pions and kaons show a much more steep decrease. We see also that vHydro predicts negative $\langle p_T \rangle$ for pions for $4\pi\eta/s \gtrsim 5$ which results from the negativity of the pion spectra at high $p_T$ since the integrand is more sensitive to the high-$p_T$ part of the spectra. In both aHydro approaches we see a clear difference between them when we compare $\langle p_T \rangle$ for protons in both cases. This means, as we have seen before, that the spectra and $\langle p_T \rangle$ of protons are sensitive to the way in which the EoS is implemented.

Finally, in Fig. 4.8 we plot the total number of charged particles as a function of $4\pi\eta/s$ predicted by all three approaches. As this figure demonstrates, all three approaches agree at small $\eta/s$. They all show that $N_{\text{chg}}$ increases, and then decreases, however, they differ
Figure 4.9: Comparisons of the spectra of neutral pions, kaons ($K^+$), and protons as a function of transverse momentum $p_T$ in p-Pb collisions obtained using aHydroQP, aHydro, and vHydro. The shear viscosity to entropy density ratio is $4\pi\eta/s = 1$, and 3 in the left, and right panels respectively.
at the point where this turnover happens. In vHydro, the turnover point happens around $4\pi\eta/s \sim 3$ due to the rapid decrease in pions number as explained before.

**p-Pb collisions:** Second, we consider the case of an asymmetric collision between a proton and a nucleus (p-Pb) and look for the predictions of all three methods. We use the same parameters as used in the Pb-Pb collisions, except the ones related to the system size. Due to the smaller number of participant nucleons, we take a lower initial temperature, $T_0 = 400$ MeV. In addition, we use a lattice spacing of $\Delta r = 0.06$ fm for both aHydro approaches, and a lattice spacing of $\Delta r = 0.02$ fm for vHydro.

In Fig. 4.9 we present our results for the primordial pion, kaon, and proton spectra produced for p-Pb collisions for $4\pi\eta/s = 1$, and 3 in the left, and right panels respectively. From both panels, we see that both aHydro approaches are in a good agreement at high $p_T$, however, there are some quantitative differences at low $p_T$. The vHydro results show the same behavior seen in the Pb-Pb collisions, that the particle spectra goes negative at high $p_T$. 

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Chapter 5

3+1d quasiparticle anisotropic hydrodynamics

As we saw in both previous Chap. 3 and 4, there are clear differences between standard aHydro and aHydroQP. The hope was that standard aHydro will give similar or very close results to aHydroQP, since we treat the non-conformality of QGP the right way in aHydroQP but not in the standard aHydro. If this is the case, using standard aHydro will be much simpler and less expensive numerically. In Refs. [31, 147], the authors try to do the standard aHydro in 3+1d, but they found clear deviations from the experimental results which they conclude to be related to the way that EoS is implemented. In this case, we proceed to use the aHydroQP in 3+1d and try to avoid difficulties related to it. The biggest problem we face is solving the integrals in $H$ functions. Eventually, we came up with an approximation method to do these numerically intense integrals where we present the details and limits of this method in App. 8.4.4. Then, we compare our aHydroQP results with with $\sqrt{s_{NN}} = 2.76$ TeV Pb-Pb collision data available from the ALICE collaboration. This chapter is mainly based on Refs. [148, 149]

5.1 Dynamical equations

In the case of full 3+1d system, one has eight independent variables, $\alpha_x$, $\alpha_y$, $\alpha_z$, $u_x$, $u_y$, $\vartheta$, $T$, and $\lambda$. So we need eight equations to solve the full 3+1d aHydroQP system which come from the equations obtained before in Sec. 2.3.4. Four equations come from the first
moment of the Boltzmann equation derived before in Eq. (2.49)

\[ D_u \mathcal{E} + \mathcal{E} \theta_u + \mathcal{P}_x u_\mu D_x X^\mu + \mathcal{P}_y u_\mu D_y Y^\mu + \mathcal{P}_z u_\mu D_z Z^\mu = 0, \]

\[ D_x \mathcal{P}_x + \mathcal{P}_x \theta_x - \mathcal{E} X_\mu D_u u^\mu - \mathcal{P}_y X_\mu D_y Y^\mu - \mathcal{P}_z X_\mu D_z Z^\mu = 0, \]

\[ D_y \mathcal{P}_y + \mathcal{P}_y \theta_y - \mathcal{E} Y_\mu D_u u^\mu - \mathcal{P}_x Y_\mu D_x X^\mu - \mathcal{P}_z Y_\mu D_z Z^\mu = 0, \]

\[ D_z \mathcal{P}_z + \mathcal{P}_z \theta_z - \mathcal{E} Z_\mu D_u u^\mu - \mathcal{P}_x Z_\mu D_x X^\mu - \mathcal{P}_y Z_\mu D_y Y^\mu = 0. \]

and three equations from the second moment derived before in Eqs. (2.56), (2.57), and (2.58)

\[ D_u \mathcal{I}_x + \mathcal{I}_x (\theta_u + 2 u_\mu D_x X^\mu) = \frac{1}{\tau_{\text{eq}}} (\mathcal{I}_{\text{eq}} - \mathcal{I}_x), \]

\[ D_u \mathcal{I}_y + \mathcal{I}_y (\theta_u + 2 u_\mu D_y Y^\mu) = \frac{1}{\tau_{\text{eq}}} (\mathcal{I}_{\text{eq}} - \mathcal{I}_y), \]

\[ D_u \mathcal{I}_z + \mathcal{I}_z (\theta_u + 2 u_\mu D_z Z^\mu) = \frac{1}{\tau_{\text{eq}}} (\mathcal{I}_{\text{eq}} - \mathcal{I}_z). \]

and the eighth equation comes from using the matching condition

\[ \mathcal{H}_3(\alpha, \tilde{m}) \lambda^4 = \mathcal{H}_{3,\text{eq}}(1, \tilde{m}_{\text{eq}}) T^4. \quad (5.1) \]

### 5.2 Transport coefficients

Since we are using the relaxation time approximation (RTA), we do not have control on all transport coefficients. Once the shear viscosity to entropy density ratio \((\eta/s)\) is fixed (to a constant in this case), the bulk viscosity \((\zeta)\) is not free parameter in our model anymore. This can be understood by noticing that both of them are function of the relaxation time \((\tau_{\text{eq}})\). Here we will show the quasiparticle expressions of both the shear viscosity and the bulk viscosity taken from Refs. [150] and [151]

\[ \eta = \frac{\tau_{\text{eq}}}{T} I_{3,2}(\tilde{m}_{\text{eq}}), \quad (5.2) \]

with \(\tilde{m}_{\text{eq}} \equiv m/T\) and

\[ I_{3,2}(x) = \frac{N_{\text{dof}} T^5 x^5}{30 \pi^2} \left[ \frac{1}{16} \left( K_3(x) - 7 K_3(x) + 22 K_1(x) \right) - K_{i,1}(x) \right], \]

\[ K_{i,1}(x) = \frac{\pi}{2} \left[ 1 - x K_0(x) S_{i-1}(x) - x K_1(x) S_0(x) \right], \quad (5.3) \]
Figure 5.1: The bulk viscosity scaled by the entropy density ($\zeta/s$) is shown as a function of the temperature obtained using quasiparticle anisotropic hydrodynamics, Bayesian analysis [21], and Black hole engineering [22].

On the other hand, the bulk viscosity $\zeta$ is given by

$$\frac{\zeta}{\eta_{eq}} = \frac{5}{3T} I_{3,2}(\hat{m}_{eq}) - c_s^2(\mathcal{E} + \mathcal{P}) + c_s^2 m \frac{dm}{dT} I_{1,1}(\hat{m}_{eq}),$$

with

$$I_{1,1}(x) = \frac{N_{dof} T^3 x^3}{6\pi^2} \left[ \frac{1}{4} \left( K_3(x) - 5K_1(x) \right) + K_{i,1}(x) \right].$$

Here, $K_n$ are modified Bessel functions of the second kind, and $S_n$ are modified Struve functions. Using Eq. (5.2), the relaxation time can be written as

$$\tau_{eq}(T) = \frac{\eta}{I_{3,2}(\hat{m}_{eq})} \frac{\mathcal{E} + \mathcal{P}}{I_{3,2}(\hat{m}_{eq})},$$

So, by sitting $\eta/s$ to a constant both $\tau_{eq}$, and $\zeta/s$ are dependent on the equation of state and not free parameters anymore.

In Fig. 5.1, we show scaled bulk viscosity, $\zeta/s$ as a function of temperature predicted by the quasiparticle model and compare it with some other recent results mainly Bayesian analysis [21], and Black hole engineering [22]. As we can see from these comparisons, we
Figure 5.2: The bulk viscosity scaled by the entropy density is shown as a function of the temperature scaled by the critical temperature \((T/T_c)\) used in one of 3+1d second order viscous hydrodynamics codes Ref. [23]. Figure taken from Ref. [23]

see that the quasiparticle prediction of the bulk viscosity agrees quite well with the other two model predictions. On the other hand, in Fig. 5.2, we show another ansätze for the bulk viscosity used in another study where the peak value is almost larger by an order of magnitude \(\zeta/s \sim 0.3\) to what quasiparticle model predicted \(\zeta/s \sim 0.05\).

5.3 3+1d aHydroQP results

In the previous section, we derived the needed dynamical equations for the hydrodynamics stage. However, there are two important ingredients that we have to include to fully compare with experimental results. First, we need to set up the appropriate initial conditions to describe the desired system, Pb-Pb system in this case. Second, we need to model what occurs after freeze-out where hydro breaks down and can not describe weakly coupled hadrons. Now, let’s consider each one of them separately.
5.3.1 Numerical setup

Next, let’s turn to the numerical setup of our 3+1d code. As can be seen from the previous sections, the dynamical equations obtained using aHydroQP are coupled partial differential equations. To solve the 3+1d dynamical equations, we use a 64³ lattice with lattice spacings $\Delta x = \Delta y = 0.5$ fm and $\Delta \varsigma = 0.375$. To do the numerics, we used fourth-order centered-differences to calculate spatial derivatives. And we used fourth-order Runge-Kutta with step size of $\Delta \tau = 0.02$ fm/c to calculate temporal updates. As always, there are some numerics instabilities in some regions which usually can be regulated in different ways. In this study, we regulate these instabilities associated with the centered-differences scheme by using weighted-LAX smoother first introduced in Ref. [100]. This regulator should not affect the physical observables since in most cases, we use a quite small weighted-LAX fraction which is 0.005. In some cases, however, for large impact parameters we used a higher value for the smoother like 0.02. This higher value will not be a problem since the system reaches the freeze-out temperature at quite early times $\lesssim 4$ fm/c before this “viscous effect” of the smoother gets to play a big role. The initial time in our model is taken to be $\tau_0 = 0.25$ fm/c and the code is stopped when the highest effective temperature is sufficiently below $T_{FO}$ in the entire volume since some areas (for example, the system edges) cool faster than others (for example, the central region).

5.3.2 Initial conditions

Heavy-ion collisions can be different due to the way that they collide, for example they can be head-on-head collisions known as central collisions or can be peripheral collisions where they barely overlap each other. The centrality class can be determined in our code by choosing the averaged impact parameter expected in each centrality class according to the optical Glauber model. The relations between different centrality classes and the corresponding averaged impact parameter are listed in Table 1.1.
Moreover, we need to set the initial energy density for the system as an initial input to the hydrodynamics code. We assume the system to be initially isotropic in momentum space through setting \(\alpha_i(\tau_0) = 1\). We also assume the system to has zero transverse flow \(\mathbf{u}_\perp(\tau_0) = 0\) and Bjorken flow in the longitudinal direction \(\vartheta(\tau_0) = \eta\).

Following the discussion in [152], we compute the initial energy density distribution in the transverse plane using a “tilted” profile. In this profile, we used a linear combination of smooth Glauber wounded-nucleon and binary-collision density profiles to set the distribution. The binary-collision mixing factor used here is \(\chi = 0.15\). On the other hand, for the longitudinal direction, we used a profile with a central plateau and Gaussian “tails” given by

\[
\rho(\varsigma) \equiv \exp \left[-(\varsigma - \Delta \varsigma)^2/(2\sigma_\varsigma^2) \Theta(|\varsigma| - \Delta \varsigma)\right].
\]

(5.7)

where \(\Delta \varsigma\) and \(\sigma_\varsigma\) set the width of the central plateau and the width of the Gaussian “tails” respectively. They both can be determined by fitting to the pseudorapidity distribution of charged hadrons. That gives \(\Delta \varsigma = 2.3\) and \(\sigma_\varsigma = 1.6\).

The resulting initial energy density as a function of \(x_\perp\) and \(\varsigma\) is given by

\[
\mathcal{E}(x_\perp, \varsigma) \propto (1 - \chi)\rho(\varsigma)\left[W_A(x_\perp)g(\varsigma) + W_B(x_\perp)g(-\varsigma)\right] + \chi\rho(\varsigma)C(x_\perp),
\]

(5.8)

where \(W_A(x_\perp)\) is the wounded nucleon density for nucleus \(A\), \(W_B(x_\perp)\) is the wounded nucleon density for nucleus \(B\), \(C(x_\perp)\) is the binary collision density, and \(g(\varsigma)\) is the “tilt function” defined through

\[
g(\varsigma) = \begin{cases} 
0 & \text{if } \varsigma < -y_N, \\
(\varsigma + y_N)/(2y_N) & \text{if } -y_N \leq \varsigma \leq y_N, \\
1 & \text{if } \varsigma > y_N.
\end{cases}
\]

(5.9)

where \(y_N = \log(2\sqrt{s_{NN}}/(m_p + m_n))\) is the nucleon momentum rapidity. For more details, I refer the reader to Ref. [152].
In panel (a), (b), and (c) we show the spatial profile of anisotropy parameters \( \alpha_x, \alpha_y, \) and \( \alpha_z \) respectively as a function of \( x \) at \( \tau = 5.25 \text{ fm/c} \).

### 5.3.3 Numerical results of the 3+1daHydro code

Before presenting our phenomenological results, let’s present some numerical results of our 3+1d hydro code to show the spatial profile of some model parameters taken in 30-40\% centrality class.

In Fig. 5.3, we show the anisotropy parameters \( \alpha_x, \alpha_y, \) and \( \alpha_z \) as a function of \( x \) at a proper time \( \tau = 5.25 \text{ fm} \). This comparison between \( \alpha \)'s is performed in the 30-40\% centrality class. As can be seen in Fig. 5.3, the evolution of the anisotropy parameters \( \alpha_x, \alpha_y, \) and \( \alpha_z \) usually do not go far away from the isotropy point \( (\alpha_x = \alpha_y = \alpha_z = 1) \). This useful behavior is used, as we will see later on, to help doing the \( \mathcal{H} \)-functions integrals which are not easy to evaluate numerically. In Fig. 5.4, we show in panel (a) the temperature profile and in panel (b) the effective temperature profile as a function of \( x \). In both panels, the proper time is taken to be \( \tau = 5.25 \text{ fm} \) in the 30-40\% centrality class. As can be seen from Fig. 5.4 the central region is much hotter than the system edges where the temperature falls fast going to the edges. In Fig. 5.5, we show the profile of the flow velocity \( u_x, \) and \( u_y \) as a function of \( x \) and \( y \) respectively. As before, this comparison is shown at \( \tau = 5.25 \text{ fm} \) in the 30-40\% centrality class.

Next, to show the relaxation of the spatial anisotropy at early times to the spatial isotropy
Figure 5.4: In panel (a) we show the effective temperature profile as a function of \( x \) where in panel (b), we show the temperature-like parameter \( \lambda \) profile as a function of \( x \). In both panels, the proper time is taken to be \( \tau = 5.25 \).}

Figure 5.5: In panel (a), and (b) we show the profile of the flow velocity \( u_x(x) \), and \( u_y(y) \) respectively at \( \tau = 5.25 \).

at late times we present in Fig. 5.6, snapshot visualizations of the temperature-like parameter at different times. As can be seen from this figure, the initial spatial anisotropy between the center of the system and the edges is quite high (factor of 5). However, at late times, the system cools down and tend to be close from isotropy (factor of 2).
Figure 5.6: Three snapshots of the spatial anisotropy of the temperature-like parameter at different times.

5.3.4 The hadronic freeze-out

The Cooper-Frye freeze-out procedure is used to compute the produced particles with specific numbers and momenta distribution. However, the produced primordial particles go through different processes before reaching the detector. There are elastic and inelastic collisions between these particles which spread their energies and momenta. Many of the particles produced are resonances which decay to other stable particles. As a result, we need another transport code to take care of the scatterings and resonance decays. The code we use in this study is THERMINATOR 2 which takes care of the inelastic collisions and resonance decays. It does not include the elastic scatterings where we think that is not going to contribute a lot to the final observables. However, we should in future include the elastic collisions using some available codes like UrQMD. THERMINATOR 2 allows us to compute many heavy-ion observables such as the spectra, multiplicity, HBT radii, etc.

Next, let’s explain how THERMINATOR 2, in practice, is connected to our 3+1d aHydro code. After running the full 3+1d evolution of the system, the freeze-out hypersurface is extracted using Cooper-Frye freeze-out based on the specified effective temperature. Then, the distribution function parameters on the freeze-out hypersurface were fed into THERMINATOR 2 where it was assumed that all hadronic species were in chemical equilibrium and
with the same distribution function parameters, namely \( \alpha \) and \( \lambda \). After the distribution function on the hypersurface is determined, THERMINATOR 2 performs sampled event-by-event hadronic production from the exported freeze-out hypersurface using Monte-Carlo sampling. Next, the hadronic feed down is performed for each sampled event. In some cases, we need to generate more hadronic events to improve the statistics of our theoretical results. In this dissertation, we used between 7,400 and 36,200 hadronic events per centrality class, for all plots shown here, depending on the observables and the centrality class under consideration. Note that for all figures, we indicate the statistical uncertainty of our model results by a shaded band surrounding the hadronic event-averaged value. This uncertainty is associated with the hadronic Monte-Carlo sampling that is used by our code.

5.3.5 Fixing free parameters

In our model, we have three free parameters and in this section we will explain how we fixed them by comparing to the experimental results. These free parameters are the initial central temperature \( T_0 \), the freeze-out temperature \( T_{\text{FO}} \), and finally the shear viscosity to entropy density ratio, \( \eta/s \) which is assumed to be a constant. In order to fix these parameters we scanned over the free parameters through a wide expected region for each of them. Then we compared our model predictions resulting from this scan with experimental data for the differential spectra of pions, kaons, and protons. We did this procedure for two different centrality classes 0-5\% and 30-40\% centrality classes to best fit the data. Finally, to minimize the fitting error, we use equal weighting for the three particle types: pions, kaons, and protons.

As a result, we find the best fit to the data by this set of parameters \( T_0 = 600 \text{ MeV} \), \( \eta/s = 0.159 \), and \( T_{\text{FO}} = 130 \text{ MeV} \).
Figure 5.7: Transverse momentum spectra of $\pi^\pm$, $K^\pm$, and $p+\bar{p}$ in different centrality classes as shown in each panel. The data shown are from the ALICE collaboration for 2.76 TeV Pb+Pb collisions [24].
5.3.6 The phenomenological results

Now, let’s turn to our model phenomenological results predicted by aHydroQP compared with ALICE data published previously in [148, 149]. In Fig. 5.7, we present the spectra of $\pi^\pm$, $K^\pm$, and $p + \bar{p}$ as a function of transverse momentum $p_T$ in six different centrality classes as shown in the figure. The agreement between our model predictions and the ALICE data is very good for $\pi^\pm$, $K^\pm$, and $p + \bar{p}$ spectra simultaneously, i.e, our models gets the mass splitting right between these hadrons. In low centrality classes the agreement between our model and the data are very good with differences appear at quite large $p_T \sim 2.5$ GeV. However, in relatively high centrality classes 30-40% and 40-50%, the differences become clear at lower $p_T \sim 1.5$ GeV.

Another important observable which can be calculated using the spectra by integrating over the transverse momenta is the $p_T$ average as a function of centrality. The mean
transverse momentum can be found from the spectra as

$$\langle p_T \rangle = \int dp_T p_T^2 \frac{1}{2\pi p_T} \frac{dN}{dp_T}. \quad (5.10)$$

In Fig. 5.8, we show the comparisons of transverse momentum of $\pi^\pm$, $K^\pm$, and $p+\bar{p}$ predicted by our model and the ALICE experiment results. As can be seen from this figure, our model was able to reproduce the data reasonably well up to quite large centrality classes 50-60%. This agreement is similar to what other hydro methods predict for example, see Ref. [23].

In Fig. 5.9, we show the charged-hadron multiplicity as a function of pseudorapidity ($\eta$) where in panel (a), we show the comparisons for 0-5%, 5-10%, 10-20%, 20-30%, and 30-40% centrality classes and in panel (b) we show the comparisons for 40-50%, 50-60%, 60-70%, 70-80%, and 80-90% centrality classes. From this figure we see that our model was able to reproduce the experimental results quite well in almost all centrality classes.

Moving ahead with our comparisons, in Fig. 5.10, we show comparisons of the identified-particle $v_2(p_T)$ predicted by our model against the ALICE results. As can be seen from this figure, the agreement between our model and the experimental data for all centrality
Figure 5.10: The elliptic flow coefficient for $\pi^\pm$, $K^\pm$, and $p+\bar{p}$ as a function of $p_T$ for different centrality classes as shown in each panel. All results are for 2.76 TeV Pb+Pb collisions and data shown are from the ALICE collaboration [27].

The agreement between the model predictions and the data from the ALICE collaboration is quite good but not at many centrality classes. For 20-30%, and 30-40% centrality classes the agreement is quite good up to $p_T \sim 2$ GeV, whereas in the other two centrality classes 10-20%, and 40-50% the agreement is good up to $p_T \sim 1$ GeV and in some cases they totally disagree as in $p+\bar{p}$ for 10-20% centrality class. This disagreement is related to using simple initial conditions, smooth Glauber initial condition. By using fluctuating initial conditions one expects the agreement to be much better.

Another interesting observable is the integrated elliptic flow coefficient $v_2$ for charged hadrons as a function of centrality. In Fig. 5.11, we present our model prediction for $v_2$ for...
charged hadrons as a function of centrality versus the $v_2\{2\}$ and $v_2\{4\}$ ALICE experimental results. As can be seen from this figure, our model is providing a good description of $v_2$, while it agrees well with $v_2\{4\}$ at most central collisions it does differ at quite large centrality classes. This kind of disagreement can be related again to using smooth initial condition since the fluctuating effects may dominate the geometrical effects especially at small centrality classes where usually the flow is almost radial. We get $v_2\{4\}$ right at most central collisions, it is going to be hard to get $v_2\{2\}$ without including fluctuating initial conditions.

Now, let’s present another observable where our model needs to be improved to describe the experimental data well. In Fig. 5.12, we show comparisons of the pseudorapidity dependence of the elliptic flow $v_2$ for charged in different centrality classes predicted by our model and the experimental data from ALICE collaboration. As can be seen from this figure, our model fails to fall fast enough to describe the data in all centrality classes. However, in some centrality classes like 20-30% centrality class our model was able to describe the mid-pseudorapidity region but still does not fall quickly enough to catch the data results behavior. Such a disagreement is seen in other hydro models where a solution to this problem is suggested to use a temperature dependent $\eta/s(T)$ [153].

Next, let’s consider, another powerful observable, the HBT radii which gives information about the space-time structure of the system. This method is based on the techniques which were originally introduced by Hanbury Brown-Twiss (HBT) [154, 155], then later on have been applied to heavy-ion collisions. This is based on calculating the correlation functions between two particles, in this case for $\pi^+\pi^+$. The mean transverse pair momentum is defined as

$$k_T = (p_T,1 + p_T,2)/2,$$

where $p_T,1$ and $p_T,1$ is the transverse momentum of both pair particles. Let’s define what
Figure 5.11: The integrated $v_2$ for charged hadrons as a function of centrality. All results are for 2.76 TeV Pb+Pb collisions and data shown are from the ALICE collaboration Ref. [28].

Figure 5.12: The pseudorapidity dependence of the elliptic flow $v_2$ for charged hadrons in different centrality classes where we take $(0 < p_T < 100 \text{ GeV})$. All results are for 2.76 TeV Pb+Pb collisions and data are from the ALICE collaboration Ref. [29].

we mean by $R_{\text{long}}$, $R_{\text{out}}$, and $R_{\text{side}}$. $R_{\text{long}}$ is the source size along the beam line, $R_{\text{out}}$ is the source size along $k_T$, and $R_{\text{out}}$ is the source size perpendicular on $k_T$.  

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Next, let’s turn to our comparisons for the HBT radii predicted by aHydroQP with experimental data from the ALICE collaboration [30]. In Fig. 5.13, we present in the left, middle, and right panels $R_{\text{out}}$, $R_{\text{side}}$, and $R_{\text{long}}$, respectively, as a function of the mean transverse momentum of the pair $\pi^+\pi^+$ in four different centrality classes, 0-5%, 5-10%, 10-20%, and 20-30%. As we can see from these figures, our model was able to describe the data quite well. To see to which extent our model agrees with data, let’s look to each radius separately. In panel (a), our model was able to reproduce the data quite well for $R_{\text{out}}$ out to $k_T \sim 0.6$ GeV. In panel (b), we see that our model shows a good agreement for $R_{\text{side}}$ out to $k_T \sim 0.9$ GeV. Finally, in panel (c), we see that the agreement we get when comparing our model to data for $R_{\text{long}}$ is less than the agreement shown in panel (a), and (b). We notice that the most disagreement happens at low $k_T \sim 0.2$ GeV.

Next, we consider the ratios of the HBT radii in Fig. 5.14. In this set of figures, in the left, middle, and right panels we show $R_{\text{out}}/R_{\text{side}}$, $R_{\text{out}}/R_{\text{long}}$, and $R_{\text{side}}/R_{\text{long}}$, respectively, as a function of the mean transverse momentum of the pair $\pi^+\pi^+$ in four different centrality classes, 0-5%, 5-10%, 10-20%, and 20-30%. We see that our model was able to reproduce the data quite well in all three panels for all centrality classes shown here.
Figure 5.13: The femtoscopic Hanbury-Brown-Twiss (HBT) radii as a function of the pair mean transverse momentum ($k_T$) for $\pi^+\pi^+$ for different centrality classes as shown in each panel. The left, middle, right panels show $R_{\text{out}}$, $R_{\text{side}}$, and $R_{\text{long}}$ respectively. All results are for 2.76 TeV Pb+Pb collisions where data shown are for $\pi^\pm\pi^\pm$ combined from the ALICE collaboration [30].
Figure 5.14: The ratios of femtoscopic HBT radii as a function of the pair mean transverse momentum ($k_T$) for $\pi^+\pi^+$ for different centrality classes as shown in each panel. The left, middle, right panels show $R_{\text{out}}/R_{\text{side}}$, $R_{\text{out}}/R_{\text{long}}$, and $R_{\text{side}}/R_{\text{long}}$ respectively. All results are for 2.76 TeV Pb+Pb collisions where data shown are for $\pi^\pm\pi^\pm$ combined from the ALICE collaboration [30].
Chapter 6

NLO anisotropic hydrodynamics

In the previous chapters, we studied leading-order quasiparticle anisotropic hydrodynamics. Though, it showed a quite good agreement with the experimental data, one must try to get the agreement much better. The first thing to do is to include next-to-leading-order corrections since before we assumed the anisotropy tensor to be diagonal. This chapter will be mainly about including the off-diagonal terms of the anisotropy tensor. However, this will be very hard and numerically intense in quasiparticle anisotropic hydrodynamics. As a result, we will consider first next-to-leading-order anisotropic hydrodynamics using the standard anisotropic hydrodynamics. We will leave the NLO quasiparticle anisotropic hydrodynamics for a future work. For now, we will Taylor-expand the anisotropy tensor assuming small off-diagonal terms. This should make the formalism easier and numerically tractable. Then, by taking moments of the Boltzmann equation, we can get the dynamical equations needed to model the full 3+1d system. In this part of this dissertation, we present only the theory setup and leave the numerical analysis for a future work.

6.1 next-to-leading-order distribution function

The leading anisotropic distribution function as defined before in Sec. 2.1 is

\[ f(x, p) = f_{\text{iso}} \left( \frac{1}{\lambda} \sqrt{p_{\mu} \Xi^{\mu\nu} p_{\nu}} \right), \quad (6.1) \]

where

\[ \Xi^{\mu\nu} = u^{\mu} u^{\nu} + \xi^{\mu\nu} - \Delta^{\mu\nu} \Phi, \quad (6.2) \]

Note that we mean here “leading order” regarding the anisotropy tensor since only diagonal terms are included. We do not mean next-to-leading-order in the distribution function, i.e., \( f = f_0 + \delta f \). Unfortunately we have no other names to distinguish them, but it should be clear from the context.
So far we only included the diagonal terms of $\xi^{\mu\nu}$. In this chapter, we will account for the non-leading terms in the anisotropy tensor $\xi^{\mu\nu}$, i.e., we will include the off diagonal terms,

$$
\xi_{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \xi_{xx} & \xi_{xy} & \xi_{xz} \\
0 & \xi_{yx} & \xi_{yy} & \xi_{yz} \\
0 & \xi_{zx} & \xi_{zy} & \xi_{zz}
\end{pmatrix}
$$

(6.3)

We note that $\xi^{\mu\nu}$ is symmetric which means that $\xi_{ij} = \xi_{ji}$. For compactness, we will call diagonal terms as $\xi_{ii} \equiv \xi_i$. In this case, expanding the argument of the square root appearing on the right-hand side of Eq. (6.1) in the LRF gives

$$
p_{\mu} \xi^{\mu\nu} p_{\nu} = \sum_i \frac{p_i^2}{\alpha_i^2} + 2\xi_{xy}p_x p_y + 2\xi_{yz}p_y p_z + 2\xi_{xz}p_x p_z .
$$

(6.4)

Then we expand the distribution function for small off-diagonal terms up to second order in $\xi$'s which results to

$$
f(x, p) = \left[ 1 - \frac{1}{\hat{p}} (\xi_{xy}\alpha_x \alpha_y \hat{p}_x \hat{p}_y + \xi_{xz}\alpha_x \alpha_z \hat{p}_x \hat{p}_z + \xi_{yz}\alpha_y \alpha_z \hat{p}_y \hat{p}_z) \\
+ \frac{(\hat{p} + 1)}{\hat{p}^3} \alpha_x \alpha_y \alpha_z (\xi_{xy}\xi_{xz}\alpha_x^2 \hat{p}_x^2 \hat{p}_y \hat{p}_z + \xi_{xy}\xi_{yz}\alpha_y^2 \hat{p}_y^2 \hat{p}_z + \xi_{xz}\xi_{yz}\alpha_z^2 \hat{p}_x \hat{p}_y^2 \hat{p}_z) \\
+ \frac{(\hat{p} + 1)}{2\hat{p}^3} (\xi_{xy}^2 \alpha_x^2 \alpha_y^2 \hat{p}_x^2 \hat{p}_y^2 + \xi_{xz}^2 \alpha_x^2 \alpha_z^2 \hat{p}_x^2 \hat{p}_z^2 + \xi_{yz}^2 \alpha_y^2 \alpha_z^2 \hat{p}_y^2 \hat{p}_z^2) \right] f_{iso}(\hat{p}) ,
$$

(6.5)

where $\hat{p} \equiv \frac{1}{\lambda} \sqrt{\sum_i \frac{p_i^2}{\alpha_i^2}}$

where the distribution function $f_{iso}(\hat{p})$ is the leading order distribution function defined as

$$
f_{iso}(x, p) = f_{eq} \left( \frac{1}{\lambda} \sqrt{\sum_i \frac{p_i^2}{\alpha_i^2}} \right) ,
$$

(6.6)

where $i \in \{x, y, z\}$ and the scale parameters $\alpha_i$ are

$$
\alpha_i \equiv (1 + \xi_i + \Phi)^{-1/2} .
$$

(6.7)
6.2 Bulk variables in NLO aHydro

Energy Density

The energy density is given by

\[ E = N_{\text{dof}} \int dPE^2 f(x,p) \]
\[ = \tilde{N} \int d^3p \sqrt{p^2} f(x,p). \quad (6.8) \]

Changing variables to \( \hat{p}_i = p_i/(\lambda \alpha_i) \), transforming to spherical coordinates, and using Eq. (6.5) one obtains

\[ E = E_{\text{eq}}(\lambda) \left( H_{3}(\alpha) + \xi_{xy}^2 H_{1101}(\alpha) + \xi_{xz}^2 H_{1110}(\alpha) + \xi_{yz}^2 H_{1011}(\alpha) \right), \quad (6.9) \]

where \( E_{\text{eq}}(\lambda) = 24\pi \tilde{N} \lambda^4 \) and all above \( H_{abcd} \) functions are defined in App. 8.5.

The pressure components

The \( x \)th-component of the pressure is given by

\[ P_x = N_{\text{dof}} \int dP p_x^2 f(x,p) \]
\[ = \tilde{N} \int d^3p \frac{p_x^2}{\sqrt{p^2}} f(x,p). \quad (6.10) \]

In a similar way, one can write the \( x \)th-component of the pressure as

\[ P_x = P_{\text{eq}}(\lambda) \left( H_{3x}(\alpha) + \xi_{xy}^2 H_{-1210}(\alpha) + \xi_{xz}^2 H_{-1201}(\alpha) + \xi_{yz}^2 H_{-1111}(\alpha) \right), \quad (6.11) \]

Where \( P_{\text{eq}}(\lambda) = 8\pi \tilde{N} \lambda^4 \).

The \( y \)th-component of the pressure is given by

\[ P_y = N_{\text{dof}} \int dP p_y^2 f(x,p) \]
\[ = \tilde{N} \int d^3p \frac{p_y^2}{\sqrt{p^2}} f(x,p). \quad (6.12) \]
It can be written as

\[ P_y = P_{\text{eq}}(\lambda) \left( \mathcal{H}_{3y}(\alpha) + \xi_{xy}^2 \mathcal{H}_{-1120}(\alpha) + \xi_{xz}^2 \mathcal{H}_{-1111}(\alpha) + \xi_{yz}^2 \mathcal{H}_{-1021}(\alpha) \right). \]  

(6.13)

The longitudinal component of the pressure is given by

\[ P_L = N_{\text{dot}} \int dP \, p^2 \, z \, f(x, p) \]
\[ = \bar{N} \int d^3p \, \frac{p^2}{\sqrt{p^2}} \, f(x, p). \]  

(6.14)

As done before, one can write it as

\[ P_L = P_{\text{eq}}(\lambda) \left( \mathcal{H}_{3L}(\alpha) + \xi_{xy}^2 \mathcal{H}_{-1111}(\alpha) + \xi_{xz}^2 \mathcal{H}_{-1102}(\alpha) + \xi_{yz}^2 \mathcal{H}_{-1012}(\alpha) \right), \]  

(6.15)

where all the \( \mathcal{H} \) functions appearing above is defined in App. 8.5.

### 6.3 First moment

In general, the energy momentum tensor can be written as [100]

\[ T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + P_{x} X^\mu X^\nu + P_{y} Y^\mu Y^\nu + P_{z} Z^\mu Z^\nu + t_{xy}(X^\mu Y^\nu + Y^\mu X^\nu) \]
\[ + t_{xz}(X^\mu Z^\nu + Z^\mu X^\nu) + t_{yz}(Y^\mu Z^\nu + Z^\mu Y^\nu), \]  

(6.16)

where,

\[ t_{xy} \equiv \int dP \, p_x \, p_y \, f(x, p), \]
\[ t_{xz} \equiv \int dP \, p_x \, p_z \, f(x, p), \]
\[ t_{yz} \equiv \int dP \, p_y \, p_z \, f(x, p). \]  

(6.17)
Using the distribution function specified in Eq. (6.5) one finds

\[ t_{xy} = \lambda^4 \left( -\xi_{xy} H_{110}(\alpha) + 2\xi_{xz}\xi_{yz} H_{-111}(\alpha) \right), \]

\[ t_{xz} = \lambda^4 \left( -\xi_{xz} H_{101}(\alpha) + 2\xi_{xy}\xi_{yz} H_{-111}(\alpha) \right), \]

\[ t_{yz} = \lambda^4 \left( -\xi_{yz} H_{011}(\alpha) + 2\xi_{xy}\xi_{xz} H_{-111}(\alpha) \right). \] (6.18)

Turning to the left hand side, using Eq. (6.16) and taking \( U-, X-, Y-, \) and \( Z-\)projections, one obtains four independent equations

\[ D_x \mathcal{E} + \mathcal{E} \theta_u + \mathcal{P}_x u_\mu D_x X^\mu + \mathcal{P}_y u_\mu D_y Y^\mu + \mathcal{P}_z u_\mu D_z Z^\mu + t_{xy} u_\mu (D_x Y^\mu + D_y X^\mu) \]

\[ + t_{xz} u_\mu (D_z Z^\mu + D_x X^\mu) + t_{yz} u_\mu (D_y Z^\mu + D_z X^\mu) = 0, \]

\[ D_x \mathcal{P}_x + \mathcal{P}_x \theta_x - \mathcal{E} X_\mu D_\mu u^\mu + \mathcal{P}_y X_\mu D_\mu Y^\mu + \mathcal{P}_z X_\mu D_\mu Z^\mu + D_y t_{xy} + t_{xy}(\theta_y - X_\mu D_x Y^\mu) \]

\[ + D_z t_{xz} + t_{xz}(\theta_x - X_\mu D_x Z^\mu) - t_{yz} X_\mu (D_y Z^\mu + D_z X^\mu) = 0, \]

\[ D_y \mathcal{P}_y + \mathcal{P}_y \theta_y - \mathcal{E} Y_\mu D_\mu u^\mu - \mathcal{P}_x Y_\mu D_x X^\mu - \mathcal{P}_z Y_\mu D_z Z^\mu + D_x t_{xy} + t_{xy}(\theta_x - Y_\mu D_y X^\mu) \]

\[ + D_z t_{xz} + t_{xz}(\theta_x - Y_\mu D_x Z^\mu) - t_{yz} Y_\mu (D_y Z^\mu + D_z X^\mu) = 0, \]

\[ D_z \mathcal{P}_z + \mathcal{P}_z \theta_z - \mathcal{E} Z_\mu D_\mu u^\mu + \mathcal{P}_x Z_\mu D_x X^\mu - \mathcal{P}_y Z_\mu D_y Y^\mu + D_x t_{xz} + t_{xz}(\theta_x - Z_\mu D_z X^\mu) \]

\[ + D_y t_{yz} + t_{yz}(\theta_y - Z_\mu D_z Y^\mu) - t_{xy} Z_\mu (D_x Y^\mu + D_y X^\mu) = 0. \] (6.19)

### 6.4 Second moment

The second moment is given by

\[ \partial_\mu I^{\mu\nu\lambda} = \frac{1}{\tau_{eq}} (u_\mu I^{\mu\nu\lambda}_{eq} - u_\mu I^{\mu\nu\lambda}), \] (6.20)

Where

\[ I^{\mu\nu\lambda} = \int dP p^\mu p^\nu p^\lambda f(x, p). \] (6.21)
\[ I = I_u [u \otimes u \otimes u] \]
\[ + \ I_x [u \otimes X \otimes X + X \otimes u \otimes X + X \otimes X \otimes u] \]
\[ + \ I_y [u \otimes Y \otimes Y + Y \otimes u \otimes Y + Y \otimes Y \otimes u] \]
\[ + \ I_z [u \otimes Z \otimes Z + Z \otimes u \otimes Z + Z \otimes Z \otimes u] \]
\[ + \ I_{xy} [u \otimes X \otimes Y + X \otimes u \otimes Y + X \otimes Y \otimes u + (x \leftrightarrow y)] \]
\[ + \ I_{xz} [u \otimes X \otimes Z + X \otimes u \otimes Z + X \otimes Z \otimes u + (x \leftrightarrow z)] \]
\[ + \ I_{yz} [u \otimes Y \otimes Z + Y \otimes u \otimes Z + Y \otimes Z \otimes u + (y \leftrightarrow z)] . \]
\[ (6.22) \]

Here we list the coefficients appearing in the second moment tensor basis

\[ I_u = \alpha I_{eq} \left[ \sum_i \alpha_i^2 + \frac{\xi_{xy}^2}{2} \alpha_x^2 \alpha_y^2 + \frac{\xi_{xz}^2}{2} \alpha_x^2 \alpha_z^2 + \frac{\xi_{yz}^2}{2} \alpha_y^2 \alpha_z^2 \right] + \frac{\xi_{xy}^2}{2} \alpha_y^2 \alpha_z^2 \alpha_x^2 + \frac{\xi_{xz}^2}{2} \alpha_x^2 \alpha_z^2 + \frac{\xi_{yz}^2}{2} \alpha_y^2 \alpha_z^2 \]
\[ I_x = \alpha I_{eq} \left[ \alpha_x^2 + \frac{\xi_{yz}^2}{2} \alpha_x^2 \alpha_y^2 + \frac{3 \xi_{xy}^2}{2} \alpha_x^2 \alpha_y^2 + \frac{3 \xi_{xz}^2}{2} \alpha_x^2 \alpha_z^2 \right] \]
\[ I_y = \alpha I_{eq} \left[ \alpha_y^2 + \frac{\xi_{xz}^2}{2} \alpha_x^2 \alpha_y^2 + \frac{3 \xi_{xy}^2}{2} \alpha_x^2 \alpha_y^2 + \frac{3 \xi_{yz}^2}{2} \alpha_y^2 \alpha_z^2 \right] \]
\[ I_z = \alpha I_{eq} \left[ \alpha_z^2 + \frac{\xi_{xy}^2}{2} \alpha_x^2 \alpha_y^2 + \frac{3 \xi_{xz}^2}{2} \alpha_x^2 \alpha_z^2 + \frac{3 \xi_{yz}^2}{2} \alpha_y^2 \alpha_z^2 \right] \]
\[ I_{xy} = \alpha \alpha_x^2 \alpha_y^2 I_{eq} \left[ -\xi_{xy} + \alpha_x^2 \xi_{xz} \xi_{yz} \right] \]
\[ I_{xz} = \alpha \alpha_x^2 \alpha_z^2 I_{eq} \left[ -\xi_{xz} + \alpha_y^2 \xi_{xy} \xi_{yz} \right] \]
\[ I_{yz} = \alpha \alpha_y^2 \alpha_z^2 I_{eq} \left[ -\xi_{yz} + \alpha_x^2 \xi_{xy} \xi_{xz} \right] \]
\[ (6.23) \]

Where \( I_{eq} \equiv 32 \pi \bar{N} \lambda^5 \).
Expanding Eq. (6.20) and taking its \( uu \)-, \( XX \)-, \( YY \)-, and \( ZZ \)-projections gives

\[
D_u \mathcal{I}_u + \mathcal{I}_u \theta_u + 2 \mathcal{I}_u u_\mu D_x X^\mu + 2 \mathcal{I}_y u_\mu D_y Y^\mu + 2 \mathcal{I}_z u_\mu D_z Z^\mu + 2 u_\mu \left[ \mathcal{I}_{x y} (D_y x^\mu + D_x y^\mu) + \mathcal{I}_{x z} (D_z x^\mu + D_x z^\mu) + \mathcal{I}_{y z} (D_z y^\mu + D_y z^\mu) \right] = \frac{1}{\tau_{eq}} (\mathcal{I}_{u,eq} - \mathcal{I}_u),
\]

(6.24)

\[
D_u \mathcal{I}_x + \mathcal{I}_x (\theta_u + 2 u_\mu D_x X^\mu) - 2 x_\mu \left[ \mathcal{I}_{x y} (D_u y^\mu + D_y u^\mu) + \mathcal{I}_{x z} (D_u z^\mu + D_z u^\mu) \right] = \frac{1}{\tau_{eq}} (\mathcal{I}_{eq} - \mathcal{I}_x),
\]

(6.25)

\[
D_u \mathcal{I}_y + \mathcal{I}_y (\theta_u + 2 u_\mu D_y Y^\mu) - 2 y_\mu \left[ \mathcal{I}_{x y} (D_u x^\mu + D_x u^\mu) + \mathcal{I}_{y z} (D_u z^\mu + D_z u^\mu) \right] = \frac{1}{\tau_{eq}} (\mathcal{I}_{eq} - \mathcal{I}_y),
\]

(6.26)

\[
D_u \mathcal{I}_z + \mathcal{I}_z (\theta_u + 2 u_\mu D_z Z^\mu) - 2 z_\mu \left[ \mathcal{I}_{x z} (D_u x^\mu + D_x u^\mu) + \mathcal{I}_{y z} (D_u y^\mu + D_y u^\mu) \right] = \frac{1}{\tau_{eq}} (\mathcal{I}_{eq} - \mathcal{I}_z).
\]

(6.27)

Also, taking \( uX \)-, \( uY \)-, and \( uZ \)-projections one can find

\[
D_x \mathcal{I}_x + \mathcal{I}_x \theta_x + (\mathcal{I}_x + \mathcal{I}_u) u_\mu D_u X^\mu - \mathcal{I}_y X_\mu D_y Y^\mu - \mathcal{I}_z X_\mu D_z Z^\mu
- \mathcal{I}_{x y} x_\mu (D_y y^\mu + D_y z^\mu) + D_y \mathcal{I}_{x y} + \mathcal{I}_{x y} (\theta_y + u_\mu D_y y^\mu - y_\mu D_y x^\mu) + D_z \mathcal{I}_{x z}
+ \mathcal{I}_{x z} (\theta_z + u_\mu D_u z^\mu - x_\mu D_x z^\mu) = 0,
\]

(6.28)

\[
D_y \mathcal{I}_y + \mathcal{I}_y \theta_y + (\mathcal{I}_y + \mathcal{I}_u) u_\mu D_u Y^\mu - \mathcal{I}_x Y_\mu D_x X^\mu - \mathcal{I}_z Y_\mu D_z Z^\mu
- \mathcal{I}_{x z} y_\mu (D_z x^\mu + D_z z^\mu) + D_z \mathcal{I}_{x y} + \mathcal{I}_{x y} (\theta_x + u_\mu D_y x^\mu - y_\mu D_y y^\mu) + D_y \mathcal{I}_{y z}
+ \mathcal{I}_{y z} (\theta_z + u_\mu D_u z^\mu - y_\mu D_y z^\mu) = 0,
\]

(6.29)

\[
D_z \mathcal{I}_z + \mathcal{I}_z \theta_z + (\mathcal{I}_z + \mathcal{I}_u) u_\mu D_u Z^\mu - \mathcal{I}_x Z_\mu D_x X^\mu - \mathcal{I}_y Z_\mu D_y Y^\mu
- \mathcal{I}_{x y} z_\mu (D_y x^\mu + D_y y^\mu) + D_y \mathcal{I}_{x z} + \mathcal{I}_{x z} (\theta_x + u_\mu D_y x^\mu - z_\mu D_y z^\mu) + D_y \mathcal{I}_{y z}
+ \mathcal{I}_{y z} (\theta_y + u_\mu D_u y^\mu - z_\mu D_z y^\mu) = 0,
\]

(6.30)

and finally projecting with \( XY \), \( XZ \), and \( YZ \) gives
\[ I_x (Y_{\mu} D_{\mu} X^\mu + Y_{\mu} D_x u^\mu) + I_y (X_{\mu} D_{\mu} Y^\mu + X_{\mu} D_y u^\mu) + I_{xz} y_{\mu} (D_{u} z^\mu + D_z u^\mu) \\
+ I_y z_{\mu} (D_{u} z^\mu + D_z u^\mu) - D_{u} I_{xy} - I_{xy} (\theta_{u} - x_{\mu} D_x u^\mu - y_{\mu} D_y u^\mu) \\
= -\frac{1}{\tau_{eq}} (I_{xy,eq} - I_{xy}), \quad (6.31) \]

\[ I_x (Z_{\mu} D_{\mu} X^\mu + Z_{\mu} D_x u^\mu) + I_z (X_{\mu} D_{\mu} Z^\mu + X_{\mu} D_z u^\mu) + I_{xy} z_{\mu} (D_{u} y^\mu + D_y u^\mu) \\
+ I_y z_{\mu} (D_{u} y^\mu + D_y u^\mu) - D_{u} I_{xz} - I_{xz} (\theta_{u} - x_{\mu} D_x u^\mu - z_{\mu} D_z u^\mu) \\
= -\frac{1}{\tau_{eq}} (I_{xz,eq} - I_{xz}), \quad (6.32) \]

\[ I_y (Z_{\mu} D_{\mu} Y^\mu + Z_{\mu} D_y u^\mu) + I_z (Y_{\mu} D_{\mu} Z^\mu + Y_{\mu} D_z u^\mu) + I_{xz} y_{\mu} (D_{u} x^\mu + D_x u^\mu) \\
+ I_x y_{\mu} (D_{u} x^\mu + D_x u^\mu) - D_{u} I_{yz} - I_{yz} (\theta_{u} - y_{\mu} D_y u^\mu - z_{\mu} D_z u^\mu) \\
= -\frac{1}{\tau_{eq}} (I_{yz,eq} - I_{yz}), \quad (6.33) \]

Note that \( I_{xy,eq} = I_{xz,eq} = I_{yz,eq} = 0. \)
Chapter 7

Summary and outlook

7.1 Summary

In this dissertation, we presented a new method for imposing a realistic equation of state in anisotropic hydrodynamics which is called quasiparticle anisotropic hydrodynamics (aHydroQP). In this method, we introduced a single finite-temperature quasiparticle mass which is fit to QCD lattice data. By taking moments of Boltzmann equation assuming an anisotropic distribution function, we obtained a set of coupled partial differential equations which can be used to describe the 3+1d spacetime evolution of the QGP. Due to the numerical difficulties and the need to understand this new method more, instead of considering the 3+1d case immediately, we considered simpler cases first, 0+1d and 1+1d systems, and compared aHydroQP with some known methods like standard anisotropic hydrodynamics and second-order viscous hydrodynamics.

First, we specialized to the case of a 0+1d system undergoing boost-invariant Bjorken expansion and compared with standard anisotropic hydrodynamics (aHydro). We found practically no differences between the two methods’ results for the temperature evolution and the scaled energy density for different shear viscosity to entropy density ratios. When we compared the pressure anisotropy ($P_L/P_T$), we found only small differences, however, we found significant differences in the evolution of the bulk pressure correction. With such differences in the bulk pressure correction, we considered azimuthally-symmetric boost-invariant (1+1d) systems. We compared the quasiparticle model with the standard aHydro model and second order viscous hydrodynamics. We also showed the three methods’ predictions for the primordial particle spectra, total number of charged particles, and average transverse
momentum for various values of the shear viscosity to entropy density ratio $\eta/s$. We showed that they agree well for small shear viscosity to entropy density ratio, $\eta/s$, however there were clear differences at large $\eta/s$.

The last two tests were only theoretical comparisons. To perform comparisons with experimental results, we needed a 3+1d simulations. In this dissertation, we presented the phenomenological predictions of 3+1d quasiparticle anisotropic hydrodynamics compared with LHC 2.76 TeV Pb-Pb collisions. We showed comparisons of charged-hadron multiplicity, identified-particle spectra, identified-particle average transverse momentum, charged-particle elliptic flow, identified-particle elliptic flow, elliptic flow as a function of pseudorapidity, and HBT radii. Overall, we found good agreement between our model predictions and ALICE data.

Finally, we presented a theoretical calculations for next-leading-order anisotropic hydrodynamics in the conformal limit. We included the off-diagonal terms in the anisotropy tensor and then expanded assuming that the off-diagonal terms are small compared with the diagonal components. We derived the general dynamical equations for 3+1d systems derived by taking moments of Boltzmann equation.

7.2 Outlook

There is a room for a lot of improvement of our code and I will list here some possible projects to improve the agreement between our code and experimental results. We need to include the off-diagonal terms of the anisotropy tensor in the quasiparticle anisotropic hydrodynamics. Moreover, we used smooth initial condition but to describe some observables like $v_2$, $v_3$, etc one needs to consider fluctuating initial conditions. The shear viscosity to entropy density ratio was assumed in this dissertation to be constant but we know that in QGP the shear viscosity to entropy density ratio is not constant but temperature dependent. We also assumed the chemical potential to be zero but for comparisons with RHIC
experiments, especially the ongoing beam-energy-scan program, one needs to consider finite chemical potential.
Appendices

8.1 Basis Vectors

In the local rest frame (LRF), the basis vectors are defined as

\[ u_{\text{LRF}}^\mu \equiv (1, 0, 0, 0), \]  
(8.1)

\[ X_{\text{LRF}}^\mu \equiv (0, 1, 0, 0), \]  
(8.2)

\[ Y_{\text{LRF}}^\mu \equiv (0, 0, 1, 0), \]  
(8.3)

\[ Z_{\text{LRF}}^\mu \equiv (0, 0, 0, 1). \]  
(8.4)

Using the appropriate Lorentz transformations/rotations one can obtain \( X_\alpha^\mu \) in the lab rest frame \((X_\alpha, \text{LAB})\) knowing the \( X_\alpha^\mu \) in the local rest frame \((X_\alpha, \text{LRF})\) [156, 100]. This can be done using

\[ X_\alpha^\mu, \text{LAB} = (L_z R_z L_z)^{-1} X_\alpha^\mu, \text{LRF} = (L_z)^{-1}(R_z)^{-1}(L_x)^{-1} X_\alpha^\mu, \text{LRF}, \]  
(8.5)

where \( L_i \) represents a boost along the \( i \)th-axis and \( R_z \) represents a rotation around the \( z \)-axis.

By specifying \( L_x, R_z, \) and \( L_z \) one can obtain [156, 100].

\[
X_\alpha^\mu, \text{LAB} = \begin{pmatrix}
\cosh \vartheta & 0 & 0 & \sinh \vartheta \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\sinh \vartheta & 0 & 0 & \cosh \vartheta
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \phi & -\sin \phi & 0 \\
0 & \sin \phi & \cos \phi & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cosh \theta_\perp & \sinh \theta_\perp & 0 & 0 \\
\sinh \theta_\perp & \cosh \theta_\perp & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
(8.6)

To get the zeroth component one can use \( X_{0, \text{LAB}}^\mu \equiv u^{\mu\nu} \) in the LRF.
\[
u^\mu = \begin{pmatrix}
\cosh \vartheta & 0 & 0 & \sinh \vartheta \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\sinh \vartheta & 0 & 0 & \cosh \vartheta
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \phi & -\sin \phi & 0 \\
0 & \sin \phi & \cos \phi & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cosh \theta_\perp & \sinh \theta_\perp & 0 & 0 \\
\sinh \theta_\perp & \cosh \theta_\perp & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix}
\]

(8.7)

And by doing the same thing for all other three components one finds (in the lab frame) that

\[
u^\mu \equiv (\cosh \theta_\perp \cosh \vartheta, \sinh \theta_\perp \cos \varphi, \sinh \theta_\perp \sin \varphi, \cosh \theta_\perp \sinh \vartheta),
\]

(8.8)

\[
X^\mu \equiv (\sinh \theta_\perp \cosh \vartheta, \cosh \theta_\perp \cos \varphi, \cosh \theta_\perp \sin \varphi, \sinh \theta_\perp \sinh \vartheta),
\]

(8.9)

\[
Y^\mu \equiv (0, -\sin \varphi, \cos \varphi, 0),
\]

(8.10)

\[
Z^\mu \equiv (\sinh \vartheta, 0, 0, \cosh \vartheta),
\]

(8.11)

where the three fields \( \vartheta, \varphi, \) and \( \theta_\perp \) introduced above are functions of \((\tau, x, y, \varsigma)\). To be general, one can introduce the temporal and transverse components of flow velocity \( u_0 = \cosh \theta_\perp \), \( u_x = u_\perp \cos \varphi \), and \( u_y = u_\perp \sin \varphi \), where \( u_\perp \equiv \sqrt{u_x^2 + u_y^2} = \sqrt{u_0^2 - 1} = \sinh \theta_\perp \), then Eq. (8.11) can be written as

\[
u^\mu \equiv (u_0 \cosh \vartheta, u_x, u_y, u_0 \sinh \vartheta),
\]

(8.12)

\[
X^\mu \equiv \left( u_\perp \cosh \vartheta, \frac{u_0 u_x}{u_\perp}, \frac{u_0 u_y}{u_\perp}, u_\perp \sinh \vartheta \right) ,
\]

(8.13)

\[
Y^\mu \equiv \left( 0, -\frac{u_y}{u_\perp}, \frac{u_x}{u_\perp}, 0 \right) ,
\]

(8.14)

\[
Z^\mu \equiv (\sinh \vartheta, 0, 0, \cosh \vartheta).
\]

(8.15)

Let’s specify Eq. (8.11), in two cases. First, for a boost-invariant and cylindrically-symmetric system (1+1d) where \( \vartheta \) equals the spacetime rapidity \((\varsigma)\) and \( \varphi \) equals the azimuthal angle
In this case, Eq. (8.11) can be simplified to
\[
\begin{align*}
    u^\mu &= (\cosh \theta \cosh \varsigma, \sinh \theta \cos \phi, \sinh \theta \sin \phi, \cosh \theta \sinh \varsigma), \\
    X^\mu &= (\sinh \theta \cosh \varsigma, \cosh \theta \cos \phi, \cosh \theta \sin \phi, \sinh \theta \sinh \varsigma), \\
    Y^\mu &= (0, -\sin \phi, \cos \phi, 0), \\
    Z^\mu &= (\sinh \varsigma, 0, 0, \cosh \varsigma).
\end{align*}
\]

Second, for a transversally-symmetric system (0+1d) where the transverse flow \( u_\perp = 0 \), i.e. \( \theta_\perp = 0 \), Eq. (8.11) can be simplified to
\[
\begin{align*}
    u^\mu &= (\cosh \varsigma, 0, 0, \sinh \varsigma), \\
    X^\mu &= (0, \cos \phi, \sin \phi, 0), \\
    Y^\mu &= (0, -\sin \phi, \cos \phi, 0), \\
    Z^\mu &= (\sinh \varsigma, 0, 0, \cosh \varsigma).
\end{align*}
\]
8.2 Second-order viscous hydrodynamics

The dynamical equations of second-order viscous hydrodynamics can be obtained from taking moments of Boltzmann equation [92, 88]. The dynamical equations obtained using first and second moments with non-vanishing bulk viscosity are

\[(\mathcal{E} + \mathcal{P} + \Pi)D_u u^\mu = \nabla^\mu (\mathcal{P} + \Pi) - \Delta^\mu_\nu \nabla_\sigma \pi^{\nu\sigma} + \pi^{\mu\nu} D_u u_\nu, \tag{8.24}\]

\[\begin{align*}
D_u \mathcal{E} & = - (\mathcal{E} + \mathcal{P} + \Pi) \theta_u + \pi^{\mu\nu} \sigma_{\mu\nu}, \\
\tau_\Pi D_u \Pi + \Pi & = - \zeta \theta_u - \delta_{\Pi\Pi} \Pi \theta_u + \varphi_1 \Pi^2 + \lambda_{\Pi\Pi} \pi^{\mu\nu} \sigma_{\mu\nu} + \varphi_3 \pi^{\mu\nu} \pi_{\mu\nu}, \tag{8.25, 8.26}\end{align*}\]

where \(\mathcal{E}\) and \(\mathcal{P}\) are the equilibrium energy density and pressure respectively obtained by the realistic equation of state, \(\tau_\Pi\) and \(\tau_\Pi\) are the shear and bulk relaxation time, and \(\tau_{\pi\pi}\) is the shear-shear-coupling transport coefficient.

Before going further, let’s define some notations used in the dynamical equations above

\[d_\mu u^\nu \equiv \partial_\mu u^\nu + \Gamma^\nu_{\mu\alpha} u^\alpha, \tag{8.28}\]

\[D_u \equiv u_\mu d^\mu, \tag{8.29}\]

\[\theta_u \equiv \nabla_\mu u^\mu, \tag{8.30}\]

\[\nabla^\mu \equiv \Delta^\mu_\nu d_\nu, \tag{8.31}\]

\[\sigma^{\mu\nu} \equiv \nabla^\mu (u^\nu), \tag{8.32}\]

\[A^{(\mu\nu)} \equiv \Delta^{\mu\nu}_{\alpha\beta} A^{\alpha\beta}, \tag{8.33}\]

\[\Delta^{\mu\nu}_{\alpha\beta} \equiv \frac{1}{2} \left( \Delta^{\mu\nu}_{\alpha} \Delta^\nu_{\beta} + \Delta^{\mu\nu}_{\beta} \Delta^\nu_{\alpha} - \frac{2}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta} \right), \tag{8.34}\]

\[\omega^{\mu\nu} \equiv \frac{1}{2} (\nabla^\mu u^\nu - \nabla^\nu u^\mu). \tag{8.35}\]
\[ d_\mu u^\nu \equiv \partial_\mu u^\nu + \Gamma^\nu_{\mu\alpha} u^\alpha, \quad \sigma^{\mu\nu} \equiv \nabla^{(\mu} u^{\nu)}, \]
\[ D_\mu \equiv u_\mu d^\mu, \quad A^{(\mu\nu)} \equiv \Delta^{\mu\nu}_{\alpha\beta} A^{\alpha\beta}, \quad \theta_\mu \equiv \nabla_\mu u^\mu, \quad \Delta^{\mu\nu}_{\alpha\beta} \equiv \frac{1}{2} (\Delta^{\mu}_{\alpha} \Delta^{\nu}_{\beta} + \Delta^{\mu}_{\beta} \Delta^{\nu}_{\alpha} - \frac{2}{3} \Delta^{\mu\nu} \Delta^{\alpha\beta}), \]
\[ \nabla^{\mu} \equiv \Delta^{\mu\nu} d_\nu, \quad \omega^{\mu\nu} \equiv \frac{1}{2} (\nabla^{\mu} u^{\nu} - \nabla^{\nu} u^{\mu}). \]

where \( \Gamma^\nu_{\mu\alpha} \) are the Christoffel symbols for polar Milne coordinates where most of them vanish and we left up only with \( \Gamma^\tau_{\xi\xi} = \tau, \Gamma^\xi_{\tau} = 1/\tau, \Gamma^r_{\phi\phi} = -r, \) and \( \Gamma^\phi_{r\phi} = 1/r. \)

Going back to the dynamical equation, we notice that in our case, many terms vanish for different reasons as we will list below. First, the vorticity tensor vanishes due to using smooth initial conditions. Second, since we are using the RTA some terms will vanish because their appearance is due to the collision term being non-linear in the distribution function. These terms are \( \varphi_1 \Pi^2, \varphi_3 \pi^{\mu\nu} \pi_{\mu\nu}, \varphi_6 \Pi \pi^{\mu\nu}, \) and \( \varphi_7 \pi^{(\mu} \pi^{\nu)\alpha} \) which mean \( \varphi_1 = \varphi_3 = \varphi_6 = \varphi_7 = 0. \) Ref. [157]. Moreover, the shear and bulk relaxation times are equal to the relaxation time, i.e., \( \tau_\Pi = \tau_\pi = \tau_{\text{eq}} \) [92].

Finally, the transport coefficients appearing in the dynamical equations are [92]

\[ \frac{\zeta}{\tau_\Pi} = \left( \frac{1}{3} - c_s^2 \right) (E + P) - \frac{2}{9} (E - 3P), \quad (8.37) \]
\[ \frac{\delta_{\Pi\Pi}}{\tau_\Pi} = 1 - c_s^2, \quad (8.38) \]
\[ \frac{\lambda_{\Pi\pi}}{\tau_\Pi} = \frac{1}{3} - c_s^2, \quad (8.39) \]
\[ \frac{\eta}{\tau_\pi} = \frac{4}{5} P + \frac{1}{15} (E - 3P), \quad (8.40) \]
\[ \frac{\delta_{\pi\pi}}{\tau_\pi} = \frac{4}{3}, \quad (8.41) \]
\[ \frac{\tau_{\pi\pi}}{\tau_\pi} = \frac{10}{7}, \quad (8.42) \]
\[ \frac{\lambda_{\pi\Pi}}{\tau_\pi} = \frac{6}{5}, \quad (8.43) \]

where \( c_s^2 \equiv dP/dE \) is the speed of sound squared and the relaxation time is give by \( \tau_{\text{eq}} = \)
\[ 15\eta (\mathcal{E} + P)/(\mathcal{E} + 9P)/T. \]

1+1d viscous hydrodynamics equations of motion

In this case, the shear tensor has the following form

\[
\pi^{\mu\nu} = \begin{pmatrix}
\pi^{\tau\tau} & \pi^{\tau r} & 0 & 0 \\
\pi^{\tau r} & \pi^{rr} & 0 & 0 \\
0 & 0 & \pi^{\phi\phi} & 0 \\
0 & 0 & 0 & \pi^{\varsigma\varsigma}
\end{pmatrix},
\]

(8.44)

Expanding Eqs. (8.24), (8.25), and (8.26) results to five independent equations with another dependent equation (total six equations)

\[
(\mathcal{E} + P + \Pi) D_u u^r = -(u^r)^2 \left[ \partial_r (P + \Pi) - d_{\nu} \pi^{\nu r} \right] - u^r u^r \left[ \partial_r (P + \Pi) - d_{\nu} \pi^{\nu r} \right],
\]

(8.45)

\[
(\mathcal{E} + P + \Pi) D_u u^r = -u^r u^r \left[ \partial_r (P + \Pi) - d_{\nu} \pi^{\nu r} \right] - (u^r)^2 \left[ \partial_r (P + \Pi) - d_{\nu} \pi^{\nu r} \right],
\]

(8.46)

\[
D_u \mathcal{E} = -(\mathcal{E} + P + \Pi) \theta_u - \pi^{\tau r} (1 - v^2) \nabla^{(r} u^{r)} - r^2 \pi^{\phi\phi} \nabla^{(\phi} u^{\phi)}
\]

\[- \tau^2 \pi^{\varsigma\varsigma} \nabla^{(\varsigma} u^{\varsigma)},
\]

(8.47)

and

\[
\tau_{\Pi} D_u \Pi + \Pi = -\zeta \theta_u - \delta_{\Pi \Pi} \Pi \theta_u - \lambda_{\Pi \pi} \left[ 2r^2 \pi^{\phi\phi} \nabla^{(\phi} u^{\phi)} + 2r^2 \pi^{\varsigma\varsigma} \nabla^{(\varsigma} u^{\varsigma)} \right]
\]

\[
+ r^2 \pi^{\varsigma\varsigma} \nabla^{(\phi} u^{\phi)} + \tau^2 \pi^{\phi\phi} \nabla^{(\varsigma} u^{\varsigma)} \right],
\]

(8.48)

\[
\tau_{\pi} D_u \pi^{\phi\phi} + \pi^{\phi} = -2r^2 \eta \nabla^{(\phi} u^{\phi)} - \delta_{\pi \pi} \pi^{\phi\phi} \theta_u - \frac{\tau_{\pi \pi}}{3} \left[ -r^2 \pi^{\phi\phi} \nabla^{(\phi} u^{\phi)} + 2r^2 \pi^{\varsigma\varsigma} \nabla^{(\varsigma} u^{\varsigma)} \right]
\]

\[
+ r^2 \pi^{\varsigma\varsigma} \nabla^{(\phi} u^{\phi)} + \tau^2 \pi^{\phi\phi} \nabla^{(\varsigma} u^{\varsigma)} \right] - r^2 \lambda_{\pi \Pi \Pi} \nabla^{(\phi} u^{\phi)},
\]

(8.49)

\[
\tau_{\pi} D_u \pi^{\varsigma\varsigma} + \pi^{\varsigma} = -2r^2 \eta \nabla^{(\varsigma} u^{\varsigma)} - \delta_{\pi \pi} \pi^{\varsigma\varsigma} \theta_u - \frac{\tau_{\pi \pi}}{3} \left[ 2r^2 \pi^{\phi\phi} \nabla^{(\phi} u^{\phi)} - r^2 \pi^{\varsigma\varsigma} \nabla^{(\varsigma} u^{\varsigma)} \right]
\]

\[
+ r^2 \pi^{\varsigma\varsigma} \nabla^{(\phi} u^{\phi)} + \tau^2 \pi^{\phi\phi} \nabla^{(\varsigma} u^{\varsigma)} \right] - r^2 \lambda_{\pi \Pi \Pi} \nabla^{(\varsigma} u^{\varsigma)},
\]

(8.50)
where

\[ \begin{align*}
-d_v \pi^\nu_r &= v^2 \partial_r \pi^\nu_r + v \partial_r \pi^\tau_r + \pi^\tau_r \left[ \partial_r v^2 + \partial_r v + \frac{v^2}{r} + \frac{v}{r} \right] + \frac{1}{r} \pi^\tau_r, \\
 d_v \pi^\tau_r &= v \partial_r \pi^\nu_r + \partial_r \pi^\tau_r + \pi^\tau_r \left[ \partial_r v + \frac{2 - v^2}{r} \right] + \frac{1}{r} \pi^\tau_r. 
\end{align*} \]  

(8.51, 8.52)

with the following identities

\[ \begin{align*}
\nabla^{(u^r)} &= -\partial_r u^r - u^r D_u u^r + \frac{1}{3} (u^r)^2 \theta_u, \\
r^2 \nabla^{(u^\phi)} &= -\frac{u^r}{r} + \frac{1}{3} \theta_u, \\
\tau^2 \nabla^{(u^\varsigma)} &= -\frac{u^\tau}{\tau} + \frac{1}{3} \theta_u, \\
\theta_u &\equiv \nabla a u^\alpha = d_a u^\alpha = \partial_r u^r + \partial_r u^r + u^r + \frac{u^r}{r}, 
\end{align*} \]  

(8.53, 8.54, 8.55)

where \( \pi^\tau_r = -v \pi^\nu_r \) and \( \pi^\phi = -\pi^\varsigma - (1 - v^2) \pi^\nu_r \).

**Viscous hydrodynamics freeze-out**

At freeze-out, the distribution function used in viscous hydrodynamics is [158, 159]

\[ f(p, x) = f_{eq}(p, x) + \delta f_{\text{shear}}(p, x) + \delta f_{\text{bulk}}(p, x), \]  

(8.57)

where

\[ \begin{align*}
\delta f_{\text{shear}}(p, x) &= f_{eq}(1 - a f_{eq}) \frac{p_\mu p_\nu \pi^{\mu\nu}}{2(E + P) T^2}, \\
\delta f_{\text{bulk}}(p, x) &= -f_{eq}(1 - a f_{eq}) \left[ \frac{m_i^2}{3 p_\mu u^\mu} - \left( \frac{1}{3} - c_s^2 \right) p_\mu u^\mu \right] \frac{\Pi}{C}, 
\end{align*} \]  

(8.58, 8.59)

with

\[ C = \frac{1}{3} \sum_{i=1}^{N} m_i^2 (2s_i + 1) \int \frac{d^3 p}{(2\pi)^3 E_i} f_{eq}(1 - a f_{eq}) \left[ \frac{m_i^2}{3 p_\mu u^\mu} - \left( \frac{1}{3} - c_s^2 \right) p_\mu u^\mu \right], \]  

(8.60)

as we defined before, \( m_i \) is the hadron mass, \( s_i \) is the hadron spin, and \( N \) is the number of hadrons included in the freeze-out. The components of the four-momentum in polar Milne
coordinates are

\[ p_\tau = \tau \cosh \varsigma - \tau \sinh \varsigma = m_{\perp} \cosh(y - \varsigma), \]
\[ p_r = p_x \cos \phi + p_y \sin \phi = p_{\perp} \cos(\phi - \varphi), \]
\[ p_\phi = -p_x \frac{\sin \phi}{r} + p_y \frac{\cos \phi}{r} = -\frac{p_{\perp}}{r} \sin(\phi - \varphi), \]
\[ p_\varsigma = -p_t \frac{\sinh \varsigma}{\tau} + p_z \frac{\cosh \varsigma}{\tau} = \frac{m_{\perp}}{\tau} \sinh(y - \varsigma). \] (8.61)

Using Eq. (8.44) and expanding \( p_\mu p_\nu \pi^{\mu\nu} \) in polar Milne coordinates one has

\[ p_\mu p_\nu \pi^{\mu\nu} = - \left( \frac{\pi^\phi}{v^2 - 1} \right) \left( m_{\perp} v \cosh(y - \varsigma) - p_{\perp} \cos(\phi - \varphi) \right)^2 \]
\[ - \pi^\phi \frac{p_{\perp}^2}{v^2} \sin^2(\phi - \varphi) - \pi^\varsigma \frac{m_{\perp}^2}{v^2} \sinh^2(y - \varsigma). \] (8.62)
8.3 Explicit formulas for derivatives

Here, we introduce the notations used in the body of the thesis when deriving the dynamical equations in 3+1d first, and then specify to 1+1d and 0+1 cases. Using the definitions

\[ \mathcal{D} \equiv \cosh(\vartheta - \varsigma)\partial_\tau + \frac{1}{\tau}\sinh(\vartheta - \varsigma)\partial_\varsigma, \]
\[ \tilde{\mathcal{D}} \equiv \sinh(\vartheta - \varsigma)\partial_\tau + \frac{1}{\tau}\cosh(\vartheta - \varsigma)\partial_\varsigma, \]  

(8.63)

\[ \nabla_\perp \cdot \mathbf{u}_\perp \equiv \partial_x u_x + \partial_y u_y, \]
\[ \mathbf{u}_\perp \cdot \nabla_\perp \equiv u_x \partial_x + u_y \partial_y, \]
\[ \mathbf{u}_\perp \times \nabla_\perp \equiv u_x \partial_y - u_y \partial_x, \]  

(8.64)

and the four-vectors defined previously in Eq. (8.15) are

\[ D_u \equiv u^\mu \partial_\mu = u_0 \mathcal{D} + \mathbf{u}_\perp \cdot \nabla_\perp, \]
\[ D_x \equiv X^\mu \partial_\mu = u_\perp \mathcal{D} + \frac{u_0}{u_\perp}(\mathbf{u}_\perp \cdot \nabla_\perp), \]
\[ D_y \equiv Y^\mu \partial_\mu = \frac{1}{u_\perp}(\mathbf{u}_\perp \times \nabla_\perp), \]
\[ D_z \equiv Z^\mu \partial_\mu = \tilde{\mathcal{D}}. \]  

(8.65)

In the general case, the divergences are given by

\[ \theta_u \equiv \partial_\mu u^\mu = \mathcal{D}u_0 + u_0 \tilde{\mathcal{D}} \vartheta + \nabla_\perp \cdot \mathbf{u}_\perp, \]
\[ \theta_x \equiv \partial_\mu X^\mu = \mathcal{D}u_\perp + u_\perp \tilde{\mathcal{D}} \vartheta + \frac{u_0}{u_\perp}(\nabla_\perp \cdot \mathbf{u}_\perp) + \frac{1}{u_\perp u_\perp^2}(\mathbf{u}_\perp \cdot \nabla_\perp)u_\perp, \]
\[ \theta_y \equiv \partial_\mu Y^\mu = -\frac{1}{u_\perp}(\mathbf{u}_\perp \cdot \nabla_\perp) \varphi, \]
\[ \theta_z \equiv \partial_\mu Z^\mu = \mathcal{D} \vartheta, \]  

(8.66)
where \( \varphi = \tan^{-1}(u_y/u_x) \).

\[
\begin{align*}
\mu D_\alpha X^\mu &= \frac{1}{u_0} D_\alpha u_\perp, \\
\mu D_\alpha Y^\mu &= u_\perp D_\alpha \varphi, \\
\mu D_\alpha Z^\mu &= u_\parallel D_\alpha \vartheta, \\
X_\mu D_\alpha Y^\mu &= u_\perp D_\alpha \varphi, \\
X_\mu D_\alpha Z^\mu &= u_\perp D_\alpha \vartheta, \\
Y_\mu D_\alpha Z^\mu &= 0, \quad \text{(8.67)}
\end{align*}
\]

where \( \alpha \in \{ u, x, y, z \} \). One should note that using the orthogonality of the basis vectors, i.e. \( D_\alpha(X^\mu u_\mu) = 0 \), contractions like \( X^\mu D_\alpha u_\mu \) can be found where \( X^\mu D_\alpha u_\mu = -u_\mu D_\alpha X^\mu \).

### 8.3.1 Simplification for 1+1d

Let’s turn now to a simpler case, the case where the system possesses a boost-invariant and cylindrically-symmetric flow. In this case, \( \varphi \rightarrow \phi \) and \( \vartheta \rightarrow \varsigma \), where \( \varsigma \) is the spatial rapidity. Using \( u_\perp \equiv \sinh \theta_\perp \), Eq. (8.64) can be written as

\[
\begin{align*}
\mathcal{D} &= \partial_r, \\
\tilde{\mathcal{D}} &= \frac{1}{\tau} \partial_\varsigma, \\
\nabla_\perp \cdot u_\perp &= \partial_r u_\perp + \frac{u_\perp}{r}, \\
u_\perp \cdot \nabla_\perp &= u_\perp \partial_r, \\
u_\perp \times \nabla_\perp &= \frac{u_\perp}{r} \partial_\phi. \quad \text{(8.68)}
\end{align*}
\]
In the same way the identities (8.66) become

\[ D_u = \cosh \theta \partial_r + \sinh \theta \partial_r, \quad \theta_u = \cosh \theta \left( \frac{1}{\tau} + \partial_r \theta \right) + \sinh \theta \left( \frac{1}{r} + \partial_r \theta \right), \]
\[ D_x = \sinh \theta \partial_r + \cosh \theta \partial_r, \quad \theta_x = \sinh \theta \left( \frac{1}{\tau} + \partial_r \theta \right) + \cosh \theta \left( \frac{1}{r} + \partial_r \theta \right), \]
\[ D_y = \frac{1}{r} \partial_\phi, \quad \theta_y = 0, \]
\[ D_z = \frac{1}{r} \partial_\kappa, \quad \theta_z = 0. \]

(8.69)

In this limit, the only non-vanishing terms in (8.67) are

\[ u^\mu D_\mu X^\mu = D_u \theta, \quad u^\mu D_\mu Z^\mu = \frac{1}{\tau} \cosh \theta, \]
\[ u^\mu D_\mu X^\mu = D_x \theta, \quad X^\mu D_\mu Y^\mu = \frac{1}{r} \cosh \theta, \]
\[ u^\mu D_\mu Y^\mu = \frac{1}{r} \sinh \theta, \quad X^\mu D_\mu Z^\mu = \frac{1}{\tau} \sinh \theta. \]

8.3.2 Simplification for 0+1d

In this rather simple case where \( \theta = 0 \), one can show that

\[ D_u = \partial_r, \quad \theta_u = \frac{1}{\tau}, \]
\[ D_x = \partial_r, \quad \theta_x = \frac{1}{r}, \]
\[ D_y = \frac{\partial_\phi}{r}, \quad \theta_y = 0, \]
\[ D_z = \frac{\partial_\kappa}{\tau}, \quad \theta_z = 0. \]

(8.70)

Finally, \( u^\mu D_\mu Z^\mu = 1/\tau \), and \( X^\mu D_\mu Y^\mu = 1/r \).
8.4 Special functions

In this section, we list the definitions of the special functions introduced above in the body of the text

\[ H_2(y, z) = \frac{y}{\sqrt{y^2 - 1}} \left[ (z^2 + 1) \tanh^{-1} \sqrt{\frac{y^2 - 1}{y^2 + z^2} + \sqrt{(y^2 - 1)(y^2 + z^2)}} \right], \quad (8.71) \]

\[ H_{2T}(y, z) = \frac{y}{(y^2 - 1)^{3/2}} \left[ (z^2 + 2y^2 - 1) \tanh^{-1} \sqrt{\frac{y^2 - 1}{y^2 + z^2} - \sqrt{(y^2 - 1)(y^2 + z^2)}} \right], \quad (8.72) \]

\[ H_{2L}(y, z) = \frac{y^3}{(y^2 - 1)^{3/2}} \left[ \sqrt{(y^2 - 1)(y^2 + z^2)} - (z^2 + 1) \tanh^{-1} \frac{y^2 - 1}{y^2 + z^2} \right], \quad (8.73) \]

and

\[ H_{2x_1}(y, z) = \frac{1}{(y^2 - 1)} \left[ \frac{2(y^2 + z^2)H_{2L}(y, z)}{1 + z^2} - y^2H_{2T}(y, z) \right], \quad (8.74) \]

\[ H_{2x_2}(y, z) = \frac{y^2}{(y^2 - 1)} \left[ 2H_{2L}(y, z) - H_{2T}(y, z) \right], \quad (8.75) \]

\[ H_{2B}(y, z) = H_{2T}(y, z) + \frac{H_{2L}(y, z)}{y^2} = \frac{2}{\sqrt{y^2 - 1}} \tanh^{-1} \frac{y^2 - 1}{y^2 + z^2}. \quad (8.76) \]

Now, let’s provide their derivatives

\[ \frac{\partial H_2(y, z)}{\partial y} = \frac{1}{y} \left[ H_2(y, z) + H_{2L}(y, z) \right], \quad (8.77) \]

\[ \frac{\partial H_2(y, z)}{\partial z} = \frac{1}{z} \left[ H_2(y, z) - H_{2L}(y, z) - H_{2T}(y, z) \right], \quad (8.78) \]

\[ \frac{\partial H_{2T}(y, z)}{\partial y} = \frac{1}{y(y^2 - 1)} \left[ 2H_{2L}(y, z) - H_{2T}(y, z) \right], \quad (8.79) \]

\[ \frac{\partial H_{2T}(y, z)}{\partial z} = \frac{-2z}{y^2(1 + z^2)} H_{2L}(y, z). \quad (8.80) \]
8.4.1 In the massive case

The definitions of $\mathcal{H}$-functions in the massive case are

\[
\mathcal{H}_3(\alpha, \hat{m}) \equiv \hat{N}\alpha_x\alpha_y \int_0^{2\pi} d\phi \alpha_\perp \int_0^\infty d\hat{p} \hat{p}^3 f_{eq}\left(\sqrt{\hat{p}^2 + \hat{m}^2}\right) \mathcal{H}_2\left(\frac{\alpha_z}{\alpha_\perp}, \frac{\hat{m}}{\alpha_\perp \hat{p}}\right), \quad (8.81)
\]

\[
\mathcal{H}_{3x}(\alpha, \hat{m}) \equiv \hat{N}\alpha_x^3\alpha_y \int_0^{2\pi} d\phi \cos^2 \phi \int_0^\infty d\hat{p} \hat{p}^3 f_{eq}\left(\sqrt{\hat{p}^2 + \hat{m}^2}\right) \mathcal{H}_{2T}\left(\frac{\alpha_z}{\alpha_\perp}, \frac{\hat{m}}{\alpha_\perp \hat{p}}\right), \quad (8.82)
\]

\[
\mathcal{H}_{3y}(\alpha, \hat{m}) \equiv \hat{N}\alpha_x\alpha_y^3 \int_0^{2\pi} d\phi \sin^2 \phi \int_0^\infty d\hat{p} \hat{p}^3 f_{eq}\left(\sqrt{\hat{p}^2 + \hat{m}^2}\right) \mathcal{H}_{2T}\left(\frac{\alpha_z}{\alpha_\perp}, \frac{\hat{m}}{\alpha_\perp \hat{p}}\right), \quad (8.83)
\]

\[
\mathcal{H}_{3T}(\alpha, \hat{m}) \equiv \frac{1}{2}\left[\mathcal{H}_{3x}(\alpha, \hat{m}) + \mathcal{H}_{3y}(\alpha, \hat{m})\right], \quad (8.84)
\]

\[
\mathcal{H}_{3L}(\alpha, \hat{m}) \equiv \hat{N}\alpha_x\alpha_y \int_0^{2\pi} d\phi \alpha_\perp \int_0^\infty d\hat{p} \hat{p}^3 f_{eq}\left(\sqrt{\hat{p}^2 + \hat{m}^2}\right) \mathcal{H}_{2L}\left(\frac{\alpha_z}{\alpha_\perp}, \frac{\hat{m}}{\alpha_\perp \hat{p}}\right), \quad (8.85)
\]

\[
\mathcal{H}_{3B}(\alpha, \hat{m}) \equiv \hat{N}\alpha_x\alpha_y \int_0^{2\pi} d\phi \int_0^\infty d\hat{p} \hat{p} f_{eq}\left(\sqrt{\hat{p}^2 + \hat{m}^2}\right) \mathcal{H}_{2B}\left(\frac{\alpha_z}{\alpha_\perp}, \frac{\hat{m}}{\alpha_\perp \hat{p}}\right), \quad (8.86)
\]

and we can define for compactness

\[
\Omega_T(\alpha, \hat{m}) \equiv \mathcal{H}_3 + \mathcal{H}_{3T}, \quad (8.87)
\]

\[
\Omega_L(\alpha, \hat{m}) \equiv \mathcal{H}_3 + \mathcal{H}_{3L}, \quad (8.88)
\]

\[
\Omega_m(\alpha, \hat{m}) \equiv \mathcal{H}_3 - \mathcal{H}_{3L} - 2\mathcal{H}_{3T} - \mathcal{H}_{3m}, \quad (8.89)
\]

Where their derivatives are

\[
\frac{\partial \mathcal{H}_3}{\partial \alpha_x} = \frac{1}{\alpha_x} (\mathcal{H}_3 + \mathcal{H}_{3x}), \quad \frac{\partial \mathcal{H}_{3x}}{\partial \alpha_x} = \frac{1}{\alpha_x} (3\mathcal{H}_{3x} - \mathcal{H}_{3x1}),
\]

\[
\frac{\partial \mathcal{H}_3}{\partial \alpha_y} = \frac{1}{\alpha_y} (\mathcal{H}_3 + \mathcal{H}_{3y}), \quad \frac{\partial \mathcal{H}_{3x}}{\partial \alpha_y} = \frac{1}{\alpha_y} (\mathcal{H}_{3x} - \mathcal{H}_{3x2}),
\]

\[
\frac{\partial \mathcal{H}_3}{\partial \alpha_z} = \frac{1}{\alpha_z} (\mathcal{H}_3 + \mathcal{H}_{3L}), \quad \frac{\partial \mathcal{H}_{3x}}{\partial \alpha_z} = \frac{1}{\alpha_z} \mathcal{H}_{3x3},
\]

\[
\frac{\partial \mathcal{H}_3}{\partial \hat{m}} = \frac{1}{\hat{m}} (\mathcal{H}_3 - \mathcal{H}_{3L} - 2\mathcal{H}_{3T} - \mathcal{H}_{3m}), \quad \frac{\partial \mathcal{H}_{3x}}{\partial \hat{m}} = \frac{1}{\hat{m}} (\mathcal{H}_{3m1} - \mathcal{H}_{3m2}),
\]

with

\[
\mathcal{H}_{3x1}(\alpha, \hat{m}) \equiv \hat{N}\alpha_x^5\alpha_y \int_0^{2\pi} d\phi \cos^4 \phi \int_0^\infty d\hat{p} \hat{p}^3 f_{eq}\left(\sqrt{\hat{p}^2 + \hat{m}^2}\right) \mathcal{H}_{2x1}\left(\frac{\alpha_z}{\alpha_\perp}, \frac{\hat{m}}{\alpha_\perp \hat{p}}\right), \quad (8.91)
\]

\[
\mathcal{H}_{3x2}(\alpha, \hat{m}) \equiv \hat{N}\alpha_x^{2}\alpha_y^2 \int_0^{2\pi} d\phi \cos^2 \phi \sin^2 \phi \int_0^\infty d\hat{p} \hat{p}^3 f_{eq}\left(\sqrt{\hat{p}^2 + \hat{m}^2}\right) \mathcal{H}_{2x1}\left(\frac{\alpha_z}{\alpha_\perp}, \frac{\hat{m}}{\alpha_\perp \hat{p}}\right), \quad (8.92)
\]

\[
\mathcal{H}_{3x3}(\alpha, \hat{m}) \equiv \hat{N}\alpha_x^{2}\alpha_y \int_0^{2\pi} d\phi \alpha_\perp^2 \cos^2 \phi \int_0^\infty d\hat{p} \hat{p} f_{eq}\left(\sqrt{\hat{p}^2 + \hat{m}^2}\right) \mathcal{H}_{2x2}\left(\frac{\alpha_z}{\alpha_\perp}, \frac{\hat{m}}{\alpha_\perp \hat{p}}\right), \quad (8.93)
\]
\( \mathcal{H}_{3m}(\alpha, \hat{m}) \equiv \tilde{N}\alpha_x\alpha_y \hat{m}^2 \int_0^{2\pi} d\phi \alpha_\perp^2 \int_0^\infty d\hat{p} \hat{p}^3 \frac{f_{eq}(\sqrt{\hat{p}^2 + \hat{m}^2})}{\sqrt{\hat{p}^2 + \hat{m}^2}} \mathcal{H}_2 \left( \frac{\alpha_z}{\alpha_\perp}, \frac{\hat{m}}{\alpha_\perp \hat{p}} \right) , \) (8.94)

\( \mathcal{H}_{3m1}(\alpha, \hat{m}) \equiv \mathcal{H}_{3x1}(\alpha, \hat{m}) + \mathcal{H}_{3x2}(\alpha, \hat{m}) - \mathcal{H}_{3x3}(\alpha, \hat{m}) , \) (8.95)

\( \mathcal{H}_{3m2}(\alpha, \hat{m}) \equiv \tilde{N}\alpha_x^3 \alpha_y \hat{m}^2 \int_0^{2\pi} d\phi \cos^2 \phi \int_0^\infty d\hat{p} \hat{p}^3 \frac{f_{eq}(\sqrt{\hat{p}^2 + \hat{m}^2})}{\sqrt{\hat{p}^2 + \hat{m}^2}} \mathcal{H}_{2T} \left( \frac{\alpha_z}{\alpha_\perp}, \frac{\hat{m}}{\alpha_\perp \hat{p}} \right) , \) (8.96)

where \( \alpha_\perp^2 \equiv \alpha_x^2 \cos^2 \phi + \alpha_y^2 \sin^2 \phi . \)

### 8.4.2 0+1d case

In the case of a 0+1d system, the transversally symmetry can be used giving \( \alpha_x = \alpha_y \) and \( \alpha_\perp = \alpha_x \). This gives \( \bar{\mathcal{H}}_{3T} \equiv \bar{\mathcal{H}}_{3x} = \bar{\mathcal{H}}_{3y} \), and the \( \mathcal{H} \) functions in this case can be simplified to

\( \bar{\mathcal{H}}_3(\alpha, \hat{m}) \equiv 2\pi \tilde{N}\alpha_x^4 \int_0^\infty d\hat{p} \hat{p}^3 f_{eq}(\sqrt{\hat{p}^2 + \hat{m}^2}) \mathcal{H}_2 \left( \frac{\alpha_z}{\alpha_x}, \frac{\hat{m}}{\alpha_x \hat{p}} \right) , \) (8.97)

\( \bar{\mathcal{H}}_{3T}(\alpha, \hat{m}) \equiv \pi \tilde{N}\alpha_x^4 \int_0^\infty d\hat{p} \hat{p}^3 f_{eq}(\sqrt{\hat{p}^2 + \hat{m}^2}) \mathcal{H}_{2T} \left( \frac{\alpha_z}{\alpha_x}, \frac{\hat{m}}{\alpha_x \hat{p}} \right) , \) (8.98)

\( \bar{\mathcal{H}}_{3L}(\alpha, \hat{m}) \equiv 2\pi \tilde{N}\alpha_x^4 \int_0^\infty d\hat{p} \hat{p}^3 f_{eq}(\sqrt{\hat{p}^2 + \hat{m}^2}) \mathcal{H}_{2L} \left( \frac{\alpha_z}{\alpha_x}, \frac{\hat{m}}{\alpha_x \hat{p}} \right) , \) (8.99)

\( \bar{\mathcal{H}}_{3m}(\alpha, \hat{m}) \equiv 2\pi \tilde{N}\alpha_x^4 \hat{m}^2 \int_0^\infty d\hat{p} \hat{p}^3 f_{eq}(\sqrt{\hat{p}^2 + \hat{m}^2}) \mathcal{H}_2 \left( \frac{\alpha_z}{\alpha_x}, \frac{\hat{m}}{\alpha_x \hat{p}} \right) , \) (8.100)

\( \bar{\mathcal{H}}_{3B}(\alpha, \hat{m}) \equiv 2\pi \tilde{N}\alpha_x^2 \hat{m} \int_0^\infty d\hat{p} \hat{p} f_{eq}(\sqrt{\hat{p}^2 + \hat{m}^2}) \mathcal{H}_{2B} \left( \frac{\alpha_z}{\alpha_x}, \frac{\hat{m}}{\alpha_x \hat{p}} \right) . \) (8.101)

with some identities of \( \bar{\mathcal{H}}_3 \) derivatives

\( \frac{\partial \bar{\mathcal{H}}_3}{\partial \alpha_x} = \frac{2}{\alpha_x} \bar{\Omega}_T , \) (8.102)

\( \frac{\partial \bar{\mathcal{H}}_3}{\partial \alpha_z} = \frac{1}{\alpha_z} \bar{\Omega}_L , \) (8.103)

\( \frac{\partial \bar{\mathcal{H}}_3}{\partial \hat{m}} = \frac{1}{\hat{m}} \bar{\Omega}_m . \) (8.104)
For simplification, we list the $H$ functions in the isotropic equilibrium case

\[
\tilde{H}_{3,\text{eq}}(\hat{m}_{\text{eq}}) = 4\pi \tilde{N} \hat{m}^2_{\text{eq}} \left[ \hat{m}_{\text{eq}}K_1(\hat{m}_{\text{eq}}) + 3K_2(\hat{m}_{\text{eq}}) \right],
\]

8.105

\[
\tilde{H}_{3T,\text{eq}}(\hat{m}_{\text{eq}}) = \tilde{H}_{3L,\text{eq}}(\hat{m}_{\text{eq}}) = 4\pi \tilde{N} \hat{m}_{\text{eq}}^2 K_2(\hat{m}_{\text{eq}}),
\]

8.106

\[
\tilde{H}_{3m,\text{eq}}(\hat{m}_{\text{eq}}) = 4\pi \tilde{N} \hat{m}_{\text{eq}}^4 K_2(\hat{m}_{\text{eq}}).
\]

8.107

8.4.3 In the massless case

In the standard case \((\lim_{m \to 0})\), the $\hat{H}$-functions used are

\[
\tilde{H}_3(\alpha) \equiv \frac{1}{24\pi N} \lim_{m \to 0} H_3(\alpha, \hat{m}) = \frac{1}{4\pi} \alpha_x \alpha_y \int_0^{2\pi} d\phi \alpha^2_\perp \bar{H}_2 \left( \frac{\alpha_z}{\alpha_\perp} \right),
\]

8.108

\[
\tilde{H}_{3x}(\alpha) \equiv \frac{1}{8\pi N} \lim_{m \to 0} H_{3x}(\alpha, \hat{m}) = \frac{3}{4\pi} \alpha^3_x \alpha_y \int_0^{2\pi} d\phi \cos^2 \phi \bar{H}_2 \left( \frac{\alpha_z}{\alpha_\perp} \right),
\]

8.109

\[
\tilde{H}_{3y}(\alpha) \equiv \frac{1}{8\pi N} \lim_{m \to 0} H_{3y}(\alpha, \hat{m}) = \frac{3}{4\pi} \alpha_x \alpha^3_y \int_0^{2\pi} d\phi \sin^2 \phi \bar{H}_2 \left( \frac{\alpha_z}{\alpha_\perp} \right),
\]

8.110

\[
\tilde{H}_{3L}(\alpha) \equiv \frac{1}{8\pi N} \lim_{m \to 0} H_{3L}(\alpha, \hat{m}) = \frac{3}{4\pi} \alpha_x \alpha_y \int_0^{2\pi} d\phi \alpha^2_\perp \bar{H}_2 \left( \frac{\alpha_z}{\alpha_\perp} \right),
\]

8.111

where $\bar{H}_{2,2T,2L}(y) \equiv H_{2,2T,2L}(y, 0)$.

Now, in 0+1d, the above $H$ functions can be simplified more by noticing that $\alpha_x = \alpha_y$ and $\tilde{H}_{3T} \equiv \tilde{H}_{3x} = \tilde{H}_{3y}$ giving

\[
\tilde{H}_3(\alpha) = 12\pi \tilde{N} \alpha^4_x \bar{H}_2 \left( \frac{\alpha_z}{\alpha_x} \right),
\]

8.112

\[
\tilde{H}_{3T}(\alpha) = 6\pi \tilde{N} \alpha^4_x \bar{H}_{2T} \left( \frac{\alpha_z}{\alpha_x} \right),
\]

8.113

\[
\tilde{H}_{3L}(\alpha) = 12\pi \tilde{N} \alpha^4_x \bar{H}_{2L} \left( \frac{\alpha_z}{\alpha_x} \right).
\]

8.114

8.4.4 A way around doing integrals in $\hat{H}$-functions

As presented before in the previous section, we were not able to do the $\hat{H}$-functions' integrals analytically. The most we can do is to integrate over $\theta$, and leave two integrals over $\phi$ and $p$ to be done numerically. This is not a problem in the 0+1d and 1+1d systems since doing these integrals in these cases are are sufficiently quick. However, if one considers 3+1d systems, then one must further optimize the evolution of these integrals. First, one could
consider simply evaluating them in the real-time, but that is numerically intensive and will slow the code too much. Second, one might consider interpolating them, however, this is not practical since each function is a function of 4 variables \((\alpha, \hat{m})\). Not only this, but one needs to interpolate beside the basic \(\hat{H}\)-functions their derivatives, 4 derivatives for each function. As you can see, this will be a memory problem that one will run to it making it impossible to use interpolation. As a result, one needs to come up with another method to do these integrals.

Before presenting the new alternative method, let’s mention two important things. First, let’s rewrite the \(\hat{H}\)-functions appearing before in Sec. 8.4.1 for aHydroQP in the following general form

\[
\hat{H}_3(\alpha, \hat{m}) \equiv \tilde{N}_\alpha \int d^3\hat{p} f_{eq}(\sqrt{\hat{p}^2 + \hat{m}^2}),
\]

\[
\hat{H}_{3i}(\alpha, \hat{m}) \equiv \tilde{N}_\alpha \alpha_i^2 \int d^3\hat{p} R_i f_{eq}(\sqrt{\hat{p}^2 + \hat{m}^2}),
\]

\[
\hat{H}_{3B}(\alpha, \hat{m}) \equiv \tilde{N}_\alpha \int d^3\hat{p} R^{-1} f_{eq}(\sqrt{\hat{p}^2 + \hat{m}^2}),
\]

where

\[
R \equiv \sqrt{\alpha_x^2 \hat{p}_x^2 + \alpha_y^2 \hat{p}_y^2 + \alpha_z^2 \hat{p}_z^2 + \hat{m}^2},
\]

\[
R_i \equiv \hat{p}_i^2 R^{-1}.
\]

Second, there is another nice property to note between \(\hat{H}\)-functions and their derivatives: they are not all independent i.e., there are some symmetries that one can use to relate them to each other. So, it turns out using theses symmetries that one just need to calculate \(\hat{H}_3\), \(\hat{H}_{3X}\) and some of their derivatives as we will demonstrate later on. So, we will explain the method in the \(\hat{H}\)-functions that we need to calculate only and then list the symmetries that one can use to relate others later on.

This alternative method is based on expanding theses integrals around the diagonal in anisotropy space where the leading term is around the isotropy point \((\alpha_x = \alpha_y = \alpha_z = 1)\).
This is justified by noticing that $\alpha_i$ in practice do not evolve too far from isotropy point. Once we expand assuming small $\alpha_i$, then the angular part of the integrals factorize out and becomes trivial. Then the only part left is the $p$ integral which still can’t be done analytically but straightforward to interpolate since it is only a function of one variable, the mass as we will see later on. To make the formalism more general and to cover more regions, we make the expansion around an arbitrary point that we will specify later on instead of $\alpha_x = \alpha_y = \alpha_z = 1$. We define this point by $\alpha_i^2 \sim \delta_0$ using $\alpha_i^2 = \delta_0 + \delta_i \epsilon$ where $\delta_0$ is the point that the expansion is made around, and $\epsilon$ is related to the order of the expansion. Using this notation, $R$ and $R_x$ can be expanded as

$$R = \sum_{n=0}^{\infty} \left( \frac{1}{2n} \right) \left( \hat{m}^2 + \delta_0 \hat{p}^2 \right)^{\frac{1}{2}-n} \hat{p}^{2n} \left[ \sin^2 \theta \left( \delta_x \cos^2 \phi + \delta_y \sin^2 \phi \right) + \delta_z \cos^2 \theta \right]^n,$$

$$R_x = \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right) \left( \hat{m}^2 + \delta_0 \hat{p}^2 \right)^{-\frac{1}{2}-n} \hat{p}^{2+2n} \cos^2 \phi \sin^2 \theta \left[ \sin^2 \theta \left( \delta_x \cos^2 \phi + \delta_y \sin^2 \phi \right) + \delta_z \cos^2 \theta \right]^n,$$

where $\delta_i = \alpha_i^2 - \delta_0$.

By plugging $R$ back in Eq. (8.115) and splitting the integral to angular part and radial ($p$) part, $H_3(\alpha, \hat{m})$ can be written as

$$H_3(\alpha, \hat{m}) = \tilde{N}\alpha \sum_{n=0}^{\infty} \left( \frac{1}{2} \right) \Omega(\delta, n) \mathcal{G}(\frac{1}{2} - n, \hat{m}, \delta_0),$$

(8.120)

where $\Omega(\delta, n)$ is the angular part of the integral

$$\Omega(\delta, n) = \int d\Omega \left[ \sin^2 \theta \left( \delta_x \cos^2 \phi + \delta_y \sin^2 \phi \right) + \delta_z \cos^2 \theta \right]^n,$$

(8.121)

and the radial part is

$$\mathcal{G}(a, \hat{m}, \delta_0) = \int d\hat{p} \hat{p}^{3-2a} \left( \hat{m}^2 + \delta_0 \hat{p}^2 \right)^a f_{eq} \left( \sqrt{\hat{p}^2 + \hat{m}^2} \right).$$

(8.122)

Similarly, using $R_X$, $H_{3x}(\alpha, \hat{m})$ can be written as

$$H_{3x}(\alpha, \hat{m}) = \tilde{N}\alpha \alpha_x^2 \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right) \Omega_x(\delta, n) \mathcal{G}(\frac{1}{2} - n, \hat{m}, \delta_0),$$

(8.123)
where $\Omega_x(\delta, n)$ is the angular part of the integral

$$
\Omega_x(\delta, n) = \int d\Omega \cos^2 \phi \sin^2 \theta \left[ \sin^2 \theta (\delta_x \cos^2 \phi + \delta_y \sin^2 \phi) + \delta_z \cos^2 \theta \right]^n. \quad (8.124)
$$

In both $H_3$ and $H_{3x}$, the integral in $\Omega$ and $\Omega_x$ is trivial to evaluate and one is left with a one dimensional integral over $p$ which is simple enough to interpolate.

Next, let’s turn to the derivatives of both $H_3$ and $H_{3x}$ with respect to $\alpha$’s. Since the radial part of the integral $G(a, \hat{m}, \delta_0)$ is independent of $\alpha$, the derivatives of both $H_3$ and $H_{3x}$ with respect to $\alpha$’s are straightforward (the integral part factorizes out). Now, it is time to use the symmetries of both $H$ and $H_X$ to calculate their derivatives. First, $H_3$ is symmetric under the exchange of $\alpha$’s, i.e.,

$$
H_3(\alpha_x, \alpha_y, \alpha_z, \hat{m}) = H_3(\alpha_y, \alpha_x, \alpha_z, \hat{m}) = H_3(\alpha_z, \alpha_y, \alpha_x, \hat{m}). \quad (8.125)
$$

As can be seen from these identities, once $\partial H_3/\partial \alpha_x$ is known, the other derivatives with respect to $\alpha_y$ and $\alpha_z$ are related by the exchange symmetry

$$
\frac{\partial H_3(\alpha_x, \alpha_y, \alpha_z, \hat{m})}{\partial \alpha_y} = \frac{\partial H_3(\alpha_y, \alpha_x, \alpha_z, \hat{m})}{\partial \alpha_x}, \quad (8.126)
$$

$$
\frac{\partial H_3(\alpha_x, \alpha_y, \alpha_z, \hat{m})}{\partial \alpha_z} = \frac{\partial H_3(\alpha_z, \alpha_y, \alpha_x, \hat{m})}{\partial \alpha_x}. \quad (8.127)
$$

In a similar way, $H_{3x}$ is symmetric under the exchange of $\alpha_y$ and $\alpha_z$ only

$$
H_{3x}(\alpha_x, \alpha_y, \alpha_z, \hat{m}) = H_{3x}(\alpha_x, \alpha_z, \alpha_y, \hat{m}). \quad (8.128)
$$

So in this case we have only one identity which can be used to relate only two derivatives, $\partial H_3/\partial \alpha_y$ and $\partial H_3/\partial \alpha_z$.

$$
\frac{\partial H_{3x}(\alpha_x, \alpha_y, \alpha_z, \hat{m})}{\partial \alpha_z} = \frac{\partial H_{3x}(\alpha_x, \alpha_z, \alpha_y, \hat{m})}{\partial \alpha_y}. \quad (8.129)
$$

As a result, we need to calculate $\partial H_3/\partial \alpha_x$ and $\partial H_3/\partial \alpha_y$ and then use Eq. (8.129) to find the third derivative $\partial H_3/\partial \alpha_z$. 

128
Next, we turn to the derivative with respect to the fourth argument, \( \hat{m} \) which has to be handled differently since it appears on the integral part \( G(a, \hat{m}, \delta_0) \). Taking the derivative with respect to \( \hat{m} \) of both \( H_3 \) and \( H_{3x} \) gives another integral which is easy to interpolate like the original integral

\[
G_m(a, \hat{m}, \delta_0) = - \int dp \frac{\hat{m}^{3-2a}(\hat{m}^2 + \delta_0 \hat{p}^2)^{\alpha-1}}{2\sqrt{\hat{m}^2 + \hat{p}^2}} \left( \hat{m}^2 - 2a \sqrt{\hat{m}^2 + \hat{p}^2 + \delta_0 \hat{p}^2} \right) f_{eq}\left( \sqrt{\hat{p}^2 + \hat{m}^2} \right).
\]

Using this new integral, the derivatives with respect to \( \hat{m} \) can be written as

\[
\frac{\partial H_3}{\partial \hat{m}} = 2\hat{m} \alpha N \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)_n \Omega(\delta, n) G_m\left(\frac{1}{2} - n, \hat{m}, \delta_0 \right),
\]

(8.130)

\[
\frac{\partial H_{3x}}{\partial \hat{m}} = 2\hat{m} \alpha^2_3 \alpha N \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)_n \Omega_x(\delta, n) G_m\left(-\frac{1}{2} - n, \hat{m}, \delta_0 \right).
\]

(8.131)

Now, we will use the symmetries between the \( H \) functions themselves to evaluate \( H_{3y} \), \( H_{3z} \), and \( H_{3B} \)

\[
H_{3y}(\alpha_x, \alpha_y, \alpha_z, \hat{m}) = H_{3z}(\alpha_y, \alpha_x, \alpha_z, \hat{m}),
\]

(8.132)

\[
H_{3z}(\alpha_x, \alpha_y, \alpha_z, \hat{m}) = H_{3x}(\alpha_z, \alpha_y, \alpha_x, \hat{m}),
\]

(8.133)

\[
H_{3B}(\alpha, \hat{m}) = \frac{1}{\hat{m}^2}(H_3(\alpha, \hat{m}) - H_{3z}(\alpha, \hat{m}) - H_{3y}(\alpha, \hat{m}) - H_{3z}(\alpha, \hat{m})).
\]

(8.134)

The \( H_{3y} \) and \( H_{3z} \) derivatives can be found in a similar way, and below we list them all

\[
\begin{align*}
\frac{\partial H_{3y}}{\partial \alpha_x}(\alpha_x, \alpha_y, \alpha_z, \hat{m}) &= \frac{\partial H_{3z}}{\partial \alpha_y}(\alpha_y, \alpha_x, \alpha_z, \hat{m}), \\
\frac{\partial H_{3y}}{\partial \alpha_y}(\alpha_x, \alpha_y, \alpha_z, \hat{m}) &= \frac{\partial H_{3x}}{\partial \alpha_x}(\alpha_y, \alpha_x, \alpha_z, \hat{m}), \\
\frac{\partial H_{3y}}{\partial \alpha_z}(\alpha_x, \alpha_y, \alpha_z, \hat{m}) &= \frac{\partial H_{3x}}{\partial \alpha_z}(\alpha_y, \alpha_x, \alpha_z, \hat{m}), \\
\frac{\partial H_{3y}}{\partial \hat{m}}(\alpha_x, \alpha_y, \alpha_z, \hat{m}) &= \frac{\partial H_{3z}}{\partial \hat{m}}(\alpha_y, \alpha_x, \alpha_z, \hat{m}).
\end{align*}
\]

The \( H_{3y} \) and \( H_{3z} \) derivatives can be found in a similar way, and below we list them all
and
\[
\begin{align*}
\frac{\partial H_3(z, \alpha_x, \alpha_y, \alpha_z, \hat{m})}{\partial \alpha_x} &= \frac{\partial H_3(x, \alpha_z, \alpha_y, \alpha_z, \hat{m})}{\partial \alpha_z}, \\
\frac{\partial H_3(z, \alpha_x, \alpha_y, \alpha_z, \hat{m})}{\partial \alpha_y} &= \frac{\partial H_3(x, \alpha_z, \alpha_y, \alpha_z, \hat{m})}{\partial \alpha_y}, \\
\frac{\partial H_3(z, \alpha_x, \alpha_y, \alpha_z, \hat{m})}{\partial \alpha_z} &= \frac{\partial H_3(x, \alpha_z, \alpha_y, \alpha_z, \hat{m})}{\partial \alpha_x}, \\
\frac{\partial H_3(z, \alpha_x, \alpha_y, \alpha_z, \hat{m})}{\partial \hat{m}} &= \frac{\partial H_3(x, \alpha_z, \alpha_y, \alpha_z, \hat{m})}{\partial \hat{m}}.
\end{align*}
\]

Finally, let’s explain the details of the expansions. Since \(\alpha_i\)’s usually evolve not far from isotropy, \(0 < \alpha_i \lesssim 3\), we expand around two points corresponding to \(\delta_0 = 1, 4\) and smoothly interpolate between the two distinct regions. So \(H_3\) can be written as
\[
H_3(\alpha, \hat{m}, \delta_0) = \begin{cases} 
H_3(\alpha, \hat{m}, 1) & \text{If } r < r_{\text{min}}, \\
H_3(\alpha, \hat{m}, 4) & \text{If } r > r_{\text{max}}, \\
\frac{r_{\text{max}} - r}{r_{\text{max}} - r_{\text{min}}} H_3(\alpha, \hat{m}, 1) + \frac{r - r_{\text{min}}}{r_{\text{max}} - r_{\text{min}}} H_3(\alpha, \hat{m}, 4) & \text{otherwise}. 
\end{cases}
\]

where \(r_{\text{min}}\) and \(r_{\text{max}}\) set the ranges of the two regions with \(r = \sqrt{\alpha_x^2 + \alpha_y^2 + \alpha_z^2}\). Here, we use \(r_{\text{min}} = 1.75\) and \(r_{\text{max}} = 1.85\) where we find the best agreement with the numerical integration. In a very similar way, \(H_{3x}\) and all needed derivatives can be calculated using exactly the same setup.

One last thing to specify, in all cases, we expand up to 12\(^{th}\) order (\(n \leq 12\)) which give a very good agreement with the direct numerical evaluation of all \(H\)-functions.
8.5 NLO special functions

In this section, we introduce definitions of special functions appearing in the next-to-leading order anisotropic hydrodynamics in Chap. 6. We start by introducing a general form for the special functions

\[ R_{abc}(y) \equiv y \int_{-1}^{1} d(cos \theta) (y^2 \cos^2 \theta + \sin^2 \theta)^{a/2} \sin^{b} \theta \cos^{c} \theta \quad (8.136) \]

\[ R_{100}(y) \equiv y \int_{-1}^{1} d(cos \theta) \sqrt{y^2 \cos^2 \theta + \sin^2 \theta} \]

\[ R_{120}(y) \equiv y \int_{-1}^{1} d(cos \theta) \sin^{4} \theta \sqrt{y^2 \cos^2 \theta + \sin^2 \theta} \]

\[ R_{111}(y) \equiv y \int_{-1}^{1} d(cos \theta) \sin^2 \theta \cos^2 \theta \sqrt{y^2 \cos^2 \theta + \sin^2 \theta} \quad (8.137) \]

\[ R_{-110}(y) \equiv y \int_{-1}^{1} d(cos \theta) \frac{\sin^2 \theta}{\sqrt{y^2 \cos^2 \theta + \sin^2 \theta}} \]

\[ R_{-101}(y) \equiv y \int_{-1}^{1} d(cos \theta) \frac{\cos^2 \theta}{\sqrt{y^2 \cos^2 \theta + \sin^2 \theta}} \]

\[ R_{-130}(y) \equiv y \int_{-1}^{1} d(cos \theta) \frac{\sin^6 \theta}{\sqrt{y^2 \cos^2 \theta + \sin^2 \theta}} \]

\[ R_{-121}(y) \equiv y \int_{-1}^{1} d(cos \theta) \frac{\sin^4 \theta \cos^2 \theta}{\sqrt{y^2 \cos^2 \theta + \sin^2 \theta}} \]

\[ R_{-112}(y) \equiv y \int_{-1}^{1} d(cos \theta) \frac{\sin^2 \theta \cos^4 \theta}{\sqrt{y^2 \cos^2 \theta + \sin^2 \theta}} \]

\[ R_{-120}(y) \equiv y \int_{-1}^{1} d(cos \theta) \frac{\sin^4 \theta \cos^2 \theta}{\sqrt{y^2 \cos^2 \theta + \sin^2 \theta}} \]

\[ R_{-111}(y) \equiv y \int_{-1}^{1} d(cos \theta) \frac{\sin^2 \theta \cos^2 \theta}{\sqrt{y^2 \cos^2 \theta + \sin^2 \theta}} \quad (8.138) \]

Now, we turn to the general functions appearing in the bulk variables

\[ H_{abcd}(\alpha) \equiv \frac{\tilde{N}}{2} \alpha \int d^3 \hat{p} \left( \frac{1}{\sum_i \alpha_i^2 \hat{p}_i^2} \right)^a (\alpha_x \hat{p}_x)^{2a} (\alpha_y \hat{p}_y)^{2b} (\alpha_z \hat{p}_z)^{2c} (\hat{p} + 1) f_{\text{iso}}(\hat{p}) \quad (8.139) \]

\[ H_{abc}(\alpha) \equiv \tilde{N} \alpha \int d^3 \hat{p} \left( \frac{1}{\sum_i \alpha_i^2 \hat{p}_i^2} \right)^{-1} (\alpha_x \hat{p}_x)^{2a} (\alpha_y \hat{p}_y)^{2b} (\alpha_z \hat{p}_z)^{2c} f_{\text{iso}}(\hat{p}) \quad (8.140) \]
\[ H_3(\alpha) \equiv \frac{1}{4\pi} \alpha_x \alpha_y \int_0^{2\pi} d\phi \, \alpha_\perp^2 R_{100} \left( \frac{\alpha_z}{\alpha_\perp} \right), \]
\[ H_{1110}(\alpha) \equiv \frac{3}{\pi} \alpha_x^3 \alpha_y \int_0^{2\pi} d\phi \, \cos^2 \phi \, \sin^2 \phi \, \alpha_\perp^2 R_{120} \left( \frac{\alpha_z}{\alpha_\perp} \right), \]
\[ H_{1101}(\alpha) \equiv \frac{3}{\pi} \alpha_x^3 \alpha_y \alpha_\perp^2 \int_0^{2\pi} d\phi \, \cos^2 \phi \, \alpha_\perp^2 R_{111} \left( \frac{\alpha_z}{\alpha_\perp} \right), \]
\[ H_{1011}(\alpha) \equiv \frac{3}{\pi} \alpha_x \alpha_y^3 \alpha_\perp^2 \int_0^{2\pi} d\phi \, \sin^2 \phi \, \alpha_\perp^2 R_{111} \left( \frac{\alpha_z}{\alpha_\perp} \right), \quad (8.141) \]

\[ H_{3z}(\alpha) \equiv \frac{3}{4\pi} \alpha_x^3 \alpha_y \int_0^{2\pi} d\phi \, \cos^2 \phi \, R_{-110} \left( \frac{\alpha_z}{\alpha_\perp} \right), \]
\[ H_{-1210}(\alpha) \equiv \frac{9}{\pi} \alpha_x^5 \alpha_y^3 \int_0^{2\pi} d\phi \, \cos^4 \phi \, \sin^2 \phi \, R_{-130} \left( \frac{\alpha_z}{\alpha_\perp} \right), \]
\[ H_{-1201}(\alpha) \equiv \frac{9}{\pi} \alpha_x^5 \alpha_y \alpha_\perp^2 \int_0^{2\pi} d\phi \, \cos^4 \phi \, R_{-121} \left( \frac{\alpha_z}{\alpha_\perp} \right), \]
\[ H_{-1111}(\alpha) \equiv \frac{9}{\pi} \alpha_x^3 \alpha_y^3 \alpha_\perp^2 \int_0^{2\pi} d\phi \, \cos^2 \phi \, \sin^2 \phi \, R_{-121} \left( \frac{\alpha_z}{\alpha_\perp} \right), \quad (8.142) \]

\[ H_{3y}(\alpha) \equiv \frac{3}{4\pi} \alpha_x \alpha_y^3 \int_0^{2\pi} d\phi \, \sin^2 \phi \, R_{-110} \left( \frac{\alpha_z}{\alpha_\perp} \right), \]
\[ H_{-1120}(\alpha) \equiv \frac{9}{\pi} \alpha_x^3 \alpha_y^5 \int_0^{2\pi} d\phi \, \cos^2 \phi \, \sin^4 \phi \, R_{-130} \left( \frac{\alpha_z}{\alpha_\perp} \right), \]
\[ H_{-1021}(\alpha) \equiv \frac{9}{\pi} \alpha_x \alpha_y^5 \alpha_\perp^2 \int_0^{2\pi} d\phi \, \sin^4 \phi \, R_{-121} \left( \frac{\alpha_z}{\alpha_\perp} \right), \quad (8.143) \]

\[ H_{3L}(\alpha) \equiv \frac{3}{4\pi} \alpha_x \alpha_y \alpha_\perp^2 \int_0^{2\pi} d\phi \, R_{-101} \left( \frac{\alpha_z}{\alpha_\perp} \right), \]
\[ H_{-1102}(\alpha) \equiv \frac{9}{\pi} \alpha_x^3 \alpha_y^4 \alpha_\perp^2 \int_0^{2\pi} d\phi \, \cos^2 \phi \, R_{-112} \left( \frac{\alpha_z}{\alpha_\perp} \right), \]
\[ H_{-1012}(\alpha) \equiv \frac{9}{\pi} \alpha_x \alpha_y^4 \alpha_\perp^4 \int_0^{2\pi} d\phi \, \sin^2 \phi \, R_{-112} \left( \frac{\alpha_z}{\alpha_\perp} \right), \quad (8.144) \]

\[ H_{110}(\alpha) \equiv 24 \tilde{N} \alpha_x^3 \alpha_y^3 \int_0^{2\pi} d\phi \, \cos^2 \phi \, \sin^2 \phi R_{-120} \left( \frac{\alpha_z}{\alpha_\perp} \right), \]
\[ H_{101}(\alpha) \equiv 24 \tilde{N} \alpha_x^3 \alpha_y \alpha_\perp^2 \int_0^{2\pi} d\phi \, \cos^2 \phi R_{-111} \left( \frac{\alpha_z}{\alpha_\perp} \right), \]
\[ H_{011}(\alpha) \equiv 24 \tilde{N} \alpha_x \alpha_y^3 \alpha_\perp^2 \int_0^{2\pi} d\phi \, \sin^2 \phi R_{-111} \left( \frac{\alpha_z}{\alpha_\perp} \right), \quad (8.145) \]
8.5.1 Calculating the conformal NLO $\mathcal{H}$-functions

In the last section, we presented the $\mathcal{H}$-functions needed in the body of the paper. There are many of them and each one of them have a three derivatives with respect to $\alpha$’s. Therefore, we need a way to avoid interpolations and numerical integrations. At the same time, we want to use as many symmetries as possible to relate the functions and their derivatives to each other. First, to avoid doing the integrations numerically which is numerically intensive or the interpolations which is a problem regarding the available space in the memory. We will use the conformality of the $\mathcal{H}$-functions and the symmetry between $\alpha$’s to find a way to analytically solve them, for example

Next, we will find a general function which all other functions can be derived from

$$G(\alpha) = \frac{\alpha}{4\pi} \int d\Omega \left( \sin^2 \theta \left( \alpha_x^2 \cos^2 \phi + \alpha_y^2 \sin^2 \phi \right) + \alpha_z^2 \cos^2 \theta \right)^{5/2}.$$  \hfill (8.146)

Clearly, $G$ is symmetric under the exchange of $\alpha$’s, for example

$$G(\alpha_x, \alpha_y, \alpha_z) = G(\alpha_y, \alpha_x, \alpha_z),$$  \hfill (8.147)

with similar results for exchanging $\alpha_x \leftrightarrow \alpha_z$ and $\alpha_y \leftrightarrow \alpha_z$.

$G$ is also conformal, i.e.

$$G(l\alpha_x, l\alpha_y, l\alpha_z) = l^8 G(\alpha_x, \alpha_y, \alpha_z).$$  \hfill (8.148)

keeping in mind these properties, one may expand $G$. So, let’s define the following functions which preserves the symmetry and conformality

$$p_1 = \frac{1}{3}(\alpha_x + \alpha_y + \alpha_z),$$

$$p_2 = \frac{1}{3}(\alpha_x\alpha_y + \alpha_x\alpha_z + \alpha_y\alpha_z),$$

$$p_3 = \alpha_x\alpha_y\alpha_z,$$ \hfill (8.149)

Hence, $G$ can be written as
\( \mathcal{G}(\alpha) = \sum_{n,l,m} p_1^n p_2^l p_3^m C_{nlm} \). 

(8.150)

Using the conformality, we must have \( n + 2l + 3m = 8 \), so we can use this to write \( n = 8 - 2l - 3m \).

\( \mathcal{G}(\alpha) = \sum_{l,m} p_1^{8-2l-3m} p_2^l p_3^m C_{nlm} \),

(8.151)

where \( -\infty \leq l \leq \infty \) and \( m \geq 1 \). Changing \( m \to m + 1 \) we find

\( \mathcal{G}(\alpha) = p_1^5 p_3^{\sum_{l,m} p_1^{8-2l-3m} p_2^l p_3^m C_{nlm}} \),

(8.152)

where \( -\infty \leq l \leq \infty \) and \( m \geq 0 \). Since we find \( \mathcal{G} \), \( H_{abcd} \)'s defined in Eq. (8.139) can be calculated, for example

\[
H_{1110}(\alpha) = \frac{4}{5} \alpha_x^2 \alpha_y^2 \alpha_z^2 \frac{\partial}{\partial \alpha_x} \frac{\partial}{\partial \alpha_y} \mathcal{G},
\]

(8.153)

\[
H_{1101}(\alpha) = \frac{4}{5} \alpha_x^2 \alpha_y^2 \alpha_z^2 \frac{\partial}{\partial \alpha_x} \frac{\partial}{\partial \alpha_z} \mathcal{G},
\]

(8.154)

\[
H_{1011}(\alpha) = \frac{4}{5} \alpha_x^2 \alpha_y^2 \alpha_z^2 \frac{\partial}{\partial \alpha_y} \frac{\partial}{\partial \alpha_z} \mathcal{G}.
\]

(8.155)

Other \( H_{abcd} \)'s can be found using similar methods. However, in practice, one doesn’t need to calculate each function from \( \mathcal{G} \), symmetries between \( H_{abcd} \) functions can be used. All what is needed is knowing \( H_{1110} \), its derivatives \( \partial H_{1110}/\partial \alpha_i \), and second derivatives \( \partial^2 H_{1110}/(\partial \alpha_i \partial \alpha_j) \) where \( (i, j) \in (x, y, z) \). Below, we will list all identities that can be used to relate other functions to \( H_{1110} \).

\[
H_{1101}(\alpha, \alpha_y, \alpha_z) = H_{1110}(\alpha_x, \alpha_z, \alpha_y),
\]

\[
H_{1011}(\alpha, \alpha_y, \alpha_z) = H_{1110}(\alpha_z, \alpha_y, \alpha_x),
\]

(8.156)

Also, only two derivatives needed for \( H_{1110} \), one of them is related

\[
\frac{\partial}{\partial \alpha_y} H_{1110}(\alpha_x, \alpha_y, \alpha_z) = \frac{\partial}{\partial \alpha_x} H_{1110}(\alpha_y, \alpha_x, \alpha_z),
\]

(8.157)
Where we used $\mathcal{H}_{1110}(\alpha_x, \alpha_y, \alpha_z) = \mathcal{H}_{1110}(\alpha_y, \alpha_x, \alpha_z)$. For shortness, we will introduce a notation for the derivatives, $\partial/\partial \alpha_i$ to be $\partial_i$ which means the derivative with respect to the $i$th argument of the function. As an example,

$$\frac{\partial}{\partial \alpha_x} \mathcal{H}_{1110}(\alpha_x, \alpha_y, \alpha_z) \equiv \partial_1 \mathcal{H}_{1110}(\alpha_x, \alpha_y, \alpha_z).$$ (8.158)

Next, we will list the derivatives of both $\mathcal{H}_{1101}$ and $\mathcal{H}_{1011}$

$$\partial_1 \mathcal{H}_{1101}(\alpha_x, \alpha_y, \alpha_z) = \partial_1 \mathcal{H}_{1110}(\alpha_x, \alpha_z, \alpha_y),$$

$$\partial_2 \mathcal{H}_{1101}(\alpha_x, \alpha_y, \alpha_z) = \partial_1 \mathcal{H}_{1110}(\alpha_x, \alpha_z, \alpha_y),$$

$$\partial_3 \mathcal{H}_{1101}(\alpha_x, \alpha_y, \alpha_z) = \partial_1 \mathcal{H}_{1110}(\alpha_x, \alpha_z, \alpha_y),$$

$$\partial_1 \mathcal{H}_{1011}(\alpha_x, \alpha_y, \alpha_z) = \partial_1 \mathcal{H}_{1110}(\alpha_z, \alpha_y, \alpha_x),$$

$$\partial_2 \mathcal{H}_{1011}(\alpha_x, \alpha_y, \alpha_z) = \partial_1 \mathcal{H}_{1110}(\alpha_z, \alpha_y, \alpha_x),$$

$$\partial_3 \mathcal{H}_{1011}(\alpha_x, \alpha_y, \alpha_z) = \partial_1 \mathcal{H}_{1110}(\alpha_z, \alpha_y, \alpha_x)$$ (8.159)

$\mathcal{H}_{-abcd}$ can be found in a similar way, and here we list all identities needed for them and their derivatives

$$\mathcal{H}_{-1210}(\alpha_x, \alpha_y, \alpha_z) = 3 \alpha_x \partial_1 \mathcal{H}_{1110}(\alpha_x, \alpha_y, \alpha_z) - 9 \mathcal{H}_{1110}(\alpha_x, \alpha_y, \alpha_z),$$

$$\mathcal{H}_{-1201}(\alpha_x, \alpha_y, \alpha_z) = 3 \alpha_x \partial_1 \mathcal{H}_{1110}(\alpha_x, \alpha_z, \alpha_y) - 9 \mathcal{H}_{1110}(\alpha_x, \alpha_y, \alpha_z),$$

$$\mathcal{H}_{-1111}(\alpha_x, \alpha_y, \alpha_z) = 3 \alpha_x \partial_3 \mathcal{H}_{1110}(\alpha_x, \alpha_y, \alpha_z) - 3 \mathcal{H}_{1110}(\alpha_x, \alpha_y, \alpha_z),$$

$$\mathcal{H}_{-1102}(\alpha_x, \alpha_y, \alpha_z) = 3 \alpha_x \partial_2 \mathcal{H}_{1110}(\alpha_x, \alpha_z, \alpha_y) - 9 \mathcal{H}_{1110}(\alpha_x, \alpha_z, \alpha_y).$$ (8.160)

By knowing the above, some other functions are related

$$\mathcal{H}_{-1120}(\alpha_x, \alpha_y, \alpha_z) = \mathcal{H}_{-1210}(\alpha_y, \alpha_x, \alpha_z),$$

$$\mathcal{H}_{-1021}(\alpha_x, \alpha_y, \alpha_z) = \mathcal{H}_{-1201}(\alpha_y, \alpha_x, \alpha_z),$$

$$\mathcal{H}_{-1012}(\alpha_x, \alpha_y, \alpha_z) = \mathcal{H}_{-1102}(\alpha_y, \alpha_x, \alpha_z).$$ (8.161)
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