A STUDY OF SUBSYSTEMS OF TOPOLOGICAL SYSTEMS
MOTIVATED BY THE QUESTION OF DISCONTINUITY

IN TopSys

A dissertation submitted
to Kent State University in partial
fulfillment of the requirements for the
degree of Doctor of Philosophy

by

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Chapter 1

Motivation and Introduction

This dissertation considers a notion of subsystem of a topological system which appears when attempting to frame a satisfactory definition of a “discontinuous function” between a pair of topological systems.

A topological system (in the sense of S. Vickers as exposited in [Vic96]) consists of an ordered triple $(X, A, \models)$, where $X$ is a set, $A$ is a frame (or locale, depending on one’s point of view), and $\models$ is a binary relation satisfying:

- $\models \subseteq X \times A$
- if $x \in X$ and $\{a_i\}_{i \in I}$ is a finite subset of $A$, then:
  \[ x \models \bigwedge_{i} a_i \text{ iff } (\forall i \in I)(x \models a_i) \]
- if $x \in X$ and $\{a_i\}_{i \in I}$ is any subset of $A$, then:
  \[ x \models \bigvee_{i} a_i \text{ iff } (\exists i \in I)(x \models a_i) \]

The relation $\models$ is read “satisfies” and is called the satisfaction relation of the system.

Note that every point $x \in X$ satisfies the top element (denoted $\top_A$ or simply $\top$), by vacuous truth since $\top$ is an empty meet, and no point can satisfy the bottom element (denoted $\bot_A$ or simply $\bot$), since it’s the empty join.

Topological systems can be considered a common “generalization” of both topological spaces and locales; further, there is a notion of continuous function between topological systems that, in a sense, generalizes continuous functions between spaces and continuous functions between locales. Thus, for example, it is
possible, in the setting of topological systems, to say what the continuous functions are between a non-
spatial locale and a non-sober topological system. Further, every continuous function in the sense of two
topological systems each coming from a topological space (resp., locale) corresponds to a continuous function
in the usual sense between the spaces (resp., locales).

Categorically, this amounts to two functorial embeddings

\[ \text{Top} \rightarrow \text{TopSys} \]

and

\[ \text{Loc} \rightarrow \text{TopSys}. \]

We are approaching what, while it is not the central theme of this dissertation, is the motivating question
behind that theme.

Notice we have a forgetful functor from \( \text{Top} \) to \( \text{Set} \):

\[
(X, S) \mapsto X
\]

\[
f \mapsto f.
\]

If we define a new category \( \text{Top}_{\text{fun}} \) with spaces as objects, and all functions between the underlying sets
as morphisms, then we have a similar functor which is now an embedding from \( \text{Top} \) into \( \text{Top}_{\text{fun}} \):

\[
(X, S) \mapsto (X, S)
\]

\[
f \mapsto f.
\]

Now, the fundamental question — How should a discontinuous function between topological systems be
defined? — basically amounts to filling in the questions marks on the following diagram of categories and
functors:

\[
\text{Top} \quad \rightarrow \quad \text{TopSys}
\]

\[
\text{Top}_{\text{fun}} \quad \rightarrow \quad ???
\]

To the uninitiated reader, it may seem that we should just take “everything that is not continuous” to be
discontinuous. In short, the reason that this does not work is there is no good definition (as of this writing)
of “function between topological systems, continuous or not,” as the following discussion explains.

First, we define forgetful functors $V_1 : \text{Top} \to \text{Set}$ and $V_2 : \text{TopSys} \to \text{Set} \times \text{Loc}$ by:

$$V_1(f : (X, \mathcal{S}) \to (Y, \mathcal{T})) = f : X \to Y$$

$$V_2((f, \phi) : (X, A, \models) \to (Y, B, \models)) = (f, \phi^{op}) : (X, A) \to (Y, B)$$

Now, consider the following:

Given a continuous function $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$ in $\text{Top}$, this includes two things:

1. Consider $f^{\leftarrow}$. We have “adjointness” of $f^{\leftarrow}$ with respect to $f$.
2. We insist $f^{\leftarrow}|_{\mathcal{S}} : \mathcal{T} \to \mathcal{S}$.

Note that in this case, being a “ground morphism” already includes (1), and (2) is added.

On the other hand, given a continuous function $(f, \phi) : (X, A, \models) \to (Y, B, \models)$ in $\text{TopSys}$ this includes two things:

1. $\phi : A \leftarrow B$
2. We further impose the adjointness condition of $\phi$ with respect to $f$.

In this case, being a ground morphism already includes (1), and (2) is additional. But we notice: (2) in the $\text{TopSys}$ case is the equivalent of (1) in the $\text{Top}$ case, and (1) in the $\text{TopSys}$ case is the equivalent of (2) in the $\text{Top}$ case. Thus, while continuous functions in both cases have similar versions of these two properties, a different part comes from the ground in each case. Thus, what we add in order to make a morphism into a continuous function in $\text{TopSys}$ is already assumed for all functions in $\text{Set}$. And, the fact that a continuous function between spaces has a backwards operator that takes open sets to open sets, has an analog that $\phi$ maps the codomain locale to the domain locale, which is always true.

Also, note that (2) is not continuity: it quantifies a relationship which holds between a powerset operator and a function, and might more appropriately be called “adjointness”. Importantly, it is not clear how to impose this condition (2) in the absence of condition (1).

Further, we note that in the $\text{Top}$ case, we have $f$ and $f^{\leftarrow}|_{\mathcal{S}}$, and the latter can be extended to $f^{\leftarrow}$ defined on the whole powerset of the codomain, $\phi(Y)$. It is not clear how to make this extension in the $\text{TopSys}$ case because we are not even given the analog of a powerset.

The observation that the lattice of subspaces of a space is in one-to-one correspondence with the lattice of subsets of the underlying set leads us to look for a lattice of “subsystmes” of a topological system, to play the role of a “generalized powerset” in this context.
This explains why we need to look beyond simply “discontinuous is whatever is not continuous,” and also shows that $\text{Set} \times \text{Loc}$ is not a suitable “ground category” for defining discontinuous functions in $\text{TopSys}$.

We now discuss some possible choices for trying to define the notion of “discontinuous function” between topological systems. We first discuss the obvious choice, which can easily be seen not to work. We then discuss a more sophisticated idea. One approach to this idea is analyzed in Chapter 9 and another mentioned in passing.

The obvious choice is to simply use for morphisms pairs $(\text{pt} f, \Omega f)$ where $\text{pt} f$ and $\Omega f$ are not related. The problem with this approach is that this definition does not agree with the notion of discontinuous function between topological spaces, and there appears to be no way to make things work.

For example, consider $(X, \mathcal{S}) = (Y, \mathcal{T}) = \text{Sierpinski space}$. Then there are four functions from $X$ to $Y$. The two constant maps and the identity map are continuous, the fourth map is not. However, there are three frame homomorphisms from $\mathcal{T}$ to $\mathcal{S}$, and so there are twelve pairs $(\text{pt} f, \Omega f)$, three of which are continuous functions; the remaining nine ordered pairs obvious cannot be put into one-to-one correspondence with the single discontinuous function between the two spaces.

A second choice follows from the observation that give topological spaces $(X, \mathcal{S})$ and $(Y, \mathcal{T})$, $\mathcal{S}$ is a subframe of $\wp(X)$ and $\mathcal{T}$ is a subframe of $\wp(Y)$. Further, one observes that the set of all functions is in one-to-one correspondence with the set of frame homomorphisms from $\wp(Y)$ to $\wp(X)$ via $f \mapsto f^\leftarrow$, and $f$ is continuous iff $f^\leftarrow (\mathcal{T}) \subseteq \mathcal{S}$.

Thus, one can try to construct a “powerset” for any given topological systems, which “contains” (a copy of) the frame of opens of the system, and which has its own satisfaction relation w.r.t. the points, extending that on the opens. If this can be done “correctly,” functions between systems should correspond to pairs $(\text{pt} f, \Omega^* f)$, with the “continuous” functions being the ones where restricting $\Omega^* f$ to opens on both domain and codomain sides yields a continuous function $(\text{pt}, \Omega f)$, and the “discontinuous” functions being the ones that are not continuous.

The author of this dissertation has attempted to perform this construction in several ways. Unfortunately, he has been unable to make any of them work. This does, however, lead to the notion of “subsystem” of a topological system, which is the focus of this work.

1.1 Contributions of this Dissertation

There are important contributions of this dissertation to the literature, including the following. The first is the construction of regular subobjects in the category $\text{TopSys}$, which significantly adds to our understanding of $\text{TopSys}$, which is known to be complete but has not been extensively studied as a category.
The second main contribution is the demonstration of the applicability of structural induction to not only prove new results but also provide insightful proofs of well-known results. Cf. also the approach of [Pul03]. While structural induction is well-known, its use in proving theorems about locales is not. In this regard, it provides an important alternative to Johnstone’s use of coverages.
Chapter 2

Historical Background and Literature

It is appropriate to discuss the history of the notion of topological system. (This background is abstracted to a large degree from [DMR14b]. I would also like to gratefully acknowledge the particular assistance of my dissertation co-director Dr. Stephen E. Rodabaugh in preparing this chapter.)

2.1 Early Background and Literature

The roots of topological systems can be traced to two threads. The first we will consider can be traced to E.W. Dijkstra’s book [Dij76], in which is emphasized: a focus on outputs as opposed to inputs; and focus on properties of outputs and properties of inputs. Specifically, he asked about finding the least restrictive property on inputs that leads to the desired output.

The second thread is the notion of Chu system, as noted below.

Suppose we have a set of inputs $X$ and a set of outputs $Y$. Suppose also we have a collection $\mathcal{P}$ of precondition predicates, which the inputs may or may not satisfy, and a collection $\mathcal{Q}$ of postcondition predicates which the outputs may or may not satisfy. For instance, $\mathcal{P}$ might be a subset of $\wp(X)$, and we might have $\mathcal{Q} \subseteq \wp(Y)$. In this case we can take set membership to be a sort of “satisfaction relation.” In what follows we will use “$x \models P$” to mean “Property $P$ is true of $x$.”

Imagine now a “program” from $(X, \mathcal{P}, \in)$ to $(Y, \mathcal{Q}, \in)$. We expect as part of this program, a function $f : X \to Y$, and since a “program” should be “computable” or “continuous,” if the collections $\mathcal{P}$ and $\mathcal{Q}$ are properly chosen, we can expect a mapping between these collections of properties — and the key is that in the spirit of Dijkstra:

- $\phi : \mathcal{Q} \to \mathcal{P}$
• \( \forall x \in X. \forall Q \in Q. x \models \phi(Q) \iff f(x) \models Q \).

The first condition says that properties get mapped to properties. The second condition, while it is described as defining an “optimal” deterministic program in [DMR14b], can also be viewed as saying “Properties are actually properties.” In the case where properties are sets, it defines a basic fact about preimages under the backwards powerset operator:

• \( \forall x \in X. \forall B \in \wp(Y). x \in f^{-1}(B) \iff f(x) \in B \).

We thus package our models of programs as “systems” \((X, \mathcal{P}, \models)\) and \((Y, \mathcal{Q}, \models)\). Replacing \(\mathcal{P}\) by \(A\) and \(\mathcal{Q}\) by \(B\), we get the standard notation of this paper: \((X, A, \models)\) and \((Y, B, \models)\). Note that we have yet to make our systems “topological.”

We note that the systems so far described (where the set of properties is unstructured) have been called in the literature “Chu spaces” (introduced in [Bar06] as well as “Chu systems” (first used in [DMR13a]). As objects, they are also identical to the “formal contexts” of Wille (see for example [GW97, GSW05]), but the morphisms considered are different. There is a huge literature on formal contexts.

Please note that the concept of predicates as open sets dates to M. Smyth [Smy83].

We call by the name of continuous function an ordered pair \((f, \phi)\) satisfying:

• \( f : X \to Y \) is a function;

• \( \phi : B \to A \) is a function

• \( \forall x \in X. \forall b \in B. x \models \phi(b) \iff f(x) \models b \).

### 2.2 Topological Part of Topological Systems

Let us now consider the topological part of topological systems. Consider a continuous function between topological spaces:

\[
f : (X, \mathcal{S}) \to (Y, \mathcal{T}).
\]

We can repackage this like the systems above:

\[
(f, f^{-1}|_{\mathcal{S}}) : (X, \mathcal{S}, \in |_{\mathcal{S}}) \to (Y, \mathcal{T}, \in |_{\mathcal{T}}),
\]

or, dropping explicit notations for domains and codomains of restricted functions and relations:

\[
(f, f^{-1}) : (X, \mathcal{S}, \in) \to (Y, \mathcal{T}, \in).
\]
Observe that any topology has the structure of a frame, i.e., a complete lattice satisfying that finite meets distribute over arbitrary joins. In particular, a topology has arbitrary joins (which are in fact unions), and finite meets (which, since we said “finite,” coincide with intersections), and we have the following facts:

- For any family of open sets \( \{a_i : i \in I\} \), and any point \( x \), we have:
  \[
  x \in \bigcup_i a_i \iff \exists i \in I. x \in a_i
  \]

- For any FINITE family of open sets \( \{a_i : i \in I\} \), and any point \( x \):
  \[
  x \in \bigcap_i a_i \iff \forall i \in I. x \in a_i.
  \]

- For any point \( x \), and noting that order-theoretically, \( X \) is the top element of the topology:
  \[
  x \in X
  \]

- For any point \( x \):
  \[
  x \notin \emptyset
  \]

These operations (arbitrary disjunction and finite conjunction) under the assumption of finite meets distributing over arbitrary joins describe propositional geometric logic, and the interchange rules described immediately above describe what one would expect if points were models. (Propositional geometric logic, aka “the logic of finite observations,” was introduced as a motivation for topological systems in [Vic96]. The notion of treating a topological system as a logical system and collection of models was studied in [DMRS16b].)

We also will we require that the “frame” parts of morphisms of topological systems to preserve arbitrary joins and finite meets.

Now, to see why we might consider a topological system that is not “spatial,” take a countable set, say \( \mathbb{N} \), and take an infinite topology \( \mathcal{T} \) on it. Then choose a (nonempty) finite subset \( F \subseteq \mathbb{N} \) and form the following “system”:

\[
(F, \mathcal{T}, \in)
\]

where \( \in \) is short for \( \in |_{\mathcal{T}} \).
Then this obviously cannot come from a topological space, since a finite set cannot have an infinite topology. (In terminology to be introduced later, this system is not homeomorphic to a topological system coming from a topological space.)

We can think of a computational-theoretic motivation for such systems: suppose we are dealing with properties that we want to keep distinguishable, but for the moment the set of “points” (inputs or outputs) doesn’t allow us to distinguish between certain predicates. Maybe we want to look at certain collections of points, which when taken as a union, allow distinguishing all the predicates, but each collection of points by itself does not.

**Example 1.** (See [DMR14b].)

Let $2 = \{0, 1\}$ and $2^*$ be the set of all finite strings (including the empty string) with entries coming from $2$.

Put $s \subseteq t$ if $s$ is a prefix of $t$. (This includes the case $s = t$.)

Set, for $s \in 2^*$ and $U \subseteq 2^*$,

$$\text{starts}(s) = \{t \in 2^* : s \subseteq t\},$$

$$\text{starts}(U) = \bigcup \{\text{starts}(s) : s \in U\}.$$

Set

$$\mathcal{A} = \{U \subset 2^* : U = \text{starts}(U)\}.$$

Then $\mathcal{A}$ is a topology on $2^*$, known as the Alexandroff topology.

Define a function $\neg$ by:

- $\neg s$ is formed by interchanging 0 and 1 in all positions in $s$, and
- for a subset $U$ of $2^*$,

$$\neg U = \{-s : s \in U\}.$$

For example, $\neg 01001110 = 10110001$, and $\neg \{011, 110, 0101\} = \{100, 001, 1010\}$. If we set

$$f(s) = \neg s, \quad \phi(U) = \neg U$$

we get a “continuous function”

$$(f, \phi) : (2^*, \mathcal{A}, \in) \rightarrow (2^*, \mathcal{A}, \in).$$
Now choose a finite subset $X \subset 2^*$, and let $Y = \{\neg s : s \in X\}$. Then we have a “continuous function”

$$(f|_X, \phi) : (X, \mathcal{A}, \models) \to (Y, \mathcal{A}, \models),$$

where in each case $\models$ stands for the appropriately restricted membership relation.

Example 2. (See page 55 of [Vic96].) The following function calculates complements of finite bit strings:

$$\text{complement}(t) = \begin{cases} 
\text{NULL} & \text{if } t = \text{NULL} \\
\text{complement}(\text{head}(t) = 0 \text{ then } 1 \text{ else } 0) \text{ :: complement(tail(t))} & \text{else}
\end{cases}$$

Where, if $t = b :: s$, then $\text{head}(t) = b$ and $\text{tail}(t) = s$.

2.3 More Literature

We now discuss the more recent history of topological systems. The first definition appears in [Vic96]. S. Vickers introduced topological systems as a pedagogical tool, because they provide a common framework for both locales and topological spaces.

In 2009, Denniston and Rodabaugh published [DR09], which related the category $\text{TopSys}$ to the well-studied category of variable-basis locale-valued topological spaces $\text{LocTop}$.

That same year, Denniston, Melton, and Rodabaugh published [DMR09], which introduced the concept of lattice-valued topological systems, where the satisfaction relation is replaced by a lattice-valued satisfaction relation. It is also possible to allow the lattice of satisfaction values to vary over, say the category $\text{Loc}$, yielding $\text{LocTopSys}$ analogous to $\text{LocTop}$. See [DMR12b].

Also in 2012, the same authors published [DMR12a], in which was studied a notion of “enriched topological system,” designed to compare to the notion of enriched category and enriched functor.

Another paper related to topological systems is [DMR14b], which traces the development from program semantics to many-valued topology.

Denniston, Melton, Rodabaugh, and S. Solovjovs worked from the analogy of points as “models” and opens as “sentences” to compare topological systems to the institutions used in logic and theoretical computer science. This was published in [DMRS15a, DMRS16b].

One important observation is that $\text{TopSys}$ is not a topological category over $\text{Set}$, $\text{Set} \times \text{Loc}$, or any other known ground. See the previously cited [DMR12b]. This is significant because the point-open interplay clearly gives topological systems some of the flavor of point-set topology. It is thus of interest to
embed TopSys into topological categories; this was explicitly done in [DR09] and in much greater detail in [DMR12b], with addition results in [DMR17c].

For other papers including the author related to topological systems, see [DMRS13, DMRS16a].

S. Solovjovs has defined variety-based topological systems by generalizing the notion of lattice-valued topological systems, as well as provided a definition of affine system, which has roughly the same relationship to that of affine space as topological system does to topological space. Works by Solovjovs related to topological systems include: [Sol10, Sol11a, Sol11b, Sol11c, Sol11d, Sol12a, Sol12b, Sol13, Sol15b, Sol15a].
Chapter 3

Preliminaries

This chapter introduces several definitions and basic results.

3.1 Categories and functors

The reader will be assumed to understand the basics of categories and of functors between categories. Good references include [AHS04, Mac71, Mac98, Mac13].

3.2 Frames and locales

Frames and locales are important in the definition of a topological system. One recalls that frames and locales are the same as objects, but the category \( \text{Loc} = \text{Frm}^{\text{op}} \) of locales and continuous functions is the opposite (or dual) category to the category \( \text{Frm} \) of frames. Thus, to each frame homomorphism \( \phi : A \to B \) there corresponds a unique continuous function of locales \( \phi^{\text{op}} : B \to A \). Similarly, if \( \phi \) is a continuous function between locales, then \( \phi^{\text{op}} \) denotes the corresponding frame homomorphism.

For convenience, we recall the definition of a frame.

**Definition 1.** A frame is an partially order set \( (A, \leq) \) satisfying the following conditions:

- Every subset \( Q \subseteq A \), including the empty subset, has a join, or least upper bound, denoted \( \bigvee Q \)
- Every finite subset \( F \subseteq A \), including the empty subset, has a meet, or greatest lower bound, denoted \( \bigwedge F \)
• For any \( a \in A \) and family \( \{ b_i : i \in I \} \subseteq A \), we have:

\[
    a \land \bigvee_i b_i = \bigvee_i (a \land b_i).
\]

Note that binary joins (respectively meets) are written with the symbol \( \lor \) (respectively \( \land \)). The empty join, which is a bottom element, is written \( \bot \); the empty meet is written \( \top \) and is a top element.

For convenience we also recall that a frame homomorphism from \( A \) to \( B \) is a function \( f : A \to B \) that preserves arbitrary joins and finite meets. In particular, top and bottom in the domain are mapped to top and bottom in the codomain.

The reader should consult the references [Joh86, Pul03, Vic96] for more on frame, locales, frame homomorphisms, and continuous functions.

### 3.3 Topological Systems

Topological systems and continuous functions between them are as in [Vic96].

For convenience, we recall the definition of topological system.

**Definition 2.** A topological system is an ordered triple \((X,A,\mathrel{\models})\) where:

- \( X \) is a set
- \( A \) is a frame
- \( \mathrel{\models} \subseteq X \times A \), satisfying
  - \( \forall x \in X. \forall \{a_i\} \subseteq A. \exists j. x \models a_j \iff x \models \bigvee_i a_i \)
  - \( \forall x \in X. \forall \text{ finite } \{a_i\} \subseteq A. \forall j. x \models a_j \iff x \models \bigwedge_i a_i. \)

In particular, for all \( x, x \models \top \) and \( x \not\models \bot \).

We will alternately use the notation \( D = (X,A,\mathrel{\models}) \) or \( D = (\text{pt}D,\Omega D,\mathrel{\models}) \). For continuous functions, we will write either \((f,\phi)\) or \( f = (\text{pt}f,\Omega f) \). The symbol \( \phi \) will be reserved for the \( \varphi \)-operator (see Chapter 7). (Thus the standard version of phi, namely \( \phi \), is used for certain morphisms, while its variant \( \varphi \) is the name of a specific operation to be introduced.)

For convenience, we recall the definition of continuous function.

**Definition 3.** The pair \((f,\phi)\) is defined to be a continuous function from \((X,A,\mathrel{\models})\) to \((Y,B,\mathrel{\models})\) if the following hold:
• \( f \) is a function from \( X \) to \( Y \)

• \( \phi \) is a frame homomorphism \( \phi : B \to A \)

• \( \forall x \in X. \forall b \in B. f(x) \models b \iff x \models \phi(b) \)

The third bullet describes the so-called adjunction condition. In papers such as [DMR13a], [DMR14b], etc., \( \phi \) is a locale morphism from \( A \) to \( B \), and \( \phi^{op} \) satisfies the adjunction condition. This approach yields the same theory and is essentially only different in notation, as well as the fact that it makes \( \text{TopSys} \), the category of topological systems and continuous functions, concrete over \( \text{Set} \times \text{Loc} \).

### 3.4 Miscellaneous Notation

If \( A \) is a set, or possibly a structured set (like a frame), then \( \Delta_A \) will denote the relation \( \{(a, a) : a \in A\} \), that is the diagonal of \( A \), or equivalently, equality considered as a set of ordered pairs. If \( A \) is clear from context, simply \( \Delta \) will be used for \( \Delta_A \) (and also for \( \Delta_B, \Delta_C \), et cetera).

The category \( \text{Loc}^{\partial} \) denotes the category of dual locales. This category is isomorphic to \( \text{Loc} \), so the notation is basically to emphasize that we are looking at “upside-down” locales.

If \( u \) is a subset of some set, say \( u \subseteq X \) or \( u \subseteq \text{pt}D \), then \( u^c \) denotes \( X \setminus u \), respectively \( \text{pt}D \setminus u \).

We specifically note that we use the convention that \( \mathbb{N} = \{0, 1, 2, 3, 4, \ldots\} \) contains 0.
Chapter 4

Examples of Topological Systems

Example 3. Let $A$ be the lattice shown in Figure 6.1, with opens $\top, \bot$ as expected, and $a, b, c$ with $a \land b = c$, and $a \lor b = \top$. Then we determine the poset of sublocales of $A$. We will see in the next chapter that this is a dual frame, and hence, because it is finite, is therefore a frame.

Example 4. Let $A$ be the lattice of the previous example. Let $X = \{x, y, z\}$ and $\models$ be determined as follows:

- $x \models \top$, $a$ only
- $y \models \top$, $b$ only
- $z \models \top$, $a, b, c$ but of course not $\bot$.

Then $(X, A, \models)$ is a topological system.

Example 5. Recall $\mathbb{N}$ is the natural numbers, which for us include 0. We define a locale $\mathcal{T}_{\text{cof}}^0$ as the following topology on $\mathbb{N}$:

$$\mathcal{T}_{\text{cof}}^0 = \{ u \subseteq \mathbb{N} : u \text{ is cofinite and contains } 0 \} \cup \{ \emptyset \}.$$  

Then let $x \models u$ iff $x \in u$. Then

$$(\mathbb{N}, \mathcal{T}_{\text{cof}}^0, \models)$$

is a topological system.

Example 6. Let $A$ be the lattice of the first example. Let $X' = \{x, y\}$ and $\models$ be determined as follows:

- $x \models \top$, $a$ only
- $y \models \top$, $b$ only.
Then \((X', A, \models)\) is a topological system.
Chapter 5

Structural Induction

Definition 4. Let $A$ be a frame, and $R$ be a subset of $A \times A$. Then $R$ is a frame congruence relation on $A$, if the following hold:

- $R$ is an equivalence relation, i.e., is reflexive, symmetric, and transitive
- For any $a, b, a', b' \in A$, if $a R b$ and $a' R b'$, then $(a \land a') R (b \land b')$
- For any families $\{a_i\}_{i \in I}$ and $\{b_i\}_{i \in I}$ of elements of $A$, if $a_i R b_i$ for all $i \in I$, then $\bigvee_i a_i R \bigvee_i b_i$

Definition 5. Let $A$ be a frame and $R$ be a subset of $A \times A$. Then the congruence relation generated by $R$, denoted by $\langle R \rangle$, is defined by

- $\langle R \rangle = \bigcap \{R' : R \subseteq R' \text{ and } R' \text{ is a congruence relation on } A\}$

Remark. Clearly, $A \times A$ works for $R'$ above, so the intersection is over a nonempty set. Observe that $\langle R \rangle$ is a congruence relation because an intersection of a family of congruence relations on $A$ is a congruence relation on $A$.

For a number of proofs in this manuscript, it will be convenient to use a special version of structural induction.

Theorem 1. Let $R$ be a subset of $A \times A$ for a frame $A$, and let $\Phi$ be a property defined for every $(b, c) \in A \times A$. Suppose the following are true:

- For all $(b, c) \in R$, $\Phi(b, c)$;
- For all $b \in A$, $\Phi(b, b)$;
- For all $b$ and $c$ in $A$, $\Phi(b, c)$ iff $\Phi(c, b)$;
• For all \( a, b, c \in A \), if \( \Phi(a, b) \) and \( \Phi(b, c) \), then \( \Phi(a, c) \);

• For every finite subset \( \{(b_i, c_i)\}_{i \in I} \) of \( A \times A \), such that \( \Phi(b_i, c_i) \) holds for all \( i \), \( \Phi(\bigwedge_{i \in I} b_i, \bigwedge_{i \in I} c_i) \) holds; and

• For every subset \( \{(b_i, c_i)\}_{i \in I} \) of \( A \times A \), such that \( \Phi(b_i, c_i) \) holds for all \( i \), \( \Phi(\bigvee_{i \in I} b_i, \bigvee_{i \in I} c_i) \) holds.

Then \( \Phi(b, c) \) holds for every \((b, c)\) in \( \langle R \rangle \).

Proof. Consider the set \( Q = \{(b, c) \in A \times A : \Phi(b, c)\} \). Clearly \( Q \) is a congruence relation containing \( R \), so \( \langle R \rangle \subseteq Q \). Thus \( \Phi(b, c) \) holds for every \((b, c) \in \langle R \rangle \). QED

Remark. It should be clear that the condition on finite meets can be weakened to one on binary meets. The case of the empty meet is taken care of by the second bullet.

Remark. To understand what is going on here, let us compare this to a special version of finite induction for groups. Suppose \( G \) is a nontrivial group, \( H \) is a nontrivial subgroup generated by \( \{g_0, ..., g_n\} \), and \( \Phi \) is a property of elements of \( G \) so that:

• for each generator \( g_i \) of \( H \), \( \Phi(g_i) \) holds;

• if \( \Phi(a) \) and \( \Phi(b) \) holds, so do \( \Phi(ab) \) and \( \Phi(a^{-1}) \).

Then, I claim \( \Phi(g) \) holds for every \( g \in H \). To prove this, let \( S \) be the set

\[ \{g \in G : \Phi(g) \text{ holds}\}. \]

By the conditions on \( \Phi \), \( S \) is closed under multiplication and taking inverses, and since \( g_0 \in S \), we can deduce that the identity \( e_G = g_0 \cdot g_0^{-1} \) is in \( S \). Thus, \( S \) is a subgroup of \( G \). Further, \( \{g_0, ..., g_n\} \subseteq S \); since \( H \) is the smallest subgroup containing \( \{g_0, ..., g_n\} \), \( H < S \). Thus, \( \Phi(g) \) holds for every \( g \in H \).

We may also compare this to the other approach to proving this claim: \( g \in H \) if and only if \( g \) can be expressed as a product of generators and inverses of generators, then induct on the length of such products. It does not matter that some elements will have more than one representation, since we are not defining something by recursion, which would require each element to be counted only once to ensure the definition be unique; it suffices that every element we care about is counted at least once.

This second approach also has an analogue for the above theorem. Start with \( R \), and for each ordinal, define:

• \( R_0 = \) the smallest equivalence relation containing \( R \)
- $R'_{\alpha+1} = \{(a, b) : (a, b) \text{ can be obtained from ordered pairs in } R_\alpha \text{ by taking pointwise arbitrary joins or finite meets}\}$

- $R_{\alpha+1} = \text{the smallest equivalence relation containing } R'_{\alpha+1}$

- $R_\alpha = \bigcup_{\beta < \alpha} R_\beta \text{ if } \alpha \text{ is a limit ordinal.}$

Clearly, for all $\alpha$, $R_\alpha \subseteq A \times A$, so the sequence of $R_\alpha$’s is eventually constant, say for all $\alpha \geq \gamma$ (for some particular $\gamma$). By transfinite induction, we can show that $\Phi(b, c)$ holds for every $(b, c)$ in $R_\alpha$. Also, since the sequence of $R_\alpha$’s is constant for $\alpha \geq \gamma$, $R_\gamma$ is closed under all the operations listed in the theorem; in particular, $R_\gamma$ is a frame congruence relation. It follows that $R \subseteq R_\gamma$, so $\Phi(b, c)$ for every $(b, c)$ in $R$.

This reduces the theorem on structural induction to transfinite induction. This is a common phenomenon in structural induction. Again, as in the case with groups, however, while there will be a minimum ordinal at which a particular ordered pair is added, there will not necessarily be a unique “representation” for this ordered pair; indeed, it may even be “added twice” at the step where it is first added. Thus, our particular version of structural induction is somewhat different than the usual one; also, it is a special case, because it applies only to frame congruences.

For more on structural induction, see one of many computer science references. For example, see [HMU06].
Chapter 6

Sublocales

A basic notion in locale theory, and also for topological systems theory, is that of sublocale. Basically, a sublocale is to a locale as subspace is to a topological space. Only in this case, instead of taking an inclusion into the point set and letting the inclusion determine a topology (which is a frame quotient of the original topology), we take a frame quotient of the lattice of “opens” and let it determine a subset of the locale points.

We will see in the next chapter that sublocales form a lattice, and later that this lattice is a dual frame. These facts will be very important when we consider the lattice of subsystems of a topological system.

The results of this chapter are very well known, but are presented here because they will be needed later. The discussion in this chapter is adapted from section 6.2 of [Vic96].

6.1 Frame surjections

One of the two most common viewpoints on sublocales used in this dissertation is that sublocales “are” frame surjections.

We remark that if $f : A \to B$ and $g : A \to C$, both surjectively, then $f$ corresponds to a “larger” sublocale than $g$ iff there exists a frame homomorphism $h : B \to C$ so that $h \circ f = g$. If $h$ exists, it is unique and it is surjective.

This does define a pre-order on frame surjections. We further choose frame quotient maps as the standard representatives of each equivalence class of surjective frame homomorphisms. Then we have a partial order on frame quotient maps from any fixed frame.

Observe that if $f$ and $g$ as above correspond to the “same” sublocale, then the map $h$ above is a frame isomorphism from $B$ to $C$. 
It should be noted that two sublocales may be isomorphic without being equal. Similarly, one sublocale may “embed” in another without being smaller in the partial order. These cases correspond to isomorphisms and frame surjections that do not form commutative triangles with the two frame surjections defining the respective sublocales. We will see examples of this below.

One may compare this to the case of subgroups, where two subgroups may be isomorphic without being the same subgroup, and a subgroup embedding need not be the inclusion. For instance, in \((\mathbb{Z}_4)^2\), there are multiple unequal subgroups isomorphic to \(\mathbb{Z}_4\), and the subgroup generated by \((2,0)\) embeds in that generated by \((0,1)\), but is not a subgroup of it.

6.2 Frame congruences

The other one of the two most common viewpoints is that a sublocale is a frame congruence. These are defined in Chapter 3. Given a sublocale \(\alpha\) of \(A\), define \(\equiv\) to be the corresponding frame congruence.

On the other hand, given a frame congruence \(\equiv\) on \(A\), we can define a frame surjection \(\alpha: A \to B\) by:

\[
[a] = \{b \in A : a \equiv b\}, \text{ for all } a \in A
\]

\[
B = \{[a] : a \in A\}
\]

with \(\leq\) on \(B\) by \([a] \leq [b] \iff a \leq b\), and

\[
\alpha(a) = [a].
\]

It can be shown that \(B\) is a frame (and the order is well-defined), and \(\alpha\) is a frame homomorphism, which is clearly surjective.

We use the notation \(B = A/\equiv\).

Finally, given \(f\), one can construct a congruence \(\equiv\) and sublocale \(\alpha\). These two constructions are “inverses” in the sense that \(f\) and \(\alpha\) represent the same sublocale; and further, if we start with a congruence, form a sublocale, and go back, we get the same congruence.

6.3 Extra relations

Given a frame, one can define a sublocale by imposing certain equivalences between elements, and generating a congruence relation.
Example 7. Let $A$ be any frame. Generate $\equiv_{\alpha}$ by imposing the single equivalence $\top \equiv_{\alpha} \bot$. Then by the fact that $\equiv_{\alpha}$ is a congruence, for any $a \in A$ we have:

\[
a = a \land \top
\]

\[
\equiv_{\alpha} a \land \bot = \bot
\]

so that

\[
a \equiv_{\alpha} \bot.
\]

This holds for all $a \in A$, so $\equiv_{\alpha} = A \times A$.

As noted below, in this case $\alpha = \bot$.

Example 8. Let $A$ be the lattice shown in Figure 6.1, with opens $\top$, $\bot$ as expected, and $a, b, c$ with $a \land b = c$, and $a \lor b = \top$. Then we determine the poset of sublocales of $A$. We will see in the next chapter that this is a dual frame, and hence, because it is finite, is therefore a frame.

We now give a formal description of the poset of all sublocales of $A$.

The first possibility is to impose no additional relations. We call this sublocale $\top$. We call the corresponding equivalence relation $\equiv_{\top}$. Note that $\equiv_{\top} = \Delta$. This is illustrated in Figure 6.2, along with the rest of the possibilities we discuss in the following.

Now let us see what happens if we impose the relations $a \equiv \top$, $b \equiv \top$, or $c \equiv \top$. First let us consider the relation generated by $a \equiv \top$; we will call this $\equiv_{a}$. Note that

\[
b = b \land \top
\]

\[
\equiv_{a} b \land a = c.
\]

Thus, we get a partition $\{\{\top, a\}, \{b, c\}, \{\bot\}\}$. It is easy to check this partition corresponds to a congruence relation, and the reader should do so, especially if she or he is on my committee.

Next, we call the relation generated by $b \equiv \top$ by $\equiv_{b}$. This leads to the partition $\{\{\top, b\}, \{a, c\}, \{\bot\}\}$, which corresponds to a congruence relation.

Next, consider what we call here $\equiv_{c}$. This is generated by $c \equiv \top$. Clearly, then, $a, b, c$, and $\top$ all must
be equivalent. To prove this formally, note that:

\[ a = a \lor c \equiv a \lor \top = \top \]

so that

\[ a \equiv \top \]

and

\[ a = a \land \top \equiv a \land c = c \]

so that

\[ a \equiv c. \]

Similarly for \( b; a \equiv b \) by transitivity.

Thus we have the partition \( \{ \{ \top, a, b, c \}, \{ \bot \} \} \), which easily is shown to correspond to a congruence relation.

Now, we consider what happens if we set \( a, b, \) or \( c \) equivalent to \( \bot \).

Imposing \( c \equiv \bot \) leads to the partition \( \{ \{ \top \}, \{ a \}, \{ b \}, \{ c, \bot \} \} \). By checking that this partition does in fact correspond to a congruence relation, we see imposing this particular equivalence does not force any others. For now, we call the corresponding equivalence relation \( \equiv_c \). In the notation \( c^c \) or \( a^c \) or \( b^c \); the superscript \( c \) stands for compliment, and it will turn out that for each element of the original locale, this element will have a Boolean complement in the lattice of sublocales.

The reader should be old hat at this by now, and can calculate that imposing \( b \equiv \bot \) forces \( a \equiv \top \) and \( c \equiv \bot \). Then one checks that the partition \( \{ \{ \top, a \}, \{ b, c, \bot \} \} \) corresponds to a congruence relation, so we have the frame shown in the appropriate diagram. We call this \( \equiv_{b^c} \).

Similarly, if we impose \( a \equiv \bot \) we get the partition \( \{ \{ \top \}, \{ a, c, \bot \} \} \). We call this \( \equiv_{a^c} \).

Finally, we may impose \( \top \equiv \bot \). By the previous example, we know all the opens will be equivalent, and we have the partition \( \{ \{ \top, a, b, c, \bot \} \} \).

One may ask what happens if we impose equivalences not involving \( \top \) or \( \bot \). As it turns out, in this example, this gives nothing more. For example, if \( a \equiv b \), one calculates that \( \top \equiv c \); since the implication works in the other direction as well, this gives the sublocale we call \( c \).

On the other hand, requiring \( a \) or \( b \) to be equivalent to \( c \) forces the other one to be equivalent to \( \top \), and the implication reverses, and we are working with \( b \) or \( a \).

Finally, we check that imposing a second equivalence in the cases where it is possible leads to another of the sublocales already considered. We remark that in the case of a more complicated locale, imposing a
second equivalence can result in a new sublocale.
The lattice $A$

\begin{align*}
\top &\quad b \quad \bot \\
a &\quad c &\quad b
\end{align*}

The lattice $A$
Figure 6.2: Sublocales of $A$
Chapter 7

Lattice of Sublocales of $A$

Given a locale $A$, Sub($A$) is the lattice of sublocales of $A$. It is well known that this lattice is a dual frame (see [Vic96]), that is, it is a complete lattice, and the following “second infinite distributivity law” holds for all $\alpha, \beta_i (i \in I)$ in Sub($A$):

$$\alpha \lor \bigwedge_{i \in I} \beta_i = \bigwedge_{i \in I} (\alpha \lor \beta_i).$$

7.1 Discussion; Sub($A$) is a complete lattice

Before getting into the details of Sub($A$), I would like to comment why I have chosen to the dual frame ordering on Sub($A$), given that we could take the opposite order, and map $a$ to the corresponding closed sublocale. To keep the correspondence between opens and open sublocales, as well as to keep the straightforward definition of a point satisfying a sublocale when we look at sublocales of the locale of opens of a system, I have chosen the open “embedding” and the dual frame ordering. This also keeps sublocales which are intuitively “bigger” (i.e., less is divided out in forming the quotient) nearer the top of the lattice of sublocales.

As before, the elements of Sub($A$) will be considered either as (equivalence classes of) frame surjections, or, more commonly, as frame congruence relations on $A$. Given $\alpha \in$ Sub($A$), we use the notation:

$$b \equiv_{\alpha} c$$

to mean $\alpha(b) = \alpha(c)$ (for $b, c \in A$).

Any element $a \in A$ determines an element of Sub($A$), denoted $\bar{a}$, or, by abuse of notation, simply $a$,
determined by

\[ b \equiv_a c \text{ iff } b \land a = c \land a \]

for all \( b, c \in A \). Similarly, verbally we may fail to distinguish between opens and the corresponding sublocales when the meaning is clear. For example, we might speak of a meet of opens when we mean a meet of the images of those opens.

We verify that \( \equiv_a \) so defined is a congruence relation. It is easy to see it is reflexive, symmetric, and transitive. Suppose both the following hold:

\[ b_1 \equiv_a c_1 \]

\[ b_2 \equiv_a c_2. \]

Then

\[ (b_1 \land b_2) \land a = (b_1 \land a) \land (b_2 \land a) \]

\[ = (c_1 \land a) \land (c_2 \land a) = (c_1 \land c_2) \land a, \]

so that

\[ b_1 \land b_2 \equiv_a c_1 \land c_2. \]

Finally, suppose for all \( i \) in some indexing set \( I \), we have:

\[ b_i \equiv_a c_i. \]

Then

\[ (\bigvee_i b_i) \land a = \bigvee_i b_i \land a \]

\[ = \bigvee_i c_i \land a = (\bigvee_i c_i) \land a, \]

so that

\[ \bigvee_i b_i \equiv_a \bigvee_i c_i. \]

We impose the following partial order on \( \text{Sub}(A) \):

\[ \alpha \leq \beta \]

iff for all \( b, c \in A \), \( b \equiv_\beta c \Rightarrow b \equiv_\alpha c. \)
Equivalently,

\[ \alpha \leq \beta \iff \equiv_\beta \subseteq \equiv_\alpha. \]

We draw the reader’s attention to the fact that the smaller a sublocale is, the larger the corresponding equivalence relation is.

We also remark that when we say a frame congruence between two opens holds “by weakening,” that a stronger congruence between those opens has been established or postulated, and the new relationship is simply a weaker version of the original one:

- if \( \alpha \leq \beta \) and \( a \equiv_\beta b \), then \( a \equiv_\alpha b \).

It is easy to see that if \( a, b \in A \), then \( a \leq_A b \) iff \( \pi \leq_{\text{Sub}(A)} \overline{b} \).

Further, under this partial order, \( \text{Sub}(A) \) is a complete lattice (as mentioned above). Suppose \( \alpha_i \in \text{Sub}(A) \) for \( i \in I \). It is easy to see that:

\[ \bigcap_{i \in I} \equiv_{\alpha_i} \]

is a congruence relation, since it is an intersection of congruence relations, and the corresponding element of \( \text{Sub}(A) \) is clearly the least possible upper bound for

\[ \{ \alpha_i \}_{i \in I} \]

Thus

\[ \equiv \bigvee_{i \in I} \alpha_i = \bigcap_{i \in I} \equiv_{\alpha_i}. \]

Recall from a previous section, that given a relation \( R \subseteq A \times A \), there is a least congruence relation containing every ordered pair in \( R \), namely the intersection of all such congruence relations, denoted by \( \langle R \rangle \).
(This intersection is over a nonempty set because \( A \times A \) is a congruence relation.)

Thus, given \( \{ \alpha_i \}_{i \in I} \), we can find the meet as the join of all lower bounds; equivalently,

\[ \equiv \bigwedge_{i \in I} \alpha_i = \bigvee_{i \in I} \equiv_{\alpha_i}. \]

We also mention the top and bottom of \( \text{Sub}(A) \):

\[ \top_{\text{Sub}(A)} = \Delta_A \]

\[ \bot_{\text{Sub}(A)} = A \times A. \]
Finally, given \( a \in A \), \( a \) has a complement \( a^c \) in \( \text{Sub}(A) \) given by

\[
b \equiv_{a^c} c \text{ iff } b \lor a = c \lor a.
\]

This can be proven to define a complement as follows:

- Show that if \( b \equiv_{a} c \) and \( b \equiv_{a^c} c \), then \( b = c \).
  
  - This shows that \( a \lor a^c = \top \) by showing \( \equiv_{a} \cap \equiv_{a^c} = \Delta_A \)

- Show that \( \bot_A \equiv \top_A \) can be proven from instances of \( \equiv_{a} \) and \( \equiv_{a^c} \).
  
  - This shows that \( a \land a^c = \bot \) by showing \( (\equiv_{a} \cup \equiv_{a^c}) = A \times A \)

First, suppose \( b \equiv_{a} c \) and \( b \equiv_{a^c} c \). Then

\[
b = b \lor (b \land a)
\]

\[
= b \lor (c \land a)
\]

\[
= (b \lor c) \land (b \lor a)
\]

\[
= (c \lor b) \land (c \lor a)
\]

\[
= c \lor (b \land a)
\]

\[
= c \lor (c \land a)
\]

\[
= c
\]

For the second item,

\[
\top \equiv_{a} a \equiv_{a^c} \bot,
\]

so that

\[
\top \equiv_{a \land a^c} \bot.
\]

Now we establish some basic properties of elements of \( \text{Sub}(A) \).
7.2 Sub\((A)\) is a dual locale

This section comprises two main results, both of which are “well-known”. For more information, see Section 6.2 of [Vic96] and Proposition 2.7 of [Joh86]. To the best of my knowledge, Lemmas 1 and 2 are new.

**Theorem 2.** Sub\((A)\) is complete lattice.

- Nonempty joins are given by \((\equiv \lor_i \alpha_i) = \bigcap_i \equiv \alpha_i\).
- The empty join is given by \((\equiv \lor \emptyset) = A \times A\).
- Meets are given by \((\equiv \land_i \alpha_i) = (\bigcup_i \equiv \alpha_i)\).

**Proof.** See previous section.

**Theorem 3.** Sub\((A)\) satisfies the second infinite distributive law. That is, finite joins distribute over arbitrary meets.

**Proof.** Let \(\alpha, \beta_i \in \text{Sub}(A)\) (where \(i \in I\) for some set \(I\)).

Since the result is clear if \(I = \emptyset\), assume \(I \neq \emptyset\).

By purely order-theoretic considerations,

\[
\alpha \lor \bigwedge_i \beta_i \leq \bigwedge_i (\alpha \lor \beta_i).
\]

The proof of the other direction:

\[
\alpha \lor \bigwedge_i \beta_i \geq \bigwedge_i (\alpha \lor \beta_i)
\]

will require several lemmas. I remark that this is my own proof of a standard result, using the technique of structural induction.

**Lemma 1.** Let \(\alpha\) be a sublocale of \(A\), so that \(\equiv_\alpha\) is a congruence relation. Then for all \(d_1, d_2, \text{ and } c \in A\), we have the following equivalence:

\[
d_1 \equiv_{c \land \alpha} d_2 \iff d_1 \land c \equiv_\alpha d_2 \land c.
\]

**Proof of Lemma** \((\Leftarrow)\) If \(d_1 \land c \equiv_\alpha d_2 \land c\), then we have:

\[
d_1 \equiv_\alpha d_1 \lor c \equiv_\alpha d_2 \lor c \equiv c d_2,
\]

whence \(d_1 \equiv_{c \land \alpha} d_2\). (By weakening and transitivity.)
(⇒) By Structural Induction. Let \( R = \equiv_c \cup \equiv_\alpha \), and \( \Phi(d_1, d_2) \) be “\( d_1 \land c \equiv_\alpha d_2 \land c \)".

Base Case \( (d_1, d_2) \in R \). Either \( d_1 \equiv_c d_2 \), so that \( d_1 \land c = d_2 \land c \), and by weakening we get the result;

OR \( d_1 \equiv_\alpha d_2 \), and meeting both sides by \( c \) gives \( d_1 \land c \equiv_\alpha d_2 \land c \).

Reflexive Clear.

Symmetric Also clear.

Transitive If \( d_1 \land c \equiv_\alpha d_2 \land c \) and \( d_2 \land c \equiv_\alpha d_3 \land c \), then transitivity of \( \equiv_\alpha \) gives \( d_1 \land c \equiv_\alpha d_3 \land c \).

Finite Meets If \( d_1^1 \land c \equiv_\alpha d_1^2 \land c \) and \( d_2^1 \land c \equiv_\alpha d_2^2 \land c \), then by the meet respecting property of \( \equiv_\alpha \) (along with associativity and idempotence of meets), we have:

\[
d_1^1 \land d_1^2 \land c \equiv_\alpha d_2^1 \land d_2^2 \land c.
\]

Arbitrary Joins If for all \( i \in I \), \( d_i^1 \land c \equiv_\alpha d_i^2 \land c \), then we have:

\[
\bigvee_i (d_i^1 \land c) \equiv_\alpha \bigvee_i (d_i^2 \land c)
\]

\[
(d_1^1) \land c \equiv_\alpha (d_1^2) \land c.
\]

QED Lemma

Lemma 2. Let \( \alpha \in \text{Sub}(A) \), and \( b, c \in A \). Then TFAE:

1. \( b \equiv_\alpha c \)

2. \( \equiv_{b \land \alpha} = \equiv_{c \land \alpha} \)

3. \( b \land \alpha = c \land \alpha \)

Proof. Clearly, we just need to show (1) ⇔ (2), because (3) is a restatement of (2).

(⇐) Suppose (2). We have:

\[
b \equiv_{b \land \alpha} \top
\]

\[
c \land b \equiv_{b \land \alpha} c \land \top = c \land c
\]

\[
c \land b \equiv_{c \land \alpha} c \land c
\]

\[
c \land b \equiv_\alpha c,
\]

where the last line is by the previous lemma.
Then similarly,
\[ c \land b \equiv_\alpha b. \]

By transitivity,
\[ b \equiv_\alpha c. \]

(⇒) Suppose \( b \equiv_\alpha c \). We show \( \equiv_{b \land \alpha} \subseteq \equiv_{c \land \alpha} \); the other direction clearly follows by interchanging \( b \) and \( c \) in the proof.

We would like to show: \( d_1 \equiv_{b \land \alpha} d_2 \implies d_1 \equiv_{c \land \alpha} d_2 \). We do Structural Induction, with \( R = \equiv_\alpha \cup \equiv_b \), and \( \Phi(d_1, d_2) \) being "\( d_1 \equiv_{c \land \alpha} d_2 \)".

**Base Case** Either \( d_1 \equiv_b d_2 \), or \( d_1 \equiv_\alpha d_2 \).

Suppose the first. Then \( d_1 \land b = d_2 \land b \). Then:

\[ d_1 \land c \equiv_\alpha d_1 \land b = d_2 \land b \equiv_\alpha d_2 \land c \]

\[ d_1 \equiv_{c \land \alpha} d_2. \]

The last line is by the previous lemma.

Suppose the second case holds. Then \( d_1 \equiv_\alpha d_2 \), so \( d_1 \equiv_{c \land \alpha} d_2 \).

**Induction Cases** These follow trivially since we already know \( \equiv_{c \land \alpha} \) is a congruence relation.

**QED Lemma**

**Continuation of Proof of Theorem**

We must show:

\[ \alpha \lor \bigwedge_i \beta_i \geq \bigwedge_i (\alpha \lor \beta_i) \]

To this end, assume \((d_1, d_2)\) is in the LHS. We will induct to show \((d_1, d_2)\) is also in the RHS. So, suppose

\[ d_1 \equiv_{\alpha \lor \bigwedge_i \beta_i} d_2. \]

Then

\[ d_1 \equiv_\alpha d_2 \text{ and } d_1 \equiv_{\bigwedge_i \beta_i} d_2. \]

We use structural induction, showing:

\[ d_1 \equiv_{\bigwedge_i \beta_i} d_2 \implies [d_1 \equiv_\alpha d_2 \implies d_1 \equiv_{\alpha \lor \bigwedge_i \beta_i} d_2]. \]
**Base Case** For some \( j \in I \), \( d_1 \equiv_{\beta_j} d_2 \). Then we have:

\[
d_1 \equiv_{\alpha \lor \beta_j} d_2
\]

\[
d_1 \equiv_{\bigwedge_i (\alpha \lor \beta_i)} d_2.
\]

**Induction Cases** Once again, these follow trivially, in this case since \( \equiv_{\bigwedge_i (\alpha \lor \beta_i)} \) is a congruence relation.

This establishes the second inequality.

Combining the two results,

\[
\alpha \lor \bigwedge_i \beta_i = \bigwedge_i (\alpha \lor \beta_i).
\]

QED

### 7.3 Representation Theorems

Cf. Section 2.7 in [Joh86], especially Proposition 2.7.

**Theorem 4.** Let \( \alpha \in \text{Sub}(A) \). Then

\[
\alpha = \bigwedge \{(a \lor d^c) \land (a^c \lor d) : a, d \in A \text{ such that } a \equiv_{\alpha} d\}.
\]

**Proof.** First, we show that \( \alpha \) is less then all such \( a \lor d^c \). So suppose:

\[
b \equiv_{a \lor d^c} c.
\]

Then:

\[
b \land a = c \land a
\]

\[
b \lor d = c \lor d.
\]

We must show \( b \equiv_{\alpha} c \). The following hold:

\[
b = b \lor (b \land a) = b \lor (c \land a) = (b \lor c) \land (b \lor a)
\]

\[
\equiv_{\alpha} (c \lor b) \land (b \lor d) = (c \lor b) \land (c \lor d)
\]

\[
= c \lor (b \land d) \equiv_{\alpha} c \lor (b \land a) = c \lor (c \land a) = c.
\]
Clearly, a similar proof holds for $a^c \lor d$. Thus the RHS is $\geq$ the LHS, in the sense that $\text{RHS} \subseteq \text{LHS}$. Suppose $b \equiv_\alpha c$. Choose $a = b$, $d = c$. It is easy to see the following:

$$
\begin{align*}
    a &\equiv_a a \lor d \\
    a &\equiv_{a^c} a \lor d \\
    a \lor d &\equiv_d d \\
    a \lor d &\equiv_{a^c} d.
\end{align*}
$$

So that

$$
b = a \equiv_{a \lor d} a \lor d \equiv_{a \lor d} d = c \quad \text{and} \\
    b \equiv (a \lor d) \land (a^c \lor d) c.
$$

Thus, the RHS is $\leq$ the LHS, so the two sides are equal. QED.

**Theorem 5.** Let $\alpha \in \text{Sub}(A)$. Then

$$
\alpha = \bigwedge \{a \lor d^c : a, d \in A \text{ such that } a \lor d^c \geq \alpha\}.
$$

**Proof.** The RHS is clearly an upper bound. On the other hand, the RHS is a lower bound for the RHS in the previous theorem, since if $a \equiv_\alpha d$, then

$$
(a \lor d^c) \land (a^c \lor d) \geq \alpha,
$$

so that

$$
a \lor d^c \geq \alpha \quad \text{and} \\
a^c \lor d \geq \alpha.
$$

QED
Chapter 8

Lattice of Subsystems of $D$

This chapter summarizes some important properties of the lattice of subsystems of a topological system $D = (ptD, \Omega D, \models)$. We denote this lattice by Sub($D$); context will make it clear whether we are talking about sublocales or subsystems. Except as noted otherwise, the definitions and results of this chapter are new.

8.1 A characterization of Sub($D$)

**Definition 6.** Let $D$ be a topological system. If $u \subseteq ptD$ and $\alpha \in \text{Sub}(\Omega D)$, then we call $u$ and $\alpha$ compatible iff for all $x \in u$ and all $a, b$ so that $a \equiv_\alpha b$ we have

$$x \models a \iff x \models b.$$  

If $x \in ptD$, we say $x$ is compatible with $\alpha$ if for all $a, b$ so that $a \equiv_\alpha b$ we have

$$x \models a \iff x \models b.$$  

**Remark 1.** Equivalently, define $x \models \alpha$ if $x$ is compatible with $\alpha$ in the above sense. If we define

$$\text{ext}(\alpha) = \{x \in ptD : x \models \alpha\},$$

then we have that $u$ and $\alpha$ are compatible iff $u \subseteq \text{ext}(\alpha)$.

**Definition 7.** We define a subsystem of $D$ to be an ordered pair $(u, \alpha)$ with $u \subseteq ptD$, $\alpha \in \text{Sub}(\Omega D)$ and $u$ compatible with $\alpha$. 

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We order \( \text{Sub}(D) \) by \( (u, \alpha) \leq (v, \beta) \) if:

\[ u \subseteq v \text{ AND } \alpha \leq \beta, \]

using the usual ordering on \( \text{Sub}(\Omega D) \), i.e.:

\[ \alpha \leq \beta \text{ iff } \beta \subseteq \alpha. \]

**Theorem 6.** Let \( D \) be a topological system. Then \( \text{Sub}(D) \) is a dual frame with the following operations:

- \( \top_{\text{Sub}(D)} = (\text{pt}(D), \top_{\text{Sub}(\Omega D)}) \)
- \( \bot_{\text{Sub}(D)} = (\emptyset, \bot_{\text{Sub}(\Omega D)}) \)
- \( \bigvee_{i \in I} (u_i, \alpha_i) = (\bigcup_i u_i, \bigvee_i \alpha_i) \)
- \( \bigwedge_{i \in I} (u_i, \alpha_i) = (\bigcap_i u_i, \bigwedge_i \alpha_i) \)

where the join and meet on the LHS are in \( \text{Sub}(D) \) and those on the RHS are in \( \text{Sub}(\Omega D) \).

**Proof.** This is clear since the order in \( \text{Sub}(D) \) is taken componentwise and the components come from frames, provided that \( \text{Sub}(D) \) is closed under the given operations.

First, note that clearly \( \text{pt}(D) \) and \( \top_{\text{Sub}(\Omega D)} \) are compatible, since \( \top_{\text{Sub}(\Omega D)} \) is just equality restricted to \( \Omega D \). Also, \( \emptyset \) is compatible with any sublocale of \( \Omega D \).

Suppose that for all \( i \), \( u_i \) is compatible with \( \alpha_i \). Since

\[ \alpha_i \leq \bigvee_{j \in I} \alpha_j, \]

it is easy to deduce that \( u_i \) is compatible with \( \bigvee_{j \in I} \alpha_j \). This holds for all \( i \in I \); thus for all \( i \),

\[ u_i \subseteq \text{ext}(\bigvee_{j \in I} \alpha_j) \]

so

\[ \bigcup_i u_i \subseteq \text{ext}(\bigvee_{j \in I} \alpha_j). \]

Re-indexing, \( \bigcup_i u_i \) is compatible with \( \bigvee_{i \in I} \alpha_i \).
The hard part, of course, is the last one. Suppose that for all $i$, $u_i$ is compatible with $\alpha_i$. We must show:

$$\bigcap_i u_i \text{ is compatible with } \bigwedge_i \alpha_i.$$ 

Fix $x \in \bigcap_i u_i$. We will proceed by structural induction. Recall that each $\alpha_i$ is a congruence relation on $\Omega D$. Set $R = \bigcup_{i \in I} \alpha_i$; then $\bigwedge_{i \in I} \alpha_i = \langle R \rangle$. Let $\Phi(b, c)$ be the statement:

$$x \models b \iff x \models c.$$ 

Clearly $\Phi(b, c)$ holds for all $(b, c) \in R$, and it is also fairly easy to see that $\Phi(b, c)$ is reflexive, symmetric, and transitive.

Now suppose $\Phi(b_1, c_1)$ and $\Phi(b_2, c_2)$. Now suppose $x \not\models (b_1 \land b_2)$. Then, for some $i$, $x \not\models b_i$. So $x \not\models c_i$, so $x \not\models (c_1 \land c_2)$. Clearly, the converse is also true; taking the “contrapositive”, $\Phi(b_1 \land b_2, c_1 \land c_2)$.

Finally, suppose that for all $i \in I$, $\Phi(b_i, c_i)$. Suppose that $x \not\models \bigvee_i b_i$. Then for all $i$, $x \not\models b_i$. So for all $i$, $x \not\models c_i$. Thus $x \not\models \bigvee_i c_i$. The logic reverses, so take the “contrapositive” to get $\Phi(\bigvee_i b_i, \bigvee_i c_i)$.

It follows by the Structural Induction Theorem that $\Phi(b, c)$ holds for all $(b, c) \in \bigwedge_i \alpha_i$. Thus $x$ is compatible with $\bigwedge_i \alpha_i$, and since $x$ was arbitrary, $\bigcap_i u_i$ is compatible with $\bigwedge_i \alpha_i$.

QED

### 8.2 Satisfaction, extent, and interior

**Opens and subsets**

Consider any topological system $D = (\text{pt} D, \Omega D, \models)$. For an open $a \in \Omega D$, we define

$$\text{ext}(a) = \{x \in \text{pt} D : x \models a \}.$$ 

Given a subset $u \subseteq \text{pt} D$, we define the interior of $u$ by

$$\text{int}(u) = \bigvee \{ a \in \Omega D : \text{ext}(a) \subseteq u \} \in \Omega D.$$
Satisfaction, extent, and interior for sublocales

Let $D = (ptD, \Omega D, |=)$ be a topological system. Recall, we extend the relation $|=\text{ to be a relation between points and sublocales of } \Omega D$ by

$$x |= \alpha \text{ iff } (\forall a, b \in \Omega D \text{ with } a \equiv \alpha b)(x |= a \Leftrightarrow x |= b).$$

Recall we also define $\text{ext}(\alpha) = \{x \in ptD : x |= \alpha\}$.

Given a sublocale $\alpha$, we define its interior by

$$\phi(\alpha) = \bigvee \{a \in \Omega D : a \leq \alpha\},$$

where the inequality is interpreted for $a \in \text{Sub}(\Omega D)$ being the sublocale determined by $a \in \Omega D$.

**Remark**

Observe that if I take $a \in A$ and consider the corresponding sublocale (call it $\alpha$), then $\text{ext}(a) = \text{ext}(\alpha)$.

**Proof** First, suppose $x |= \alpha$. Then for all $b, c \in A$ with $b \land a = c \land a$, $x |= b$ iff $x |= c$. Let $b = \top$ and $c = a$. Since $x |= \top$, $x |= a$.

On the other hand, suppose $x |= a$. We must show that $(b \land a = c \land a) \Rightarrow (x |= b \Leftrightarrow x |= c)$. So suppose $(b \land a = c \land a)$. If $x |= b$, then $x |= b \land a$, so $x |= c \land a$, so $x |= c$; and similarly, if $x |= c$, then $x |= b$. Thus $x |= \alpha$.

Satisfaction, extent, and interior for subsystems

Again, $D = (ptD, \Omega D, |=)$ is a topological system. Satisfaction is easy to define: $x |= (u, \alpha)$ iff $x \in u$. Thus, using the obvious definition:

$$\text{ext}(u, \alpha) = \{x \in ptD : x \in u\} = u.$$

Finally, I define the interior of $(u, \alpha)$ by

$$\text{int}(u, \alpha) = \phi(\alpha) \land \text{int}(u).$$
Remark

There is a natural way in which sublocales may be interpreted as subsystems. If $\alpha$ is a sublocale, then the corresponding system is $(\text{ext}(\alpha), \alpha)$; notice:

$$\text{ext}(\text{ext}(\alpha), \alpha) = \text{ext}(\alpha)$$

$$\text{int}(\text{ext}(\alpha), \alpha) = \varphi(\alpha) \land \text{int}(\text{ext}(\alpha)) = \varphi(\alpha)$$

where the last equality holds because if $a \leq \alpha$, then $\text{ext}(a) \subseteq \text{ext}(\alpha)$, so $\varphi(\alpha) \leq \text{int}(\text{ext}(\alpha))$.

8.3 Order-theoretic adjunctions

Certain order-theoretic adjunctions hold for different combinations of $\text{ext}$, $\text{int}$, and $\varphi$. This section summarizes six adjunction, including two involving concepts introduced in the next chapter. The proofs are straightforward.

$\text{ext}$ and $\text{int}$

$\text{int} : \varphi(\text{pt} D) \to \Omega D$ and $\text{ext} : \varphi(\text{pt} D) \leftarrow \Omega D$

$$\text{ext}(a) \subseteq u \text{ iff } a \leq \text{int}(u)$$

$(\overline{\text{int}})$ and $\varphi$

$(\overline{\text{int}}) : \Omega D \to \text{Sub}(\Omega D)$ and $\varphi : \Omega D \leftarrow \text{Sub}(\Omega D)$

$$a \leq \varphi(\alpha) \text{ iff } \overline{\alpha} \leq \alpha$$

and $\varphi(\overline{\alpha}) = a$.

$\Rightarrow$ and $\text{int}$

$\Rightarrow : \Omega D \to \text{Sub}(D)$ and $\text{int} : \Omega D \leftarrow \text{Sub}(D)$

$$a \leq \text{int}(u, \alpha) \text{ iff } (\text{ext}(a), a) \leq (u, \alpha)$$

We also have two adjunctions involving concepts introduced later.
scge and ext

\[ \text{scge} : \wp(\text{pt} D) \to \text{Sub}(\Omega D) \quad \text{and} \quad \text{int} : \wp(\text{pt} D) \leftarrow \text{Sub}(\Omega D) \]

\[ u \subseteq \text{ext}(\alpha) \iff \text{scge}(u) \leq \alpha \]

swge and ext

\[ \text{swge} : \wp(\text{pt} D) \to \text{Sub}(D) \quad \text{and} \quad \text{ext} : \wp(\text{pt} D) \leftarrow \text{Sub}(D) \]

\[ \text{swge}(u) \leq (v, \alpha) \iff u \subseteq \text{ext}(v, \alpha) \]

For definitions of scge and swge, please see Definition 8 and Definition 10.

Between Sublocales and Subsystems

Finally, an adjunction between Sub(\Omega D) and Sub(D):

\[ (v, \beta) \leq (\text{ext}(\alpha), \alpha) \iff \beta \leq \alpha \]

8.4 Basic properties of extent and interior

Properties as defined on \( \Omega D \)

The following are easy to show:

- For any collection \( \{a_i\} \subseteq \Omega D \), \( \text{ext}(\bigvee_i a_i) = \bigcup_i \text{ext}(a_i) \).
- For any finite collection \( \{a_i\} \subseteq \Omega D \), \( \text{ext}(\bigwedge_i a_i) = \bigcap_i \text{ext}(a_i) \).
- For any collection \( \{u_i\} \subseteq \wp(X) \), \( \bigvee_i \text{int}(u_i) \leq \text{int}(\bigcup_i u_i) \).

It should be noted that the inequality in the third item above need not reverse. To see this consider a system with frame of opens given by \( \{\top, a, b, c, \bot\} \) with \( a > b, a > c \) and \( b \) incomparable with \( c \); and set of points \( X = \{x, y\} \) where both \( x \) and \( y \) satisfies \( \top \) and \( a \), only \( x \) satisfies \( b \) and only \( y \) satisfy \( c \) (and of course nothing satisfies \( \bot \)). Then:

\[ \text{int}(\{x\}) \vee \text{int}(\{y\}) = a \neq \top = \text{int}(\{x\} \cup \{y\}) \]

The following also holds:
• For any collection \( \{ u_i \} \subseteq \wp(X), \text{int}(\bigcap u_i) = (\bigwedge_{\Omega D} \text{int}(u_i)), \)

where the subscript “\( \Omega D \)” on the RHS indicates the meet is to be taken in \( \Omega D \) (and not \( \text{Sub}(\Omega D) \)).

**Proof.** Clearly, “\( \leq \)” holds (by order-theoretic considerations alone).

To show the reverse, suppose

\[
a \leq (\bigwedge_{\Omega D} \text{int}(u_i)).
\]

Then for all \( i \),

\[
a \leq \text{int}(u_i),
\]

so that for all \( i \),

\[
\text{ext}(a) \subseteq u_i
\]

and

\[
\text{ext}(a) \subseteq \bigcap_i u_i
\]

so

\[
a \leq \text{int}(\bigcap u_i).
\]

Taking \( a = (\bigwedge_{\Omega D} \text{int}(u_i)) \) give the desired result.

QED

**Remark 2.** Note that this last equality still holds even if the indexing set is infinite.

**Properties as defined on** \( \text{Sub}(\Omega D) \)

**Lemma 3.** The following holds:

• For any collection \( \{ \alpha_i \} \subseteq \text{Sub}(\Omega D), \, \varphi(\bigwedge_i \alpha_i) = (\bigwedge_{\Omega D} \varphi(\alpha_i)) \).

**Proof.** Clearly, \( \varphi(\bigwedge_i \alpha_i) \leq (\bigwedge_{\Omega D} \varphi(\alpha_i)) \) holds (since \( \varphi \) preserves order, and \( \varphi(\bigwedge_i \alpha_i) \), is, by definition, a member of \( \Omega D \)).

Suppose (where \( a \in \Omega D \)):

\[
a \leq (\bigwedge_{\Omega D} \varphi(\alpha_i))
\]

Then:

\[
\forall i, \, a \leq \varphi(\alpha_i)
\]

\[
\forall i, \, a \leq \alpha_i
\]
\[ a \leq \bigwedge_i \alpha_i \]

\[ a \leq \varphi(\bigwedge_i \alpha_i). \]

Taking \( a = (\bigwedge_\Omega D)_i \varphi(\alpha_i) \) gives the result.

QED

**Lemma 4.** The following holds:

- For any collection \( \{ \alpha_i \} \subseteq \text{Sub}(\Omega D) \), \( \text{ext}(\bigwedge_\alpha \alpha_i) = \bigcap_\alpha \text{ext}(\alpha_i) \).

**Proof.** Clearly, \( \subseteq \) holds since \text{ext} preserves order.

Now suppose \( x \models \alpha_i \) for all \( i \in I \). Then for all \( i \),

\[ b \equiv_{\alpha_i} c \Rightarrow (x \models b \iff x \models c). \]

We must show \( x \models \bigwedge_\alpha \alpha_i \), i.e.:

\[ b \equiv_{\bigwedge_\alpha \alpha_i} c \Rightarrow (x \models b \iff x \models c). \]

Proceed by structural induction on pairs \((b, c)\), taking as the base case \((b, c) \in \bigcup_i \equiv_{\alpha_i} \). In the base case, \( x \models b \iff x \models c \) holds by assumption (since \( b \equiv_{\alpha_i} c \) for some \( i \)).

The reflexivity, symmetry, and transitivity cases follow since \( x \models b \iff x \models c \) is reflexive, symmetric and transitive as a relation between \( b \) and \( c \).

Finite Meets. Suppose \( x \models b_i \iff x \models c_i \) for \( i \in \{0, 1\} \). Then:

\[ x \models (b_1 \land b_2) \iff \]

\( \{x \models b_1 \text{ and } x \models b_2\} \iff \)

\( \{x \models c_1 \text{ and } x \models c_2\} \iff \)

\( x \models (c_1 \land c_2). \)

Arbitrary joins. Suppose for some set \( I \), \( x \models b_i \iff x \models c_i \) for all \( i \in I \). Then:

\[ x \models (\bigvee b_i) \iff \]

\( \exists i, x \models b_i \iff \)
\[ \exists i, x \models c_i \text{ iff } x \models (\bigvee c_i). \]

Therefore, by the Structural Induction Theorem, for all \( b, c \) in \( \Omega D \),

\[ b \equiv_{\bigwedge \alpha_i} c \Rightarrow \{ x \models b \leftrightarrow x \models c \}. \]

Thus, \( x \models \bigwedge \alpha_i \).

QED

**Remark.** Since the corresponding equality need not hold for meets in \( \Omega D \), this shows that infinite meets in \( \Omega D \) need not “equal” the meets in \( \text{Sub}(\Omega D) \) of the corresponding elements in \( \text{Sub}(\Omega D) \).

**Lemma 5.** The following holds:

- Let \( \alpha_1, \alpha_2 \in \text{Sub}(\Omega D) \). Then

\[ \text{ext}(\alpha_1 \lor \alpha_2) = \text{ext}(\alpha_1) \cup \text{ext}(\alpha_2). \]

**Proof.** Clearly “\( \supseteq \)" holds since \( \text{ext} \) preserves order. So suppose (towards a contradiction) that \( x \in X \) so that \( x \models (\alpha_1 \lor \alpha_2) \) but \( x \not\models \alpha_1 \) and \( x \not\models \alpha_2 \). Then there exist \( a_i \) and \( b_i \) for \( i = 1, 2 \) so that (for \( i \in \{1, 2\} \)):

\[ a_i \equiv_{\alpha_i} b_i \]

\[ x \models a_i \]

\[ x \not\models b_i; \]

further, by replacing \( b_1 \) by \( a_i \land b_i \), we may assume:

\[ a_i \geq b_1. \]

Now:

\[ a_1 \equiv_{\alpha_1} b_1 \]

\[ a_1 \lor b_1 \lor b_2 \equiv_{\alpha_1} b_1 \lor b_2 \]

\[ (a_1 \lor b_1 \lor b_2) \land (a_1 \land a_2) \equiv_{\alpha_1} (b_1 \lor b_2) \land (a_1 \land a_2) \]

\[ (a_1 \land a_1 \land a_2) \lor [(b_1 \lor b_2) \land (a_1 \land a_2)] \equiv_{\alpha_1} (b_1 \lor b_2) \land (a_1 \land a_2). \]
But the LHS of the last equation simplifies to $a_1 \land a_2$, so

$$a_1 \land a_2 \equiv_{\alpha_1} (b_1 \lor b_2) \land (a_1 \land a_2).$$

A similar argument starting with $a_2 \equiv_{\alpha_2} b_2$ yields:

$$a_1 \land a_2 \equiv_{\alpha_2} (b_1 \lor b_2) \land (a_1 \land a_2)$$

so that

$$a_1 \land a_2 \equiv_{\alpha_1 \lor \alpha_2} (b_1 \lor b_2) \land (a_1 \land a_2).$$

But $x$ satisfies the LHS but not the RHS. Therefore, by the definition of $|=\text{ for Sub}(\Omega D)$, $x \not|= \alpha_1 \lor \alpha_2$.

QED

**Remark.** The following inequalities are consequences of the fact that $\varphi$ and $\text{ext}$ preserve order:

- For any collection $\{\alpha_i\} \subseteq \text{Sub}(\Omega D)$, $\bigvee_i \varphi(\alpha_i) \leq \varphi(\bigvee_i \alpha_i)$

- For any collection $\{\alpha_i\} \subseteq \text{Sub}(\Omega D)$, $\bigcup_i \text{ext}(\alpha_i) \subseteq \text{ext}(\bigvee_i \alpha_i)$.

The following examples show that in both cases, equality need not hold.

**Example 9.** Let $A = \{\bot, a, \top\}$ with $\bot < a < \top$. Then some quick calculations with the Representation Theorems shows $\text{Sub}(A)$ is isomorphic to the four-point diamond, having the elements corresponding to the three of $A$ as well as one we will call $a^c$. So:

$$\varphi(a) \lor \varphi(a^c) = a \lor \bot = a$$

$$\varphi(a \lor a^c) = \varphi(\top) = \top \neq a.$$  

QED

**Example 10.** In this example, arbitrary joins do not interchange with $|=\text{.}$

**Content of Example**

Let $\mathbb{N}$ be the natural numbers. Define a topological system $D$ as follows:

$$\text{pt} D = \mathbb{N}$$

$$\Omega D = \{u \subseteq \mathbb{N} : 0 \in u \text{ and } u \text{ is cofinite}\} \cup \{\emptyset\}$$
$\models_D$ is membership.

Since $\Omega D$ is a topology on $ptD$, this does define a topological system.

First, observe that for any $u \in \Omega D$, $\text{ext}(u^c) = ptD \setminus \text{ext}(u)$. This follows from the lemmas in this subsection, along with the facts that $u \wedge u^c = \bot$ and $u \vee u^c = \top$. In particular, consider a cofinite $u$ in $\Omega D$. Then $0 \nmid u^c$, since $0 \nmid u$.

Now let $\gamma = \bigvee\{u^c : u \in \Omega D \text{ is cofinite}\}$. I claim $0 \models \gamma$. Suppose $a, b \in \Omega D$ satisfy $a \equiv \gamma b$. Then for all cofinite $u$ containing 0,

$$a \equiv_{u^c} b$$

$$a \lor u = b \lor u$$

$$a \cup u = b \cup u \text{ (taking extents of both sides)}$$

For any nonzero $n \in \mathbb{N}$, I can find $u$ so $n \not\in$ $u$, so that $n \in a$ iff $n \in b$. Thus $a$ and $b$ agree (as sets) for any $n \neq 0$. Looking back at $\Omega D$, if $a = \emptyset$, then $b$ must be empty as well, so $0 \in a \iff 0 \in b$ (since both sides are false). If $a$ is cofinite, then $b$ cannot be empty and thus must be cofinite, so by definition of $\Omega D$, $0 \in a$ and $0 \in b$, so $0 \in a \iff 0 \in b$. (Actually, $a = b$, but this would probably take a sentence or two longer proof.)

Thus:

$$0 \models \gamma = \bigvee\{u^c : u \in \Omega D \text{ is cofinite}\}$$

but

$$0 \nmid u^c \text{ for all cofinite } u \text{ in } \Omega D.$$ 

So $\models$ need not interchange with arbitrary joins in $\text{Sub}(\Omega D)$.

QED

**Properties as defined on $\text{Sub}(D)$**

**Lemma 6.** The following hold:

- For any collection $\{(u_i, \alpha_i)\} \subseteq \text{Sub}(D)$, $\text{ext}(\bigvee_i (u_i, \alpha_i)) = \bigcup_i \text{ext}(u_i, \alpha_i)$.

- For any collection $\{(u_i, \alpha_i)\} \subseteq \text{Sub}(D)$, $\text{ext}(\bigwedge_i (u_i, \alpha_i)) = \bigcap_i \text{ext}(u_i, \alpha_i)$.

**Proof.** These equalities follow directly from the definition of $\text{ext}$ on $\text{Sub}(D)$, together with the characterization of joins and meets in $\text{Sub}(D)$.

QED
Lemma 7. The following holds:

• For any collection \( \{(u_i, \alpha_i)\} \subseteq \text{Sub}(D) \), \( \text{int}(\bigwedge_i (u_i, \alpha_i)) = \bigwedge_i \text{int}(u_i, \alpha_i) \).

Proof.

\[
\text{int}(\bigwedge_i (u_i, \alpha_i)) = \text{int}(\bigcap_i u_i, \bigcap_i \alpha_i)
\]
\[
= \varphi(\bigcap_i \alpha_i) \land \text{int}(\bigcap_i u_i)
\]
\[
= (\bigwedge_{\Omega D} \varphi(\alpha_i)) \land (\bigwedge_{\Omega D} \text{int}(u_i))
\]
\[
= (\bigwedge_{\Omega D} (\varphi(\alpha_i) \land \text{int}(u_i)))
\]
\[
= (\bigwedge_{\Omega D} \text{int}(u_i, \alpha_i)).
\]

Remark: One can double-check the empty case directly, just to be sure.

QED

Corollary 1. The operation \( \text{int} : \text{Sub}(D) \rightarrow \Omega D \) preserves order.

Proof. Any function between lattices which preserves meets also preserves order. QED

Lemma 8. The following holds:

• For any collection \( \{(u_i, \alpha_i)\} \subseteq \text{Sub}(D) \), \( \bigvee_i \text{int}(u_i, \alpha_i) \leq \text{int}(\bigvee_i (u_i, \alpha_i)) \).

Proof. This is a consequence of the fact that \( \text{int} \) preserves order.

QED
Chapter 9

Example of $\text{Sub}(\Omega D)$ and $\text{Sub}(D)$ for Specific $D$

Example 11. Subsystems of a Simple System

Let $D$ be as in Example 4.

Content of Example.

The the extents of various sublocales may be computed as follows:

- $\text{ext}(\top) = \{x, y, z\}$
- $\text{ext}(a) = \{x, z\}$
- $\text{ext}(b) = \{y, z\}$
- $\text{ext}(c) = \{z\}$
- $\text{ext}(\bot) = \emptyset$
- $\text{ext}(a^\circ) = \{y\}$
- $\text{ext}(b^\circ) = \{x\}$
- $\text{ext}(c^\circ) = \{x, y\}$. 

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Thus the following lists all subsystems of this $D$:

$$(\{x, y, z\}, \top)$$

$$(\{x, y\}, \top)$$

$$(\{x, z\}, \top)$$

$$(\{y, z\}, \top)$$

$$(\{x\}, \top)$$

$$(\{y\}, \top)$$

$$(\{z\}, \top)$$

$$(\emptyset, \top)$$

$$(\{x, z\}, a)$$

$$(\{x\}, a)$$

$$(\{z\}, a)$$

$$(\emptyset, a)$$

$$(\{y, z\}, b)$$

$$(\{y\}, b)$$

$$(\{z\}, b)$$

$$(\emptyset, b)$$

$$(\{x, y\}, c^c)$$

$$(\{x\}, c^c)$$

$$(\{y\}, c^c)$$

$$(\emptyset, c^c)$$
Example 12. Sublocales of $T_{\text{cof}}^0$

This example will explain the lattice of sublocales of $T_{\text{cof}}^0$.

Content of Example. Let $D$ be the topological system of Example 5. Then $\Omega D = T_{\text{cof}}^0$.

The dual locale $\text{Sub}(\Omega D)$ is generated by elements of the form $\overline{\pi}$ and $\overline{\pi^c}$, where $u \in \Omega D$, and for the purposes of this example, $\overline{\pi^c}$ denotes the complement of $\overline{\pi}$ in $\text{Sub}(\Omega D)$. Observe the following:

$$\overline{\pi} \land \overline{\pi^c} = \bot$$

$$\overline{\pi} \lor \overline{\pi^c} = \top.$$ 

Now all sublocales are of the form $\bigwedge_{i \in I} (\overline{u_i} \lor \overline{v_i^c})$ for some families $\{u_i\}_{i \in I}$ and $\{v_i\}_{i \in I}$.

Consider the following claim.

CLAIM 1. If $u_i \neq \emptyset$, then:

$$\overline{u_i} \lor \overline{v_i^c} = \overline{(u_i \cup v_i^c)}$$

where $v_i^c$ is the set theoretic complement of $v_i$ (with respect to $\text{pt} D$).

Proof of Claim. Observe:

$$(\overline{v_i} \lor \overline{v_i^c}) \land \overline{v_i}$$

$$= \overline{u_i} \land \overline{v_i}$$

$$= u_i \land v_i,$$

where the second line follows from distributing and cancelling, and the third line follows because we are working in a topology, so finite meets are actually intersections.
Observe also:

\[(\overline{u_i} \lor \overline{v_i}) \lor \overline{u_i} = \top.\]

Also, clearly:

\[
(\overline{u_i} \cup \overline{v_i}) \land \overline{u_i} \\
= \overline{u_i} \lor \overline{v_i}
\]

and:

\[
(\overline{u_i} \cup \overline{v_i}) \lor \overline{u_i} = \top.
\]

Recall that in any distributive lattice, if \(a \lor c = b \lor c\), and \(a \land c = b \land c\), then \(a = b\).

Thus, because \(\text{Sub}(\Omega D)\) is a distributive lattice (as any frame is),

\[
\overline{u_i} \lor \overline{v_i} = (\overline{u_i} \cup \overline{v_i}).
\]

Thus the claim holds.

CLAIM 2. Let \(\{u_i\}_{i \in I}\) and \(\{u_j\}_{j \in J}\) be two collections of nonempty elements of \(\Omega D\). Then: If \(\bigcap_{i \in I} u_i = \bigcap_{j \in J} u_j\), then \(\bigwedge_{i \in I} \overline{u_i} = \bigwedge_{j \in J} \overline{u_j}\).

Proof of claim. Under the hypothesis, the following is true for each \(x \in \text{pt}(D)\):

\[
\exists i \in I.x \notin u_i \iff \exists j \in J.x \notin u_j.
\]

It suffices to show:

\[
\forall j \in J. \bigwedge_{i \in I} \overline{u_i} \leq \overline{u_j}.
\]

So fix \(j \in J\). Now, \(u_j\) is cofinite, so let \(x_1, \ldots, x_K\) be the elements of \(\text{pt}D\) not in \(u_j\). Choose \(u_{i_1}, \ldots u_{i_K}\) so that \(\forall k.x_k \notin u_{i_k}\), so that \(u_{i_1} \cap \cdots \cap u_{i_K} \subseteq u_j\).

Then:

\[
\bigwedge_{i \in I} \overline{u_i} \leq \overline{u_{i_1} \land \cdots \land u_{i_K}}
\]

\[
= \overline{u_{i_1} \cap \cdots \cap u_{i_K}}
\]

\[
\leq \overline{u_j}.
\]

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Continuation of Example. Consider again that any sublocale has the form:

$$\bigwedge_{i \in I} (\overline{u_i} \lor \overline{v_i}'').$$

We have shown (Claim 1) that unless $u_i = \emptyset$ then $\overline{u_i} \lor \overline{v_i}''$ may be replaced by $\overline{u_i} \cup \overline{v_i}''$. In particular, we can rewrite this as:

$$\bigwedge_{j \in J} \overline{u_j} \land \bigwedge_{l \in L} \overline{v_l}''.$$

Now the first term is determined by $\bigcap_{j \in J} \overline{u_j}$ by Claim 2. (The case where $u_j = \emptyset$ for some $j$ is sufficiently trivial to dismiss.) Now, if $L$ is a finite set, then:

$$\bigwedge_{l \in L} \overline{v_l}'' = (\bigvee_{l \in L} \overline{v_l})^c$$

$$= (\bigcup_{l \in L} v_l)^c,$$

and the second term is reduced to complement of an image of a single open.

On the other hand, suppose $L$ is an infinite set. Let $v = \bigcup_{l \in L} v_l$. Then $v$ is clearly a member of $\Omega D$. Then:

$$\forall l. v \geq v_l$$

$$\forall l. \overline{v} \geq \overline{v_l}$$

$$\forall l. \overline{v'} \leq \overline{v_l}''$$

$$\overline{v'} \leq \bigwedge_{l \in L} \overline{v_l}.''$$

This inequality reverses: Since $v_l$'s are cofinite, $v = \bigcup_{l \in M} v_l$ for some finite subset $M$ of $L$. Thus, we have:

$$v = \bigcup_{l \in M} v_l$$

$$\overline{v} = \bigcup_{l \in M} \overline{v_l} = \bigvee_{l \in M} \overline{v_l}$$

$$\overline{v'} = \bigwedge_{l \in M} \overline{v_l}'' \geq \bigwedge_{l \in L} \overline{v_l}''.$$

Thus, in any event, our sublocale is one of the following:

- A meet of opens.
• The complement of an open.

• A meet of opens, met with the complement of an open.

CLAIM 3: Suppose \( v_1, v_2 \neq ptD \), and also that:

\[
\bigcap_{i \in I} u_i \cap v_1^c = \bigcap_{j \in J} u_j \cap v_2^c,
\]

for some (possibly infinite) families \( \{u_i\}_{i \in I} \) and \( \{u_j\}_{j \in J} \) of opens.

Then:

\[
\bigwedge_{i \in I} u_i \land v_1^c = \bigwedge_{j \in J} u_j \land v_2^c.
\]

Proof of claim: It suffices to show that for all \( u_j \),

\[
\bigwedge_{i \in I} u_i \land v_1^c \leq u_j
\]

and also

\[
\bigwedge_{i \in I} u_i \land v_1^c \leq v_2^c.
\]

The first inequality follows similarly to the second using a variation of the proof of Claim 2 above; the second is more complicated so I prove it:

\[
\bigcap_{i \in I} u_i \cap v_1 \subseteq v_2^c
\]

\[
\bigcap_{i \in I} u_i \cap v_2 \subseteq v_1
\]

As above, we get that for some finite \( I' \),

\[
\bigcap_{i \in I'} u_i \cap v_2 \subseteq v_1
\]

so that

\[
\bigwedge_{i \in I'} u_i \land v_2 \subseteq v_1
\]

\[
\bigwedge_{i \in I'} u_i \land v_1^c \subseteq v_2^c
\]

\[
\bigwedge_{i \in I} u_i \land v_1^c \subseteq v_2^c.
\]

This establishes the claim.

Observe, however, that \( I \) or \( J \) could be empty. Also, since \( v_1 \) is cofinite, its complement (as a set) is
finite. Thus, \(\bigcap_{i \in I} v_i \cap v_1^c\) must be finite, so there exists a \(v_3\) so \(\bigcap_{i \in I} u_i \cap v_1^c = v_3^c\), and pairing \(v_3\) with an empty collection of \(u_k\)'s, we get:

\[
\bigwedge_{i \in I} u_i \land v_1^c = v_3^c.
\]

Thus, any sublocale is one of the following:

- A meet of opens.
- The complement of an open.

Clearly these are distinct except in trivial cases, because in general 0 satisfies a meet of opens, but 0 does not satisfy any complements of opens (except \(\emptyset^c\)). Thus, we get a lattice isomorphic to the following one: The full collection of all subsets of \(\mathbb{N}\) which contain 0 as the upper portion, and sitting below each finite subset \(F\), the subset \(F \setminus \{0\}\). By considering the partial order, we see the join of an infinite collection \(F_i\) for \(i \in I\) is given by \(\bigcup_i F_i \cup \{0\}\).

QED Example

**Example 13.** This example continues the previous example, describing the lattice of subsystems of the system from Example 12.

**Content of Example.**

Let \(D\) be as in Example 12. Then to describe \(\text{Sub}(D)\), we compute the extents of each element of \(\text{Sub}(\Omega D)\).

Notice:

1. \(\text{ext}(\top) = \mathbb{N}\)
2. \(\text{ext}(\bigwedge_i \pi_i) = \bigcap_i u_i\)
3. \(\text{ext}(\pi^c) = \nu^c\)
4. \(\text{ext}(\bot) = \emptyset\).

Recall that \(\text{Sub}(D) = \{(u, \alpha) : \alpha \in \text{Sub}(\Omega D) \text{ and } u \subseteq \text{ext}(\alpha)\}\).

Thus, if one thinks of \(\text{Sub}(D)\) as an order-theoretic structure lying in the plane, then one can picture \(\text{Sub}(D)\) as a three-dimensional structure with subsystems of the form \((\text{ext}(\alpha), \alpha)\) in one plane and those subsystems \((u, \alpha)\) with \(u \subset \text{ext}(\alpha)\) lying below this plane.

QED Example

Please see the illustrations at the end of the chapter.
Figure 9.1: Sublocales of $T_{cof}$
Figure 9.2: Subsystems of \((\mathbb{N}, T_{cof}, \models)\)
Chapter 10

The frame $\Omega_{\mathcal{P}D}$

In this chapter we define an interesting frame which is a quotient of the dual frame $\text{Sub}(D)$. It was originally intended as a frame of “arbitrary” subsets of a topological system. However, we will see in the next chapter that continuous functions need not preserve the equivalence relation from the quotient in the codomain system to the quotient in the domain system.

10.1 The operation $\text{scge}$

It is useful to have a notation for the smallest sublocale whose extent contains a given subset of the set of points. The existence of such a smallest sublocale is guaranteed by the fact that $\text{ext}$ takes meets to intersections, together with the fact that $\text{ext}(\top)$ is the set of all points.

**Definition 8.** Given $u \in \wp(\text{pt}D)$, we define

\[ \text{scge}(u) = \bigwedge \{ \alpha \in \text{Sub}(\Omega D) : u \subseteq \text{ext}(\alpha) \}. \]

**Remark 3.** (1) Given $u \in \wp(\text{pt}D)$, $\text{scge}(u)$ is the smallest member of $\text{Sub}(\Omega D)$ whose extent contains $u$.

(2) Given $u \in \wp(\text{pt}D)$ and $\alpha \in \text{Sub}(\Omega D)$, $u$ is compatible with $\alpha$ if and only if $\alpha \geq \text{scge}(u)$.

(3) It should be clear that $\text{scge}(u)$ preserves order.

**Lemma 9.** Let $\{u_i\}_{i \in I} \subseteq \wp(\text{pt}D)$, for a topological system $D$. Then

\[ \text{scge}\left( \bigcup_i u_i \right) = \bigvee_i \text{scge}(u_i). \]

**Proof.** By remark part (3) above, LHS $\geq$ RHS. On the other hand, $\bigvee_i \text{scge}(u_i)$ is the meet of all elements
greater than or equal to it. Suppose that

$$\bigvee_i \text{scge}(u_i) \leq \alpha.$$ 

Then

$$\bigcup_i u_i \subseteq \bigcup_i \text{ext(}\text{scge}(u_i))$$

$$= \text{ext(}\bigvee_i \text{scge}(u_i))$$

$$\subseteq \text{ext}(\alpha).$$

It follows that \(\alpha\) is an upper bound for \(\text{scge}(\bigcup_i u_i)\). Since \(\alpha\) could be \(\bigvee_i \text{scge}(u_i)\), the RHS of the lemma is greater than or equal to the LHS. Combining the two inequalities yields the lemma.

QED

Lemma 10. Suppose \(u \in \wp(ptD)\) is such that for some \(a \in \Omega D\), \(\text{ext}(a) = u\). Then

$$\text{scge}(u) \land \text{scge}(u^c) = \bot.$$ 

Proof. Clearly, since \(\text{ext}(a^c) = u^c\), we have:

$$\text{scge}(u) \leq a;$$

$$\text{scge}(u^c) \leq a^c.$$ 

Since \(a \land a^c = \bot\), the equation follows.

QED

Lemma 11. If \(D\) is a spatial topological system, then

$$\text{scge}(\text{ext}(a)) = a.$$ 

Proof. The direction “\(\leq\)’’ is clear. So consider the converse direction.

Since \(\text{scge}(\text{ext}(a))\) is a meet of elements of the form \(b_1 \lor b_2^c\), suppose

$$\text{scge(}\text{ext}(a)) \leq b_1 \lor b_2^c.$$ 

Then we have:

$$\text{ext}(a) = \text{ext(}\text{scge}(\text{ext}(a))) \leq \text{ext}(b_1) \cup \text{ext}(b_2)^c$$

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\[
\text{ext}(a) \cap \text{ext}(b_2) \subseteq \text{ext}(b_1)
\]

\[
\text{ext}(a \wedge b_2) \subseteq \text{ext}(b_1).
\]

Then by spatiality,
\[
a \wedge b_2 \leq b_1
\]
\[
a \leq b_1 \lor b_2.
\]

Taking a meet over all elements of the form of the RHS that are greater than or equal to \(\text{scge}(\text{ext}(a))\) we get:
\[
a \leq \text{scge}(\text{ext}(a)).
\]

QED

**Definition 9.** If a topological system \(D\) satisfies:
\[
\forall u \subseteq \text{pt}D, \exists a \in \Omega D, \text{ext}(a) = u
\]

then we say \(D\) is discrete.

**Remark 4.** Note that a spatial discrete system is homeomorphic to a discrete space.

**Lemma 12.** If \(D\) is discrete, then for all \(u,v \subseteq \text{pt}D\),
\[
\text{scge}(u) \wedge \text{scge}(v) = \text{scge}(u \cap v).
\]

**Proof.** Clearly, “\(\geq\)” holds.

So, suppose
\[
\text{scge}(u \cap v) \leq \theta
\]
for some \(\theta \in \text{Sub}(\Omega D)\). Choose \(b,c \in \Omega D\) so that \(\text{ext}(b) = u\) and \(\text{ext}(c) = v\). Such \(b\) and \(c\) exist by discreteness of \(D\).

Then set
\[
\theta_1 = (\theta \lor b^c) \wedge b
\]
\[
\theta_2 = (\theta \lor b^c) \wedge c.
\]

Then
\[
\text{ext}(\theta_1) = \text{ext}[(\theta \lor b^c) \wedge b]
\]
\[ \{ \text{ext}(\theta) \cup \text{ext}(c^c) \} \cap \text{ext}(b) = \text{ext}(b) = u \]

and similarly

\[ \text{ext}(\theta_2) = v. \]

So

\[ \text{scge}(u) \land \text{scge}(v) \leq \theta_1 \land \theta_2 \]

\[ = (\theta \lor c^c) \land b \land (\theta \lor b^c) \land c = \theta \land b \land c \leq \theta. \]

So every upper bound for \( \text{scge}(u \cap v) \) is also an upper bound for \( \text{scge}(u) \land \text{scge}(v) \). It follows that:

\[ \text{scge}(u) \land \text{scge}(v) \leq \text{scge}(u \cap v) \]

and therefor

\[ \text{scge}(u) \land \text{scge}(v) = \text{scge}(u \cap v). \]

QED

10.2 Complements in \( \text{Sub}(D) \)

**Notation** If \( u \subseteq \text{pt}D \), for some topological system \( D \), then \( u^c \) denotes \( \text{pt}D \setminus u \).

**Lemma** If \( D \) is a topological system and \( (u, \alpha) \) is a subsystem, then \( (u^c, \alpha^c \lor \text{scge}(u^c)) \) is the dual Heyting complement of \( (u, \alpha) \) in \( \text{Sub}(D) \), i.e. the smallest element of \( \text{Sub}(D) \) that when joined with \( (u, \alpha) \) yields \( (\text{pt}D, \top) \).

**Proof.** Clearly,

\[ (u, \alpha) \lor (u^c, \alpha^c \lor \text{scge}(u)) = (\text{pt}D, \top). \]

Now, suppose \( (u, \alpha) \lor (v, \beta) = (\text{pt}D, \top) \). Then \( u \cup v = \text{pt}D \), so \( u^c \subseteq v \). Further, \( \alpha \lor \beta = \top \), so \( \beta \geq \alpha^c \). But \( \beta \) must be compatible with \( v \), so \( \beta \geq \text{scge}(v) \geq \text{scge}(u^c) \). Thus we have:

- \( v \supseteq u^c \)
- \( \beta \geq \alpha^c \lor \text{scge}(u^c) \)

These show that \( (u^c, \alpha^c \lor \text{scge}(u^c)) \) is the smallest possible candidate, and since it actually “works,” it must the complement.

QED
10.3 The operation swge

For completeness, we include the operation swge: \( \wp(\text{pt}D) \rightarrow \text{Sub}(D) \).

**Definition 10.** Let \( D \) be a topological system. For any \( \mathcal{u} \in \wp(\text{pt}D) \), we define:

\[
\text{swge}(\mathcal{u}) = (\mathcal{u}, \text{scge}(\mathcal{u})).
\]

The subsystem \( \text{swge}(\mathcal{u}) \) has extent exactly \( \mathcal{u} \), and is the smallest subsystem with that extent.

10.4 General properties of the dual Heyting complement on a dual frame

The result of this section is just the dual of well-known results about complete Heyting algebras. They are included because they are useful in manipulating elements of \( \text{Sub}(D) \) and \( \text{Sub}(\Omega D) \).

**Lemma** Let \( ^c \) be the dual Heyting complement on a dual frame \( A \). Then:

- \( (-)^c \) reverses order and \( (-)^{cc} \) preserves order.
- For any \( a \in A \), \( a^{cc} \leq a \).
- For any \( \{a_i\}_{i \in I} \subseteq A \), \( (\bigwedge a_i)^c = \bigvee a_i^c \).
- For any \( \{a_i\}_{i \in I} \subseteq A \), \( (\bigvee a_i)^c \leq \bigwedge a_i^c \).
- For any \( \{a_i\}_{i \in I} \subseteq A \), \( (\bigvee a_i)^{cc} = (\bigvee a_i^c)^{cc} \).
- For any \( \{a_i\}_{i \in I} \subseteq A \), \( (\bigwedge a_i)^{cc} = (\bigwedge a_i^c)^{cc} \).

**Proof.** (1) If \( a \leq b \), the \( a^c \vee b \geq a^c \vee a = \top \), so \( a^c \geq b^c \). The second claim follows from the first (apply the first twice).

(2) By definition of \( ^c \), \( a^c \vee a = \top \). But \( a^{cc} \) is the least \( \gamma \) such that \( \gamma \vee a^c = \top \). Thus, \( a \geq a^{cc} \).

(3)(\geq) Since \( \bigwedge a_i \leq a_j \), for all \( j \),

\[
(\bigwedge a_i)^c \geq a_j^c \\
(\bigwedge a_i)^c \geq \bigvee a_j^c.
\]
(3)(≤)

\[(\bigvee a_j^c) \lor \bigwedge a_i\]

\[= \bigwedge_i (a_i \lor \bigvee_j a_j^c)\]

\[\geq \bigwedge_i (a_i \lor a_i^c)\]

\[= \top.\]

Thus,

\[(\bigwedge a_i)^c \leq \bigvee a_i^c.\]

(4) The proof is dual to that of (3)(≥).

(5) First, observe that:

\[(\bigvee a_i^{cc})^{cc} = (\bigwedge a_i^{cc})^{cc}\]

\[= (\bigwedge a_i^{cc})^c\]

\[= (\bigvee a_i^{cc}).\]

Thus, we have:

\[(\bigvee a_i^{cc}) = (\bigvee a_i^{cc})^{cc} \leq (\bigvee a_i)^{cc}.\]

On the other hand,

\[(\bigvee a_i)^c = (\bigvee a_i)^{ccc}\]

\[= (\bigvee a_i^{ccc})^c\]

\[= (\bigvee a_i^{cc})^{cc}\]

\[= (\bigvee a_i^{cc})^c,\]

So that

\[(\bigvee a_i)^c \lor (\bigvee a_i^{cc})\]

\[= (\bigvee a_i^{cc})^c \lor (\bigvee a_i^{cc})\]

\[= \top.\]
It follows that

$$\bigvee a_i^{cc} \leq \left( \bigvee a_i \right)^{cc}.$$  

(6) One inequality is established above:

$$\left( \bigvee a_i^{cc} \right) \leq \left( \bigvee a_i \right)^{cc}.$$  

For the other direction, observe that:

$$\bigvee_i a_i^{cc} = \left( \bigwedge_j a_j \right)^c$$

and

$$\left( \bigvee_i a_i^{cc} \right) \lor \bigwedge_j a_j^c$$

$$= \bigwedge_j [\left( \bigvee_i a_i^{cc} \right) \lor a_j^c]$$

$$\geq \bigwedge_j (a_j^{cc} \lor a_j^c)$$

$$= T.$$

It follows from the definition of $^c$ that:

$$\left( \bigvee a_i^{cc} \right) \geq \left( \bigvee a_i \right)^{cc}.$$  

(7) Observe

$$\left( \bigwedge a_i \right)^{cc} = \left( \bigvee a_i \right)^c$$

$$= [\left( \bigwedge a_i^{cc} \right)^c]^c$$

$$= \left( \bigwedge a_i^{cc} \right)^{cc}.$$  

QED

10.5 The relation $\equiv$

Given a topological system $D$, we define an equivalence relation on Sub($D$). This is essentially the construction of a complete Boolean algebra from a complete Heyting algebra, only we start with a dual complete Heyting algebra.
Definition 11. Let $D$ be a topological system. Let $(u, \alpha)$ and $(v, \beta)$ be subsystems of $D$. Then $(u, \alpha) \equiv (v, \beta)$ if and only if:

- $(u, \alpha)^{cc} = (v, \beta)^{cc}$.

Remarks. (1) Observe that $(u, \alpha) \equiv (v, \beta)$ automatically forces $u = v$ (since in this case $u = u^{cc} = v^{cc} = v$).

(2) Note that $(u, \alpha)^{cc} = (v, \beta)^{cc}$ is equivalent to $(u, \alpha)^c = (v, \beta)^c$. This follows because Sub($D$) is a dual frame, and $^c$ is the dual Heyting complement, so therefore $(u, \alpha)^{cc} = (u, \alpha)^c$.

10.6 The lattice $\Omega_{\mathbb{P}D}$ and the system $\mathbb{P}D$

Definition 12. We define $\Omega_{\mathbb{P}D} = \text{Sub}(D)/\equiv$.

Theorem 7. Let $D$ be a topological system. Then:

- $\equiv$ is an equivalence relation and respects $\wedge$ and $\vee$
- $(u, \alpha) \equiv (v, \beta)$ iff $(u, \alpha)^c \equiv (v, \beta)^c$

Proof. Clearly $\equiv$ is an equivalence relation.

Arbitrary meets. Suppose $(u_i, \alpha_i) \equiv (u_i, \beta_i)$ for $i \in I$. Then:

$$(u_i, \alpha_i)^{cc} = (u_i, \beta_i)^{cc}$$

$$(u_i, \alpha_i)^c = (u_i, \beta_i)^c$$

$$\bigvee_i (u_i, \alpha_i)^c = \bigvee_i (u_i, \beta_i)^c$$

$$\bigwedge_i (u_i, \alpha_i)^{cc} = \bigwedge_i (u_i, \beta_i)^{cc}$$

Arbitrary joins. Suppose $(u_i, \alpha_i) \equiv (u_i, \beta_i)$ for $i \in I$. Then by Lemma in Section 3, we have:

$$\left( \bigvee_i (u_i, \alpha_i) \right)^{cc} = \left( \bigvee_i (u_i, \alpha_i)^{cc} \right)^{cc}$$
= \left( \bigvee_{i} (u_i, \beta_i)^{cc} \right)^{cc}

= \left( \bigvee_{i} (u_i, \beta_i) \right)^{cc}.

The second item follows from Remark (2) above.

QED

10.7 Definition of \( \mathbb{P}D \)

Definition 13. We define \( \mathbb{P}D = (ptD, \Omega \mathbb{P}D, \models) \), where

\[ x \models [(u, \alpha)] \text{ if and only if } x \in u. \]

10.8 Examples of scge and swge

Example 14. Consider the topological system \( \{x, y, z\}, A, \models \) of Example 4. Then

\[ \text{scge}(\{x, y\}) = \bigwedge \{\alpha : \{x, y\} \subseteq \text{ext}(\alpha)\} \]

\[ = \top \land c^c = c^c. \]

Example 15. Consider the previous example, and modify it so that \( y \models a \) and \( y \not\models b \). Then

\[ \text{scge}(\{x\}) = \bigwedge \{\alpha : \{x\} \subseteq \text{ext}(\alpha)\} \]

\[ = \top \land a \land c^c \land b^c = b^c. \]

Observe that

\[ \text{ext(scge)}(\{x\}) = \{x, y, z\}. \]

Also, in this example,

\[ \text{swge}(\{x\}) = (\{x\}, \{x, y, z\}). \]
Chapter 11

An attempt at NNC Functions

11.1 NNC Functions: This attempt’s idea

The observation is that given a function between topological spaces, we can define a backwards powerset operator between the topologies of the discrete spaces based on the same underlying sets. Thus, given that I can define a cBa based on a topological system, and work that cBa to be the frame of a new system, it is natural to try to use a systems with frames $\Omega_f(D)$ and $\Omega_f(E)$ in trying to construct NNCF’s from system $D$ to system $E$. This will be seen to fail, however, when we look at Conjecture 1.

**Definition 14.** Let $(pt_f, \Omega) : (pt_E, \Omega_E) \to (pt_D, \Omega_D)$.

- We define $\Omega_f : \text{Sub}(\Omega_E) \to \text{Sub}(\Omega_D)$ as follows:

  \[ \Omega_f(\alpha) = \langle (\Omega_f(a), \Omega_f(b)) : (a, b) \in \alpha \rangle. \]

**Lemma** $\Omega_f$ preserves arbitrary meets and finite joins (including $\top$ and $\bot$).

**Proof**

\[
\begin{align*}
\Omega_f(\top_{\Omega E}) &= \langle (\Omega_f(a), \Omega_f(b)) : a = b \rangle \\
&= \top_{\Omega D}.
\end{align*}
\]

Notice $\top_{\Omega E} \equiv \top \bot_{\Omega E}$. Therefore:

\[
\begin{align*}
\Omega_f(\top_{\Omega E}) &\equiv_{\Omega_f(\top)} \Omega_f(\bot_{\Omega E})
\end{align*}
\]
Arbitrary meets.

The proof for meets is by structural induction, with one structural induction argument nested in another. First, note that simply by order-theoretics (and the easy fact that $\Omega f$ preserves order), we have:

$$\bigwedge_i \Omega f(\alpha_i) \geq \Omega f \left( \bigwedge_i \alpha_i \right).$$

To show the other direction, we recall ‘$\preceq$’ is ‘$\supseteq$’, and show that

$$a \equiv_{\Omega f(\bigwedge_i \alpha_i)} b$$ implies

$$a \equiv_{\bigwedge_i \Omega f(\alpha_i)} b.$$

**Induct.** Let

$$S = \{(a, b) \in \Omega D \times \Omega D : a = \Omega f(a'), b = \Omega f(b') \text{ for some } a', b' \text{ with } a' \equiv_{\bigwedge_i \alpha_i} b' \}.$$  

We take $\Phi_S(a, b)$ to be the statement:

$$a \equiv_{\bigwedge_i \Omega f(\alpha_i)} b.$$

For the **BASE CASE**, we show $\Phi_S(a, b)$ for all $(a, b) \in S$. This can be done by showing $\Phi_S(\Omega f(a'), \Omega f(b'))$ for all $(a', b')$ such that

$$a' \equiv_{\bigwedge_i \alpha_i} b'.$$

• **Sub-induction** As the ‘Induction Hypothesis’, we take $\Phi_R(a', b')$ to be the statement:

$$\Omega f(a') \equiv_{\bigwedge_i \Omega f(\alpha_i)} \Omega f(b').$$

Let $R = \{(a', b') : \exists j \in I \text{ such that } a' \equiv_{\alpha_j} b' \}.$

• **Sub-induction **BASE CASE: Now suppose $(a', b') \in R$. Then for some $j$, $a' \equiv_{\alpha_j} b'$. Thus:

$$\Omega f(a') \equiv_{\Omega f(\bigwedge_i \alpha_i)} \Omega f(b').$$
so that by \( \overline{\Omega f}(\alpha_j) \geq \bigwedge_i \overline{\Omega f}(\alpha_i) \),

\[ \Omega f(a') \equiv \bigwedge_i \overline{\Omega f}(\alpha_i) \Omega f(b'). \]

• Sub-induction INDUCTION STEP: Since \( \{(a', b') : \Phi_R(a', b')\} \) is, by definition, a frame congruence relation, the Induction Step is automatically taken care of.

• Thus, we have shown that:

\[ \langle R \rangle \subseteq \{(a', b') : \Phi_R(a', b')\}. \]

Notice \( \langle R \rangle = \bigwedge_i \alpha_i \), and \( \Phi_R(a', b') \) holds iff \( \Phi_S(\Omega f(a'), \Omega f(b')) \) does. Thus, the BASE CASE of the main induction argument holds.

Finally, for the induction step of the main argument, \( \{(a, b) : \Phi_S(a, b)\} \) is a frame congruence relation. Thus, \( \Phi_S(a, b) \) holds for every \((a, b) \in \langle S \rangle \). Observe

\[ \langle S \rangle = \langle (\Omega f \times \Omega f)^{-\tau}(\bigwedge_i \alpha_i) \rangle \]

\[ = \overline{\Omega f}(\bigwedge_i \alpha_i) \]

and

\[ \{(a, b) : \Phi_S(a, b)\} = \bigwedge_i \overline{\Omega f}(\alpha_i). \]

Thus, \( \bigwedge_i \overline{\Omega f}(\alpha_i) \leq \overline{\Omega f}(\bigwedge_i \alpha_i) \).

Combining this with the ‘trivial’ other direction yields the desired equality.

**Finite joins** Empty joins are covered above, and singleton joins are trivial. Given these, it suffices to prove that \( \overline{\Omega f} \) preserve binary joins, for then the case of three or more will follow by Mathematical Induction.

Suppose \( \alpha, \beta \in \text{Sub}(\Omega E) \). We must show:

\[ \overline{\Omega f}(\alpha \lor \beta) = \overline{\Omega f}(\alpha) \lor \overline{\Omega f}(\beta). \]

By order-theoretics:

\[ \overline{\Omega f}(\alpha \lor \beta) \geq \overline{\Omega f}(\alpha) \lor \overline{\Omega f}(\beta). \]

Thus, it suffice to show:

\[ \overline{\Omega f}(\alpha \lor \beta) \leq \overline{\Omega f}(\alpha) \lor \overline{\Omega f}(\beta). \]
This can be done by letting \((a, b) \in \overline{\Omega f}(\alpha) \lor \overline{\Omega f}(\beta)\) and showing

\[(a, b) \in \overline{\Omega f}(\alpha \lor \beta).\]

Note that it suffices to show the result for \(a \leq b\). To see this, suppose the implication holds for all \((c, d)\) with \(c \leq d\), and further suppose \((a, b) \in \overline{\Omega f}(\alpha) \lor \overline{\Omega f}(\beta)\). Since \(\overline{\Omega f}(\alpha) \lor \overline{\Omega f}(\beta)\) is a congruence relation, \((a \land b, a)\) and \((a \land b, b)\) are also in this set, and further, \(a \land b \leq a\) and \(a \land b \leq b\). Thus by the special case, \((a \land b, a)\) and \((a \land b, b)\) are in \(\overline{\Omega f}(\alpha \lor \beta)\).

Then the result follows for \((a, b)\) follows by symmetry and transitivity of \(\equiv_{\overline{\Omega f}(\alpha \lor \beta)}\).

So, suppose \((a, b) \in (\overline{\Omega f}(\alpha) \lor \overline{\Omega f}(\beta))\) with \(a \leq b\).

Then \((a, b) \in \overline{\Omega f}(\alpha)\) and \((a, b) \in \overline{\Omega f}(\beta)\). Therefore, there exist \(a', a'', b',\) and \(b'' \in \Omega E\) such that:

- \(\Omega f(a') = \Omega f(a'') = a\)
- \(\Omega f(b') = \Omega f(b'') = b\)
- \(a' \equiv_\alpha b'\) and
- \(a'' \equiv_\beta b''\)

Then:

\[
(a' \lor a'') \land (b' \land b'')
\]

\[
= (a' \land b' \land b'') \lor (a'' \land b' \land b'')
\]

\[
\equiv_\alpha (b' \land b'') \lor (a'' \land b' \land b'')
\]

\[
= b' \land b''.
\]

Similarly,

\[
(a' \lor a'') \land (b' \land b'')
\]

\[
\equiv_\beta b' \land b''.
\]

So

\[
(a' \lor a'') \land (b' \land b'') \equiv_{\alpha \lor \beta} b' \land b''.
\]

Applying \(\overline{\Omega f}\), we get:

\[
a = a \land b
\]
So \( a \equiv_{\Omega f(\alpha \lor \beta)} b \). Thus, \((a, b) \in \Omega f(\alpha \lor \beta)\).

QED

**Conjecture 1. FALSE** Let \( \Omega D \) and \( \Omega E \) be frames and \( \Omega f : \Omega E \to \Omega D \) be a frame homomorphism. Further, let \( \alpha \in \text{Sub}(\Omega D) \). Then:

\[ \Omega f(\alpha)_{\text{cc}} = \Omega f(\alpha_{\text{cc}})_{\text{cc}}. \]

**Proof.**

Let \( X = \mathbb{N} \) and \( A = \{ u \subseteq \mathbb{N} : u \text{ is finite} \} \cup \{ \emptyset \} \), and \( B = 2 \) (the two element locale), where \( A \) has the subset ordering. Since \( A \) is a topology on \( \mathbb{N} \), it is a locale when equipped with this ordering. Consider \( \Omega f : A \to B \) defined by

\[ \Omega f(u) = \begin{cases} \top & \text{if } u \neq \emptyset \\ \bot & \text{if } u = \emptyset \end{cases} \]

Let \( \alpha = \bigwedge\{u \in A : u \neq \bot\} \).

Observe that \( A \) may be paired with the point set \( X \cup \{ * \} \) where \( * \) is an object not in \( X \), by using the satisfaction relation:

\[ x \models u \iff x \in u \text{ for } x \in X \]

\[ * \models u \iff u \neq \emptyset. \]

Then the extent of \( \alpha \) in \( (X \cup \{ * \}, A, \models) \) is \( \{ * \} \) and in particular not empty, so \( \alpha \neq \bot \).

Notice that \( \text{ext}(\alpha) = \emptyset \), so \( \text{ext}(\alpha_{\text{cc}}) = X \). Set \( \gamma = \alpha_{\text{cc}} \). By definition, for all \( x \in \mathbb{N} \), if \( u \equiv_{\gamma} v \) then \( [x \in u \iff x \in v] \). Thus, for all \( u \) and \( v \), if \( u \equiv_{\gamma} v \) then \( u = v \). Therefore, \( \alpha_{\text{cc}} = \gamma = \top \), and \( \alpha_{\text{cc}} = \bot \).

Thus:

\[ \Omega f(\alpha)_{\text{cc}} = \Omega f(\bigwedge\{u \in A : u \neq \bot\})_{\text{cc}} = \]

\[ [\bigwedge\{\Omega f(u) : u \in A \text{ and } u \neq \bot\}]_{\text{cc}} = [\bigwedge_{u \neq \bot} (\top)]_{\text{cc}} = \top_{\text{cc}} = \top \]

\[ \Omega f(\alpha_{\text{cc}})_{\text{cc}} = \Omega f(\bot)_{\text{cc}} = \bot_{\text{cc}} = \bot. \]
Remark 5. Given \( f : D \to E \) continuous, we would like to define a map \( \Omega_f : \mathbb{P}D \to \mathbb{P}E \). However, the above result (falseness of the conjecture) shows that two subsystems may be equal in \( \mathbb{P}E \) but their images not equal in \( \mathbb{P}D \). It is worth noting that while the example deals with frame and frame homomorphisms, we can easily translate the result to the case of pointless systems and continuous functions of systems.

The possible definition of \( \Omega_f \):

- \( (\Omega_f)((u, \alpha)) = [(pt f^\leftarrow (u), \overline{\Omega f}(\alpha))] \)

11.1.1 Functoriality of Sublocale Construction

Theorem 8. The mapping

\[
(\Omega f : A \leftarrow B) \mapsto (\overline{\Omega f} : \text{Sub}(A) \leftarrow \text{Sub}(B))
\]

defines a functor \( \text{Sub} : \text{Loc} \to \text{Loc}^\partial \).

Proof. That this works on objects is clear. That this works on each hom-set is established above. Now we need functoriality, in particular, preservation of identities and composition.

Identities:

\[
\overline{\text{Id}}(\alpha) = (\langle \text{Id}(a), \text{Id}(b) : a \equiv_\alpha b \rangle) = \alpha
\]

Composition.

We must show:

\[
\overline{\Omega g}(\overline{\Omega f}(\alpha)) = (\overline{\Omega g \circ \Omega f})(\alpha)
\]

for all \( \alpha \in \text{Sub}(C) \), given \( A \xleftarrow{\Omega f} B \xleftarrow{\Omega g} C \) in \( \text{Frm} \).

What is to be shown may be rewritten:

\[
(\langle \Omega g(a'), \Omega g(b') \rangle : a' \equiv_{\Omega f(\alpha)} b') = (\langle \Omega g \Omega f(a), \Omega g \Omega f(b) : a \equiv_\alpha b \rangle).
\]

Now, if \( (c, d) \) is a generator in the RHS, we have \( c = \Omega g \Omega f(a) \) and \( d = \Omega g \Omega f(b) \) for some \( a \) and \( b \) so that \( a \equiv_\alpha b \). Then set \( a' = \Omega f(a) \) and \( b' = \Omega f(b) \). Clearly \( a' \equiv_{\Omega f(\alpha)} b' \), and we can take this \( a' \) and \( b' \) on the LHS, showing that the generators on the RHS are also generators on the LHS. Thus \( \text{RHS} \subseteq \text{LHS} \).

It remains to show the converse inclusion. Write RHS and LHS as names for the relations on the right and left sides, respectively. Work by structural induction.
Assume \( a'' \) LHS \( b'' \) for some \( a'', b'' \in A \). The only case to check is the Base Case, since the other cases are automatic as the RHS is a congruence relation. In the base case, \( a'' = \Omega g(a') \) and \( b'' = \Omega g(b') \) for some \( a', b' \in B \) satifying \( a' \equiv_{\Omega f} (\alpha) b' \).

Do a subinduction. We want to show:

- If \( a' \equiv_{\Omega f} \Omega f (a') \) b', then \( \Omega g(a') \equiv_{\Omega g \circ \Omega f (\alpha)} \Omega g(b') \).

Base case of subinduction. Then \( a \equiv_{\alpha} b \) where \( a' = \Omega f (a) \) and \( b' = \Omega f (b) \). Then \( \Omega g(a') = (\Omega g \circ \Omega f)(\alpha) \equiv_{\Omega g \circ \Omega f (\alpha)} (\Omega g \circ \Omega f (\alpha)) = \Omega g(b') \), and we are done.

The reflexive, symmetric and transitive cases follow because the relation of having \( (\Omega g \circ \Omega f)(\alpha) \)-equivalent images under \( \Omega g \) is easily seen to be an equivalence relation.

Finite meets. Suppose we have both of the following:

\[
\Omega g(a'_1) \equiv_{\Omega g \circ \Omega f (\alpha)} \Omega g(b'_1)
\]

\[
\Omega g(a'_2) \equiv_{\Omega g \circ \Omega f (\alpha)} \Omega g(b'_2).
\]

Then:

\[
\Omega g(a'_1 \land a'_2) = \Omega g(a'_1) \land \Omega g(a'_2)
\]

\[
\equiv_{\Omega g \circ \Omega f (\alpha)} \Omega g(b'_1 \land b'_2) = \Omega g(b'_1 \land b'_2).
\]

Arbitrary joins. This is similar to finite meets, only with an indexed join rather than a binary meet.

End of sub-induction. The reflexive, symmetric, transitive, finite meet, and join cases follow because the relation in question is by definition a congruence relation.

Thus

\[
\text{LHS} \subseteq \text{RHS}
\]

\[
\text{LHS} = \text{RHS}.
\]

QED

11.1.2 Functoriality of Subsystem Construction

Definition 15. Let \( f : D \to E \) in TopSys. Then define

\[
\Omega(f^{-}) : \text{Sub}(D) \leftarrow \text{Sub}(E)
\]
Lemma 13. The mapping defined has the indicated domain and codomain, and is a dual locale morphism.

Proof. The domain is obvious. To verify the codomain, we must verify that \((ptf)^\rightarrow (u)\) is compatible with \(\overline{\Omega f}(\alpha)\). Once this is verified, that it is a dual locale morphisms will follow since \((ptf)^\rightarrow\) and \(\overline{\Omega f}\) are dual locale morphisms.

It remains to verify that \((ptf)^\rightarrow (u)\) is compatible with \(\overline{\Omega f}(\alpha)\). To that end, suppose:

\[ x \in (ptf)^\rightarrow (u). \]

Further suppose \(a \equiv_{\overline{\Omega f}(\alpha)} b\). We must show:

\[ x \models a \iff x \models b. \]

Note that \(ptf(x) \in u\).

Do structural induction. Recall \(\overline{\Omega f}(\alpha) = (\langle \Omega f(d), \Omega f(e) \rangle : d \equiv_{\alpha} e)\).

BASE CASE. Then \(a = \Omega f(d)\) and \(b = \Omega f(e)\) for some \(d, e\) with \(d \equiv_{\alpha} e\). Since \(ptf(x) \in u\), we have

\[ ptf(x) \models d \iff ptf(x) \models e \]

and therefore

\[ x \models \Omega f(d) \iff x \models \Omega f(e) \]

\[ x \models a \iff x \models b. \]

Reflexive case. This is trivial.

Symmetric case. This is also straightforward.

Transitive case. If \(x \models a \iff x \models b\), and \(x \models b \iff x \models c\), then it follows that \(x \models a \iff x \models c\).

Finite meets. Suppose \(x \models a_i \iff x \models b_i\) for \(i = 1, 2\). Then TFAE:

\[ x \models a_1 \land a_2 \]

\[ x \models a_1\text{ and }x \models a_2 \]

\[ x \models b_1\text{ and }x \models b_2 \]
Arbitrary joins. Suppose $x \models a_i \iff x \models b_i$ for all $i \in I$. Then TFAE:

$$x \models \bigvee_i a_i,$$

$$\exists i \in I. x \models a_i,$$

$$\exists i \in I. x \models b_i,$$

$$x \models \bigvee_i b_i.$$

QED

**Theorem 9.** The mapping

$$(f : D \to E) \mapsto (\Omega(f^\rightarrow) : \text{Sub}(D) \leftarrow \text{Sub}(E))$$

defines a contravariant functor $\text{Sub} : \text{TopSys} \to \text{Loc}^\partial$.

**Proof.**

Preservation of identities and composition follows from the properties of $(ptf)^\leftarrow$ and $\Omega f$ and in particular, one should use the functoriality for sublocales established above.

QED

### 11.2 Another Attempt at Discontinuous Functions (Summary)

Another way to try to form the “powerset” is to use $\text{Sub}(D)$ as the powerset for a system $D$. This doesn’t work either, since there may be more locale morphisms from $\text{Sub}(X, S, \in)$ to $\text{Sub}(Y, T, \in)$ (where these are spaces) than locale morphisms from $\wp(X)$ to $\wp(Y)$. 
Chapter 12

Topology of Topological Systems

12.1 Open and Closed subsets

Definition 16. A subsystem \((u, \alpha) \in \text{Sub}(D)\) is open if \(\alpha = \varnothing\) for some \(a \in \Omega D\) and \(u = \text{ext}(\alpha)\).

A subsystem \((v, \beta)\) is closed if it is of the form \((u, \alpha)^c\) for some open \((u, \alpha) \in \text{Sub}(D)\).

Lemma. An arbitrary join of open subsystems is open. A finite meet of open subsystems is open.

Proof. The first statement follows from the fact that joins are computed componentwise, together with the join interchange law (for the topological system \(D\)). The second statement follows from the fact that meets are computed componentwise, that that finite meets agree in \(\Omega D\) and \(\text{Sub}(\Omega D)\), and the meet interchange law for \(D\).

Definition 17. If \(D\) is a topological system, and \((u, \alpha) \in \text{Sub}(D)\), then \((u, \alpha)\) is compact if for any collection of open subsets

\[
\{(v_i, \beta_i)\}_{i \in I} \subseteq \text{Sub}(D)
\]

satisfying

\[
(u, \alpha) \leq \bigvee_{i \in I} (v_i, \beta_i),
\]

there exists a finite \(J \subseteq I\) such that

\[
(u, \alpha) \leq \bigvee_{i \in J} (v_i, \beta_i).
\]

Definition 18. \(D\) is a topological system, then \(D\) is compact if the subsystem \((\text{pt}D, \top_{\Omega D})\) is compact as a subsystem of \(D\).

Lemma. A closed subset of a compact topological system is compact.
Proof. Let $D$ be compact and choose $(u, a^c)$ closed in $\text{Sub}(D)$. Let $\{(v_i, \beta_i)\}_{i \in I}$ be an open cover of $(u, a^c)$. Note that $(u^c, a)$ is open. Now,

$$\{(v_i, \beta_i)\}_{i \in I} \text{ together with } (u^c, a)$$

covers $(ptD, \top_{\Omega_D})$. Since $D$ is compact, we can choose a finite collection of these subsets that also covers $D$. In particular, there is a finite collection $\{(v_i, \beta_i)\}_{i \in J}$ with $J \subseteq I$ which together with $(u^c, a)$ covers $D$.

Thus,

$$\begin{align*}
(ptD, \top_D) & \leq \bigvee_{i \in J} (v_i, \beta_i) \lor (u^c, a) \\
(u, a^c) & \leq [\bigvee_{i \in J} (v_i, \beta_i) \lor (u^c, a)] \land (u, a^c) \\
(u, a^c) & \leq \bigvee_{i \in J} (v_i, \beta_i) \land (u, a^c) \leq \bigvee_{i \in J} (v_i, \beta_i).
\end{align*}$$

Since $J$ is finite, the result follows.

QED

Conjecture 2. Let $f : D \to E$ in $\text{TopSys}$. Then the forward image of a compact subset of $D$ is compact.

(Where the forward image operator is defined by the equation:

$$(\Omega(f^\to))^{-1}(u, \alpha) = \bigvee \{(v, \beta) : \Omega_f(v, \beta) \leq (u, \alpha) \}.$$ )

12.2 Spatiality

Definition 19. A topological system $D$ is called spatial if there exists a homeomorphism $f : D \cong (X, T, \in)$ where $(X, T)$ is a topological space.

Lemma 14. If $D$ is a topological system, then $D$ is spatial if and only if for all $a, b \in \Omega D$, if $a \neq b$, then $\text{ext}_D(a) \neq \text{ext}_D(b)$.

Lemma 15. If $D$ is a spatial topological system, then $\text{scge}(ptD) = \top$.

Proof. Suppose $\text{scge}(ptD) = \alpha$. Then for all $x \in ptD$,

$$a \equiv_{\alpha} b \Rightarrow [x \models a \leftrightarrow x \models b].$$

Therefore, if $a \equiv_{\alpha} b$, then $\text{ext}(a) = \text{ext}(b)$, and since $D$ is spatial, $a = b$. Thus $a \equiv_{\alpha} b$ iff $a = b$, so $\text{scge}(ptD) = \alpha = \top$. 

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Lemma 16. Let $A$ be a Boolean algebra. Suppose $a, b \in A$ and:

- $a^c \lor b = \top$
- $a \lor b^c = \top$

Then $a = b$.

Proof.

\[ a \land (a^c \lor b) = a \land \top \]
\[ (a \land a^c) \lor (a \land b) = a \]
\[ a \land b = a. \]

Similarly:
\[ b \land a = b. \]

Therefore:
\[ a = a \land b = b \land a = b. \]

QED

Theorem 10. Spatial System Theorem Let $D$ be a spatial topological system. Then the extent mapping $\text{ext} : \Omega_f D \to P(\text{pt} D)$ is an order-isomorphism.

Proof. Consider two subsets $[(u, \alpha)]$ and $[(u, \beta)]$ with the same points. Recall that it makes sense to talk about the points of a subset, since equivalent subsystems have the same points. I will show:

\[ (u, \alpha) \lor (u, \beta)^c = (\text{pt} D, \top), \]

and since we can switch the systems, the same proof yields:

\[ (u, \alpha)^c \lor (u, \beta) = (\text{pt} D, \top). \]
Taking equivalence classes, and applying the Lemma immediately above:

\[ [(u, \alpha)] = [(u, \beta)]. \]

**Proof of Claim.**

First, observe that \( u \subseteq \text{ext}(\alpha) \), so that

\[
\text{ext}(\alpha \lor \text{scge}(u^c)) = \text{pt}D
\]

\[
\text{ext}(\alpha \lor \beta^c \lor \text{scge}(u^c)) = \text{pt}D.
\]

Then:

\[
(u, \alpha) \lor (u, \beta)^c
\]

\[
= (u, \alpha) \lor (u^c, \beta^c \lor \text{scge}(u^c))
\]

\[
= (u \cup u^c, \alpha \lor \beta^c \lor \text{scge}(u^c))
\]

\[
= (\text{pt}D, \top).
\]

QED
Chapter 13

New Definitions and Open Directions

13.1 Relatively sober topological subsystems

Definition 20. Let \((X, A, \models)\) be a topological system. Let \((u, \alpha)\) be an element of \(\text{Sub}(X, A, \models)\). We define relatively sober by \((u, \alpha)\) is relatively sober in \((X, A, \models)\) iff given the points map \(pt : X \to \text{Pts}(A)\), the simultaneous restriction and corestriction:

\[
pt|_\text{Pts}(A)|_\alpha : u \to \text{Pts}(A)|_\alpha
\]

is one-to-one and onto.

We may ask the following types of questions: (1) When are relatively sober subsystems sober? (2) How is the best way to get a sober (resp. relatively sober) subsystem if we specify what points we want? (3) How is the best way to get a sober (resp. relatively sober) subsystem if we specify what we want the sublocale to be?

13.2 Miscellaneous fact and open questions

Consider the following easy lemma, which is included for the purpose of asking open questions.
Lemma 17. Let $\gamma \in \text{Sub}(A)$ have a Boolean complement in $\text{Sub}(A)$, i.e., $\gamma = \gamma^c$. Then:

$$\gamma \land \bigvee_i \alpha_i = \bigvee_i (\gamma \land \alpha_i)$$

for any collection $\{\alpha_i\}_{i \in I} \subseteq \text{Sub}(A)$.

Proof The $\geq$ direction is clear.

For the $\leq$ direction, suppose

$$\delta \geq \bigvee_i (\gamma \land \alpha_i).$$

Then:

$$\forall i \, \delta \geq \gamma \land \alpha_i$$

$$\forall i \, \delta \lor \gamma^c \geq (\gamma \land \alpha_i) \lor \gamma^c = \alpha_i \lor \gamma^c \geq \alpha_i$$

$$\delta \lor \gamma^c \geq \bigvee \alpha_i$$

$$(\delta \lor \gamma^c) \land \gamma \geq \gamma \land \bigvee \alpha_i$$

$$\delta \geq \delta \land \gamma \geq \gamma \land \bigvee \alpha_i.$$  

Taking $\delta = \bigvee_i (\gamma \land \alpha_i)$ yields

$$\bigvee_i (\gamma \land \alpha_i) \geq \gamma \land \bigvee \alpha_i.$$  

The equality follows by combining the two inequalities. QED

Remark If $\gamma$ doesn’t have a Boolean complement, this result need not hold.

Proof

Let $\mathbb{N}$ be the natural numbers. Define a topological system $D$ as follows:

$$\text{pt}D = \mathbb{N}$$

$$\Omega D = \{ u \subseteq \mathbb{N} : u \text{ is cofinite and } 0 \in u \} \cup \{ \emptyset \}$$

$$\models_D \text{ is membership.}$$

Since $\Omega D$ is a topology on $\text{pt}D$, this does define a topological system.

Define

$$\gamma = \bigwedge \{ u \in \Omega D : 0 \models u \}.$$
Clearly, $\gamma \neq \bot$, since $0 \models \gamma$.

However,

$$\gamma \wedge u^c = \bot$$

for any $u \neq \bot$.

On the other hand, recall from a previous example that:

$$0 \models \bigvee\{u^c : u \neq \bot\}$$

so

$$0 \models \gamma \wedge \bigvee\{u^c : u \neq \bot\} \neq \bot$$

and

$$\gamma \wedge \bigvee\{u^c : u \neq \bot\} \neq \bigvee\{\gamma \wedge u^c : u \neq \bot\}.$$

QED

**Remark** Possibly open questions If $D = (\text{pt} D, \Omega D, \models)$ is topological system, need the following hold in (1) Sub($\Omega D$), and (2) Sub($D$):

- $\alpha^c \wedge \alpha^{cc} = \bot$?

### 13.3 Possible Research Directions

In this section, I list several possible research directions in which to potentially take research on topological systems.

#### 13.3.1 Quotient Systems

Completing the analogy with spaces and locales, subsystems turn out to correspond to regular epimorphisms (see Appendix). There is also a notion of quotient space, and presumably this could be generalized to “quotient locales,” which we expect to be subframes. Thus, one could examine monomorphisms in TopSys and see if they give a suitable notion of *quotient of a topological system*.

#### 13.3.2 Monomorphisms, Epimorphism, etc.

Another question that could be examined along the lines of the previous question is to see how monomorphisms, epimorphisms, regular monomorphisms, regular epimorphisms, extremal monomorphisms, and ex-
trenal epimorphisms are treated by the standard embeddings of \( \mathsf{Top} \) and \( \mathsf{Loc} \) into \( \mathsf{TopSys} \), and by the adjoints to these functors. For example, are these types of morphisms preserved or reflected? A good starting place here would be to look up the behavior of the adjoint pair between \( \mathsf{Top} \) and \( \mathsf{Loc} \) w.r.t. various types of morphisms.

### 13.3.3 Topology of Topological Systems

Follow the ideas in Chapter 10, we can develop of theory of points, open subsystems, closed subsystems, compact subsystems, etc. and compare and contrast it to the corresponding results for spaces and locales. This could also be specialized to the cases of spatial and sober topological systems.

### 13.3.4 Computer Science Applications

Right now it is clear that subsystems of topological systems are “related” to computer science in the sense that they have connections to areas that have applications to computer science. It would be interesting to look for direct applications.

### 13.3.5 Generalizations or Variations of Topological Systems

First, we can look at, say Chu systems, or the “measurable systems” or the “sup-topological systems” mused upon by Denniston, Melton, and Rodabaugh. That is, we can vary the structure of the set of “opens” and correspondingly replace the join and meet interchange laws by laws appropriate to the algebraic structure used.

Second, we can do this in a more systematic way, such as has been done by S. Solovyyov, and prove theorems about whole classes of categories formed with various types of co-algebras. For work by S. Solovyyov on this topic, see [Sol10, Sol11a, Sol11b, Sol11c, Sol11d, Sol12a, Sol12b, Sol13, Sol15b, Sol15a].

### 13.3.6 Discontinuous Functions Problem

Finally, we can look at the original motivation of this paper. Towards the end of solving this, we can examine each proposed solution, see what is “good” about it – that is, examine the motivation for trying that approach and what can be proven; and for each counterexample we can further examine to see why it fails and see what this says about modifying our approach.

A second way this question could be “solved” would be to use category-theoretic considerations to prove the diagram involving \( \mathsf{Top} \), \( \mathsf{Top}_{fun} \) and \( \mathsf{TopSys} \) cannot be completed and still have certain reasonable criteria satisfied.
13.4 Concrete Research Plans

Eventually, I would like to write a book about topological systems. This should be a mathematically rigorous development and motivation of the subject. There should be a discussion of and clear motivation for the discontinuous function problem, along with detailed examples of how different approaches fail. Such a book should give a clear account of the loose ends that are part of the current state of the art.

Additionally, it would be useful to develop an extensive categorical understanding of the category \textbf{TopSys}.

Also, there should be a discussion of various types of morphisms and limits and colimits, and a discussion of which of these are preserved, reflected, detected, etc. under well-known functors introduced in [Vic96].

Further, I would like to develop the topology of topological systems. This is a totally undeveloped area with connections to both general topology and locale theory.

Finally, it would be interesting to explore homotopy theory of topological systems. Doing this would nicely allow me to study another interest of mine, Homotopy Type Theory, or HoTT.
Appendix

This appendix written by the author was prepared as a stand-alone document and is included here. It contains a proof that subsystems of topological systems correspond to regular monomorphisms.

**Proposition 1.** Let \( \Omega f^{\text{op}} : A \to B \) in \( \text{Loc} \). TFAE:

1. \( \Omega f : B \to A \) is surjective as a function
2. \( \Omega f \) is a regular epimorphism in \( \text{Frm} \)
3. \( \Omega f^{\text{op}} \) is a regular monomorphism in \( \text{Loc} \)

**Proof.** See [AHS04] or [Joh86]. Also, cf. Exercise 6.2 of [Vic96].

QED

**Proposition 2.** Let \( (f, \Omega f) : (X, A, \models) \to (Y, B, \models) \) in \( \text{TopSys} \). The if \( f \) is injective and \( \Omega f \) is surjective, then \( (f, \Omega f) \) is a regular monomorphism in \( \text{TopSys} \).

**Proof.**

We proceed by constructing two morphisms in \( \text{TopSys} \) which are equalized by \( (f, \Omega f) \).

First, observe that by the Proposition above, \( \Omega f^{\text{op}} \) is a regular monomorphism. Let \( C \in \text{Loc} \) and \( \Omega g^{\text{op}}, \Omega h^{\text{op}} : B \to C \) be such that \( \Omega f^{\text{op}} \) is an equalizer of \( \Omega g^{\text{op}} \) and \( \Omega h^{\text{op}} \).

Now define \( Z = (\{0,1\} \times \text{Pts}(C)) \cup X \) (we can assume without loss of generality that this union is disjoint), and define a satisfaction relation on \( (Z, C) \) by:

\[ (i, p) \models_{Z,C} b \iff p \models b \] (for each of \( i = 0,1 \)), and

\[ x \models_{Z,C} b \iff f(x) \models \Omega g(b). \]

I make the following claims:

1. \( (Z, C, \models) \) is a topological system.
2. For \( x \in X \), \( f(x) \models \Omega g(b) \) iff \( f(x) \models \Omega h(b) \).

(Item 1): That satisfaction interchanges with joins and finite meets for points of the form \((i,p)\) follows from the definition of \(\models\) for such points. That satisfaction interchanges with joins and finite meets for points of the form \( x \in X \) follows by rewriting \( x \models b_j \), where \( j \in J \) for some indexing set \( J \), as \( f(x) \models_{Y,B} \Omega g(b_j) \), \( j \in J \), and applying the facts that \( \Omega g \) is a frame homomorphism and \((Y,B,\models)\) satisfies the interchange laws.

(Item 2): We have the following chain of equivalences:

\[
f(x) \models \Omega g(b) \quad \text{IFF} \quad x \models_{X,A} \Omega f(\Omega g(b)) \quad \text{IFF} \quad x \models_{X,A} \Omega f(\Omega h(b)) \quad \text{IFF} \quad f(x) \models \Omega h(b).
\]

Define functions \( g, h : Y \to Z \) as follows. First, for \( y \in Y \setminus f^{-1}(X) \), define points \( p_g(y) \) and \( p_h(y) \) of \( C \) by:

\[
p_g(y)(b) = \begin{cases} \top & \text{if } y \models \Omega g(b) \\ \bot & \text{if } y \not\models \Omega g(b) \end{cases}
\]

\[
p_h(y)(b) = \begin{cases} \top & \text{if } y \models \Omega h(b) \\ \bot & \text{if } y \not\models \Omega h(b) \end{cases}.
\]

Then define

\[
g(y) = \begin{cases} f^{-1}(y) & \text{if } y \in f^{-1}(X) \\ (0, p_g(y)) & \text{otherwise} \end{cases}
\]

\[
h(y) = \begin{cases} f^{-1}(y) & \text{if } y \in f^{-1}(X) \\ (1, p_h(y)) & \text{otherwise} \end{cases}.
\]

I claim the following:

1. \((g, \Omega g)\) and \((h, \Omega h)\) are continuous functions.

2. \(g \circ f = h \circ f\).

3. \((g, \Omega g) \circ (f, \Omega f) = (h, \Omega h) \circ (f, \Omega f)\).
(Items 2 and 3): Item 2 can be verified by noting that if \( x \in X \), then \( f(x) \in f^{-1}(X) \) so \( g(f(x)) = h(f(x)) \) (comparing the definitions of \( g \) and \( h \)). Item 3 follows from Item 2 and the fact that \( \Omega f^{\text{op}} \) equalizes \( \Omega g^{\text{op}} \) and \( \Omega h^{\text{op}} \) (once it is established that the pairs in question are morphisms).

(Item 1): The cases of \( g \) and \( h \) are similar, so we will only treat \( (g, \Omega g^{\text{op}}) \). Suppose \( y \in Y \) and \( b \in C \). First consider the case \( y \in f^{-1}(X) \). Then we have the following chain of equivalences:

\[
g(y) \models_{Z,C} b \iff \neg \exists Z,C b \iff f^{-1}(y) \models_{Z,C} b \iff f(f^{-1}(y)) \models_{Y,B} \Omega g(b) \iff y \models_{Y,B} \Omega g(b).
\]

Now consider the case \( y \notin f^{-1}(X) \). Then we have:

\[
g(y) \models_{Z,C} b \iff \neg \exists Z,C b \iff \neg \exists (0, p_g(y)) \models_{Z,C} b \iff \neg \exists p_g(y) \models_{C,b} \iff \neg \exists y \models_{Y,B} \Omega g(b).
\]

Thus \( (g, \Omega g) \) is a continuous function, and by an analogous argument (using the second item following the definition of \( \models_{Z,C} \)) one shows \( (h, \Omega h) \) is a continuous function.

**Claim.** The morphism \( (f, \Omega f) \) is an equalizer of \( (g, \Omega g) \) and \( (h, \Omega h) \) in \textbf{TopSys}.

**Proof of claim.** First, note (Item 3) immediately above, which says

\[
(g, \Omega g) \circ (f, \Omega f) = (h, \Omega h) \circ (f, \Omega f).
\]

Now, let

\[
(f', \Omega f') : (X', A', \models) \to (Y, B, \models)
\]
in TopSys be any continuous function satisfying

\[(g, \Omega g) \circ (f', \Omega f') = (h, \Omega h) \circ (f', \Omega f').\]

Then we have

\[\Omega g^{\text{op}} \circ \Omega f'^{\text{op}} = \Omega h^{\text{op}} \circ \Omega f'^{\text{op}},\]

so there is a unique \(\Omega f'^{\text{op}} : A' \to A\) so that

\[\Omega f^{\text{op}} \circ \Omega f'^{\text{op}} = \Omega f'^{\text{op}},\]

by the fact that \(\Omega f^{\text{op}}\) equalizes \(\Omega g^{\text{op}}\) and \(\Omega h^{\text{op}}\).

Now we need a function \(\overline{f} : X' \to X\) to go with \(\Omega f'^{\text{op}}\). Now, \(\overline{f}\) must satisfy \(f \circ \overline{f} = f'\). Thus, given \(x' \in X'\), \(f\) must take \(\overline{f}(x')\) to \(f'(x')\). Note that since \(g(f'(x')) = h(f'(x'))\), \(f'(x) \in f^{-1}(X)\), so this can in fact be accomplished, by setting

\[\overline{f}(x') = f^{-1}(f'(x')),\]

and this is the unique assignment that works.

Finally, everything will work once we show \((\overline{f}, \Omega f)\) is a continuous function. Let \(x' \in X'\), and suppose \(a \in A\). By hypothesis of the proposition, \(\Omega f\) is surjective, so \(a = \Omega f(b)\) for some \(b \in B\). Then:

\[\overline{f}(z) \models a \iff \overline{f}(z) \models \Omega f(b) \iff f(\overline{f}(z)) \models b \iff f'(z) \models b \iff z \models \Omega f'(b) \iff z \models \Omega \overline{f}(\Omega f(b)) \iff z \models \Omega \overline{f}(a).\]
This shows \((\overline{f}, \Omega \overline{f})\) is a continuous function, and the above work shows that
\[
(f, \Omega f^{\text{op}}) \circ (\overline{f}, \Omega \overline{f})^{\text{op}} = (f', \Omega f'^{\text{op}}),
\]
and the unique function so this is true. Thus the claim holds.

QED

Proposition 3. Let \((f, \Omega f) : (X, A, \models) \rightarrow (Y, B, \models)\) in \(\text{TopSys}\). Then if \((f, \Omega f)\) is a regular monomorphism in \(\text{TopSys}\), then \(f\) is injective and \(\Omega f\) is surjective.

Proof. We prove surjectivity of \(\Omega f\) first, since it is useful in proving \(f\) injective. Suppose \((f, \Omega f)\) is as given in the statement of the proposition. Then there exists a topological system \((Z, C, \models)\) and continuous functions \((g, \Omega g), (h, \Omega h) : (Y, B, \models) \rightarrow (Z, C, \models)\) in \(\text{TopSys}\) such that \((f, \Omega f)\) equalizes \((g, \Omega g)\) and \((h, \Omega h)\).

I claim \(\Omega f^{\text{op}}\) equalizes \(\Omega g^{\text{op}}\) and \(\Omega h^{\text{op}}\). Suppose \(A'\) is a locale and \(\Omega f'^{\text{op}} : A' \rightarrow B\) is a continuous function in \(\text{Loc}\). Consider the topological system \((\emptyset, A', \models)\), where \(\models\) is the empty relation. If \(f'\) is the unique function from \(\emptyset\) to \(Y\), then \((f', \Omega f')\) is a continuous function in \(\text{TopSys}\). Also, clearly, \(g \circ f' = h \circ f'\), so that:
\[
(g, \Omega g) \circ (f', \Omega f') = (h, \Omega h) \circ (f', \Omega f').
\]
Therefore, there is a unique continuous function \((\overline{f}, \Omega \overline{f})\) so that
\[
(f', \Omega f') = (f, \Omega f) \circ (\overline{f}, \Omega \overline{f}).
\]
Since \(\overline{f} : \emptyset \rightarrow X\), \(\overline{f}\) is the “empty function from \(\emptyset\) to \(X\)”, and \((\overline{f}, \Omega e)\) is a continuous function for any continuous function \(\Omega e^{\text{op}}\) from \(A'\) to \(A\) in \(\text{Loc}\), it follows that \(\Omega f^{\text{op}}\) is the unique solution for \(\Omega e^{\text{op}}\) in
\[
\Omega f'^{\text{op}} = \Omega f^{\text{op}} \circ \Omega e^{\text{op}}.
\]
Thus \(\Omega f^{\text{op}}\) satisfies the definition of the equalizer of \(\Omega g^{\text{op}}\) and \(\Omega h^{\text{op}}\).

Thus \(\Omega f^{\text{op}}\) is a regular monomorphism, and \(\Omega f\) is surjective.

Now we show injectivity of \(f\). We have assumed that \((f, \Omega f)\) is a regular monomorphism; it therefor follow that \((f, \Omega f)\) is a monomorphism and left-cancelable. Suppose towards a contradiction that \(f\) is not injective. Then choose \(x_0 \neq x_1\) so that \(f(x_0) = f(x_1)\). I claim \(x_0\) and \(x_1\) satisfy the same elements of \(A\). Suppose
Since $\Omega f$ is surjective, $a = \Omega f(b)$ for some $b$. Then

\[
x_0 \models a \iff x_0 \models \Omega f(b) \iff f(x_0) \models b \iff f(x_1) \models b \iff x_1 \models \Omega f(b) \iff x_1 \models a.
\]

Define a system $\langle \{q_0\}, A, \models \rangle$ where $q_0 \models a \iff x_0 \models a$. Let $g, h : \{q_0\} \to X$ by $g(q_0) = x_0$ and $h(q_0) = x_1$. Define $\Omega g = \Omega h = \Omega f$. Then clearly $(g, \Omega g^\text{op})$ and $(h, \Omega h^\text{op})$ are continuous functions, also, clearly,

\[
(f, \Omega f) \circ (g, \Omega g) = (f, \Omega f) \circ (h, \Omega h),
\]

but

\[
(g, \Omega g) \neq (h, \Omega h),
\]

a contradiction.

Therefore, $f$ is injective.

This completes the proof.

QED
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Publications of the Author


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