INVESTIGATION OF FRACTION SCHEMES AND MODELS
AS A MEANS TO UNDERSTAND HOW SIXTH GRADE STUDENTS
MAKE SENSE OF FRACTIONS

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This study aimed to develop models of students’ ways of interacting with and understandings of fraction models. The purpose of the study was to understand how students interact with items that require them to employ their fraction schemes involving area, length, and set models. The study attempted to address how using fraction schemes as a theoretical framework (Steffe & Olive, 2010) across different fraction models helps identify differences in students’ ways of thinking in relation to these models.

The current study differed from prior studies in that it simultaneously attended to four fraction schemes (i.e., partitioning, part-whole, partitive unit fraction, and partitive fraction schemes) and their representations with area, length, and set models (i.e., continuous and discrete models). In addition, the study investigated potential interactions between fraction schemes and models. Results from this study inform instructional practices that allow for variation in model types and ways of operating described in fraction schemes. Furthermore, detailed models of students’ fraction understandings presented in the findings would be valuable for practitioners when making instructional decisions.

A convergent parallel mixed methods design was employed in that both quantitative and qualitative analyses were conducted to interpret the students’ understanding of fractions. The Friedman test statistics and interpretive qualitative
approach allowed the examination of different schemes, different models types, and particular schemes across types of models. Data were gathered through one-on-one clinical interviews with 18 sixth grade students at a suburban public middle school in the Midwestern United States.
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CHAPTER I
INTRODUCTION

Rationale for the Study

When it comes to mathematics, it is common knowledge that even adults may feel challenged to make sense of some concepts, procedures, or relationships within. It may be, however, due to the cumulative nature upon which mathematical ideas are built. I was trained to be a mathematics teacher for 5th to 8th grades and have taught mathematics in Turkey and the Czech Republic for two years. As part of my teacher preparation program and employment, I was fortunate enough to be introduced to what is called “reform mathematics” in the U.S. mathematics literature. Reform mathematics calls for greater attention to student-centered and conceptually-oriented instructional methods. Direct experience with my own students’ struggles and my professional interest as a mathematics educator have focused my research interest on fractions. Students’ continued difficulties with learning fractions (National Mathematics Advisory Panel, 2008) could indicate that traditional teaching of fractions is still pervasive in U.S. classrooms. My interest in students’ conceptions of fractions also stems from thinking about what teachers expect students to learn from fraction models and in reality how students make sense of the concept. The purpose of this study was to unpack the complex nature that fraction models embody and to inquire into how students operate when they are provided with certain fraction models.

I begin with a clarification of conventional fraction models and traditional teaching of fractions. I define models as concrete or pictorial representations of abstract
mathematical concepts such as fractions. I define traditional teaching of fractions as introducing representations through fraction notations (e.g., \( \frac{a}{b}, \frac{a}{b'}, a:b \)); the meaning of fractions as confined to part-whole; and, fraction models limited to ubiquitous area models such as tape diagrams and circular models. I believe multiple types of models are essential for building students’ conceptual understanding of fractions; therefore, procedures should be connected to mathematical concepts, meaning (e.g., fraction subconstructs and schemes), and students’ thinking. Using models in teaching fractions have long been touted as a best practice (e.g., Cramer, Behr, Post, & Lesh, 2009; Cramer & Henry, 2002; Petit, Laird, & Marsden, 2010; Siebert & Gaskin, 2006); however, how students understand these models has not been adequately investigated. Furthermore, even less attention has focused specifically on fraction models and how students’ misconceptions of fractions might emerge from using these models.

**Purpose of the Study**

The study aimed to develop models of students’ ways of interacting with and understandings of fraction models. The purpose of the study was to understand how students interact with items that require them to employ their fraction schemes involving area, length, and set models. The study attempted to address how using fraction schemes as a theoretical framework (Steffe & Olive, 2010) across different fraction models helps identify differences in students’ ways of thinking in relation to these models.

The study yields results that support teachers in designing and implementing instruction productive to students’ developing fraction understandings. Furthermore, detailed models of students’ fraction understandings should be valuable for practitioners
when making instructional decisions. Previous seminal studies on models and students’ fraction schemes have attended to various aspects of students’ cognitions, such as: students’ fraction learning trajectories and development of fraction schemes, hierarchy of fraction schemes and operations, the analysis of schemes and their associated operations, efficiency of initial emphasis on different subconstructs of fractions, and fraction knowledge represented with continuous models (Busi et al., 2015; Lamon, 2001; Norton & Wilkins, 2009, 2010; Olive & Steffe, 2002; Steffe, 2002, 2004; Steffe & Olive, 2010; Tunc-Pekkan, 2015). The current study differs from these prior studies in that it simultaneously attended to four fraction schemes (i.e., partitioning, part-whole, partitive unit fraction, and partitive fraction schemes) and their representations with area, length, and set models (i.e., continuous and discrete models). In addition, the study investigated potential interactions between fraction schemes and fraction models.

**Research Questions**

The study was designed to address whether students’ actions are compatible with fraction schemes. Furthermore, the study examined students’ fraction schemes as they interacted with different model types and was guided by the following three research questions:

1. How do sixth grade students’ responses to items that correspond to the development of a particular scheme compare and contrast?

2. How do sixth grade students’ responses to items involving a particular model type compare and contrast?

3. Do certain types of models facilitate the development of certain schemes?
CHAPTER II
REVIEW OF THE LITERATURE

A comprehensive review of relevant literature provided considerable insight into students’ struggles to make sense of fractions. Among the factors that make rational numbers, and therefore fractions, difficult to understand are their many representations and interpretations (Kilpatrick, Swafford, & Findell, 2001). Wu (1999) claimed, “The teaching of fractions is the first major bottleneck in school mathematics, and yet it has remained problematic” (p. 22). The difficulties encountered by students involving fractions are such that some scholars suggest deemphasizing teaching of fractions in the middle school curriculum (e.g., Groff, 1994, 1996). In addition, Groff (1996) implied that incidental teaching of fractions, done whenever the use for fractions arises in the school day, would suffice. Moreover, there is a strong positive relationship between students’ understanding of fractions and their overall success in mathematics (Lortie-Forgues, Tian, & Siegler, 2015; McNamara & Shaughnessy, 2010; Siegler et al., 2012).

Relational understanding of fractions is crucial in terms of conceptualizing topics such as ratio, proportional relationship, decimals, percentages, probability and algebra. For example, relational understanding of fraction division calls for knowing not only how the reciprocal multiplication algorithm (i.e., “invert and multiply” algorithm) is applied but also why it works. Therefore, the teaching and learning of fractions merit researchers’ attention. On the other hand, traditional teaching of the fraction concept
does not seem to lead to a robust understanding of fractions (Lamon, 2001; Moss & Case, 1999; Watanabe, 2002).

Prior research also contributed to this study’s design and theoretical framework by identifying specific misconceptions and challenges students may possess or encounter. However, students’ misconceptions as identified in the literature are dominantly fragmented. In other words, these misconceptions are introduced in a very limited context and scale. For example, the reason why students add numerators and denominators of fractions is claimed to be due to overgeneralization of whole numbers to fraction addition (Mack, 1995). However, as researchers “we are in the business of embedding, of putting things in relation to the circumstances, both immediate and broad, of which they are a part” (Schram, 2006, p. 10). Although literature can inform discussions on effective ways to teach fractions, further research should explore students’ interpretations of fractions through models in more comprehensive ways and support the development of conceptual understandings of fractions.

**Fraction Models**

In teaching fractions, three different kinds of models—area, length, and set—are typically used to represent fractions. However, when a single type of model is used dominantly, it falls short in terms of attending to students’ cognitive levels and to idiosyncratic fraction meanings. Moreover, teachers need to be mindful that models and “representations do not embody mathematical meanings independently” (Watanabe, 2002, p. 463). Rather, students conceive of models in the way that make sense to them and sometimes the meaning they attribute to these models may not be mathematically
correct. Furthermore, physical objects (e.g., models) do not impose mathematical relationship on students (Suh, 2007; Thompson, 1994). In other words, as characterized by Glasersfeld (1996),

The physical signals that travel from one communicator to another—for instance the sounds of speech and the visual patterns of print or writing in linguistic communication—do not actually carry on or contain what we think of as meaning. (Emphasis in original, p. 11)

Therefore, following Glasersfeld (1996) and my own experiences, I argue that the meanings attributed to fraction models are interpreted “within the subject’s range of experience” (p. 6). Teachers and researchers should draw on students’ ways of interacting with and interpreting models to develop and extend their students’ thinking. Similarly, according to Maher and Davis (1990), “For instruction, adults choose certain representations because they can see in them an embodiment of something they already understand—the representations are media for what the teacher already knows” (p. 155).

Following Lamon (2001), it could still be claimed that “both presentational models (used by adults in instruction) and representational models (produced by students in learning) play significant roles in instruction and its outcomes” (emphasis in original, p. 146). In Table 1, model types are compared in a way that displays what constitutes the whole, parts, and the fraction within the model.
Table 1

Model Types for Fractions

<table>
<thead>
<tr>
<th>Model</th>
<th>What Defines the Whole</th>
<th>What Defines the Parts</th>
<th>What the Fraction Means</th>
</tr>
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<tbody>
<tr>
<td>Area</td>
<td>The area of the defined region</td>
<td>Equal area</td>
<td>The part of the area covered, as it relates to the whole or the unit</td>
</tr>
<tr>
<td>Length</td>
<td>The unit of length</td>
<td>Equal length</td>
<td>The length of the part in relation to the length of the whole</td>
</tr>
<tr>
<td>Set</td>
<td>Collection of objects as a whole</td>
<td>Equal number of objects</td>
<td>The number of objects in the subset, as it relates to the defined whole</td>
</tr>
</tbody>
</table>


Fraction Subconstructs

Five interpretations of fractions introduced by Kieren (1976) inform the conceptual framework of this study in regards to the meanings fractions might represent to students. To be considered proficient in understanding the fraction concept and to have profound rational number knowledge requires understanding five interpretations of fractions in terms of whole-part comparison, measure [measurement], quotient [division], operator, and ratio and rate (Kieren, 1976).

Fraction as Part-Whole

Part-whole interpretation of the fraction concept is basically about partitioning the whole. For example, when a whole is divided into five equal parts, each part is one-fifth. It is mostly followed by activities such as shading parts of the whole. This construct is
seen as the most common interpretation emphasized in instruction (Van de Walle, Karp, & Bay-Williams, 2013). Clarke (2011) also stated, “Part-whole comparison dominates the way in which fractions are presently taught” (p. 37). Furthermore, Lamon (2001) argued that, “mathematically and psychologically, the part-whole interpretation of fraction is not sufficient as a foundation for the system of rational numbers” (p. 150). Finally, the part-whole interpretation is particularly applicable to fractions, decimals, percentages, and ratio.

**Fraction as Measure**

The measurement subconstruct is the basis of iterating fractional parts. That is, two-fifths is constructed by iterating two one-fifths. This construct underscores how much (numerator) rather than how many parts (denominator), which is the case in part-whole interpretations (Van de Walle et al., 2013).

**Fraction as Quotient**

Examples such as “How much of a pizza does each child get when 2 pizzas are shared equally among 5 children?” can be given for the quotient subconstruct. Kilpatrick et al. (2001) argued that, “in some ways, [equal] sharing can play the role for rational numbers that counting does for whole numbers” (p. 232). Clarke (2011) claimed that the notion of “fraction as division” is *the forgotten notion* in teaching of fractions.

**Fraction as Operator**

The fraction can also be an operator, as in 2/5 of 10 as thought of as “2/5 × 10.” The common misconception associated with this construct is that the number multiplied
by the fraction operator “always makes bigger” and division “always makes smaller” (Clarke, Roche, & Mitchell, 2008).

**Fraction as Ratio-Rate**

The ratio of number of girls to the number of students in a class is another context of which fractions are a part. Post, Cramer, Behr, Lesh, and Harel (1993) stated that “ratio, measure, and operator constructs are not given nearly enough emphasis in the school curriculum” (p. 328). The Common Core State Standards for Mathematics argues that students need to understand and use all these meanings for fractions (National Governors Association [NGA] Center & CCSSO, 2010).

The current study attempted to reveal the complexity of the fraction concept and supplemental cognitive schemes students are required to develop to understand fractions. Therefore, it was expected that students experiencing different developmental stages will struggle with understanding fractions since the nature of the fraction constructs are developmental. As Lamon (2001) described, “Each of these [subconstructs], in turn, has its own set of representations and operations, models that capture some—but not all—of the characteristics of the field of rational numbers” (p. 150). Furthermore, Lamon asserted, “Basing instruction on a single interpretation and selectively introducing only some of its representations in instruction can leave the student with an inadequate foundation to support her or his understanding of the field of rational numbers” (p. 150).

**Fraction Operations**

In this section, several crucial operations are outlined that relate to the four particular fraction schemes that I employed to explain students’ actions as they responded
to fraction items in one-to-one interviews (see Appendix A). Operations constitute a crucial role for schemes, since schemes are in fact ways of operating (Norton & McCloskey, 2008). Glasersfeld (1995) described operations as mental actions abstracted from previous experience to become available for acting on various situations.

The Unitizing Operation

The unitizing operation produces a whole, or a unit by establishing it as a separate entity (Olive, 1999; Steffe, 2002; Steffe & Olive, 2010). Unitizing can be described as treating an entity as a unit or uniting a collection of objects to be acted on as a whole (Norton & McCloskey, 2008). In the latter case, the whole would be a composite unit. Three candy bars taken as a whole could be an example of a composite unit. Depending on the models used to represent a fraction, the unit can be discrete or continuous.

The Partitioning Operation

The partitioning operation produces equal parts within the unit. Partitioning can be identified as students marking the unit to generate equal parts. Equal sharing tasks (e.g., sharing a candy bar among three people) are an example in which partitioning is employed as an operation.

The Disembedding Operation

As an operation, disembedding involves envisioning a fractional part of a whole without altering the partitioned whole. Imagining how three fifths of a whole partitioned into five equal parts looks like is an example of disembedding.
The Iterating Operation

The iterating operation is repetition of a part to produce identical parts to the original. For instance, using one fourth of a whole to construct three-fourths requires three iterations of one-fourths.

Table 2 summarizes the operations outlined in this section (Norton & McCloskey, 2008; Norton & Wilkins, 2009; Olive, 1999; Steffe, 2002).

Table 2

*Fractional Operations*

<table>
<thead>
<tr>
<th>Operation</th>
<th>Explanation</th>
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<tr>
<td>Unitizing</td>
<td>Considering an object or collection of objects as a unit, or a whole</td>
</tr>
<tr>
<td>Partitioning</td>
<td>Dividing the unit, or the whole, into equal parts</td>
</tr>
<tr>
<td>Disembedding</td>
<td>Imaginatively taking out a fraction from the whole, while keeping the whole intact</td>
</tr>
<tr>
<td>Iterating</td>
<td>Repeating a part to create identical copies of it</td>
</tr>
</tbody>
</table>


Fraction Schemes

Schemes are structures composed of three parts: a recognition template, operations (i.e., mental actions) to act on the recognized situation, and expected results of operating (Glasersfeld, 1995). They are considered hypothetical ways of operating that explain and model students’ actions and verbalizations (Norton & McCloskey, 2008; Norton & Wilkins, 2009; Steffe, 2002; Steffe & Olive, 2010). In that regard, fraction
schemes are constructs that are attributed to students in order to explain students’ actions on fractional tasks. Before attending to the design of this study, four of these theorized ways of operating (i.e., fraction schemes) are described to articulate which actions are considered to be compatible with a corresponding fraction scheme.

**The Partitioning Schemes (PS)**

Partition schemes consist of a simultaneous partitioning scheme and equi-partitioning scheme. A simultaneous partitioning scheme could be considered as unitizing the whole and partitioning it. On the other hand, students who have constructed an equi-partitioning scheme understand parts in the partitioned whole as equal and identical; therefore, an equi-partitioning scheme can be iterated to reestablish the whole. The following schemes are considered to be built upon the foundation with which partition schemes provide students.

**The Part-Whole Scheme (PWS)**

Students with part-whole scheme employ unitizing and partitioning operations followed by disembedding some numbers of fractional parts from a partitioned whole. In fact, disembedding is the key operation on which part-whole schemes are grounded (Steffe & Olive, 1996). Students attributed with a part-whole scheme conceive that the pieces both within the fraction and within the whole are equal and identical. For example, the pieces both in four-fifths and the ones in the whole are equal and identical. Students who have not constructed an equi-partitioning scheme are limited to employing a simultaneous partitioning scheme. Therefore, students without equi-partitioning scheme are not considered to possess a part-whole scheme.
The Partitive Unit Fraction Scheme (PUFS)

Students attributed with a partitive unit fraction scheme conceive of an unpartitioned whole as some number of iterations of the unit fraction. The partitive unit fraction scheme relies on the equi-partitioning scheme. The primary role of an equi-partitioning scheme is to produce equal and identical pieces; and, for the partitive unit fraction scheme, the role of an equi-partitioning scheme is to find out the relation between the fractional part and unpartitioned whole through iterations (Norton & McCloskey, 2008).

The Partitive Fraction Scheme (PFS)

The partitive fraction scheme is a reorganization of the partitive unit fraction scheme to non-unit fractions. It involves two levels of unit coordination (abbreviated as 2UC). Students with the partitive fraction scheme are able to coordinate between the fractional unit and the whole as well as between the fractional unit and the proper fraction. For instance, students must coordinate two-thirds (proper fraction) as two iterations of the fraction unit (one-third) and the whole as three iterations of the fraction unit. The unit coordination at two levels occurs in that two-thirds is a unit of two fraction units, and the whole is a unit of three fractional units.

Table 3 summarizes those schemes outlined in this section (Norton & McCloskey, 2008; Norton & Wilkins, 2009; Olive, 1999; Steffe, 2002).
Table 3

*Fractional Schemes*

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Actions Associated with the Scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simultaneous Partitioning</td>
<td>Unitizing the whole, partitioning the whole using a composite unit as a template</td>
</tr>
<tr>
<td>Part-Whole Scheme</td>
<td>Unitizing, producing ( m/n ) by partitioning a whole into ( n ) parts and disembedding ( m ) of those parts</td>
</tr>
<tr>
<td>Equi-partitioning Scheme</td>
<td>Unitizing, partitioning and iterating any part to determine its identity with other parts (production of equal and identical parts)</td>
</tr>
<tr>
<td>Partitive Unit Fraction</td>
<td>Determining the size of a unit fraction relative to a given, unpartitioned whole, by iterating the unit fraction to produce a partitioned whole</td>
</tr>
<tr>
<td>Scheme</td>
<td></td>
</tr>
<tr>
<td>Partitive Fraction Scheme</td>
<td>Unitizing, determining the size of a proper fraction relative to a given, unpartitioned whole, by partitioning the proper fraction to produce a unit fraction and iterating the unit fraction to reestablish the proper fraction and the whole (two levels of unit coordination)</td>
</tr>
</tbody>
</table>


In developing models of students’ fractions understanding, two seminal theoretical frameworks are typically used: Kieren’s fraction subconstructs and Steffe’s fraction schemes. Fraction subconstructs, and fraction schemes were provided to establish a ground for the proposed study. In addition, a brief comparison of these seminal frameworks could help with conceptualizing fraction knowledge. Although Kieren (1976) did not attempt to treat subconstructs in a hierarchical order when introducing the subconstructs, related studies (e.g., Lamon, 2001) indicate that basing instruction on the measure subconstruct may afford students with more opportunities to
develop proportional reasoning, computational achievement and making a transition to other fraction subconstructs than the other four subconstructs could do. On the other hand, fraction schemes are theorized through longitudinal clinical experiments with children in that they are inductive in nature. That is, the construction of schemes follows a learning trajectory as children reorganize their existing schemes to form more complex schemes. Therefore, fraction schemes form a hierarchy of students’ progression in the development of fraction understanding.

The current study utilized fraction scheme theory, rather than fraction subconstructs for the following reasons: (a) In Kieren’s subconstructs, how the five subconstructs should be incorporated in the design of the fractional models is not clear; (b) The study attempted to conceptualize students’ ways of interacting with fraction models as opposed to expecting them to develop what appears to be proficiency requirements prescribed by Kieren (1976).

Norton and Wilkins (2012) claimed the subconstructs introduced by Kieren (1976) align with the fraction schemes described by Steffe and Olive (2010). Furthermore, Norton and Wilkins (2012) argued, “The part-whole scheme aligns with the part-whole subconstruct; the partitive unit fraction scheme aligns with the measure subconstruct; and the iterative fraction scheme aligns with the operator subconstruct” (p. 576). Although these alignments, and in particular the iterative fraction scheme, are beyond the scope of this study, conceptualizing these alignments is crucial to understanding how these two distinct ways of interpreting fractions can inform one another.
CHAPTER III

METHODOLOGY

Research Site and Sample

The research site for the study is Oak Grove Middle School, a suburban public middle school (Grades 6–8) in the Midwestern United States. The choice of this particular school is mainly due to convenience. I negotiated entry into the school through my thesis advisor who has worked previously with teachers in this district and school. As of the Fall 2015 semester, 735 children attended Oak Grove Middle School with the following demographics: Caucasian (76.2%), African American (11.8%), Hispanic (2.3%), Asian (1.6%), American Indian (0.4%), and multi-ethnic descent (7.4%). Approximately 45% of students are classified as “economically disadvantaged” and are on free or reduced lunch program. The ratio of students to teachers at Oak Grove is 14:1.

The Common Core State Standards for Mathematics (NGA Center & CCSSO, 2010), the standards used at Oak Grove, require students to have a robust understanding of fractions by sixth grade (e.g., four operations with fractions, and understandings of fraction equivalence and ordering). Participants were told that the study was designed to make their conceptions of fractions explicit. Potential participants included students enrolled in the integrated class, the general mathematics curriculum and those enrolled in an advanced placement course. The advanced placement course covers all of the sixth grade content along with roughly half of the seventh grade content. Potential participants included students from both genders and various mathematics achievement levels.
The sixth grade student population involved 243 students. However, I had access to only 145 of these students taught by two teachers Mr. Parker and Mrs. Watkins. Randomly selected students were given a solicitation letter and consent forms to take home and discuss with their parents or legal guardians. Participation in the study was completely voluntary—randomly selected students were under no obligation to participate. Moreover, only those students who gave their consent, both their own and their parents’ (or legal guardians’), were interviewed for the study. Forty-one students, out of 145 students, were randomly selected with a number generator and distributed consent forms to probe a range of mathematical understanding (12 forms to integrated, 22 forms to regular and seven forms to students in advanced placement course).

The recruiting efforts resulted in 18 participants taught by two teachers, Mr. Parker and Mrs. Watkins. Of these 18 students, three were enrolled in the integrated class (Mrs. Watkins); nine were in the regular mathematics course (Mr. Parker and Mrs. Watkins), and six were enrolled in an advanced placement sixth grade course (Mr. Parker). The overall response rate was approximately 44% (18/41). The response rate for each tier was 25% (3/12), 40.9% (9/22), and 85.7% (6/7) respectively.

Participating students were comprised equally of both genders—nine boys and nine girls. Demographically, participants were comprised of 15 Caucasian (83.3% of the sample), two African-American (11.1% of the sample) and one student with multi-ethnic descent (5.6% of the sample).
Description of the Instrument

Piaget (1970) argued that unless students are presented tasks with misleading characteristics, their true understanding cannot be judged but merely their skills to imitate the way they had been taught. Glasersfeld (1996) contended that, “cognitive change and learning take place when a scheme, instead of producing the expected result, leads to perturbations.” (p. 8). Therefore, creating activities that engage students in productive struggle helped me identify the boundaries of students’ understandings and support students’ development of robust conceptual understandings.

The Fraction Schemes Tasks and Interview Protocol for Clinical Interviews (see Appendix A) employed in this study included four different sets of items. These items were developed from a perspective of “gathering, discovering, and creating knowledge in the course of some activity having a purpose” (Romberg, 1992, p. 61) and were used to indicate whether students had constructed a particular fraction scheme. Furthermore, the items were selected for the purpose of directing students’ attention to particular aspects of the fraction concept (i.e., fraction schemes). Items on the Fraction Schemes Tasks are modified versions of tasks from previous studies that examined students’ cognitions of fractions (Battista, 2012; Norton & McCloskey, 2008). Items were adapted in a way to include three different types of models for each fraction scheme. Each set of fraction scheme items is a combination of three items with area, length, and set models. Those three items addressing a particular scheme were pooled to create evidence of a fraction scheme. In this study, the ideas described earlier in the fraction schemes and operations
sections were also used to guide item construction. Table 4 shows the allocation of the items corresponding to different types of models and fraction schemes.

Table 4

*Distribution of Items Across Model Types and Schemes*

<table>
<thead>
<tr>
<th>Item</th>
<th>Model Type</th>
<th>Fraction Scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>Area</td>
<td>Partitioning Schemes</td>
</tr>
<tr>
<td>8</td>
<td>Length</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Set</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>Area</td>
<td>Part-Whole Scheme</td>
</tr>
<tr>
<td>6</td>
<td>Length</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Set</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Area</td>
<td>Partitive Unit Fraction Scheme</td>
</tr>
<tr>
<td>12</td>
<td>Length</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Set</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Area</td>
<td>Partitive Fraction Scheme</td>
</tr>
<tr>
<td>5</td>
<td>Length</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Set</td>
<td></td>
</tr>
</tbody>
</table>

As described in the scoring rubric (Appendix B), each interview item was assigned a score of 0, 0.5, or 1 based on the amount of written explanation and student work articulated for the item (i.e., amount of evidence demonstrated for a given scheme). The content validity of items in the measurement instrument and how those items align with the fraction content for sixth grade students was validated through previous research (Battista, 2012; Norton & McCloskey, 2008) and requirements of the sixth grade Common Core State Standards for Mathematics.
Models used in interview items were labeled to describe the specific scheme and model type to which they attend. Such labeling assisted subsequent data analysis. For instance, $A$ stands for the area model, $L$ stands for the length model, and $S$ stands for the set model. In that regard, $L.PWS$ stands for the length model addressing the part-whole scheme.

**Data Collection: Clinical Interviews**

The first week in the field consisted of class observations and recruitment efforts. IRB approved consent forms were distributed to potential participants during the first week of fieldwork. Since potential participants were minors, a parental (legal guardian) consent form was sent home with students (see Appendix C). It was made clear to participants and their parents that students’ participation in the study would not affect their grade in the course or their relationship with their mathematics teachers, and they were free to withdraw from the study at any time without negative consequence. Pseudonyms (e.g., S1 refers to a particular student in the sample) were used in reporting data to eliminate potential ethical dilemmas associated with participant’s identity.

The central data gathering mechanism for this study involved engaging participants in one-on-one interviews to elicit evidence of their ways of operating as they solved fraction items (Appendix A). Each participant was given 12 items to solve during the interview sessions. Items listed in Appendix A are hierarchically ordered in regards to the rigor of the fraction schemes. However, items were randomized in a specific order as demonstrated in Table 4 so that all students engaged with the items in the same order.
in the interviews. Data consisted of students’ drawings, written explanations, visual representations, and verbalizations made in response to each fraction item.

A pilot interview took place in April 2015, where I had the chance to try out the preliminary version of the items as well as my interview skills. The aim of the pilot study was “to gain a preliminary sense of the meaning that experiences have for the participants involved in them” through this pilot interview (Schram, 2006, p. 37). The interviewee was a fifth grade student who now attends the school of interest in this study. The choice of this particular participant was mainly due to ease of accessibility and being age appropriate for learning fraction concepts.

I served as interviewer. Interviews took place at the beginning of 2016 Spring semester, involved students’ interactions with a 12-item fraction scheme task assessment (Appendix A), and lasted approximately 45 minutes per student. One-on-one task-based clinical interviews with each participant were the primary data source for this study (Clement, 2000; Ginsburg, 1997). These interviews provided me the opportunity for more in-depth probing of students’ conceptions regarding the four fraction schemes and three models. My primary aim in conducting these interviews was to understand participants’ ways of thinking rather than to make in-the-moment judgments relative to the correctness of their responses. As Doyle (1983) asserted, “tasks influence learners by directing their attention to particular aspects of content and by specifying ways of processing information” (p. 161).

Holding myself restricted to the data gathered through interviews, I attempted to develop “accurate descriptive account[s] of what the participants did and said” (Hatch,
2002, p. 79) or in other words “reportable data” (Wolcott, 1995, p. 98). Since written marks are limited as sources of indication or counter-indication of students’ ways of operating (Norton & Wilkins, 2010), participants were asked to “think aloud” and verbalize their thought processes as they attempted to make sense of and answer each item. The interview sessions were audio- and videotaped to secure a record of interviews for subsequent data analysis. The video camera captured the participant’s written work and the exchanges between myself and the participant. The video- and audiotapes do not include students’ names; rather, pre-determined pseudonyms were used to address each participant. Prior to taping, I read the assent (Appendix C) to make sure students knew they could ask me to stop recording at any time. After interviews were completed, I described in detail components of students’ responses in regards to schemes and models so as to include the mathematical reasoning students applied, and the challenge(s) students encountered—the locus of the inquiry (Schram, 2006, p. 29).

The interview method was derived from Piaget’s clinical interview methodology, which enables the researcher to investigate the processes by which a single child makes sense of mathematical knowledge (Cobb, Wood, & Yackel, 1990). However, limitations of this method include the claim that, “in the course of analysis, for example, the researcher focuses almost exclusively on what the child might be thinking and implicitly takes the social process of mutually negotiating the interview situation for granted” (Cobb et al., 1990, p. 128).
Data Analysis

The study addressed students’ conceptions regarding four fraction schemes and three fraction models (i.e., representations) through a convergent parallel mixed methods design (Creswell & Plano Clark, 2011). According to Creswell and Plano Clark (2011), such a design requires the collection of both quantitative and qualitative data since each present partial understanding of the phenomenon. Therefore, in order to develop robust models of students’ understandings, both quantitative (i.e., item scores) and qualitative data (students’ conjectures and remarks through interviews) were collected in parallel and then analyzed separately.

Quantitative data was used to test the hierarchical Fraction Scheme Theory (Steffe & Olive, 2010), whereas qualitative data were used to investigate the fractional reasoning of sixth-grade students at this particular middle school. The main reason for collecting both quantitative and qualitative data was to corroborate results in order to better understand students’ ways of operating regarding the three distinct fraction models.

Interrater Reliability

In order to determine the reliability with which participants’ assessments were scored, I collaborated with a fellow graduate student in Evaluation and Measurement master’s degree program at Kent State University. The second evaluator had worked as a middle school mathematics teacher for four years in Turkey. Therefore, he is familiar with students’ fraction understanding. He also read a few articles to familiarize himself with the Fractional Scheme Theory for the scope of this study. Two raters, myself and the second evaluator, independently scored responses to the 12-item assessment collected
from 18 students based on the scoring rubric. A Cohen’s linearly weighted Kappa statistic (Cohen, 1968) was calculated, $K = 0.76$ ($p < 0.001$), 95% CI (0.680, 0.836), indicating substantial inter-rater consistency (Landis & Koch, 1977). Weighted Kappa takes into account the degree of disagreement between two raters. In other words, discrepancies that are further apart (e.g., 0–1) are penalized more heavily than close discrepancies (e.g., 0–0.5 or 0.5–1). Both raters discussed any differences in scoring and reconciled these differences to create one item score for each student. The resulting scores were used as quantitative data from the clinical interviews.

**Quantitative Analysis**

There are 12 items addressing four different schemes. Each set of scheme items is composed of three items with distinct model types (i.e., a set of three items for each scheme type). Each item has a scoring scale from 0 to 1 (see Appendix B). Of the 12 items, three were pooled to create evidence of a fraction scheme, resulting in a score for the corresponding scheme ranging from 0 to 3. For instance, scores pooled from $A.PS$, $L.PS$, and $S.PS$ items (see Appendix A) constituted students’ overall scores for partitioning schemes. The overall scores were then used to determine whether or not students had constructed a particular scheme.

Of the 12 items, four involve area models, four involve length models, and four consist of set models. Consequently, since each item has a scoring scale from 0 to 1, each type of model had a resultant overall score ranging from 0 to 4. For example, scores pooled from $S.PS$, $S.PWS$, $S.PUFS$, and $S.PFS$ items constituted students’ overall scores for the set model type. The overall scores were then used to determine whether or not
scores for a specific model type showed a statistically significant difference. In addition, each scheme score was analyzed to determine if there was a difference in students’ scores for a particular type of model.

The mean and median scores and standard deviation for each set of items attending to a particular scheme and type of model were reported in Chapter 4. A Friedman two-way analysis of variance by ranks test was run to determine whether there is:

1. A statistically significant difference in students’ scores in terms of fraction scheme (partition, part-whole, partitive unit fraction, and partitive fraction schemes).
2. A statistically significant difference in students’ scores in terms of type of model (area, length and set models).
3. A statistically significant difference in students’ scores for a particular fraction scheme in terms of type of model.

The Friedman test allowed the examination of different schemes, different models types, and a particular scheme across types of model. The Friedman test is an overall test of whether the scores vary as a function of the observed conditions (i.e., fraction schemes, and types of the model; Siegel & Castellan, 1988).

**Qualitative Analysis**

Simultaneous data collection and analysis were conducted in a manner that Seidel (1998) described as “a process of Noticing, Collecting, and Thinking about interesting things” (emphasis in original, p. 1). Following Seidel, data gathered through interviews,
such as interview transcripts, were analyzed with respect to potential patterns that emerged (i.e., emergent codes or categories). In addition, prior substantive theories and existing fraction literature (e.g., Steffe & Olive, 2010; Thompson & Saldanha, 2003; Tunc-Pekkan, 2015) informed the development of potential explanations for any emergent patterns. The challenge here was to be “purposeful without being predictive” (Schram, 2006, p. 67) and avoid “prematurely shut[ting] down avenues of meaningful questioning” (p. 60). Even though any emergent categories or codes were not rigidly prescribed, I anticipated that some codes would be informed by existing literature. These tentative codes emerged from students’ additive reasoning versus multiplicative reasoning, emphasis on congruent parts, understanding of area, or strategies relied on counting, and so forth. However, I needed to be mindful of “forcing explanations on the data or making an early appeal to existing concepts in the literature” (Schram, 2006, p. 102). In particular, constant comparison for similarities and differences between, for instance, accounts of interviewees relative to different fraction models was made.

During the analysis, attending to the language (i.e., verbalization) used to name fractions also communicated the way students made sense of fractions. For example, given a fraction, say 3/5, whether the students read “three slash five,” “three fifths,” “three out of five,” “three over five,” or “three one-fifths,” and so forth provided insights into students’ ways of understanding fractions (i.e., “sensitizing concept to focus;” Denzin, 1978, and Patton, 1990, as cited in Hatch, 2002, p. 80).
CHAPTER IV

RESULTS

Analysis of the Findings

The following section presents the quantitative and qualitative analyses conducted to address the three guiding research questions of this study. The statistics tests were conducted to examine the quantitative data and the strategies that were apparent in students’ responses were elaborated under each research question. The qualitative analysis of students’ strategies provided the opportunity to probe students’ understanding of fractions.

Research Question 1

Research question 1 is: How do sixth grade students’ responses to items that correspond to the development of a particular scheme compare and contrast?

Scores from three items measuring each of four fraction schemes were combined to generate a maximum total score of three for each student. After items corresponding to each particular scheme were scored (Partitioning Schemes, Items #3, #8, #9; Part-Whole Scheme, Items #6, #10, #11; Partitive Unit Fraction Scheme, Items #2, #4, #12; Partitive Fraction Scheme, Items #1, #5, #7), the resultant scores for each individual scheme (from all 18 students) were pooled to generate an overall scheme score. Table 5 presents item numbers corresponding to certain fraction schemes and model types.
Table 5

*Items Across Model Types and Schemes*

<table>
<thead>
<tr>
<th>Scheme/Model (Item number)</th>
<th>Partitioning Schemes</th>
<th>Part-Whole Scheme</th>
<th>Partitive Unit Fraction Scheme</th>
<th>Partitive Fraction Scheme</th>
<th>Model Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area</td>
<td>Item 9</td>
<td>Item 11</td>
<td>Item 2</td>
<td>Item 7</td>
<td>4</td>
</tr>
<tr>
<td>Length</td>
<td>Item 8</td>
<td>Item 6</td>
<td>Item 12</td>
<td>Item 5</td>
<td>4</td>
</tr>
<tr>
<td>Set</td>
<td>Item 3</td>
<td>Item 10</td>
<td>Item 4</td>
<td>Item 1</td>
<td>4</td>
</tr>
<tr>
<td>Scheme Total</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>12</td>
</tr>
</tbody>
</table>

These scores were then converted to percentages to assist in the interpretation of students’ overall performance for each fraction scheme. As mentioned earlier, during the interviews, items were asked in a randomized order rather than the hierarchical trajectory that researchers hypothesized (Steffe & Olive, 2010). The results of quantitative and qualitative analyses are elaborated on in the next section for the purpose of answering the first research question.

**Quantitative analysis.** Since the data were related across four scheme types (i.e., four different scores from the same participants), I initially attempted a One-Factor Repeated Measures ANOVA. However, upon further examination, data were non-normal, which necessitates the use of an alternative procedure—the Friedman (1937) test. The Friedman test is a nonparametric procedure based on ranks when there are more than two related samples. This omnibus test was conducted before the post hoc examination of specific pairs of schemes to look for an overall difference across the four fraction schemes.
The Friedman test was significant, \( \chi^2(3, N=18) = 22.50, p < .001 \). Post hoc analysis with Wilcoxon Signed-Rank tests was conducted (Wilcox, 1996) with a Bonferroni correction applied, resulting in a significance level set at \( p < .008 \). The Bonferroni procedure requires the alpha (i.e., .05) to be divided by the number of paired comparisons (six pairwise comparisons of four fraction schemes in this study) in order to control for Type I error. There were statistically significant differences between Partitive Fraction Scheme and Partitioning Scheme (\( Z = -3.05, p < .008 \)), Partitive Fraction Scheme and Part-Whole Scheme (\( Z = -2.88, p < .008 \)), and Partitive Unit Fraction Scheme and Partitioning Schemes (\( Z = -2.65, p = .008 \)). In addition, the mean ranks were higher for the PS and PWS. There was, however, no significant difference between all other pairs of schemes (e.g., Partitioning Schemes and Part-Whole Scheme).

Table 6 includes descriptive statistics for students’ scores for the four types of fraction schemes. The distribution of the scores also indicates apparent differences between the above specified pairs. For example, lower standard deviation in scores for Partitioning Schemes indicate that the scores are less spread out compared to scores for Partitive Fraction Scheme. Therefore, there is a greater variability in PFS scores compared to those of PS. With higher mean and median scores, along with less variation in PS scores, students’ overall performances were higher in PS than in PFS.
Table 6

Descriptive Statistics for Fraction Scheme Scores

<table>
<thead>
<tr>
<th></th>
<th>PS</th>
<th>PWS</th>
<th>PUFS</th>
<th>PFS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>2.67</td>
<td>2.39</td>
<td>1.94</td>
<td>1.58</td>
</tr>
<tr>
<td>Median</td>
<td>3.00</td>
<td>3.00</td>
<td>2.00</td>
<td>1.50</td>
</tr>
<tr>
<td>Std. Deviation</td>
<td>.59</td>
<td>.92</td>
<td>1.16</td>
<td>1.23</td>
</tr>
</tbody>
</table>

Qualitative analysis. Students’ scores were also examined to see whether or not a student had constructed a particular scheme. Such examination required that cutoff scores be determined to ascertain whether or not each student had constructed a particular scheme. If a student’s scheme score was in the interval $[0, 1]$ —out of a total of three— that student was inferred not to have constructed that particular scheme. For instance, if the overall resultant score for partitioning schemes was 0.5, this indicated the student had not constructed the scheme yet. If the score was in the interval $(1, 2)$, the scores were to be negotiated between the two raters. However, there was not such a case in the sample. If the student’s score was in the interval $[2, 3]$, the student was inferred to have constructed that particular scheme.

Table 7 shows the distribution of students with developed schemes differentiated by tier (where a “+” indicates the student had developed that scheme). As illustrated in Table 7, advanced students’ scores accumulated towards the bottom of the table where all four schemes are deemed to have been constructed. The triangular shape in Table 7 demonstrates that the less complex schemes were available to almost all the students in
the sample; however, all the students in the integrated tier and some students in the regular tier had yet to construct more sophisticated schemes, such as PUFS and PFS.

Table 7

*Distribution of Fraction Schemes Construction of Students in the Sample*

<table>
<thead>
<tr>
<th>Tier</th>
<th>PS</th>
<th>PWS</th>
<th>PUFS</th>
<th>PFS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integrated</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Integrated</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Integrated</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular</td>
<td>+</td>
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<td>Regular</td>
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<tr>
<td>Regular</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Advanced</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Advanced</td>
<td>+</td>
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<tr>
<td>Advanced</td>
<td>+</td>
<td></td>
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<tr>
<td>Advanced</td>
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<td>Advanced</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Advanced</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>17</td>
<td>15</td>
<td>12</td>
<td>9</td>
</tr>
</tbody>
</table>
According to the criteria mentioned above, 94% (17 of 18) of the students had constructed the Partitioning Scheme, whereas 83% (15 of 18) of them had constructed the Part-Whole Scheme. On the other hand, the percentage of students who constructed PUFS and PFS was considerably lower at 67% (12 of 18) and 50% (9 of 18), respectively. Based on these percentages, the schemes seem to align as the proposed learning trajectory, with PS and PWS developing earlier followed later by PUFS and PFS (Norton & Wilkins, 2009, 2010; Steffe & Olive, 2010). However, according to Table 7, there seems to be operational differences between PS and PWS (i.e., part-whole situations including unit fractions and proper fractions) unlike the opposite claim by Norton and Wilkins (2009). Otherwise, we would expect all students that had developed PS to have developed PWS as well.

**Research Question 2**

Research question 2 is: How do sixth grade students’ responses to items involving a particular model type compare and contrast?

Similar to the total resultant score for each fraction scheme, scores on four items related to a particular model type (i.e., area model, Items #2, 7, 9, 11; length model, Items #5, 6, 8, 12; set model, Items #1, 3, 4, 10) were pooled to generate a score for a particular model type across fraction schemes (see Table 5). In order to answer the second research question, these scores were analyzed and responses to items represented with a particular model were probed.

**Quantitative analysis.** The Friedman test was conducted to evaluate differences among area, length and set models. The Friedman test was significant, \( \chi^2(2, N = 18) = \)
7.17, \( p < .05 \). The Kendall coefficient of concordance of .20 indicated reasonably strong differences among all three types of fraction models. Post hoc analysis with Wilcoxon Signed-Rank tests was conducted with a Bonferroni correction applied resulting in a significance level set at \( p < .017 \). There was a statistically significant difference between the length and set models (\( Z = -2.75, p < .01 \)). In addition, the mean ranks were higher for the length model. There was, however, no significant difference between length model and area model (\( p > .02 \)), and set and area models (\( p > .10 \)).

Table 8 presents descriptive statistics for students’ scores for the three types of fraction models. The distribution of the scores also indicates apparent differences between length and set models. Lower standard deviation in scores for length models points out that the length model scores are less spread out compared to scores for set models. Consequently, there is a greater variability in scores for set models compared to that of length models. Higher mean and median scores, along with less variation in length model scores, indicate higher overall student performance on length models than that on set models.

Table 8

*Descriptive Statistics for Model Type Scores*

<table>
<thead>
<tr>
<th></th>
<th>Area</th>
<th>Length</th>
<th>Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>2.81</td>
<td>3.39</td>
<td>2.39</td>
</tr>
<tr>
<td>Median</td>
<td>3.00</td>
<td>3.75</td>
<td>2.50</td>
</tr>
<tr>
<td>Std. Deviation</td>
<td>1.20</td>
<td>1.05</td>
<td>1.64</td>
</tr>
</tbody>
</table>
Qualitative analysis. Area and set models in the study can be conceived as composite units even before any partitioning operation conducted. However, before partitioning, the length models are “continuous items of a single unit” (Wilkins, Norton, & Boyce, 2013, p. 35). This means that students need to use the unitizing operation to be able to work with composite units. Such a difference might have resulted in lower achievement scores in the area and set models in PWS, PUFS, and PFS items compared to length models.

The follow-up test indicated statistically insignificant differences between scores for set and area models, and length and area models. However, I cannot claim that there are no operational differences between situations involving these pairs of models. That is because these pairs (i.e., set and area models, and length and area models) were not treated in the same way by students in the sample. Participating students in the study attended to different features of the models while working with these items, as summarized in Table 9. As articulated in Chapter 2, the models differ in terms of what defines the unit, what defines the parts, and what the fraction represents.
Investigating students’ strategies for different models indicated that students’ capabilities to coordinate the features of a particular model with the fraction given in the item statement might have confounded with their in-activity schemes. For instance, all students in the sample correctly answered A.PS, while 72% of them correctly answered S.PS (see Figure 1). Even though both items addressed the Partitioning Scheme, the overall achievement was not the same. Moreover, regardless of the fraction scheme, students’ performance on items with set models was reasonably lower (i.e., at least 20% lower) compared to items with length models. On the other hand, students performed way better on length models except for the Partitioning Scheme. Figure 1 presents percentages of correct responses across three models types differentiated by fraction schemes.
Figure 1. Performance across model types. This figure illustrates the percentages of correct responses across three models differentiated by the fraction scheme.

For L.PUFS and L.PFS items, students’ articulated thinking and reasoning were mostly related to measuring, estimation and size comparison; whereas, set and area models corresponding to the same scheme types did not seem to generate such a shift in students’ discourse. On the other hand, students’ reliance on number sentences seems to have been prompted when working with area and set models for PUFS and PFS.

Students’ performance for A.PWS was lower than that on A.PS. The reason for this might be that the number of rows and columns in A.PS are equal to the magnitude of the denominator given in the item statement. On the other hand, for A.PWS, the number of rows and columns were not the same as the magnitude of the given fraction. This situation seems to be more challenging to some students, resulting in relatively low
achievement for the A.PWS item. Length and set models did not demonstrate such a decrease in percentages from PS to PWS.

Research Question 3

Research question 3 is: Do certain types of models facilitate the development of certain schemes?

The overall percentage of correct responses for each item can be seen in Figure 2. Addressing the third research question entails taking a closer look at student responses across model types for each fraction scheme. Therefore, three different items for each scheme type were analyzed to investigate whether there were statistically significant differences between their scores through quantitative analysis. Moreover, qualitative analysis provided the opportunity to find out whether there were operational differences in students’ responses (i.e., verbalizations, strategies, operations, etc.) for three model types within each fraction scheme.

Figure 2. Performance across fraction schemes. This figure illustrates the percentages of correct responses across fraction schemes differentiated by the model type.
**Partitioning scheme across model types.** The subsequent section presents quantitative and qualitative analyses conducted to address the third research question in the context of Partitioning Scheme. The statistics tests were conducted to examine students’ Partitioning Scheme scores and the strategies that were apparent in students’ responses to three items corresponding to Partitioning Scheme were elaborated. The qualitative analysis of students’ strategies provided the opportunity to investigate students’ understanding of partitioning across model types.

**Quantitative analysis.** The Friedman test at the item level (Items #3, 8 and 9) was conducted to evaluate differences in PS scores among area, length and set models. The Friedman test was significant, \( \chi^2(2, N=18) = 8.40, p < .05 \). The Kendall coefficient of concordance is .23. Wilcoxon tests were used to follow up the finding by controlling for the Type I errors at the .05 level using Bonferroni procedure, resulting in a significance level set at \( p < .017 \). However, there was no significant difference between any pairs of model types for Partitioning Scheme.

**Qualitative analysis.** Items #3, 8, and 9 were designed to elicit the Partitioning Scheme. Responses to these items were coded either as 0 or 1 meaning that there were no partial scores for these responses. Consequently, all students gave correct answers to A.PS item, 94% (17 of 18) correctly answered L.PS item, and 72% (13 of 18) correctly answered S.PS item.

**Area model for PS.** Item #9 (A.PS) was the only item that all students correctly answered. A square on a dot paper was given as the A.PS item as demonstrated in Figure
3. Most students were not as interested in how many unit squares there were inside the model as they were in partitioning the model to get five parts within the model.

![Diagram](Insert Diagram)

Figure 3. The area model for PS (A.PS). Item #9 designed to elicit Partitioning Scheme with the area model.

Three strategies stood out among students’ responses to A.PS as shown in Table 10. These strategies can be categorized as partitioning into five equal parts, treating 25 unit squares as the whole, and finding the equivalent fraction.

Students with the partitioning into five equal parts strategy conceived the shaded piece as “one out of five equal sized pieces.” Moreover, the dot paper might have facilitated the partitioning for some students. S15 made a comment about how she used the line of dots to divide the model into five pieces: “All I had to do is to draw the lines because the dots were already there.” Students’ conceptions of unitizing also appeared to impact their strategies; that is, their decisions to operate with either fives parts or 25 unit squares within the model.
Table 10

*Examples of the Strategies and Unitizing Used for A.PS*

<table>
<thead>
<tr>
<th>Response</th>
<th>Strategy</th>
<th>Unit of 1</th>
<th>Sample Student Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>Partitioning into five equal parts</td>
<td>Five parts (coded 1)</td>
<td><img src="image1.png" alt="Image" /></td>
</tr>
<tr>
<td>R2</td>
<td>Treating 25 unit squares as the whole</td>
<td>25 unit squares (coded 1)</td>
<td><img src="image2.png" alt="Image" /></td>
</tr>
<tr>
<td>R3</td>
<td>Finding the equivalent fraction</td>
<td>25 unit squares (coded 1)</td>
<td><img src="image3.png" alt="Image" /></td>
</tr>
</tbody>
</table>

*Length model for PS.* For Item #8, one-fifth of the bar was asked to be shaded (Figure 4). Ninety four percent of the students correctly answered this item. The only response incompatible with PS was R3 as shown in Table 11. This response does not seem to satisfy the requirement of equally sized pieces resulted from the partitioning operation. Therefore, this particular response was coded as 0.
Shade $\frac{1}{5}$ of the bar.

Figure 4. The length model for PS (L.PS). Item #8 designed to elicit Partitioning Scheme with the length model.

Table 11

Examples of the Strategies Used for L.PS

<table>
<thead>
<tr>
<th>Response</th>
<th>Description of the Strategy</th>
<th>Sample Student Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>Partitioning the bar into five equal pieces and relating the fractional language to the model (coded 1)</td>
<td><img src="image" alt="Student Response" /></td>
</tr>
<tr>
<td>R2</td>
<td>Partitioning the bar into five equal pieces and counting the partitioning marks (coded 1)</td>
<td><img src="image" alt="Student Response" /></td>
</tr>
<tr>
<td>R3</td>
<td>Separating the bar into unequal pieces (coded 0)</td>
<td><img src="image" alt="Student Response" /></td>
</tr>
</tbody>
</table>

A halving strategy to partition the model was not available for L.PS item. The partitioning into five pieces took a little bit more time and thought for students who relied on the halving strategy. One such student expressed this problematic situation as: “It is harder when you are doing it with an odd number because you cannot cut the bar in half and then cut those in half as many times.” Moreover, students employed different ways of counting the pieces within the length model. Some students counted the pieces by
fifths (e.g., R1) whereas some counted the partitioning marks (e.g., R2) to keep track of the partitioning as demonstrated in Table 11.

For this L.PS item, there were no students who partitioned the bar into multiples of five, instead of just five pieces. Such a result could be attributed to the fractional language that students employed. Most students referred to the fractional meaning that one-fifth communicated to them. These remarks indicate that the denominator always provides the number of pieces in the whole and the numerator tells the number of pieces in fractional part.

*Set model for PS.* Participating students demonstrated lower achievement in S.PS (i.e., Item #3) when compared to A.PS and L.PS items. Noticeable differences in students’ achievement on S.PS entails close analysis of the reasoning that students employed in their work, and in particular, the strategies they employed to answer the item. Ten balls were given as the set model and students were asked to shade one-fifth of this set. Figure 5 shows the S.PS item.

![Shade 1/5 of the balls.](image)

*Figure 5.* The set model for PS (S.PS). Item #3 designed to elicit Partitioning Scheme with the set model.
Students employed three main strategies to answer this item: *treating five balls as an operational unit, partitioning into five groups, and finding the equivalent fraction*. Six of the students in the sample were able to justify their answers by using two of these strategies at the same time. Before moving to the particular aspects of these strategies, the unitizing operation is elaborated to provide clarity to the reader.

Students’ conceptions of unitizing (i.e., considering an item or collection of items as a whole) were based on the strategy they used. As such, students’ conceptions of unitizing can be elaborated upon by looking beyond students’ overall performance on S.PS. Students may use composite units (e.g., 10 balls as unit of 1) by uniting a collection of discrete objects into a whole (Norton & McCloskey, 2008), in other words, by producing units of units (Hackenberg, 2013). I defined the operational unit as the unit by which the students drew upon to answer the item. Even though 10 balls constitute the whole (i.e., unit of 1), some students used a sub-unit that allowed them to treat some numbers of discrete objects (i.e., balls) as an operational unit. For S.PS, if the operational unit of five balls was used, the disembedding of two shaded-in balls should follow (from an observer’s perspective). However, following Lamon (1996) it could be claimed that “reunitizing of a quantity in terms of more composite units” seem to be a complicated operation for students (p.188). This seems to be the case for participating students who finalized their answer after shading in only one of the balls (e.g., R2 in Table 12).

In the first strategy, students treated five of the balls as an operational unit, and of these students, some interpreted the item as if it required them to shade in one-fifth of five balls. One of these students was perplexed about what to do and commented:
Table 12

*Examples of the Strategies and Unitizing Used for S.PS*

<table>
<thead>
<tr>
<th>Response</th>
<th>Strategy</th>
<th>Operational Unit</th>
<th>Unit of 1</th>
<th>Sample Student Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>Treating five balls as an</td>
<td>Operational unit of five balls</td>
<td>Two groups of five balls (coded 1)</td>
<td>![Image 1]</td>
</tr>
<tr>
<td></td>
<td>operational unit</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R2</td>
<td>Partitioning into five groups</td>
<td>Operational unit of two balls</td>
<td>One group of five balls (coded 0)</td>
<td>![Image 2]</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R3</td>
<td>Finding the equivalent fraction</td>
<td>Operational unit of ten balls</td>
<td>Five groups of two balls (coded 1)</td>
<td>![Image 3]</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R4</td>
<td></td>
<td>Operational unit of ten balls</td>
<td>One group of eight balls (coded 1)</td>
<td>![Image 4]</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

“Usually, when it says one-fifth there is five and you only shade in one but there is ten.”

However, students treating five balls as an operational unit responded correctly (i.e., coded 1) when their operations were followed by the disembedding of one shaded-in ball in each operational unit.

On the other hand, students who used the equivalent fraction strategy claimed to *extend* the fraction (i.e., 1/5) to the equivalent fraction whose denominator was 10 (i.e., the number of balls given in the model). In either case, there is an underlying notion...
communicated in such responses that “the denominator is the whole” or “the denominator shows the number of items/pieces/parts in the whole.” In order to clarify the strategies, illustrated sample responses are given with examples of how responses were coded (coded 0 and coded 1) in Table 12.

R2 in Table 12 is an example of a response coded as 0. This particular instance facilitated the interpretation of “five balls as unit of 1” reasoning for students. As demonstrated in Table 12, the student whose response is shown as R2 unitized five balls as the unit of 1 and shaded in only one of 10 balls. Furthermore, the student claimed that one shaded-in ball from each group would be two-fifths (i.e., 2/5). This particular case suggests the student was challenged to treat the collection of 10 balls as a unit (i.e., whole). In other words, the unitizing operation was challenging to the student. However, she did not seem to have a problem with unitizing in A.PS and L.PS. Accordingly, students find conceptualizing discrete composite units problematic even though they are able to correctly answer items with continuous units. The following excerpt is from the interview session with this student and helps make explicit her unitizing of five balls as the unit of 1:

We just shade in one [ball]. That’s one fifth. Here is another fifth [pointing at the second group of five balls]. But if we shaded in two of them, we would have two-fifths. If we did not shade in this [the shaded-in ball in the second group of five] we would have one-fifth shaded. (S13, 02/02/2016)

**Part-whole scheme across model types.** The following section involves quantitative and qualitative analyses conducted to address the third research question in
the context of Part-Whole Scheme. The statistics tests allowed for examination of students’ Part-Whole Scheme scores and the strategies that were apparent in students’ responses to three Part-Whole Scheme items were articulated. The qualitative analysis of students’ strategies presented the opportunity to examine students’ Part-Whole Scheme across model types.

**Quantitative analysis.** The Friedman test at item level (Items #6, 10, and 11) was conducted to investigate differences in Part-Whole Scheme (PWS) scores among area, length, and set models. The Friedman test was significant, \( \chi^2(2, N = 18) = 6.33, p < .05 \) with the Kendall coefficient of concordance of .18 indicating fairly strong differences among three types of fraction models for PWS. Post hoc analysis with Wilcoxon Signed-Rank tests were conducted with a Bonferroni correction applied, resulting in a significance level set at \( p < .017 \). There was, however, no significant difference between pairs of model types for the Part-Whole Scheme.

**Qualitative analysis.** In general, students’ performances on PWS items were compatible with that on PS. Scores from Items #6, 10, 11 were used to evaluate whether students’ responses differ in respect to the type of the model. The overall scores for PWS items for each model type were: Area - 14; Length - 17; Set - 12.

Responses to these items were coded either as 0 or 1 meaning that there were no partial scores for the responses to PWS items. Therefore, the overall scores for PWS items over the model types can also be interpreted as frequencies of correct responses to PWS items. In other words, 78% (14 of 18) of the students gave correct answers to
A.PWS item, 94% (17 of 18) correctly answered L.PWS item, and 67% (12 of 18) correctly answered S.PWS item.

_Area model for PWS._ Item #11 (A.PWS) was developed to evaluate students’ part-whole scheme in the context of an area model. A rectangle on the dot paper was given as shown in Figure 6. Two main strategies were observed among students’ responses to A.PWS: _partitioning into four equal parts followed by the disembedding of three parts and finding the equivalent fraction._ Students who partitioned the model into four parts appeared to conceive the shaded piece as “three out of four equal sized parts.” Students’ conceptions of unitizing seemed to impact their strategies in their decisions to operate with either four parts or 24 unit squares within the model.

![Area model for PWS](image)

_Shade $\frac{3}{4}$ of the shape.

*Figure 6.* The area model for PWS (A.PWS). Item #11 designed to elicit Part-Whole Scheme with the area model.
Students’ performances on A.PWS were lower than that on A.PS. This could result from the fact that relating the fraction language to the model was not readily available to some students. In other words, some students were puzzled because the number of columns and rows were not the same as the number representing the denominator of the fraction (i.e., 3/4). Moreover, 50% of the students utilized partitioning the model into four parts, whereas nearly 28% used equivalent fraction strategy in which the model was partitioned into eight or 24 equal sized pieces.

An instance was encountered in which a student separated the model into four unequal parts and attempted to relate the fractional language with the model (see R4 in Table 13). The student separated the model into four parts as she construed the denominator as the number of parts within the whole, but she seemed to ignore the requirement that those parts have to be of the same size. This student, however, claimed that parts have to be equal sized for the length model (L.PWS).

Another response, which was coded 0, exemplified the effort to relate the fractional language to the model. As demonstrated in Table 13, R5 counted the number of rows and then shaded in four of the unit squares (i.e., three rows and four unit squares in regards to 3/4). In other words, R5 was trying to operate with numbers corresponding to the numerator and denominator of the fraction given in the item. However, it was not compatible with fractional language.
Table 13

*Examples of Strategies and Unitizing Used for A.PWS*

<table>
<thead>
<tr>
<th>Response</th>
<th>Strategy</th>
<th>Unit of 1</th>
<th>Sample Student Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>Partitioning into four equal parts followed by the disembedding of three parts</td>
<td>Four parts (coded 1)</td>
<td><img src="image1" alt="Sample Student Response" /></td>
</tr>
<tr>
<td>R2</td>
<td>Finding the equivalent fraction</td>
<td>Eight parts (coded 1)</td>
<td><img src="image2" alt="Sample Student Response" /></td>
</tr>
<tr>
<td>R3</td>
<td>Finding the equivalent fraction</td>
<td>24 unit squares (coded 1)</td>
<td><img src="image3" alt="Sample Student Response" /></td>
</tr>
<tr>
<td>R4</td>
<td>Separating into unequal parts</td>
<td>Four parts (coded 0)</td>
<td><img src="image4" alt="Sample Student Response" /></td>
</tr>
<tr>
<td>R5</td>
<td>Unable to relate to fractional language to the model</td>
<td>Not clear (coded 0)</td>
<td><img src="image5" alt="Sample Student Response" /></td>
</tr>
</tbody>
</table>

*Length model for PWS.* The L.PWS item was used to measure students’ part-whole reasoning with a length model. Participating students were asked to shade in three-fourths of the bar as illustrated in Figure 7. Students’ performance on the L.PWS item and their responses were quite similar to that on L.PS (Item #8).
Figure 7. The length model for PWS (L.PWS). Item #6 designed to elicit Part-Whole Scheme with the length model.

Ninety four percent of the students correctly answered L.PWS. The only response incompatible with PWS reasoning was provided by R3 as shown in Table 14. The student claimed that pieces are not equally sized. Therefore, this particular response was coded as 0.

Most students employed the halving strategy to partition the bar into four pieces. These students first partitioned the bar in half and then divided the two resultant pieces in half. For the L.PWS item, I did not encounter any students who partitioned the bar into multiples of four, instead of four pieces. The reason could be the fractional language that students employed. Most students referred to the fractional meaning that three-fourths communicated to them in a way that the denominator always indicated the number of pieces in the whole and the numerator indicated the number of pieces in the fractional part.

Participating students employed different ways of counting the pieces within the length model similar to how they operated in L.PS. Some students counted the pieces by fourths (e.g., R1 in Table 14) whereas some counted the partitioning marks to keep track of the disembedding of three pieces shaded-in (e.g., R2 in Table 14).
Table 14

*Examples of the Strategies Used for L.PWS*

<table>
<thead>
<tr>
<th>Response</th>
<th>Description of the Strategy</th>
<th>Sample Student Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>Partitioning the bar into four equal pieces and relating the fractional language to the model (coded 1)</td>
<td><img src="image1" alt="Image" /></td>
</tr>
<tr>
<td>R2</td>
<td>Partitioning the bar into four equal pieces and counting partitioning marks (coded 1)</td>
<td><img src="image2" alt="Image" /></td>
</tr>
<tr>
<td>R3</td>
<td>Separating the bar into unequal pieces followed by the disembedding operation (coded 0)</td>
<td><img src="image3" alt="Image" /></td>
</tr>
</tbody>
</table>

*Set model for PWS.* Students’ ways of operating in the context of S.PWS was similar to that of S.PS; that is, the strategies they employed were quite comparable. Three main strategies emerged through analysis of the S.PWS data: *treating four balls as an operational unit, partitioning into four groups followed by the disembedding of three groups, and finding the equivalent fraction.* Four of the students in the sample were able to justify their answers by treating four balls as an operational unit and finding the equivalent fraction. Of these four students, three were in the advanced placement course and one was in the general mathematics curriculum. Figure 8 presents the item designed to elicit PWS with a set model.
Figure 8. The set model for PWS (S.PWS). Item #10 designed to elicit Part-Whole Scheme with the set model.

Students who used the equivalent fraction strategy claimed to extend the fraction (i.e., $\frac{3}{4}$) to the equivalent fraction whose denominator was eight (i.e., the number of balls given in the model). On the other hand, some students treated four of the balls as an operational unit and some of them interpreted the item as if it was asking them to shade in three-fourths of four balls. Students treating four balls as an operational unit responded correctly (coded 1) when their operations were followed by disembedding of three shaded-in balls in each operational unit. With either strategy, there was a strong underlying notion communicated in students’ responses that “the denominator is the whole” or “the denominator shows the number of items/pieces/parts in the whole.” Table 15 attempts to clarify students’ strategies by illustrating sample student responses and their respective coding (coded 0 and coded 1).
Table 15

*Examples of the Strategies and Unitizing Used for S.PWS*

<table>
<thead>
<tr>
<th>Response</th>
<th>Strategy</th>
<th>Operational Unit</th>
<th>Unit of 1</th>
<th>Sample Student Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>Treating four balls as an operational unit</td>
<td>Operational unit of four balls</td>
<td>Two groups of four balls (coded 1)</td>
<td><img src="image1.png" alt="Image" /></td>
</tr>
<tr>
<td>R2</td>
<td>Partitioning into four groups followed by the disembedding of three groups</td>
<td>Operational unit of two balls</td>
<td>Four groups of two balls (coded 1)</td>
<td><img src="image2.png" alt="Image" /></td>
</tr>
<tr>
<td>R3</td>
<td>Finding the equivalent fraction</td>
<td>Operational unit of eight balls</td>
<td>One group of eight balls (coded 1)</td>
<td><img src="image3.png" alt="Image" /></td>
</tr>
<tr>
<td>R4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Students who were unable to correctly answer Item #10 might have been challenged because they had conceptualized that the denominator of the fraction and unit of 1 were always the same for part-whole tasks. For instance, S11, a student who treated four of the balls as the unit of 1 and did not employ the disembedding operation, expressed her frustration as: “It is kind of confusing because there is *[sic]* four in a row and there is *[sic]* eight in all.” It seems that her reasoning aligns with the “out of”
interpretation of fractions (i.e., three balls are shaded out of four balls) uttered as “three-fourths tells me there is [sic] four in a row and you shade in three.” Had the student (i.e., S11) conceived that the sub-unit of four balls was not necessarily the unit of 1, then she might have been able to disembed three shaded-in balls in each sub-units.

Similar to S11, S13 unitized four balls as the unit of 1 and shaded in only three of eight balls. Furthermore, S13 claimed that six shaded-in balls would be six-fourths (i.e., 6/4) as shown in Figure 9.

Figure 9. An example of the unitizing of four balls as the unit of 1. S13’s answer (on the left) and account of six shaded balls as “6/4 = 1\frac{2}{4}” (on the right).

In addition, the following excerpt from my interview with S13 demonstrates how she treated four balls as the unit of 1 (i.e., the whole). It also indicates that the student is challenged to treat the collection of eight balls as a unit (i.e., whole) and the unitizing operation is problematic for the student.

Protocol I: S13’s explanation of four balls as the unit of 1.
S13: So, here is four, here is four (draws a circle around each group of four balls given in the model). (After shading in three of eight balls) This is three-fourths.

I: OK.

S13: And over here we shaded in three fourths.

I: OK. So, are you shading three from each? Or just three?

S13: Just three. If we did that, this here would be six-fourths. But that (6/4) is a bigger number. So, it would be one and one half. If six balls were shaded, that would be six-fourths or one and two-fourths which would be one and one-half.

**Partitive unit fraction scheme across model types.** The quantitative and qualitative analyses conducted to attend to the third research question in the context of Partitive Unit Fraction Scheme were presented in the following section. The statistics tests were conducted to probe students’ Partitive Unit Fraction Scheme scores and the strategies that were evident in students’ responses to three items corresponding to Partitive Unit Fraction Scheme were elaborated. The qualitative analysis of students’ strategies provided the opportunity to examine students’ Partitive Unit Fraction Scheme across three model types.

**Quantitative analysis.** The Friedman test at the item level (Items #2, 4, and 12) was conducted to investigate differences in PUFS scores among area, length, and set models. The Friedman test was significant, \( \chi^2(2, N=18) = 7.14, p < .05 \). The Kendall coefficient of concordance of .20 indicated a reasonably strong difference among three
types of fraction models for PUFS. Wilcoxon tests were used to follow up this finding with a Bonferroni procedure applied, resulting in a significance level set at $p < .017$. However, there was no significant difference between any pairs of models for Partitive Unit Fraction Scheme.

**Qualitative analysis.** Items #2, 4, and 12 were created to evaluate whether students’ PUFS responses differed in the context of different model types. The overall scores for PUFS items for each model type were: Area – 10, Length – 15, and Set –10. Responses to these items were coded either as 0 or 1 meaning that there were no partial scores for the responses to PUFS items. Consequently, 56% (10 of 18) of the student correctly answered A.PUFS and S.PUFS items, while 83% (15 of 18) of them correctly answered L.PUFS item.

**Area model for PUFS.** A.PUFS item demonstrated in Figure 10 entails replicating the given unit fractional piece four times to construct the unit of 1. In order to avoid fostering the part-whole understanding of the fraction and to prompt size comparison between the fractional unit and the whole, only the unit fraction was given in the item. Moreover, the dot paper allows counting of the unit squares for students who can interpret the area model as some number of unit squares. Otherwise, simply counting the number of unit squares in the fractional unit and the whole, and writing these numbers of unit squares as numerator and denominator respectively, would suffice to be coded as the correct answer. However, such reasoning is far from justified as being PUFS, which entails iterating in order to produce the whole.
Figure 10. The area model for PUFS (A.PUFS). Item #2 designed to elicit Partitive Unit Fraction Scheme with the area model.

Various types of strategies were observed which could be classified as: (a) replicating the given piece representing the fractional unit, (b) complementary interpretation of the other piece, (c) part-whole interpretation of the other piece, and (d) complementary part-whole interpretation of the other piece. Iteration of unit fraction occurred non-contextually or contextually in the responses. In order to produce the other piece (i.e., the whole), the student iterated the unit fractional part by replicating it until four of those pieces would produce the other piece. In this case, the student understands that denominator of the fraction indicates the number of iterations needed to construct the whole. This way of thinking was named non-contextual because the student did not refer to the particulars of the model such as comparisons of the number of the unit squares and the dimensions of the fractional unit and the whole. In Table 16, R1 is an instance of replicating the given piece (non-contextual). Use of the given piece as replicable units was also apparent in responses in which the context afforded by the area model was

The shaded piece is $\frac{1}{4}$ as big as another piece. Draw the other piece.
leveraged. In Table 16, R2 demonstrates replicating the given piece (contextual). R2 included replicating the fractional unit four times, and counting the unit squares inside to make sure there are four times as many as the given shaded piece. Nonetheless, both of these approaches utilized the unit fraction as a measurement unit to produce the unit of 1.

The analysis of responses to A.PUFS showed that some students constructed a separate notion of the whole and the other piece given in the item statement. For such students, the other piece asked in the A.PUFS item was the piece that makes a whole when added to the given shaded piece. In other words, when the other piece was asked, such students referred to the piece that corresponded to three-fourths instead of four-fourths. Responses compatible with this reasoning were gathered under *complementary interpretation of the other piece*. In Table 16, R3 is an example of this particular theme. S2 articulated her way of thinking as “You just add three more pieces around the square and that makes a whole.” However, she was able to acknowledge that the whole is the “four pieces altogether.”
Table 16

Examples of the Strategies Used for A.PUFS

<table>
<thead>
<tr>
<th>Response</th>
<th>Strategy</th>
<th>Sample Student Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>Replicating the given piece representing the fractional unit (non-contextual) (coded 1)</td>
<td><img src="image1" alt="Diagram" /></td>
</tr>
<tr>
<td>R2</td>
<td>Replicating the given piece representing the fractional unit (contextual) (coded 1)</td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
<tr>
<td>R3</td>
<td>Complementary interpretation of the other piece (coded 1)</td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
<tr>
<td>R4</td>
<td>Part-whole interpretation of the other piece (coded 0)</td>
<td><img src="image4" alt="Diagram" /></td>
</tr>
<tr>
<td>R5</td>
<td>Complementary part-whole interpretation of the other piece (coded 0)</td>
<td><img src="image5" alt="Diagram" /></td>
</tr>
</tbody>
</table>
One finding that emerged from the analysis of responses to A.PUFS was that approximately 39% of the students conceived the item as a part-whole problem and failed to reverse their part-whole understanding. R4 demonstrated in Table 16 is an instance of this reasoning. The item statement that includes “as big as another piece” was also challenging for some students. This statement communicated to students that the other piece should be smaller than the given shaded piece. S11 expressed her notion of the other piece as: “It [the other piece] can’t be bigger than this [the given shaded piece].” This particular way of reasoning could have resulted from students’ generalization of part-whole tasks in which they are asked to shade in a smaller part of the whole namely a proper fractional part. Similarly, R5 in Table 16 indicates the part-whole understanding; however, the interpretation of the other piece intermingles with the part-whole understanding. This theme was titled complementary part-whole interpretation of the other piece. S14 justified his operation as “the other piece is one fourth of the given piece taken away and therefore it is smaller of a piece.” Therefore, the other piece is what is left after one-fourth is taken away as illustrated by R5 in Table 16.

Length model for PUFS. As demonstrated in Figure 11, two bars were given so that students could find the size of the shorter bar relative to the longer bar. In order to identify the size of the shorter bar, students with PUFS iterate the smaller bar by repeating it within the longer bar until they realize that five of those shorter bars would produce the longer bar (i.e., the whole). These students understand that the number of iterations determine the size of the fractional unit relative to the unpartitioned whole (Steffe & Olive, 2010). According to Steffe (2002), PUFS “establishes one-to-many
relation between the part and the partitioned whole” (p. 292). Therefore, PUFS includes unit coordination at one level between the unit fraction and the whole. For the length model, 83% of the students’ work indicated ways of operating that are consistent with PUFS.

If the longer bar is considered one whole, what fraction is the shorter bar?

Figure 11. The length model for PUFS (L.PUFS). Item #12 designed to elicit Partitive Unit Fraction Scheme with the length model.

Table 17 demonstrates some of the sample work gathered through interviews. Three main strategies were apparent in correct responses: (a) iterating the shorter bar, (b) partitioning the shorter bar followed by the iteration of the new unit, and (c) partitioning the longer bar followed by “two-count strategy.” Several responses that did not utilize only the iteration of the smaller bar, but showed a strong evidence of PUFS, were also given a full score (i.e., coded 1; see R3 and R4 in Table 17).

Students who iterated the smaller bar to see how many times it would fit in the longer bar used the smaller bar (i.e., unit fraction) to reestablish the whole so that they could measure the whole bar in terms of the shorter bar. In this regard, it was observed that these students used the shorter bar as a nonstandard measurement unit. Students with iteration strategy expressed an awareness of the connection between the iteration and
Table 17

*Examples of the Strategies Used for L.PUFS*

<table>
<thead>
<tr>
<th>Response</th>
<th>Strategy</th>
<th>Sample Student Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>Iterating the shorter bar (coded 1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td><img src="image1.png" alt="Image of student response" /></td>
<td></td>
</tr>
<tr>
<td>R2</td>
<td></td>
<td><img src="image2.png" alt="Image of student response" /></td>
</tr>
<tr>
<td>R3</td>
<td>Partitioning the shorter bar followed by the iteration of “the new unit” (coded 1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td><img src="image3.png" alt="Image of student response" /></td>
<td></td>
</tr>
<tr>
<td>R4</td>
<td>Partitioning the longer bar into ten followed by “two-count strategy” (coded 1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td><img src="image4.png" alt="Image of student response" /></td>
<td></td>
</tr>
<tr>
<td>R5</td>
<td>Partitioning both of the bars (coded 0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td><img src="image5.png" alt="Image of student response" /></td>
<td></td>
</tr>
<tr>
<td>R6</td>
<td>(coded 0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td><img src="image6.png" alt="Image of student response" /></td>
<td></td>
</tr>
<tr>
<td>R7</td>
<td>Partitioning the shorter bar (coded 0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td><img src="image7.png" alt="Image of student response" /></td>
<td></td>
</tr>
</tbody>
</table>
equi-partitioning. In other words, those students were able to identify that the iteration of the shorter bar resulted in identical pieces within the longer bar. In Table 17, R1 and R2 are examples of iteration reasoning used in the context of L.PUFS.

Some students did not start with the iteration of the shorter bar; instead, they first partitioned the shorter bar in half. Then, they iterated this resultant half bar and used it as a measurement unit to decide the size of the shorter bar. In Table 17, R3 indicates such an instance of partitioning followed by iteration. Even though the answer was incorrect, her reasoning allowed her to estimate size of the shorter bar relative to the given whole. Therefore, R3 was given a full score (i.e., coded 1). A student who used partitioning followed by the iteration of the new unit explained her reasoning as:

Technically, I could have as many as parts I wanted to. It could have been one. I just decided to make it two. And then you just keep dividing up by about the size of the thing [the half bar resulted from partitioning] and count them up. (S8, 01/27/2016)

Student R4 in Table 17 is an example of a student who divided the longer bar into 10. She started by marking the ends of the longer bar 0 and 10. When asked the reason why she partitioned the longer bar into 10, she claimed naming the fraction was easy when the whole was 10 or 100. One of the limitations of this particular strategy is that the whole is limited to being partitioned into 10, and so it works for fractions which their denominators are multiples or factors of 10. For the L.PUFS item, however, this strategy helped the student obtain a correct answer. After partitioning the longer bar, the student partitioned the shorter bar so that the shorter bar aligned with first two pieces within the
longer bar as demonstrated in Table 17 (see R4). Then, the student counted the number of pieces in the unit fractional part (i.e., two) and in the whole (i.e., 10) by which she concluded that the smaller bar was two-tenths. This strategy could be described as “two-count strategy” (Saxe et al., 2007) in the context of L.PUFS.

A particular response (see R2 in Table 17) warrants further consideration. R2 was the only response in which both percents and fraction forms were used. One possible reason why the student preferred to give her answer in percents could be the proximity of the tutoring she had been given on percents, decimals, and fractions. However, there was another student who was also in the same tutoring program, but he did not use percents in his responses at all.

One of the main qualitative differences between L.PUFS and the other two model types was students’ discourse. The context of the students’ discourse shifted to include such terms as “estimate, predict, measure” uttered by students while working on L.PUFS item. The actions they performed also reflected those references to estimation and measurement. For instance, some students used their knuckles or the tip of their pencils to estimate the longer bar while trying to draw the bars identical to the shorter bar. In Table 17, R4 also indicates how the student conceptualized the given bar to have measurement units such as inch and centimeter. Hence, the student whose response demonstrated in R4 argued that the longer bar might represent a length of 10 inches or centimeters, so the 2-cm or 2-inch shorter bar would be two-tenths or “two out of ten.”

Set model for PUFS. In order to accommodate the diversity in representing fractions for Partitive Unit Fraction Scheme, a set of six balls was given as a fractional
unit (Figure 12). S.PUFS allows students to use the unit as an iterable unit to find the whole (Steffe & Olive, 2010). However, for similar reasons articulated in the area model for PUFS section, the whole was referred to as the other set and the only given was the unit fraction represented by a set of six balls.

![Diagram](image)

If these balls are \( \frac{3}{4} \) as many as another set of balls, how many balls are there in the other set?

**Figure 12.** The set model for PUFS (S.PUFS). Item #4 designed to elicit Partitive Unit Fraction Scheme with the set model.

Approximately 56% of the students were able to answer S.PUFS correctly. The overall achievement on S.PUFS and A.PUFS items were similar. Six students answered neither of these two items whereas eight students correctly answered both. Four of the students were successful in either S.PUFS or A.PUFS. Several strategies were observed which could be classified as: (a) replicating the number of balls representing the unit fraction, (b) partitioning the unit fraction in half followed by replicating the half, (c) complementary interpretation of the other set, (d) part-whole interpretation of the other set, and (e) complementary part-whole interpretation of the other set.

Iterating the set of six balls six times by drawing (see R1 in Table 18) or writing number sentences representing the iteration (see R3 and R4 in Table 18) were two common strategies observed in correct responses. Treating the set of six balls as a unit
Table 18

*Examples of the Strategies Used for S.PUFS*

<table>
<thead>
<tr>
<th>Response</th>
<th>Strategy</th>
<th>Sample Student Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>Replicating the number of balls representing the unit fraction (coded 1)</td>
<td><img src="image1" alt="Image" /></td>
</tr>
<tr>
<td>R2</td>
<td>Partitioning the unit fraction in half followed by replicating the half of the unit fraction (coded 1)</td>
<td><img src="image2" alt="Image" /></td>
</tr>
<tr>
<td>R3</td>
<td>Replicating the unit fraction represented with a number sentence (coded 1)</td>
<td><img src="image3" alt="Image" /></td>
</tr>
<tr>
<td>R4</td>
<td>Complementary interpretation of the other piece represented with number sentences (coded 1)</td>
<td><img src="image4" alt="Image" /></td>
</tr>
<tr>
<td>R5</td>
<td>Representing with number sentences (coded 1)</td>
<td><img src="image5" alt="Image" /></td>
</tr>
<tr>
<td>R6</td>
<td>Representing with number sentences (coded 1)</td>
<td><img src="image6" alt="Image" /></td>
</tr>
<tr>
<td>R7</td>
<td>Part-whole interpretation of the other set (coded 0)</td>
<td><img src="image7" alt="Image" /></td>
</tr>
<tr>
<td>R8</td>
<td>Complementary part-whole interpretation of the other set (coded 0)</td>
<td><img src="image8" alt="Image" /></td>
</tr>
</tbody>
</table>
and then iterating the unit six times, the students arrived at the whole set of 36 balls. There is a particular response that is different than iterating the set of balls six times. Instead, the student first partitioned the given set in half and iterated the half set (i.e., three balls) to produce the whole (see R2 in Table 18).

Students with the correct responses seemed to have an understanding of coordination between the unit fraction and the whole. However, for students who treated the set of six balls as a unit of 1 (i.e., the whole), the part-whole interpretation took over and no iterations occurred. Therefore, the other set was concluded to be a set of one ball by those students (see R7 in Table 18). The complementary interpretation of the other set was also apparent in S.PUFS. R4 in Table 18 indicates the separate notion of the whole and the other set. Even though students with such reasoning concluded that there were 30 balls in the other set, they were able to identify that 36 balls comprised the whole. Moreover, complementary part-whole interpretation of the other piece was observed in which five balls were conceived as the other set after one ball was taken away (see R8).

The S.PUFS item seems to have prompted more arithmetic responses than A.PUFS and L.PUFS. Number sentences provided in Table 18 demonstrate some of these arithmetic responses. As seen in Table 18, some number sentences were made of whole numbers, whereas some involved fractions. Students who used whole numbers were able to explain their reasoning in more comfortable way. When asked to justify their answers, some students appeared to struggle with representing the situation given in the item to their number sentences with fractions. For example, R6 includes a number sentence coded as 1 (i.e., correct response). However, it was not clear whether the
student construed the problem situation as partitive division (i.e., the amount per group) or measurement division (i.e., the number of groups; Jansen & Hohensee, 2015). The second line of operation in R6, which the student described as “backwards” operation, served as a verification of former division operation for the student.

**Partitive fraction scheme across model types.** The following part involves quantitative and qualitative analyses conducted to address the third research question in the context of Partitive Fraction Scheme. The statistics tests allowed for examination of students’ Partitive Fraction Scheme scores and the strategies that were apparent in students’ responses to three items corresponding to Partitive Fraction Scheme were elaborated. The qualitative analysis of students’ strategies presented the opportunity to investigate students’ Partitive Fraction Scheme across three types of model.

**Quantitative analysis.** The Friedman test at the item level (Items #1, 5, and 7) was conducted to investigate differences in PFS scores among area, length and set models. The Friedman test was significant, $\chi^2(2, N=18) = 6.20, p < .05$ and the Kendall coefficient of concordance of .17 indicated fairly strong differences among three model types for PFS. Wilcoxon tests were used to follow up this finding by using a Bonferroni procedure, resulting in a significance level set at $p < .017$. However, there was no significant difference between pairs of model types for Partitive Fraction Scheme.

**Qualitative analysis.** Items #1, 5, 7 entail reconstructing the unit of 1 (i.e., the whole) from proper fractions. The overall scores for PFS items for each model type were: Area – 8.5, Length – 12, and Set – 8. Responses to these items were coded as 0, .5 or 1 meaning that there were partial scores for the responses to PFS items. Consequently,
47% of the student were able to correctly answer A.PFS item, 44% gave a correct answer to S.PFS item and 67% correctly answered L.PFS item.

*Area model for PFS.* The A.PFS item entail partitioning the given piece to find the unit fraction and replicating the unit fraction five times to construct the whole piece. In order to avoid triggering the part-whole understanding of the fraction and to prompt size comparison between fractional part and the whole, only the fractional part represented by the given shaded piece was given as demonstrated in Figure 13.

![The shaded piece is \( \frac{3}{5} \) as big as another piece. Draw the other piece.](image)

*Figure 13.* The area model for PFS (A.PFS). Item #7 designed to elicit Partitive Fraction Scheme with the area model.

Various types of strategies were observed which could be categorized as: (a) *partitioning followed by iteration (PFI)*, (b) *complementary interpretation of the other piece*, (c) *part-whole interpretation of the other piece*, and (d) *complementary part-whole interpretation of the other piece*. Table 19 demonstrates sample responses that could be classified into one of these categories. Students with the first strategy partitioned the shaded piece to see the number of iterations (corresponding to the numerator of the fraction) of the fractional unit (i.e., \( 1/5 \)). Then, they iterated the piece representing the
unit fraction five times to produce the other piece (i.e., whole piece; see Figure 14). Several students operated in a similar way by representing their operations with number sentences with fractions or whole numbers. These number sentences (e.g., R2 and R3 in Table 19) were written by students who interpreted the area of given model as the number of unit squares within the shape or the field covered by some numbers of unit squares.

![Partitioning followed by iteration (PFI) in A.PFS.](image)

*Figure 14.* Partitioning followed by iteration (PFI) in A.PFS. Partitioning the given shaded piece into three to find the unit fraction followed by the iteration of the unit five times to produce the whole.

Responses to A.PFS showed that some students constructed a separate notion of the whole and the other piece given in the item statement. It meant, to them, that the other piece makes a whole when added to the given piece. Responses compatible with this reasoning were gathered under *complementary interpretation of the other piece.* In Table 19, R4 is an example of this particular theme. S10 elaborated her interpretation of the other piece as:
Table 19

*Examples of the Strategies Used for A.PFS*

<table>
<thead>
<tr>
<th>Response</th>
<th>Strategy</th>
<th>Sample Student Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>Partitioning followed by the iteration (PFI) (coded 1)</td>
<td><img src="image1.jpg" alt="Diagram 1" /></td>
</tr>
<tr>
<td>R2</td>
<td>PFI-Represented by a number sentence with fractions (coded 1)</td>
<td><img src="image2.jpg" alt="Diagram 2" /></td>
</tr>
<tr>
<td>R3</td>
<td>PFI-Represented by number sentences with whole numbers (coded 1)</td>
<td><img src="image3.jpg" alt="Diagram 3" /></td>
</tr>
<tr>
<td>R4</td>
<td>PFI-Complementary interpretation of the other piece (coded 1)</td>
<td><img src="image4.jpg" alt="Diagram 4" /></td>
</tr>
<tr>
<td>R5</td>
<td>Complementary interpretation of the other piece (coded 0)</td>
<td><img src="image5.jpg" alt="Diagram 5" /></td>
</tr>
<tr>
<td>R6</td>
<td>Part-whole interpretation of the other piece (coded 0)</td>
<td><img src="image6.jpg" alt="Diagram 6" /></td>
</tr>
<tr>
<td>R7</td>
<td>Complementary part-whole interpretation of the other piece (coded 0)</td>
<td><img src="image7.jpg" alt="Diagram 7" /></td>
</tr>
</tbody>
</table>
When I read this question, I think that they want me to draw the other piece that makes it a whole. So, I would think this [pointing at the unshaded piece shown in R4 in Table 25] would be the other piece that makes it a whole. (S10, 01/28/2016)

Several students recognized the other piece as the complementary piece were coded as 0 since these students were focused on the other piece being two-fifths without considering the particulars of the given shape (see R5 in Table 19).

One of the challenges students encountered in A.PFS was recognizing that the given shaded piece referred to three-fourths of the other piece (i.e., of the whole piece). Approximately 22% of the students, however, conceived the given piece as the whole and moved forward to find the three-fifths of it (e.g., R6 in Table 19). Such conceptions indicate that these students interpreted the item as a part-whole task in which they are asked to find the fractional part of a given whole. Similarly, R7 in Table 19 demonstrates the part-whole understanding; however, the interpretation of the other piece seems to intervene with the part-whole understanding. This particular strategy was classified as complementary part-whole interpretation of the other piece. Therefore, the other piece was resulted from taking three-fifths of the given piece away.

One particular response required further consideration (see Figure 15). The student whose response is shown in Figure 15 seems to operate in a consistent way with PFS. The response included partitioning followed by iteration represented by the number sentences. The student, however, operated with the total number of dots and concluded that the other piece should consist of 40 dots.
Students operated with the given area model in three different ways by attending to three different aspects of the model: (a) the number of the unit squares, (b) the distance between the dots (i.e., dimension of the model), and (c) the number of dots on the sides or total number of dots within the model. Overall analysis of the results (see Table 20) shows that interpretation of area as the number of squares or interpretation of the dimension is necessary but not sufficient to be able to solve A.PFS item.

Table 20

*The response in this cell was coded as 0.5 (see Figure 15).
Length model for PFS. The longer bar was given as a whole and the relative size of the shorter bar was asked to evaluate students’ Partitive Fraction Scheme as shown in Figure 16. Students who have constructed PFS can operate with unit fractions to find out the relative size of a proper fractional part.

If the longer bar is considered one whole, what fraction is the shorter bar?

Figure 16. The length model for PFS (L.PFS). Item #5 designed to elicit Partitive Fraction Scheme with the length model.

Two main strategies apparent in students’ correct responses were iterating the “extra piece” and partitioning followed by “two-count strategy.” Table 21 demonstrates some of the sample responses. Several responses that used only the partitioning operation, but showed partial evidence of PFS were given a partial score (i.e., coded .5) (see R3, R4, R5, and R6 in Table 21). The crucial difference between the two strategies was the operation by which students determined the unit fraction. The former strategy aims at partitioning through iteration by using the unit fraction as a measurement unit. On the other hand, the latter strategy intends to partition without a reference point to determine the size of the unit.
Students who iterated the “extra piece” to see how many times it would fit in the longer bar used it as a unit fraction to find out the relative size of the smaller bar. In this regard, it was observed that these students used the length difference between bars as a nonstandard measurement unit. Students with iteration strategy expressed an awareness of the connection between the iteration and equi-partitioning. In other words, those students were able to identify that the iteration resulted in identical pieces within the longer bar. In Table 21, R1 and R2 are examples of the iteration of the unit fraction reasoning. Even though R2 is not the correct answer, it was given a full score because the student was able to coordinate unit fraction with proper fraction (i.e., smaller bar) and the whole (i.e., longer bar) simultaneously. Such reasoning demonstrates unit coordination at two levels. The student articulated his way of thinking as:

I am trying to think of something [that] I can use to determine how wide this [shorter bar] is. I kind of estimated the space in between the longer bar and the shorter bar. This [shorter bar] obviously has to be broken down to something. That piece [referring to the length difference] could go into the shorter bar five times. (S8, 01/27/2016)
### Examples of the Strategies Used for L.PFS

<table>
<thead>
<tr>
<th>Response</th>
<th>Strategy</th>
<th>Sample Student Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>Iterating the “extra piece” (coded 1)</td>
<td><img src="image1.png" alt="Diagram" /></td>
</tr>
<tr>
<td>R2</td>
<td></td>
<td><img src="image2.png" alt="Diagram" /></td>
</tr>
<tr>
<td>R3</td>
<td>Partitioning the longer bar into ten followed by “two-count strategy” (coded .5)</td>
<td><img src="image3.png" alt="Diagram" /></td>
</tr>
<tr>
<td>R4</td>
<td>Partitioning both of the bars simultaneously followed by “two-count strategy” (coded .5)</td>
<td><img src="image4.png" alt="Diagram" /></td>
</tr>
<tr>
<td>R5</td>
<td>Partitioning the bars sequentially followed by “two-count strategy” (coded .5)</td>
<td><img src="image5.png" alt="Diagram" /></td>
</tr>
<tr>
<td>R6</td>
<td>25% used as a benchmark to measure the “extra piece” (coded .5)</td>
<td><img src="image6.png" alt="Diagram" /></td>
</tr>
<tr>
<td>R7</td>
<td>No partitioning or no iterating (coded 0)</td>
<td><img src="image7.png" alt="Diagram" /></td>
</tr>
</tbody>
</table>
Some students did not start with the iteration of the length difference; instead, they started with partitioning the bars sequentially (i.e., first the longer bar and then the shorter bar) or both of the bars simultaneously (e.g., R4 in Table 21). Then, they counted the number of pieces within each of the bars (i.e., two-count strategy) to determine the size of the shorter bar. Even though R3, R4, R5, and R6’s strategies resulted in an incorrect answer, the underlying reasoning might be taken as a partial evidence for PFS. Their strategies did not iterate the unit fraction that needs to be determined before partitioning. Nonetheless, those responses were not far from being a descent estimation of the size of the shorter bar relative to the given whole. Therefore, they were given a partial score (i.e., coded .5).

Two students divided the longer bar into 10 smaller pieces and marked the ends of the each piece. Similar to what was observed in L.PUFS item, these students utilized ten-tenths as the whole bar as a benchmark. One of the limitations of this particular strategy is that the whole is limited to be partitioned into 10, and so it works for proper fractions whose denominators are multiples or factors of 10. For the L.PFS item, however, this strategy did not result in a correct answer. One of the students explained her reasoning as:

I started with zero [then drew partition marks until the end of the longer bar and she was trying to have ten pieces]. Eight would be the measure of the shorter bar and then ten would be the whole. The measure [of the shorter bar] is eight out of ten. But, to figure out the shorter bar, we have to figure out this bar [the longer
bar] and I put the bar at ten and I measured it and the length is eight and it would be eight out of ten. (S4, 01/26/2016)

R6 was the only response that was given in percents by the same student mentioned in L.PUFS. Therefore, one might think L.PFS might have also prompted the use of percents, considering that this particular student did not utilize percents in any of the items except L.PUFS and L.PFS. In addition, estimating efforts was also evident in L.PFS as well as in L.PUFS. Some students even claimed that their answer could not be definite. The following is a remark from a student who conceived L.PFS item as an estimation task: “So, you would get five-sixths or around there . . . since it is estimating I cannot be exactly positive” (S8, 01/27/2016).

Set model for PFS. The set of 15 balls was given as a fractional part that allows students to find the unit fraction and then iterate it to find the whole as demonstrated in Figure 17. However, for similar reasons articulated in the area model for PFS, the whole was referred to as the other set and those 15 balls were given as the proper fractional part (i.e., three-fifths) of the other set.

![Diagram](image)

If these balls are \( \frac{2}{5} \) as many as another set of balls, how many balls are there in the other set?

Figure 17. The set model for PFS (S.PFS). Item #1 designed to elicit Partitive Fraction Scheme with the set model.
Students employed different strategies that could be categorized as: (a) *partitioning followed by iteration (PFI)*, (b) *complementary interpretation of the other piece*, and (c) *part-whole interpretation of the other piece*. Table 22 demonstrates sample responses that could be classified into one of those categories. Students with the partitioning followed by iteration strategy partitioned the given set to find the number of iterations of the fractional unit (i.e., one-fifth). Then, they iterated the balls representing the unit fraction five times to produce the other set (i.e., the whole; see Figure 18). It demonstrates unit coordination at two levels. In other words, students coordinate three-fifths as three iterations of one-fifth and the whole set as five iterations of one-fifth.

![Figure 18](image_url)

*Figure 18.* Partitioning followed by iteration (PFI) in S.PFS. Partitioning the given set into three to find the unit fractional part followed by iteration of the unit fraction five times to produce the whole.

Several students operated in a similar way by representing their operations with number sentences with fractions or whole numbers. These number sentences (e.g., R2, R3, and R4 in Table 22) were used by students who interpreted the given balls as the three-fifths of the other set. However, it was not always the case that students construed the set of 15 balls as a unit of three fractional units. Instead, some students interpreted
the given set of 15 balls as the unit of 1 and found the three-fifths of the set (e.g., R6 in Table 22).

Table 22

*Examples of the Strategies Used for S.PFS*

<table>
<thead>
<tr>
<th>Response</th>
<th>Strategy</th>
<th>Sample Student Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>Partitioning followed by iteration (PFI) (coded 1)</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>R2</td>
<td>PFI-represented by a number sentence with fractions (coded 1)</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>R3</td>
<td>PFI-represented by number sentences with whole numbers (coded 1)</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>R4</td>
<td>PFI-complementary interpretation of the other piece (coded 1)</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>R5</td>
<td>Complementary interpretation of the other piece (coded .5)</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>R6</td>
<td>Part-whole interpretation of the other piece (coded 0)</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
</tbody>
</table>

Responses to S.PFS also showed that some students constructed a separate notion of the whole and the other set given in the item statement. Similar to above mentioned
items, for these students the other set makes a whole when added to the given 15 balls. Responses compatible with this reasoning were gathered under complementary interpretation of the other set. In Table 22, R4 and R5 are examples of this particular conception. The response given in R5 was coded as 0.5 by both of the raters since she was did not produce the whole set (i.e., the set of 25 balls).

Reliance on number sentences was more abundant in responses to S.PFS compared to L.PFS. Moreover, some students were observed to reason in ways that allows them to find the answer by writing two number sentences and then compare the size of the resultant sets. Figure 19 shows those number sentences. These students stated that the given set was already three-fifths and there were 15 balls, so the other set should be more than 15 balls. After writing two number sentences, which gave them a set of nine balls and a set of 25 balls, they compared the number of balls in these sets with 15 balls. Then, they moved forward with 25 balls as their ultimate answers.

Number sentence I:

\[
\frac{15}{3} \times 3 = \frac{45}{5} = 9
\]

Number sentence II:

\[
\frac{16}{2} \times \frac{5}{3} = \frac{76}{3} = 25
\]

*Figure 19.* Number sentences used to answer S.PFS. Interpretation of the other set as a bigger set by comparing sets of 9 and 25 balls with the given set of 15 balls.
CHAPTER V
DISCUSSION, IMPLICATIONS AND LIMITATIONS

In this study, data analyses indicated that students’ fraction schemes form a developmental trajectory as theorized in complementary research (Olive, 1999; Steffe, 2002; Steffe & Olive, 2010). Existing fraction scheme and operation frameworks, described in Chapter 2, were used to construct the items utilized in this study and to analyze the student interviews. More sophisticated fraction schemes, such as Partitive Unit Fraction and Partitive Fraction Schemes, appeared unavailable to some sixth grade students in the sample. On the other hand, the three model types (i.e., length, area, and set) provided diverse contexts for students that triggered different fractional operations and ways of thinking. Furthermore, certain models were more accessible to students who had yet to develop specific fraction schemes. Individual student interviews, and models of students’ ways of thinking as they engaged with and solved the 12 items, revealed aspects of students’ conceptions that had not previously been identified by studies that focused on students’ written performance on tasks (Norton & Wilkins, 2009, 2010; Tunc-Pekkan, 2015).

This study was guided by the following three research questions:

1. How do sixth grade students’ responses to items that correspond to the development of a particular scheme compare and contrast?

2. How do sixth grade students’ responses to items involving a particular model type compare and contrast?

3. Do certain types of models facilitate the development of certain schemes?
Research Question 1

The overall decrease in the percentage of correct responses to Partitive Unit Fraction and Partitive Fraction Scheme items, compared to Partitioning and Part-Whole Scheme items, requires elaboration. Regardless of the model type, students’ performance on items with more advanced fraction schemes (i.e., PUFS and PFS), defined by the Fractional Scheme Theory, was lower than their performance on items related to Partitioning and Part-Whole Schemes. However, the dramatic decline in students’ performance on items referring to partitive fraction knowledge in comparison to Partitioning and Part-Whole Schemes items is not surprising. Prior research has also marked relatively low achievement in more complex schemes (Norton & Wilkins, 2009, 2010; Steffe & Olive, 2010; Tunc-Pekkan, 2015).

As seen in Table 7 in Chapter 4, all four schemes were not available to all sixth grade students in the sample. For instance, all students with Partitive Fraction Scheme seem to have also constructed Partitive Unit Fraction, Part-Whole and Partitioning Schemes as projected in the learning trajectory. On the other hand, some of the students who had already constructed the Part-Whole Scheme did not seem to develop their Partitive Unit Fraction and Partitive Fraction Schemes. Partial scores (i.e., coded 0.5) were more abundant for responses to PFS items suggesting these students were in the process of constructing their Partitive Fraction Scheme. Moreover, even those students with Partitioning and Part-Whole Schemes—deemed to be operationally similar schemes (Norton & Wilkins, 2009)—appeared to conceptualize these two ways of operating in different ways depending on the model type. Based on the overall percentages of correct
responses and analysis of students’ responses, sixth grade students in this study performed differently with the Part-Whole Scheme items than they did with Partitioning Scheme. However, the difference was not statistically significant.

**Research Question 2**

Students’ performance on set models was lower compared to their corresponding performances on area and length models, regardless of the corresponding fraction scheme. Unitizing of set models (i.e., treating some numbers of objects as a unit of 1) seemed to be more challenging for students, especially when the number of objects that constituted the whole differed from the magnitude of the denominator given in the item (e.g., “Shade 1/5 of ten balls” as in S.PS, item #3, in this study). Such students might have been challenged because they had reasoned that the denominator of the fraction and unit of 1 were always the same for partitioning and part-whole tasks.

On the other hand, length models seemed to be more accessible to students, resulting in higher performance on length model items. Regardless of the fraction schemes being measured, students’ performances were significantly different for length and set models across scheme types. Percentages of correct scores for length model items were consistently higher (at least 22%) than those of set models. Analyses of students’ written work and verbal responses suggest the use of length models might trigger students’ fraction schemes across all four types. The results of previous research (i.e., Tunc-Pekkan, 2015) also indicated that linear continuous representation (i.e., corresponding to the length model in this study) could be used as a model that facilitates
the activation of fraction schemes. However, such results do not indicate that the length model is the only model which teachers should build their entire curriculum.

Different features (i.e., features depending on how the whole and the fraction are defined within the model) of model types might activate students’ fraction reasoning and help develop their capacities to coordinate these features with their fraction knowledge addressed in the item. Furthermore, it seems reasonable to expect that the use of a single type of model might be insufficient to afford students the opportunities to operate in ways consistent with fractional schemes across all model types. Working with one type of fraction representation could lead students to construct their fractions knowledge based on particular aspects of that model type.

Students’ frequent remarks about the meaning of the numerator and the denominator (i.e., numerator *always* indicates the number of pieces shaded and the denominator shows the number of pieces in the whole) might be an instance of overgeneralization with length models (e.g., tape diagrams) frequently used in part-whole tasks. Results presented here indicate that students’ conceptions resulted from working with a single model type; furthermore, such conceptions might hinder their capability to work with composite discrete units such as the set models. Therefore, providing students with opportunities to interact with different kinds of models (i.e., area, length, and set models) has the potential to help students develop more robust understandings of the fraction concept due to experiencing a wider variation of situations.
Research Question 3

Strategies used by students were considered as indications of their fraction conceptions and used to analyze the data to answer the third research question. Investigating students’ responses to distinct models representing the same fraction scheme indicated that students’ overall performance and strategies differed relative to the type of fraction model. For example, students performed better on the area model for Partitioning Schemes; however, their performances on the set model used to elicit Partitioning Schemes were considerably lower (i.e., 28% lower). Similar discrepancies were observed across the other three fraction schemes. This suggests the differences in performance resulted from the model types used to represent the scheme.

The item statements with set model representation for schemes at the lower level of hierarchy, such as Partitioning and Part-Whole Schemes, required students to operate with units of units—operations not expected of them in PS and PWS Schemes according to Fractional Scheme Theory. Accordingly, students’ conceptions of unitizing appeared to impact their strategies; that is, their decisions to operate with either some number of parts or unit squares (e.g., five parts or 25 unit squares as the unit of 1) within area models or groups of balls (e.g., two groups of five balls as the unit of 1) within set models for Partitioning and Part-Whole Schemes. In other words, these composite units triggered students’ unitizing operation as evidenced in their responses to A.PS, S.PS, A.PWS and S.PWS (items #9, 3, 11, and 10, respectively). Students appeared challenged to conceptualize composite units since the use of composite units (i.e., units of units) entailed operating with more than one level of units, even though they correctly answered
items with continuous single units (i.e., length models for Partitioning and Part-Whole Schemes).

Most of the responses to the S.PWS item (item #10) constitute a way of operating that does not seem to follow PWS’s description afforded by Fractional Scheme Theory; specifically, that students “produce $m/n$ by partitioning a whole into $n$ parts” and “disembed $m$ of those parts” (Norton & Wilkins, 2009, p. 154). The way students treating four balls as an operational unit operated can be described as dividing the whole set into the sets of $n$ (i.e., four) and then disembedding $m/n$th of each set.

The reason for the operational deviation observed in S.PWS (item #10) may be that in developing the Fraction Scheme Theory, students participating in the teaching experiments had worked with continuous models rather than discrete models. Only one student in this study operated in a way that could be described as compatible with the Fractional Scheme Theory (see Table 3 in Chapter 2). This particular response is R3 in Table 15 in Chapter 4.

In the context of Partitive Unit Fraction and Partitive Fraction Schemes, length models seem to activate students’ estimation skills and reasoning, whereas area and set models seem to foster the use of students’ whole number knowledge. Number sentences with whole numbers might indicate a developing progression from whole numbers to fraction knowledge as theorized by Steffe and Olive (2010). Approximately 67% of the students who had not yet constructed the PFS (i.e., their total scores were 1 out of 3 or lower) were able to get a partial or a full score for their responses to L.PFS (item #5)—
22.2% received full and 44.4% received partial scores. Such results suggest using length models as stepping stones toward the construction of the PFS.

There was overarching part-whole reasoning demonstrated in students’ responses to Partitive Unit Fraction and Partitive Fraction Schemes in that some students counted the number of parts in the fractional part and the whole. Even though relatively higher percentage of the students were able to answer L.PUFS and L.PFS (items #12 and 5, respectively), there is not enough evidence to claim these students had already constructed a multiplicative relationship between the unit fraction and the whole or the unit fraction and the fractional part. Some students still relied on their part-whole reasoning in that they count the number of pieces in fractional part and the whole. On the other hand, a fewer number of students who correctly answered L.PUFS and L.PFS demonstrated the multiplicative use of the unit fraction and used it as a measurement unit.

**Implications for Future Fraction Instruction**

Results from this study inform instructional practices that allow for variation in model types and ways of operating described in fraction schemes. In that regard, results presented here suggest the frequent use of set models to develop students’ unitizing operation. In particular, the length model for Partitive Unit Fraction Scheme was observed to trigger students’ estimation reasoning and elicit size comparisons between the unit fraction and the whole.

For students who are able to interpret the relationship between the number of iterations and the size of the unit fraction, the length model in the context of Partitive Unit Fraction Scheme could be utilized to compare the sizes of different unit fractions.
Teachers can introduce length models for Partitive Unit Fraction Scheme as a task to help their students determine the relative size of unit fractions. In addition, motivating students to compare two unit fractions at a time should help them conceptualize the reason why the bigger the denominator is, the smaller the unit fraction is. Such tasks, as exemplified in Figure 20, could help activate students’ unit coordinations and size comparisons between two unit fractions.

![Figure 20](image)

*Figure 20.* A sample length model for activating students’ iterating operation and size comparison of unit fractions. Such length models could help students build on their PUFS and compare two unit fractions represented by two short bars in reference to the longer bar.

Item statements (i.e., A.PUFS, A.PFS, S.PUFS and S.PFS) that included the idioms “as many as” and “as big as” puzzled a considerable number of students in this study. Consequently, performance on items containing these idioms was at least 20% lower than their length model counterparts. Therefore, results from this study suggest that teachers should foster the use of multiplicative language (e.g., “five times . . .”, “three-fifths as big as . . .”) as opposed to additive language (e.g., “five more pieces . . .”) to explain the relationship between the fractional quantities.
A.PUFS, S.PUFS, A.PFS and S.PFS (items #2, 4, 7, and 1, respectively) were used to elicit size comparisons between the fractional part and the whole. Students’ responses to these items indicated that some students constructed separate notions of the whole and the other set or piece given in the item statement. Such results suggest teachers should provide students with tasks that involve students recognizing the whole themselves without explicit referral to the whole in the task statement.

**Limitations**

Although results from this study help to make students’ conceptions of fractions explicit, there are some limitations that must be detailed. One of the most important limitations in the results of this study involves the lack of generalizability due to sample size. The study was conducted on a small percentage (7.4%) of the overall sixth grade student population at Oak Grove Middle School. However, the ethnical demographics of participating students were close to being representative of the entire student population at Oak Grove Middle School (See Chapter 3). On the other hand, the sample was not necessarily representative of sixth grade students in the United States in terms of students’ proficiency level, ethnic, and socio-economic factors. Therefore, there is a need for a follow up study involving a larger, more representative, sample to determine whether the findings presented here remain consistent.

Another limitation involves coding of the qualitative data. Although scoring of the responses to items was triangulated with a second rater, a similar procedure was not conducted for the coding of the qualitative data (i.e., students’ verbal responses, strategies, and actions). Involving a second investigator to develop themes through
independent analyses would enhance the trustworthiness of the study. Furthermore, complementary sources of data (e.g., multiple interview sessions with each participating student or a comprehensive written assessment involving more items) could promote the reliability of the results.

Based on the number sentences produced by students for Partitive Unit Fraction and Partitive Fraction Schemes items, results from this study suggest further research of students’ use of number sentences with fractions versus whole numbers to investigate possible conceptual differences between them. Moreover, items corresponding to Iterative Fraction Scheme (IFS) could be included to probe whether there are differences in students’ IFS scores and conceptions of improper fractions represented with different model types.

**Conclusion**

This study contributes new insights into students’ conceptions of fractions and can be used by mathematics teachers and teacher educators to inform future fraction instruction and instructional interventions. Students’ conceptions, as presented here, can provide researchers and teachers with a more in-depth understanding of students’ fraction schemes and students’ interpretations of fraction models “within the subject’s range of experience” (Glasersfeld, 1996). Middle grades teachers should be aware of the affordances and constraints of the fraction models they employ in their instruction in the context of each particular scheme. The quantitative and qualitative analyses presented in this study indicate that students’ scores and responses differ in regards to the fraction
scheme, the type of the model used, and in particular, across different model types within each scheme.

Giving a second thought to models used in fraction teaching, an element of fraction teaching usually taken for granted and accompanied with ubiquitous models (e.g., tape diagrams, pizza, and pies) could contribute to the scholarly understanding and students’ learning of fractions. These models cannot assure student learning; however, they may help teachers create opportunities for students to learn and better understand their students’ mathematical thinking. As Cobb et al. (1990) claimed, learning opportunities are not embedded in tasks, but are realized while interacting with students. Therefore, the instruction at schools should allow students to interact with models that provide opportunities for students to develop different fractional meanings and schemes.
APPENDIX A

FRACTION SCHEMES TASKS AND INTERVIEW PROTOCOL FOR

CLINICAL INTERVIEWS
Appendix A

Fraction Schemes Tasks and Interview Protocol for Clinical Interviews

Pseudo-Name: ___________________________ Date: __________________

I would like you to complete a few math tasks about fractions. I am interested in how you come up with the answers, so it is important for you to tell me what you are thinking about. You may draw and write about how you solved these problems. I may ask some questions as you work through tasks or about what you wrote and drew. The interview will not be graded, so you do not have to worry about wrong answers. While we do this, I will be audio and video recording so I can look back over what we did later. Is that OK? You can ask me stop recording anytime. Are you ready?

Partitioning Schemes (PS) Tasks

1. [Provide a copy of L.PS with the question. Read the question to the student. Ask student to explain the reasoning for the answer. Ask clarifying questions as needed.]

   ![L.PS Diagram]

   Shade $\frac{1}{5}$ of the bar.

2. [Provide a copy of S.PS with the question. Read the question to the student. Ask student to explain the reasoning for the answer. Ask clarifying questions as needed.]

   ![S.PS Diagram]

   Shade $\frac{1}{5}$ of the balls.
3. [Provide a copy of A.PS with the question. Read the question to the student. Ask student to explain the reasoning for the answer. Ask clarifying questions as needed.]

Shade $\frac{1}{5}$ of the shape.

Part-Whole Scheme (PWS) Tasks

4. [Provide a copy of L.PWS with the question. Read the question to the student. Ask student to explain the reasoning for the answer. Ask clarifying questions as needed.]

Shade $\frac{3}{4}$ of the bar.
5. [Provide a copy of S.PWS with the question. Read the question to the student. Ask student to explain the reasoning for the answer. Ask clarifying questions as needed.]

Shade $\frac{3}{4}$ of the balls.

6. [Provide a copy of A.PWS with the question. Read the question to the student. Ask student to explain the reasoning for the answer. Ask clarifying questions as needed.]

Shade $\frac{3}{4}$ of the shape.
Partitive Unit Fraction Scheme (PUFS) Tasks

7. [Provide a copy of L.PUFS with the question. Read the question to the student. Ask student to explain the reasoning for the answer. Ask clarifying questions as needed.]

L.PUFS
If the longer bar is considered one whole, what fraction is the shorter bar?

8. [Provide a copy of S.PUFS with the question. Read the question to the student. Ask student to explain the reasoning for the answer. Ask clarifying questions as needed.]

S.PUFS
If these balls are \( \frac{1}{6} \) as many as another set of balls, how many balls are there in the other set?
9. [Provide a copy of A.PUFS with the question. Read the question to the student. Ask student to explain the reasoning for the answer. Ask clarifying questions as needed.]

A.PUFS

The shaded piece is $\frac{1}{4}$ as big as another piece. Draw the other piece.

Partitive Fraction Scheme (PFS) Tasks

10. [Provide a copy of L.PFS with the question. Read the question to the student. Ask student to explain the reasoning for the answer. Ask clarifying questions as needed.]

L.PFS

If the longer bar is considered one whole, what fraction is the shorter bar?
11. [Provide a copy of S.PFS with the question. Read the question to the student. Ask student to explain the reasoning for the answer. Ask clarifying questions as needed.]

12. [Provide a copy of A.PFS with the question. Read the question to the student. Ask student to explain the reasoning for the answer. Ask clarifying questions as needed.]
APPENDIX B

SCORING RUBRIC
## Appendix B

### Scoring Rubric

<table>
<thead>
<tr>
<th>Score</th>
<th>Evidence Shown</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Obvious indication that the student cannot operate in a way compatible with that particular scheme. Evidence might include incorrect responses and markings that are incompatible with actions that would fit the scheme.</td>
</tr>
<tr>
<td>0.5</td>
<td>Weak indication that the student can operate in a way compatible with that particular scheme.</td>
</tr>
<tr>
<td>1</td>
<td>Obvious indication that the student can operate in a way compatible with that particular scheme. Evidence might include correct responses, markings, partitioning and iterating operations.</td>
</tr>
</tbody>
</table>

APPENDIX C

CONSENT FORMS
Appendix C

Consent Forms

AUDIOTAPE/VIDEO PARENT CONSENT FORM

Investigation of Fraction Models as a Means to Understand How Sixth Grade Students Make Sense of Fractions
Principal Investigator: Dr. Scott A. Courtney
Co-Investigator: Ezra Eliustaoglu

I agree to allow my child to participate in audio- and videotaped interview sessions about using fraction models to represent fractions as part of Dr. Courtney and Ms. Eliustaoglu’s study. I agree to allow Ezra Eliustaoglu (co-investigator) to audio- and videotape these interview sessions. The date, time and place of the interview will be mutually agreed upon.

I agree to allow Ezra Eliustaoglu to use my child’s audio and video comments in this study. I understand that my child’s real name will not be used. I understand that the information gathered from audio- and videotape may be presented as part of Ms. Eliustaoglu’s master’s thesis, at professional conferences, or published in professional journals, both online and in print. The sole purpose of sharing this information with others will be to further scholarly understanding of productive fraction learning for students through different types of fraction models.

I have read this consent form. I agree to allow my child to participate in this study. By signing below, I give my consent.

Child’s Name (please print) ________________________________

________________________________________________________

Parent or Guardian Signature Date

Nov 10, 2006
To
Nov 15, 2006

APPROVED
Informed Consent to Participate in a Research Study

**Study Title:** Investigation of Fraction Models as a Means to Understand How Sixth Grade Students Make Sense of Fractions

**Principal Investigator:** Dr. Scott A. Courtney  
**Co-Investigator:** Esra Eliustaoglu

Your child is being invited to participate in a research study. This consent form will provide you with information on the research project, what your child will need to do, and the associated risks and benefits of the research. Your child's participation is voluntary and your child was randomly selected to participate in the study out of sixth grade students at Stanton Middle School. Only those who gave their consent to participate will be observed and interviewed for the study. Please read this form carefully. It is important that you ask questions and fully understand the research in order to make an informed decision. You will receive a copy of this document to take with you.

**Purpose:**  
The purpose of this study is to gain insight into the complex decision to use fraction models to support students’ understandings of the fraction concept and inquire into the meaning different types of fraction models (i.e., area, length and set models) have for students.

**Procedures**  
This study will involve student observations and interviews. Participating students will be observed in their mathematics classrooms, as they engage in mathematics, so that the investigators can attend to students’ direct experiences as they occur. One-on-one interviews will be conducted to help investigators interpret students' understandings of fraction models and to better understand how students interact with these models. The central purpose of these interviews will be to elicit students’ understandings of fractions as they interact with the fraction models. During the interviews, students will be asked to work on several problems employing specific fraction models. Study participants will be expected to engage in three 45-minute interview sessions.

Data collection, consisting of student observations and interviews, will last from November 15, 2015 to March 14, 2016. Since this study will comprise Ms. Eliustaoglu’s master’s thesis, which will be written throughout the fall and spring semesters, we ask that study participants be available for potential follow-up interviews through May 15, 2016.

Participation in the study is voluntary and your child is free to leave the study at any time. It is possible that copies of participants’ written work (e.g., the written explanations, drawings, visual representations used to solve the fraction tasks) may be displayed in Ms. Eliustaoglu’s culminating master's thesis and at professional conferences.

INVESTIGATION OF FRACTION MODELS AS A MEANS TO UNDERSTAND HOW SIXTH GRADE STUDENTS MAKE SENSE OF FRACTIONS
conferences or professional journals through dissemination of Ms. Eliustaoglu’s work, pseudonyms will be used for all participant work samples.

**Audio and Video Recording**

This study will employ video/audiotape recordings of students as they interact (e.g., their drawings and written work that shows their strategies to solve the task) with different fraction models. These video/audiotapes will be utilized by the investigators to explore how students interact and engage with different types of fraction models and their ways of developing strategies in engaging with fraction models. All written work and video/audiotapes will be kept under lock and key in Dr. Courtney’s work office. Taped recordings will be used for Ms. Eliustaoglu’s master's thesis as well as research and education presentations at professional meetings. The video will not include student's name (pseudo-names will be used to address the corresponding student). Prior to audio and video taping Ms. Eliustaoglu will read them the assent. She will make sure that students knows that they can ask her to stop recording.

It is possible that copies of your child's written work, along with transcripts from the video/audiotaped activities, may be presented at professional conferences or published in professional journals as they relate to using different types of fraction models in mathematics education. Pseudonyms will be used for all work samples and video/audiotaped activities.

**Benefits**

Participants may benefit from this study as they engage with different types of fraction models. More specifically, reflecting on different types of fraction models will require students to employ different strategies and communicate relevant aspects of fraction concepts. The result of the study is expected to support teachers in implementing instruction productive to students developing fraction understanding.

**Risks and Discomforts**

There are no anticipated risks beyond those encountered in everyday life.

**Privacy and Confidentiality**

Participant related information will be kept confidential within the limits of the law. Any identifying information will be kept in a secure location and only the researchers will have access to the data. Research participants will not be identified in any publication or presentation of research results. Pseudonyms will be used for all work samples and video/audiotaped activities. This signed parental consent form will be kept separate from study data, and responses will not be linked to your child.

The research information may, in certain circumstances, be disclosed to the Institutional Review Board (IRB), which oversees research at Kent State University, or to certain federal agencies. Your child’s confidentiality may not be maintained if there is an indication that if he/she may harm themselves or others.

**Compensation**

Your child will not receive compensation for his/her participation in this study. In addition, participation or non-participation will have no effect on your child’s grade in the classroom.

INVESTIGATION OF FRACTION MODELS AS A MEANS TO UNDERSTAND HOW SIXTH GRADE STUDENTS MAKE SENSE OF FRACTIONS
Voluntary Participation
Taking part in this research study is entirely up to you and your child. You and/or your child may choose not to participate or may discontinue their participation at any time without penalty or loss of benefits to which he/she is otherwise entitled. You will be informed of any new, relevant information that may affect your child’s health, welfare, or willingness to continue participation in this study. Participation or non-participation will have no effect on your child’s grade in their mathematics classroom.

Contact Information
If you have any questions or concerns about this research, you may contact Dr. Scott A. Courtney (scourt5@kent.edu). This project has been approved by the Kent State University Institutional Review Board. If you have any questions about your child's rights as a research participant or complaints about the research, you may call the IRB at 330-672-2704.

Consent Statement and Signature
I have read this consent form and have had the opportunity to have my questions answered to my satisfaction. I voluntarily agree to grant permission for my child to participate in this study. I understand that a copy of this consent will be provided to me for future reference.

________________________________  _____________________
Child's Name                               Date

Parent or Guardian Signature
Informed Consent Requirements for Minors

Documented informed consent must be obtained from children age 13 to 18. The parent(s) or guardian(s) must also sign the informed consent document. In research that involves greater than minimal risk to subjects, the signature of both parents may be required.

Assent is required of children age 12 years and younger, as well as parental permission. Assent is a child's affirmative agreement to participate in research. An assent script must be provided with the IRB application.

Assent Script:
(Investigation of Fraction Models as a Means to Understand How Sixth Grade Students Make Sense of Fractions)

Procedure for obtaining assent from children
1. Hi, [child's name].
2. My name is Esra Eliustaoglu, and I am trying to learn more about sixth grade students' understanding of fractions.
3. I would like to meet with you and some other children to talk about fractions. I would like you to complete a few math tasks and then draw and write about how you solved the problems. I may ask some questions as you work through tasks or about what you wrote and drew. While we do this, I’ll be video recording so I can look back over what we did later.
4. Do you want to do this?
5. Do you have any questions before we start?
6. If you want to stop at any time just tell me.
REFERENCES


