REMOVABLE SINGULARITIES
FOR HÖLDER CONTINUOUS SOLUTIONS OF THE FRACTIONAL LAPLACIAN

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CHAPTER 1

Introduction

The purpose of this thesis is to find a way to judge whether a set $E$ is a removable set for a certain class of solutions of differential operator called the fractional Laplacian. We shall see that a natural gauge by which to measure the size of a removable set is its Hausdorff measure. Removable singularities for solutions of differentiable operators have been heavily studied ever since Riemann proved that a single point is removable for a bounded analytic function. In 1958, L. Carleson [1] showed that a set $E$ is removable for Hölder continuous solutions of the Laplacian operator if and only if its $(n - 2 + \alpha)$ dimensional Hausdorff measure equals zero, where $\alpha$ is the Hölder exponent of the solution. In this thesis, we extend Carleson’s result to a certain of non-local pseudo-differential operator called the fractional Laplacian.

In the second chapter, we present some concepts and lemmas that are related to smooth functions. We also introduce an important tool, which is the partition of unity. We shall first review the classical construction. Then, we shall describe the Harvey-Polking partition of unity which provides a sharp estimate on the derivatives of the functions involved in the partition of unity. We conclude this chapter by providing a lemma concerning the spherical average of smooth functions.

The third chapter addresses the basic properties of the fractional Laplacian. We introduce a natural notion of distributional solution to the fractional Laplacian.

Chapter 4 begins by presenting some basic results about the Fourier transform.
We also follow [3] to calculate the Fourier transform of fractional Laplacian of a smooth function. Moreover, in this chapter, we calculate the Fourier transform of a Gaussian following [4]. We introduce an important result about the inverse of Fourier transform. The last section of this chapter provides the definition and some properties of Hausdorff measure. In this section, we show that we can cover any set $E$ that has zero Hausdorff measure with dyadic cubes where the sum of their sidelengths is arbitrary small. We also present two fundamental results concerning Hausdorff measure, which are the Mass distribution principle and the Frostman’s lemma.

In the final chapter, we introduce our main theorem, which shows that a set $E$ is removable for Hölder continuous solutions of the fractional Laplacian if and only if its $n - 2\alpha + \gamma$ dimensional Hausdorff measure equals zero, where $\gamma$ is the Hölder exponent of the solution and $2\alpha$ is the order of the fractional Laplacian.
CHAPTER 2
Smooth Functions

In this section, we follow [6] in providing definitions of some concepts and examples that are related to smooth functions. We also present important theorems regarding to smooth functions. Let’s first introduce some important definitions and examples.

**Definition 2.0.1.** The space $C^0(\Omega)$, where $\Omega$ is an open subset of $\mathbb{R}^n$, is a linear space of continuous functions on $\Omega$.

**Definition 2.0.2.** Let $\Omega \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$. The *support* of $\phi$, where $\phi \in C^0(\Omega)$, is the closure of the set $\{x : x \in \Omega, \phi(x) \neq 0\}$ in $\mathbb{R}^n$. We write $\text{supp}(\phi) = \{x : x \in \Omega, \phi(x) \neq 0\}$.

**Example 2.0.3.** Choose $\Omega = (0, \pi) \subset \mathbb{R}$ and define $\phi \in C^0(\Omega)$ to be $\phi(x) = \sin x$. Then,

$$\text{supp}(\phi) = \{x : x \in \Omega, \phi(x) \neq 0\} = (0, \pi) = [0, \pi]$$

**Definition 2.0.4.** The space $C^\infty(\Omega)$ is the set of all continuous functions whose partial derivatives of all orders with respect to all variables are continuous in $\Omega$.

The space $C^\infty_0(\Omega)$ is the set of all continuous functions that supported inside compact set in $\Omega$.

i.e. $C^\infty_0(\Omega) = \{\phi : \phi \in C^\infty(\Omega), \text{supp}(\phi) \subset \subset \Omega\}$.

Then, $C^\infty_0(\Omega) = \{\phi : \phi \in C^\infty(\Omega), \text{supp} \phi \subset \subset \Omega\} = C^\infty(\Omega) \cap C^\infty(\Omega)$. The space $C^\infty_0(\Omega)$ is called the *test space* (or, alternatively, the *space of test functions*) and is also denoted by $\mathcal{D}(\Omega)$. 

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Example 2.0.5. Let $\Omega = \mathbb{R} = (-\infty, \infty)$ and let $\phi$ be a function defined as the following,

$$
\phi(x) = \begin{cases} 
  e^{-\frac{1}{1-x^2}} & \text{for } |x| < 1 \\
  0 & \text{for } |x| \geq 1
\end{cases}.
$$

Observe that $\phi$ is compactly supported in $\mathbb{R}$ with $\text{supp}(\phi) = [-1, 1] \subset \mathbb{R}$. Thus, we can say $\phi \in C^\infty_0(\mathbb{R})$.

Definition 2.0.6. We define the class of distributions of $\Omega$, denoted by $\mathcal{D}'(\Omega)$ to be the set of all linear continuous functionals defined on $\mathcal{D}(\Omega)$. We express a distribution $T$ acting on $\phi \in \mathcal{D}(\Omega)$ as $T(\phi)$ or equivalently $\langle T, \phi \rangle$.

Example 2.0.7. We can associate a distribution to a function $f \in L^1(\mathbb{R}^n)$ by the following

$$
T_f(\phi) = \langle T_f, \phi \rangle = \int_{\Omega} f(x)\phi(x)dx,
$$

for $\phi \in \mathcal{D}(\mathbb{R}^n)$.

2.1 Partition of Unity

2.1.1 Preliminaries

Let $A$ and $B$ are closed sets in $\mathbb{R}^n$ with $A \cap B = \emptyset$. Define $f \in C(\mathbb{R}^n)$ by

$$
f(x) = \frac{d(x, B)}{d(x, A) + d(x, B)}.
$$

Then $0 \leq f(x) \leq 1$, and $f(x) = \begin{cases} 
  1 & \text{if } x \in A \\
  0 & \text{if } x \in B
\end{cases}$. 


2.1.2 Construction of Partition of Unity

We begin by reviewing the classical construction of the partition of unity.

**Theorem 2.1.1. (partition of unity)** Let \( E \subset \mathbb{R}^n \) is a compact set and let \((U_j)_{j \in \mathbb{N}}\) be an open cover of \( E \). Then there exist continuous functions \((\varphi_j)_{j \in \mathbb{N}}\) supported in \( U_j \), with \( \sum_j \varphi_j(x) = 1 \) for every \( x \in E \).

**Proof.** For \( x \in E \), let \( r_x > 0 \) be such that \( B(x, r_x) \subset U_k \) for some \( k \). Thus, the collection \((B(x, r_x))_{x \in E}\) form an open cover of \( E \). Since \( E \) is a compact set, we can construct a finite subcover of \( E \), say, \( B(x_1, r_1), \ldots, B(x_N, r_N) \).

Let \( \widetilde{U}_j = \bigcup \{B(x_k, r_k) : B(x_k, r_k) \subset U_j\} \). Then, \( \widetilde{U}_j \subset U_j \) and \( \overline{\widetilde{U}_j} = Cl(\widetilde{U}_j) \subset U_j \).

Using the function \( f \) from the preliminaries, choosing \( A = \overline{U}_j \) and \( B = U_j^c \), we conclude the existence of functions, say, \( \eta_j \in C(\mathbb{R}^n) \) with \( \eta_j \equiv 1 \) on \( \widetilde{U}_j \) and \( \text{supp}(\eta_j) \subset U_j \). Define the functions \( \varphi_j \) for every \( j \in \mathbb{N} \) as the following

\[
\varphi_1 = \eta_1,
\varphi_2 = \eta_2(1 - \eta_1),
:\vdots:
\varphi_N = \eta_N(1 - \eta_{N-1})\ldots(1 - \eta_1) = \eta_N \prod_{j=1}^{N-1} (1 - \eta_j).
\]

Observe that \( \varphi_1 = \eta_1 \equiv 1 \) on \( U_1 \) and it is identically to 0 everywhere else. By summing over \( j \), we have

\[
\sum_j \varphi_j = \varphi_1 + \varphi_2 + \ldots + \varphi_N = \eta_1 + \eta_2(1 - \eta_1) + \ldots + \eta_N(1 - \eta_1)\ldots(1 - \eta_{N-1}).
\]
By an easy induction, we conclude that

\[ \sum_j \varphi_j = 1 - \prod_j (1 - \eta_j). \]

Indeed, when \( j = 1 \), then we have that \( \varphi_1 = \eta_1 \) as defined. Assuming that for \( k \leq N - 1 \), \( \sum_{j=1}^{k} \varphi_j = 1 - \prod_{j=1}^{k} (1 - \eta_j) \), we write

\[
\sum_{j=1}^{k+1} \varphi_j = 1 - \prod_{j=1}^{k+1} (1 - \eta_j) \sum_{j=1}^{k+1} \varphi_j = \sum_{j=1}^{k} \varphi_j + \varphi_{k+1}
\]

\[
= 1 - \prod_{j=1}^{k} (1 - \eta_j) + \eta_{k+1} \prod_{j=1}^{k} (1 - \eta_j)
\]

\[
= 1 - \prod_{j=1}^{k} (1 - \eta_j)(1 - \eta_{k+1}) = 1 - \prod_{j=1}^{k+1} (1 - \eta_j).
\]

If \( x \in E \), then \( x \in \overline{U}_j \) for some \( j \) which implies that \( \eta_j(x) = 1 \). On the other hand, we will have \( \eta_i = 0 \) for every \( i \neq j \). Hence,

\[
\sum_{j=1}^{N} \varphi_j(x) = 1 \quad \forall x \in E.
\]

\[\square\]

### 2.2 Refined Partition of Unity

**Definition 2.2.1.** Let \( \phi : \mathbb{R}^n \rightarrow \mathbb{R} \) be a mollifier, which is a function satisfying

- \( \phi \in C^\infty(\mathbb{R}^n) \)
- \( \phi \geq 0 \) in \( \mathbb{R}^n \)
- \( \phi(x) = 0 \) whenever \( |x| \geq 1 \) and
- \( \int_{\mathbb{R}^n} \phi dm_n = 1 \)
In the following lemma we will construct the Harvey-Polking partition of unity, from [2].

**Lemma 2.2.2.** Let \( \{Q_i: 1 \leq i \leq N\} \) be a finite disjoint collection of dyadic cubes of length \( l_i \) such that \( l_1 \geq l_2 \geq \ldots \geq l_N \). For each \( i \), there is a function \( \varphi_i \in C_0^\infty(\mathbb{R}^n) \) with \( \text{supp} \varphi_j \subset 3Q_j \) such that \( \sum_{i=1}^N \varphi_i(x) = 1 \) for all \( x \in \bigcup_{i=1}^N Q_i \). Furthermore, for each multi-index \( \alpha \), there is a constant \( C_\alpha \), depending only on \( n \) and \( \alpha \), for which

\[
|D^\alpha \varphi_i(x)| \leq \frac{C_\alpha}{l_i^{|\alpha|}} \text{ for all } x \text{ and } 1 \leq i \leq N.
\]

**Proof.** Firstly, we choose \( \psi \in C_0^\infty(\mathbb{R}^n) \) such that \( \psi(x) = 1 \) if \( x \in [-1, 1]^n \) and \( \psi(x) = 0 \) if \( x \not\in [-3, 3]^n \). We will show here how to construct such a function explicitly. Assume that we are in 1-dimensional case. Define a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that

\[
f(t) = \begin{cases} 
e^{-\frac{1}{1-t^2}} & |t| < 1 \\ 0 & |t| \geq 1 \end{cases}
\]  

as we saw in example (2.0.5), \( f \) is a smooth function.

Now let \( \tilde{f}(t) = \frac{f(t)}{\int_{\mathbb{R}} f(s)ds} \). Then, \( \text{supp} \tilde{f} \subset [-1, 1] \), \( \tilde{f} \in C^\infty(\mathbb{R}) \), and \( \int_{\mathbb{R}} \tilde{f}(t)dt = 1 \).

For \( \epsilon > 0 \), set \( \tilde{f}_\epsilon(t) = \frac{1}{\epsilon} \tilde{f}(\frac{t}{\epsilon}) \). Then, \( \tilde{f}_\epsilon \in C^\infty(\mathbb{R}) \) and \( \int_{\mathbb{R}} \tilde{f}_\epsilon = 1 \). Also, we have that \( \text{supp} \tilde{f}_\epsilon \subset [-\epsilon, \epsilon] \).

Choose \( \epsilon = \frac{1}{2} \). Define,

\[
\eta(t) = \left( \tilde{f}_{\frac{1}{2}} \ast \chi_{[-2,2]} \right)(t), \text{ where for a set } E, \chi_E \text{ is the characteristic function defined by }
\]

\[
\chi_E = \begin{cases} 
1 & \text{if } x \in E \\
0 & \text{otherwise}
\end{cases}
\]

Then,

\[
\eta(t) = \left( \tilde{f}_{\frac{1}{2}} \ast \chi_{[-2,2]} \right)(t) = \int_{-2}^{2} \tilde{f}_{\frac{1}{2}}(t-s)ds.
\]
With the change of variable $\tilde{s} = t - s$, we see that

$$\eta(t) = \int_{t-2}^{t+2} \tilde{f}_{\frac{1}{2}}(\tilde{s})d\tilde{s}.$$ 

Note that if $|t| < 1$, then $[-\frac{1}{2}, \frac{1}{2}] \subset [t - 2, t + 2]$. Therefore,

$$\int_{t-2}^{t+2} \tilde{f}_{\frac{1}{2}}(\tilde{s})d\tilde{s} = 1.$$ 

On the other hand, if $|t| \geq 3$, then $[-\frac{1}{2}, \frac{1}{2}] \cap [t - 2, t + 2] = \emptyset$. Thus,

$$\int_{t-2}^{t+2} \tilde{f}_{\frac{1}{2}}(\tilde{s})d\tilde{s} = 0.$$ 

Now in $n$-dimensional case, define $\psi : \mathbb{R}^n \to \mathbb{R}$ such that for $x \in \mathbb{R}^n$.

$$\psi(x) = \eta(x_1)\eta(x_2) \cdots \eta(x_n) = \prod_{j=1}^{n} \eta(x_j). \quad (2.2.2)$$ 

Note that this function $\psi$ has the required properties.

Next, set $x_k$ to be the center of a cube $Q_k$. Put $\psi_k(x) = \psi \left( \frac{x-x_k}{t_k} \right)$.

Then, $\psi_k \equiv 1$ on $Q_k$ and supp($\psi_k$) $\subset 3Q_k$.

We will construct the partition of unity as in theorem (2.1.1).

$$\varphi_1 = \psi_1,$$

$$\varphi_2 = \psi_2(1 - \psi_1),$$

$$\vdots$$

$$\vdots$$

$$\varphi_N = \psi_N \left( 1 - \psi_{N-1} \right) \cdots (1 - \psi_1) = \psi_N \prod_{j=1}^{N-1} (1 - \psi_j).$$

Therefore,

$$\varphi_1 + \varphi_2 + \cdots + \varphi_N = \sum_{j=1}^{N} \varphi_j = 1 - \prod_{j=1}^{N} (1 - \psi_j).$$
And hence,
\[
\sum_{j=1}^{N} \varphi_j(x) = 1 \text{ for all } x \in \bigcup_{j=1}^{N} Q_j.
\]

In order to prove the estimate on the derivatives of \( \varphi_j \), we will follow the same technique as in [2]. Let \( \theta_k = \sum_{j=1}^{k} \varphi_j = 1 - \prod_{j=1}^{k} (1 - \psi_j) \).

Since \( l_k \geq l_{k+1} \), it suffices to prove the estimate for \( \theta_k \). Indeed, once we show that \(|D^\alpha \theta_k| \leq \frac{C_\alpha}{l_k^{|\alpha|}} \) for every \( k, \alpha \), then, as \( \varphi_k = \theta_k - \theta_{k-1} \), we have
\[
|D^\alpha \varphi_k| \leq |D^\alpha \theta_k| + |D^\alpha \theta_{k-1}| \leq \frac{2C_\alpha}{l_k^{|\alpha|}}.
\]

For integers \( v_1, \ldots, v_k \) where \( 1 \leq v_i \leq k \) define
\[
g_{v_1, \ldots, v_r} = \begin{cases} 
0 & \text{if } v_i = v_j \text{ for some } i \neq j \\
\prod_{i \neq v_1, \ldots, v_r} (1 - \psi_j) & \text{if all } v_i \text{ are distinct}
\end{cases}.
\]

Then, there are constants \( C_{\beta^1, \ldots, \beta^r} \) depending only on multi-index subscripts such that \( C_{\beta^1, \ldots, \beta^r} = \frac{\alpha!}{\beta_1! \beta_2! \ldots \beta^r!} \). We can estimate the derivative of \( \theta_k \) as follows
\[
D^\alpha \theta_k = \sum_{\beta^1, \ldots, \beta^r} C_{\beta^1, \ldots, \beta^r} \left( \sum_{v_1, \ldots, v_r=1}^{k} g_{v_1, \ldots, v_r} (D^{\beta^1} \psi_{v_1})(D^{\beta^2} \psi_{v_2}) \ldots \ldots \ldots (D^{\beta^r} \psi_{v_r}) \right),
\]
where the outer sum is over all sets of multi-indices \( \{\beta^1, \beta^2, \ldots, \beta^r\} \) for which \( |\beta^i| \geq 1 \) and \( \beta^1 + \beta^2 + \ldots + \beta^r = \alpha \).

Therefore, as \( 0 \leq g_{v_1, \ldots, v_r} \leq 1 \) for any choice of integers \( v_1, \ldots, v_r \), we obtain
\[
|D^\alpha \theta_k(x)| \leq \sum_{\beta^1, \beta^2, \ldots, \beta^r} C_{\beta^1, \beta^2, \ldots, \beta^r} \left( \sum_{v_1=1}^{k} |D^{\beta^1} \psi_{v_1}(x)| \right) \ldots \ldots \left( \sum_{v_r=1}^{k} |D^{\beta^r} \psi_{v_r}(x)| \right).
\]

Here is a lemma helping us to complete the proof.

**Lemma 2.2.3.** There exists \( C = C(n) \), a constant depending only on dimension, such that for each \( l > 0 \) every \( x \in \mathbb{R}^n \) lies in at most \( C(n) \) cubes \( 3Q \) where \( Q \) is a dyadic cube of length \( l \). Moreover, we can say that \( C(n) = 6^n \).
Now consider a typical sum $\sum_{v=1}^{k} |D^\beta \psi_v(x)|$. Note that $D^\beta \psi_v(x) = 0$ unless $x \in 3Q_v$. Furthermore, if $x \in 3Q_v$, then $|D^\beta \psi_v(x)| \leq C_\beta l_v^{-|\beta|}$. Thirdly, by Lemma (2.2.3), we have that for each $l > 0$ every $x \in \mathbb{R}^n$ can only lie in at most $6^n$ cubes $3Q_v$ with $l(Q_v) = 2^l l_k$. Thus,

$$\sum_{v=1}^{k} |D^\beta \psi_v(x)| = \sum_{m=0}^{\infty} \sum_{l_v = 2^m l_k}^{v} |D^\beta \psi_v|$$

$$\leq 6^n \sum_{m=0}^{\infty} \frac{C(|\beta|)}{2^m |\beta| l_k^{|\beta|}}$$

$$\leq \frac{1}{1 - 2^{-|\beta|}} \frac{C(|\beta|)}{l_k^{|\beta|}}$$

$$\leq \frac{C(|\beta|)}{l_k^{|\beta|}}.$$

Therefore, we have

$$|D^\alpha \theta_k| \leq \sum C_{\beta_1, \ldots, \beta_r} \left( C(|\beta_1|) l_k^{-|\beta_1|} \right) \ldots \left( C(|\beta_r|) l_k^{-|\beta_r|} \right) \leq \frac{C(\alpha)}{l_k^{|\alpha|}}, \quad (2.2.3)$$

as required. \qed

Let's now prove Lemma (2.2.3).

**Proof.** Let's label by $Q_1, \cdots, Q_M$ the dyadic cubes of sidelength $l$ with $x \in 3Q_j$, where $j \in \{1, 2, \cdots, M\}$. Then, each such cube $3Q_j$ lies inside a cube, say, $Q'$ centered at $x$ of sidelength atmost $6l$. Since they are dyadic, the cubes $(Q_j)_j$ are pairwise disjoint. We want to show that $M \leq 6^n$. But,

$$M \cdot l^n = \sum_{Q} m_n(Q) = m_n \left( \bigcup Q \right) \leq m_n(Q') \leq 6^n l^n \quad (2.2.4)$$

Therefore, we have $M \leq 6^n$ as required. \qed
We conclude this section by presenting an important property regarding the spherical average of a smooth function.

**Lemma 2.2.4.** For a smooth function \( f \) we have

\[
\left| \int_{\partial B(0,1)} [f(r\omega) - f(0)]dS(\omega) \right| \leq cr^2 \max_{B(0,1)} |\Delta f|,
\]

where \( \Delta \) is the Laplacian operator.

**Proof.** We know from divergence theorem that,

\[
\int_{\partial B(0,1)} \nabla f \cdot v dS = \int_{B(0,1)} \Delta f \, dx \tag{2.2.5}
\]

Let \( I = \int_{\partial B(0,1)} [f(r\omega) - f(0)]dS(\omega) \). The fundamental theorem of calculus yields that,

\[
I = \int_0^r \frac{d}{dt} \left[ \int_{S^{n-1}} f(t\omega)dS(\omega) \right] dt
\]

\[
\frac{d}{dt} \int_{S^{n-1}} f(t\omega)dS(\omega) = \lim_{h \to 0} \int_{S^{n-1}} \frac{f((t + h)\omega) - f(t\omega)}{h} dS(\omega)
\]

\[
= \lim_{h \to 0} \int_{S^{n-1}} \omega \cdot \nabla f((t + \theta)\omega)dS(\omega).
\]

where \( |\theta| < h \). But,

\[
\omega \cdot \nabla f((t + \theta)\omega) \to \omega \cdot \nabla f(t\omega) \text{ uniformly as } h \to 0 \tag{2.2.6}
\]

Applying dominated convergence theorem gives us

\[
\frac{d}{dt} \int_{S^{n-1}} f(t\omega)dS(\omega) = \int_{S^{n-1}} \omega \cdot \nabla f(t\omega)dS(\omega)
\]

\[
= t \int_{B(0,1)} \text{div}(\nabla f(tx))dx
\]

\[
= t \int_{B(0,1)} \Delta f(tx)dx.
\]
Therefore, we have that

$$\left| \int_{S^{n-1}} [f(t\omega) - f(0)]dS(\omega) \right| = \left| \int_0^r t \int_{B(0,1)} \triangle f(tx) dx dt \right|$$

$$\leq \int_0^r t \cdot \sup_{x \in B(0,1)} |\triangle f(tx)| m_n(B(0,1)) dt$$

$$\leq \frac{r^2}{2} m_n(B(0,1)) \sup_{B(0,1)} |\triangle f|.$$

(2.2.7)
CHAPTER 3

Fractional Laplacian

**Definition 3.0.1.** Define the fractional Laplacian of a function $\phi \in C^2_0(\mathbb{R}^n)$ by the following

$$(-\triangle)^{\alpha} \phi(x) = P.V. \int_{\mathbb{R}^n} \frac{\phi(x) - \phi(y)}{|x - y|^{n+2\alpha}} dm_n(y)$$

$$= \lim_{r \to 0} \int_{\mathbb{R}^n \setminus B(x,r)} \frac{\phi(x) - \phi(y)}{|x - y|^{n+2\alpha}} dm_n(y).$$

Now we want to verify that the fractional Laplacian is well-defined. By using polar co-ordinates, let $y = x + t\omega$.

$$\int_{\mathbb{R}^n \setminus B(x,r)} \frac{\phi(x) - \phi(y)}{|x - y|^{n+2\alpha}} dm_n(y) = \int_r^{\infty} \frac{t^{n-1}}{t^{n+2\alpha}} \int_{S^{n-1}} [\phi(x) - \phi(x + t\omega)] dS(\omega) dt. \quad (3.0.3)$$

But then we know from Lemma (2.2.4) that

$$\left| \int_{S^{n-1}} [\phi(x) - \phi(x + t\omega)] dS(\omega) \right| \leq \begin{cases} ct^2 \sup_{B(x,t)} |\nabla \phi| & \text{if } t \leq 1 \\ c(\phi) & \text{if } t > 1 \end{cases} \quad (3.0.4)$$

Therefore, we have that

$$\left| \int_r^{\infty} \frac{t^{n-1}}{t^{n+2\alpha}} \int_{S^{n-1}} [\phi(x) - \phi(x + t\omega)] dS(\omega) dt \right| \leq \int_r^{1} \frac{1}{t^{1+2\alpha}} \sup_{B(x,t)} |\nabla \phi| dt$$

$$+ \int_1^{\infty} c(\phi) \frac{1}{t^{1+2\alpha}} dt \quad (3.0.5)$$

$$\leq c_1(\phi) \int_r^{1} \frac{1}{t^{1+2\alpha}} dt + c_2(\phi) \int_1^{\infty} \frac{1}{t^{1+2\alpha}} dt.$$
Hence, the fractional Laplacian is well-defined!

Now we will use the definition of fractional Laplacian and try to rewrite it in different way. For $u \in C^2_0(\mathbb{R}^n)$

$$(-\triangle)^{\alpha} u(x) = P.V. \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|y-x|^{n+2\alpha}} dy.$$ 

Choosing $z = y - x$ we will have

$$(-\triangle)^{\alpha} u(x) = P.V. \int_{\mathbb{R}^n} \frac{u(x + z) - u(x)}{|z|^{n+2\alpha}} dz.$$ 

Moreover, by substituting $z = -\tilde{z}$

$$P.V. \int_{\mathbb{R}^n} \frac{u(x + z) - u(x)}{|z|^{n+2\alpha}} dz = P.V. \int_{\mathbb{R}^n} \frac{u(x - \tilde{z}) - u(x)}{|\tilde{z}|^{n+2\alpha}} d\tilde{z}.$$ 

Consequently, after relabeling $\tilde{z}$ as $z$,

$$2P.V. \int_{\mathbb{R}^n} \frac{u(x + z) - u(x)}{|z|^{n+2\alpha}} dz = P.V. \int_{\mathbb{R}^n} \frac{u(x + z) - u(x)}{|z|^{n+2\alpha}} dz + P.V. \int_{\mathbb{R}^n} \frac{u(x - z) - u(x)}{|z|^{n+2\alpha}} dz$$

$$= P.V. \int_{\mathbb{R}^n} \frac{u(x + z) + u(x - z) - 2u(x)}{|z|^{n+2\alpha}} dz.$$ 

In conclusion, we have that

$$(-\triangle)^{\alpha} u(x) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n+2\alpha}} dy. \quad (3.0.7)$$

**Lemma 3.0.2.** Let $\gamma \in (0,2\alpha)$ and let $u$ be a function such that

$$|u(x) - u(y)| \leq |x - y|^{\gamma} \quad \forall x, y \in \mathbb{R}^n. \quad (3.0.8)$$

Define a function $\phi$ with some properties $\phi \in C^2_0(B(x,r))$, and $|\triangle \phi| \leq \frac{1}{r^2}$. Then,

$$\left| \int_{\mathbb{R}^n} [u(y) - u(x)] \cdot ((-\triangle)^{\alpha} \phi)(y) dm_n(y) \right| \leq C \ r^{n-2\alpha+\gamma}. \quad (3.0.9)$$
Proof. We will separate the integral over two parts which are $B(x,3r)$ and $\mathbb{R}^n \setminus B(x,3r)$. Then we will have the following,

\[
\int_{\mathbb{R}^n} [u(y) - u(x)]((-\triangle)^\alpha \phi)(y)dm_n(y) = \int_{\mathbb{R}^n \setminus B(x,3r)} [u(y) - u(x)]((-\triangle)^\alpha \phi)(y)dm_n(y) \\
+ \int_{B(x,3r)} [u(y) - u(x)]((-\triangle)^\alpha \phi)(y)dm_n(y).
\]

Let $B = B(x,r)$, and

\[
I_1 = \int_{\mathbb{R}^n \setminus 3B} [u(y) - u(x)]((-\triangle)^\alpha \phi)(y)dm_n(y). \quad (3.0.10)
\]

However, since we are given that $\phi$ is compactly supported inside $B$, we have $\phi(y) = 0$ if $y \in \mathbb{R}^n \setminus 3B$. Also, as $|y - z|$ is never zero, we can write

\[
((-\triangle)^\alpha \phi)(y) = \frac{\phi(y) - \phi(z)}{|y - z|^{n+2\alpha}}dz = \int_B \frac{-\phi(z)}{|y - z|^{n+2\alpha}}dz. \quad (3.0.11)
\]

By triangle inequality,

\[
|y - z| \geq |x - y| - |x - z| \geq \frac{2}{3}|x - y|
\]

Indeed, $|x - y| \geq 3r \geq 3|x - z|$, and so $|y - z|$ is about the same as $|x - y|$. Thus, we have

\[
|((-\triangle)^\alpha \phi)(y)| = \left| \int_B \frac{-\phi(z)}{|x - y|^{n+2\alpha}}dz \right| \\
\leq \frac{1}{|x - y|^{n+2\alpha}}m_n(B(x,r)) \sup_B |\phi(z)| \\
= \frac{r^n}{|x - y|^{n+2\alpha}}m_n(B(0,1)) \sup_B |\phi(z)|
\]
Using the previous estimate (3.0.5) and polar co-ordinates yields that

$$
|I_1| = \left| \int_{\mathbb{R}^n \setminus B} [u(y) - u(x)]((-\Delta)\phi)(y) \, dy \right|
$$

$$
\leq \int_{\mathbb{R}^n \setminus B} |u(y) - u(x)| |(-\Delta)\phi| \, dy
$$

$$
\leq \int_{\mathbb{R}^n \setminus B} |x - y|^\gamma \frac{r^n}{|x - y|^{n+2\alpha}} m_n(B(0,1)) \sup_B |\phi(z)| \, dy
$$

$$
= r^n \int_0^\infty t^{n-1} \int_{S^{n-1}} t^{\gamma - n - 2\alpha} m_n(B(0,1)) \sup_B |\phi(z)| \, d\sigma(\omega) \, dt
$$

$$
= 2m_n(B(0,1))3^{\gamma - 2\alpha} r^{\gamma + n - 2\alpha} \max_B |\phi(z)|.
$$

Therefore,

$$
|I_1| \leq C r^{n + \gamma - 2\alpha} \quad \text{where} \quad C \quad \text{is a constant. (3.0.12)}
$$

Now for the second part of the integral we have

$$
|I_2| = \left| \int_{B(x,3r)} [u(x) - u(y)]((-\Delta)\phi)(y) \, dm_n y \right|
$$

By using polar co-ordinates, $y = x + t\omega$

$$
|I_2| = \left| \int_0^{3r} \int_{\partial B(x,3r)} [u(x + t\omega) - u(x)]((-\Delta)\phi)(x + t\omega) dS(\omega) \, dt \right|
$$

$$
= \left| \int_0^{3r} n m_n(B(0,1)) t^{n-1} \int_{\partial B(x,3r)} [u(x + t\omega) - u(x)]((-\Delta)\phi)(x + t\omega) dS(\omega) \, dt \right|
$$

$$
\leq \int_0^{3r} n m_n(B(0,1)) t^{n-1} \int_{\partial B(x,3r)} |u(x + t\omega) - u(x)| |(-\Delta)\phi(x + t\omega)| dS(\omega) \, dt
$$

$$
\leq \int_0^{3r} n m_n(B(0,1)) t^{n-1} \int_{\partial B(x,3r)} t^{\gamma} |(-\Delta)^{\alpha}\phi(x + t\omega)| dS(\omega) \, dt.
$$

But we know from the estimate of fractional Laplacian (3.0.5) that

$$
|(-\Delta)^{\alpha}\phi(y)| \leq C(\phi) \int_r^\infty \frac{dt}{t^{1+2\alpha}} = C r^{-2\alpha}.
$$

Hence,

$$
|I_2| \leq C(n, \phi) r^{n - 2\alpha + \gamma}. \quad (3.0.13)
$$
Combining (3.0.12) and (3.0.13) we have that

\[ |I| \leq |I_1| + |I_2| \leq C(n, \phi) \, r^{n-2\alpha+\gamma}. \]

We will end this chapter by introducing an important definition talking about the distribution acting on a smooth function.

**Definition 3.0.3.** If \( u \) is a \( \gamma \)-Hölder continuous function, then we define the fractional Laplacian of \( u \) as a distribution acting on smooth functions by

\[ \langle (-\Delta)^\alpha u, \varphi \rangle = \int_{\mathbb{R}^n} u(x) [(-\Delta)^\alpha \varphi(x)] \, dx \]

for \( \varphi \in C_0^\infty(\mathbb{R}^n) \).

Note that the previous lemma ensures that the right hand side is an absolutely convergent integral.
CHAPTER 4

Fourier Transform

Definition 4.0.1. Let $f$ be a function in $L^1(\mathbb{R}^n)$. The Fourier transform of $f$, denoted by $\hat{f}$ or $\mathcal{F}(f)(\zeta)$, defined as the following

$$\mathcal{F}(f)(\zeta) = \hat{f}(\zeta) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x \cdot \zeta}dx,$$  \hspace{1cm} (4.0.1)

where $\zeta \in \mathbb{R}^n$ and $x \cdot \zeta = \sum_{i=1}^{n} \zeta_i \cdot x_i$.

We shall introduce some important properties about the Fourier transform for $f \in L^1(\mathbb{R}^n)$.

The map $f \rightarrow \hat{f}$ is linear in $f$. \hspace{1cm} (4.0.2)

Next properties for Fourier transform talking about translation and dilation.

$$\mathcal{F}(\tau_h f)(\zeta) = e^{-2\pi i \zeta \cdot h} \hat{f}(\zeta), \ h \in \mathbb{R}^n.$$ \hspace{1cm} (4.0.3)  

$$\mathcal{F}(\delta_\lambda f)(\zeta) = \lambda^n \hat{f}(\lambda \zeta), \ \lambda > 0.$$ \hspace{1cm} (4.0.4)

where $\tau_h$ is the translation operator, $(\tau_h f)(x) = f(x - h)$, and $\delta_\lambda$ is the scaling operator, $(\delta_\lambda f)(x) = f \left( \frac{x}{\lambda} \right)$.

Two other facts about Fourier transform are

$\hat{f} \in L^\infty(\mathbb{R}^n)$ and $\|\hat{f}\|_\infty \leq \|f\|_1,$ \hspace{1cm} (4.0.5)

$\hat{f}$ is a continuous function. \hspace{1cm} (4.0.6)
If \( u \in C_0^\infty(\mathbb{R}^n) \), then

\[
\mathcal{F}(\partial_{x_j} u)(\zeta) = \int_{\mathbb{R}^n} (\partial_{x_j} u)(x) \, e^{-2\pi i x \cdot \zeta} \, dx.
\]

But now integration by parts formula yields

\[
\mathcal{F}(\partial_{x_j} u)(\zeta) = -\int_{\mathbb{R}^n} u(x) \frac{\partial}{\partial x_j} (e^{-2\pi i x \cdot \zeta}) \, dx.
\]

Note that

\[
\frac{\partial}{\partial x_j} (e^{-2\pi i x \cdot \zeta}) = \frac{\partial}{\partial x_j} \sum_k [-2\pi i x_k \cdot \zeta_k] e^{-2\pi i x \cdot \zeta}
\]

\[
= -2\pi i \zeta_j e^{-2\pi i x \cdot \zeta},
\]

which implies that

\[
\mathcal{F}(\partial_{x_j} u)(\zeta) = -\int_{\mathbb{R}^n} -2\pi i \zeta_j u(x) e^{-2\pi i x \cdot \zeta} \, dx
\]

\[
= 2\pi i \zeta_j \mathcal{F}(u)(\zeta).
\]

(4.0.7)

Similarly, we can estimate the Fourier transform of Laplacian for any function \( u \) in \( C_0^\infty(\mathbb{R}^n) \).

\[
\mathcal{F}(\Delta u) = \int_{\mathbb{R}^n} (\Delta u)(x) \, e^{-2\pi i x \cdot \zeta} \, dx.
\]

Integration by parts yields

\[
\mathcal{F}(\Delta u) = \int_{\mathbb{R}^n} u(x) \Delta (e^{-2\pi i x \cdot \zeta}) \, dx
\]

\[
= \int_{\mathbb{R}^n} u(x)[-4\pi^2|\zeta|^2] e^{-2\pi i x \cdot \zeta} \, dx
\]

\[
= -4\pi^2|\zeta|^2 \mathcal{F}(u)(\zeta).
\]

(4.0.8)
4.0.1 Fourier Transform of the Fractional Laplacian

Lemma 4.0.2. Let \( \alpha \in (0, 1) \), and define \( u \in C^2_0(\mathbb{R}^n) \). Then

\[
\mathcal{F}((-\Delta)^{\alpha}u(x))(\zeta) = C(n, \alpha)|\zeta|^{2\alpha}\mathcal{F}(u)(\zeta).
\]

Proof. Firstly, we need to show that \((-\Delta)^{\alpha}u \in L^1(\mathbb{R}^n \times \mathbb{R}^n)\).

In other words, by using the estimate (3.0.7) we need to prove that

\[
I = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x + y) + u(x - y) - 2u(x)|}{|y|^{n+2\alpha}} \, dy \, dx < +\infty.
\]

We will take the first integral with respect to \( y \) and split it over \( B(0, 1) \) and \( \mathbb{R}^n \setminus B(0, 1) \).

Then, the first part of the integral \( I \) will be

\[
I_1 = \int_{\mathbb{R}^n} \int_{B(0, 1)} \frac{|u(x + y) + u(x - y) - 2u(x)|}{|y|^{n+2\alpha}} \, dy \, dx. \tag{4.0.9}
\]

The second part of integral will be

\[
I_2 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B(0, 1)} \frac{|u(x + y) + u(x - y) - 2u(x)|}{|y|^{n+2\alpha}} \, dy \, dx. \tag{4.0.10}
\]

Now we can write the numerator as the following way

\[
u(x + y) + u(x - y) - 2u(x) = [u(x + y) - u(x)] + [u(x - y) - u(x)].
\]

Let's recall the Mean Value Theorem which says that if \( f : \mathbb{R}^n \to \mathbb{R} \) then,

\[
f(x_1) - f(x_2) = \nabla f(\theta) \cdot (x_1 - x_2)
\]

for some \( \theta = tx_1 + (1-t)x_2, t \in (0, 1) \). Applying this for \( \lambda_i \in (0, 1) \) where \( i \in \{1, 2, 3\} \) to our situation yields

\[
[u(x + y) - u(x)] - [u(x) - u(x - y)] = \nabla u(x + \lambda_1 y).
\]

\[
= \sum_{j=1}^{n} \left[ \frac{\partial u}{\partial x_j}(x + \lambda_1 y) - \frac{\partial u}{\partial x_j}(x - \lambda_2 y) \right] y_j.
\]
Using Mean Value Theorem again yields
\[
\sum_{j=0}^{n} \left[ \nabla \frac{\partial u}{\partial x_j} \left( (x + \lambda_3 y)_j \right) \right] \sum_{j,k} \frac{\partial^2 u}{\partial x_j \partial x_k} (x + \lambda_3 y) y_j y_k
= \langle (D^2u) y, y \rangle.
\]

Therefore,
\[
|u(x + y) + u(x - y) - 2u(x)| \leq |y|^2 \sup_{B(x,1)} |D^2u|.
\]

Now let’s go back to the first part of the integral \(I\) which is \(I_1\).

We need to show that both \(|I_1| < \infty\) and \(|I_2| < \infty\) which means \(|I| < \infty\) as required.

\[
I_1 = \int_{\mathbb{R}^n} \int_{B(0,1)} \frac{|u(x + y) + u(x - y) - 2u(x)|}{|y|^{n+2\alpha}} \, dy \, dx
\leq \int_{B(0,1)} \left| \frac{y}{|y|^{n+2\alpha}} \right| dy \int_{\mathbb{R}^n} \max_{B(x,1)} |D^2u| \, dx.
\]

Assuming that
\[
\int_{\mathbb{R}^n} |D^2u| \, dx < +\infty.
\]

Also we know that
\[
\int_{B(0,1)} \frac{1}{|y|^{n+2\alpha-2}} \, dy < +\infty
\]
since \(n + 2\alpha - 2 < n\).

Therefore, we have \(|I_1| < +\infty\).

For the second part of the integral \(I_2\), we have
\[
|I_2| = \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B(0,1)} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n+2\alpha}} \, dy \, dx \right|
\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B(0,1)} \frac{|u(x + y) + u(x - y) - 2u(x)|}{|y|^{n+2\alpha}} \, dy \, dx
= \int_{\mathbb{R}^n \setminus B(0,1)} \frac{1}{|y|^{n+2\alpha}} \int_{\mathbb{R}^n} |u(x + y) + u(x - y) - 2u(x)| \, dy \, dx.
\]
By changing variable we will get
\[ |I_2| \leq \int_{\mathbb{R}^n \setminus B(0,1)} \frac{1}{|y|^{n+2\alpha}} dy \int_{\mathbb{R}^n} 4|u(x)| \, dx. \]

But since \( u \) is integrable, we have
\[ \int_{\mathbb{R}^n} |u(x)| \, dx < +\infty. \]

Note that since \( n + 2\alpha > n \),
\[ \int_{\mathbb{R}^n \setminus B(0,1)} \frac{1}{|y|^{n+2\alpha}} < +\infty. \]

Then, \( |I_2| < +\infty \). Therefore, \( |I| < +\infty \).

But this implies that,
\[ (-\Delta)^\alpha u \in L^1(\mathbb{R}^n \times \mathbb{R}^n) \text{ as desired} \]

Next step, we want to show that the Fourier transform of the fractional Laplacian is given by
\[ \mathcal{F}((-\Delta)^\alpha u)(\zeta) = C(n,\alpha)|\zeta|^{2\alpha} \mathcal{F}(u)(\zeta). \] (4.0.11)

This is a standard calculation, and we shall follow [3]. Since
\[ (x,y) \rightarrow \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2\alpha}} \in L^1(\mathbb{R}^n \times \mathbb{R}^n), \]
we may apply Fubini’s theorem to get that
\[ \mathcal{F}((-\Delta)^\alpha u(x)(\zeta)) = \frac{1}{2} \int_{\mathbb{R}^n} \left[ \mathcal{F}(u(x+y) + u(x-y) - 2u(x)) \right] (\zeta) \, dy. \]
By linearity of Fourier transform

\[
\begin{align*}
\mathcal{F}(u(x+y))(\zeta) + \mathcal{F}(u(x-y))(\zeta) - 2\mathcal{F}(u(x))(\zeta) &= \frac{1}{2} \int_{\mathbb{R}^n} [F(u(x+y))(\zeta) + F(u(x-y))(\zeta) - 2F(u(x))(\zeta)] dy \\
\mathcal{F}(u(x+y))(\zeta) + e^{2\pi i y \cdot \zeta} - 2u(x) e^{2\pi i x \cdot \zeta} dx &= \frac{1}{2} \int_{\mathbb{R}^n} (e^{2\pi i y \cdot \zeta} + e^{-2\pi i y \cdot \zeta} - 2) dy F(u)(\zeta) \\
&= -\int_{\mathbb{R}^n} \frac{1}{|y|^{n+2\alpha}} \left[ \frac{2 - (e^{2\pi i y \cdot \zeta} + e^{-2\pi i y \cdot \zeta})}{2} \right] dy F(u)(\zeta) \\
&= -\int_{\mathbb{R}^n} \frac{1 - \cos \zeta \cdot y}{|y|^{n+2\alpha}} dy F(u)(\zeta). \quad (4.0.12)
\end{align*}
\]

It suffices to show that

\[
-\int_{\mathbb{R}^n} \frac{1 - \cos \zeta \cdot y}{|y|^{n+2\alpha}} dy = |\zeta|^{2\alpha} C(n, \alpha). \quad (4.0.13)
\]

But \(\zeta \in \mathbb{R}^n\), so we can write it as \(\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{R}^n\). Note that near \(\zeta = 0\) we have

\[
\frac{1 - \cos \zeta_1}{|\zeta|^{n+2\alpha}} \leq \frac{|\zeta_1|^2}{|\zeta|^{n+2\alpha}} \leq \frac{1}{|\zeta|^{n-2+2\alpha}}.
\]

Therefore,

\[
\int_{\mathbb{R}^n} \frac{1 - \cos \zeta_1}{|\zeta|^{n+2\alpha}} d\zeta \text{ is finite and positive.} \quad (4.0.14)
\]

Consider a function \(\mathcal{I} : \mathbb{R}^n \to \mathbb{R}\) defined as follows

\[
\mathcal{I}(\zeta) = \int_{\mathbb{R}^n} \frac{1 - \cos \zeta \cdot y}{|y|^{n+2\alpha}} dy.
\]

Now \(\mathcal{I}\) is rotationally invariant, that is

\[
\mathcal{I}(\zeta) = \mathcal{I}(|\zeta|e_1),
\]

where \(e_1 = \left(1, \ 0, \ \cdots \ , 0\right)\).

For any \(x, z \in \mathbb{R}^n\) with \(|x| = |z|\) then there exists a rotation \(R_{x,z}\) such that

\[
R_{x,z}(x) = z.
\]
Applying this fact yields that

\[ R(|\zeta|e_1) = \zeta \quad \text{since} \quad ||\zeta|e_1| = |\zeta| \]

where \( R \) is \( n \times n \) matrix.

Also, if we have that \( A \) is a matrix and \( x, y \) are vectors, then

\[
(Ax).y = \langle Ax, y \rangle = \sum_{i,j=1} A_{ij}x_jy_i
\]
\[
= \sum_{i,j=1} x_j(A_{ij}y_i)
\]
\[
= \langle x, A^T y \rangle,
\]

where \( A^T \) is \( A \)'s transpose.

Consider \( R \) as a rotation and denote its transpose by \( R^T \) such that \( R(|\zeta|e_1) = \zeta \).

Then, by substituting \( \tilde{y} = R^T y \)

\[
\mathcal{I}(\zeta) = \int_{\mathbb{R}^n} \frac{1 - \cos(\zeta \cdot y)}{|y|^{n+2\alpha}} dy
\]
\[
= \int_{\mathbb{R}^n} \frac{1 - \cos(R(|\zeta|e_1) \cdot y)}{|y|^{n+2\alpha}} dy
\]
\[
= \int_{\mathbb{R}^n} \frac{1 - \cos \left( (|\zeta|e_1) \cdot (R^T y) \right)}{|y|^{n+2\alpha}} dy.
\]

Then,

\[
\mathcal{I}(\zeta) = \int_{\mathbb{R}^n} \frac{1 - \cos \left( (|\zeta|e_1) \cdot \tilde{y} \right)}{||\tilde{y}||^{n+2\alpha}} d\tilde{y}
\]

Note that \( dm_n(\tilde{y}) = |\det R^T|.dm_n(y) = dm_n(y) \). Since rotation doesn’t change length, we will have \(|y| = ||\tilde{y}||\). Hence,

\[
\mathcal{I}(\zeta) = \int_{\mathbb{R}^n} \frac{1 - \cos ((|\zeta|e_1) \cdot y)}{|y|^{n+2\alpha}} dy = \mathcal{I}(|\zeta|e_1)
\]

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the substitution $\xi = |\zeta|y$ gives that

$$I(\zeta) = I(|\zeta|e_1)$$

$$= \int_{\mathbb{R}^n} \frac{1 - \cos(|\zeta|y_1)}{|y|^{n+2\alpha}} \, dy$$

$$= \int_{\mathbb{R}^n} \frac{1 - \cos(\xi_1)}{|\xi||\zeta|^{n+2\alpha} |\zeta|^n} \, d\xi$$

$$= |\zeta|^{2\alpha} C(n, \alpha),$$

which proves (4.0.13) and therefore (4.0.12) as we want to show.

4.1 Fourier Transform of a Gaussian

**Theorem 4.1.1.** For $\lambda > 0$, denote by $g_\lambda$ the **Gaussian function** on $\mathbb{R}^n$ given by

$$g_\lambda(x) = e^{-\pi \lambda |x|^2}$$

for $x \in \mathbb{R}^n$. Then, the Fourier transform for $g_\lambda(x)$ is given by

$$\hat{g}_\lambda(\zeta) = \lambda^{-\frac{n}{2}} e^{-\pi |\zeta|^2/\lambda}.$$

**Proof.** In this proof we will use the same technique as in chapter five of [4]. We define $g_\lambda$ to be the Gaussian function on $\mathbb{R}^n$ so that for $x \in \mathbb{R}^n$ we have

$$g_\lambda(x) = e^{-\pi \lambda |x|^2}$$

By (4.0.4), it suffices to consider $\lambda = 1$. Since $x \in \mathbb{R}^n$, $|x|^2 = \sum_{j=1}^n x_j^2$. Thus,

$$g_1(x) = \prod_{j=1}^n e^{-\pi (x_j)^2}.$$ 

It suffices to consider $n = 1$.

$$g_1(x) = e^{-\pi x^2}.$$
Because $g_1 \in L^1(\mathbb{R})$, we can apply Fourier transform for $g_1$

$$\tilde{g}_1(\zeta) = \int_{\mathbb{R}} e^{-2\pi i x \cdot \zeta} e^{-\pi x^2} \, dx$$

$$= \int_{\mathbb{R}} e^{-\pi (x^2 + 2ix \cdot \zeta + (i\zeta)^2)} e^{-\pi \zeta^2} \, dx$$

$$= \int_{\mathbb{R}} e^{-\pi (x + i\zeta)^2} e^{-\pi \zeta^2} \, dx$$

$$= e^{-\pi \zeta^2} \int_{\mathbb{R}} e^{-\pi (x + i\zeta)^2} \, dx$$

$$= g_1(\zeta) f(\zeta),$$

where

$$f(\zeta) = \int_{\mathbb{R}} e^{-\pi (x + i\zeta)^2} \, dx.$$

The exponential decay of the integrand in the $x$-variable allows us to differentiate $f$ with respect to $\zeta$ as many times as we like. Therefore, $f \in C^\infty(\mathbb{R}^n)$ and

$$\frac{d}{d\zeta} f(\zeta) = -2\pi i \int_{\mathbb{R}} (x + i\zeta) e^{-\pi (x + i\zeta)^2} \, dx$$

$$= i \int_{\mathbb{R}} \frac{d}{dx} e^{-\pi (x + i\zeta)^2} \, dx = i e^{-\pi (x + i\zeta)^2} \bigg|_{-\infty}^{\infty} = 0,$$

i.e., $f(\zeta)$ is constant. However, note that

$$f^2(0) = \left( \int_{\mathbb{R}} e^{-\pi x^2} \, dx \right)^2$$

$$= \int_0^\infty e^{-\pi x_1^2} \, dx_1 \int_0^\infty e^{-\pi x_2^2} \, dx_2$$

$$= \int_0^\infty \int_0^\infty e^{-\pi (x_1^2 + x_2^2)} \, dx_1 \, dx_2 = \int_{\mathbb{R}^2} e^{-\pi |x|^2} \, dm_2 x.$$

By polar co-ordinates

$$= 2\pi \int_0^\infty e^{-\pi r^2} \, dr$$

$$= 2\pi \left( \frac{-1}{2\pi} e^{-\pi r^2} \right) \bigg|_0^\infty = 1.$$
We will provide now an important fact about the Fourier transform.

**Lemma 4.1.2.** If $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then

$$f = \mathcal{F}^{-1}(\hat{f}).$$

**Proof.** We know that

$$\mathcal{F}^{-1}(\hat{f}) = \int_{\mathbb{R}^n} e^{2\pi ix \cdot \zeta} \hat{f}(\zeta) dm_n(\zeta).$$

By dominated convergence theorem for fixed $\epsilon > 0$

$$\mathcal{F}^{-1}(\hat{f}) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} e^{2\pi ix \cdot \zeta} e^{-\pi \epsilon |\zeta|^2} \hat{f}(\zeta) dm_n(\zeta)$$

$$= \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} e^{-\pi \epsilon |\zeta|^2} e^{2\pi ix \cdot \zeta} \int_{\mathbb{R}^n} e^{-2\pi iy \cdot \zeta} f(y) dm_n(y) \ dm_n(\zeta).$$

By using Fubini’s theorem

$$\mathcal{F}^{-1}(\hat{f}) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} f(y) \int_{\mathbb{R}^n} e^{-\pi \epsilon |\zeta|^2} e^{-2\pi i(y-x) \cdot \zeta} dm_n(\zeta) \ dm_n(y).$$

By (4.1.1), we get that

$$\mathcal{F}^{-1}(\hat{f}) = \lim_{\epsilon \to 0} \epsilon^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(y) e^{-\frac{|y-x|^2}{\epsilon}} dm_n(y).$$

Now claim that

$$f(x) = \lim_{\epsilon \to 0} \epsilon^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(y) e^{-\frac{|y-x|^2}{\epsilon}} dm_n(y).$$

To see this first notice that

$$\epsilon^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|y-x|^2}{\epsilon}} dm_n(y) = 1.$$
Since $\hat{f} \in L^1(\mathbb{R}^n)$, we have that $f$ is continuous. Then there exists $\delta > 0$ such that $|f(y) - f(x)| < \frac{\tau}{2}$ for some $\tau > 0$ if $y \in B(x, \delta)$. Therefore,

$$|I| = \lim_{\epsilon \to 0} \epsilon^{-\frac{n}{2}} \left| \int_{\mathbb{R}^n} [f(y) - f(x)] e^{-\frac{\pi |y - x|^2}{\epsilon}} dm_n(y) \right|$$

$$\leq \lim_{\epsilon \to 0} \epsilon^{-\frac{n}{2}} \int_{B(x, \delta)} |f(y) - f(x)| e^{-\frac{\pi |y - x|^2}{\epsilon}} dm_n(y)$$

$$+ \lim_{\epsilon \to 0} \epsilon^{-\frac{n}{2}} \int_{\mathbb{R}^n \backslash B(x, \delta)} |f(y) - f(x)| e^{-\frac{\pi |y - x|^2}{\epsilon}} dm_n(y)$$

$$= I_1 + I_2.$$

By continuity and since $f \in L^\infty(\mathbb{R}^n)$, there exists $M > 0$ such that $|f(y) - f(x)| < M$

$$I_1 + I_2 \leq \frac{\tau}{2} + M \lim_{\epsilon \to 0} \epsilon^{-\frac{n}{2}} \int_{|x-y| \geq \delta} e^{-\frac{\pi |z|^2}{\epsilon}} dy.$$  

By changing variable (choose $z = \frac{x-y}{\sqrt{\epsilon}}$) and using monotone convergence theorem as $\epsilon \to 0^+$

$$I_2 \leq M \lim_{\epsilon \to 0} \epsilon^{-\frac{n}{2}} \int_{|z| \geq \frac{\delta}{\sqrt{\epsilon}}} e^{-\pi |z|^2} dz \to 0.$$  

Then,

$$|I| \leq \lim_{\epsilon \to 0} \epsilon^{-\frac{n}{2}} \int_{\mathbb{R}^n} |[f(y) - f(x)]| e^{-\frac{\pi |y - x|^2}{\epsilon}} dm_n(y) \to 0,$$

which means that

$$f(x) = \int_{\mathbb{R}^n} f(y) e^{-\frac{\pi |y - x|^2}{\epsilon}} dm_n(y).$$
4.2 Hausdorff Measure

**Definition 4.2.1.** For $s > 0$, choose $\delta > 0$ define the Hausdorff measure for a set $E$ by
\[
\mathcal{H}_s^\delta(E) = \inf \left\{ \sum_{j=1}^{\infty} \text{diam}(U_j)^s : E \subset \bigcup_j U_j, \text{diam}(U_j) \leq \delta \right\},
\]
where $(U_j)_{j \in \mathbb{N}}$ are open sets cover $E$. Note that $\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}_s^\delta(E)$.

We shall now introduce some important properties about the Hausdorff measure.

- Define the Hausdorff measure for $E$ covered by a collection of balls $B(x_j, r_j)$ by
\[
\mathcal{H}_s^{\delta, \text{balls}}(E) = \lim_{\delta \to 0} \mathcal{H}_s^\delta(E) = \lim_{\delta \to 0} \inf \left\{ \sum_{j=1}^{\infty} r_j^s : \text{where } E \subset \bigcup_{j=1}^{\infty} B(x_j, r_j), r_j \leq \delta \right\}.
\]

Since one can place a ball of radius $r_j$ around a set of diameter $(r_j - \epsilon)$ for any $\epsilon > 0$, there exists a constant $C_0$ such that
\[
\mathcal{H}_s^{\text{balls}}(E) \leq C_0 \mathcal{H}^s(E).
\]

- **Hausdorff measure is invariant under translations**
\[
\mathcal{H}_s^\delta(E + h) = \mathcal{H}_s^\delta(E) \quad \forall h \in \mathbb{R}^n.
\]

- **Hausdorff measure is invariant under rotations**
\[
\mathcal{H}_s^\delta(rE) = \mathcal{H}_s^\delta(E)
\]
where $r$ is a rotation in $\mathbb{R}^n$.

**Lemma 4.2.2.** Let $E \subset \mathbb{R}^n$ is a compact set with $\mathcal{H}^s(E) = 0$. Then, $E$ can be covered by a finite number of disjoint dyadic cubes $(Q_j)^N_{j=1}$ with
\[
\sum_{j=1}^{N} l(Q_j)^s \leq \epsilon.
\]
Proof. Fix \( \epsilon, \delta > 0 \). We have that \( \mathcal{H}^s_{\text{balls}}(E) = 0 \). Therefore, there is a cover of \( E \) by balls \((B_j)_{j=1}^\infty \) such that \( \sum_{j=1}^\infty r_j < \delta \). But since \( E \) is compact, it is in fact covered by some finite numbers of the balls \( B_j \). Each of these balls \( B_j \) can be covered by at most \( 2^n \) dyadic cubes of sidelengths at most \( 2r_j \). We therefore obtain a collection of dyadic cubes, say, \( \widetilde{Q}_1, \cdots, \widetilde{Q}_M \) such that

\[
\sum_j \widetilde{l}_j^s \leq \sum_j 2^n (2r_j)^s \leq 2^{n+s} \sum_j r_j^s \leq 2^{n+s} \delta^s < \epsilon
\]

whenever \( \delta^s < \frac{\epsilon}{2^{n+s}} \).

Recall that if we have two dyadic cubes, say \( Q, Q' \), with \( Q \cap Q' \neq \emptyset \), then either \( Q \subseteq Q' \) or \( Q' \subseteq Q \). Thus, we can choose for each \( x \in E \) the maximal dyadic cube that contains \( x \) is \( \widetilde{Q}_j \) \( \forall j \) so then this collection is disjoint with \( \sum_j \widetilde{l}_j^s < \epsilon \). We may now order them by sidelength as \( \widetilde{l}_1 \geq \widetilde{l}_2 \geq \cdots \geq \widetilde{l}_M \).

We need to discuss now an important principle called **Mass Distribution Principle**.

**Lemma 4.2.3. Mass Distribution Principle** Suppose that \( \mu \) is a Borel measure on \( X \), and \( K \subset X \) satisfies \( \mu(K) > 0 \). If there is a constant \( C > 0 \) so that \( \mu(B(x,r)) \leq Cr^s \) for every \( x \in X, r > 0 \). Then

\[
\mathcal{H}^s(K) \geq \frac{\mu(K)}{C}.
\]

Proof. Suppose that there is a constant \( C > 0 \) with \( \mu(B(x,r)) \leq Cr^s \) for each \( x \in X \).

Let \((B(x_j,r_j))_{j \in \mathbb{N}}\) be a collection of balls which covers \( K \). Then,

\[
K \subset \bigcup_j B(x_j, r_j) \text{ by monotonicity } \mu(K) \leq \sum_{j=1}^\infty \mu(B(x_j, r_j)).
\]

By the assumption which is \( \mu(B(x_j, r_j)) \leq Cr_j^s \), we have that

\[
\mu(K) \leq \sum_{j=1}^\infty \mu(B(x_j, r_j)) \leq C \sum_{j=1}^\infty r_j^s
\]
By first property of Hausdorff measure

\[ \mathcal{H}^s_{balls}(K) \leq C_0 \mathcal{H}^s(K) \]

for some constant \(C_0\). But then we have, for any \(\delta > 0\)

\[ \frac{\mu(K)}{C} \leq \mathcal{H}^s_\delta(K) \]

for some constant \(C\).

Now we will state an important lemma called **Frostman’s Lemma**.

**Lemma 4.2.4. (Frostman’s Lemma)**

Let \(K \subset \mathbb{R}^n\) be a compact set. Then,

\( \mathcal{H}^s(K) > 0 \) if and only if there is a measure \(\mu \in \mu(K)\) such that \(\mu(B(x,r)) \leq Cr^s\) for any \(x \in \mathbb{R}^n, r > 0\).

**Proof.** Assuming that there exists a measure \(\mu \in \mu(K)\) such that \(\mu(B(x,r)) \leq Cr^s\).

We want to show \(\mathcal{H}^s(K) > 0\). But then from (4.2.3) we see that \(\mathcal{H}^s(K) \geq \frac{\mu(K)}{C} > 0\).

Conversely, Assume that \(\mathcal{H}^s(K) > 0\).

Note that there exists a cube \(Q\) of sidelength 1 with \(K \cap Q \neq \emptyset\) and \(\mathcal{H}^s(K \cap Q) > 0\).

By translation invariance of Hausdorff measure in \(\mathbb{R}^n\), we may assume that \(K \cap Q_0\) satisfies \(\mathcal{H}^s(K \cap Q_0) > 0\) where \(Q_0 = [0,1]^n\). Then there exists \(\alpha > 0\) such that \(\sum_j l(Q_j)^s \geq \alpha\) whenever \((Q_j)_j\) is a cover of \(K\) by cubes. By replacing \(K\) by \(K \cap Q_0\), we shall assume that \(K \subset [0,1]^n = Q_0\).

Now let’s start to build the measure, say, \(\mu^{(m)}\).
For each $m \in \mathbb{N}$, consider the collection of dyadic subcubes of $Q_0$ with sidelength $2^{-m}$, say, $D_m$. Define $\mu^{(m)}$ to be

$$\mu^{(m)} = \sum_{Q \in D_m} l(Q)^s \frac{\chi_{Q} m_n(Q)}{m_n(Q)}.$$ 

Since $l(Q)^n = m_n(Q)$, we have

$$\mu^{(m)} = \sum_{Q \in D_m} \frac{m_n(Q)^{\frac{s}{n}}}{m_n(Q)} \chi_{Q} m_n(Q).$$

Hence, for every $Q \in D_m$, it follows that

$$\mu^{(m)}|_Q = \frac{m_n(Q)^{\frac{s}{n}}}{m_n(Q)} \chi_{Q} m_n(Q).$$

If $Q \cap K = \emptyset$, then $\mu^{(m)}(Q) = 0$. On the other hand, if $Q \cap K \neq \emptyset$, we have $\chi_Q m_n(Q) = m_n(Q)$. Thus, $\mu^{(m)}(Q) = l(Q)^s$.

Now for $D_{m-1}$ which is a collection of dyadic subcubes of $Q_0$ of sidelength $2^{-(m-1)}$ we will define $\mu^{(m)}_{m-1}$ for every $Q \in D_{m-1}$ by

$$\mu^{(m)}_{m-1}|_Q = \begin{cases} 
\mu^{(m)}|_Q & \text{if } \mu^{(m)}(Q) \leq l(Q)^s \\
l(Q)^s \frac{1}{\mu^{(m)}(Q)} \mu^{(m)}|_Q & \text{if } \mu^{(m)}(Q) > l(Q)^s
\end{cases},$$

where $\mu|_Q$ is the restriction of a measure $\mu$ to $Q$.

Notice that if $\mu^{(m)}(Q) > l(Q)^s$ for some $Q \in D_{m-1}$, then $\mu^{(m)}_{m-1}(Q) = l(Q)^s$.

Observe that we still have for each $Q \in D_m$ that $\mu^{(m)}_{m-1}(Q) \leq l(Q)^s$, and also we have $\mu^{(m)}_{m-1}(Q) \leq l(Q)^s$ for every $Q \in D_{m-1}$.

We conclude that $\mu^{(m)}_{m-1}(Q) \leq l(Q)^s$ for every $Q \in D_m \cup D_{m-1}$.

Similarly, we will define $\mu^{(m)}_{j-1}$ for $Q \in D_{j-1}$ based on $\mu^{(m)}_j$ as

$$\mu^{(m)}_{j-1}|_Q = \begin{cases} 
\mu^{(m)}_j|_Q & \text{if } \mu^{(m)}_j(Q) \leq l(Q)^s \\
l(Q)^s \frac{1}{\mu^{(m)}_j(Q)} \mu^{(m)}_j|_Q & \text{if } \mu^{(m)}_j(Q) > l(Q)^s
\end{cases}. $$

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Arrive at a measure $\mu^{(m)}(Q) = \mu^{(m)}_0(Q)$ we have

$$\mu^{(m)}(Q) \leq l(Q)^s \text{ for every } Q \in \bigcup_{j=0}^m D_j.$$  

Furthermore, by construction for each $x \in K$, there is a cube $Q \in \bigcup_{j=0}^m D_k$ such that $\mu^{(m)}(Q) = l(Q)^s$. Recall that, if $Q, Q'$ are any two dyadic cubes whose interiors intersect, then either $Q \subseteq Q'$ or $Q' \subseteq Q$.

Let $Q_x$ be the largest cube that contains $x \in K$ satisfying $\mu^{(m)}(Q_x) = l(Q_x)^s$. Let $x, x' \in K$ with $Q_x \cap Q_{x'} \neq \emptyset$. By maximality we have $Q_x = Q_{x'}$.

Therefore, we find a pairwise disjoint collection $Q_1, ..., Q_N$ of such maximal cubes with $K \subset \bigcup_j Q_j$. Then,

$$\mu^{(m)} \left( \bigcup_j Q_j \right) = \sum_j \mu^{(m)}(Q_j) = \sum_j l(Q_j)^s \geq \alpha.$$  

Since we know that $K \subset [0,1]^n$, $\mu^{(m)}(K) \leq \mu^{(m)}([0,1]^n) \leq 1$.

Thus, the weak compactness of measures, see [5], allows us to pass to a subsequence of the measure $\mu^{(m)}$ that converges weakly to a measure $\mu$ satisfying

$$1 \geq \mu(K) \geq \alpha.$$  

Weak convergence here means that

$$\lim_{m \to \infty} \int_{\mathbb{R}^d} \varphi \, d\mu^{(m)} = \int_{\mathbb{R}^d} \varphi \, d\mu \text{ for every } \varphi \in C_0(\mathbb{R}^d).$$  

Let $B(x, r)$ be a ball centered at $x$ with radius $r < 1$, then $B(x, r)$ is contained in a union of $2^n$-dyadic cubes of sidelength between $r$ and $2r$.

But then if $j$ is large enough, the $\mu^{(j)}$-measure of each of these cubes $Q_j$ satisfies $\mu^{(j)}(Q) \leq l(Q)^s$. Hence,

$$\mu^{(j)}(B(x, r)) \leq 2^n (2r)^s = 2^{n+s} r^s.$$  

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But then, by weak lower semicontinuity of the weak limit, we get that

\[\mu(B(x,r)) \leq 2^{n+s}r^s\] for every \(B(x,r)\).

This completes the proof. \qed
CHAPTER 5

Main Theorem

We will state in this chapter the main result which relates Hausdorff measure and removable set.

**Theorem 5.0.1.** Suppose that \( u : \mathbb{R}^n \to \mathbb{R} \) is \( \gamma \)-Hölder continuous solution where \( \gamma \in (0, 2\alpha), \gamma < 1 \). Then,

\[
|u(y) - u(x)| \leq C|y - x|^\gamma
\]

for some constant \( C \). Also, assume that \( (-\Delta)^\alpha u = 0 \) in \( D'(\mathbb{R}^n \setminus E) \) for some compact set \( E \). Then, \( (-\Delta)^\alpha u = 0 \) in \( D'(\mathbb{R}^n) \) so that

\[
\int u(-\Delta)^\alpha \varphi dx = 0 \quad \text{for every } \varphi \in C_0^\infty(\mathbb{R}^n)
\]

if and only if \( \mathcal{H}^{n-2\alpha+\gamma}(E) = 0 \), which means \( E \) is removable.

**Proof.** Let’s first assume that \( \mathcal{H}^{n-2\alpha+\gamma}(E) = 0 \). By (4.2.2) we can find a finite collection of disjoint dyadic cubes \( Q_1, \cdots, Q_N \) have sidelengths \( l(Q_1) \geq l(Q_2) \geq \cdots \geq l(Q_N) \) with \( E \subset \bigcup_j Q_j \) such that \( \sum_j l(Q_j)^{n-2\alpha+\gamma} < \epsilon \), for some \( \epsilon > 0 \).

Define a distribution \( T \) such that,

\[
T(\varphi) = \langle u, (-\Delta)^\alpha \varphi \rangle \in \mathbb{R}.
\]

It is given that \( (-\Delta)^\alpha u = 0 \) in \( D'(\mathbb{R}^n \setminus E) \) (i.e. if \( \text{supp}(\varphi) \cap E = \emptyset \), then \( T(\varphi) = 0 \)). The Harvey-Polking partition of unity provides us with functions \( \varphi_j \in C_0^\infty(3Q_j) \) with
\[ \sum_j \varphi_j(x) = 1 \text{ for every } x \in E. \]

Since it is true that we can cover each \(3Q_j\) by a ball of radius \(3\sqrt{n} l(Q_j)\), we have that \(\varphi_j \in C_0^\infty(B(x_j, 3\sqrt{n} l(Q_j)))\). Therefore, we can use the estimate of derivative from (2.2.2), which yields that \(|\triangle \varphi_j| \leq \frac{C}{j}\). Pick \(\chi \in C_0^\infty(\mathbb{R}^n)\) an arbitrary function.

We can write the distribution of the function \(\chi\) as

\[ T(\chi) = T\left(\chi - \sum_j \chi \varphi_j + \sum_j \chi \varphi_j\right). \]

Since \(T\) is a distribution, by linearity we have

\[ T(\chi) = \left[ \sum_j T(\chi \varphi_j) \right] + T\left[ \chi \left(1 - \sum_j \varphi_j\right) \right]. \]

But from the assumptions, we know that \(\sum_j \varphi(x) = 1\) for each \(x \in E\). Since both \(\chi\) and \(\varphi\) are in \(C_0^\infty(\mathbb{R}^n)\), \(\chi(1 - \sum_j \varphi_j)\) is also in \(C_0^\infty(\mathbb{R}^n)\) and its supported is disjoint with \(E\). But then,

\[ T\left[ \chi \left(1 - \sum_j \varphi_j\right) \right] = 0. \]

Now since we have \(|\triangle (\chi \varphi_j)| \leq \frac{C}{j}\) with \(l_j\) is arbitrary small and \(\chi \varphi_j\) is supported in \(B(x_j, 3\sqrt{n} l_j)\), using (3.0.2) yields

\[ \sum_j |T(\chi \varphi_j)| \leq C(3\sqrt{n})^{n-2\alpha+\gamma} \sum_j l_j^{n-2\alpha+\gamma} \leq C\epsilon. \]

Therefore,

\[ |T(\chi)| < C\epsilon \quad \text{for an arbitrary function } \chi \in C_0^\infty(\mathbb{R}^n). \quad (5.0.1) \]

For the converse, let’s assume that \(E\) is removable so we want to show that \((n-2\alpha+\gamma)\)-dimensional Hausdorff measure of \(E\) is identically 0.

Suppose that \(\mathcal{H}^{n-2\alpha+\gamma}(E) > 0\) so that by (4.2.4) we can construct a measure \(\mu\) with
some properties, which are \( \mu(E) = 1 \) and \( \forall x \in \mathbb{R}^n \) we have that

\[ \mu(B(x, r)) \leq Cr^{n-2\alpha+\gamma}. \]

Now we want to create \( u \) such that \( T = (-\Delta)^{\alpha} u \neq 0 \) in \( \mathbb{R}^n \) with \( \text{supp}(T) \subset E \).

Set \( u \) as the following

\[ u = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2\alpha}} d\mu(y). \]

Firstly, we need to show that \( u \) is \( \gamma \)-Hölder continuous function i.e. \( u \) satisfies that, for \( x, x' \in E \) we want to show that \( |u(x) - u(x')| \leq C|x - x'|^\gamma \).

To show that, we are going to follow the same manner as in Carleson’s paper [1].

Assume that \( |x - x'| = \delta \).

\[
\begin{align*}
  u(x) &= \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2\alpha}} d\mu(y) \\
  &= \int_{\mathbb{R}^n} (n-2\alpha) \int_{|x-y|}^\infty \frac{1}{t^{n-2\alpha}} \frac{1}{t} d\mu(y) \\
  &= (n-2\alpha) \int_{\mathbb{R}^n} \int_0^\infty \chi_{\{y,t\}: |x-y|<t} \frac{1}{t^{n-2\alpha+1}} dt d\mu(y)
\end{align*}
\]

By Fubini’s theorem, we can switch the order of integration, so then

\[
\begin{align*}
  u(x) &= (n-2\alpha) \int_0^\infty \frac{1}{t^{n-2\alpha+1}} \int_{\mathbb{R}^n} \chi_{\{y,t\}: |x-y|<t} d\mu(y) dt \\
  &= (n-2\alpha) \int_0^\infty \frac{1}{t^{n-2\alpha+1}} \mu(B(x,t)) dt.
\end{align*}
\]

Therefore,

\[
\begin{align*}
  u(x) - u(x') &= (n-2\alpha) \int_{2\delta}^\infty \frac{1}{t^{n-2\alpha+1}} \left[ \mu(B(x,t)) - \mu(B(x',t)) \right] dt.
\end{align*}
\]

Since we have that \( B(x, t - \delta) \subset B(x', t) \), we obtain

\[
\begin{align*}
  u(x) - u(x') &\leq (n-2\alpha) \int_{2\delta}^\infty \frac{1}{t^{n-2\alpha+1}} \left[ \mu(B(x,t)) - \mu(B(x, t - \delta)) \right] dt.
\end{align*}
\]
By changing variable (choose $t' = t - \delta$)

$$u(x) - u(x') \leq (n - 2\alpha) \left[ \int_\delta^\infty \frac{\mu(B(x,t))}{t^{n-2\alpha+1}} dt - \int_\delta^\infty \frac{\mu(B(x,t'))}{(t' + \delta)^{n-2\alpha+1}} dt' \right].$$

Therefore, by relabeling $t$ as $t'$

$$|u(x) - u(x')| \leq (n - 2\alpha) \int_\delta^\infty \mu(B(x,t)) \left| \frac{1}{t^{n-2\alpha+1}} - \frac{1}{(t + \delta)^{n-2\alpha+1}} \right| dt.$$

By Mean Value Theorem for a function $g$ defined as $g(t) = \frac{1}{t^{n-2\alpha+1}}$, we know that for some $\theta \in (t, t + \delta)$

$$|g(t + \delta) - g(t)| \leq C\delta|g'(\theta)| \leq \frac{C\delta}{\theta^{n-2\alpha+2}}.$$

Also, we know by Frostman’s lemma (4.2.4) that $\mu(B(x,t)) \leq t^{n-2\alpha+\gamma}$ then,

$$|u(x) - u(x')| \leq C\delta \int_\delta^\infty t^{n-2\alpha+\gamma} \frac{dt}{t^{n-2\alpha+2}} = C\delta \int_\delta^\infty t^{\gamma-2} dt \leq C\delta^\gamma.$$

But $\delta$ is given by the distance between $x$ and $x'$. Thus,

$$|u(x) - u(x')| \leq C|x - x'|^\gamma.$$

Therefore, $u$ is $\gamma$-Hölder continuous function.

Now we need to prove that the distribution that is given by

$$\langle u, (-\triangle)^\alpha \varphi \rangle = \int_{\mathbb{R}^n} u(y)(-\triangle)^\alpha \varphi(x) d\mu(y)$$

for every $\varphi \in C_0^\infty(\mathbb{R}^n)$

is not zero in $E$. This will be done if we show that

$$\int_{\mathbb{R}^n} \frac{1}{|x - y|^{n - 2\alpha}} (-\triangle)^\alpha \varphi(y) dm_n(y) = c_\alpha \varphi(x).$$

In this proof, we are going to use the same technique as in [4].
Claim 5.0.2. Suppose that $\chi \in L^\infty(m_n) \cap L^1(m_n)$ and $\hat{\chi} \in L^\infty(m_n) \cap L^1(m_n)$. Then for $\beta \in (0, n)$

$$\mathcal{F}^{-1} \left( \frac{1}{|\zeta|^\beta} \hat{\chi}(\zeta) \right)(x) = c_{\beta,n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\beta}} \chi(y) dm_n(y).$$

Consider the integral

$$I = \int_0^\infty e^{-\pi|\zeta|^2} \lambda^{\beta/2-1} d\lambda.$$ 

By changing variable, let $\tilde{\lambda} = \lambda|\zeta|^2$ then $\lambda^{\beta/2-1} = \frac{\tilde{\lambda}^{\beta/2-1}}{|\zeta|^2|\zeta|^2}$ and $d\tilde{\lambda} = |\zeta|^2 d\lambda$.

Therefore,

$$I = \int_0^\infty e^{-\pi\tilde{\lambda}} \tilde{\lambda}^{\beta/2-1} d\tilde{\lambda}$$

$$= |\zeta|^{-\beta} \int_0^\infty e^{-\pi\tilde{\lambda}} \tilde{\lambda}^{\beta/2-1} d\tilde{\lambda}$$

$$= |\zeta|^{-\beta} C_\beta. \quad (5.0.2)$$

Next step, we need to verify that $|\zeta|^{-\beta} \hat{\chi}(\zeta)$ is integrable, i.e. $\int_{\mathbb{R}^n} |\zeta|^{-\beta} |\hat{\chi}(\zeta)| d\zeta < +\infty$.

$$\int_{\mathbb{R}^n} |\zeta|^{-\beta} |\hat{\chi}(\zeta)| d\zeta = \int_{B(0,1)} |\zeta|^{-\beta} |\hat{\chi}(\zeta)| d\zeta + \int_{\mathbb{R}^n \setminus B(0,1)} |\zeta|^{-\beta} |\hat{\chi}(\zeta)| d\zeta.$$ 

Since we have $\hat{\chi} \in L^\infty(m_n)$, then it is bounded inside the ball $B(0,1)$ which means that there exists $M > 0$ such that $|\hat{\chi}(\zeta)| \leq M$. By polar co-ordinates, we know that

$$\int_{B(0,1)} |\zeta|^{-\beta} d\zeta < +\infty.$$ 

Now let’s see the second part of the integral. Since $|\zeta|^{-\beta} \leq 1$ for $\zeta \in \mathbb{R}^n \setminus B(0,1)$, we have as $\hat{\chi} \in L^1(m_n)$

$$\int_{\mathbb{R}^n \setminus B(0,1)} |\zeta|^{-\beta} |\hat{\chi}(\zeta)| d\zeta \leq \int_{\mathbb{R}^n \setminus B(0,1)} |\hat{\chi}(\zeta)| d\zeta.$$
But $\hat{\chi} \in L^1(m_n)$ which means that
\[
\int_{\mathbb{R}^n \setminus B(0,1)} |\hat{\chi}(\zeta)| d\zeta \leq \int_{\mathbb{R}^n} |\hat{\chi}(\zeta)| d\zeta < +\infty.
\]
Then,
\[
\int_{\mathbb{R}^n} |\zeta|^{-\beta} |\hat{\chi}(\zeta)| d\zeta < +\infty.
\]
Hence, $|\zeta|^{-\beta} |\hat{\chi}(\zeta)|$ is integrable. Therefore,
\[
C_{\beta} F^{-1} \left( |\zeta|^{-\beta} \hat{\chi}(\zeta) \right)(x) = \int_{\mathbb{R}^n} e^{2\pi i\zeta \cdot x} \left\{ \int_0^{\infty} e^{-\pi |\zeta|^2 \lambda^{\beta/2} - 1} d\lambda \right\} \hat{\chi}(\zeta) d\zeta.
\]
By Fubini’s theorem, we have
\[
C_{\beta} F^{-1} \left( |\zeta|^{-\beta} \hat{\chi}(\zeta) \right)(x) = \int_0^{\infty} \left\{ \int_{\mathbb{R}^n} e^{2\pi i\zeta \cdot x} e^{-\pi |\zeta|^2 \lambda^{\beta/2} - 1} \hat{\chi}(\zeta) d\zeta \right\} \lambda^{\beta/2 - 1} d\lambda.
\]
Choose $g_\lambda(\zeta) = e^{-\pi |\zeta|^2 \lambda}$. Then, using (4.1.1) yields
\[
g_\lambda(x) = \lambda^{\frac{n}{2}} e^{-\pi |x|^2 / \lambda} \in L^1(m_n) \cap L^\infty(m_n).
\]
Since we have
\[(g_\lambda \ast \chi)(x) = \int_{\mathbb{R}^n} \lambda^{\frac{n}{2}} e^{-\pi |x-y|^2 / \lambda} \chi(y) dy,
\]
we can say that
\[
(g_\lambda \ast \chi)(\zeta) = \int_{\mathbb{R}^n} e^{-2\pi i \zeta \cdot x} \int_{\mathbb{R}^n} g_\lambda(x-y) \chi(y) dm_n y dm_n x
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-2\pi i (x-y) \cdot \zeta} g_\lambda(x-y) dm_n(x) e^{-2\pi i y \cdot \zeta} \chi(y) dm_n(y)
\]
\[
= \hat{g_\lambda}(\zeta) \cdot \hat{\chi}(\zeta).
\]
Hence,
\[
C_{\beta} F^{-1} \left( |\zeta|^{-\beta} \hat{\chi}(\zeta) \right)(x) = \int_0^{\infty} \lambda^{\frac{n}{2}} e^{-\pi |x-y|^2 / \lambda} \chi(y) dy.
\]
We therefore know from (5.0.2) that

$$C_\beta \mathcal{F}^{-1} (|\zeta|^{-\beta} \hat{\chi} (\zeta)) (x) = C_{n-\beta} \int_{\mathbb{R}^n} |x - y|^{-n+\beta} \chi (y) dy.$$ 

Choose $\beta = 2\alpha$ and $\chi (x) = (-\Delta)^{\alpha} \varphi (x)$ which implies by (4.0.2) that

$$\hat{\chi} (\zeta) = c |\zeta|^{2\alpha} \hat{\varphi} (\zeta)$$ 

for some nonzero constant $c$. Thus,

$$\mathcal{F}^{-1} \left( \frac{1}{|\zeta|^{2\alpha}} \hat{\chi} (\zeta) \right) = c \mathcal{F}^{-1} (\hat{\varphi} (\zeta)) = c \varphi (x)$$

But then we have,

$$c_{\alpha, n} \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-2\alpha}} (-\Delta)^{\alpha} \varphi (y) dm_n (y) = c \varphi (x), \text{ with } c \neq 0. \quad (5.0.3)$$

Thus $\langle u, (-\Delta)^{\alpha} \varphi \rangle = C_\alpha \int_{\mathbb{R}^n} \varphi d\mu \neq 0$, and this completes the proof. \qed


