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by

Najat B. Almutairi

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Thesis written by
Najat B. Almutairi
B.S., Qassim University, 2007
M.S., Kent State University, 2016

Approved by

Dr. Mikhail Chebotar, Advisor

Dr. Andrew Tonge, Chair, Department of Mathematical Sciences

Dr. James L. Blank, Dean, College of Arts and Sciences
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CHAPTER 1

Introduction

1.1 Preliminaries

The purpose of this thesis is to describe the image of quaternion algebra under multilinear polynomials. In it, we provide a proof that the image of multilinear non-central polynomials contains $ai + bj + ck$ for all real numbers $a$, $b$ and $c$.

In the first section, we prove that for every two elements $x_1$ and $x_2$ in the quaternion algebra, $x_1 x_2 - x_2 x_1 = ai + bj + ck$ for all real numbers $a$, $b$, and $c$. From here it is shown that for any element in quaternion algebra with an invertible element $h$ in $\mathbb{H}$, there is a real number $d$ of the form $h^{-1}(ai + bj + ck)h = di$.

In the next section, we prove that for every permutation $\sigma$ in the symmetric group on $n$ letters and $x_t \in \{i, j, k, 1\}$, $t = 1, ..., n$, there exists $\epsilon \in \{-1, 1\}$ such that $x_{\sigma(1)} x_{\sigma(2)} ... x_{\sigma(n)} = \epsilon x_1 x_2 ... x_n$. By using technical lemmas we show our main result, that is the image of multilinear non-central polynomial contains $ai + bj + ck$ for all real numbers $a$, $b$ and $c$.

Following [2], [3], [6] and [7] we start with the history of quaternions. In 1748, Euler described four-square identity and in 1840 Olinde Rodrigues described parameterization of general rotations by four parameters, but neither one of them treated the four-parameter rotations as an algebra. In addition, Carl Friedrich Gauss had actually discovered quaternions in 1819, but had never bothered to publish his work. It was eventually released in 1900.

The quaternion algebra was first publicly described by Irish mathematician William Rowan Hamilton in 1843. He knew the complex numbers could be viewed as points in plane, but he wanted to extend this method to points in three-dimensional space. For many
years he didn’t find any problem with addition and subtraction of triples of numbers, that is representation points by their coordinates in space. However, the problem of multiplication and division was difficult for a long time.

On Monday the 16th of October 1843, he discovered a great idea of quaternions. While he was walking on his way to meeting of the Royal Irish Academy, the concepts of quaternions were taking organization in his mind. When he arrived to the answer, he had a desire to carve formula for the quaternions into the stone of Brougham Bridge in Dublin as the following:

\[ i^2 = j^2 = k^2 = ijk = -1 \]

Moreover, Hamilton had founded a school of ”quaternionists”, and he spent most of the remainder of his life to studying and teaching them. Also, he had contributed to spread quaternions in several books. The last of his books, was Elements of Quaternions. It was longest of his works, 800 pages long and it was published after his death.

In fact, the quaternions are number system that extends the complex numbers. They were the first non-commutative division algebra to be discovered. Hamilton was credited to add a vast new ideas and structures in a real way for the development of abstract algebra as field of study.

In the recent years, there are many studies for the images of non-commutative polynomials. The image of \( p = x_1x_2 - x_2x_1 \), was described in [1], by Albert and Muckenhoupt. They proved the following theorem:

**Theorem 1.** Let \( M \in M_n(F) \) be an \( n \times n \) matrix with elements in an arbitrary field \( F \) and with zero trace. Then there exist \( n \times n \) matrices \( A \) and \( B \) with elements in \( F \) such that \( M = AB - BA \).

In [5], Chuang described the ranges of arbitrary polynomials in finite matrix rings. He proved the following:
Theorem 2. Assume that $F$ is a finite field. A subset $A$ of $M_n(F)$ ($n \geq 1$) is the range of a polynomial without constant term in $F(X)$, that is the free non-commutative $F$-algebra (with the identity) generated by an infinite set of noncommuting indeterminates $X = \{x_0, x_1, x_2, \ldots\}$, if and only if $0 \in A$ and $uAu^{-1} \subseteq A$ for any invertible element $u \in M_n(F)$.

Chuang also observed that for an infinite field $F$, if $\text{Im } p$ consists only of nilpotent matrices, then $p$ is a polynomial identity (PI). Recall that for any algebra $R$ over a field $F$, a polynomial $p \in F\langle x_1, \ldots, x_m \rangle$ is called a polynomial identity (PI) of the algebra $R$ if $p(a_1, \ldots, a_m) = 0$ for all $a_1, \ldots, a_m \in R$.

In [10], Lee and Zhou described right ideals generated by an idempotent of finite rank in infinite division rings. Recall that a division ring is a ring in which every non-zero element has a multiplicative inverse and a commutative division ring is called a field. We observe that there are infinite division rings that are not fields, the quaternions being such an example. An element $x$ of a ring $R$ is called idempotent if it satisfies $x^2 = x$. Lee and Zhou proved the following theorem:

Theorem 3. Let $D$ be an infinite division ring with center $E$. Suppose that $f(x_1, \ldots, x_t) \in E\langle x_1, \ldots, x_t \rangle$ is not a polynomial identity for $M_n(D)$. Then $f(x_1, \ldots, x_t)$ takes a unit in $M_n(D)$.

Kaplansky conjectured that for any $n$, the image of $p$ evaluated on the set $M_n(K)$ of $n$ by $n$ matrices is either zero, the set of scalar matrices, the set $sl_n(K)$ of matrices of trace 0, or all of $M_n(K)$. In [8] Kanel-Belov, Malev, and Rowen were then able to describe the images of non-commutative polynomials evaluated on $2 \times 2$ matrices. They proved the following result for $n = 2$:

Theorem 4. If $p$ is a multilinear polynomial evaluated on the matrix ring $M_2(K)$ (where $K$ is a quadratically closed field), then $\text{Im } p$ is either $\{0\}$, $K$, $sl_2$, or $M_2(K)$.
Recall that a field $k$ is quadratically closed if every non-constant polynomial of degree $\leq 2 \deg p$ in $K[x]$ has a root in $K$.

In [9] Kanel-Belov, Malev, and Rowen continued their work by describing the images of multilinear polynomials evaluated on $3 \times 3$ matrices:

**Theorem 5.** If $p$ is a multilinear polynomial evaluated on $3 \times 3$ matrices then $\text{Im} \ p$ is one of the following:

- $\{0\}$,
- the set of scalar matrices,
- $\text{sl}_3(K)$,
- a dense subset of $M_3(K)$,
- the set of 3-scalar matrices, or
- the set of scalars plus 3-scalar matrices.

In our thesis, we study the image of multilinear non-central polynomials evaluated on quaternion algebra. Then we continue our discussion by showing example that explain the essential requirement in our theorem. Now, we complete this section by mentioning some significant definitions.

1.2 Definitions

**Definition 1.** Quaternions are an extension of complex numbers, with 4 parts instead of 2. It is often defined as $q = a + bi + cj + dk$, where $a, b, c, \text{ and } d$ are real numbers. The set of all quaternions is denoted by $\mathbb{H}$.

**Definition 2.** Conjugation of quaternions is similar to conjugation of complex numbers to define it, let $q = a + bi + cj + dk$ be a quaternion. The conjugate of $q$ is denoted by $q^*$, and is defined as a form $q^* = a - bi - cj - dk$ where $a, b, c, \text{ and } d$ are real numbers.
Definition 3. Conjugation is an involution, meaning that it is its own inverse, so conjugating an element twice returns the original element.

Definition 4. The conjugate of a product of two quaternions is the product of the conjugates in the reverse order. That is, if \(p\) and \(q\) are quaternions, then \((pq)^* = q^*p^*\), not \(p^*q^*\).

Definition 5. The square root of the product of a quaternion with its conjugate is called its norm and is denoted by \(\|q\|\). We expressed it as follows:

\[
\|q\| = \sqrt{qq^*} = \sqrt{q^*q} = \sqrt{a^2 + b^2 + c^2 + d^2}
\]

It is always a positive real number, and is the same as the Euclidean norm on \(H\) considered as the vector space \(\mathbb{R}^4\).

Definition 6. The reciprocal of a quaternion is defined by using conjugation and the norm. The product of a quaternion with its reciprocal should equal 1, which imply that the product of \(q\) and \(q^*/\|q\|^2\) (in either order) is 1. So the reciprocal of \(q\) is defined to be \(q^{-1} = \frac{q^*}{\|q\|^2}\).

Definition 7. For each positive integer \(n\), let \(X = \{1, 2, \ldots, n\}\). A bijective function from \(X\) to \(X\) is called a permutation of \(n\) letters.

Definition 8. The set of all permutations of \(n\) letters is called the symmetric group on \(n\) letters, and is denoted by \(S_n\).

Definition 9. A multilinear polynomial is a polynomial that is linear in each of its variables.

Definition 10. A polynomial is called a central polynomial if it takes only values from the center.

Definition 11. For any polynomial \(p \in K\langle x_1, \ldots, x_m \rangle\), the image of \(p\) (in \(R\)) is defined as

\[
\text{Im } p = \{ r \in R : \text{there exist } a_1, \ldots, a_m \in R \text{ such that } p(a_1, \ldots, a_m) = r \}.
\]
Remark 1. Im $p$ is invariant under conjugation, since

$$ap(x_1, ..., x_m)a^{-1} = p(ax_1a^{-1}, ax_2a^{-1}, ..., ax_ma^{-1}) \in Im \ p,$$

for any nonsingular $a \in M_n(K)$. 
CHAPTER 2

Images of Multilinear Polynomials

2.1 Quaternion Algebra

In this section we shed light on the quaternion algebra. It can be seen as generalization of the Hamilton quaternion. Every quaternion algebra over any field $F$ of char $\neq 2$ can be described as a four-dimensional $F$-vector with basis $\{1, i, j, k\}$.

According to Hamilton, the quaternion algebra is denoted by $\mathbb{H}$, and defined as

$$\mathbb{H} = \{ q = a + bi + cj + dk : a, b, c, d \in \mathbb{R} \}$$

where $i$, $j$, and $k$ are called three imaginary quaternion units satisfying the following:

$$ij = k \quad ji = -k$$

$$jk = i \quad kj = -i$$

$$ki = j \quad ik = -j$$

$$i^2 = j^2 = k^2 = -1$$

In this chapter we show that the image of every multilinear non-central polynomial over $\mathbb{R}$ contains $ai + bj + ck$ for all real numbers $a, b$ and $c$. To obtain our this result we need to prove the following lemmas.

**Lemma 1.** Let $\mathbb{H}$ be the quaternion algebra and let $x_1$ and $x_2$ are elements in $\mathbb{H}$. Then

$$x_1x_2 - x_2x_1 = ai + bj + ck \quad (1)$$

where $a$, $b$, and $c$ are real numbers.
Proof. Let $x_1, x_2 \in \mathbb{H}$. Then we can write the following:

$$
x_1 = a_1 + b_1 i + c_1 j + d_1 k \quad \text{for some } a_1, b_1, c_1, d_1 \in \mathbb{R}
$$

$$
x_2 = a_2 + b_2 i + c_2 j + d_2 k \quad \text{for some } a_2, b_2, c_2, d_2 \in \mathbb{R}
$$

Thus we have

$$
x_1 x_2 - x_2 x_1 = (a_1 + b_1 i + c_1 j + d_1 k)(a_2 + b_2 i + c_2 j + d_2 k) - (a_2 + b_2 i + c_2 j + d_2 k)(a_1 + b_1 i + c_1 j + d_1 k)
$$

$$
= a_1 a_2 + a_1 b_2 i + a_1 c_2 j + a_1 d_2 k + b_1 a_2 i + b_1 b_2 i^2 + b_1 c_2 j + b_1 d_2 ik
$$

$$
+ c_1 a_2 j + c_1 b_2 ji + c_1 c_2 j^2 + c_1 d_2 jk + d_1 a_2 k + d_1 b_2 ki + d_1 c_2 kj + d_1 d_2 k^2
$$

$$
- (a_2 a_1 + a_2 b_1 i + a_2 c_1 j + a_2 d_1 k + b_2 a_1 i + b_2 b_1 i^2 + b_2 c_1 ij + b_2 d_1 ik
$$

$$
+ c_2 a_1 j + c_2 b_1 ji + c_2 c_1 j^2 + c_2 d_1 jk + d_2 a_1 k + d_2 b_1 ki + d_2 c_1 kj + d_2 d_1 k^2)
$$

$$
= (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2 - a_2 a_1 + b_2 b_1 + c_2 c_1 + d_2 d_1) + (a_1 b_2 + b_1 a_2
$$

$$
+ c_1 d_2 - d_1 c_2 - a_2 b_1 - b_2 a_1 - c_2 d_1 + d_2 c_1) i + (a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2
$$

$$
- a_2 c_1 + b_2 d_1 - c_2 a_1 - d_2 b_1) j + (a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2 - a_2 d_1 - b_2 c_1
$$

$$
+ c_2 b_1 - d_2 a_1) k
$$

$$
= (2c_1 d_2 - 2c_2 d_1) i + (2b_2 d_1 - 2b_1 d_2) j + (2b_1 c_2 - 2b_2 c_1) k
$$

$$
= ai + bj + ck
$$

One can check that the system of equations

$$
\begin{align*}
2c_1 d_2 - 2c_2 d_1 &= a \\
2b_2 d_1 - 2b_1 d_2 &= b \\
2b_1 c_2 - 2b_2 c_1 &= c
\end{align*}
$$

is solvable for any choice of $a$, $b$ and $c$. However, in our opinion it is not obvious, so we will use a different technique in the general case. \qed

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In the following lemma we follow the technique of [4, lemma 3] to get the result.

**Lemma 2.** Let $ai + bj + ck$ be an element of $\mathbb{H}$. Then there is a real number $d$ and an element $h$ in $\mathbb{H}$ such that $h^{-1}(ai + bj + ck)h = di$.

**Proof.** Our first goal is to show that if $q = ai + bj + ck \in [\mathbb{H}, \mathbb{H}]$, then there exists $r \in \mathbb{H}$ such that $|r| = 1$, $r^{-1}qr = xi + yj$.

- If $c = 0$, take $r = 1$, then we are done.

- If $c \neq 0$, set $r = \frac{\alpha}{|\alpha|}$ such that
  \[\alpha = b - \sqrt{b^2 + c^2} + ci.\]

Let $u = \sqrt{b^2 + c^2}$. Then we have $r^{-1} = \frac{b-u-ci}{m}$ and $r = \frac{b-u+ci}{m}$ where $m = \sqrt{(b-u)^2 + c^2}$.

So we get

\[
r^{-1}qr = \frac{1}{m^2} \left\{ \left[ (b - u - ci) \cdot (ai + bj + ck) \right] \cdot (b - u + ci) \right\} = \frac{1}{m^2} \left\{ \left[ (b^2 + b^2j - uai - ubj - uck + ca + c^2j) \cdot (b - u + ci) \right] \right\} = \frac{1}{m^2} \left\{ b^2ai - baui - bac + b^3j - b^2uj - b^2ck - uabi + u^2ai + uac - ub^2j + u^2bj + ubck - uc^2j - c^2k \right\} = \frac{1}{m^2} \left\{ (2ab^2 - 2abu + 2ac^2)i + (2b^3 - 2b^2u + 2bc^2 - 2uc^2)j \right\} = xi + yj\]

where $x, y \in \mathbb{R}$. Hence we have that $r^{-1}qr = xi + yj$.

Now let $p = xi + yj$. Then there exists $s \in \mathbb{H}$ such that $|s| = 1$, $s^{-1}ps = di$.

- If $y = 0$, take $s = 1$, then we are done.
If $y \neq 0$, set $s = \frac{\beta}{|p|}$ such that

$$\beta = -1 + \left(\frac{x}{y} + \sqrt{\frac{x^2}{y^2} + 1}\right)i + j + \left(\frac{x}{y} - \sqrt{\frac{x^2}{y^2} + 1}\right)k.$$ 

Take $v = \sqrt{x^2y^{-2} + 1}$. Then we have $s^{-1} = n^{-1}[-1 - (xy^{-1} + v)i - j - (xy^{-1} - v)k]$ and $s = n^{-1}[-1 + (xy^{-1} + v)i + j + (xy^{-1} - v)k]$ where $n = \sqrt{1 + (xy^{-1} + v)^2 + 1 + (xy^{-1} - v)^2}$.

So we obtain

$$s^{-1}ps = \frac{1}{n^2}\{-1 - (xy^{-1} + v)i - j - (xy^{-1} - v)k \cdot [(xi + yj) \cdot (-1 + (xy^{-1} + v)i + j + (xy^{-1} - v)k)]
+ (xy^{-1} + v)j - yvk\}
= \frac{1}{n^2}\{-1 - (xy^{-1} + v)i - j - (xy^{-1} - v)k \cdot [(x^2y^{-1} - xv - y) - yvi + (-x^2y^{-1} + xv - y)j - yvk]\}
= \frac{1}{n^2}\{x^3y^{-1} - x + yv + (x^2y^{-1} - xv + y)j + yvk + (x^3y^{-2} + x^2y^{-1}v)i + (x^2y^{-1}v + xv^2)i + (x + yv)i - xv - y^2 - (xy^{-1} + v)(-x^2y^{-1} + xv - y)k + (-xv - y^2j + (x^2y^{-1} + xv + y)j - yvk - x^2y^{-1} + xv - y + yvi + x^2y^{-1}(xy^{-1} - v)k + xv(xy^{-1} - v)k + y(xy^{-1} - v)k + yv(xy^{-1} - v)j + (xy^{-1} - v)(-x^2y^{-1} + xv - y)i - xv + yv^2\}
= \frac{1}{n^2}\{(4yv + 4x^2y^{-1}v)i + (2x^2y^{-1} + 2y - 2yv^2)j + (2x^3y^{-2} + 2x - 2xv^2)k\}.$$

Since $v^2 = x^2y^{-2} + 1$ we get that

$$s^{-1}ps = \frac{1}{n^2}[i(4yv + 4x^2y^{-1}v)] = id$$

where $d \in \mathbb{R}$. Hence $s^{-1}ps = s^{-1}r^{-1}qrs = di$. If we take $h = rs$, then we have that $h^{-1}qh = di$. \hfill \Box

### 2.2 Multilinear Polynomials

For any field $F$ and a positive integer $n$, we denote by $F(x_1, \ldots, x_n)$ the $F$-algebra freely generated by the (non-commuting) variables $x_1, \ldots, x_n$. A polynomial $f(x_1, \ldots, x_n) \in$
$F(x_1, ..., x_n)$ is called multilinear of degree $n$ if it is of the form

$$ f(x_1, ..., x_n) = \sum_{\sigma \in S_n} a_{\sigma} x_{\sigma(1)} x_{\sigma(2)} ... x_{\sigma(n)} $$

where $S_n$ is the group of all permutations of $\{1, ..., n\}$ and $a_{\sigma} \in F$.

**Lemma 3.** Let $\mathbb{H}$ be the quaternion algebra and let $S_n$ be the symmetric group on $n$ letters. Let each $x_i; i = 1, ..., n$, be an element from the set $\{i, j, k, 1\} \subset \mathbb{H}$. Then for every permutation $\sigma \in S_n$, there exists $\epsilon \in \{-1, 1\}$ such that

$$ x_{\sigma(1)} x_{\sigma(2)} ... x_{\sigma(n)} = \epsilon x_1 x_2 ... x_n. \quad (2) $$

**Proof.** We proceed by induction on $n$, with $n \geq 2$.

The base step, $n = 2$, there are two cases:

(a) $x_{\sigma(1)} x_{\sigma(2)} = x_1 x_2$, and

(b) $x_{\sigma(1)} x_{\sigma(2)} = x_2 x_1$.

Since $x_1, x_2 \in \{i, j, k, 1\}$, $x_2 x_1 = \pm x_1 x_2$ because

$$ 1 \cdot 1 = 1 \cdot 1, \ i \cdot i = i \cdot i, \ j \cdot j = j \cdot j, \ k \cdot k = k \cdot k, $$

$$ 1 \cdot i = i \cdot 1, \ 1 \cdot j = j \cdot 1, \ 1 \cdot k = k \cdot 1, $$

$$ i \cdot 1 = 1 \cdot i, \ i \cdot j = -j \cdot i, \ i \cdot k = -k \cdot i, $$

$$ j \cdot 1 = 1 \cdot j, \ j \cdot i = -i \cdot j, \ j \cdot k = -k \cdot j, $$

$$ k \cdot 1 = 1 \cdot k, \ k \cdot i = -i \cdot k, \ k \cdot j = -j \cdot k. $$

Now, assume equation (2) holds when $n = k$.

Then we have

$$ x_{\sigma(1)} x_{\sigma(2)} ... x_{\sigma(k)} = \epsilon x_1 x_2 ... x_k. $$
Hence it remains to show that equation (2) holds when \( n = k + 1 \). Note that

\[
x_{\sigma(1)}x_{\sigma(2)}...x_{\sigma(i)}...x_{\sigma(2)}x_{\sigma(k)}x_{\sigma(k+1)} = x_{\sigma(1)}x_{\sigma(2)}...x_{\sigma(2)}x_{\sigma(k+1)}
\]

\[
= \epsilon' x_{\sigma(1)}x_{\sigma(2)}...x_{\sigma(i-1)}x_{\sigma(i+1)}...x_{\sigma(k)}x_{\sigma(k+1)}x_{k+1}
\]

\[
= \epsilon' \epsilon'' x_{1}x_{2}...x_{k+1}
\]

where \( \epsilon', \epsilon'' \in \{-1, 1\} \). If we take \( \epsilon' \epsilon'' = \epsilon \), then we have

\[
x_{\sigma(1)}x_{\sigma(2)}...x_{\sigma(k)}x_{\sigma(k+1)} = \epsilon x_{1}x_{2}...x_{k+1}.
\]

As a result, equation (2) holds when \( n = k + 1 \). \( \square \)

The following observation is useful for the proof of our main theorem.

Let \( f(x_1, ..., x_n) \) be a nonzero polynomial with real coefficients which is linear in each variable. Then

\[
f(x_1, ..., x_n) = \sum_{\sigma \in S_n} a_{\sigma} x_{\sigma(1)}x_{\sigma(2)}...x_{\sigma(n)}.
\]

From lemma 3, we know that for any \( x_1, x_2, ..., x_n \in \{i, j, k, 1\} \) and for any \( \sigma \in S_n \) we have that

\[
x_{\sigma(1)}x_{\sigma(2)}...x_{\sigma(n)} = \epsilon x_{1}x_{2}...x_{n} \quad \text{with} \quad \epsilon \in \{1, -1\}.
\]

So, for any \( x_1, x_2, ..., x_n \in \{i, j, k, 1\} \) we have

\[
f(x_1, ..., x_n) = \sum_{\sigma \in S_n} a_{\sigma} \epsilon_{\sigma} x_{1}x_{2}...x_{n}.
\]

Now we are ready to prove our main result.

**Theorem 6.** Let \( \mathbb{H} \) be the algebra of quaternions. Let \( f(x_1, ..., x_n) \) be a polynomial with real coefficients which is linear in each variable. Suppose that \( f \) is not central. Then the image of \( f \) contains \( ai + bj + ck \) for all real numbers \( a, b \) and \( c \).

**Proof.** Let \( f(x_1, x_2, ..., x_n) \) be a nonzero multilinear polynomial. Then we have

\[
f(x_1, x_2, ..., x_n) = \sum_{\sigma \in S_n} a_{\sigma} x_{\sigma(1)}x_{\sigma(2)}...x_{\sigma(n)}
\]
By using lemma 3, we get that

$$f(x_1, x_2, \ldots, x_n) = \left( \sum_{\sigma \in S_n} a_{\sigma} \epsilon_{\sigma} \right) x_1 x_2 \ldots x_n$$

for all $x_1, x_2, \ldots, x_n \in \{i, j, k, 1\}$.

If $f(x_1, x_2, \ldots, x_n) \in \mathbb{R}$, for all $x_1, x_2, \ldots, x_n \in \{i, j, k, 1\}$, then $f(x_1, x_2, \ldots, x_n) \in \mathbb{R}$, for all $x_1, x_2, \ldots, x_n \in \mathbb{H}$, and hence $f$ is a central polynomial which is a contradiction with our assumption. So, $f(x_1, x_2, \ldots, x_n) \notin \mathbb{R}$ for all $x_1, x_2, \ldots, x_n \in \{i, j, k, 1\}$. Now by lemma 3 we have

$$f(x_1, x_2, \ldots, x_n) = \left( \sum_{\sigma \in S_n} a_{\sigma} \epsilon_{\sigma} \right) x_1 x_2 \ldots x_n$$

If we have one of the following

$$f(x_1, x_2, \ldots, x_n) = ri,$$

$$f(x_1, x_2, \ldots, x_n) = rj,$$

$$f(x_1, x_2, \ldots, x_n) = rk$$

for some $r \in \mathbb{R}$, then we will get the same result. Say, if we choose $f(x_1, x_2, \ldots, x_n) = ri$, and let $ai + bj + ck \in \mathbb{H}$, then by lemma 2 there exist a real number $d$ and an element $h \in \mathbb{H}$ such that

$$h^{-1}(ai + bj + ck)h = di \quad (3)$$

Now by using the equation (3), we get $ai + bj + ck = hdih^{-1}$. It can be rewritten as $ai + bj + ck = dr^{-1}(hrith^{-1})$. Then by using this substitution $ri = f(x_1, x_2, \ldots, x_n)$, we can write $ai + bj + ck = dr^{-1}(hf(x_1, x_2, \ldots, x_n)h^{-1})$. It is known that since $h$ is invertible element, we can use this property $hf(x_1, x_2, \ldots, x_n)h^{-1} = f(hx_1h^{-1}, hx_2h^{-1}, \ldots, hx_nh^{-1})$. By applying this property we can write $ai + bj + ck = dr^{-1}f(hx_1h^{-1}, hx_2h^{-1}, \ldots, hx_nh^{-1})$. That leads to $ai + bj + ck = f(dr^{-1}hx_1h^{-1}, hx_2h^{-1}, \ldots, hx_nh^{-1})$. Therefore we conclude that the image of $f$ contains $ai + bj + ck$ for all real numbers $a, b$ and $c$. 

\[\square\]
2.3 Example

The following example shows that a non-central is essential in our theorem.

**Example 1.** Let $x_1, x_2, x_3, x_4$ are elements of $\mathbb{H}$. Then

$$f(x_1, x_2, x_3, x_4) = [x_1, x_2][x_3, x_4] + [x_3, x_4][x_1, x_2]$$

is a central.

*Proof.* Since $x_1, x_2, x_3, x_4 \in \mathbb{H}$, then we can write the following:

$$x_1 = a_1i + b_1j + c_1k + d_1 \quad \text{for some } a_1, b_1, c_1, d_1 \in \mathbb{R}$$

$$x_2 = a_2i + b_2j + c_2k + d_2 \quad \text{for some } a_2, b_2, c_2, d_2 \in \mathbb{R}$$

$$x_3 = a_3i + b_3j + c_3k + d_3 \quad \text{for some } a_3, b_3, c_3, d_3 \in \mathbb{R}$$

$$x_4 = a_4i + b_4j + c_4k + d_4 \quad \text{for some } a_4, b_4, c_4, d_4 \in \mathbb{R}.$$ 

First we want to calculate $[x_1, x_2]$ and $[x_3, x_4]$ as follows:

$$[x_1, x_2] = x_1x_2 - x_2x_1$$

$$= (a_1i + b_1j + c_1k + d_1)(a_2i + b_2j + c_2k + d_2) - (a_2i + b_2j$$

$$+ c_2k + d_2)(a_1i + b_1j + c_1k + d_1)$$

$$= -a_1a_2 + a_1b_2k - a_1c_2j + a_1d_2i - b_1a_2k - b_1b_2 + b_1c_2i$$

$$+ b_1d_2j + c_1a_2j - c_1b_2i - c_1c_2 + c_1d_2k + d_1a_2i + d_1b_2j$$

$$+ d_1c_2k + d_1d_2 + a_2a_1 - a_2b_1k + a_2c_1j - a_2d_1i + b_2a_1k$$

$$+ b_2b_1 - b_2c_1i - b_2d_1j - c_2a_1j + c_2b_1i + c_2c_1 - c_2d_1k$$

$$- d_2a_1i - d_2b_1j - d_2c_1k - d_2d_1$$

$$= (2b_1c_2 - 2c_1b_2)i + (2c_1a_2 - 2a_1c_2)j + (2a_1b_2 - 2b_1a_2)k$$

$$= \alpha_1i + \beta_1j + \gamma_1k.$$
Hence we get that \([x_1, x_2] = \alpha_1 i + \beta_1 j + \gamma_1 k\) for some \(\alpha_1, \beta_1\) and \(\gamma_1 \in \mathbb{R}\). Now we will do the same previous calculate with \([x_3, x_4]\) and we get that

\[
[x_3, x_4] = x_3 x_4 - x_4 x_3
\]

\[
= (a_3 i + b_3 j + c_3 k + d_3)(a_4 i + b_4 j + c_4 k + d_4) - (a_4 i + b_4 j + c_4 k + d_4)(a_3 i + b_3 j + c_3 k + d_3)
\]

\[
= -a_3 a_4 + a_3 b_4 k - a_3 c_4 j + a_3 d_4 i - b_3 a_4 k - b_3 b_4 + b_3 c_4 i
\]

\[
+ b_3 d_4 j + c_3 a_4 j - c_3 b_4 i - c_3 c_4 + c_3 d_4 k + d_3 a_4 i + d_3 b_4 j
\]

\[
+ d_3 c_4 k + d_3 d_4 + a_4 a_3 - a_4 b_3 k + a_4 c_3 j - a_4 d_3 i + b_4 a_3 k
\]

\[
+ b_4 b_3 - b_4 c_3 i - b_4 d_3 j - c_4 a_3 j + c_4 b_3 i + c_4 c_3 - c_4 d_3 k
\]

\[
- d_4 a_3 i - d_4 b_3 j - d_4 c_3 k - d_4 d_3
\]

\[
= (2b_3 c_4 - 2c_3 b_4)i + (2c_3 a_4 - 2a_3 c_4)j + (2a_3 b_4 - 2b_3 a_4)k
\]

\[
= \alpha_2 i + \beta_2 j + \gamma_2 k.
\]

Then we have \([x_3, x_4] = \alpha_2 i + \beta_2 j + \gamma_2 k\) for some \(\alpha_2, \beta_2\) and \(\gamma_2 \in \mathbb{R}\). Now by substitution in equation (4) we get that

\[
f(x_1, x_2, x_3, x_4) = (\alpha_1 i + \beta_1 j + \gamma_1 k)(\alpha_2 i + \beta_2 j + \gamma_2 k) + (\alpha_2 i + \beta_2 j + \gamma_2 k)(\alpha_1 i + \beta_1 j + \gamma_1 k)
\]

\[
= -\alpha_1 \alpha_2 + \alpha_1 \beta_2 k - \alpha_1 \gamma_2 j - \beta_1 \alpha_2 k - \beta_1 \beta_2 + \beta_1 \gamma_2 i + \gamma_1 \alpha_2 j - \gamma_1
\]

\[
= \beta_2 i - \gamma_1 \gamma_2 - \alpha_2 \alpha_1 + \alpha_2 \beta_1 k - \alpha_2 \gamma_1 j - \beta_2 \alpha_1 k - \beta_2 \beta_1 + \beta_2 \gamma_1 i + \gamma_2
\]

\[
= \alpha_1 j - \gamma_2 \beta_1 i - \gamma_2 \gamma_1
\]

\[
= -2\alpha_1 \alpha_2 - 2\beta_1 \beta_2 - 2\gamma_1 \gamma_2
\]

\[
= d.
\]

For some \(d \in \mathbb{R}\) and hence we conclude that \(f\) is a central polynomial.
CHAPTER 3

Conclusion

We started this thesis by mentioning some of the important studies and results that have related to our argument. There are many recent of studies for the images of non-commutative polynomials. In 1957, Albert and Muckenhoupt described the image of the polynomial $f(x, y) = xy - yx$ on $M_n(K)$ contains all matrices with trace 0. In 1990, Chuang showed that if $K$ is finite, then every subset $S$ of $M_n(K)$ that contains 0 and is closed under conjugation is the image of some polynomial. In 2009, Lee and Zhou proved that when $K$ is an infinite division ring, for any non-identity $p$ with coefficients in the center of $K$ then the image of $p$ contains an invertible matrix. In 2012, Kanel-Belov, Malev, and Rowen described the images of non-commutative polynomials evaluated on $2 \times 2$ matrices. Also, in 2015, Kanel-Belov, Malev, and Rowen continued their work by described the images of multilinear polynomials evaluated on $3 \times 3$ matrices.

In our thesis we described the image of quaternion algebra under multilinear polynomials. Our main goal was to prove that the image of every multilinear non-central polynomial over $\mathbb{R}$ contains $ai + bj + ck$ for all real numbers $a, b$ and $c$. Unfortunately, our result does not hold for a central polynomial as it is justified by an example which we provided in our thesis. An important open problem that has been related to our main theorem is to describe the images of multilinear polynomials over finite-dimensional division algebras.
BIBLIOGRAPHY


