CHARACTERIZING MAPS OF MATRIX RINGS BY ACTION ON ZERO PRODUCTS

A thesis submitted
to Kent State University in
partial fulfillment of the requirements
for the degree of Master of Science

by

Fawziah M. Alharthi

December, 2015
Thesis written by

Fawziah M. Alharthi

B.S., Umm Al-Qura University, 2010
M.S., Kent State University, 2015

Approved by

Dr. Mikhail Chebotar, Advisor

Dr. Andrew Tonge, Chair, Department of Mathematical Sciences

Dr. James L. Blank, Dean, College of Arts and Sciences
# TABLE OF CONTENTS

Acknowledgements .......................................................... iv

1 Introduction ................................................................. 1
   1.1 Preliminaries ......................................................... 1
   1.2 Functional Identities .............................................. 5

2 Characterizing maps through zero products ........................... 11
   2.1 Maps preserving zero products .................................... 11
   2.2 Maps acting like derivations ...................................... 18
   2.3 Example ............................................................. 23

3 Conclusion ................................................................. 26

BIBLIOGRAPHY ............................................................... 28
Acknowledgements

I would like to thank Kent State University, and the Department of Mathematical Sciences for the pleasure of being educated by this distinguished faculty.

I am wholeheartedly grateful to the members of my defense committee- Dr. Jenya Soprunova, and Dr. Dmitry Ryabogin for their valuable input, discussion and suggestions. Sincere appreciation is extended to the graduate coordinator at our department Dr. Artem Zvavitch for his advice and guidance through two years. My thanks is also extended to Matthew Alexander for offering his time and helping me to edit my thesis.

I would like to express my deep respect and thanks to my great advisor Dr. Mikhail Chebotar. This thesis is due to his support, advice, guidance, and valuable comments. Dr. Chebotar was always available to answer my questions with kindness and shared generously his time and knowledge. I am truly indebted to him.

I am especially thankful to those patient souls- my family members who stood beside me to accomplish this study. Words cannot express how grateful I am to my Dad and my Mom. I will never forget their endless sacrifice since I was a little girl. I could not imagine myself graduating today without my husband Khaled’s constant support through the hard times. I do not know how to thank my sisters and brothers enough for their love and encouragement. My heart-felt thanks to my lovely children, Bader and Soulaff, for their patience. Their smiles and big hearts gave me the strength to face the obstacles and continue forward.
CHAPTER 1

Introduction

1.1 Preliminaries

The purpose of this thesis is to describe the surjective additive maps preserving zero products, and the additive maps that behave like derivation when acting on zero products. In both parts, we consider the maps from a ring \( A = M_n(R); n \geq 2 \) into itself, where \( R \) is a unital ring.

In the first section, we show that \( \theta \) which preserves zero products is of the form \( \theta = \lambda \varphi \) where \( \lambda \) lies in the center of \( A = M_n(R) \), that is \( C(A) := \{ a \in A : ax = xa \text{ for all } x \in A \} \), and \( \varphi \) is a homomorphism map from \( A \) into itself, that is, for every \( x \) and \( y \) are in \( A \) we have \( \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) \).

In the following section, we show a result analogous to the previous one, that is, \( \delta(x) = d(x) + cx \) where \( c \) belongs to the center of \( A \) and \( d \) is a derivation.

Over the last couple of years there has been a lot of work characterizing maps done for many classes of rings by using three types of products. Consider any ring \( A \) and any two elements \( x, y \in A \), and let us present the following types of multiplications. One of them is called Jordan multiplication which is defined by \( x \circ y = xy + yx \). Another type is known as Lie multiplication that is defined by \( [x, y] = xy - yx \). Finally, the ordinary multiplication which is denoted by \( x \cdot y \) or \( xy \).

In [3], Wong described maps on simple algebras preserving zero products. In particular, [6, corollary D] implies the following result.
Corollary 1. Let $A$ be a simple associative algebra over a field $\mathbb{K}$, which is not a division algebra, and suppose that $A$ contains a nonzero idempotent $e$ such that $eAe$ is finite-dimensional over $\mathbb{K}$. Let $f$ be a bijective linear map on $A$ which preserves zero products. Then $f$ is the product of an element of the unit group of the extended centroid of $A$ with a linear automorphism of $A$.

Recently, there are many studies that conclude similar results for several kinds of rings. In [4], Chebotar, Ke, and Lee described maps preserving zero products for prime rings containing nontrivial idempotent. Recall that a ring $A$ is called a prime ring if $aAb = 0$ implies $a = 0$ or $b = 0$ for $a, b \in A$, and an element $a$ is called a nontrivial idempotent if it is not 0 or 1 and satisfies that $a^2 = a$. They proved the following theorem.

Theorem 1. Let $A$ and $B$ be prime rings and $\theta : A \rightarrow B$ a bijective additive map such that $\theta(x)\theta(y) = 0$ for all $x, y \in A$ with $xy = 0$. Suppose that $A$ contains a nontrivial idempotent $e$.

(i) If $1 \in A$, then $\theta(xy) = \lambda \theta(x)\theta(y)$ for all $x, y \in A$, where $\lambda = \frac{1}{\theta(1)}$ and $\theta(1) \in Z(B)$, the center of $B$. In particular, if $\theta(1) = 1$, then $\theta$ is a ring isomorphism from $A$ onto $B$.

(ii) If $\deg B \geq 3$, then there exists $\lambda \in C(B)$, the extended centroid of $B$, such that $\theta(xy) = \lambda \theta(x)\theta(y)$ for all $x, y \in A$.

Also, in [5] Stopar extended these results and he considered surjective (not necessarily injective) additive map preserving zero products. Then he proved the following theorem:

Theorem 2. Let $A$ be a ring and $B$ a prime ring. Let $\theta : A \rightarrow B$ be a surjective additive map such that $\theta(x)\theta(y) = 0$ for all $x, y \in A$ with $xy = 0$. Suppose that $R$ is a unital ring that contains $A$ as a subring, and let $e$ be an idempotent in $R$ such that $eA \cup Ae \subseteq A$. Denote $f = 1 - e$. If either $e \in A, f \in A$, or $A = A^2$, then one of the following holds:
(a) $\theta(eA + Ae + AeA) = 0$;

(b) $\theta(fA + Af + AfA) = 0$;

(c) There exists $\lambda \in C(B)$, the extended centroid of $B$, such that $\theta(xy) = \lambda \theta(x)\theta(y)$ for all $x, y \in A$.

In [6] Chebotar, Ke, Lee, and Zhang characterized maps preserving zero Jordan product, that is, $\theta(x)\theta(y) + \theta(y)\theta(x) = 0$ whenever $xy + yx = 0$ on matrix rings.

In the thesis, we expand these results and shed light on the class of the ring in [6], and the type of multiplication in [3,4,5] to describe $\theta$. That means we describe the maps preserving zero products for matrix rings. We follow the technique of [6] with a modification that is resulted of changing the type of product. Also, we consider a more general situation in which our result covers the case when $n = 2$ in the ring $A = M_n(R)$ by providing a new method to prove some theorems. An interesting thing occurs when we examine our ring while omitting the condition “$A$ contains 1”, we find the desired result, $\theta$ is the product of a central element with a homomorphism map, is not satisfied. Then, we provide an example to illustrate this point.

To get the results we want, we will use a tool that plays a significant role to prove some theorems. This tool is known as Functional Identities. Let us start with some elementary notations on matrices.

**Definition 1.** A matrix ring is a set of matrices forming a ring under matrix addition, and matrix multiplication. The set of $n \times n$ matrices with entries from another ring $R$ is a matrix ring, denoted $M_n(R)$.

**Definition 2.** A matrix unit is a matrix whose entries are all 0 except in one cell where it is one. We denote a matrix whose $(i,j)$-entry is 1 and 0 elsewhere by $e_{ij}$.
Definition 3. The unit matrix or identity matrix of size $n$ is the $n \times n$ square matrix with ones on the main diagonal and zeros elsewhere. We shall denote it by $1$.

Now, we continue with some basic concepts on maps.

Definition 4. A map $\theta$ from a ring $R$ to itself is called an additive map if it satisfies $\theta(x + y) = \theta(x) + \theta(y)$ for any $x$ and $y$ in $R$.

Definition 5. An additive map $\theta$ from a ring $R$ to itself preserves zero product if $\theta(x) \cdot \theta(y) = 0$ whenever $x \cdot y = 0$ for $x, y \in R$.

Definition 6. An additive map $\delta$ from a ring $R$ to itself is called a derivation if $\delta(x \cdot y) = \delta(x) \cdot y + x \cdot \delta(y)$ for all $x, y \in R$.

Finally, we introduce a vector space over a field, algebra and an annihilator.

Definition 7. A vector space over a field $\mathbb{K}$ is a nonempty set $V$ with two operations:

(i) Vector addition: If $u, v \in V$, then $u + v \in V$

(ii) Scalar multiplication: If $u \in V$, $\lambda \in \mathbb{K}$, then $\lambda u \in V$,

and satisfies the following eight axioms for any vectors $u, v, w \in V$:

(a) $(u + v) + w = u + (v + w)$.

(b) There exists a vector $0 \in V$, such that, for every $u \in V$, $u + 0 = 0 + u = u$.

(c) For every $u \in V$, there is a vector $-u \in V$, such that,

$$u + (-u) = (-u) + u = 0.$$  

(d) $u + v = v + u$.

(e) $\lambda(u + v) = \lambda u + \lambda v$, for any scalar $\lambda \in \mathbb{K}$. 

4
(f) \((\lambda_1 + \lambda_2)u = \lambda_1 u + \lambda_2 u\) for any scalars \(\lambda_1, \lambda_2 \in K\).

(g) \((\lambda_1 \lambda_2)u = \lambda_1 (\lambda_2 u)\) for any scalars \(\lambda_1, \lambda_2 \in K\).

(h) \(1u = u\), for the unit scalar \(1 \in K\).

**Definition 8.** An algebra is a ring which is also a vector space over a filed.

**Definition 9.** A left annihilator of an algebra \(A\) is defined as,

\[ L(A) = \{l \in A \text{ such that } lx = 0 \text{ for all } x \in A\}, \]

and a right annihilator of an algebra \(A\) is defined as,

\[ R(A) = \{r \in A \text{ such that } xr = 0 \text{ for all } x \in A\}. \]

### 1.2 Functional Identities

A functional identity on a ring \(R\) is an expression consisting of arbitrary elements and unknown functions. In this theme we will talk briefly about the basic functional identities, and the quasi-polynomials. Now, we shall describe the notations of the basic functional identities in general as given in [1],[2],[6] and [7]. Then, we will study a special case when the functional identity involves three variables, which is really important for us to reach our aim.

Throughout this section assume that \(m\) is a positive integer, \(Q\) is a unital ring with center \(C\), \(R\) is nonempty subset of \(Q\) and \(E : R^{m-1} \rightarrow Q\), \(p : R^{m-2} \rightarrow Q\) are arbitrary maps. When \(m = 1\), \(E\) will be an element in \(Q\) and \(p = 0\). For nonempty subsets \(R_1, R_2, \ldots, R_m\) of \(Q\), we set \(\hat{R} = R_1 \times R_2 \times \ldots \times R_m\), and for \(1 \leq i < j \leq m\), we have

\[ \hat{R}^i = R_1 \times R_2 \times \ldots \times R_{i-1} \times R_{i+1} \times \ldots \times R_m, \]

\[ \hat{R}^{ij} = \hat{R}^{ji} = R_1 \times R_2 \times \ldots \times R_{i-1} \times R_{i+1} \times \ldots \times R_{j-1} \times R_{j+1} \times \ldots \times R_m. \]
For \( x_1 \in \mathbb{R}_1, x_2 \in \mathbb{R}_2, \ldots, x_m \in \mathbb{R}_m \), we have

\[
\bar{x}_m = (x_1, \ldots, x_m) \in \hat{\mathbb{R}}
\]

\[
\bar{x}_m^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \in \hat{\mathbb{R}}^i
\]

\[
\bar{x}_m^{ij} = (\bar{x}_m^i) = (x_1, \ldots, x_i, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m) \in \hat{\mathbb{R}}^{ij}.
\]

For example, if \( m = 3 \), then we have \( \hat{\mathbb{R}}^1 = \mathbb{R}_2 \times \mathbb{R}_3, \hat{\mathbb{R}}^2 = \mathbb{R}_1 \times \mathbb{R}_3, \hat{\mathbb{R}}^3 = \mathbb{R}_1 \times \mathbb{R}_2 \).

Also we have \( \hat{\mathbb{R}}^1 = \mathbb{R}_2 \times \mathbb{R}_3, \hat{\mathbb{R}}^2 = \mathbb{R}_1 \times \mathbb{R}_3, \hat{\mathbb{R}}^3 = \mathbb{R}_1 \times \mathbb{R}_2 \).

For \( \bar{x}_1 \in \mathbb{R}_1, \bar{x}_2 \in \mathbb{R}_2, \bar{x}_3 \in \mathbb{R}_3 \), we can write \( \bar{x}_3 = (x_1, x_2, x_3) \in \hat{\mathbb{R}}, \bar{x}_3^1 = (x_2, x_3) \in \hat{\mathbb{R}}^1, \bar{x}_3^2 = (x_1, x_3) \in \hat{\mathbb{R}}^2, \bar{x}_3^3 = (x_1, x_2) \in \hat{\mathbb{R}}^3, \bar{x}_3^{12} = (x_2, x_3) \in \hat{\mathbb{R}}^{12}, \bar{x}_3^{13} = (x_3) \in \hat{\mathbb{R}}^{13}, \bar{x}_3^{23} = (x_3) \in \hat{\mathbb{R}}^{23}, \bar{x}_3^{123} = (x_3) \in \hat{\mathbb{R}}^{123}. \)

Now let \( I, J \subseteq \{1, 2, \ldots, m\} \), and for each \( i \in I, j \in J \), let \( E_i : \hat{\mathbb{R}}^i \rightarrow \mathbb{Q} \), and \( F_j : \hat{\mathbb{R}}^j \rightarrow \mathbb{Q} \) be arbitrary maps. The basic functional identities are

\[
\sum_{i \in I} E_i(\bar{x}_m^i) x_i + \sum_{j \in J} x_j F_j(\bar{x}_m^j) = 0 \quad \text{for all} \quad \bar{x}_m \in \hat{\mathbb{R}},
\]

and a slightly more general one,

\[
\sum_{i \in I} E_i(\bar{x}_m^i) x_i + \sum_{j \in J} x_j F_j(\bar{x}_m^j) \in C \quad \text{for all} \quad \bar{x}_m \in \hat{\mathbb{R}}.
\]

The goal in the theory of functional identities is to describe the form of the maps that appear in the identity. A natural possibility when (1) (and hence also (2)) is fulfilled is when there exist maps \( p_{ij} : \hat{\mathbb{R}}^{ij} \rightarrow \mathbb{Q}, i \in I, j \in J, i \neq j \), and \( \lambda_k : \mathbb{R}^{m-1} \rightarrow C, k \in I \cup J \), as the following.

\[
E_i(\bar{x}_m^i) = \sum_{j \in J, j \neq i} x_j p_{ij}(\bar{x}_m^i) + \lambda_i(\bar{x}_m^i), i \in I,
\]

\[
F_j(\bar{x}_m^j) = - \sum_{i \in I, j \neq i} p_{ij}(\bar{x}_m^i)x_i - \lambda_j(\bar{x}_m^j), j \in J,
\]

(3)
\[ \lambda_k = 0 \text{ if } k \notin I \cap J. \]

In fact, (3) satisfies the equation (1). In this case, we can say (3) is a standard solution, the solution which satisfies the functional identity of non-reliance on properties of the ring, of (1) (and of (2)).

For example, according to our choice \( m = 3 \), let \( I = \{3\} \), and \( J = \{1\} \). Then we can write (1) as the following

\[
E_3(x_1, x_2)x_3 + x_1F_1(x_2, x_3) = 0,
\]

and its standard solution is

\[
E_3(x_1, x_2) = x_1p_{31}(x_2) + \lambda_3(x_1, x_2),
\]

\[
F_1(x_2, x_3) = -p_{31}(x_2)x_3 - \lambda_1(x_2, x_3).
\]

Since 1 and 3 \( \notin I \cap J \), \( \lambda_3(x_1, x_2) = 0 = \lambda_1(x_2, x_3) \). So we have

\[
E_3(x_1, x_2) = x_1p_{31}(x_2),
\]

\[
F_1(x_2, x_3) = -p_{31}(x_2)x_3.
\]

If \( I \) or \( J \) is empty, then according to our convention the sum over \( \emptyset \) is 0, and for all \( \pi_m \in \hat{R} \) the identity (1) can be written as

\[
\sum_{i \in I} E_i(\pi_m^i)x_i = 0, \quad (4)
\]

\[
\sum_{j \in J} x_jF_j(\pi_m^j) = 0. \quad (5)
\]

Similarly, special cases of (2) are

\[
\sum_{i \in I} E_i(\pi_m^i)x_i \in C, \quad (6)
\]
\[ \sum_{j \in J} x_j F_j(\tau^j_m) \in C. \] (7)

It follows from the definition that the standard solution of (4) and (6) is \( E_i = 0 \) for each \( i \).

Similarly, the standard solution of (5) and (7) is \( F_j = 0 \) for each \( j \).

**Definition 10.** A nonempty subset \( R \) of \( Q \) is said to be a \( d \)-free subset of \( Q \), where \( d \) is a positive integer, if for all \( I, J \subseteq \{1, 2, ..., m\} \) the following two conditions are satisfied:

(a) If \( \max\{|I|, |J|\} \leq d \), (1) has (3) as a standard solution and it is unique.

(b) If \( \max\{|I|, |J|\} \leq d - 1 \), (2) has (3) as a standard solution and it is unique.

So we can say \( R \subseteq Q \) is a \( d \)-free set if any functional identity has a standard solution as a unique solution.

Now as mentioned in [6], for applications we need more involved functional identities than (1) and (2). Let \( S \) be an arbitrary set, and let \( \theta: S \to Q, E_i, F_j: S^{m-1} \to Q \) where \( i \in I, j \in J \) be maps of sets. We have the following identities.

\[ \sum_{i \in I} E_i(\tau^i_m)\theta(x_i) + \sum_{j \in J} \theta(x_j)F_j(\tau^j_m) = 0 \quad \text{for all } \tau^i_m \in \hat{S}, \] (8)

\[ \sum_{i \in I} E_i(\tau^i_m)\theta(x_i) + \sum_{j \in J} \theta(x_j)F_j(\tau^j_m) \in C \quad \text{for all } \tau^i_m \in \hat{S}. \] (9)

When \( S = R \subseteq Q \) and \( \theta \) is the identity map, then (8) and (9) are exactly (1) and (2). Observe that the standard solution of (8) is

\[ E_i(\tau^i_m) = \sum_{j \in J, j \neq i} \theta(x_j)p_{ij}(\tau^i_m) + \lambda_i(\tau^i_m); i \in I, \]

\[ F_j(\tau^j_m) = -\sum_{i \in I, j \neq i} p_{ij}(\tau^j_m)\theta(x_i) - \lambda_j(\tau^j_m); j \in J, \] (10)

For all \( x_m \in S^m \) where

\[ p_{ij}: \hat{S}^{ij} \to Q, i \in I, j \in J, j \neq i, \]
\( \lambda_k : S^{m-1} \rightarrow C, k \in I \cup J, \) with \( \lambda_k = 0 \) if \( k \notin I \cap J. \)

In our case, when \( m = 3, I = \{3\}, J = \{1\}, \) we have \( \theta : S \rightarrow Q, E_i, F_j : S^2 \rightarrow Q \)
where \( i \in I, \) and \( j \in J. \) Then we can write (8) as the following

\[
E_3(x_1, x_2)\theta(x_3) + \theta(x_1)F_1(x_2, x_3) = 0,
\]

and its standard solution is

\[
E_3(x_1, x_2) = \theta(x_1)p_{31}(x_2) + \lambda_3(x_1, x_2),
\]
\[
F_1(x_2, x_3) = -p_{31}(x_2)\theta(x_3) - \lambda_1(x_2, x_3).
\]

Since 1 and 3 \( \notin I \cap J, \) \( \lambda_3(x_1, x_2) = 0 = \lambda_1(x_2, x_3). \) So we get

\[
E_3(x_1, x_2) = \theta(x_1)p_{31}(x_2),
\]
\[
F_1(x_2, x_3) = -p_{31}(x_2)\theta(x_3).
\]

Now, let us introduce a quasi-polynomial. As we said before, let \( \theta \) be a fixed map from \( S \) into \( Q. \) Simply, the degree of a quasi polynomial is just the number of variables involved.

A quasi- polynomial \( P \) of degree 1 is a nonzero function from \( S \) to \( Q \) such that

\[
P(x_1) = \lambda \theta(x_1) + \mu(x_1)
\]

where \( \lambda \in C \) and \( \mu : S \rightarrow C. \)

A quasi- polynomial of degree 2 is also a nonzero function but from \( S^2 \) to \( Q \) that can be written as

\[
P(x_1, x_2) = \lambda_1 \theta(x_1)\theta(x_2) + \lambda_2 \theta(x_2)\theta(x_1) + \mu_1(x_1)\theta(x_2) + \mu_2(x_2)\theta(x_1) + \nu(x_1, x_2)
\]

with \( \lambda_1, \lambda_2 \in C, \mu_1, \mu_2 : S \rightarrow C, \nu : S \times S \rightarrow C. \)

A quasi- polynomial of degree 3 consists of summands such that

\[
\lambda_1 \theta(x_1)\theta(x_2)\theta(x_3), \mu_1(x_1)\theta(x_2)\theta(x_3), \nu_1(x_1, x_2)\theta(x_3), etc.
\]
Now we shall consider a general type of functional identities when one equates a special case of the basic functional identity in which $E = E_i = F_j$ with a quasi-polynomial, that is,

$$\sum_{i \in I} E(x_i^m)\theta(x_i) + \sum_{j \in J} \theta(x_j)E(x_j^m) = P.$$ 

Then under suitable $d$-freeness circumstance $E(x_i^m), E(x_j^m)$ must be quasi-polynomials. In our example when $m = 3, I = \{3\}$, and $J = \{1\}$, we can write this identity as

$$E(x_1, x_2)\theta(x_3) + \theta(x_1)E(x_2, x_3) = P.$$ 

Then if those appropriate $d$-freeness conditions hold, $E$ is a quasi polynomials of degree 2, and we will talk about this later.

It is worth mentioning that using functional identities is a good technique, but in our case of the matrix ring over a unital ring, it is not the best technique since we can get the required result by easier and more general method that also covers the $M_2(R)$ case. We will use it as another way to prove some theorems.
CHAPTER 2

Characterizing maps through zero products

2.1 Maps preserving zero products

Let \( \theta \) be a map preserving zero products. In this section we will show \( \theta = \lambda \phi \) where \( \lambda \) is a central element, and \( \phi \) is a homomorphism. The steps taken to reach that result are as follows.

1. We will find a connection between all possible products of the form \( \theta(a_{ij}) \cdot \theta(b_{jk}) \) by Lemma 1.

2. By Theorem 3, we will prove that the map preserving zero products preserves equal products.

3. In Theorem 4, we will use the functional identities to get the desired result.

4. In Theorem 5, we will provide a different technique to get our goal of this section.

Lemma 1. Let \( R \) be a ring with 1, and \( A = M_n(R) \) where \( n \geq 2 \), and \( \theta : A \to A \) is an additive map which preserves zero products. Then, for \( a, b \in R \) and \( i, j, k, l \in \{1, 2, 3, \ldots, n\} \), we have:

(a) \( \theta(a_{ij}) \cdot \theta(b_{kl}) = 0 \) if \( j \neq k \).

(b) \( \theta(a_{ij}) \cdot \theta(b_{jk}) = \theta(e_{ii}) \cdot \theta(ab)_{ik} = \theta(ab)_{ik} \cdot \theta(e_{kk}) \).

Proof. (a) We know that for \( j \neq k \), we have via direct calculation \( a_{ij} \cdot b_{kl} = 0 \). Since \( \theta \) preserves zero products, we obtain \( \theta(a_{ij}) \cdot \theta(b_{kl}) = 0 \).

(b) Notice that if \( j \neq i \) we have \( (a_{ij} + e_{ii}) \cdot (b_{jk} - (ab)_{ik}) = a_{ij} \cdot b_{jk} - a_{ij} \cdot (ab)_{ik} + e_{ii} \cdot b_{jk} - e_{ii} \cdot (ab)_{ik} \). Since \( a_{ij} \cdot (ab)_{ik} = 0 \) and \( e_{ii} \cdot b_{jk} = 0 \), we get \( a_{ij} \cdot b_{jk} - e_{ii} \cdot (ab)_{ik} = 0 \).
Thus \((a_{ij} + e_{ii}) \cdot (b_{jk} - (ab)_{ik}) = 0\). That leads directly to \(\theta(a_{ij} + e_{ii}) \cdot \theta(b_{jk} - (ab)_{ik}) = 0\) since \(\theta\) preserves zero products. Observe that by using the additive property which is given in the lemma, we can write the last step as \([\theta(a_{ij}) + \theta(e_{ii})] \cdot [\theta(b_{jk}) - \theta(ab)_{ik}] = 0\). Then, \(\theta(a_{ij}) \cdot \theta(b_{jk}) - \theta(a_{ij}) \cdot \theta(ab)_{ik} + \theta(e_{ii}) \cdot \theta(b_{jk}) - \theta(e_{ii}) \cdot \theta(ab)_{ik} = 0\). By (a) we have \(\theta(a_{ij}) \cdot \theta(ab)_{ik} = 0\), and \(\theta(e_{ii}) \cdot \theta(b_{jk}) = 0\). So, we have \(\theta(a_{ij}) \cdot \theta(b_{jk}) - \theta(e_{ii}) \cdot \theta(ab)_{ik} = 0\). Consequently,

\[\theta(a_{ij}) \cdot \theta(b_{jk}) = \theta(e_{ii}) \cdot \theta(ab)_{ik}\]

If \(j \neq k\), \(\theta(a_{ij}) \cdot \theta(b_{jk}) = \theta(ab)_{ik} \cdot \theta(e_{kk})\). This can be derived from \((a_{ij} + (ab)_{ik}) \cdot (b_{jk} - e_{kk})\).

Indeed, \((a_{ij} + (ab)_{ik}) \cdot (b_{jk} - e_{kk}) = a_{ij} \cdot b_{jk} - a_{ij} \cdot e_{kk} + (ab)_{ik} \cdot b_{jk} - (ab)_{ik} \cdot e_{kk}\). Since \(a_{ij} \cdot e_{kk} = 0\), and \((ab)_{ik} \cdot b_{jk} = 0\), we get \(a_{ij} \cdot b_{jk} - (ab)_{ik} \cdot e_{kk} = 0\). Thus, \((a_{ij} + (ab)_{ik}) \cdot (b_{jk} - e_{kk}) = 0\). This implies \(\theta(a_{ij} + (ab)_{ik}) \cdot \theta(b_{jk} - e_{kk}) = 0\) since \(\theta\) preserves zero product.

Again by the additive property we have \([\theta(a_{ij}) + \theta(ab)_{ik}] \cdot [\theta(b_{jk}) - \theta(e_{kk})] = 0\). Then, \(\theta(a_{ij}) \cdot \theta(b_{jk}) - \theta(a_{ij}) \cdot \theta(e_{kk}) + \theta(ab)_{ik} \cdot \theta(b_{jk}) - \theta(ab)_{ik} \cdot \theta(e_{kk}) = 0\). Since \(\theta(a_{ij}) \cdot \theta(e_{kk}) = 0\), and \(\theta(ab)_{ik} \cdot \theta(b_{jk}) = 0\) by (a), we get \(\theta(a_{ij}) \cdot \theta(b_{jk}) - \theta(ab)_{ik} \cdot \theta(e_{kk}) = 0\). Therefore,

\[\theta(a_{ij}) \cdot \theta(b_{jk}) = \theta(ab)_{ik} \cdot \theta(e_{kk})\]

Now we want to prove that \(\theta(a_{ii}) \cdot \theta(b_{ii}) = \theta(e_{ii}) \cdot \theta(ab)_{ii}\). For \(l \neq i\), we have \((a_{ii} - a_{il}) \cdot (b_{li} + b_{ii}) = (ab)_{ii} - (ab)_{ii} = 0\). Since \(\theta\) preserves zero products, we have \(\theta(a_{ii} - a_{il}) \cdot \theta(b_{li} + b_{ii}) = 0\). Then we can write \([\theta(a_{ii}) - \theta(a_{il})] \cdot [\theta(b_{li}) + \theta(b_{ii}) = 0\) since \(\theta\) is additive. As \(\theta(a_{ii}) \cdot \theta(b_{li}) = 0\) and \(\theta(a_{il}) \cdot \theta(b_{ii}) = 0\), we obtain that \(\theta(a_{ii}) \cdot \theta(b_{li}) - \theta(a_{il}) \cdot \theta(b_{ii}) = 0\). As a result, \(\theta(a_{ii}) \cdot \theta(b_{ii}) = \theta(a_{il}) \cdot \theta(b_{ii}) = \theta(e_{ii}) \cdot \theta(ab)_{ii}\), and this concludes the proof.

The above lemma allows us to show that a map preserving zero products preserves equal products.

**Theorem 3.** Let \(R\) be a ring with 1, and \(A = M_n(R)\) where \(n \geq 2\). Let \(\theta: A \longrightarrow A\) be an additive map such that \(\theta(x) \cdot \theta(y) = 0\) whenever \(x \cdot y = 0\) for \(x, y \in A\). Then, for \(x_i, y_i \in A\),
with $\sum_{i=1}^{t} x_i \cdot y_i = 0$, we have $\sum_{i=1}^{t} \theta(x_i) \cdot \theta(y_i) = 0$. In particular, for $x, y, u, v \in A$ with $x \cdot y = u \cdot v$, we have $\theta(x) \cdot \theta(y) = \theta(u) \cdot \theta(v)$.

Proof. Any element in $A$ can be written as a sum of elements represented by the set $B = \{a_{ij}|a \in R, 1 \leq i, j \leq n\}$. So since $\theta$ is an additive map, we find $\theta(A)$ is generated additively by this set $\theta(B) = \{\theta(a_{ij})|a \in R, 1 \leq i, j \leq n\}$. Therefore, if $x_1, y_1, ..., x_t, y_t \in A$, we get $x_1 \cdot y_1 + ... + x_t \cdot y_t$ can be expressed as a sum of elements of the form $x \cdot y$ with $x, y \in B$, and $\theta(x_1) \cdot \theta(y_1) + ... + \theta(x_t) \cdot \theta(y_t)$ can be written as a sum of the image of these elements, that is, $\theta(x) \cdot \theta(y)$. Observe that the element $x \cdot y$ where $x, y \in B$ is of one of the following forms: $0, c_{ij}$, or $c_{ii}$, where $i \neq j$ and $c \in R$. Now let us study every form separately.

For the terms $x \cdot y$ with $x, y \in B$, and $x \cdot y = 0$, we consider two possibilities:

- If the terms are of the form $a_{ij} \cdot b_{kl}$, with $j \neq k$, then the corresponding terms $\theta(a_{ij}) \cdot \theta(b_{kl}) = 0$ by (a) in the previous lemma.

- If the terms are of the form $a_{ij} \cdot b_{jk}$, with $ab = 0$, then by (b) in the previous lemma, we can write the corresponding terms such that

$$\theta(a_{ij}) \cdot \theta(b_{jk}) = \theta(c_{ii}) \cdot \theta((ab)_{ik}) = \theta((ab)_{ik}) \cdot \theta(e_{kk}),$$

but $\theta((ab)_{ik}) = \theta(0)$, where $0$ denotes to a matrix whose all entries are zeros. Notice that since the map $\theta$ is additive, we have $\theta(0) = \theta((0 + 0)) = \theta(0) + \theta(0)$. So $\theta(0) = 0$. Thus, we can write

$$\theta(a_{ij}) \cdot \theta(b_{jk}) = \theta(c_{ii}) \cdot \theta((ab)_{ik}) = \theta((ab)_{ik}) \cdot \theta(e_{kk}) = \theta((0)_{ik}) \cdot \theta(e_{kk}) = 0.$$

In both cases, when $j \neq k$ and when $j = k$ with $ab = 0$, we get the following.

$$\sum \theta(a_{ij}) \cdot \theta(b_{kl}) = 0 \quad (11)$$

Now let us see the second case when $x \cdot y = c_{ij}$ with $i \neq j \in \{1, \ldots, n\}$. These terms arise from terms of the form $a_{ik} \cdot b_{kj}$ with $c = ab \neq 0$. Consider the sum $x_1 \cdot y_1 + x_2 \cdot y_2 + \ldots + x_t \cdot y_t$, and notice that every term in it is a sum of nonzero terms of the form $c_{ij}$. Since, from our
assumption, \( x_1 \cdot y_1 + x_2 \cdot y_2 + \ldots + x_t \cdot y_t = 0 \), we must have the sum of nonzero terms \( c_{ij} \) equals to zero. Observe that \( \theta \) is an additive map, so we have the corresponding term of this sum equals to the sum of the corresponding terms for every term, and then equal to 0. By (b) in the previous lemma we can write the corresponding terms of \( a_{ik} \cdot b_{kj} \) as 
\[
\theta(a_{ik}) \cdot \theta(b_{kj}) = \theta(e_{ii}) \cdot \theta(c_{ij}) = \theta(c_{ij}) \cdot \theta(e_{jj}); \quad c = ab.
\]
Since the sum of these terms equals to zero, we get
\[
\sum \theta(a_{ik}) \cdot \theta(b_{kj}) = 0 \tag{12}
\]

The third case is when \( x \cdot y = c_{ii} \) with \( i \in \{1, \ldots, n\} \). These terms arise from terms of the form \( a_{ik} \cdot b_{ki} \) with \( c = ab \neq 0 \). Consider the sum \( x_1 \cdot y_1 + x_2 \cdot y_2 + \ldots + x_t \cdot y_t \), and notice that every term in it is a sum of nonzero terms of the form \( c_{ii} \). Since, from our assumption, \( x_1 \cdot y_1 + x_2 \cdot y_2 + \ldots + x_t \cdot y_t = 0 \), we must have the sum of nonzero terms \( c_{ii} \) equals to zero. Observe that \( \theta \) is an additive map, so we have the corresponding term of this sum is equal to the sum of the corresponding terms for every term, and then equal to 0. By (b) in the previous lemma we can write the corresponding terms of \( a_{ik} \cdot b_{ki} \) as 
\[
\theta(a_{ik}) \cdot \theta(b_{ki}) = \theta(e_{ii}) \cdot \theta(c_{ii}) = \theta(c_{ii}) \cdot \theta(e_{ji}); \quad c = ab.
\]
Since the sum of these terms is equal to zero, we get
\[
\sum \theta(a_{ik}) \cdot \theta(b_{ki}) = 0 \tag{13}
\]

From (11), (12) and (13), we find that \( \theta(x_1) \cdot \theta(y_1) + \ldots + \theta(x_t) \cdot \theta(y_t) = 0 \) for all three forms of the element \( x \cdot y \).

In particular, if \( x, y, u, v \in M_n(A) \) such that \( x \cdot y = u \cdot v \), then we can write \( x \cdot y + (-u) \cdot v = 0 \). By the result of this theorem, we find \( \theta(x) \cdot \theta(y) + \theta(-u) \cdot \theta(v) \) also vanishes. So we get
\[
\theta(x) \cdot \theta(y) = \theta(u) \cdot \theta(v)
\]

As a consequence of the previous theorem, our map \( \theta \) satisfies the following property:
\[ \theta(1) \cdot \theta(x \cdot y) = \theta(x) \cdot \theta(y) \quad \text{for all } x, y \in A \]

Indeed, if \( x_1 \cdot y_1 + x_2 \cdot y_2 = 0 \), then we have \( \theta(x_1) \cdot \theta(y_1) + \theta(x_2) \cdot \theta(y_2) = 0 \). Set \( x_1 = 1, y_1 = x \cdot y, x_2 = x \) and \( y_2 = (-y) \). Since \( 1 \cdot (x \cdot y) + x \cdot (-y) = 0 \), we have \( \theta(1) \cdot \theta(x \cdot y) + \theta(x) \cdot \theta(-y) = 0 \). Then,

\[ \theta(1) \cdot \theta(x \cdot y) = \theta(x) \cdot \theta(y) \quad \text{for all } x, y \in A. \]

So we can see obviously that \( \theta \) is close to being a homomorphism multiplied by a scalar.

In the next theorem, we shall use the functional identities which we presented in the introduction. According to our notations, [1, Theorem 4.13] implies the following result.

**Corollary 2.** Let \( 0 \leq n < m \), and let \( E: S^n \rightarrow Q \). Suppose

\[ \sum_{i \in I} E(x^i_m)\theta(x_i) - \sum_{j \in J} \theta(x_j)E(x^j_m) = P. \]

Here \( P \) is a quasi-polynomial with central coefficient \( \lambda_1 \). Suppose that either \( R \) is \( m \)-free and \( \lambda_1 = 0 \), or \( R \) is \( (m+1) \)-free. Then all \( E \) are quasi-polynomials.

Also, [1, Lemma 4.4] implies the following result “Let \( P \) be a quasi-polynomial of degree \( \leq m \), and suppose that either \( \hat{R} \) is \( m \)-free and \( \lambda_1 = 0 \) or \( \hat{R} \) is \( (m+1) \)-free. Then \( P = 0 \) if and only if each of its coefficients equals to zero”

**Theorem 4.** Let \( R \) be a ring with \( 1 \), and \( A = M_n(R); \ n \geq 3 \). Let \( \theta: A \rightarrow A \) be a surjective additive map preserving zero products, and let \( \theta(A) \) be a 3-free subset of \( A \). Then \( \theta(x) = \lambda \varphi(x) \) where \( \lambda \) is a central element and \( \varphi \) is a homomorphism.

**Proof.** In the light of the preceding theorem we can write

\[ \theta(x \cdot y) \cdot \theta(z) = \theta(x) \cdot \theta(y \cdot z) \]
That yields to \( \theta(x \cdot y) \cdot \theta(z) - \theta(x) \cdot \theta(y \cdot z) = 0 \). Observe that we can rewrite it as the following functional identity

\[
E(x, y) \cdot \theta(z) - \theta(x) \cdot E(y, z) = 0
\]  

(14)

to be in the standard form where \( E : A^2 \rightarrow A \) is defined by \( E(x, y) = \theta(x \cdot y) \). Since \( \theta(A) \) is a 3-free subset of \( A \), we can apply Corollary 2. By putting \( P = 0, m = 3, n = 2 \), we get that \( E(x, y) = \theta(x \cdot y) \) must be equal to a quasi polynomial, and since there are two variables, the polynomial will be of degree 2. So there exist elements \( \lambda_1, \lambda_2 \in C \), maps \( \mu_1, \mu_2 : A \rightarrow C \), and \( \nu : A^2 \rightarrow C \) such that

\[
E(x, y) = \theta(x \cdot y) = \lambda_1 \theta(x)\theta(y) + \lambda_2 \theta(y)\theta(x) + \mu_1(x)\theta(y) + \mu_2(y)\theta(x) + \nu(x, y)
\]  

(15)

Similarly, we can write

\[
E(y, z) = \theta(y \cdot z) = \lambda_1 \theta(y)\theta(z) + \lambda_2 \theta(z)\theta(y) + \mu_1(y)\theta(z) + \mu_2(z)\theta(y) + \nu(y, z)
\]  

(16)

By substituting from (15) and (16) in (14), we get the following

\[
[\lambda_1 \theta(x)\theta(y) + \lambda_2 \theta(y)\theta(x) + \mu_1(x)\theta(y) + \mu_2(y)\theta(x) + \nu(x, y)]\theta(z)
\]

\[
- \theta(x)[\lambda_1 \theta(y)\theta(z) + \lambda_2 \theta(z)\theta(y) + \mu_1(y)\theta(z) + \mu_2(z)\theta(y) + \nu(y, z)] = 0.
\]

Then we get

\[
\lambda_1 \theta(x)\theta(y)\theta(z) + \lambda_2 \theta(y)\theta(x)\theta(z) + \mu_1(x)\theta(y)\theta(z) + \mu_2(y)\theta(x)\theta(z)
\]

\[
+ \nu(x, y)\theta(z) - [\theta(x)\lambda_1 \theta(y)\theta(z) + \theta(x)\lambda_2 \theta(z)\theta(y) + \theta(x)\mu_1(y)\theta(z)
\]

\[
+ \theta(x)\mu_2(z)\theta(y) + \theta(x)\nu(y, z)] = 0.
\]
After we cancel some terms, we get

\[ \lambda_2 \theta(y) \theta(x) \theta(z) - \theta(x) \lambda_2 \theta(z) \theta(y) - \theta(x) \mu_2(z) \theta(y) + [\mu_2(y) - \mu_1(y)] \theta(x) \theta(z) + \mu_1(x) \theta(y) \theta(z) + \nu(x,y) \theta(z) - \theta(x) \nu(y,z) = 0. \]

Since \( \theta(A) \) is a 3-free subset of \( A \), we can apply [1, Lemma 4.4]. Since \( P = 0 \) in the last identity, we have \( \lambda_2 = 0, \mu_1 = \mu_2 = 0 \), and \( \nu = 0 \). As a consequence, we get

\[ \theta(x \cdot y) = \lambda_1 \theta(x) \cdot \theta(y) \]  

(17)

where \( \lambda_1 \) is a central element. Take \( \varphi(x) = \lambda_1 \theta(x) \). Accordingly, \( \varphi(x \cdot y) = \lambda_1 \lambda_1 \theta(x) \cdot \theta(y) \). So \( \varphi(x \cdot y) = (\lambda_1 \theta(x)) \cdot (\lambda_1 \theta(y)) \).

That yields, \( \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) \). That means \( \varphi \) is a homomorphism. Observe that \( \lambda_1 \) is invertible. Indeed, if we take \( y = 1 \) in (17), we will get \( \theta(x) = \lambda_1 \theta(x) \theta(1) \). Since \( \theta \) is surjective, we can find \( x \) such that \( \theta(x) = 1 \). Then, we can write \( 1 = \lambda_1 \theta(1) \). As a result, \( \theta(x) = \lambda_1^{-1} \varphi(x) \). Put \( \lambda_1^{-1} = \lambda \), and since the inverse of a central element is also central, \( \lambda \) is a central element. Now we have \( \theta(x) = \lambda \varphi(x) \) as required.

An interesting thing is that we can prove this theorem in an easier and more general way by the following.

**Theorem 5.** Let \( R \) be a ring with 1, and \( A = M_n(R); n \geq 2 \). Let \( \theta : A \rightarrow A \) be a surjective additive map preserving zero products, then \( \theta(x) = \lambda \varphi(x) \) where \( \lambda \) belongs to the center of \( A \), and \( \varphi \) is a homomorphism.

*Proof.* From Theorem 3, since \( (x \cdot y) \cdot z = x \cdot (y \cdot z) \), we have

\[ \theta(x \cdot y) \cdot \theta(z) = \theta(x) \cdot \theta(y \cdot z) \]  

(18)
Observe that if we set \( x = z = 1 \) in (18), we will get \( \theta(y) \cdot \theta(1) = \theta(1) \cdot \theta(y) \), and since \( \theta \) is a surjective map, we find that \( \theta(1) \) lies in the center of \( A \). Now, by taking \( z = 1 \) in (18), we get
\[
\theta(x \cdot y) \cdot \theta(1) = \theta(x) \cdot \theta(y) \quad (19)
\]
Notice that also since \( \theta \) is a surjective map, every element in the co-domain is an image for an element in the domain. So there is an element \( u \) in \( A \) such that \( \theta(u) = 1 \). Now, setting \( x = y = u \) in (19), we get
\[
\theta(u \cdot u) \cdot \theta(1) = \theta(u) \cdot \theta(u)
\]
So we have \( \theta(u^2) \cdot \theta(1) = 1 \). That is \( \theta(u^2) \) is the multiplicative inverse of \( \theta(1) \). As a result for that, we find \( \theta(1) \) is an invertible element. So we can write this equation \( \theta(x \cdot y) \cdot \theta(1) = \theta(1) \cdot \theta(x \cdot y) = \theta(x) \cdot \theta(y) \) as the following
\[
\theta(x \cdot y) = \frac{1}{\theta(1)} \cdot \theta(x) \cdot \theta(y)
\]
where \( \theta(1) \neq 0 \) and it belongs to the center of \( A \). Putting \( \mu = \frac{1}{\theta(1)} \), we get
\[
\theta(x \cdot y) = \mu \theta(x) \cdot \theta(y) \quad (20)
\]
Now take \( \varphi(x) = \mu \theta(x) \). Accordingly, \( \varphi(x \cdot y) = \mu \theta(x \cdot y) \). Then by (20), we can write
\[
\varphi(x \cdot y) = \mu \theta(x) \cdot \theta(y).
\]
So \( \varphi(x \cdot y) = (\mu \theta(x)) \cdot (\mu \theta(y)) \). That yields to \( \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) \), that is \( \varphi \) is a homormphism. As a result, \( \theta(x) = \mu^{-1} \varphi(x) \). Put \( \mu^{-1} = \lambda \). Now we have \( \theta(x) = \lambda \varphi(x) \) as required.

2.2 Maps acting like derivations

This section is organized in a similar way to the previous one. It describes a kind of maps that behaves like derivation when acting on zero products, and provides a result analogous to what we got in Theorem 5.
Lemma 2. Let \( R \) be a ring with 1, \( A = M_n(R) \) where \( n \geq 2 \), and let \( \delta : A \to A \) be an additive map such that \( \delta(x) y + x \delta(y) = 0 \) whenever \( xy = 0 \). Then, for \( a,b \in R \) and \( i,j,k,l \in \{1,2,3,\ldots,n\} \), we have:

(a) \( \delta(a_{ij}) \cdot b_{kl} + a_{ij} \cdot \delta(b_{kl}) = 0 \) if \( j \neq k \).

(b) \( \delta(a_{ij}) \cdot b_{jk} + a_{ij} \cdot \delta(b_{jk}) = \delta(e_{ii}) \cdot (ab)_{ik} + e_{ii} \cdot \delta((ab)_{ik}) = \delta((ab)_{ik}) \cdot e_{kk} + (ab)_{ik} \cdot \delta(e_{kk}) \).

Proof. (a) We know that for \( j \neq k \), we have via direct calculation \( a_{ij} \cdot b_{kl} = 0 \). We can obtain \( \delta(a_{ij}) \cdot b_{kl} + a_{ij} \cdot \delta(b_{kl}) = 0 \).

(b) Notice that if \( j \neq i \), \( (a_{ij} + e_{ii}) \cdot (b_{jk} - (ab)_{ik}) = a_{ij} \cdot b_{jk} - a_{ij} \cdot (ab)_{ik} + e_{ii} \cdot b_{jk} - e_{ii} \cdot (ab)_{ik} \).

Since \( a_{ij} \cdot (ab)_{ik} = 0 \) and \( e_{ii} \cdot b_{jk} = 0 \), we get \( a_{ij} \cdot b_{jk} - (ab)_{ik} = 0 \). Thus \( (a_{ij} + e_{ii}) \cdot (b_{jk} - (ab)_{ik}) = 0 \). That leads directly to \( \delta(a_{ij} + e_{ii}) (b_{jk} - (ab)_{ik}) + (a_{ij} + e_{ii}) \delta(b_{jk} - (ab)_{ik}) = 0 \).

Observe that by using the additive property of the map which is given in the lemma, we can write the last step as \( [\delta(a_{ij}) + \delta(e_{ii})] \cdot (b_{jk} - (ab)_{ik}) + [a_{ij} + e_{ii}] \delta((ab)_{ik}) = 0 \).

Then, \( \delta(a_{ij}) \cdot b_{jk} - \delta(a_{ij}) \cdot (ab)_{ik} + \delta(e_{ii}) \cdot b_{jk} - \delta(e_{ii}) \cdot (ab)_{ik} + a_{ij} \cdot \delta(b_{jk}) - a_{ij} \cdot \delta(ab)_{ik} + e_{ii} \cdot \delta(b_{jk}) - e_{ii} \cdot \delta(ab)_{ik} = 0 \). So we get \( [\delta(a_{ij}) \cdot b_{jk} + a_{ij} \cdot \delta(b_{jk})] - [\delta(a_{ij}) \cdot (ab)_{ik} + a_{ij} \cdot \delta(ab)_{ik}] = 0 \). By part (a) of this lemma, we have \( [\delta(a_{ij}) \cdot (ab)_{ik} + a_{ij} \cdot \delta(ab)_{ik}] = 0 \), and \( [\delta(e_{ii}) \cdot b_{jk} + e_{ii} \cdot \delta(b_{jk})] = 0 \). We also have \( [\delta(a_{ij}) \cdot b_{jk} + a_{ij} \cdot \delta(b_{jk})] - [\delta(e_{ii}) \cdot (ab)_{ik} + e_{ii} \cdot \delta(ab)_{ik}] = 0 \). Consequently,

\[
\delta(a_{ij}) \cdot b_{jk} + a_{ij} \cdot \delta(b_{jk}) = \delta(e_{ii}) \cdot (ab)_{ik} + e_{ii} \cdot \delta(ab)_{ik}
\]

For \( j \neq k \), we obtain

\[
\delta(a_{ij}) \cdot b_{jk} + a_{ij} \cdot \delta(b_{jk}) = \delta((ab)_{ik}) \cdot e_{kk} + (ab)_{ik} \cdot \delta(e_{kk}).
\]

This can be derived from \( (a_{ij} + (ab)_{ik}) \cdot (b_{jk} - e_{kk}) \). Indeed, \( (a_{ij} + (ab)_{ik}) \cdot (b_{jk} - e_{kk}) = a_{ij} \cdot b_{jk} - a_{ij} \cdot e_{kk} + (ab)_{ik} \cdot b_{jk} - (ab)_{ik} \cdot e_{kk} \). Since \( a_{ij} \cdot e_{kk} = 0 \), and \( (ab)_{ik} \cdot b_{jk} = 0 \), we get \( a_{ij} \cdot b_{jk} - (ab)_{ik} \cdot e_{kk} = 0 \). Thus, \( (a_{ij} + (ab)_{ik}) \cdot (b_{jk} - e_{kk}) = 0 \). This implies
\[\delta(a_{ij} + (ab)_{ik}) \cdot (b_{jk} - e_{kk}) + (a_{ij} + (ab)_{ik}) \cdot \delta(b_{jk} - e_{kk}) = 0.\]  
Again by the additive property of the map, we have \[\delta(a_{ij}) + \delta(ab)_{ik} \cdot [b_{jk} - e_{kk}] + [a_{ij} + (ab)_{ik}] \cdot \delta(b_{jk} - e_{kk}) = 0.\]  
Then, \[\delta(a_{ij}) \cdot b_{jk} - (a_{ij}) \cdot e_{kk} + \delta(ab)_{ik} \cdot b_{jk} - \delta(ab)_{ik} \cdot e_{kk} + a_{ij} \cdot \delta(b_{jk}) - a_{ij} \cdot \delta(e_{kk}) + (ab)_{ik} \cdot \delta(b_{jk}) - (ab)_{ik} \cdot \delta(e_{kk}) = 0.\]  
Now we can write \[\delta(a_{ij}) \cdot b_{jk} + a_{ij} \cdot \delta(b_{jk}) - \delta((ab)_{ik}) \cdot b_{jk} + (ab)_{ik} \cdot \delta(b_{jk}) - \delta((ab)_{ik}) \cdot e_{kk} + (ab)_{ik} \cdot \delta(e_{kk}) = 0.\]  
Since \[\delta(a_{ij}) \cdot e_{kk} + a_{ij} \cdot \delta(e_{kk}) = 0,\]  
and \[\delta((ab)_{ik}) \cdot b_{jk} + (ab)_{ik} \cdot \delta(b_{jk}) = 0\] by (a), we get \[\delta(a_{ij}) \cdot b_{jk} + a_{ij} \cdot \delta(b_{jk}) - \delta((ab)_{ik}) \cdot (e_{kk}) + (ab)_{ik} \cdot \delta(e_{kk}) = 0.\]  
Consequently,  
\[\delta(a_{ij}) \cdot b_{jk} + a_{ij} \cdot \delta(b_{jk}) = \delta((ab)_{ik}) \cdot e_{kk} + (ab)_{ik} \cdot \delta(e_{kk})\]

Now we want to prove that \[\delta(a_{ii}) \cdot b_{ii} + a_{ii} \cdot \delta(b_{ii}) = \delta(e_{ii}) \cdot (ab)_{ii} + e_{ii} \cdot \delta((ab)_{ii}).\]  
Notice that for \(l \neq i,\) we have \((a_{ii} - a_{il}) \cdot (b_{ii} + b_{il}) = (ab)_{ii} - (ab)_{ii} = 0.\) We obtain \[
\delta(a_{ii} - a_{il}) \cdot (b_{ii} + b_{il}) + (a_{ii} - a_{il}) \cdot \delta(b_{ii} + b_{il}) = 0.\]

Since \(\delta\) is an additive map, we have \[\delta(a_{ii}) - \delta(a_{il}) \cdot [b_{ii} + b_{il}] + [a_{ii} - a_{il}] \cdot \delta(b_{ii} + b_{il}) = 0.\] Then we get \[
\delta(a_{ii}) \cdot b_{ii} + \delta(a_{ii}) \cdot b_{il} - \delta(a_{il}) \cdot b_{ii} + \delta(a_{il}) \cdot b_{il} + a_{ii} \cdot \delta(b_{ii}) + a_{ii} \cdot \delta(b_{il}) - a_{il} \cdot \delta(b_{ii}) - a_{il} \cdot \delta(b_{il}) = 0.\]

This implies \[\delta(a_{ii}) \cdot b_{il} + a_{ii} \cdot \delta(b_{il}) = \delta(a_{il}) \cdot b_{il} + a_{il} \cdot \delta(b_{il}) = 0.\]  
Since \(\delta(a_{ii}) \cdot b_{ii} + a_{ii} \cdot \delta(b_{ii}) = 0 = \delta(a_{il}) \cdot b_{il} + a_{il} \cdot \delta(b_{il}),\) we get \[
\delta(a_{ii}) \cdot b_{ii} + a_{ii} \cdot \delta(b_{ii}) = \delta(a_{ii}) \cdot b_{il} + a_{il} \cdot \delta(b_{il}) = \delta(e_{ii}) \cdot (ab)_{ii} + e_{ii} \cdot \delta((ab)_{ii}).\]

This completes the proof.

\[\square\]

This lemma enables us to provide the following theorem.

**Theorem 6.** Let \(R\) be a ring with 1, and \(A = M_n(R)\) where \(n \geq 2.\) Let \(\delta: A \to A\) be an additive map such that if \(x, y \in A\) and \(\delta(x) \cdot y + x \cdot \delta(y) = 0\) whenever \(x \cdot y = 0.\) Then, for \(x_i, y_i \in A\) with \(\sum_{i=1}^{l} x_i \cdot y_i = 0,\) we have \(\sum_{i=1}^{l} (\delta(x_i) \cdot y_i + x_i \cdot \delta(y_i)) = 0.\) In particular, for \(x, y, u, v \in A\) with \(x \cdot y = u \cdot v,\) we have \(\delta(x) \cdot y + x \cdot \delta(y) = \delta(u) \cdot v + u \cdot \delta(v)\)

**Proof.** Let \(B = \{a_{ij} | a \in R, 1 \leq i, j \leq n\}.\) As we said before, any element in \(A\) can be written as a sum of elements that are represented by \(B.\) Since \(\delta\) is an additive map, any element
in $\delta(A)$ can be expressed as a sum of elements of $\delta(B) = \{\delta(a_{ij})|a \in R, 1 \leq i, j \leq n\}$. Now, for $x_1, y_1, ..., x_t, y_t \in A$, we have $x_1 \cdot y_1 + ... + x_t \cdot y_t$ is a sum of elements of the form $x \cdot y$ with $x, y \in B$, and $\delta(x_1) \cdot y_1 + x_1 \cdot \delta(y_1) + ... + \delta(x_t) \cdot y_t + x_t \cdot \delta(y_t)$ is a sum of corresponding elements $\delta(x) \cdot y + x \cdot \delta(y)$. Observe that the element $x \cdot y$ is of one of the following forms: 0, $c_{ij}$, or $c_{ii}$, where $i \neq j$ and $c \in R$.

For the terms $x \cdot y$ with $x, y \in B$, and $x \cdot y = 0$, we consider two possibilities:

- If the terms are of the form $a_{ij} \cdot b_{kl}$, with $j \neq k$, then the corresponding terms $\delta(a_{ij}) \cdot b_{kl} + a_{ij} \cdot \delta(b_{kl}) = 0$ by (a) in the previous lemma.

- If the terms are of the form $a_{ij} \cdot b_{jk}$, with $ab = 0$, then by (b) in the previous lemma, we can write the corresponding terms such that

$$\delta(a_{ij}) \cdot b_{jk} + a_{ij} \cdot \delta(b_{jk}) = \delta(e_{ii}) \cdot (ab)_{ik} + e_{ii} \cdot \delta((ab)_{ik}) = \delta((ab)_{ik}) \cdot e_{kk} = 0$$

In both cases when $j \neq k$ and when $j = k$ with $ab = 0$, we get the following.

$$\sum (\delta(a_{ij}) \cdot b_{kl} + a_{ij} \cdot \delta(b_{kl})) = 0 \quad (21)$$

The terms $x \cdot y = c_{ij}$ with $i \neq j \in \{1, 2, ..., n\}$ are resulted from terms of the form $a_{ik} \cdot b_{kj}$ with $c = ab \neq 0$. Since $\sum_{i=1}^{t} x_i \cdot y_i = 0$, the sum of nonzero terms of the form $c_{ij}$ equals to zero. Notice that $\delta$ is an additive map, so the corresponding term of this sum equals to the sum of the corresponding terms for every term, and then equal to zero. So by part (b) in Lemma 2, we can write the corresponding terms as $\delta(a_{ik}) \cdot b_{kj} + a_{ik} \cdot \delta(b_{kj}) = \delta(e_{ii}) \cdot c_{ij} + e_{ii} \cdot \delta(c_{ij}); c = ab$. Since the sum of these terms vanishes, we get

$$\sum (\delta(a_{ik}) \cdot b_{kj} + a_{ik} \cdot \delta(b_{kj})) = 0 \quad (22)$$

By the same way, the terms $x \cdot y = c_{ii}$ with $i \neq j \in \{1, 2, ..., n\}$ are resulted from terms of the form $a_{ik} \cdot b_{ki}$ with $c = ab \neq 0$. Since $\sum_{i=1}^{t} x_i \cdot y_i = 0$, the sum of nonzero
terms of the form $c_{ii}$ equals to zero. Notice that $\delta$ is an additive map, so the corresponding term of this sum equals to the sum of the corresponding terms for every term, and then equal to zero. Again by part (b) in Lemma 2, we can write the corresponding terms as $\delta(a_{ik}) \cdot b_{ki} + a_{ik} \cdot \delta(b_{ki}) = \delta(e_{ii}) \cdot c_{ii} + e_{ii} \cdot \delta(c_{ii})$; $c = ab$. Since the sum of these terms equals to zero, we get

$$\sum (\delta(a_{ik}) \cdot b_{ki} + a_{ik} \cdot \delta(b_{ki})) = 0 \quad (23)$$

From (21), (22) and (23), we can see that $\delta(x_1) \cdot y_1 + x_1 \cdot \delta(y_1) + \ldots + \delta(x_i) \cdot y_i + x_i \cdot \delta(y_i)$.

In particular, if $x, y, u, v \in M_n(A)$ such that $x \cdot y = u \cdot v$, then $x \cdot y - u \cdot v = 0$. According to this theorem, the corresponding term $\delta(x) \cdot y + x \cdot \delta(y) - \delta(u) \cdot v - u \cdot \delta(v)$ also vanishes. Thus, we get $\delta(x) \cdot y + x \cdot \delta(y) = \delta(u) \cdot v - u \cdot \delta(v)$.

**Theorem 7.** Let $R$ be a ring with 1, and $A = M_n(R)$; $n \geq 2$. Let $\delta : A \to A$ be an additive map such that $\delta(x) \cdot y + x \cdot \delta(y) = 0$ whenever $x \cdot y = 0$. Then, $\delta(x) = d(x) + c \cdot x$ where $c = \delta(1)$ belongs to the center of $A$, and $d : A \to A$ is a derivation.

**Proof.** By applying the previous theorem, since $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, we get $\delta(x) \cdot (y \cdot z) + x \cdot \delta(y \cdot z) = (x \cdot y) \cdot z + (x \cdot y) \cdot \delta(z)$. Then, we can write

$$\delta(x) \cdot (y \cdot z) + x \cdot \delta(y \cdot z) - \delta(x \cdot y) \cdot z - (x \cdot y) \cdot \delta(z) = 0 \quad (24)$$

By taking $x = z = 1$ in (24), we get $\delta(1) \cdot y + \delta(y) - \delta(y) - y \cdot \delta(1) = 0$. That leads to $\delta(1) \cdot y = y \cdot \delta(1)$. So $\delta(1)$ belongs to the center of $A$. Now, by taking $z = 1$, we get $\delta(x) \cdot y + x \cdot \delta(y) - \delta(x \cdot y) - x \cdot y \cdot \delta(1) = 0$. Then, we have

$$\delta(x) \cdot y + x \cdot \delta(y) - x \cdot y \cdot c = \delta(x \cdot y) \quad \text{where } c = \delta(1) \quad (25)$$

Suppose $d(x) = \delta(x) - c \cdot x$. Accordingly, $d(x \cdot y) = \delta(x \cdot y) - c \cdot (x \cdot y)$. Then, by equation (25), we obtain $d(x \cdot y) = \delta(x) \cdot y + x \cdot \delta(y) - c \cdot x \cdot y - c \cdot x \cdot y$. So we get $d(x \cdot y) = (\delta(x) - c \cdot x) \cdot y + x \cdot (\delta(y) - c \cdot y)$. That yields to $d(x \cdot y) = d(x) \cdot y + x \cdot d(y)$. That
means \( d \) is a derivation. Thus, \( \delta(x) = d(x) + c \cdot x \) where \( c = \delta(1) \) is a central element, and \( d : A \rightarrow A \) is a derivation.

### 2.3 Example

Let \( R \) be a commutative algebra over \( \mathbb{Q} \) generated by two elements \( a \) and \( b \), and satisfies that \( a^3 = b^3 = a^2b = ab^2 = 0 \) for all \( a, b \in R \). Then, \( S = \{ a, b, a^2, b^2, ab = ba \} \) is a \( \mathbb{Q} \)-basis of \( R \). So the left and the right annihilator of \( R \) contain \( a^2, b^2, ab \) and their linear combinations.

Now let us define the matrix ring \( M_2(R) \) such that for any \( u \in M_2(R) \) we have \( u = u_1e_{11} + u_2e_{12} + u_3e_{21} + u_4e_{22} \), where \( u_i \in R \). So every \( u_i \in R \) is written as a linear combination of the elements of \( S \) and scalars of \( \mathbb{Q} \) such that

\[
\begin{align*}
  u &= (q_{11}a + q_{12}b + q_{13}a^2 + q_{14}b^2 + q_{15}ab)e_{11} + (q_{21}a + q_{22}b + q_{23}a^2) \\
  &\quad + q_{24}b^2 + q_{25}ab)e_{12} + (q_{31}a + q_{32}b + q_{33}a^2 + q_{34}b^2 + q_{35}ab)e_{21} \\
  &\quad + (q_{41}a + q_{42}b + q_{43}a^2 + q_{44}b^2 + q_{45}ab)e_{22}.
\end{align*}
\]

Consider the linear map \( \theta : M_2(R) \rightarrow M_2(R) \), and for \( i, j \in \{1, 2\} \) we have the following

\[
\begin{align*}
  \theta(ae_{ij}) &= ae_{ij} \\
  \theta(be_{ij}) &= be_{ij} \\
  \theta(a^2e_{ij}) &= b^2e_{ij} \\
  \theta(b^2e_{ij}) &= a^2e_{ij} \\
  \theta(abe_{ij}) &= abe_{ij}
\end{align*}
\]

First, we can say directly \( \theta \) is additive since it is a linear map. It is also a surjective map. Indeed, for any \( \theta(u) \in M_2(R) \), from (26) we can write

\[
\begin{align*}
  \theta(u) &= \theta((q_{11}a + q_{12}b + q_{13}a^2 + q_{14}b^2 + q_{15}ab)e_{11}) + \theta((q_{21}a + q_{22}b + q_{23}a^2) \\
  &\quad + q_{24}b^2 + q_{25}ab)e_{12}) + \theta((q_{31}a + q_{32}b + q_{33}a^2 + q_{34}b^2 + q_{35}ab)e_{21}) \\
  &\quad + \theta((q_{41}a + q_{42}b + q_{43}a^2 + q_{44}b^2 + q_{45}ab)e_{22}) \\
  &= \theta(q_{11}ae_{11} + q_{12}be_{11} + q_{13}a^2e_{11} + q_{14}b^2e_{11} + q_{15}abe_{11}) + \theta(q_{21}ae_{12}
\end{align*}
\]
If we look carefully to the last equality, we find it is nothing but an element in $M_2(R)$. As a consequence, $\theta$ is a surjective map.

Moreover, $\theta$ preserves zero products, that is, $\theta(u)\theta(v) = 0$ whenever $uv = 0$ for $u, v \in M_2(R)$. Indeed, from (27) we can write the following

\[
\begin{align*}
\theta(ace_{ij}) &= ace_{ij} + 0 \\
\theta(bce_{ij}) &= bce_{ij} + 0 \\
\theta(a^2e_{ij}) &= a^2e_{ij} + (b^2e_{ij} - a^2e_{ij}) \\
\theta(b^2e_{ij}) &= b^2e_{ij} + (a^2e_{ij} - b^2e_{ij}) \\
\theta((ab)e_{ij}) &= (ab)e_{ij} + 0
\end{align*}
\]
Notice that $a^2e_{ij}, b^2e_{ij}, abe_{ij},$ and any linear combination of them annihilate all elements of $M_2(R)$. This is because the elements $a^2, b^2,$ and $ab$ belong to the annihilator of $R$. So the second term of every equality is an annihilator element. Then, by linearity of $\theta$, we find every $\theta(u) \in M_2(R)$ can be written as a sum of $u$ and an annihilator element. So for any $u, v \in M_2(R)$, we have $\theta(u)\theta(v) = uv + ur + lv + lr$, where $r$ denotes the right annihilator element, and $l$ denotes the left annihilator element, and then the last three terms equal to 0. Thus, $\theta(u)\theta(v) = 0$ whenever $uv = 0$ for $u, v \in M_2(R)$, and $\theta$ preserves zero products.

Now let us show $\theta \neq \lambda \varphi$, where $\lambda$ is a central element and $\varphi$ is a homomorphism map. To seek a contradiction, assume $\theta$ is of the standard form. So we have $\theta(\alpha e_{11}ae_{11}) = \lambda \varphi(\alpha e_{11}ae_{11})$. Since $\varphi$ is a homomorphism, we can write $\theta(\alpha e_{11}ae_{11}) = \lambda \varphi(\alpha e_{11})\varphi(\alpha e_{11})$. By multiplying both sides by $\lambda$, we get $\lambda \theta(\alpha e_{11}ae_{11}) = [\lambda \varphi(\alpha e_{11})][\lambda \varphi(\alpha e_{11})]$. Then, by our assumption since $\theta(x) = \lambda \varphi(x)$, we can write $\lambda \theta(\alpha e_{11}ae_{11}) = \theta(\alpha e_{11})\theta(\alpha e_{11})$. So

$$
\lambda \theta(a^2e_{11}) = \theta(\alpha e_{11})\theta(\alpha e_{11}).
$$

By (27), we have

$$
\lambda b^2e_{11} = a^2e_{11}
$$

On the other hand, consider $\theta(\alpha e_{11}be_{11}) = \lambda \varphi(\alpha e_{11}be_{11})$. Since $\varphi$ is a homomorphism map, we can write $\theta(\alpha e_{11}be_{11}) = \lambda \varphi(\alpha e_{11})\varphi(\alpha e_{11})$. By multiplying both sides by $\lambda$, we get $\lambda \theta(\alpha e_{11}be_{11}) = [\lambda \varphi(\alpha e_{11})][\lambda \varphi(\alpha e_{11})]$. Then, by our assumption since $\theta(x) = \lambda \varphi(x)$, we have $\lambda \theta(\alpha e_{11}be_{11}) = \theta(\alpha e_{11})\theta(\alpha e_{11})$. We get

$$
\lambda \theta((ab)e_{11}) = \theta(\alpha e_{11})\theta(\alpha e_{11}).
$$

As a result by (27), we have

$$
\lambda(ab)e_{11} = (ab)e_{11}
$$

From (28) and (29), we find $a^2 = b^2$, which is a contradiction.

As a consequence, “$A$ contains 1” is a necessary condition for $\theta$ to be in the standard form.
CHAPTER 3

Conclusion

In 1980, Wong proved an important theorem mentioned in the introduction. There are many recent results generalizing the Wong’s result. In 2004, Chebotar, Ke, and Lee described maps preserving zero products for prime rings containing a nontrivial idempotent. In 2006, Chebotar, Ke, Lee, and Zhang characterized maps preserving zero Jordan products for matrix rings. In 2012, Stopar described a surjective additive map which preserves zero products from a ring with a nontrivial idempotent to a prime ring. This thesis expands the previous work in this field by presenting three results. We consider two types of additive maps on matrix rings. The goal is to prove that one kind of them which is a surjective additive map preserving zero products can be written as the product of a central element with some homomorphism, and the other type which is additive map acting like derivation through zero products can be represented in a parallel form. The third result shows that our standard form does not hold for nonunital rings.

We divided this thesis into sections. We began by mentioning some contributions that have been made in the area related to our argument, and provided the necessary definitions. The following section was devoted to the preliminary study of the theory of functional identities. Our concern was not to study the theory of functional identities exhaustively, but to provide an abbreviated treatment that supported our proofs of some theorems. In the next section we proved the standard form for surjective additive maps preserving zero products. We then continued our discussion by focusing on additive maps $\delta$ that behave like derivation when acting on zero products. Using similar methods to those used in the preceding section, we proved that $\delta(x) = cx + d(x)$, where $c$ is a central element and $d$ is a
derivation. Finally, we provided a counterexample for a nonunital ring.

The purpose of this study was to describe maps for matrix rings by action on zero products. Our results may be extended by considering maps preserving zero products with other kinds of restrictions, and by considering such maps on other types of algebraic structures.
BIBLIOGRAPHY


