OPTION PRICING WITH STOCHASTIC VOLATILITY AND RECOMBINING TREE

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by
Yuzhang Chen
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Thesis written by

Yuzhang Chen
B.S., Beijing Normal University
M.Sc., Kent State University

Approved by

Oana Macioalca, Ph.D., Master’s Advisor, Department of Mathematical Sciences,
Andrew Tonge, Ph.D., Chair, Department of Mathematical Sciences
James L.Blank, Ph.D., Dean, College of Arts & Sciences
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Yuzhang Chen

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CHAPTER 1 Introduction

The history of modeling financial markets with stochastic processes starts in 1900, when Bachelier assumed the stock prices to be a Brownian motion with drift. Ever since, various empirical formulas and pricing models have come out in succession, but none were widely accepted until the significant breakthrough by Fischer Black and Myron Scholes during the 1970s. They demonstrated the method to price options based on geometric Brownian motion. The Black-Scholes formula is still frequently used today due to its fair accuracy and convenience. However, despite landmark success in option pricing theory by the publication of Black-Scholes model, this formula has significant biases. The model fails to describe the structure of observed option prices from the assumption that volatility remains constant. In practice, the implied volatility tends to vary in terms of the strike price and expiry, which is known as the volatility skew in its graph.

A natural improvement is to regard the volatility process as a stochastic process. However, accounting for such a stochastic volatility (SV) within a valuation formula is not a straightforward task. Among the various stochastic volatility models the most notable ones are Hull and White Model (1987), Heston Model (1993) and Schöbel-Zhu model (1999). Even so, there are no simple formulas for the price of options on SV-driven stocks. Approximations have been constructed to these specific volatility models, e.g. Hilliard and Schwartz (1996), and Ritchken and Trevor (1999). In the effort of
proposing the pricing scheme not restricted to seeking explicit formulas, the tree-based Binomial model (Sharpe, 1978) was applied in late 1970s to approximate the lognormal Black-Scholes model and associated pricing formulas discretely. For European options without dividends, the binomial model converges on the Black–Scholes formula as the number of time steps goes to infinity. Later, Cox and Rubinstein (1985) used similar approach to value American options on dividend paying stock, with some relaxed assumptions of the original Black-Scholes model, consolidating the deemed power and versatility of tree-based pricing.

The structure of this thesis is as follows. In chapter 2, we introduce the basic concepts in stochastic processes, including the Brownian motion, stochastic integrals and some important lemmas such as Itô’s Lemma. In chapter 3, we provide a brief introduction to the Binomial tree model. We describe the procedure of creating a tree binomial tree and calculating the value of the simulation. In chapter 4, we’ll illustrate the classical Black-Scholes Model. In chapter 5, we’ll discuss our method to price options, and make comparisons to the classical models. In chapter 6 we end with conclusions.
CHAPTER 2 Stochastic Processes and Option pricing

2.1 Brownian Motion

Definition 2.1.1 A sequence of random variables, \( B_t \), for \( t \geq 0 \), is called a Brownian Motion or a Wiener Process if it satisfies the following properties:

- \( B_0 = 0 \)

- For every \( t \) and \( s \), with \( s < t \), the distribution of \( B_t - B_s \) is independent of \( B_r \), for every \( r \leq s \)

- For every \( t \) and \( s \), with \( s < t \), we have that \( B_t - B_s \) is distributed as a normal distribution with variance \( t - s \).

- \( B_t \) has continuous trajectories.

\( B_t \) is a Gaussian process since the finite dimensional distributions of Brownian motion are multivariate Gaussian. From the definition we know that \( B_t - B_s \) has the same distribution as \( B_{t-s} - B_0 = B_{t-s} \), which is a normal distribution with mean zero and variance \( t - s \).
It can be deduced that Brownian motion has expectation

$$E(B_t) = 0$$  \hspace{1cm} (2.1)

and the covariance

$$\text{Cov}(B_s, B_t) = E[(B_t - B_s + B_s)B_t] = E[B_tB_s] + E(B^2_s)$$

$$= E(B_t - B_s)E(B_s) + s = 0 + s = s = \min(s, t)$$  \hspace{1cm} (2.2)

2.2 Stochastic Integrals

**Definition** 2.1.2 The collection \(\{\mathcal{F}_t\}\) is called a filtration if for all \(s < t\), we have
\[ \mathcal{F}_s \subset \mathcal{F}_t \]

**Definition 2.1.3** If \( Y \) is a stochastic process such that for all \( t \geq 0 \) we have

\[ Y_t \in \mathcal{F}_t \]

then we say that \( Y \) is adapted to the \( \mathcal{F}_t - filtration \)

We now consider a stochastic process \( X \) and a given Brownian Motion \( B \), to construct the stochastic integral.

**Definition 2.1.4** We say that the process \( X \) belongs to the class \( \mathcal{L}^2 [a, b] \) if the following conditions are satisfied.

- \( \int_a^b E[X_s^2] ds < \infty \)
- The process \( X \) is adapted to the \( \mathcal{F}_t - filtration \)

For a general process \( X \in \mathcal{L}^2 [a, b] \), we define the stochastic integral as follows.

1. Approximate \( X \) with a sequence of simple processes \( X_n \) such that

\[ \int_a^b [ (X_{n,s} - X_s)^2] ds \rightarrow 0 \]

2. For each \( n \) define \( Z_n = \int_a^b X_{n,s} d\mathbb{B}_s \), and there exists a stochastic variable \( Z \)

such that \( \lim_{n \rightarrow \infty} Z_n = Z \).

3. Now we define the stochastic integral by

\[ \int_a^b X_s d\mathbb{B}_s = \lim_{n \rightarrow \infty} \int_a^b X_{n,s} d\mathbb{B}_s \]

**Definition 2.1.5** A stochastic process \( X \) is called an \( (\mathcal{F}_t) - martingale \) if the following conditions are satisfied.
• X is adapted to the $\mathcal{F}_t$ - filtration

• $E[|X_t|] < \infty$ for all t

• For every $s \leq t$, $E[X_t | \mathcal{F}_s] = X_s$

2.3 Itô’s Lemma

Let $X$ be a stochastic process and suppose that there exists a real number $C$ and two adapted processes $\mu$ and $\sigma$ such that following relation holds for all $t \geq 0$

$$X_t = C + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dB_s$$  \hfill (2.3)

where B is a Brownian Motion and C is a real number. This equation is often written as

$$dX_t = \mu(t)dt + \sigma(t)dB_t$$  \hfill (2.4)

$$X_0 = C$$  \hfill (2.5)

Here we say $X$ has a stochastic differential given by (2.4) with an initial condition given by (2.5). Note that the formal equation $dX_t = \mu(t)dt + \sigma(t)dB_t$ has no independent meaning. It’s just short for the expression (2.3) above.

Stochastic differential equation (SDE) is often used to model fluctuating stock prices, it can be thought of as the derivative of Brownian motion, We’ll use the fact $(dB)^2 = dt$ and give the main result in the theory of stochastic calculus ---- the Itô’s Lemma, which is basically the chain rule for stochastic processes.

**Theorem 2.1.1 (Itô’s lemma)** Assume that the process X has a stochastic differential given by

$$dX_t = \mu(t)dt + \sigma(t)dB_t$$  \hfill (2.6)
where $\mu$ and $\sigma$ are adapted processes to a $\mathcal{F}_t$ - filtration, and let $f$ be a twice differentiable function. Define the process $Z$ by $Z_t = f(t, X_t)$, then $Z$ has the following stochastic differential:

$$dZ_t = \left\{ \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial x^2} \right\} dt + \sigma \frac{\partial f}{\partial x} dB_t$$

(2.7)

A heuristic proof is based on a second-order Taylor expansion of $f(t, X_t)$. A formal proof of the lemma relies on taking the limit of a sequence of random variables, which is beyond the scope of this thesis.

CHAPTER 3 Option Pricing with Binomial Tree Model

By creating a binomial tree, for a number of time steps between the valuation and expiration dates, the binomial pricing model traces the evolution of the option's underlying asset in discrete-time. Each node in the tree represents a possible price of the underlying at a given point in time.

Valuation is performed iteratively, starting at each of the final nodes (those that may be reached at the time of expiration), and then working backwards towards the first node (valuation date). The value computed at each stage is the value of the option at that point in time.
At each step, we assume that the price of the underlying asset will move up or down by a specific factor \( u \) or \( d \) (determined by underlying volatility \( \sigma \) and the time duration of a step \( t \)). We assume the tree is recombinant, thus creating a binomial tree.

![Figure 3.1 A 2-step binomial tree](image)

then we start from the penultimate node and calculate the value of the option \( C \) at each node. The price of the final node is equal to \( \text{Max}[S_t - K, 0] \) (Call option), and the value of its predecessor is calculated through the following equation

\[
C_{t-\Delta t,i} = e^{-r\Delta t} \left( pC_{t,i} + (1-p)C_{t,i+1} \right)
\]  

(2.8)

where \( p \) is determined by the specific tree method. For example, the Cox, Ross, & Rubinstein Method assumes \( u = e^{\sigma \sqrt{\Delta t}} \), \( d = e^{-\sigma \sqrt{\Delta t}} \) and \( p = \frac{e^{r\Delta t} - d}{u - d} \). Repeat the process and work all the way back to the starting point we’ll get the Binomial value of the option.
CHAPTER 4 Option Pricing with Black-Scholes Model

4.1 Black-Scholes Model and Arbitrage

**Definition 2.3.1** The price process \( Y \) is the price of a risk free asset if it has the dynamics

\[
dY_t = r(t)Y_t dt
\]  

(2.9)

where \( r \) is any adapted process.

It can also be written as \( \frac{dY_t}{dt} = r(t)Y_t \) since it has no driving \( dB \) term.

Assume the stock price \( S \) is given by

\[
dS_t = S_t \alpha(t,S_t)dt + S_t \sigma(t,S_t)dB_t
\]  

(2.10)

The function \( \alpha \) is the local mean rate of return of \( S \), and \( \sigma \) is the volatility of \( S \).

Since the “white noise” \( dB \) is random, the stock has a stochastic rate of return, which is opposite to the prevailing short rate \( r(t) \) of a risk free asset. The famous Black-Scholes model is the special case of the above model when \( r, \alpha \) and \( \sigma \) are deterministic constants.

**Definition 2.3.2** The Black-Scholes Model consists of two assets with the following dynamics.

\[
dY_t = rY_t dt
\]  

(2.11)

\[
dS_t = \alpha S_t dt + \sigma S_t dB_t
\]  

(2.12)

where \( r, \alpha \) and \( \sigma \) are deterministic constants.
Definition 2.3.3 An arbitrage opportunity is a self-financed portfolio $h$ such that

- $V^h(0) = 0$
- $V^h(T) \geq 0$ almost surely
- $P(V^h(T) > 0) > 0$

We say that the market is arbitrage-free if there are no arbitrage opportunities.

4.2 Black-Scholes Equation

Now suppose we wish to price a call option $C$ on a stock $S$, with strike $K$ and expiry $T$. Assume $S$ follows a geometric Brownian motion with drift $\mu$ and volatility $\sigma$, that is to say, $dS_t = \mu dt + \sigma dB_t$, using Itô’s lemma to deduce the SDE for $(C(t, S_t))$.

$$dC = \left(\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt + \sigma S \frac{\partial C}{\partial S} dB_t$$

If we consider the portfolio of the option and $\alpha$ stocks, we obtain

$$d(C + \alpha S) = \left(\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \alpha \mu S\right) dt + \sigma S \frac{\partial C}{\partial S} + \alpha dB_t$$

(2.13)

Take $\alpha$ equal to the value of the derivative with spot, we have a deterministic portfolio.

Since a risk-free portfolio must grow at the risk-free rate, the drift above must be equal to $r(C + \alpha S)$. Therefore we conclude that

$$\frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = r(C - S \frac{\partial C}{\partial S})$$

(2.14)

A little bit rearranging we get

Theorem 2.2.1 (Black-Scholes Equation)
\[
\frac{\partial C}{\partial S} + rS \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC
\]  
(2.15)

4.3 Black-Scholes Formula

Of course we want the solution of the Black-Scholes equation. Let \( S = e^Z \), the equation then becomes

\[
\frac{\partial C}{\partial t} + (r - \frac{1}{2} \sigma^2) \frac{\partial C}{\partial Z} + \frac{1}{2} \sigma^2 \frac{\partial^2 C}{\partial Z^2} = rC
\]  
(2.16)

Let \( \tau = T - t \), and \( D = e^{\tau r} C \), we obtain

\[
\frac{\partial D}{\partial \tau} - (r - \frac{1}{2} \sigma^2) \frac{\partial D}{\partial Z} - \frac{1}{2} \sigma^2 \frac{\partial^2 D}{\partial Z^2} = 0
\]  
(2.17)

Continue simplifying by letting \( \rho = Z + (r - \frac{1}{2} \sigma^2) \tau \) and our equation becomes

\[
\frac{\partial D}{\partial \tau} = \frac{1}{2} \sigma^2 \frac{\partial^2 D}{\partial \rho^2}
\]  
(2.18)

which is a one-dimensional heat-equation. By solving the heat-equation and transforming we obtain

**Theorem 2.2.2 (Black’s Formula)** The price, at time \( t \), of a European call option with expiry \( T \) and strike \( K \), is given by

\[
C(t, S_t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)
\]  
(2.19)

where \( N(x) \) denotes the cumulative standard normal distribution and

\[
d_1 = \frac{\log(S / K) + (r + \frac{1}{2} \sigma^2)(T - t)}{\sigma \sqrt{T - t}}
\]  
(2.20)
In the Black-Scholes model the input data contains the string \( S, r, T, t, K \) and \( \sigma \). Since the former five parameters can be directly observed, the problem left is to estimate the volatility \( \sigma \). One way of finding the market expectation of the volatility is by getting price data for another "benchmark" option, written on the same stock. Suppose the price of the benchmark option is \( p \), then we obtain the implied volatility by solving the equation

\[
p = C(S, r, T, t, K, \sigma)
\]

then we use the implied volatility \( \sigma \) to price our original option.

\[
d_2 = \frac{\log(S / K) + (r - \frac{1}{2} \sigma^2)(T - t)}{\sigma \sqrt{T - t}}
\]

(2.21)

### CHAPTER 5 Option Pricing with Stochastic Volatility Model

Despite the popularity of the Black-Scholes Model, it disagrees with the reality in a number of ways. One of the most significant limitations is that the model assumes that the risk free rate and the volatility are constant, failing to adequately describe the stock returns. It has been observed that the volatility empirically shows some characteristics:

- Volatility is correlated with stock returns
- Volatility tends to fluctuate around a long term value
• Volatility clustering phenomenon: large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes.

The better solution is to assume that volatility follows a stochastic process, and models in this framework are called stochastic volatility models.

A general stochastic volatility (SV) model is defined as

\[ dS_t = \mu S_t dt + \sigma(t)S_t dB_t \]  
\[ \sigma(t) = f(Y_t) \]  
\[ dY_t = \mu_f(t,Y_t) dt + \sigma_f(t,Y_t) dZ_t \]  
\[ dB_t dZ_t = \rho dt \]  

where \( S_t \) is the asset price, \( f(g) \) is some deterministic function, \( B_t \) and \( Z_t \) are two Brownian Motions with correlation \( \rho \).

\( Y \) can be described by the following stochastic differential equations:

• Log-Normal process: \( dY_t = a Y_t dt + b Y_t dZ_t \)

• Feller process: \( dY_t = K(\nu - Y_t) dt + \nu \sqrt{Y_t} dZ_t \)

With different assumptions of the volatility process \( Y \) and the function \( \sigma(t) \), various stochastic models have been established. For example, the Heston model has \( \sigma(y) = \sqrt{y} \) and \( Y_t \) is a Feller process, while the Hull-White model assumes \( \sigma(y) = \sqrt{y} \) and log-normal \( Y \). Generally, none of these models are widely accepted. In this chapter, we’ll focus on a static model.
We’ll deal with the logarithm of the price process \( X_t = \log S_t \). Under an equivalent martingale measure, the returns satisfy the following equations

\[
dX_t = (r - \frac{\sigma^2(t)}{2})dt + \sigma(t)dB_t,
\]

\[
dY_t = \alpha(\mu - Y_t)dt + \beta dZ_t
\]

(2.26)

Where \( r \) is the short-term risk-free rate of interest, \( B_t \) and \( Z_t \) denote the corresponding Brownian Motions under the martingale measure.

Even though the above model is continuous in time, what we actually have in real world are discrete observations of historical stock prices \( S_{1t}, S_{2t}, \ldots, S_{Kt} \), where \( t_k \) is the time today. We will use the historical prices to estimate the volatility distribution by using a particle filter system in the first section, then compute the option price by means of a multinomial recombining tree.

### 5.1 Empirical Distribution of the Volatility Process

Now we describe the so-called stochastic volatility particle filter, a method to find the empirical distribution given stock price observations. We work under the objective probability measure \( P \), which is based on all past prices. Define

\[
p_i = P[Y_i \in dy | X_{h}, \ldots, X_{h}]
\]

as the probability measure for all \( i = 1, \ldots, K \), which is also the filtered stochastic volatility process of \( Y \) at time \( t_i \) given discrete values of \( X \). Since \( X_{h}, \ldots, X_{h} \) are assumed to be known, \( p_i \) is depended explicitly on the observed values.
This algorithm refers to the work of Del Moral, Jacod and Protter (Del Moral et al., 2001). We modified it so that it works for our specific case. The proof of convergence is beyond the scope of this thesis, Here we illustrate the method in detail.

Assume we have K+1 past price observations at times $t_0 < t_1 < \ldots < t_K$ with equal distance $\Delta t = t_j - t_{j-1} = h$, where $t_K$ is the time today. Let $\phi$ be a bounded integrable function in $L^1(\mathbb{R})$ such that

$$\int \phi(x)dx = 1, \text{and} \int |x| \phi(x)dx < +\infty$$

We will use the function:

$$\phi(x) = e^{-2|x|}$$

For $n > 0$ we also define the contraction in terms of $\phi$ as $\phi_n(x) = \sqrt[n]{n(x^{2/n})}$.

Define $M$ as the number of Euler steps in each time interval $[t_{i-1}, t_i]$. For theoretical reasons, $M = n^{1/3}$ is a good choice due to Del Moral et al.

**Step 1:**

We start the iteration with initial values of X and Y:

- $X_{0,t_0}$, where $x_0$ is the historical value of X at time $t_0$.
- $Y_{0,t_0}$, where $y_0$ is the implied volatility.

**Mutation Step:** We simulate the stochastic differential equations (2.26) recursively starting from the initial data $(X_{0,0}, Y_{0,0}) = (x_0, y_0)$: for $i=0$ to $M-1$,

$$X_{0,i+1} = X_{0,i} + \left(\mu - \frac{\sigma^2(Y_{0,i})}{2}\right) \frac{h}{M} + \sigma(Y_{0,i}) N_i \sqrt{\frac{h}{M}}$$
\[ Y_{t_{i+1}} = Y_{t_i} + \alpha (y_0 - Y_{t_i}) \frac{h}{M} + \beta N^j_i \sqrt{\frac{h}{M}} \]  

(2.27)

where \( N_j \) and \( N^j_i \) are iid Normal random variables with mean 0 and variance 1.

We’ll keep the final values from the recursive step, which are the Euler approximations of the values \( X_{t_i} \) and \( Y_{t_i} \):

\[
X_{t_i} = X_{t_0,M} \\
Y_{t_i} = Y_{t_0,M}
\]

We repeat the above mutation steps \( n \) times independently so that in the end we’ll obtain \( n \) pairs \( \{X^j_{t_i}, Y^j_{t_i}\}_{j=1,2,\ldots,n} \).

**Selection Step:** Introduce the following discrete probability measure, constructed using the pairs obtained from the last step.

\[
\Phi^n_i = \frac{1}{c^n} \sum_{j=1}^{n} \phi_n(X^j_{t_i} - x_i)\delta(Y^j_{t_i})
\]  

(2.28)

Where \( C = \sum_{j=1}^{n} \phi_n(X^j_{t_i} - x_i) \) such that \( \Phi^n_i \) is a probability measure, and \( \delta() \) is the dirac delta function. At the end of the selection step we build an estimate of the original \( Y_{t_i} \). The simulated articles that are closer to the observed value of return will have a higher weight.

**Steps 2 to K:**

For each step \( i = 2,3,\ldots,K \), first we apply the mutation step for the Euler scheme starting with the observed value \( x_{t_{i-1}} \), which is known at time \( t_{i-1} \) and the volatility value \( Y^{j}_{t_{i-1}} \), which is sampled from the \( \Phi^n_{i-1} \). For each \( j = 1,2,\ldots,n \), we begin with
(X_{t_{i+1}}, Y_{t_{i+1}}) \), and end with \((X_{t_i}, Y_{t_i})\). Similarly, we apply the selection step to these pairs and obtain

$$\Phi_i^n = \frac{1}{c} \sum_{j=1}^{n} \phi_n(X_{t_i}^j - x_i)\delta(Y_{t_i}^j)$$

where

$$C = \sum_{j=1}^{n} \phi_n(X_{t_i}^j - x_i)$$

**Output:**

At each step i, the output would be the discrete distribution \(\Phi_i^n\), which is the estimate for \(p_i(dy)\), i.e. the distribution of \(Y_i\) at time \(t_i\), given the historical stock prices and the implied volatility. We’ll use only the last estimated probability distribution to construct the quadrinomial tree in the next section.

### 5.2 Multinomial Recombining Tree

Here we discuss the construction of the multinomial recombining tree using the previously obtained probability distribution of volatility. We wish to price an option with maturity \(T\) starting from today, so our time interval is \([0, T]\).

We divide the time interval into \(N\) periods with equal lengths, that makes our tree branch at times \(i\Delta t\) with depth of \(N\). Each node represent a possible log returns of the stock \(X_i = \log S_i\). The successors of a certain node at step \(i\) is determined by the following method:

**Step 1:**
Sample a value from the particle filter distribution as the volatility, denoted as $Y_i$,

**Step 2:**

For a parent node with value $x$, consider a grid of points of the form $lY_i\sqrt{\Delta t}$ with $l \in \mathbb{N}$. Let $j = \left\lfloor \frac{x}{Y_i\sqrt{\Delta t}} \right\rfloor + 1$, where $\left\lfloor \cdot \right\rfloor$ is the floor function, take the four successors of $x$ to be

$$x_1 = (j + 1)Y_i\sqrt{\Delta t} + (r - \frac{Y_i^2}{2})\Delta t, \quad x_2 = jY_i\sqrt{\Delta t} + (r - \frac{Y_i^2}{2})\Delta t$$

$$x_3 = (j - 1)Y_i\sqrt{\Delta t} + (r - \frac{Y_i^2}{2})\Delta t, \quad x_4 = (j - 2)Y_i\sqrt{\Delta t} + (r - \frac{Y_i^2}{2})\Delta t$$

To find out the weight of each successor, compare $x - jY_i\sqrt{\Delta t}$ with $x - (j - 1)Y_i\sqrt{\Delta t}$, let $\delta$ be equal to the one with smaller absolute value, i.e. the one closer to $x$. Denote $q = \delta / (Y_i\Delta t)$, depending on the value of $q$ we have the following two cases:

- $q \in [0, \frac{1}{2}]$, the probability of the successors are

$$p_1 = p, \quad p_2 = \frac{1}{2}(1 - q + q^2) - 3p$$

$$p_3 = 3p - q^2, \quad p_4 = \frac{1}{2}(1 + q + q^2) - p$$

- $q \in [-\frac{1}{2}, 0]$, then we have
\[ p_1 = \frac{1}{2}(1 + q + q^2) - p, \quad p_2 = 3p - q^2 \]
\[ p_3 = \frac{1}{2}(1 - q + q^2) - 3p, \quad p_4 = p \]

Here \( p \in [\frac{1}{12}, \frac{1}{6}] \) and both cases above define an equivalent martingale measure.

**Step 3:**

Starting with the initial value \( x_0 \), we compute the four corresponding successors with their weights using a sampled value \( Y_1 \). After this, we sample another value \( Y_2 \) and compute the respective successors for each of the nodes. Continuing this process we’ll construct a N-period recombining quadrinomial tree. And since the tree is recombining, the level of computation is polynomial and no greater than \( N^a \) for some \( a < 3 \). Once we completed building the tree we can apply a standard pricing technique in consistent with the no-arbitrage condition: Assume there are \( m \) final nodes on the tree, and the values and weights of these nodes are denoted by \((X_i)_{i=1,2,...,m}\) and \((P_i)_{i=1,2,...,m}\). The price of a European call option is determined by the following equation:

\[
C(T, K, r) = e^{-rT} \sum_{i=1}^{m} \text{Max}[X_i - K, 0] * P_i
\]

**Step 4:**

We can repeat this entire procedure above for a 1000 times, and then use a Monte-Carlo Method to find the expectation of all prices obtained from each generated tree. Thus it reduces the variance of the process and is consistent with the fact that the expected value represents arbitrage-free price.
5.3 Real data

In this application we used The dataset contains 255 datapoints (one year period) with initial values \( x_0 = 6.795 \) and \( y_0 = 0.13 \). The parameters for our model are \( \alpha = 0.027 \), \( \beta = 0.012 \), \( \mu = -0.0014 \), \( M = 10 \) and \( n = 1000 \). The picture below is the estimated volatility distribution.

![Simulated volatility distribution](image)

Figure 5.1 Simulated volatility distribution

We compute the price of a European call option using the quadrinomial tree algorithm described previously, with the parameters \( S_0 = 1124.09 \), risk-free rate \( r = 0.01 \), maturity \( T = 29 \) days, time steps \( N = 50 \) and \( p = 0.135 \). The monte-carlo
simulation with the corresponding strike price is listed in table 2.1, together with the values computed using the Black-Scholes model and the Binomial model.

<table>
<thead>
<tr>
<th>Strike Price</th>
<th>Static Tree Method</th>
<th>Black-Scholes with const. volatility=0.13</th>
<th>Binomial Tree Model</th>
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<td>Strike</td>
<td>Our Simulated Model</td>
<td>Black-Scholes Model</td>
<td>Binomial Model</td>
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<td>0.2021211</td>
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</tbody>
</table>

Table 5.1 Option Prices calculated by our simulated model for a range of strikes, the Black-Scholes Model and the Binomial model.
CHAPTER 6 Conclusions

In the thesis we developed the methods for pricing options with a multinomial recombining tree and stochastic volatility. The option prices calculated by the quadrinomial tree model are close to the observed prices on market and the prices obtained through classic pricing models. The procedure of determining the market’s current volatility is based on the construction of largely non-parametric particle filter for the empirical distribution of the volatility using historical prices of the underlying. This distribution is used for arbitrage-free option pricing, based on a multinomial tree. This pricing algorithm has the advantage of employing a tree model, which implies that it can be used for pricing any path-dependent option. In addition, since the tree is highly recombining, the computational time needed for pricing is minimized. As we see from the comparison with the classical models, our method is as accurate as previously successful models. Even though there are still discussions and modifications to be made about issues such as the estimation of coefficients, it is applicable and referable to the current financial market.
R CODES

• Simulation of the empirical distribution

Iteration<-function(x0,y0,alpha,beta,miu,M,n,h,K){
    tempx<-0
    tempy<-0
    px<-0
    py<-0
    tempx[1]<-x0
    tempy[1]<-y0
    for (j in 1:n){
        for (i in 1:M){
            tempy[i+1]<-tempy[i]+alpha*(y0-tempy[i])*h/M+beta*(h/M)^(1/2)
            tempx[i+1]<-tempx[i]+(miu-tempy[i]^2/2)*h/M+tempy[i]*rnorm(1,0,1)*h/M
        }
    }
    py[j]<-round(tempy[M+1],3)
    px[j]<-round(tempx[M+1],3)
}

C<-0
for(i in 1:n){
    C<-C+n^(1/3)*exp(-2*abs(px[i]-Price_Vol$StockPrice[2]))}
for(i in 1:1000)
    phi[i]<-0

for(i in 1:n)
    phi[py[i]*1000]<-phi[py[i]*1000]+n^(1/3)*exp(-2*abs(px[i]-Price_Vol$StockPrice[2]))/C

VolSpace<-seq(0.001,1,by=0.001)

for(j in 2:K)
    pairy<-0
    pairx<-0
    tempy<-0
    tempx<-0

for(g in 1:n)
    tempy[1]<-sample(x=VolSpace,size=1, prob=phi)

for (i in 1:M)
    tempy[i+1]<-tempy[i]+alpha*(tempy[1]-tempy[i])*h/M+rnorm(1)*beta*(h/M)^(1/2)
    tempx[i+1]<-tempx[i]+(miu-tempy[i]^2/2)*h/M+tempy[i]*rnorm(1)*(h/M)^(1/2)
pairy[g]<-round(tempy[M+1],3)
pairx[g]<-round(tempx[M+1],3)

C<-0
for(i in 1:n){
  C<-C+n^(1/3)*exp(-2*abs(pairx[i]-Price_Vol$StockPrice[j+1]))
}
for(i in 1:1000){
  phi[i]<-0
}
for(i in 1:n){
  phi[pairy[i]*1000]<-phi[pairy[i]*1000]+round(n^(1/3)*exp(-2*abs(pairx[i]-Price_Vol$StockPrice[j+1]))/C,3)
}
}
plot(x=VolSpace,y=phi,xlim=c(0.10,0.16),xlab="Volatility",ylab="Probability",type="h")
return(phi)

• Simulation of the recombining tree
QuadTree<-function(step,start,rate,expiry,strike,p,phi){

price<-c()
prob<-c()
newnodes<-4
oldnodes<-1
total<-1
price[1]<-log(start)
prob[1]<-1
VolSpace<-seq(0.001,1,by=0.001)
for(i in 1:step){
    vol<-sample(x=VolSpace,size=1, prob=phi)
    sv<-vol*(expiry/step)^(1/2)
    max<-floor(price[total-oldnodes+1]/sv)+2
    min<-floor(price[total]/sv)-1
    newnodes<-max-min+1
    for(j in 1:newnodes){
        total<-total+1
        price[total]<-(max-j+1)*sv+(rate-vol^2/2)*(expiry/step)
        prob[total]<-0
    }
}
for(k in 1:oldnodes){
    m<-total-newnodes+1
}
father<-total-newnodes-oldnodes+k

while(price[m]>price[father]){
    m<-m+1
}

if(abs(price[m]-price[father])>abs(price[m-1]-price[father])){
    q<-(price[father]-price[m-1])/sv
    p4<-p
    p1<-0.5*(1+q+q^2)-p
    p2<-3*p-q^2
    p3<-0.5*(1-q+q^2)-3*p
}
else {
    q<-(price[father]-price[m])/sv
    p1<-p
    p2<-0.5*(1+q+q^2)-3*p
    p3<-3*p-q^2
    p4<-0.5*(1-q+q^2)-p
}

prob[m-2]<-prob[m-2]+prob[father]*p1
prob[m-1]<-prob[m-1]+prob[father]*p2
prob[m]<-prob[m]+prob[father]*p3
prob[m+1]<-prob[m+1]+prob[father]*p4
oldnodes <- newnodes

sim <- 0

for (i in 1:newnodes) {
    if (exp(price[total - newnodes + i]) > strike) {
        sim <- sim + (exp(price[total - newnodes + i]) - strike) * prob[total - newnodes + i]
    }
}

OptionPrice <- sim * exp(-rate * expiry)

return (OptionPrice)
REFERENCES


