PARTITIONING THE SET OF SUBGROUPS OF A FINITE GROUP USING THOMPSON’S GENERALIZED CHARACTERS

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by

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Chapter 1

Introduction

In this dissertation we expand upon the work of Thompson [7] and Gagola [2] to determine if certain types of counting functions that are associated with sets of subgroups of a finite group are generalized characters. In particular, we investigate ways to partition the set of subgroups of a finite group so that each set in the partition is associated with a generalized character. The goal is to produce as many generalized characters as we can by making the partition as fine as possible. In this chapter we start with some basic terminology that we use in the chapters that follow and then discuss the history and motivation behind the main theorems of this paper.

We start with some basic concepts from group theory. For a more in depth coverage of the topic see [3]. A group is a nonempty set together with an associative binary operation (often denoted by juxtaposition) such that there exists an identity element and every element of the set has an inverse. When the group operation is commutative we say that the group is abelian. (An abelian group with a second associative binary operation that respects the distributive laws is called a ring.) We will oftentimes refer to a group’s binary operation as its product. Given two groups $G$ and $H$ we define a homomorphism between them to be a map $\varphi : G \to H$ such that for all $x, y \in G$, $\varphi(xy) = \varphi(x)\varphi(y)$. In other words a group homomorphism is a function that preserves products.

Let $G$ be a group and $A$ a nonempty set such that for all $g \in G$ and $a \in A$ there is a unique element $a \cdot g$. Suppose the following conditions hold:

1) $a \cdot 1 = a$ for all $a \in A$ and
2) \((a \cdot g) \cdot h = a \cdot (gh)\) for all \(a \in A\) and \(g, h \in G\).

Then we say that \(G\) acts on \(A\) or that \(\cdot\) is an action of \(G\) on \(A\). We will use exponential notation for group actions and write \(a^g\) in place of \(a \cdot g\). An important example of a group action is when a group \(G\) acts on itself by conjugation where the action is \(h^g = g^{-1}hg\). We define the conjugacy class of an element, \(h \in G\), by \(cl(h) = \{g^{-1}hg : g \in G\}\).

If a subset, \(H\), of a group, \(G\), forms a group under the same operation as \(G\) then we say \(H\) is a subgroup of \(G\), denoted \(H \leq G\). Given a nonempty subset \(A\) of a group \(G\) we introduce some important subgroups. We define the centralizer of \(A\) in \(G\) as \(C_G(A) = \{g \in G : g^{-1}ag = a \text{ for all } a \in A\}\). This is the set of elements of \(G\) that commute with every element of \(A\). The special case when \(A = G\) is given a slightly different name due to its importance. The center of \(G\) is defined as \(Z(G) = \{g \in G : g^{-1}xg = x \text{ for all } x \in G\}\) and we note that \(Z(G) = C_G(G)\). Another important subgroup is the normalizer of \(A\) in \(G\) which is defined as \(N_G(A) = \{g \in G : g^{-1}Ag = A\}\) where \(g^{-1}Ag = \{g^{-1}ag : a \in A\}\). There is no special definition for when \(A = G\) here as \(N_G(G) = G\) always since groups are closed under their product. On the other hand, \(Z(G) = G\) if and only if \(G\) is abelian. The subgroup generated by \(A\) is denoted by \(\langle A \rangle\) and is the closure of the set \(A\) under the group operation and the process of taking inverses. When \(A = \{a\}\) consists of only one element we say the subgroup generated by \(A\), denoted \(\langle a \rangle\), is cyclic. The rank of a group \(G\), denoted \(r(G)\), is equal to the minimum size of a generating set. It follows that if \(G\) is a nontrivial cyclic group, then \(r(G) = 1\). If \(|G| = p^am\), for some prime \(p\) such that \(p \nmid m\), then a subgroup with order \(p^a\) is called a Sylow \(p\)-subgroup.

If \(H \leq G\) and \(N_G(H) = G\), then we say \(H\) is a normal subgroup of \(G\), denoted \(H \trianglelefteq G\). We define the conjugacy class of \(H\) in \(G\) as \(cl_G(H) = \{g^{-1}Hg : g \in G\}\) = \(\{H^g : g \in G\}\). It follows that \(H \trianglelefteq G\) if and only if \(cl_G(H) = \{H\}\). For \(x, y \in G\) we define the commutator of \(x\) and \(y\) by \([x, y] = x^{-1}y^{-1}xy\). An important example of a normal subgroup is the commutator subgroup (also known as the derived subgroup), defined as \(G' = \langle [x, y] : x, y \in G \rangle\).

Now we discuss some topics from character theory that will be used in later chapters.
For a standard reference of character theory see [4]. A representation of a group $G$ is a homomorphism $X : G \rightarrow GL(n, \mathbb{C})$ where $GL(n, \mathbb{C})$ is the group of all invertible $n \times n$ matrices over the complex numbers, $\mathbb{C}$. If $X$ is a representation of $G$, then the character, $\chi$, afforded by $X$ is given by $\chi(g) = tr(X(g))$, the sum of the diagonal entries of the matrix $X(g)$. Characters that cannot be written as sums of other characters are called irreducible characters. The set of irreducible characters of a group $G$ is denoted by $Irr(G)$. A class function of $G$ is a function that is constant on the conjugacy classes of $G$. That is $\varphi : G \rightarrow \mathbb{C}$ is a class function if $\varphi(y^{-1}xy) = \varphi(x)$ for all $x, y \in G$. All characters have this property. The set $Irr(G)$ is an orthonormal basis for the vector space of class functions of $G$ over $\mathbb{C}$ using a natural inner product. This means that not only can we write every class function as a complex combination of elements from $Irr(G)$ but the coefficients can be found using the aforementioned inner product. For class functions $\varphi$ and $\psi$ this natural inner product is defined as $[\varphi, \psi] = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}$, where $\overline{\psi(g)}$ is the complex conjugate of $\psi(g)$. (It will be clear from context whether the bracket notation is being used for the commutator of two group elements or for the inner product of two class functions.) Therefore, if $\varphi$ is a class function of $G$, then $\varphi = \sum_{\chi \in Irr(G)} [\varphi, \chi] \chi$. A nonzero class function $\varphi$ is a character if and only if $[\varphi, \chi]$ is a nonnegative integer for all $\chi \in Irr(G)$. A class function $\varphi$ is a generalized character (sometimes referred to as a virtual character) if $[\varphi, \chi]$ is an integer for all $\chi \in Irr(G)$. In other words, a generalized character is a class function of $G$ that can be written as the difference of two characters of $G$. The set of generalized characters of a finite group $G$ forms a ring referred to as the character ring.

If a group acts on a set $A$ then we can define the permutation character, $\chi$, by $\chi(g) = |\{a \in A : a^g = a\}|$. That is $\chi$ counts the number of fixed points under the action of $g$ for each $g \in G$. A generalized permutation character is a class function that can be written as the difference of two permutation characters. The set of generalized permutation characters is a subring of the character ring. Generalized characters are the focus of this dissertation and we now discuss the history and motivation behind this work.
If $G$ is a finite group and $\mathcal{P}$ is a collection of subgroups of $G$, define $\psi_{\mathcal{P}} : G \to \mathbb{Z}$ by $\psi_{\mathcal{P}}(g) = |S_{\mathcal{P}}(g)|$ where $S_{\mathcal{P}}(g) = \{x \in G : \langle g, x \rangle \in \mathcal{P} \}$. When $\mathcal{P}$ is the collection of subgroups of $G$ that have abelian Sylow $p$-subgroups for some fixed prime $p$, Thompson [7] shows that $\psi_{\mathcal{P}}$ is a generalized character. He does this by showing $|C_G(g)| \mid \psi_{\mathcal{P}}(g)$ for all $g \in G$ and then citing Theorem 1.1, for which we include the proof in the appendix.

**Theorem 1.1:** If $\psi$ is an integer-valued class function of a finite group $G$ such that $\psi(g) = \psi(g')$ whenever $\langle g \rangle = \langle g' \rangle$ and $|C_G(g)| \mid \psi(g)$ for all $g \in G$, then $\psi$ is in the character ring of $G$.

Thompson also observes that $\psi_{\mathcal{P}}$ is a generalized character when $\mathcal{P}$ is the collection of all solvable subgroups or the collection of all nilpotent subgroups of a finite group $G$. At the end of [7] Thompson remarks that these generalized characters are “quite possibly” characters. However, Moretó [6] responds in the negative, citing a different counterexample for each property. Gagola then expands upon Thompson’s idea in [2] to show when $\psi_{\mathcal{P}}$ is a generalized permutation character. First he uses a definition originally appearing in [5]:

**Definition 1.2:** Two subgroups $H_1$ and $H_2$ of a finite group $G$ are linked, denoted $H_1 \bowtie H_2$, if there exists a subgroup $N \leq H_1 \cap H_2$ and elements $x_1, x_2 \in N_G(N)$ so that $H_1 = \langle N, x_1 \rangle$, $H_2 = \langle N, x_2 \rangle$ and $x_1^{-1}x_2 \in C_G(N)$. In other words $x_1$ and $x_2$ each normalize $N$ and induce the same automorphism on $N$. (The subgroup $N$ is forced to be normal in both $H_1$ and $H_2$ as each $H_i$ is generated by $N$ and an element normalizing $N$.)

A slightly stronger definition follows:

**Definition 1.3:** Two subgroups $H_1$ and $H_2$ of a finite group $G$ are strongly linked, denoted $H_1 \bowtie_s H_2$, if there exists a subgroup $N \leq H_1 \cap H_2$ and elements $x_1, x_2 \in N_G(N)$ so that $H_1 = \langle N, x_1 \rangle$, $H_2 = \langle N, x_2 \rangle$, and $x_1^{-1}x_2 \in C_G(N)$ and $N = \langle n^y : y \in \langle x_1 \rangle \rangle$ for some $n \in N$. (It follows that if $H_1 \bowtie_s H_2$, then $H_1 \bowtie H_2$.)

Then Gagola uses a modified version of Definition 1.4 below to prove his main result:
**Definition 1.4:** A collection of subgroups $\mathcal{P}$ is said to be *admissible* if:

1) $\mathcal{P}$ is closed under conjugation, and
2) $\mathcal{P}$ is closed under strong linkage.

In other words, if $H_1$ is conjugate to or strongly linked to $H_2$, then $H_1 \in \mathcal{P}$ if and only if $H_2 \in \mathcal{P}$.

We note that in [2], Gagola refers to admissibility with linkage in place of strong linkage. However, looking back at his Lemma G, we see that he only considers admissible collections of subgroups as in Definition 1.4. Therefore, his main result is true using either definition of admissibility. From now on, whenever we refer to an admissible collection of subgroups of a finite group $G$, we will always be referring to Definition 1.4. This includes the statement of Gagola’s main result from [2] which follows.

**Theorem 1.5(Gagola):** If $\mathcal{P}$ is an admissible collection of subgroups of $G$, then $\psi_\mathcal{P}$ is in the permutation character ring of $G$.

Theorem 1.5 provides us access to some of the elements of the permutation character ring without needing to know the representations or the irreducible characters of $G$. Our main goal is to get access to even more elements of the permutation (or at least generalized) character ring in this way.

In Chapter 2 we set up some of the notation that will be used in the remaining chapters and investigate some properties of linkage. If two subgroups are strongly linked then they are certainly linked. Therefore, any property that holds for linkage also holds for strong linkage. The most important property that we prove is if two subgroups are linked then they have the same commutator subgroup. We also show by way of counterexample that one cannot always choose the normal subgroup to be $H \cap K$ in order to link $H$ and $K$.

In Chapter 3 we define an equivalence relation, called *clinked*, on a finite group $G$. Borrowing vocabulary from lattice theory we refer to the equivalence classes as *atoms*. Clinked
is defined so that each atom is admissible and every admissible collection of subgroups of $G$ is the union of atoms. We partition the set of subgroups in this way so that every atom in the partition has a generalized permutation character associated with it. Thus, we have a purely group theoretic way of producing elements of the permutation character ring. In particular we do not have to know the representations that afford these elements. Also included in this chapter are results that help classify the atoms of a given group. A rather handy result is Theorem 3.7, where we prove that the set of all abelian subgroups with rank less than or equal to two is an atom of a finite group.

Lastly, in Chapter 4 we investigate our main goal of finding more elements of the character ring. We classify the atoms of three infinite families of groups and prove that there are atoms that can be partitioned so that each set in this finer partition is associated with a generalized character. In Theorem 4.4 we consider the family of elementary abelian $p$-groups with order $p^3$ for some prime $p$.

**Theorem 4.4:** Let $G = C_p \times C_p \times C_p$ for some prime $p$ and let $\mathcal{G}$ be the collection of all proper subgroups of $G$. If $\emptyset \neq \mathcal{P} \subsetneq \mathcal{G}$ then $\psi_\mathcal{P}$ is a generalized character if and only if $\mathcal{P}$ or $\mathcal{G} \setminus \mathcal{P}$ is

1. a collection of any $p^2$ rank 2 subgroups of $G$, or
2. a collection of $p^2$ rank 1 subgroups of $G$ that complement a rank 2 subgroup of $G$, together with any $p$ rank 2 subgroups of $G$, or
3. a collection of $p^2$ rank 1 subgroups of $G$ that complement a rank 2 subgroup of $G$, together with any $p^2 + p$ rank 2 subgroups of $G$.

Since $G$ is abelian, Theorem 3.7 gives us that the set of all proper subgroups of $G$ is an atom of $G$. However, we can see that there are three types of ways in which we can partition this atom further and produce more generalized characters. In case (i) of Theorem 4.4 there lacks a group theoretic reason behind the finer partition.

In an effort to move away from purely combinatorial results we classify nonabelian atoms of finite groups. Hence, we turn our attention to dihedral groups of order $2n$, denoted $D_{2n}$.
The interesting case to consider here is when $n$ is even since when $n$ is odd, the nonabelian atoms are comprised of only one conjugacy class of subgroups. When $n$ is even there are exactly two dihedral subgroups of order $2m$ in $D_{2n}$ up to conjugation for each $m \mid n$ with $\frac{n}{m}$ even. We will distinguish between these two conjugation types by using a tilde so that $D_{2m}$ is not conjugate to $\widetilde{D_{2m}}$. For ease of notation we use $\text{cl}(H)$ in place of $\text{cl}_{D_{2n}}(H)$ in the statement of the next theorem.

**Theorem 4.12:** Let $G = D_{2n}$ with $2 < n = 2^\alpha \cdot c$, $1 \leq \alpha \in \mathbb{Z}$, $1 \leq c \in \mathbb{Z}$ and $c$ odd. Let $D_{2m} \leq G$ such that $2 < m = 2^\beta \cdot d$ with $1 \leq \beta \leq \alpha$ and $d \mid c$. Then the atom

- i) $\text{cl}(D_{2m})$ does not split.

- ii) $\text{cl}(D_{2m}) \cup \text{cl}(\widetilde{D_{2m}})$ splits as $\text{cl}(D_{2m})$ and $\text{cl}(\widetilde{D_{2m}})$ if and only if $\beta \geq 3$ or $\beta = 2$ with $d \geq 3$.

- iii) $\text{cl}(D_m) \cup \text{cl}(\widetilde{D_m}) \cup \text{cl}(D_{2m})$ splits uniquely as $\text{cl}(D_m) \cup \text{cl}(\widetilde{D_m})$ and $\text{cl}(D_{2m})$.

- iv) $\text{cl}(D_m) \cup \text{cl}(\widetilde{D_m}) \cup \text{cl}(D_{2m}) \cup \text{cl}(\widetilde{D_{2m}})$ splits non-uniquely as

  - $\text{cl}(D_m) \cup \text{cl}(\widetilde{D_m})$ and $\text{cl}(D_{2m}) \cup \text{cl}(\widetilde{D_{2m}})$, or
  - $\text{cl}(D_m) \cup \text{cl}(D_{2m})$ and $\text{cl}(\widetilde{D_m}) \cup \text{cl}(\widetilde{D_{2m}})$, or
  - $\text{cl}(D_m) \cup \text{cl}(\widetilde{D_{2m}})$ and $\text{cl}(\widetilde{D_m}) \cup \text{cl}(D_{2m})$.

The last and possibly strongest of the results in Chapter 4 involves classifying the splitting of atoms of the direct product of a dihedral group and a cyclic group.

**Theorem 4.14:** Let $G = D_{2p} \times C_n$ such that $p$ is an odd prime and $n > 1$. Then there are only two atoms of $G$, namely the abelian atom and the nonabelian atom. Furthermore, the nonabelian atom splits if and only if $2n$ is divisible by a square.

In this dissertation we find collections of subgroups, $\mathcal{P}$, of a finite group so that the Thompson class function, $\psi_{\mathcal{P}}$, is a generalized character. Another natural (and still open)
problem is to find which collections of subgroups are associated with characters. As we see from [6], some of the “nice” collections of subgroups such as nilpotent or solvable do not yield characters. However, we show in Theorem 3.8 that the Thompson class function associated with the collection of abelian subgroups of a finite group is a character. The collection of all subgroups of a finite group is also associated with a character (the order of the group times the principal character). It is unknown whether these two are the only collections of subgroups that are associated with a character for all groups or if there are more.
Chapter 2

Linkage

In this chapter we investigate the property of linkage. Often times there are many ways in which two subgroups $H$ and $K$ can be linked. (Consider a nontrivial cyclic group as in Example 2.2 below.) With this in mind we define a shorthand way to describe a specific normal subgroup, $N$, and specific elements $x_i$ that normalize $N$ and induce the same automorphism on $N$ that shows $H \cong K$.

**Definition 2.1:** Let $H, K \leq G$ with $H = \langle N, x_1 \rangle$ linked to $K = \langle N, x_2 \rangle$ where $N, x_1,$ and $x_2$ are as in Definition 1.2. Then we say the triple $(N, x_1, x_2)$ is a link between $H$ and $K$. Moreover, if $(N, x_1, x_2)$ is a link between $H$ and $K$ such that $N = \langle n^y : y \in \langle x_1 \rangle \rangle = \langle n^y : y \in \langle x_2 \rangle \rangle$ for some $n \in N$ as in Definition 1.3, we say $(N, x_1, x_2)$ is a strong link.

If $(N, x_1, x_2)$ is a link between $H$ and $K$ then $(N, x_2, x_1)$ is a link between $K$ and $H$. To avoid any confusion that this symmetry may cause we shall stick to the conventions of Definition 2.1 when referring to a link, $(N, x_1, x_2)$, between $H$ and $K$ by choosing $x_1 \in H$ and $x_2 \in K$. In most cases, it is clear from the context which element belongs to which subgroup.

**Example 2.2:** Let $H = \langle x \rangle$ be a nontrivial cyclic group. Then $H$ is linked to itself through $(H, 1, 1)$, $(H, 1, x)$, and $(\langle 1 \rangle, x, y)$ where $y$ is any generator of $H$. This example shows that if two groups are linked, then there may be more than one normal subgroup, $N$, to link them.

For a nonabelian example of this fact, consider the possible links between $G = S_3 \times C_3$ and the natural embedding of $S_3 \leq G$. Here the normal subgroup to link $S_3$ and $G$ can be
either $S_3$ or $A_3$. For instance $(A_3, x, x'y)$ is a link, where $x$ is an element of order 2 in $S_3$, $x' \in cl_{S_3}(x)$ and $y$ is a generator of $C_3$. Also, $(S_3, x, xy)$ is a link where $x$ is a nonidentity element of $S_3$ and $y$ is a generator of $C_3$. All of the links we have just described are strong links. There exist links between $S_3$ and $G$ that are not strong, namely, $(S_3, 1, y)$ where $y$ is a generator of $C_3$.

Even though there can be many different ways to link two subgroups $H$ and $K$, the fact that they are linked depends only on their embedding in $G = \langle H, K \rangle$. This follows since if $(N, x_1, x_2)$ is a link between $H$ and $K$ then $N \leq H \cap K$, $x_1 \in H$, and $x_2 \in K$. Therefore, if $H \bowtie K$ in $G = \langle H, K \rangle$, then $H \bowtie K$ in $\hat{G}$ for any $\hat{G} \geq G$.

For $H, K \leq G$, it is possible for there to exist $N \leq H \cap K$ with $H = \langle N, x_1 \rangle$ and $K = \langle N, x_2 \rangle$ and $x_1 x_2^{-1} \in C_{\langle H,K \rangle}(N)$ but with $N \not\leq H \cap K$ and $H \nmid K$. We show this in the next example.

**Example 2.3:** Let $H = A_5$ and $K = S_5$ and consider the triple

$$(N = \langle (23)(45) \rangle, x_1 = (12345), x_2 = (12)(354)).$$

Notice $x_1 \notin N_{\langle H,K \rangle}(N)$ and $x_2 \notin N_{\langle H,K \rangle}(N)$, but $A_5 = \langle N, x_1 \rangle$, $S_5 = \langle N, x_2 \rangle$ and $x_1 x_2^{-1} \in C_{\langle H,K \rangle}(N)$. It remains to show that $A_5$ is not linked to $S_5$. Since $A_5$ is simple and neither $A_5$ nor $S_5$ is cyclic, the only possible subgroup to link $A_5$ and $S_5$ is $N = A_5$. However if $(A_5, x_1, x_2)$ is a link between $A_5$ and $S_5$ then $x_1 \in A_5$ and $x_2 \in S_5 \setminus A_5$. Since $C_{S_5}(A_5) = \langle 1 \rangle \leq A_5$ and $x_1 x_2^{-1} \notin A_5$ we know $x_1 x_2^{-1} \notin C_{S_5}(A_5)$. Therefore $A_5$ is not linked to $S_5$.

Now that we have stressed the importance of the common subgroup being normal in order to get a link we mention that if $H_1 \bowtie H_2$ through $(N, x_1, x_2)$ then not only is $N$ normal in both $H_1$ and $H_2$ but the factor groups $H_i/N$ are cyclic. This is because the $x_i$ normalize $N$ and so we have $H_i = \langle N, x_i \rangle = N \langle x_i \rangle$ so that $H_i/N = \langle x_i N \rangle$ is cyclic. It follows that if $(N, x_1, x_2)$ is a link between $H$ and $K$ then the commutator subgroups $H'$ and $K'$ are contained in $N$. Moreover, we have (as Gagola remarked in [2]) the following:
**Theorem 2.4:** Let $H, K \leq G$. If $H \cong K$, then $H' = K'$.

**Proof:** Let $(N, x_1, x_2)$ be a link between $H$ and $K$. Then $H = N\langle x_1 \rangle$ and $K = N\langle x_2 \rangle$. We have

$$H' = (N\langle x_1 \rangle)' = \langle [a, b] : a, b \in N\langle x_1 \rangle \rangle = \left\langle [nx_1^\alpha, nx_1^\beta] : n, n' \in N, \alpha, \beta \in \mathbb{Z} \right\rangle.$$  

Now let $[nx_1^\alpha, nx_1^\beta]$ be an arbitrary generator for $H'$. Then

$$[nx_1^\alpha, nx_1^\beta] = x_1^{-\alpha}n^{-1}x_1^{-\beta}(n')^{-1}nx_1^\alpha n'x_1^\beta$$

$$= x_1^{-\alpha}\beta(x_1^\alpha n^{-1}x_1^{-\beta})(n')^{-1}n(x_1^\alpha n'x_1^{-\alpha})x_1^{\alpha+\beta}$$

$$= x_1^{-\alpha+\beta}(x_1^\beta n^{-1}x_1^{-\beta})(n')^{-1}n(x_2^\alpha n'x_2^{-\alpha})x_1^{\alpha+\beta} \text{ (since $x_1^{-1}x_2 \in C_G(N)$)}$$

$$= x_2^{-\alpha+\beta}(x_2^\beta n^{-1}x_2^{-\beta})(n')^{-1}n(x_2^\alpha n'x_2^{-\alpha})x_2^{\alpha+\beta} \text{ (since $x_1^{-1}x_2 \in C_G(N)$)}$$

$$= x_2^{-\alpha}n^{-1}x_2^{-\beta}(n')^{-1}nx_2^\alpha n'x_2^\beta$$

$$= [nx_2^\alpha, nx_2^\beta],$$

which is a generator for $K'$. Hence, $H' \leq K'$. Similarly starting with an arbitrary generator of $K'$ we can get the reverse inclusion so that $H' = K'$.$\square$

Another natural question about linkage is which normal subgroups with cyclic factor groups can be used for linkages. Reading the definition of $H \cong K$ one might try to prove that if $H \cong K$ then $N := H \cap K$ can always be used to link $H$ and $K$. The next example of a group of order 32 and two non-conjugate isomorphic subgroups of order 16 shows that this is not true in general.

**Example 2.5:** Let $G = \langle h, x, g : h^8 = x^2 = g^2 = 1, h^x = h^5, h^g = xh, x^g = x \rangle$ and let $k := gh$. We consider $H := \langle h, x \rangle$ and $K := \langle k, x \rangle$. Note that $h \notin K$ so that $H \neq K$. Also, $H$
and $K$ both have index 2 so we know they are both normal in $G$ and therefore not conjugate to each other. We show $H \cong K$ but $(H \cap K, a, b)$ is not a link between $H$ and $K$ for any $a, b \in G$.

First, we find a link between $H$ and $K$. Observe $k^4 = (gh)^4 = xhhxhh = h^4$ and that $\langle h^4, x \rangle \cong C_2 \times C_2$. Let $N = \langle h^4, x \rangle$, which is normal in $H \cap K$. Then $H = \langle N, h \rangle$, $K = \langle N, k \rangle$, and $hk^{-1} = h(gh)^{-1} = g$. Since $x^g = x$ and $(h^4)^g = (xh)^4 = h^4$ we get $g \in C_G(N)$ so that $\langle (h^4, x), h, k \rangle$ is a link between $H$ and $K$.

Next, we compute $H \cap K$ and show $C_G(H \cap K) = H \cap K$. Notice $h^2 = x^2h^2 = x(gh)^2 = xk^2 \in \langle k, x \rangle = K$ and since $H \neq K$ we have $H \cap K = \langle h^2, x \rangle \cong C_4 \times C_2$. Observe that this implies $\langle h^4, x \rangle \neq H \cap K$. As $g \in N_G(H)$ and $G = \langle H, g \rangle$ we know if $y \in G \setminus H$ then $y = gh^i$ or $y = gxh^i$ for some $i \in \{0, 1, 2, ..., 7\}$. Now $(h^2)^{gh^i} = gh^ih^2h^{-i}g^{-1} = gh^2g^{-1} = xhxh = h^5h = h^6$. Similarly $(h^2)^{gxh^i} = h^6$ and so $C_G(H \cap K) \leq H$. Since $H \cap K$ is abelian we know $H \cap K \leq C_G(H \cap K) \leq H$. Then $x^h = xh^4 \neq x$ and $|H : H \cap K| = 2$ implies $C_G(H \cap K) = H \cap K$.

Finally, suppose $(H \cap K, a, b)$ is a link between $H$ and $K$. Then $a(H \cap K)$ and $b(H \cap K)$ are nonidentity elements of the quotient group $G/(H \cap K)$, which is isomorphic to $V_4$, the Klein four group. The structure of $V_4$ implies $ab(H \cap K) = ab^{-1}(H \cap K) \neq 1(H \cap K)$. Therefore, $ab^{-1} \notin H \cap K = C_G(H \cap K)$ and so $(H \cap K, a, b)$ is not a link between $H$ and $K$ for any $a \in H, b \in K$, as desired.

Since linkage is reflexive and symmetric, one might anticipate that it is an equivalence relation. We end this chapter using $V_4$ to show that $\cong$ is not an equivalence relation.

**Example 2.6:** Consider the Klein four group $V_4 = \langle a, b \rangle$. Then $V_4 \cong \langle a \rangle$ through $(\langle a \rangle, b, a)$ and $\langle a \rangle \cong \langle 1 \rangle$ through $(\langle 1 \rangle, a, 1)$. However, as $V_4$ is not cyclic, $V_4 \neq \langle \langle 1 \rangle, x \rangle = \langle x \rangle$ for any $x \in V_4$. Hence, $V_4 \not\cong \langle 1 \rangle$.

We note that the links we found in Example 2.6 are strong links. This shows that strong linkage is also not transitive. Not only that, if a group requires more than two generators,
it cannot be strongly linked to itself and so strong linkage is not even reflexive. In the next chapter we create an equivalence relation involving strong linkage rectifying the absence of reflexivity and transitivity. We do this in order to partition the set of subgroups of a finite group in such a way that each equivalence class is associated with a generalized character.
Chapter 3

Clinkage

In this chapter we define an equivalence relation so that each equivalence class is admissible and every admissible collection of subgroups of a finite group $G$ is a union of equivalence classes. We form this equivalence relation so that it creates a coarser partition than the partition of $G$ into conjugacy classes of subgroups. This seems reasonable since at the end of the day we are trying to get access to generalized characters of $G$. If we were to make a finer partition than conjugacy classes of subgroups, then the Thompson counting functions $\psi_p$ associated with these equivalence classes would not likely be class functions, let alone generalized characters.

We note before the next definition that if two subgroups $H, K \leq G$ are conjugate then it is standard notation to write $H \simeq K$. With this in mind, the choice of notation for our equivalence relation should be a little more intuitive.

**Definition 3.1:** We say two subgroups $H, K \leq G$ are *clinked*, denoted $H \simeq_{cl} K$, if there exists a sequence of subgroups $H = L_1, L_2, \cdots, L_n = K$, where $L_i$ is conjugate to or strongly linked to $L_{i+1}$ for $i = 1, 2, \cdots, n - 1$.

We have defined clinked so that it is transitive. It is reflexive since conjugation is reflexive, and it is symmetric since both conjugation and strong linkage are symmetric. Therefore, clinked is an equivalence relation.

The set of all admissible collections of subgroups of a finite group $G$ is a boolean algebra under union, intersection, and complementation. The atoms of this lattice are clinked classes.
Each atom is made up of a union of conjugacy classes of subgroups of \( G \). For ease of notation, if \( H \leq G \) we use \( cl(H) \) instead of \( cl_G(H) \) to denote the conjugacy class of \( H \) in \( G \). In cases where it is not clear which groups the conjugacy class is coming from we will be sure to include the subscript. Therefore, if \( \mathcal{P} \) is an atom of \( G \), then we can write \( P = cl(H_1) \cup cl(H_2) \cup \cdots \cup cl(H_n) \), where the \( H_i \) are representatives of distinct conjugacy classes of subgroups that comprise \( \mathcal{P} \). For example, if \( H \leq G \) such that \( H \) requires more than two generators, then the atom containing \( H \) is \( cl_G(H) \).

**Proposition 3.2:** Let \( A, B, C \leq G \). If \( A \triangleright B \sim C \), then \( A \sim \tilde{B} \triangleright C \) for some \( \tilde{B} \leq G \).

**Proof:** Let \( (N, x_1, x_2) \) be a link between \( A \) and \( B \) and suppose \( B^g = C \) for some \( g \in G \). Then \( C = B^g = \langle N, x_2 \rangle^g = \langle N^g, x_2^g \rangle \). Now \( A \) is conjugate to \( A^g = \langle N, x_1 \rangle^g = \langle N^g, x_1^g \rangle \), so \( N^g \leq A^g \cap C \). Since \( x_1, x_2 \in N_G(N) \), \( x_1^g, x_2^g \in N_G(N^g) \). Similarly, \( x_1x_2^{-1} \in C_G(N) \) implies \( x_1^g(x_2^g)^{-1} \in C_G(N^g) \). Thus letting \( \tilde{B} = A^g \) gives the desired result. \( \square \)

Proposition 3.2 implies that if \( H \triangleright \bowtie_s K \) then there exists a sequence of subgroups \( H = L_1 \sim L_2 \bowtie_s L_3 \bowtie_s \cdots \bowtie_s L_n = K \). In other words, conjugation is done first and then everything else is strong linkage. In fact, the converse of Proposition 3.2 is also true so that conjugation can be saved for last if desired. However, as the next proposition involving an extraspecial group shows, conjugation might not be necessary at all (even in a nonabelian group) when partitioning a group into its clinked classes.

Recall from Chapter 1 that for any finite group \( G \), the rank of \( G \), denoted \( r(G) \), is equal to the minimum size of a generating set of \( G \).

**Proposition 3.3:** Let \( G \) be an extraspecial group of order \( p^n \) for some prime \( p \) and positive integer \( n \). Then the nonabelian rank 2 subgroups of \( G \) are all contained in the same atom.

Before we prove this we make a couple of observations. First, the definition of an extraspecial group implies \( G' = Z(G) \), \( |Z(G)| = p \) and \( G/Z(G) \) is an elementary abelian \( p \)-group of even dimension. It follows that if \( H \) is a nonabelian subgroup of \( G \), then \( Z(G) \leq H \).
Indeed, $G'$ has order $p$ and as $\langle 1 \rangle < H' \leq G'$ we see $Z(G) = G' = H' \leq H$. We now proceed with the proof.

**Proof:** Let $H, K \leq G$ be nonabelian rank 2 subgroups of $G$. Then $|H| = |K| = p^3$ and $Z(G) \leq H \cap K$ so that $|H \cap K| \geq p$. Hence $|H \cap K| \in \{p, p^2, p^3\}$.

**Case 1:** $|H \cap K| = p^3$.

As $r(H) = r(K) = 2$, we have $H = K$ and so $H \cong K$ by the reflexive property.

**Case 2:** $|H \cap K| = p^2$.

Let $x_1 \in H \setminus (H \cap K)$ and $x_2 \in K \setminus (H \cap K)$. First, we show that if $y \in (H \cap K) \setminus Z(G)$, then $y^{x_1} = yz$ and $y^{x_2} = yz'$ for some $z, z' \in Z(G) \setminus \{1\}$.

Let $y \in (H \cap K) \setminus Z(G)$. Then $(H \cap K)/Z(G) = \langle yZ(G) \rangle$ since $|H \cap K : Z(G)| = p$. Now $H$ acts on $H \cap K$ by conjugation so we know $H$ acts on $(H \cap K)/Z(G)$ by conjugation, and since $p$ divides the number of fixed points under the latter action we have that the action of $H$ on $(H \cap K)/Z(G)$ is trivial. Thus, $(yZ(g))^{x_1} = yZ(G)$ which implies $y^{x_1}z_1 = yz_2$ for some $z_1, z_2 \in Z(G)$. Hence, $y^{x_1} = yz$, where $z = z_2z_1^{-1} \in Z(G)$. This same argument with $K$ replacing $H$ and $x_2$ replacing $x_1$ gives us $y^{x_2} = yz'$ for some $z' \in Z(G)$, where $z$ may not be the same as $z'$.

Now suppose $z = 1$. Then $y^{x_1} = y1 = y$ so $x_1$ commutes with $y$. Since $y$ was arbitrarily chosen from $(H \cap K) \setminus Z(G)$ we have that $x_1$ commutes with $(H \cap K) \setminus Z(G)$. Thus by the definition of the center of a group, $x_1$ commutes with all of $H \cap K$. But then $\langle H \cap K, x_1 \rangle$ is abelian since $H \cap K$, a group of order $p^2$ by assumption, is abelian. This is a contradiction as $\langle H \cap K, x_1 \rangle = H$ is nonabelian. Therefore $z$, and by the same argument $z'$, are not equal to 1.

As $G$ is an extraspecial group, $Z(G)$ is cyclic of prime order and so $Z(G) = \langle z \rangle$. Thus $z' = z^i$ for some $i \in \{1, 2, 3, \ldots, p - 1\}$. Now $y^{x_1} = yz^i = y^{x_2}$. Also, $z^{x_1} = z = z^{x_2}$ so that $x_1$ and $x_2$ induce the same automorphism on $H \cap K$. Since $|H : H \cap K| = p$ and $i \in \{1, 2, 3, \ldots, p - 1\}$ we have $H = \langle H \cap K, x_1 \rangle$. Similarly $K = \langle H \cap K, x_2 \rangle$. This implies
\((H \cap K, x_1^i, x_2)\) is a link between \(H\) and \(K\). Notice that \(H \cap K = \langle y^a : a \in \langle x_2 \rangle \rangle\), which implies \((H \cap K, x_1^i, x_2)\) is a strong link. Therefore, \(H\) and \(K\) are in the same atom, as desired.

**Case 3:** \(|H \cap K| = p\).

Since \(G\) is an extraspecial group and \(Z(G) \leq H \cap K\) we know that \(Z(G) = H \cap K\). Note that \(H/Z(G)\) and \(K/Z(G)\) are not cyclic so we know \(H\) is not linked to \(K\). Therefore, we will need to find some intermediate subgroups to show \(H\) is clinked to \(K\).

**Subcase 3.1:** \(H \leq C_G(K)\).

Let \(h, h' \in H\) such that \(h\) and \(h'\) do not commute and \(k, k' \in K\) such that \(k\) and \(k'\) do not commute. Let \(L_1 = \langle z, h', hk \rangle\) and \(L_2 = \langle z, k', hk \rangle\). Since \(h\) and \(h'\) do not commute and \(H \leq C_G(K)\), we have \(h'\) and \(hk\) do not commute so that \(L_1\) is nonabelian. Similarly, \(L_2\) is nonabelian. Now we have \(H \triangleright\triangleleft_s L_1 \triangleright\triangleleft_s L_2 \triangleright\triangleleft_s K\) by Case 2 and so \(H \triangleright\triangleleft_s K\).

**Subcase 3.2:** \(H \notin C_G(K)\).

Let \(Z(G) = \langle z \rangle, h \in H\), and \(k \in K\) such that \(h\) and \(k\) do not commute. Then \(L := \langle z, h, k \rangle\) is a nonabelian rank 2 subgroup of \(G\). Hence \(H \triangleright\triangleleft_s L \triangleright\triangleleft_s K\) by case 2 and therefore \(H \triangleright\triangleleft_s K\).

This exhausts all possible cases so the proposition is proved. □

Even though examples like this do not take advantage of Proposition 3.2, its usefulness is shown in the next proposition, which provides a quick way to see if two subgroups \(H, K \leq G\) are not clinked.

**Proposition 3.4:** If \(H \triangleright\triangleleft K\), then \(H' \sim K'\).

**Proof:** Since \(H \triangleright\triangleleft K\), Proposition 3.2 implies there exists a sequence of subgroups \(H = L_1 \sim L_2 \triangleright\triangleleft_s L_3 \triangleright\triangleleft_s \cdots \triangleright\triangleleft_s L_n = K\). Then Theorem 2.4 implies \(L'_i = L'_{i+1}\) for \(i = 2, 3, \ldots, n - 1\), so that \(L'_2 = K'\). Since \(H\) is conjugate to \(L_2\), \(H'\) is conjugate to \(L'_2 = K'\). □
Proposition 3.4 implies that if $H'$ is not conjugate to $K'$, then $H$ is not clinked to $K$. We illustrate this in the next example.

**Example 3.5:** If $A \leq G$ is abelian and $H \leq G$ is nonabelian, then $A$ is not clinked to $H$ since $A' = \langle 1 \rangle$ and $H' \neq \langle 1 \rangle$.

The converse of Proposition 3.4 is not true. That is, $H'$ is conjugate to $K'$ does not imply $H$ is clinked to $K$. We can generalize Example 2.3 to $A_n \leq S_n$ with $n \geq 5$ and note $A'_n = S'_n = A_n$. However, as we saw in the example, $C_{S_n}(A_n) = \langle 1 \rangle$ and so $A_n$ is not linked to $S_n$. The centralizer of the derived subgroup has control over linkage as we see in the next proposition.

**Proposition 3.6:** Let $H$ be a subgroup of a finite group $G$. If $C_G(H') \leq H'$, then $H$ can only be linked to itself. In particular, the atom containing $H$ is $\text{cl}(H)$.

**Proof:** Assume $C_G(H') \leq H'$ and assume $(N, x_1, x_2)$ is a link between $H$ and $K$. We show $H = K$ using double inclusion. Since $H/N$ is cyclic $H' \leq N$ and so $C_G(N) \leq C_G(H')$. By assumption $C_G(H') \leq H' \leq H$ so that $C_G(N) \leq H$. Now $x_1x_2^{-1} \in C_G(N)$ implies $x_1x_2^{-1} \in H$ and since $x_1 \in H$ we have $x_2 \in H$. Hence $K = \langle N, x_2 \rangle \leq H$.

For the reverse inclusion, we use Theorem 2.4 to get $C_G(N) \leq H' = K' \leq K$. Then $x_1x_2^{-1} \in C_G(N)$ implies $x_1x_2^{-1} \in K$ and $x_2^{-1} \in K$ implies $x_1 \in K$, so that $H \leq K$. Therefore $H = K$.

As $C_G(H') \leq H'$ is preserved under conjugation, the atom containing $H$ is $\text{cl}(H)$.

The task of classifying atoms of a group is not trivial and can be quite laborious at times. The next theorem alleviates some of the checking that needs to be done since it takes care of the abelian subgroups of rank less than or equal to two. As remarked earlier any subgroup of rank greater than 2 is contained in an atom consisting of only its conjugacy class.

**Theorem 3.7:** If $A$ is an abelian group with $r(A) \leq 2$, then $A \cong \langle 1 \rangle$. In particular, the set of abelian subgroups with rank less than or equal to 2 is an atom of a group.
Proof: Assume $A = \langle a, b \rangle$. Then $(\langle a \rangle, b, a)$ is a strong link between $A$ and $\langle a \rangle = \langle a^y : y \in \langle b \rangle \rangle$ so that $A \cong_s \langle a \rangle$. Then $(\langle 1 \rangle, a, 1)$ is a strong link between $\langle a \rangle$ and $\langle 1 \rangle$ so that $\langle a \rangle \cong_s \langle 1 \rangle$. As $\cong_s$ is transitive, $A \cong_s \langle 1 \rangle$.

In particular, if $A, B \leq G$ such that $r(A) \leq 2$ and $r(B) \leq 2$, then $A \cong_s \langle 1 \rangle \cong_s B$ so that $A \cong_s B$. Thus $A$ and $B$ are in the same atom. If $H$ is any nonabelian subgroup of $G$ then Example 3.5 and Proposition 3.4 imply $H$ is not clinked to any abelian subgroup of $G$. □

Even though our main goal is to find a partition of subgroups of a finite group that provides us with generalized characters, we end this chapter with a more ambitious endeavor. Theorem 3.8 provides us with the motivation.

**Theorem 3.8:** If $\mathcal{P}$ is the atom of abelian subgroups with rank less than or equal to 2 of a finite group $G$, then $\psi_{\mathcal{P}}$ is a permutation character of $G$.

**Proof:** By definition

\[
\psi_{\mathcal{P}}(g) = |\{x \in G : \langle g, x \rangle \text{ is abelian}\}|
\]

\[
= |\{x \in G : x \in C_G(g)\}|
\]

\[
= |C_G(g)|.
\]

Thus, $\psi_{\mathcal{P}}(g)$ counts the number of fixed points under the action of $g$ on $G$ by conjugation. That is, $\psi_{\mathcal{P}}$ is the permutation character of the action. □

The Thompson counting function in Theorem 3.8 is a character and not just a generalized character. It is natural to ask if there are any other collections of subgroups, $\mathcal{P}$, of a finite group $G$ such that $\psi_{\mathcal{P}}$ is a character. Of course Moretó [6] has shown that some of the “nice” collections of subgroups such as the collection of nilpotent subgroups or the collection of solvable subgroups do not yield a character in general.

Any collection $\mathcal{P}$ that yields $\psi_{\mathcal{P}}$ as a character must include at least one conjugacy class from the abelian atom and in particular a conjugacy class of a cyclic subgroup, otherwise
\( \psi_P(1) = 0 \). To that end, unless \( G = \langle 1 \rangle \), \( \mathcal{P} \) must include more than just \( cl(\langle 1 \rangle) \) or else \( [\psi_P, 1_G] = \frac{1}{|G|} \notin \mathbb{Z} \).

A peculiar example happens when a finite group has a normal subgroup whose order squared is a multiple of the order of \( G \). If \( G \) is a finite group with a normal subgroup \( N \) such that \( |G| \cdot |N|^2 \), then \( \psi_P \) is a character for \( \mathcal{P} = \{ H \leq G : H \leq N \} \). Indeed, \( \psi_P = \frac{|N|^2}{|G|} \rho_{G/N} \).
Chapter 4

Splitting Atoms

We defined the relation clinked so that every atom (clinked class) is admissible and every admissible collection of subgroups of a finite group $G$ is a union of atoms of $G$. In some groups $G$, atoms can be split, by which we mean they can be partitioned (still keeping conjugacy classes intact) so that the Thompson counting function associated with each subset is a generalized character. In fact, some atoms can split in more than one way, producing even more generalized characters. We start with a couple of results that speak to the additivity of Thompson counting functions over disjoint unions. Then we will discuss three infinite families of groups that have atoms which split in Theorems 4.4, 4.12, and 4.14. This leads to the open question:

Is there a refinement of clinked so that the Thompson counting function of each equivalence class of the refinement is a generalized character for a generic finite group $G$?

**Proposition 4.1:** Let $\mathcal{P}$ and $\mathcal{Q}$ be disjoint collections of subgroups of a finite group $G$. Then $\psi_{\mathcal{P} \cup \mathcal{Q}} = \psi_{\mathcal{P}} + \psi_{\mathcal{Q}}$.

For this proof recall that for a collection of subgroups $\mathcal{P}$ of a finite group $G$ we defined

$S_\mathcal{P}(g) = |\{x \in G : \langle g, x \rangle \in \mathcal{P} \}|$.

**Proof:** Let $g \in G$. Then $x \in S_{\mathcal{P} \cup \mathcal{Q}}(g)$ if and only if $\langle g, x \rangle \in \mathcal{P} \cup \mathcal{Q}$, if and only if $\langle g, x \rangle \in \mathcal{P}$ or $\langle g, x \rangle \in \mathcal{Q}$ but not both, as the union is disjoint. But this is equivalent to $x \in S_\mathcal{P}(g)$ or
\( x \in S_Q(g) \) but not both, as \( S_P(g) \) and \( S_Q(g) \) are disjoint. Therefore,

\[
\psi_{P \cup Q}(g) = |S_{P \cup Q}(g)| = |S_P(g) \cup S_Q(g)| = |S_P(g)| + |S_Q(g)| = \psi_P(g) + \psi_Q(g). \]

It follows from Proposition 4.1 that if \( P \) and \( Q \) are collections of subgroups of a finite group \( G \) with \( Q \subseteq P \) then \( \psi_Q = \psi_P - \psi_{P \setminus Q} \). Indeed, \( \psi_P = \psi_{Q \cup (P \setminus Q)} = \psi_Q + \psi_{P \setminus Q} \). From this, the next useful corollary is immediate.

**Corollary 4.2:** Let \( G \) be a finite group and suppose \( \psi_Q \) is a generalized character of \( G \) for some collection of subgroups \( Q \). If \( Q \subseteq P \), then \( \psi_P \) is a generalized character if and only if \( \psi_{P \setminus Q} \) is a generalized character.

Before getting to the main results of this chapter, we mention that we will be ignoring subgroups whose rank is greater than 2. If \( H \leq G \) such that \( r(H) > 2 \), then \( H \nsubseteq K \) for any \( K \neq H \), as \( H \) cannot be generated by \( x_1 \in H \) and a normal subgroup \( N = \langle n^y : y \in \langle x_1 \rangle \rangle \) for any \( n \in N \), while at the same time having cyclic quotient group \( H/N \). It follows that if \( r(H) > 2 \) then \( cl(H) \) is an atom of \( G \). We ignore these atoms as we are not interested in splitting conjugacy classes apart. Not only that, as Proposition 4.3 below states, the generalized character associated with these atoms is not particularly interesting.

**Proposition 4.3:** If \( H \leq G \), and \( r(H) > 2 \), then \( \psi_{cl(H)}(g) = 0 \) for all \( g \in G \).

**Proof:** Since the rank of a subgroup is preserved under conjugation, every element of \( cl(H) \) has rank equal to the rank of \( H \), which is greater than 2. As \( \langle g, x \rangle \) has rank less than or equal to 2 for all \( g, x \in G \), \( \langle g, x \rangle \notin cl(H) \) for any \( g, x \in G \). Therefore, \( \psi_{cl(H)}(g) = 0 \) for all \( g \in G \). \( \square \)
Section 4.1 \( C_p \times C_p \times C_p \)

In this section we classify the ways in which the atoms of an elementary abelian \( p \)-group of order \( p^3 \), say \( G = C_p \times C_p \times C_p \), split in Theorem 4.4. By Theorem 3.8 we know that the set of all proper subgroups of \( G \) forms an atom of \( G \). (The only other atom is \( cl \{ G \} = \{ G \} \).) There are three ways that this atom can be split. Before we state this result, we digress to discuss the subgroup structure of \( G \).

The subgroup structure corresponds to the subspace structure of a 3-dimensional vector space over a field with \( p \) elements, say \( V \). As there are \( p \) elements in the field, there are \( p^3 - 1 \) nonzero vectors in \( V \). If \( \overrightarrow{v} \in V \setminus \{ \overrightarrow{0} \} \), then \( \langle \overrightarrow{v} \rangle \) contains \( p \) elements. It follows that there are \( \frac{p^3 - 1}{p - 1} = p^2 + p + 1 \) distinct rank 1 subspaces of \( V \). Suppose \( \overrightarrow{w} \in V \setminus \langle \overrightarrow{v} \rangle \), then \( \langle \overrightarrow{v}, \overrightarrow{w} \rangle \) is a rank 2 subspace of \( V \). For any \( \overrightarrow{u} \in \langle \overrightarrow{v}, \overrightarrow{w} \rangle \setminus \langle \overrightarrow{v} \rangle \), \( \langle \overrightarrow{v}, \overrightarrow{u} \rangle = \langle \overrightarrow{v}, \overrightarrow{w} \rangle \). It follows that there are \( \frac{(p^3 - 1)(p^3 - p)}{(p^2 - 1)(p^2 - p)} = p^2 + p + 1 \) distinct rank 2 subspaces of \( V \).

Hence, \( G \) has \( p^2 + p + 1 \) distinct rank 1 subgroups and \( p^2 + p + 1 \) distinct rank 2 subgroups. In addition to these, we have the trivial subgroup and the whole group \( G \). It also follows from the subspace structure that any two rank 2 subgroups intersect at exactly one rank 1 subgroup and that any two distinct rank 1 subgroups will determine a unique rank 2 subgroup. Lastly, it is also important to note that the rank 2 subgroups of \( G \) are the kernels of nonprincipal irreducible characters of \( G \).

Theorem 4.4: Let \( G = C_p \times C_p \times C_p \) for some prime \( p \) and let \( \mathcal{G} \) be the collection of all proper subgroups of \( G \). If \( \emptyset \neq \mathcal{P} \subset \mathcal{G} \) then \( \psi_\mathcal{P} \) is a generalized character if and only if \( \mathcal{P} \) or \( \mathcal{G} \setminus \mathcal{P} \) is

i) a collection of any \( p^2 \) rank 2 subgroups of \( G \), or

ii) a collection of \( p^2 \) rank 1 subgroups of \( G \) that complement a rank 2 subgroup of \( G \), together with any \( p \) rank 2 subgroups of \( G \), or

iii) a collection of \( p^2 \) rank 1 subgroups of \( G \) that complement a rank 2 subgroup of \( G \), together with any \( p^2 + p \) rank 2 subgroups of \( G \).

In order to prove Theorem 4.4 we calculate the Thompson counting function values for
the subgroups of rank less than or equal to 2 in Lemma 4.5. Then in Lemma 4.6 we calculate inner products of the Thompson counting functions with irreducible characters of \( G \). In these lemmas we stay consistent by using the notation \( \psi_{cl(H)} \) but since these results are about an abelian group we keep in mind that \( cl(H) = \{ H \} \).

**Lemma 4.5:** Let \( G = C_p \times C_p \times C_p \) for some prime \( p \). Let \( H \) be a rank 1 subgroup of \( G \) and \( K \) be a rank 2 subgroup of \( G \). Then

\[
\begin{align*}
\text{(i)} & \quad \psi_{cl(H)}(g) = \begin{cases} 
0 & \text{if } g \neq 1 \\
1 & \text{if } g = 1,
\end{cases} \\
\text{(ii)} & \quad \psi_{cl(H)}(g) = \begin{cases} 
p - 1 & \text{if } g = 1 \\
p & \text{if } 1 \neq g \in H \\
0 & \text{if } g \in G \setminus H, \text{ and}
\end{cases} \\
\text{(iii)} & \quad \psi_{cl(K)}(g) = \begin{cases} 
0 & \text{if } g = 1 \\
p^2 - p & \text{if } 1 \neq g \in K \\
0 & \text{if } g \in G \setminus K.
\end{cases}
\end{align*}
\]

**Proof:** (i) This is immediate.

(ii) If \( g = 1 \), then \( \langle g, x \rangle = H \) if and only if \( 1 \neq x \in H \) and there are \(|H| - 1 = p - 1 \) such elements, so \( \psi_{cl(H)}(1) = p - 1 \). If \( 1 \neq g \in H \) then \( \langle g, x \rangle = H \) if and only if \( x \in H \), so that \( \psi_{cl(H)}(g) = |H| = p \). If \( g \notin H \), then \( \langle g, x \rangle \neq H \) for all \( x \in G \) so \( \psi_{cl(H)}(g) = 0 \).

(iii) Since \( K \) is not cyclic, \( \psi_{cl(K)}(1) = 0 \). If \( 1 \neq g \in K \) then \( \langle g \rangle \lhd K \) and \( |\langle g \rangle| = p \). For any \( x \in \langle g \rangle \), \( \langle g, x \rangle = \langle g \rangle \neq K \). If \( x \in K \setminus \langle g \rangle \), then \( \langle g, x \rangle = K \) since \( |K : \langle g \rangle| = p \) is prime. If \( x \notin K \), then \( \langle g, x \rangle \neq K \). Therefore when \( 1 \neq g \in K \), \( \psi_{cl(K)}(g) = |K| - |\langle g \rangle| = p^2 - p \). If \( g \in G - K \), then \( \langle g, x \rangle \neq K \) for all \( x \in G \), so \( \psi_{cl(K)}(g) = 0 \).

**Lemma 4.6:** Let \( G = C_p \times C_p \times C_p \) for some prime \( p \). Let \( H \) be a rank 1 subgroup of \( G \) and \( K \) be a rank 2 subgroup of \( G \). If \( \lambda \in Irr(G) \), then
\[ [\lambda, \psi_{cl(i)}] = \frac{1}{p^3}, \]

\[ [\lambda, \psi_{cl(H)}] = \begin{cases} \frac{p^2-1}{p^3} & \text{if } H \leq \ker \lambda \\ -\frac{1}{p^3} & \text{if } H \not\leq \ker \lambda, \text{ and} \end{cases} \]

\[ [\lambda, \psi_{cl(K)}] = \begin{cases} p - 1 - \frac{p^2}{p^3} & \text{if } K \leq \ker \lambda \\ -\frac{p^2}{p^3} & \text{if } K \not\leq \ker \lambda. \end{cases} \]

**Proof:** Let \( \lambda \in \text{Irr}(G). \)

(i) Since \( G \) is abelian, \( \lambda(1) = 1. \) Then Lemma 4.5 implies

\[
[\lambda, \psi_{cl(i)}] = \frac{1}{p^3} \sum_{g \in G} \lambda(g) \psi_{cl(i)}(g) \\
= \frac{1}{p^3} \lambda(1) \psi_{cl(i)}(1) \\
= \frac{1}{p^3}.
\]

(ii) As \( \psi_{cl(H)}(g) = 0 \) for \( g \in G \setminus H, \)

\[
[\lambda, \psi_{cl(H)}] = \frac{1}{p^3} \sum_{g \in H} \lambda(g) \psi_{cl(H)}(g).
\]

Then we manipulate the right hand side of the equation to write it in the form of an inner product. We start with some elementary algebra so the right hand side becomes

\[
\frac{1}{p^2} \left( \frac{1}{p} \cdot \sum_{g \in H} \lambda(g) \psi_{cl(H)}(g) \right).
\]

Since \( \psi_{cl(H)}(g) = p \) for \( g \in H \setminus \{1\} \) and \( \psi_{cl(H)}(1) = p - 1, \) this is equal to

\[
\frac{1}{p^2} \left( \sum_{g \in H} \lambda(g) - \frac{1}{p} \right).
\]
Now we replace \( \sum_{g \in H} \lambda(g) \) with \( p[1_H, \lambda|_H] \) to get

\[
\frac{1}{p}[1_H, \lambda|_H] - \frac{1}{p^3} = \begin{cases} 
\frac{p^2 - 1}{p^3} & \text{if } H \leq \ker \lambda \\
-\frac{1}{p^3} & \text{if } H \nsubseteq \ker \lambda 
\end{cases}
\]

as desired.

(iii) Lemma 4.5 says that \( \psi_{d(K)}(g) = 0 \) for \( g \in G \backslash K \) so that

\[
[\lambda, \psi_{d(K)}] = \frac{1}{p^3} \sum_{g \in K} \lambda(g) \psi_{d(K)}(g).
\]

Again we use elementary algebra in order to rewrite the right hand side so that it involves an inner product. Doing so we have the right hand side equals

\[
\frac{1}{p} \left( \frac{1}{p^2} \cdot \sum_{g \in K} \lambda(g) \psi_{d(K)}(g) \right).
\]

Since \( \psi_{d(K)}(g) = p^2 - p \) for \( g \in K \backslash \{1\} \) and \( \psi_{d(K)}(1) = 0 \) this is equal to

\[
\frac{1}{p} \left( (p^2 - p) \cdot [1_K, \lambda|_K] - \frac{(p^2 - p)}{p^2} \right)
= (p - 1) \cdot [1_K, \lambda|_K] - \frac{1}{p} + \frac{1}{p^2}
= \begin{cases} 
p - 1 - \frac{p^2 - p}{p^3} & \text{if } K \leq \ker \lambda \\
-\frac{p^2 - p}{p^3} & \text{if } K \nsubseteq \ker \lambda 
\end{cases}
\]

as desired.\( \square \)

Note: Using Theorem 1.1 and Lemma 4.5, it is easy to see for example that any collection of \( p^2 \) rank 2 subgroups would yield a generalized character. This is because \( |C_G(g)| = p^3 \) for all \( g \in G \) as \( G \) is abelian and \( p^3 \mid p^2(p^2 - p) \). The real work of proving Theorem 4.4 is to prove that collections other than those appearing in (i), (ii), and (iii) do not produce generalized characters.
Proof of Theorem 4.4: Let $G = \{H \leq G : H \neq G\}$. Theorem 1.5 implies $\psi_G$ is a generalized character since $G$ is an atom and hence an admissible collection of subgroups of $G$. Suppose $\emptyset \neq \mathcal{P} \subsetneq G$. Using Corollary 4.2 we will also assume $(1) \notin \mathcal{P}$, otherwise we can replace $\mathcal{P}$ with $G \setminus \mathcal{P}$. We break this argument into 3 cases:

Case 1: $\mathcal{P}$ is a collection of rank 1 subgroups of $G$.

We show $\psi_{\mathcal{P}}$ is not a generalized character. Let $\mathcal{P} = \{H_1, H_2, ..., H_m\}$, where $1 \leq m \leq p^2 + p + 1$ and each $H_i$ is a rank 1 subgroup of $G$. Lemma 4.6 implies $[1_G, \psi_{d(H_i)}] = \frac{p^2 - 1}{p^3}$ for all $i = 1, 2, ..., m$. Applying Proposition 4.1, we get $[1_G, \psi_{\mathcal{P}}] = m \cdot \frac{p^2 - 1}{p^3}$. Since $0 < m < p^3$, $p^3 \nmid m (p^2 - 1)$. Hence, $[1_G, \psi_{\mathcal{P}}] \notin \mathbb{Z}$ and so $\psi_{\mathcal{P}}$ is not a generalized character.

Case 2: $\mathcal{P}$ is a collection of rank 2 subgroups of $G$.

We show $\psi_{\mathcal{P}}$ is a generalized character if and only if $\mathcal{P}$ is as in (i). Let $\mathcal{P} = \{K_1, K_2, ..., K_n\}$, where $1 \leq n \leq p^2 + p + 1$ and each $K_i$ is a rank 2 subgroup of $G$. Let $\lambda \in Irr(G)$. Lemma 4.6 and Proposition 4.1 imply $[\lambda, \psi_{\mathcal{P}}] \in \mathbb{Z}$ if and only if $p^3 \mid np (p - 1)$. Since $p^2 + p + 1 < 2p^2$ for all primes $p$, we have $p^3 \mid np (p - 1)$ if and only if $n = p^2$. Thus $\psi_{\mathcal{P}}$ is a generalized character if and only if $n = p^2$.

If $\mathcal{P}$ contains $n$ rank 2 subgroups of $G$, then case 2 and Corollary 4.2 allow us to proceed with $n \leq p^2 - 1$. Indeed if $n \geq p^2$, then $\mathcal{P} = \mathcal{Q} \cup \mathcal{R}$, where $\mathcal{Q}$ is a collection of $p^2$ rank 2 subgroups of $G$ and, as there are only $p^2 + p + 1$ rank 2 subgroups of $G$, $\mathcal{R}$ must contain less than or equal to $p + 1$ rank 2 subgroups of $G$. Case 2 implies $\psi_{\mathcal{Q}}$ is a generalized character. Then Corollary 4.2 implies $\psi_{\mathcal{P}}$ is a generalized character if and only if $\psi_{\mathcal{P} \setminus \mathcal{Q}} = \psi_{\mathcal{R}}$ is a generalized character.

Case 3: $\mathcal{P}$ is a collection containing both rank 1 and rank 2 subgroups of $G$.

We show $\psi_{\mathcal{P}}$ is a generalized character if and only if $\mathcal{P}$ is as in (ii) (or (iii)). Let $\mathcal{P} = \{H_1, H_2, ..., H_m, K_1, K_2, ..., K_n\}$, where $1 \leq m \leq p^2 + p + 1$ and $1 \leq n \leq p^2 - 1$, and each $H_i$ is a rank 1 subgroup of $G$ and each $K_i$ is a rank 2 subgroup of $G$. Lemma 4.6 and Proposition 4.1 imply
Assume \([1_G, \psi_p] \in \mathbb{Z}\). Then \(p^3 \cdot j = m(p+1)-np\) for some \(j \in \mathbb{Z}\) and we see \(p \mid m\), so that \(p \leq m \leq p^2 + p\). Since \(n \leq p^2 - 1\), we have

\[
p^3 \cdot j = m(p+1) - np \
\geq p(p+1) - (p^2 - 1)p \
= p^2 + p - p^3 + p \
> -p^3,
\]

so in particular \(j \geq 0\). On the other hand, \(m \leq p^2 + p\) and \(1 \leq n\), so

\[
p^3 \cdot j = m(p+1) - np \
\leq (p^2 + p)(p+1) - (1)p \
= p^3 + 2p^2.
\]

This is equivalent to

\[
j \leq 1 + \frac{2}{p}
\]

and since \(j\) is an integer and \(p \geq 2\), we see either \(j \in \{0, 1\}\) or \(j = 2\) and \(p = 2\).

Let \(m = pk\) where \(k \in \mathbb{Z}\) such that \(1 \leq k \leq p + 1\).

**Subcase 3.1: \(j = 0\)**

We show \(\psi_p\) is not a generalized character. We have

\[
m(p+1) - np = p^3 \cdot j = 0,
\]
and so  
\[ p^2k + pk - np = 0, \]
which is equivalent to  
\[ pk + k = n. \]
Since \( n \leq p^2 - 1, \)
\[ k(p + 1) \leq p^2 - 1, \]
which is equivalent to  
\[ k \leq p - 1. \]
Thus,  
\[ m \leq p(p - 1). \]

Let \( 1_G \neq \lambda \in Irr(G), \) \( m_1 = |\{H \in \mathcal{P} : H \text{ is rank } 1, \ H \leq \ker \lambda\}|, \) and  
\( m_2 = |\{H \in \mathcal{P} : H \text{ is rank } 1, \ H \not\leq \ker \lambda\}|, \) so that \( m_1 + m_2 = m = pk. \) Proposition 4.1 and Lemma 4.6 imply

\[ [\lambda, \psi_p] \in \mathbb{Z} \quad \Leftrightarrow \quad m_1 \frac{p^2-1}{p^3} - m_2 \frac{1}{p^2} - n \frac{p^2 - p}{p^3} \in \mathbb{Z} \]
\[ \Leftrightarrow \quad m_1 \frac{p^2-1}{p^3} - (pk - m_1) \frac{1}{p^2} - (pk + k) \frac{p^2 - p}{p^3} \in \mathbb{Z} \]
\[ \Leftrightarrow \quad \frac{m_1}{p} \in \mathbb{Z}. \]

Assume \([\lambda, \psi_p] \in \mathbb{Z}, \) so that \( p \mid m_1. \) Since \( \ker \lambda \) contains precisely \( p + 1 \) rank 1 subgroups of \( G, \)
\[ m_1 \in \{0, p\}. \]

Hence, \( \psi_p \) is a generalized character if and only if \( |\{H \in \mathcal{P} : H \text{ is rank } 1, \ H \leq \ker \lambda\}| \in \{0, p\} \) for all \( 1_G \neq \lambda \in Irr(G). \)

We assume \( |\{H \in \mathcal{P} : H \text{ is rank } 1, \ H \leq \ker \lambda\}| \in \{0, p\} \) for all \( 1_G \neq \lambda \in Irr(G) \) and seek a contradiction. Let \( \mathcal{H} = \{H_1, H_2, ..., H_m\} \) be the set of rank 1 subgroups of \( G \) contained in
\( \mathcal{P} \) and \( \mathcal{H}^c \) be the collection of rank 1 subgroups of \( G \) that are not contained in \( \mathcal{P} \).

Suppose \( X \) and \( Y \) are distinct elements of \( \mathcal{H}^c \) and \( K \) is the rank 2 subgroup of \( G \) that contains \( X \) and \( Y \). Now \( K \) contains exactly \( p + 1 \) rank 1 subgroups of \( G \) and \( K = \ker \lambda \) for some \( 1_G \neq \lambda \in \text{Irr}(G) \). Thus our current assumption that \( K \) contains either 0 or \( p \) rank 1 subgroups of \( \mathcal{P} \) implies \( H \nsubseteq K \) for all \( H \in \mathcal{H} \). In other words \( K = XY \) is a rank 2 subgroup that contains \( p + 1 \) elements of \( \mathcal{H}^c \).

Since there are \( p^2 + p + 1 \) rank 1 subgroups of \( G \) and \( |\mathcal{H}| = m \leq p^2 - p \), we have

\[
|\mathcal{H}^c| = p^2 + p + 1 - m \geq 2p + 1.
\]

Therefore, if \( \{ H(i) \}_{i=1}^{p+1} \) denotes the set of rank 1 subgroups of \( G \) in \( K \), then
\( \mathcal{H}^c \setminus \{ H(i) \}_{i=1}^{p+1} \neq \emptyset \). Fix \( L \in \mathcal{H}^c \setminus \{ H(i) \}_{i=1}^{p+1} \). Then \( LH(r) := K(r) \) is a rank 2 subgroup of \( G \) such that \( K \neq K(r) \), and \( K \cap K(r) = H(r) \) for all \( 1 \leq r \leq p + 1 \). If \( s \neq r \) such that \( 1 \leq s \leq p + 1 \), then \( K(r) \cap K(s) = L \). By construction \( K(r) \) contains \( (p + 1) - 2 = p - 1 \) rank 1 subgroups of \( G \) that are not in \( \{ H(i) \}_{i=1}^{p+1} \cup \{ L \} \) for all \( r \). Thus,

\[
|\mathcal{H}^c| \geq (p + 1) + (p + 1)(p - 1) = p^2 + p.
\]

This implies \( |\mathcal{H}| = 0 \) or 1. The assumption of case 3 implies \( |\mathcal{H}| \neq 0 \) and if \( |\mathcal{H}| = 1 \), then \( p \nmid m \), which is a contradiction. Thus \( \psi_\mathcal{P} \) is not a generalized character in Subcase 3.1.

Subcase 3.2: \( j = 1 \)

We start with

\[
m(p + 1) - np = p^3,
\]

which, since \( m = pk \), is equivalent to

\[
k(p + 1) - p^2 = n.
\]
Since $n \geq 1$ we get

$$k(p+1) - p^2 \geq 1,$$

and so

$$k \geq p,$$

which implies $k = p$ or $k = p + 1$, as $kp = m \leq p^2 + p$. We deal with each case separately.

First, suppose $k = p$. That is, $m = p^2$. Since $k(p + 1) - p^2 = n$ we know $n = p$. We show $\psi_\mathcal{P}$ is a generalized character if and only if $\mathcal{P}$ is comprised of $(ii)$ or $(iii)$. Let $\mathcal{P} = \{H_1, H_2, ..., H_{p^2}, K_1, K_2, ..., K_p\}$, where each $H_i$ is a rank 1 subgroup of $G$ and each $K_i$ is a rank 2 subgroup of $G$.

First, suppose each subgroup in $\{H_1, H_2, ..., H_{p^2}\}$ complements a rank 2 subgroup of $G$, say $K := \ker \lambda$ for some $1_G \neq \lambda \in \text{Irr}(G)$. In other words, assume $H_i \nsubseteq K$ for all $1 \leq i \leq p^2$. Then Proposition 4.1 and Lemma 4.6 imply $[\psi_\mathcal{P}, \lambda] \in \mathbb{Z}$ if and only if $-\frac{p^2}{p^2} - p\left(\frac{p^2-1}{p^2}\right) \in \mathbb{Z}$ and a routine calculation shows the latter is equal to $-1 \in \mathbb{Z}$. Now let $\theta \in \text{Irr}(G) \setminus \langle \lambda \rangle$, where $\langle \lambda \rangle = \{\lambda^l : l \in \mathbb{Z}\}$, and say $L := \ker \theta$. Then since $L$ contains $p + 1$ rank 1 subgroups of $G$ and $K \cap L$ is one of these, $L$ must contain exactly $p$ rank 1 subgroups of $G$ that complement $K$. That is, $p$ of $H_1, H_2, ..., H_{p^2}$ are in $\ker \theta$. Thus, Proposition 4.1 and Lemma 4.6 imply $[\psi_\mathcal{P}, \theta] \in \mathbb{Z}$ if and only if $p\left(\frac{p^2-1}{p^2}\right) + (p^2 - p)\left(-\frac{1}{p}\right) - p\left(\frac{p^2-1}{p^2}\right) \in \mathbb{Z}$, and a routine calculation shows the latter is $0 \in \mathbb{Z}$. Thus $\psi_\mathcal{P}$ is a generalized character when $\mathcal{P}$ is as in $(ii)$. Case 2 and Corollary 4.2 imply $\psi_\mathcal{P}$ is also a generalized character when $\mathcal{P}$ is as in $(iii)$.

Now suppose $\{H_1, H_2, ..., H_{p^2}\}$ does not complement any rank 2 subgroup of $G$. Consider a bipartite graph between the $p^2 + p + 1$ rank 2 subgroups of $G$ and $\{H_1, H_2, ..., H_{p^2}\}$ where the edges are defined by inclusion. Then the degree of each vertex representing a rank 2 subgroup of $G$ is greater than or equal to 1. On the other hand, each of the $p^2$ rank 1 subgroups is contained in $p + 1$ rank 2 subgroups of $G$ and so there are $p^2(p + 1) = p^3 + p^2$ edges on this graph. As $p^3 + p^2 < p(p^2 + p + 1)$, there must exist a rank 2 subgroup, say $K := \ker \lambda$, such that the degree of the vertex representing $K$ is less than $p$. (In other words,
one of the rank 2 subgroups contains less than $p$ of the chosen $p^2$ rank 1 subgroups.) If we let $l$ be the degree of this vertex, then Lemma 4.5 and Lemma 4.6 imply $[\psi_p, \lambda] \in \mathbb{Z}$ if and only if $l \cdot \frac{p^2 - 1}{p^3} + (p^2 - l) \cdot \left( -\frac{1}{p^3} \right) + p \left( -\frac{p^2 + p}{p^3} \right) = \frac{l - p}{p} \in \mathbb{Z}$. As $1 \leq l < p$ we conclude $\psi_p$ is not a generalized character when $\{H_1, H_2, ..., H_{p^2}\}$ does not complement a rank 2 subgroup of $G$.

We now consider $k = p + 1$. That is, $m = p^2 + p$. Since $n = k(p + 1) - p^2$, we know $n = 2p + 1$. We show $\psi_p$ is not a generalized character.

Let $\mathcal{P} = \{H_1, H_2, ..., H_{p^2+p}, K_1, K_2, ..., K_{2p+1}\}$, where each $H_i$ is a rank 1 subgroup of $G$ and each $K_i$ is a rank 2 subgroup of $G$. Since there are $p^2 + p + 1$ rank 1 subgroups of $G$, there is exactly one rank 1 subgroup of $G$, say $H$, that is not in $\mathcal{P}$. Now $H$ is only contained in $p + 1$ of the $p^2 + p + 1$ rank 2 subgroups of $G$, so let $1_G \neq \lambda \in \text{Irr}(G)$ be chosen so that $H \not\subseteq \ker \lambda := K \in \mathcal{P}$. As $K$ contains $p + 1$ rank 1 subgroups of $G$, we know that $p + 1$ of $H_1, H_2, ..., H_{p^2+p}$ are contained in $K$ while $p^2 - 1$ of them are not contained in $K$. Then Proposition 4.1 and Lemma 4.6 imply

$$[\psi_p, \lambda] \in \mathbb{Z} \text{ if and only if } (p + 1) \cdot \left( \frac{p^2 - 1}{p^3} \right) + (p^2 - 1) \cdot \left( -\frac{1}{p^3} \right) + (2p + 1) \cdot \left( -\frac{p^2 - p}{p^3} \right) \in \mathbb{Z}$$

A routine calculation shows the latter is equal to $\frac{1 - p}{p} \notin \mathbb{Z}$. Thus $\psi_p$ is not a generalized character.

**Subcase 3.3:** $p = 2$ and $j = 2$.

We show $\chi_p$ is not a generalized character.

We have

$$m(p + 1) - np = 2p^3,$$

which is equivalent to

$$3k - 8 = n.$$
As $n \geq 1$,

$$k \geq 3.$$ 

On the other hand, $m \leq p^2 + p = 6$ implies

$$k \leq 3,$$

and so $k = 3$ implying $m = 6$ and $n = 1$.

Let $\mathcal{P} = \{H_1, H_2, \cdots, H_6, K_1\}$, where each $H_i$ is a rank 1 subgroup of $G$ and $K_1$ is a rank 2 subgroup of $G$. Since there are 7 rank 1 subgroups of $G$, there is one rank 1 subgroup of $G$, say $H$, that is not in $\mathcal{P}$. Now $H$ is only contained in 3 of the 7 rank 2 subgroups of $G$, so let $K := \ker \lambda (1_G \neq \lambda \in \text{Irr}(G))$ be chosen so that $H \not\leq K$. Then since each rank 2 subgroup of $G$ contains 3 rank 1 subgroups of $G$ we have that 3 of $H_1, H_2, ..., H_6$ are contained in $\ker \lambda$ while 3 of them are not. Then $[\psi, \lambda] \in \mathbb{Z}$ if and only if $(3) \cdot \left( \frac{3}{8} \right) + (3) \cdot \left( -\frac{1}{8} \right) + 1 \cdot \left( -\frac{2}{8} \right) \in \mathbb{Z}$. However, the latter is equal to $-\frac{1}{4} \notin \mathbb{Z}$. Thus $\psi_{\mathcal{P}}$ is not a generalized character in Subcase 3.3.

It follows from Corollary 4.2 that all other possible cases for $\mathcal{P}$ have been considered as the collection $G \setminus \mathcal{P}$ in the previous three cases and hence we are done. □

One of the discouraging things about Theorem 4.4 is that when $\mathcal{P}$ is of the form in (i), the choice of the rank 2 subgroups does not depend on the group structure. It is purely combinatorial. It follows that there are $\binom{p^2+p+1}{p^2}$ possible combinations of $p^2$ rank 2 subgroups that give us a generalized character.

Contrary to this, we have that the rank 1 subgroups must be chosen to complement a rank 2 subgroup in $(ii)$ and $(iii)$. Still we have this purely combinatorial reason for choosing the number of rank 2 subgroups. It follows that there are $(p^2 + p + 1) \binom{p^2+p+1}{p}$ ways to choose $\mathcal{P}$ to be of the form in $(ii)$ and $(p^2 + p + 1) \binom{p^2+p+1}{p^2+p}$ ways to choose $\mathcal{P}$ to be of the form in $(iii)$ to give us a generalized character.
Section 4.2  \( D_{2n} \)

In an attempt to get away from these purely combinatorial reasons to split atoms of a group \( G \), we shift our attention to nonabelian groups and disregard the abelian atoms of these groups.

The next collection of results culminates to classify all atoms of a given dihedral group and shows which nonabelian atoms can be split. This is done in Theorem 4.12. Fortunately, the only types of subgroups of a dihedral group are cyclic or dihedral, so the classification will not be unwieldy. We will use the notation \( D_{2n} = \langle r, s : r^n = s^2, rs = sr^{-1} \rangle \) so that the order of \( D_{2n} \) is \( 2n \).

The derived subgroup of \( D_{2n} \) is \( \langle r^2 \rangle \). Thus, \(|D_{2n}'| = |\langle r^2 \rangle| = \frac{n}{\text{gcd}(n,2)} = \begin{cases} n & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases} \)

We will use this fact to help classify the atoms of \( D_{2n} \).

**Theorem 4.7:** Let \( G = D_{2n} \), where \( n \) is odd. Then the abelian atom is \( \{ \langle r^l \rangle : l \mid n \} \cup \text{cl} (\langle s \rangle) \) and all nonabelian atoms are of the form \( \text{cl} (D_{2m}) \) where \( m > 1 \) and \( m \mid n \).

**Proof:** Let \( G = D_{2n} \), where \( n \) is odd. The conjugacy classes of \( G \) are \( \{ r^j \} \) for all integers \( 0 \leq j < n \), and \( \{ sr^k : 0 \leq k < n, k \in \mathbb{Z} \} \). The abelian subgroups of \( G \) are subgroups of \( \langle r \rangle \) and conjugates of \( \langle s \rangle \). Thus, the abelian atom is \( \{ \langle r^l \rangle : l \mid n \} \cup \{ \text{cl} (\langle s \rangle) \} \) by Theorem 3.7. For each \( m > 1 \) and \( m \mid n \), \( D_{2m} = \langle r^\frac{n}{m}, s \rangle \) is the only dihedral subgroup of its order up to conjugation. We know \( m \) is odd since it divides an odd number and so \(|D_{2m}'| = m \). Thus \( D_{2m}' \) is conjugate to \( D_{2l}' \) if and only if \( m = l \). Therefore \( \text{cl} (D_{2m}) \) comprises an atom for all \( m > 1 \) with \( m \mid n \).

Theorem 4.7 is not very interesting when trying to split nonabelian atoms as the nonabelian atoms in this case consist of only one conjugacy class of subgroups. Therefore, for the remainder of this section we will be dealing with dihedral groups of order \( 2n \) where \( n \) is even. In this situation, the conjugacy classes of rotations of \( D_{2n} \) are still \( \{ r^{\pm j} \} \) for all integers \( 0 \leq j < n \), but the class of reflections breaks apart into two classes. They are
cl_{D_{2n}}\langle s \rangle = \{ sr^{2k} : 0 \leq k < \frac{n}{2}, k \in \mathbb{Z} \} \text{ and } cl_{D_{2n}}\langle sr \rangle = \{ sr^{2k+1} : 0 \leq k < \frac{n}{2}, k \in \mathbb{Z} \}. This means that there are exactly two dihedral subgroups of order $2m$ in $D_{2n}$ up to conjugation for each $m \mid n$ with $\frac{n}{m}$ even. We will distinguish between these two conjugation types by using a tilde so that $D_{2m} := \langle r^{\frac{n}{m}}, s \rangle$ is not conjugate to $\widetilde{D}_{2m} := \langle r^{\frac{n}{m}}, sr \rangle$.

**Lemma 4.8:** Let $G = D_{2n}$ with $n = 2^\alpha \cdot c$, $\alpha \geq 1$, and $c$ odd such that $n > 2$. Then the abelian atom of $G$ is:

(i) $\{ \langle r^l \rangle : l | n \} \cup cl(\langle s \rangle) \cup cl(\langle sr \rangle) \cup cl(D_4)$ if $\alpha = 1$, or

(ii) $\{ \langle r^l \rangle : l | n \} \cup cl(\langle s \rangle) \cup cl(\langle sr \rangle) \cup cl(D_4) \cup cl(\widetilde{D}_4)$ if $\alpha \geq 2$.

The proof of Lemma 4.8 follows from the paragraph preceding it and Theorem 3.7.

**Lemma 4.9:** Let $G = D_{2n}$ with $2 < n = 2^\alpha \cdot c$, $1 \leq \alpha \in \mathbb{Z}$, $1 \leq c \in \mathbb{Z}$, and $c$ odd. Let $D_{2m} \leq G$ such that $2 < m = 2^\beta \cdot d$ with $1 \leq \beta \leq \alpha$ and $d | c$. Then $D_{2m}$ is contained in one of the following atoms:

(i) $cl(D_{2m})$ if $2 \leq \beta = \alpha$

(ii) $cl(D_{2m}) \cup cl(\widetilde{D}_{2m})$ if $2 \leq \beta < \alpha$

(iii) $cl(D_m) \cup cl(D_{2m}) \cup cl(D_4)$ if $1 = \beta = \alpha$

(iv) $cl(D_m) \cup cl(\widetilde{D}_m) \cup cl(D_{2m}) \cup cl(\widetilde{D}_{2m})$ if $1 = \beta < \alpha$

Observe that all nonabelian subgroups are accounted for in the list above. This is so because the subgroups $D_{2k}$, where $k$ is odd, appear in parts (iii) and (iv) where $\beta = 1$ and $m = 2k$.

**Proof:** Since $2 < n$ and $2 < m$, neither $D_{2n}$ nor $D_{2m}$ is abelian.

(i) and (ii): For $2 \leq \beta$, $|D'_{2m}| = \frac{m}{2}$. Let $D_{2l} \leq D_{2n}$ be dihedral with $D'_{2l}$ conjugate to $D'_{2m}$. Then

\[
\frac{m}{2} = |D'_{2l}| = \begin{cases} 
l & \text{if } l \text{ is odd} \\
l/2 & \text{if } l \text{ is even}. \end{cases}
\]

Thus, $D_{2l}$ is isomorphic to either $D_{2m}$ or $D_m$. Now $m = 2 \cdot (2^{\beta-1} \cdot d)$, and $2 \leq \beta$ means $2^{\beta-1} \cdot d$ is even. Therefore, $|D'_m| = \frac{2^{\beta-1} \cdot d}{2} = 2^{\beta-2} \cdot d \neq 2^{\beta-1} \cdot d = |D'_{2m}|$, and so the only possible
subgroups of $D_{2n}$ to which $D_{2m}$ can be clinked are those that are isomorphic to $D_{2m}$. When $eta = \alpha$ there is only one conjugacy class of subgroups isomorphic to $D_{2m}$ (since they contain a full Sylow 2-subgroup) and we get (i). When $\beta < \alpha$ there are two classes. In this case, \((\langle r^{\frac{n}{m}} \rangle, s, sr)\) is a strong link between $D_{2m} = \langle r^{\frac{n}{m}}, s \rangle$ and $\widetilde{D_{2m}} = \langle r^{\frac{n}{m}}, sr \rangle$ and we get (ii).

(iii) and (iv): Let $1 = \beta$. Notice $|D_{2n} : D_m|$ is even, so $D_m = \langle r^{\frac{2n}{m}}, s \rangle$ and $\widetilde{D_m} = \langle r^{\frac{2n}{m}}, sr \rangle$ are not conjugate and \((\langle r^{\frac{n}{m}} \rangle, s, sr)\) is a strong link between them. Also, $(D_m, r^{\frac{n}{m}+2\frac{n}{m}}, r^{2\frac{n}{m}})$ is a strong link between $D_{2m} = \langle D_m, r^{\frac{n}{m}+2\frac{n}{m}} \rangle$ and $D_m = \langle D_m, r^{2\frac{n}{m}} \rangle$. Now if $\beta = \alpha$, then there is only one conjugacy class of subgroups isomorphic to $D_{2m}$ and (iii) is proved. If $\beta < \alpha$, then $D_{2m} = \langle r^{\frac{n}{m}}, s \rangle$ is not conjugate to $\widetilde{D_{2m}} = \langle r^{\frac{n}{m}}, sr \rangle$ and \((\langle r^{\frac{n}{m}} \rangle, s, sr)\) is a strong link between them, proving (iv). □

Note: In the proof of (iii) we did not need to show $\widetilde{D_m} \bowtie_s D_{2m}$ because of the transitivity of $\bowtie_s$ even though $\left(\widetilde{D_m}, r^{\frac{n}{m}+2\frac{n}{m}}, r^{2\frac{n}{m}}\right)$ is a strong link between them. In the proof of (iv) we did not need to show $\widetilde{D_{2m}} \bowtie_s \widetilde{D_m}$, although it is indeed the case as $\left(\widetilde{D_m}, r^{\frac{n}{m}+2\frac{n}{m}}, r^{2\frac{n}{m}}\right)$ is a strong link between them. Also in case (iv), $\widetilde{D_m}$ is not linked to $D_{2m}$ and similarly $D_m$ is not linked to $\widetilde{D_{2m}}$. A graphical representation of linkages in the atoms occurring in cases (iii) and (iv) is shown below:

![Graphical representation](image)

Case (iii) Case (iv)

We remark here that for $D_{2n}$ (where $n$ is even), the converse of Proposition 3.4 holds, so that every atom has a distinct common derived subgroup. In other words, for $H, K \leq D_{2n}$, we have $H \bowtie_s K$ if and only if $H' \sim K'$. In fact, the set of divisors of $n$ gives a nice indexing set of all the atoms of $D_{2n}$. This also happens in the case where $n$ is odd, even though the
nonabelian atoms are single conjugacy classes.

**Lemma 4.10:** Let $D_{2m} \leq D_{2n} = G$ be dihedral. Then

$$\psi_{cl(D_{2m})}(g) = \begin{cases} 
0 & \text{if } g \text{ is not an element of any conjugate of } D_{2m} \\
0 & \text{if } g \in \langle r \rangle \text{ such that } \langle g \rangle < \langle r^{\frac{n}{m}} \rangle \\
n & \text{if } g \in \langle r \rangle \text{ such that } \langle g \rangle = \langle r^{\frac{n}{m}} \rangle \text{ and } \frac{n}{m} \text{ is odd} \\
\frac{n}{2} & \text{if } g \in \langle r \rangle \text{ such that } \langle g \rangle = \langle r^{\frac{n}{m}} \rangle \text{ and } \frac{n}{m} \text{ is even} \\
2\varphi(m) & \text{if } g \text{ is contained in a conjugate of } D_{2m} \text{ and } g \notin \langle r \rangle ,
\end{cases}$$

where $\varphi$ is Euler’s totient function.

**Proof:** For ease of notation we will use $cl(x)$ instead of $cl_{D_{2m}}(x)$ and let $\psi = \psi_{cl(D_{2m})}$. Also, when considering $\frac{n}{m}$ even we will assume that $cl(s) \cap D_{2m} \neq \emptyset$ since the same reasoning holds if $cl(sr) \cap D_{2m} \neq \emptyset$. An alternate explanation of why we can assume $cl(s) \cap D_{2m} \neq \emptyset$ is that $\psi_{cl(D_{2m})}$ and $\psi_{cl(D_{2m})}$ are exchanged by an outer automorphism.

First, if $g$ is not in any conjugate of $D_{2m}$, then $\langle g, x \rangle$ is not conjugate to $D_{2m}$ for any $x \in D_{2n}$. Hence, $\psi(g) = 0$ in this case.

Second, let $g \in \langle r \rangle$ such that $\langle g \rangle < \langle r^{\frac{n}{m}} \rangle$, so that $o(g) < m$. If $x \in \langle r \rangle$, then $\langle g, x \rangle \leq \langle r \rangle$ is cyclic and so it is not conjugate to $D_{2m}$. If $x \in D_{2n} \setminus \langle r \rangle$, then $\langle g, x \rangle$ is dihedral but $|\langle g, x \rangle| = 2o(g) < 2m = |D_{2m}|$ and so $\langle g, x \rangle$ is not conjugate to $D_{2m}$. Thus, $\psi(g) = 0$ in this case.

Third and fourth, let $\langle g \rangle = \langle r^{\frac{n}{m}} \rangle$. If $x \in \langle r \rangle$, then $\langle g, x \rangle \leq \langle r \rangle$ is cyclic and so it is not conjugate to $D_{2m}$. If $x$ is not contained in any conjugate of $D_{2m}$, then $\langle g, x \rangle \notin cl(D_{2m})$.

If $x$ is in some conjugate of $D_{2m}$ and $x \notin \langle r \rangle$, then $\langle g, x \rangle$ is dihedral of order $2m$. If $\frac{n}{m}$ is odd, then $D_{2m}$ is the unique dihedral group of order $2m$ in $D_{2n}$ up to conjugacy. Hence, $\psi(g) = |D_{2m} \setminus \langle r \rangle| = n$. If $\frac{n}{m}$ is even, then there are two dihedral subgroups of order $2m$ in $D_{2n}$ up to conjugacy and $cl(sr) \cap (\bigcup_{H \in cl(D_{2m})} H) = \emptyset$. Therefore, $\psi(g) = |D_{2n} \setminus (\langle r \rangle \cup cl(sr))| = 2n - (n + \frac{n}{2}) = \frac{n}{2}$.

Lastly, we consider $g \in cl(s)$. We know $\psi$ is a class function, and so we work with
\( g = s \). If \( x \) is not contained in any conjugate of \( D_{2m} \), then \( \langle g, x \rangle \not\in cl(D_{2m}) \). If \( x \in \langle r \rangle \) such that \( \langle x \rangle < \langle r^{\frac{n}{m}} \rangle \), then \( \langle g, x \rangle \) is dihedral but \( |\langle g, x \rangle| = 2o(x) < |D_{2m}| \). If \( x \in \langle r \rangle \) such that \( \langle x \rangle = \langle r^{\frac{n}{m}} \rangle \), then \( \langle g, x \rangle = \langle s, r^{\frac{n}{m}} \rangle \), so \( \langle g, x \rangle \in cl(D_{2m}) \) and there are \( \varphi(o(r^{\frac{n}{m}})) = \varphi(m) \) such \( xs \) in \( D_{2n} \). Note that \( \langle g, r^{\frac{n}{m}} \rangle \) is the unique subgroup in \( cl(D_{2m}) \) that contains \( g \) and so if \( x \in cl(s) \) with \( \langle g, x \rangle \in cl(D_{2m}) \), then \( \langle g, x \rangle = \langle g, r^{\frac{n}{m}} \rangle \). Therefore, \( x = sr^{\frac{n}{m}} \) for some integer \( 1 \leq j < m \). Notice \( \langle g, x \rangle = \langle s, r^{\frac{n}{2j}} \rangle \) and \( o(r^{\frac{n}{2j}}) = \frac{m}{(m,j)} \). Thus, \( o(r^{\frac{n}{2j}}) = m \) if and only if \( (m,j) = 1 \), so \( \langle g, x \rangle \in cl(D_{2m}) \) if and only if \( (m,j) = 1 \). There are \( \varphi(m) \) such \( js \) and therefore \( \varphi(m) \) such \( xs \). Hence, \( \psi(g) = 2\varphi(m) \) in this case.

In order to calculate inner products we will need the character table of \( D_{2n} \) for even \( n > 2 \). A construction of this table appears in the Appendix. In the table, \( \epsilon \) is a primitive \( n^{th} \) root of unity.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
D_{2n} & r^{\pm 1} & r^{\pm 2} & \ldots & r^{\frac{n}{2}} & s & sr \\
\hline
\lambda_1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\lambda_2 & 1 & 1 & 1 & -1 & -1 & -1 \\
\lambda_3 & 1 & -1 & 1 & (-1)^{\frac{n}{2}} & 1 & -1 \\
\lambda_4 & 1 & -1 & 1 & (-1)^{-\frac{n}{2}} & -1 & 1 \\
\chi_1 & 2 & \epsilon + \epsilon^{-1} & \epsilon^2 + \epsilon^{-2} & \epsilon^{\frac{n}{2}} + \epsilon^{-\frac{n}{2}} & 0 & 0 \\
\chi_2 & 2 & \epsilon^2 + \epsilon^{-2} & \epsilon^{2} + \epsilon^{-2} & \epsilon^{\frac{n}{2}} + \epsilon^{-\frac{n}{2}} & 0 & 0 \\
\vdots & & & & & & \\
\chi_{\frac{n}{2}-1} & 2 & \epsilon^{\left(\frac{n}{2}-1\right)} + \epsilon^{-\left(\frac{n}{2}-1\right)} & \epsilon^{\left(\frac{n}{2}-1\right)^2} + \epsilon^{-\left(\frac{n}{2}-1\right)^2} & \epsilon^{\left(\frac{n}{2}-1\right)^{-\frac{n}{2}}} + \epsilon^{-\left(\frac{n}{2}-1\right)^{-\frac{n}{2}}} & 0 & 0 \\
\hline
\end{array}
\]

**Lemma 4.11:** Let \( G = D_{2n} \) with \( 2 < n = 2^\alpha \cdot c \), \( 1 \leq \alpha \in \mathbb{Z}, 1 \leq c \in \mathbb{Z}, \) and \( c \) odd. Let \( D_{2m} \leq D_{2n} \) such that \( 2 < m = 2^\beta \cdot d \) with \( 0 \leq \beta \leq \alpha \) and \( d|c \). Assume \( cl(s) \cap D_{2m} \neq \emptyset \).
when considering \( \frac{n}{m} \) even. Let \( \psi = \psi_{cl(2m)} \) Then,

\[
[\lambda_1, \psi] = \begin{cases} 
3 \cdot \frac{\varphi(m)}{2} & \text{if } \frac{n}{m} \text{ is odd} \\
3 \cdot \frac{\varphi(m)}{4} & \text{if } \frac{n}{m} \text{ is even,}
\end{cases}
[\lambda_2, \psi] = \begin{cases} 
-\frac{\varphi(m)}{2} & \text{if } \frac{n}{m} \text{ is odd} \\
-\frac{\varphi(m)}{4} & \text{if } \frac{n}{m} \text{ is even,}
\end{cases}
\]

\[
[\lambda_3, \psi] = \begin{cases} 
-\frac{\varphi(m)}{2} & \text{if } \frac{n}{m} \text{ is odd} \\
3 \cdot \frac{\varphi(m)}{4} & \text{if } \frac{n}{m} \text{ is even,}
\end{cases}
[\lambda_4, \psi] = \begin{cases} 
-\frac{\varphi(m)}{2} & \text{if } \frac{n}{m} \text{ is odd} \\
-\frac{\varphi(m)}{4} & \text{if } \frac{n}{m} \text{ is even, and}
\end{cases}
\]

\[
[\lambda_k, \psi] = \begin{cases} 
\left( \frac{\varphi(m)}{\varphi(D_2m)} \right)^{\mu \left( \frac{m}{\gcd(m, k)} \right)} & \text{if } \frac{n}{m} \text{ is odd} \\
\frac{1}{2} \left( \frac{\varphi(m)}{\varphi(D_2m)} \right)^{\mu \left( \frac{m}{\gcd(m, k)} \right)} & \text{if } \frac{n}{m} \text{ is even,}
\end{cases}
\]

where \( \mu \) is the Möbius function.

Note: We make the extra assumption that \( cl(s) \cap D_{2m} \neq \emptyset \) when considering \( \frac{n}{m} \) even since the only thing that changes when \( cl(sr) \cap D_{2m} \neq \emptyset \) is that \([\lambda_3, \psi]\) and \([\lambda_4, \psi]\) are exchanged.

**Proof**: We use the above character table of \( D_{2n} \). As \( \lambda_1 \) and \( \lambda_2 \) differ only on the reflections and \( \lambda_3 \) and \( \lambda_4 \) also differ only on the reflections, we omit the calculation of \([\lambda_2, \psi]\) and \([\lambda_4, \psi]\).

First, we compute \([\lambda_1, \psi]\):

\[
[\lambda_1, \psi] = \frac{1}{|G|} \sum_{g \in G} \lambda_1(g) \psi(g)
= \frac{1}{2n} \left( \sum_{g \in (r)} \lambda_1(g) \psi(g) + \sum_{g \in D_{2n} \setminus (r)} \lambda_1(g) \psi(g) \right),
\]
as \( \psi(g) = 0 \) for either \( g \notin \langle r \rangle \) and \( cl(g) \cap D_{2m} = \emptyset \) or \( g \in \langle r \rangle \) such that \( \langle g \rangle \neq \langle r^{\frac{a}{m}} \rangle \). This yields

\[
\frac{1}{2n} \left( \sum_{g \in \langle r \rangle \atop (g) = \langle r^{\frac{a}{m}} \rangle} 1 \cdot \psi(g) + \sum_{g \in D_{2m} \setminus \langle r \rangle \atop cl(g) \cap D_{2m} \neq \emptyset} 1 \cdot \psi(g) \right)
\]

which gives us

\[
\begin{cases}
\frac{1}{2n} \left( \varphi(m) \cdot n + n \cdot 2\varphi(m) \right) & \text{if } \frac{n}{m} \text{ is odd} \\
\frac{1}{2n} \left( \varphi(m) \cdot \frac{n}{2} + \frac{n}{2} \cdot 2\varphi(m) \right) & \text{if } \frac{n}{m} \text{ is even}
\end{cases}
\]

\[
= \begin{cases}
3 \cdot \frac{\varphi(m)}{2} & \text{if } \frac{n}{m} \text{ is odd} \\
3 \cdot \frac{\varphi(m)}{4} & \text{if } \frac{n}{m} \text{ is even.}
\end{cases}
\]

(For \( [\lambda_2, \psi] \), if \( \frac{n}{m} \) is even then the + changes to −.)

Next, we compute \( [\lambda_3, \psi] \):

\[
[\lambda_3, \psi] = \frac{1}{|G|} \sum_{g \in G} \lambda_3(g) \psi(g)
\]

\[
= \frac{1}{2n} \left( \sum_{g \in \langle r \rangle \atop (g) = \langle r^{\frac{a}{m}} \rangle} \lambda_3(g) \psi(g) + \sum_{g \in cl(s)} \lambda_3(g) \psi(g) + \sum_{g \in cl(sr)} \lambda_3(g) \psi(g) \right),
\]

as \( \psi(g) = 0 \) for \( g \in \langle r \rangle \) such that \( \langle g \rangle \neq \langle r^{\frac{a}{m}} \rangle \). Then we get

\[
\begin{cases}
\frac{1}{2n} \left( \sum_{g \in \langle r \rangle \atop (g) = \langle r^{\frac{a}{m}} \rangle} \lambda_3(g) \cdot n + n \cdot 1 \cdot 2\varphi(m) - n \cdot 1 \cdot 2\varphi(m) \right) & \text{if } \frac{n}{m} \text{ is odd} \\
\frac{1}{2n} \left( \sum_{g \in \langle r \rangle \atop (g) = \langle r^{\frac{a}{m}} \rangle} \lambda_3(g) \cdot \frac{n}{2} + \frac{n}{2} \cdot 1 \cdot 2\varphi(m) \right) & \text{if } \frac{n}{m} \text{ is even,}
\end{cases}
\]

as \( \chi(g) = 0 \) for \( g \notin \langle r \rangle \) and \( cl(g) \cap D_{2m} = \emptyset \). (For \([\lambda_4, \psi]\), if \( \frac{n}{m} \) is even, then the + changes
to \(-\). From this we get

\[
\begin{cases}
\frac{1}{2} \left( \sum_{1 \leq j < m} \lambda_3 \left( \frac{j}{m} \right) \right) & \text{if } \frac{n}{m} \text{ is odd} \\
\frac{1}{4} \left( \sum_{1 \leq j < m} \lambda_3 \left( \frac{j}{m} \right) + 2 \varphi \left( \frac{m}{n} \right) \right) & \text{if } \frac{n}{m} \text{ is even}
\end{cases}
\]

\[
= \begin{cases}
-\frac{\varphi(m)}{2} & \text{if } \frac{n}{m} \text{ is odd} \\
3 \cdot \frac{\varphi(m)}{4} & \text{if } \frac{n}{m} \text{ is even},
\end{cases}
\]

since when \(\frac{n}{m}\) is odd, \(m\) is even. Thus \((j, m) = 1\) implies \(j\) is odd. When \(\frac{n}{m}\) is even, \(j \frac{n}{m}\) is even. Then \(\lambda_3(r^{\text{even}}) = 1\) while \(\lambda_3(r^{\text{odd}}) = -1\).

Lastly, we compute \([\chi_k, \psi]\):

\[
[\chi_k, \psi] = \frac{1}{|G|} \sum_{g \in G} \chi_k(g) \psi(g)
= \frac{1}{2n} \sum_{1 \leq j < m \atop (j, m) = 1} \chi_k\left(\left(\frac{n}{m}\right)^j\right) \psi\left(\left(\frac{n}{m}\right)^j\right),
\]

as \(\chi_k(g) = 0\) if \(g \in D_{2n} \setminus \langle r \rangle\), and \(\psi(g) = 0\) if \(g \in \langle r \rangle\) and \(\langle g \rangle \neq \langle r^{\frac{n}{m}} \rangle\). Then we get

\[
\begin{cases}
\frac{1}{2n} \sum_{1 \leq j < m \atop (j, m) = 1} \chi_k\left(\left(\frac{n}{m}\right)^j\right) n & \text{if } \frac{n}{m} \text{ is odd} \\
\frac{1}{2n} \sum_{1 \leq j < m \atop (j, m) = 1} \chi_k\left(\left(\frac{n}{m}\right)^j\right) \frac{n}{2} & \text{if } \frac{n}{m} \text{ is even}.
\end{cases}
\]

Factoring out \(n\) when \(\frac{n}{m}\) is odd, and \(\frac{n}{2}\) when \(\frac{n}{m}\) is even gives us

\[
\begin{cases}
\frac{1}{2} \sum_{1 \leq j < m \atop (j, m) = 1} \left( \left(\epsilon^{\frac{n}{m}} k\right)^j + \left(\epsilon^{\frac{n}{m}} k\right)^{-j} \right) & \text{if } \frac{n}{m} \text{ is odd} \\
\frac{1}{4} \sum_{1 \leq j < m \atop (j, m) = 1} \left( \left(\epsilon^{\frac{n}{m}} k\right)^j + \left(\epsilon^{\frac{n}{m}} k\right)^{-j} \right) & \text{if } \frac{n}{m} \text{ is even}.
\end{cases}
\]
As \( j \) runs through the numbers less than and relatively prime to \( m \) so does \(-j\) modulo \( m \) and so we factor out a 2 and change the limit of the sum to \( \frac{m}{2} \) instead of \( m \) to get

\[
\begin{cases}
\sum_{1 \leq j \leq \frac{m}{2}} \left( (\epsilon_{\frac{m}{m}})^j + (\epsilon_{\frac{m}{m}})^{-j} \right) & \text{if } \frac{n}{m} \text{ is odd} \\
\frac{1}{2} \sum_{1 \leq j \leq \frac{m}{2}} \left( (\epsilon_{\frac{m}{m}})^j + (\epsilon_{\frac{m}{m}})^{-j} \right) & \text{if } \frac{n}{m} \text{ is even.}
\end{cases}
\]

Since \((j, m) = 1\) implies \( \left(j, \frac{m}{GCD(m, k)}\right) = 1\), \((\epsilon_{\frac{m}{m}})^j\) is a primitive \( \left(\frac{m}{GCD(m, k)}\right)^{th} \) root of unity. Therefore we are summing over all of the primitive \( \left(\frac{m}{GCD(m, k)}\right)^{th} \) roots of unity \( \frac{\varphi(m)}{\varphi\left(\frac{m}{GCD(m, k)}\right)} \) times. In other words, we have

\[
\begin{cases}
\frac{\varphi(m)}{\varphi\left(\frac{m}{GCD(m, k)}\right)} \mu \left(\frac{m}{GCD(m, k)}\right) & \text{if } \frac{n}{m} \text{ is odd} \\
\frac{1}{2} \frac{\varphi(m)}{\varphi\left(\frac{m}{GCD(m, k)}\right)} \mu \left(\frac{m}{GCD(m, k)}\right) & \text{if } \frac{n}{m} \text{ is even.}
\end{cases}
\]

This last statement is really von Sterneck’s formula and appears as Corollary 3.9 of [1].

For positive even integers \( n \) we have established what the atoms of \( D_{2n} \) are in Lemma 4.9. We have also calculated the Thompson counting function associated with each conjugacy class of subgroups in each nonabelian atom in Lemma 4.10 and calculated their inner products with the irreducible characters of \( D_{2n} \) in Lemma 4.11. In the next theorem we determine if any of the nonabelian atoms of \( D_{2n} \) can be split.

**Theorem 4.12:** Let \( G = D_{2n} \) with \( 2 < n = 2^\alpha \cdot c, 1 \leq \alpha \in \mathbb{Z}, 1 \leq c \in \mathbb{Z}, \) and \( c \) odd. Let \( D_{2m} \leq G \) such that \( 2 < m = 2^\beta \cdot d \) with \( 1 \leq \beta \leq \alpha \) and \( d|c \). Consider cases (i) through (iv) as in Lemma 4.9. Then in case
i) \( \text{cl}(D_{2n}) \) does not split,

ii) \( \text{cl}(D_{2m}) \cup \text{cl}(\tilde{D}_{2m}) \) splits as \( \text{cl}(D_{2m}) \) and \( \text{cl}(\tilde{D}_{2m}) \)
    if and only if \( \beta \geq 3 \) or \( \beta = 2 \) with \( d \geq 3 \),

iii) \( \text{cl}(D_m) \cup \text{cl}(\tilde{D}_m) \cup \text{cl}(D_{2m}) \) splits uniquely as \( \text{cl}(D_m) \cup \text{cl}(\tilde{D}_m) \) and \( \text{cl}(D_{2m}) \),

iv) \( \text{cl}(D_m) \cup \text{cl}(\tilde{D}_m) \cup \text{cl}(D_{2m}) \cup \text{cl}(\tilde{D}_{2m}) \) splits non-uniquely as
    \( \text{cl}(D_m) \cup \text{cl}(\tilde{D}_m) \) and \( \text{cl}(D_{2m}) \cup \text{cl}(\tilde{D}_{2m}) \), or
    \( \text{cl}(D_m) \cup \text{cl}(D_{2m}) \) and \( \text{cl}(\tilde{D}_m) \cup \text{cl}(\tilde{D}_{2m}) \), or
    \( \text{cl}(D_m) \cup \text{cl}(\tilde{D}_{2m}) \) and \( \text{cl}(\tilde{D}_m) \cup \text{cl}(D_{2m}) \).

**Proof:** Case (i) is trivial.

In case (ii), \( \alpha > \beta \geq 2 \) and there are only two conjugacy classes of subgroups in the atom. Thus, \( \psi := \psi_{\text{cl}(D_{2m})} \) is a generalized character if and only if \( \psi_{\text{cl}(\tilde{D}_{2m})} \) is a generalized character.

We consider \( [\chi_k, \psi] = \frac{1}{2} \left( \frac{\varphi(m)}{\varphi(D(m,k))} \mu \left( \frac{m}{\varphi(GCD(m,k))} \right) \right) \) by Lemma 4.11. If \( 4 \mid \frac{m}{GCD(m,k)} \), then \( \mu \left( \frac{m}{GCD(m,k)} \right) = 0 \) and so \( [\chi_k, \psi] = 0 \). If \( 4 \nmid \frac{m}{GCD(m,k)} \) then \( \frac{m}{GCD(m,k)} = 2^\gamma d e \) for some odd integer \( e \) dividing \( d \) and \( \gamma \in \{0,1\} \). Hence, \( \frac{\varphi(m)}{\varphi(D(m,k))} = 2^{\beta-1} \frac{\varphi(d)}{\varphi(e)} \). Now, \( \beta \geq 2 \), \( \frac{\varphi(d)}{\varphi(e)} \in \mathbb{Z} \), and \( \mu \left( \frac{m}{GCD(m,k)} \right) \in \{-1,0,1\} \) so \( \frac{1}{2} \left( 2^{\beta-1} \frac{\varphi(d)}{\varphi(e)} \mu \left( \frac{m}{GCD(m,k)} \right) \right) \in \mathbb{Z} \). Thus, \( [\chi_k, \psi] \in \mathbb{Z} \) for all \( 1 \leq k \leq n-1 \). However, Lemma 4.11 states \( [\lambda_i, \psi] \in \left\{ 3 \cdot \frac{\varphi(m)}{4}, -\frac{\varphi(m)}{4} \right\} \) and \( \frac{\varphi(m)}{4} = 2^{\beta-3} \varphi(d) \). Now \( 2 \mid \varphi(d) \) for odd \( d > 1 \), so \( [\lambda_i, \psi] \in \mathbb{Z} \) if and only if either \( \beta \geq 3 \) or \( \beta = 2 \) and \( d \geq 3 \) and so (ii) holds.

Note that in case (ii), if \( \langle g \rangle = \langle r^{\frac{n}{2}} \rangle \), then Lemma 4.10 implies \( \psi_{\text{cl}(D_{2m})}(g) = \frac{n}{2} \). But \( |\text{cl}(g)| = 2 \) and so \( |C_G(g)| = n \). Thus \( |C_G(g)| \nmid \frac{n}{2} \) and so the converse of Theorem 1.1 is not true.

In case (iii), \( 1 = \beta = \alpha \). To show that the atom splits it is enough to show \( [\chi, \psi_{\text{cl}(D_{2m})}] \in \mathbb{Z} \) for all \( \chi \in \text{Irr}(G) \). To show uniqueness it is enough by Corollary 4.2 to show that
that and the factor valued function, \[ \psi_{\psi(D_m)} \notin \mathbb{Z} \] for some \( \chi \in \text{Irr}(G) \).

From Lemma 4.11, we have \( \lambda, \psi_{\psi(D_m)} \in \mathbb{Z} \) if and only if \( \frac{\varphi(m)}{2} = \frac{\varphi(d)}{2} \in \mathbb{Z} \). Since \( D_{2m} \) is nonabelian, \( d \geq 3 \) and so \( 2|\varphi(d) \). Thus, \( \lambda, \psi_{\psi(D_{2m})} \in \mathbb{Z} \). Again from Lemma 4.11, we have
\[
\left[ x_k, \psi_{\psi(D_{2m})} \right] = \varphi\left( \frac{\varphi(m)}{GCD(m,k)} \right) \mu\left( \frac{m}{GCD(m,k)} \right).
\]
As \( \varphi\left( \frac{\varphi(m)}{GCD(m,k)} \right) | \varphi(m) \) and \( \mu \) is an integer valued function, \( \left[ x_k, \psi_{\psi(D_{2m})} \right] \in \mathbb{Z} \) for all \( k \). Thus, the atom splits into \( cl(D_{2m}) \) and \( cl(D_m) \). Now,
\[
\left[ x_k, \psi_{\psi(D_m)} \right] = \left( \frac{\varphi(d)}{2} \right) \left( \frac{d}{GCD(d,k)} \right)^\mu \left( \frac{d}{GCD(d,k)} \right)
\]
de \( \beta = 1 \) implies \( \frac{m}{2} = d \). We factor \( d \) into its prime factorization, say \( d = p_1^{\omega_1} p_2^{\omega_2} \cdots p_r^{\omega_r} \) where the \( p_i \) are distinct primes and the \( \omega_i \) are positive integers. As \( 1 \leq k \leq c - 1 \) we can choose \( k = p_1^{\omega_1 - 1} p_2^{\omega_2 - 1} \cdots p_r^{\omega_r - 1} \) so that \( \frac{d}{GCD(d,k)} = p_1 p_2 \cdots p_r \). Then, for this choice of \( k \),
\[
\frac{1}{2} \left( \frac{\varphi(d)}{\varphi\left( \frac{d}{GCD(d,k)} \right)} \right)^\mu \left( \frac{d}{GCD(d,k)} \right) = \frac{1}{2} \cdot \frac{p_1^{\omega_1 - 1}(p_1 - 1)p_2^{\omega_2 - 1}(p_2 - 1) \cdots p_r^{\omega_r - 1}(p_r - 1)}{(p_1 - 1)(p_2 - 1) \cdots (p_r - 1)}
\cdot \mu(p_1 p_2 \cdots p_r)
\]
\[
= \pm \frac{1}{2} \cdot p_1^{\omega_1 - 1} p_2^{\omega_2 - 1} \cdots p_r^{\omega_r - 1}.
\]
Now, \( d \) odd implies \( 2 \nmid p_1^{\omega_1 - 1} p_2^{\omega_2 - 1} \cdots p_r^{\omega_r - 1} \) so \( \left[ x_k, \psi_{\psi(D_m)} \right] \notin \mathbb{Z} \).

In case \((iv)\), \( 1 = \beta < \alpha \) (and \( d > 2 \)) and we can use the same argument as in case \((iii)\), letting \( d = p_1^{\omega_1} p_2^{\omega_2} \cdots p_r^{\omega_r} \) to show the atom cannot be split into any individual classes. Therefore, the only possible splits are those that split the atom into pairs of conjugacy classes. Lemma 4.11 implies

\[
\psi_{\psi(D_m)} = \left( \frac{3\varphi(d)}{4} \lambda_1 - \frac{\varphi(d)}{4} \lambda_2 + \frac{3\varphi(d)}{4} \lambda_3 - \frac{\varphi(d)}{4} \lambda_4 \right) + \sum_{k=1}^{n-1} \frac{1}{2} \cdot \frac{\varphi(d)}{GCD(d,k)} \mu\left( \frac{d}{GCD(d,k)} \right)
\]

\[
\psi_{\psi(D_m)} = \left( \frac{3\varphi(d)}{4} \lambda_1 - \frac{\varphi(d)}{4} \lambda_2 - \frac{\varphi(d)}{4} \lambda_3 + \frac{3\varphi(d)}{4} \lambda_4 \right) + \sum_{k=1}^{n-1} \frac{1}{2} \cdot \frac{\varphi(d)}{GCD(d,k)} \mu\left( \frac{d}{GCD(d,k)} \right)
\]

\[
\psi_{\psi(D_{2m})} = \left( \frac{3\varphi(2d)}{4} \lambda_1 - \frac{\varphi(2d)}{4} \lambda_2 + \frac{3\varphi(2d)}{4} \lambda_3 - \frac{\varphi(2d)}{4} \lambda_4 \right) + \sum_{k=1}^{n-1} \frac{1}{2} \cdot \frac{\varphi(2d)}{GCD(d,k)} \mu\left( \frac{d}{GCD(d,k)} \right)
\]

\[
\psi_{\psi(D_{2m})} = \left( \frac{3\varphi(2d)}{4} \lambda_1 - \frac{\varphi(2d)}{4} \lambda_2 - \frac{\varphi(2d)}{4} \lambda_3 + \frac{3\varphi(2d)}{4} \lambda_4 \right) + \sum_{k=1}^{n-1} \frac{1}{2} \cdot \frac{\varphi(2d)}{GCD(d,k)} \mu\left( \frac{d}{GCD(d,k)} \right).
\]

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Since \( d > 2 \), \( 2|\varphi(d) \) so \( \varphi(d) = 2j \) for some \( j \in \mathbb{Z} \) and we have

\[
\begin{align*}
\psi_{cl(D_m)} &= \frac{3j}{2} \lambda_1 - \frac{j}{2} \lambda_2 + \frac{3j}{2} \lambda_3 - \frac{j}{2} \lambda_4 + \frac{1}{2} \sum_{k=1}^{\frac{d}{2}} \varphi(d) \left( \frac{d}{\text{GCD}(d,k)} \right) \\
\psi_{cl(D_m^-)} &= \frac{3j}{2} \lambda_1 - \frac{j}{2} \lambda_2 - \frac{j}{2} \lambda_3 + \frac{3j}{2} \lambda_4 + \frac{1}{2} \sum_{k=1}^{\frac{d}{2}} \varphi(d) \left( \frac{d}{\text{GCD}(d,k)} \right) \\
\psi_{cl(D_{2m})} &= \frac{3j}{2} \lambda_1 - \frac{j}{2} \lambda_2 + \frac{3j}{2} \lambda_3 - \frac{j}{2} \lambda_4 + \frac{1}{2} \sum_{k=1}^{\frac{d}{2}} \varphi(d) \left( \frac{d}{\text{GCD}(d,k)} \right) \\
\psi_{cl(D_{2m}^-)} &= \frac{3j}{2} \lambda_1 - \frac{j}{2} \lambda_2 - \frac{j}{2} \lambda_3 + \frac{3j}{2} \lambda_4 + \frac{1}{2} \sum_{k=1}^{\frac{d}{2}} \varphi(d) \left( \frac{d}{\text{GCD}(d,k)} \right)
\end{align*}
\]

so that adding any two of these class functions results in a generalized character. □

Note that since all subgroups of the nonabelian atoms are dihedral, they all have rank 2. Therefore, any splitting will have a non-zero generalized character as its associated Thompson counting function.

Before moving on to classifying the splitting of atoms for our next family of groups, we mention the next proposition which deals with embedding a dihedral group into a symmetric group. This proposition involves case (iii) of Theorem 4.12.

**Proposition 4.13:** Let \( k \geq 3 \) be odd and \( D_{4k} = \langle r, s \rangle \leq S_{k+2} = G \) where \( r = (1 2 \ldots k)(k+1 \ldots k+2) \) and \( s = (2 k)(3 k - 1) \ldots \left( \frac{k+1}{2} \right) \left( \frac{k+3}{2} \right) \). Then \( cl(D_{2k}) \cup cl(D_{2k}^-) \cup cl(D_{4k}) \) is an atom of \( G \). Furthermore, the atom splits uniquely into \( cl(D_{2k}) \cup cl(D_{2k}^-) \) and \( cl(D_{4k}) \).

**Proof:** The common derived subgroup between \( D_{2k}, D_{2k}^-, \) and \( D_{4k} \) is \( \langle r^2 \rangle \). The lattice of subgroups of \( D_{4k} \) that contain \( \langle r^2 \rangle \) is shown below:
Let $N$ be a normal subgroup of $D_{2k}$, $\overline{D_{2k}}$, or $D_{4k}$ with a cyclic quotient. Then $\langle r^2 \rangle \leq N$ and so $C_G(N) \leq C_G(\langle r^2 \rangle) = \langle r \rangle \leq D_{4k}$. Suppose $\langle N, x_1 \rangle := H \leq G$ such that $H$ is linked to $K \in \big\{ D_{2k}, \overline{D_{2k}}, D_{4k} \big\}$. Then $K = \langle N, x_2 \rangle$ where $x_1 x_2^{-1} \in C_G(N) \leq D_{4k}$. Hence, $x_1 \in D_{4k}$ and so $H \leq D_{4k}$. We have already seen in Lemma 4.10 that the only subgroups of $D_{4k}$ that are strongly linked to $D_{4k}$ are in $\big\{ D_{2k}, \overline{D_{2k}}, D_{4k} \big\}$. Therefore, $cl(D_{2k}) \cup cl(\overline{D_{2k}}) \cup cl(D_{4k})$ is an atom of $G$.

Let $\psi = \psi_{cl(D_{4k})}$. We show

$$
\psi(g) = \begin{cases} 
0 & \text{if } g \text{ is not contained in a conjugate of } D_{4k}, \\
0 & \text{if } \langle g \rangle < \langle r^x \rangle \text{ for some } x \in G, \\
|C_G(g)| & \text{otherwise}. 
\end{cases}
$$

First, if $g$ is not contained in a conjugate of $D_{4k}$ then $\langle g, \sigma \rangle \notin cl(D_{4k})$ for any $\sigma \in G$ and so $\psi(g) = 0$ in this case.

Second, let $\langle g \rangle < \langle r^x \rangle$ for some $x \in G$. Since $r^x$ is contained in a unique $D_{4k}$, if $\sigma$ is not contained in $(D_{4k})^x$, then $\langle g, \sigma \rangle \notin cl(D_{4k})$. If $\sigma \leq \langle r^x \rangle$, then $\langle g, \sigma \rangle \leq \langle r^x \rangle \notin cl(D_{4k})$. If $\sigma \in (D_{4k})^x \setminus \langle r^x \rangle$, then $\langle g, \sigma \rangle$ is dihedral with order less than $D_{4k}$, so that $\langle g, \sigma \rangle \notin cl(D_{4k})$. Hence, $\psi(g) = 0$ in this case.

Third, let $\langle g \rangle = \langle r^x \rangle$ for some $x \in G$. Since $r^x$ is contained in a unique $D_{4k}$, if $\sigma$ is not
contained in \((D_{4k})^x\), then \(\langle g, \sigma \rangle \not\in cl(D_{4k})\). If \(\sigma \leq \langle r^x \rangle\), then \(\langle g, \sigma \rangle \leq \langle r^x \rangle \not\in cl(D_{4k})\). If \(\sigma \in (D_{4k})^x \setminus \langle r^x \rangle\), then as \(\langle r^x \rangle\) is maximal in \((D_{4k})^x\), \(\langle g, \sigma \rangle = (D_{4k})^x \in cl(D_{4k})\). Therefore, 
\[\psi (g) = |(D_{4k})^x \setminus \langle r^x \rangle| = 2k = |C_G(g)|.\]

Fourth, let \(g \in (D_{4k})^x \setminus \langle r^x \rangle\) for some \(x \in G\) such that \(g \in cl_G(s)\). Since \(\psi\) is a class function we may assume that \(g = s\). (Unlike \(r^x\), \(s\) may be contained in more than one conjugate of \(D_{4k}\).) If \(\sigma\) is not contained in any conjugate of \(D_{4k}\), then \(\langle g, \sigma \rangle \not\in cl(D_{4k})\). Assume \(\sigma \in \langle r^y \rangle\) for some \(y \in G\). If \(\langle \sigma \rangle \leq \langle r \rangle\), then \(\langle s, \sigma \rangle\) is dihedral of order less than \(4k\); so that \(\langle s, \sigma \rangle \not\in cl(D_{4k})\). If \(\sigma \in cl_G(r)\) but \(\sigma^* \not= \sigma^{-1}\), then \(\langle g, \sigma \rangle \not\in cl(D_{4k})\). For each \(\sigma \in cl_G(r)\) with \(\sigma^* = \sigma^{-1}\), it follows that \(\langle g, \sigma \rangle \in cl(D_{4k})\). For each \(\sigma \in \langle r^y \rangle\), we have \(\langle s, \sigma \rangle = \langle s, s\sigma \rangle\). Therefore, \(\psi (g) = 2 |A|\) where \(A := \{\sigma \in G : \sigma \in cl_G(r)\text{ and }\sigma^* = \sigma^{-1}\}\). It remains to calculate \(|A|\).

If \(\sigma \in A\), then \(\sigma = r't\) where \(r'\) is a \(k\)-cycle and \(t\) is a transposition. Moreover, since \(\sigma^* = \sigma^{-1}\), \(t\) must consist of two of the three points that are fixed by \(s\) with the remaining point fixed by \(s\) somewhere in \(r'\). Now rewrite \(r'\) so that the fixed point is in the first position. Then we have \(k - 1\) choices for what to put in position 2 of \(r'\), forcing what is put in position \(k\). Next we have \(k - 3\) choices for what to put in position 3 of \(r'\), forcing what is put in position \(k - 1\). Continuing in this fashion we see that \(|A| = 3 \cdot (k - 1) (k - 3) \cdots (2)\). Therefore,
\[\psi (g) = 2 \cdot 3 \cdot (k - 1) (k - 3) \cdots (2) = |C_G(s)|.\]

Lastly, define \(c := (k + 1 \ k + 2)\), and let \(g \in (D_{4k})^x \setminus \langle r^x \rangle\) for some \(x \in G\) such that \(g \in cl_G(sc)\). Since \(\psi\) is a class function we may assume that \(g = sc\). Using a similar argument as in the previous two paragraphs, we see that \(\psi (g) = 2 |B|\) where \(B := \{\sigma \in G : \sigma \in cl(r)\text{ and }\sigma^{sc} = \sigma^{-1}\}\). Note that \(|B|\) is slightly different than \(|A|\) since \(sc\) only fixes one point while \(s\) fixed three points. However, similar reasoning shows \(|B| = (k + 1) (k - 1) (k - 3) \cdots (4)\). Therefore,
\[\psi (g) = 2 \cdot (k + 1) (k - 1) (k - 3) \cdots (4) = |C_G(sc)|.\]
Now we have $|C_G(g)| \mid \psi(g)$ for all $g \in G$ and so $\psi$ is a generalized character by Theorem 1.1.

A similar calculation as for $\psi_{d(D_{4k})}$ shows

$$\psi_{d(D_{2k})}(g) = \begin{cases} 0 & \text{if } g \text{ is not contained in any conjugate of } D_{2k} \\ 0 & \text{if } \langle g \rangle < \langle (r^2)^x \rangle \text{ for some } x \in G \\ \frac{|C_G(g)|}{2} & \text{if } \langle g \rangle = \langle (r^2)^x \rangle \text{ for some } x \in G \\ |C_G(g)| & \text{otherwise.} \end{cases}$$

Then $[\psi_{d(D_{2k})}, 1_G] = \frac{3}{2}$ and so $\psi_{d(D_{2k})}$ is not a generalized character and the atom does not split further. □

Section 4.3 $D_{2p} \times C_n$

Keeping on the same line of thought with embedding dihedral groups into “bigger” groups, we now come to the third infinite family of groups in which some nonabelian atoms split. The family of groups we work with are direct products of a dihedral group of order $2p$ and a cyclic group of order $n$, denoted $D_{2p} \times C_n$, where $p$ is an odd prime and $n > 1$. This is an example of a family of groups with just two atoms, namely, the abelian and the nonabelian atom. We focus only on how to split the nonabelian atom. We begin with a brief discussion of the nonabelian subgroups of such a group.

Let $G = D_{2p} \times C_n := \langle (r, x), (s, 1) : r^p = s^2 = x^n = 1, rs = sr^{-1}, r^x = r, s^x = s \rangle$. Then $\langle (r, x^{\frac{n}{2}}), (s, 1) \rangle \cong D_{2p} \times C_d$ is a nonabelian subgroup of $G$ for all divisors, $d$, of $n$. When $n$ is odd these are the nonabelian subgroups. When $n$ is even there are additional nonabelian subgroups with even index in $G$. For each $D_{2p} \times C_d$ with even index we associate the nonabelian subgroup $\widehat{D_{2p} \times C_d} := \langle (r, 1), (s, x^{\frac{n}{2}}) \rangle$ which is not conjugate to $D_{2p} \times C_d$. This completely describes the nonabelian subgroups of $G$ and so we are ready for the main result of this section.
Theorem 4.14: Let \( G = D_{2p} \times C_n \) such that \( p \) is an odd prime and \( n > 1 \). Then there are only two atoms of \( G \), namely the abelian atom and the nonabelian atom. Furthermore, the nonabelian atom splits if and only if \( 2n \) is divisible by a square.

Proof: Let \( D_{2p} \times (1) = \langle (r, 1), (s, 1) : (r, 1)^p = (s, 1)^2 = 1, (r, 1)(s, 1) = (s, 1)(r, 1)^{-1} \rangle \) and \( (1) \times C_n = \langle (1, x) \rangle \). From the discussion before the theorem we know \( D_{2p} \times C_d = \langle (r, x^{\frac{n}{d}}), (s, 1) \rangle \) is a nonabelian subgroup of \( G \) for each divisor, \( d \), of \( n \). When \( n \) is odd these are the only nonabelian subgroups of \( G \), but when \( n \) is even we also have the nonabelian subgroups \( \overline{D_{2p} \times C_d} = \langle (r, 1), (s, x^{\frac{n}{d}}) \rangle \) for all divisors \( d \) of \( n \) with \( \frac{n}{d} \) even.

The first part of the theorem now follows from the observation that \( D_{2p} \times C_d = \langle D_{2p} \times (1), (r, x^{\frac{n}{d}}) \rangle \) and \( D_{2p} \times (1) = \langle D_{2p} \times (1), (r, 1) \rangle \) with \( (r, x^{\frac{n}{d}})(r, 1)^{-1} = (1, x^{\frac{n}{d}}) \in C_G(D_{2p} \times (1)) \). Then, if necessary, \( D_{2p} \times C_d = \langle \langle r x^{\frac{n}{d}} \rangle, (s, 1) \rangle \) and \( D_{2p} \times C_d = \langle \langle r x^{\frac{n}{d}} \rangle, (s, x^{\frac{n}{d}}) \rangle \) with \( (s, x^{\frac{n}{d}})(s, 1)^{-1} = (1, x^{\frac{n}{d}}) \in C_G(\langle r x^{\frac{n}{d}} \rangle) \). (Note that \( \langle r x^{\frac{n}{d}} \rangle = (D_{2p} \times C_d) \cap \overline{(D_{2p} \times C_d)} \).)

Thus, if \( \mathcal{N} \) is the nonabelian atom of \( G \), then

\[
\mathcal{N} = \left\{ \langle (r, x^{\frac{n}{d}}), (s, 1) \rangle : d \mid n \right\} \cup \left\{ \langle (r, 1), (s, x^{\frac{n}{d}}) \rangle : 2d \mid n \right\}.
\]

Of course when \( n \) is odd the second set is empty so that \( \mathcal{N} = \left\{ \langle (r, x^{\frac{n}{d}}), (s, 1) \rangle : d \mid n \right\} \).

To classify how the nonabelian atom splits, we will calculate inner products with the irreducible characters of \( G \). The character table of \( G \) is described below:

| \( \lambda_j \) \( j \in \{1, 2, \ldots, n\} \) | \( \epsilon^{ij} \) | \( \epsilon^{ij} \) |
| \( \text{sgn} \lambda_j \) \( j \in \{1, 2, \ldots, n\} \) | \( \epsilon^{ij} \) | \( -\epsilon^{ij} \) |
| \( \lambda_j \chi_k \) \( j \in \{1, 2, \ldots, n\}, k \in \{1, 2, \ldots, \frac{p-1}{2} \} \) | \( \epsilon^{ij} (\zeta^{-hk} + \zeta^{-hk}) \) | 0 |

where \( \epsilon \) is a primitive \( n^{th} \) root of unity and \( \zeta \) is a primitive \( p^{th} \) root of unity.
The $\lambda$s come from the irreducible characters of $G/(D_{2p} \times \langle 1 \rangle) \cong C_n$ and the $\chi$s come from the non-linear irreducible characters of $G/(\langle 1 \rangle \times C_n) \cong D_{2p}$. The non-linear irreducible characters of $D_{2p}$ can be found by inducing irreducible characters of $C_p$. The character $sgn$ is the sign character that comes from the non-principal character of $G/(C_p \times C_n)$. We now prove that the nonabelian atom splits if and only if $2n$ is divisible by a square.

$(\Rightarrow)$ (Contrapositive) Assume $2n$ is square-free. This implies $n$ is odd and so

$\mathcal{N} = \{ \langle (r, x^n) \rangle, (s, 1) : d \mid n \}$. We want to show that $\mathcal{N}$ does not split.

Let $\emptyset \neq \mathcal{P} \subset \mathcal{N}$. Then $\mathcal{P} := \{ D_{2p} \times C_d : d \in D \}$ where $D$ is an arbitrary, nonempty set of divisors of $n$. We want to show that $\psi_{\mathcal{P}}$ is not a generalized character.

First, we calculate the Thompson counting function associated with all nonabelian subgroups of $D_{2p} \times C_d$ for a fixed $d \in D$. Let $\sigma_d := \sum_{d|d} \chi_{cl(D_{2p} \times C_d)}$. Then

<table>
<thead>
<tr>
<th>$\sigma_d(g)$</th>
<th>$g \in (1) \times C_n$</th>
<th>$g \in (C_p \setminus \langle 1 \rangle) \times C_d$</th>
<th>$g \in (D_{2p} \setminus C_p) \times C_d$</th>
<th>$g \in (D_{2p} \setminus \langle 1 \rangle) \times (C_n \setminus C_d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_d(g)$</td>
<td>0</td>
<td>$pd$</td>
<td>$2d(p - 1)$</td>
<td>0</td>
</tr>
</tbody>
</table>

Let $\theta_d := \psi_{cl(D_{2p} \times C_d)}$ be the Thompson counting function associated with the single conjugacy class, $cl(D_{2p} \times C_d)$. Since the collection of all subgroups of a group forms a partially ordered set under inclusion, we can use Möbius inversion to see $\theta_d$ is an integer linear combination of the $\sigma_{d'}$ functions. (Observe $\sigma_d = \sum_{d'|d} \theta_{d'}$. Applying Möbius inversion we get $\theta_d = \sum_{d'|d} \mu(d/d') \sigma_{d'}$ and it is known that $\mu$ is integer valued.)
For \( j \in \{1, 2, \ldots, n\} \) and \( k \in \{1, \ldots, \frac{p-1}{2}\} \),

\[
\begin{align*}
[\lambda_j \chi_k, \sigma_d] &= \frac{1}{2pn} \sum_{g \in (C_p \setminus \{1\}) \times C_d} \lambda_j \chi_k (g) \sigma_d (g) \\
&= \frac{d}{2n} \sum_{(r^h, x^{\frac{np}{2} i}) \in (C_p \setminus \{1\}) \times C_d} \lambda_j \chi_k \left((r^h, x^{\frac{np}{2} i})\right) \\
&= \frac{d}{2n} \left( \sum_{h=1}^{p-1} (\zeta^{hk} + \zeta^{-hk}) \right) \left( \sum_{i=1}^{d} (\varepsilon^{\frac{np}{2} j})^i \right) \\
&= \frac{d}{n} \left( \sum_{h=1}^{p-1} (\zeta^k)^h \right) \left( \sum_{i=1}^{d} (\varepsilon^{\frac{np}{2} j})^i \right).
\end{align*}
\]

Let \( \varepsilon_d := \varepsilon^{\frac{np}{2}} \). Then \( \varepsilon_d \) is a primitive \( d^{th} \) root of unity. Then, as \( j \in \{1, 2, \ldots, n\} \),

\[
\frac{d}{n} \left( \sum_{h=1}^{p-1} (\zeta^k)^h \right) \left( \sum_{i=1}^{d} (\varepsilon_d)^i \right) = \begin{cases} 
0 & \text{if } \text{GCD}(d, j) < d \\
\frac{d^2}{n} \sum_{h=1}^{p-1} (\zeta^k)^h & \text{if } \text{GCD}(d, j) = d.
\end{cases}
\]

Now \( k \in \{1, 2, \ldots, \frac{p-1}{2}\} \), and so \( \zeta^k \) is still a primitive \( p^{th} \) root of unity. Thus, we have:

\[
[\lambda_j \chi_k, \sigma_d] = \begin{cases} 
0 & \text{if } \text{GCD}(d, j) < d \\
-\frac{d^2}{n} & \text{if } \text{GCD}(d, j) = d.
\end{cases}
\]

Note: \( \text{GCD}(d, j) = d \) means that \( C_d \leq \ker(\lambda_j \chi_k) \) and \( \text{GCD}(d, j) < d \) means that \( C_d \not\subseteq \ker(\lambda_j \chi_k) \).

Now \( \theta_d = \sum_{d' | d} \mu \left(\frac{d}{d'}\right) \sigma_{d'} \), and so

\[
[\lambda_j \chi_k, \theta_d] = \sum_{d' | d} \mu \left(\frac{d}{d'}\right) [\lambda_j \chi_k, \sigma_{d'}] = -\frac{1}{n} \sum_{d' | \text{GCD}(d, j)} \mu \left(\frac{d}{d'}\right) d'^2.
\]
For any positive integer \( t \), define

\[
D_{1,t} = \{ d \in D : t \text{ divides } d \text{ and } \mu(d) = 1 \} \quad \text{and} \\
D_{-1,t} = \{ d \in D : t \text{ divides } d \text{ and } \mu(d) = -1 \} .
\]

Notice that \( D_{1,t} \) or \( D_{-1,t} \) may be empty. (For example if \( t \) does not divide any element of \( D \), then \( D_{1,t} \) and \( D_{-1,t} \) are both empty.)

Now, for any \( j \in \{1, 2, \ldots, n\} \),

\[
\left[ \lambda_j \chi_k, \psi_P \right] = \sum_{d \in D} \left[ \lambda_j \chi_k, \theta_d \right] = -\frac{1}{n} \sum_{d \in D} \sum_{d' | \gcd(d, j)} \mu\left( \frac{d}{d'} \right) d'^2
\]

\[
= -\frac{1}{n} \sum_{d' | j} \left( \sum_{d \in D_{1,d'}} \mu\left( \frac{d}{d'} \right) d'^2 + \sum_{d \in D_{-1,d'}} \mu\left( \frac{d}{d'} \right) d'^2 \right)
\]

\[
= -\frac{1}{n} \left( \sum_{d' | j, \mu(d') = 1} \left( \sum_{d \in D_{1,d'}} \mu\left( \frac{d}{d'} \right) d'^2 + \sum_{d \in D_{-1,d'}} \mu\left( \frac{d}{d'} \right) d'^2 \right) + \sum_{d' | j, \mu(d') = -1} \left( \sum_{d \in D_{1,d'}} \mu\left( \frac{d}{d'} \right) d'^2 + \sum_{d \in D_{-1,d'}} \mu\left( \frac{d}{d'} \right) d'^2 \right) \right)
\]

\[
= -\frac{1}{n} \left( \sum_{d' | j, \mu(d') = 1} d'^2 (|D_{1,d'}| - |D_{-1,d'}|) + \sum_{d' | j, \mu(d') = -1} d'^2 (|D_{-1,d'}| - |D_{1,d'}|) \right) .
\]

Assume to the contrary that \( \psi_P \) is a generalized character. If we let

\( \mathcal{P}^c := \{ D_{2n} \times C_n : d \mid n \text{ and } d \notin D \} \) which is the complement of \( \mathcal{P} \) in \( \mathcal{N} \), then Corollary 4.2 implies \( \psi_{\mathcal{P}^c} \) is also a generalized character as \( \mathcal{P} \cup \mathcal{P}^c \) is an admissible family. (Note that \( \mathcal{P} \neq \mathcal{N} \) and so \( \mathcal{P}^c \) is nonempty.) Now we may assume without loss of generality that \( n \notin D \), since if not we can replace \( \mathcal{P} \) with \( \mathcal{P}^c \).

We prove \(|D_{-1,j}| = |D_{1,j}| \) for all \( j \in \{1, 2, \ldots, n - 1\} \). Notice that if \( j \) does not divide
any $d \in D$ then $|D_{1,j}| = |D_{-1,j}| = 0$ and the result is true, so we may further assume $j$ divides some $d \in D$ so that $j$ is a proper divisor of $n$. For a fixed such $j$, let $\omega(j)$ denote the number of distinct prime factors of $j$. We proceed by induction on $\omega(j)$.

If $\omega(j) = 0$, then $j = 1$ and so

$$\left[\lambda_j \chi_k, \psi_p\right] = \left[\lambda_1 \chi_k, \psi_p\right] = \frac{1}{n} (|D_{-1,1}| - |D_{1,1}|).$$

As $|D_{1,1}| - |D_{-1,1}|$ is always less than $n$ for $n > 1$, we must have $|D_{-1,1}| = |D_{1,1}|$.

For the inductive step, assume $\omega(j) > 0$ and that $|D_{1,l}| = |D_{-1,l}|$ for all $l$ such that $\omega(l) < \omega(j)$. Then

$$\left[\lambda_j \chi_k, \psi_p\right] = -\frac{1}{n} \left( \sum_{d'|j, \mu(d') = 1} d'^2 (|D_{1,d'}| - |D_{-1,d'}|) + \sum_{d'|j, \mu(d') = -1} d'^2 (|D_{-1,d'}| - |D_{1,d'}|) \right)$$

$$= \begin{cases} 
-\frac{1}{n} \left( \sum_{d'|j, d' \neq j, \mu(d') = 1} d'^2 (|D_{1,d'}| - |D_{-1,d'}|) 
+ \sum_{d'|j, d' \neq j, \mu(d') = -1} d'^2 (|D_{-1,d'}| - |D_{1,d'}|) \right) & \text{if } \mu(j) = 1 \\
-\frac{1}{n} \left( \sum_{d'|j, d' \neq j, \mu(d') = 1} d'^2 (|D_{1,d'}| - |D_{-1,d'}|) 
+ \sum_{d'|j, d' \neq j, \mu(d') = -1} d'^2 (|D_{-1,d'}| - |D_{1,d'}|) \right) + j^2 (|D_{1,j}| - |D_{-1,j}|) & \text{if } \mu(j) = -1,
\end{cases}$$

and by the induction hypothesis we get

$$\left[\lambda_j \chi_k, \psi_p\right] = \begin{cases} 
-\frac{j^2(|D_{1,j}| - |D_{-1,j}|)}{n} & \text{if } \mu(j) = 1 \\
-\frac{j^2(|D_{-1,j}| - |D_{1,j}|)}{n} & \text{if } \mu(j) = -1.
\end{cases}$$
It follows that \([\lambda_j, \chi_k, \psi_p] \in \mathbb{Z}\) if and only if \(\frac{j(D_{1,j} - |D_{1,j}|)}{2} \in \mathbb{Z}\). Since \(j\) is a proper divisor of \(n\), then as the number of divisors of \(n\) that \(j\) divides is less than \(\frac{n}{j}\), we have \(|D_{1,j}| - |D_{1,j}| < \frac{n}{j}\). Since \(\text{GCD} \left(j, \frac{n}{j}\right) = 1\), we see that \(\frac{j(D_{1,j} - |D_{1,j}|)}{2} \in \mathbb{Z}\) if and only if \(|D_{1,j}| = |D_{1,j}|\). Thus, assuming \(\psi_p\) is a generalized character we get \(|D_{1,j}| = |D_{1,j}|\) for all \(j \in \{1, 2, \ldots, n - 1\}\) and in particular \(|D_{1,d}| = |D_{1,d}|\) for all \(d \in D\).

Now fix a maximal element \(d_0\) of \(D\) so that \(d_0 \not| d\) for all \(d_0 \neq d \in D\). Since \(n \notin D\), we know \(d_0 \in \{1, 2, \ldots, n - 1\}\) and so \(|D_{1,d_0}| = |D_{1,d_0}|\). On the other hand, the only element of \(D\) that \(d_0\) can divide itself since it is maximal. But then as \(\mu(d_0)\) cannot be equal to both 1 and \(-1\), we see \(|D_{1,d_0}| \neq |D_{1,d}|\). This is a contradiction, which implies that \(\psi_p\) is not a generalized character for any \(\emptyset \neq \mathcal{P} \subset \mathcal{N}\). In other words, \(\mathcal{N}\) does not split.

\((\Leftarrow)\) Assume \(2n\) is divisible by a square. We prove that

\[
\mathcal{N} = \{\langle (r, x^n), (s, 1) \rangle : d \mid n \} \cup \{\langle (r, 1), (s, x^n) \rangle : 2d \mid n \}\] splits in some way.

First suppose \(n\) is even and in particular \(n = 2^m \cdot m\), where \(m\) is odd. Consider \(\mathcal{P} = \{\langle (r, x^l), (s, 1) \rangle : l \mid m \} \subset \mathcal{N}\). In other words, \(\mathcal{P} = \{D_{2p} \times C_{2^l} : l \mid m \}\). We calculate the values of \(\text{Thompson counting function}\ \psi_p\).

Notice \(\psi_p((1, x^i)) = 0\) since \((1, x^i) \in Z(G)\) and every subgroup in \(\mathcal{P}\) is nonabelian.

Next, given an element \(g = (r, x^i)\), the only possible \(y \in G\) such that \(\langle g, y \rangle \in \mathcal{P}\) are those of the form \((s', x^k)\), where \(s' \in D_{2p}\backslash C_p\) and \(x^k\) is any element of \(C_n\). Otherwise, \(\langle g, y \rangle \leq C_p \times C_n\), which is an abelian group. If \(i\) is odd, then \(2^m) \mid \frac{2^m}{GCD(2^m, i)} = o(x^i)\) and so we have \(\langle g, y \rangle = \langle (r, x^i), (s', x^k) \rangle = D_{2p} \times C_{2^l}\) for some \(l\) dividing \(m\). That is, \(\langle g, y \rangle \in \mathcal{P}\). Since there are \(pn\) elements in \(G\) of the form \((s', x^k)\), we see that when \(i\) is odd, \(\psi_p((r, x^i)) = pm\).

On the other hand, if \(i\) is even we still need elements of the form \((s', x^k)\) if we have any hope of generating a group in \(\mathcal{P}\). However, when \(i\) is even, \(g \in (D_{2p} \times C_{2^l}) \cap (D_{2p} \times C_{2^l})\). Now \((s', x^k) \in D_{2p} \times C_{2^l}\) when \(k\) is even and \((s', x^k) \in D_{2p} \times C_{2^l}\) when \(k\) is odd. This implies \(\langle g, y \rangle \leq D_{2p} \times C_{2^l}\) or \(\langle g, y \rangle \leq D_{2p} \times C_{2^l}\). Hence, \(\text{Lagrange's Theorem implies} \ 2^{n+1} \mid \langle g, y \rangle\) and in particular \(\langle g, y \rangle \notin \mathcal{P}\). So if \(i\) is even, then \(\psi_p((r, x^i)) = 0\).

Next, suppose \(g = (s', x^i)\), where \(s' \in D_{2p}\backslash C_p\). We note \(\langle s', y_1 \rangle = D_{2p}\) if and only if
$y_1 \in D_{2p} \setminus \langle s' \rangle$, of which there are $2(p - 1)$ elements. Therefore we only consider $y \in G$ of the form $y = (y_1, x^k)$, where $x^k$ is any element of $C_n$. Now when $k$ is odd, $2^\alpha \mid o(x^k)$ and we have $\langle y, (s', x^k) \rangle = D_{2p} \times C_{2^\alpha l}$ for some $l$ dividing $m$. That is, $\langle y, (s', x^k) \rangle \in \mathcal{P}$. When $k$ is even, $y \in D_{2p} \times C_{2}^\alpha$ or $y \in D_{2p} \times C_{2}$. This means that for even $k$, $\langle y, (y_1, x^k) \rangle \leq D_{2p} \times C_{2}^\alpha$ or $\langle y, (y_1, x^k) \rangle \leq D_{2p} \times C_{2}$, and again Lagrange’s theorem implies $\langle y, (s', x^k) \rangle \notin \mathcal{P}$. Hence $\psi_\mathcal{P} ((s', x^k)) = \frac{2(p-1)n}{2} = (p - 1)n$.

Since Thompson class functions are constant on elements generating the same cyclic group, the values of $\psi_\mathcal{P}$ can be summarized as:

<table>
<thead>
<tr>
<th>$g$</th>
<th>$(1, x^i)$</th>
<th>$(r, x^i)$</th>
<th>$(r, x^i)$</th>
<th>$(s, x^i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$ is odd</td>
<td>$i$ is even</td>
<td>$0$</td>
<td>$pn$</td>
<td>$(p - 1)n$</td>
</tr>
</tbody>
</table>

Notice $|C_G ((r, x^i))| = pn$ and $|C_G ((s, x^i))| = 2n$, so that $|C_G (g)| \mid \psi_\mathcal{P} (g)$ for all $g \in G$. Thus Theorem 1.1 implies $\psi_\mathcal{P}$ is a generalized character.

It remains to show that $\mathcal{N}$ can be split for $n$ odd and divisible by a square. Suppose $n = q^\alpha \cdot m$, where $q$ is a prime, $\alpha \geq 2$, and $m$ is a positive integer. We show $\psi_\mathcal{P}$ is a generalized character for $\mathcal{P} := \{D_{2p} \times C_d : q^\alpha \mid d \} \subseteq \mathcal{N}$. We start by calculating the values of $\psi_\mathcal{P}$.

First, if $g \in (1) \times C_n$, then $\psi_\mathcal{P} (g) = 0$. Indeed, $g \in Z(G)$, which implies $\langle g, y \rangle$ is abelian for all $y \in G$, but every subgroup in $\mathcal{P}$ is nonabelian.

Next, if $g \in (C_p \setminus \{1\}) \times C_n$, say $g = (r^h, x^i)$, then the only possible $y \in G$ such that $\langle y, (s', x^k) \rangle \in \mathcal{P}$ are those of the form $\langle s', x^k \rangle$, where $s' \in D_{2p} \setminus C_p$ and $x^k \in C_n$. Otherwise $\langle g, y \rangle \leq C_p \times C_n$, which is an abelian group. Now if $x^i \in C_n \setminus C_{\frac{n}{2}}$, then $q^\alpha \mid o(x^i)$ and so Lagrange’s theorem implies $\langle g, (s', x^k) \rangle \in \mathcal{P}$ for any $x^k \in C_n$. Thus, $\psi_\mathcal{P} (g) = pn$ for $g \in (C_p \setminus \{1\}) \times \left(C_n \setminus C_{\frac{n}{2}}\right)$.

On the other hand, if $x^i \in C_{\frac{n}{2}}$, then we must make sure $x^k \in C_n \setminus C_{\frac{n}{2}}$. Otherwise $\langle g, y \rangle \leq D_{2p} \times C_{q^\alpha - 1}$, which implies $\langle g, y \rangle \notin \mathcal{P}$. It follows that $\psi_\mathcal{P} (g) = pn (|C_n \setminus C_{q^\alpha - 1}|) = pn \left(\frac{q-1}{q}\right)$.
for \( g \in (C_p \langle 1 \rangle) \times C_n^q. \)

Next, if \( g \in (D_{2p} \backslash C_p) \times C_n \), say \( g = (sr^{h}, x^i) \), then the only possible \( y \in G \) such that \( \langle g, y \rangle \in \mathcal{P} \) are those \( y \in (D_{2p} \backslash \langle sr^{h} \rangle) \times C_n \). Otherwise, \( \langle g, y \rangle \leq C_2 \times C_n \), which is abelian.

Now if \( x^i \in C_n \backslash C_n^q \), then \( \langle g, y \rangle \in \mathcal{P} \) for all \( y \in (D_{2p} \backslash \langle sr^{h} \rangle) \times C_n \) and so \( \psi_\mathcal{P}(g) = 2(p-1)n \) for \( g \in (D_{2p} \backslash C_p) \times (C_n \backslash C_n^q). \)

However, if \( \langle x^i \rangle \leq C_n^q \), then we must restrict our choice of \( y \) to \( y \in (D_{2p} \backslash \langle sr^{h} \rangle) \times (C_n \backslash C_n^{q^a-1}, m) \) and so \( \psi_\mathcal{P}(g) = 2(p-1)|C_n \backslash C_n^{q^a-1}, m| = 2(p-1)n \left( \frac{q-1}{q} \right) \) for \( g \in (D_{2p} \backslash C_p) \times C_n^q. \)

We summarize these calculations in the following table, whose column headings partition \( G \) into sets where \( \psi_\mathcal{P} \) have distinct values:

<table>
<thead>
<tr>
<th>( \psi_\mathcal{P}(g) )</th>
<th>( \langle 1 \rangle \times C_n )</th>
<th>( (C_p \backslash \langle 1 \rangle) \times (C_n \backslash C_n^q) )</th>
<th>( (C_p \backslash \langle 1 \rangle) \times C_n^q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \psi_\mathcal{P}(g) )</td>
<td>0</td>
<td>( pm )</td>
<td>( pm \left( \frac{q-1}{q} \right) )</td>
</tr>
</tbody>
</table>

For any subgroup \( H \) of \( G \), let \( \chi_H \) denote the characteristic function of \( H \). That is,

\[
\chi_H(g) = \begin{cases} 
1 & \text{if } g \in H \\
0 & \text{if } g \notin H.
\end{cases}
\]

Furthermore, if \( H \leq G \), then \( \chi_H = \frac{|H|}{|G|} (1_H)^G. \) Thus,

\[
\psi_\mathcal{P} = 0 \cdot \chi_{\langle 1 \rangle \times C_n} + pm \left( \chi_{C_p \times C_n} - \chi_{\langle 1 \rangle \times C_n} - \chi_{C_p \times C_n^q} + \chi_{\langle 1 \rangle \times C_n^q} \right) \\
+ pm \left( \frac{q-1}{q} \right) \left( \chi_{C_p \times C_n^q} - \chi_{\langle 1 \rangle \times C_n^q} \right) \\
+ 2n(p-1) \left( \chi_{(D_{2p} \backslash C_n)} - \chi_{C_p \times C_n} - \chi_{D_{2p} \times C_n^q} + \chi_{C_p \times C_n^q} \right) \\
+ 2(p-1)n \left( \frac{q-1}{q} \right) \left( \chi_{D_{2p} \times C_n^q} - \chi_{C_p \times C_n^q} \right).
\]
The derived subgroup of $G$ is $C_p \times \langle 1 \rangle$. Therefore, $C_p \times C_n$, $C_p \times C_\frac{n}{q}$, $D_{2p} \times C_n$, and $D_{2p} \times C_\frac{n}{q}$ are all normal subgroups of $G$ since they contain $G'$. Also, the subgroups $\langle 1 \rangle \times C_n$ and $\langle 1 \rangle \times C_\frac{n}{q}$ are normal as they are central subgroups of $G$. Hence we rewrite and simplify $\psi_p$ as

$$
\psi_p = \frac{pn}{2pn} \left( \frac{pn}{1C_p \times C_n} \right)^G - \frac{n}{2pn} \left( \frac{1(1) \times C_n}{1C_p \times C_n} \right)^G - \frac{pn}{2pnq} \left( \frac{1C_p \times C_\frac{n}{q}}{1C_p \times C_n} \right)^G + \frac{n}{2pnq} \left( \frac{1(1) \times C_\frac{n}{q}}{1C_p \times C_n} \right)^G
$$

which simplifies to

$$
\psi_p = c_1 \left( \frac{1C_p \times C_n}{1C_p \times C_n} \right)^G + c_2 \left( \frac{1(1) \times C_n}{1C_p \times C_n} \right)^G + c_3 \left( \frac{1C_p \times C_\frac{n}{q}}{1C_p \times C_n} \right)^G + c_4 \left( \frac{1(1) \times C_\frac{n}{q}}{1C_p \times C_n} \right)^G + c_5 \left( \frac{1D_{2p} \times C_\frac{n}{q}}{1C_p \times C_n} \right)^G + c_6 1_G,
$$

where $c_1 = -\frac{n(p-2)}{2}$, $c_2 = -\frac{n}{2}$, $c_3 = \frac{n(p-2)}{2q^2}$, $c_4 = \frac{n}{2q^2}$, $c_5 = -\frac{2n(p-1)}{q^2}$, and $c_6 = 2n(p-1)$.

Hence for any $\chi \in Irr(G)$, applying Frobenius reciprocity gives us:

$$
[\psi_p, \chi] = c_1 \left[ 1C_p \times C_n, \chi |_{C_p \times C_n} \right] + c_2 \left[ 1(1) \times C_n, \chi |_{(1) \times C_n} \right] + c_3 \left[ 1C_p \times C_\frac{n}{q}, \chi |_{C_p \times C_\frac{n}{q}} \right] + c_4 \left[ 1C_p \times C_\frac{n}{q}, \chi |_{(1) \times C_\frac{n}{q}} \right] + c_5 \left[ 1D_{2p} \times C_\frac{n}{q}, \chi |_{D_{2p} \times C_\frac{n}{q}} \right] + c_6 \left[ 1G, \chi \right].
$$

Now for any subgroup $H$ of $G$,

$$
[1_H, \chi |_H] = \begin{cases} 
\chi(1) & \text{if } H \not\leq \ker(\chi) \\
0 & \text{if } H \not\in \ker(\chi) 
\end{cases}
$$

Therefore, as $2c_l$ is an integer for all $c_l$, we know $[\psi_p, \lambda_j \chi_k] \in \mathbb{Z}$ for every $j \in \{1, 2, \ldots, n\}$.
and \( k \in \{1, 2, \ldots, \frac{n-1}{2}\} \). That takes care of the non-linear characters of \( G \).

For the linear characters, we have \( \ker (\lambda_j) = D_{2p} \times C_{GCD(j,n)} \) and \( \ker (\text{sgn}\lambda_j) = C_p \times C_{GCD(j,n)} \). The following lattice shows the inclusion of the subgroups that occur in the inner product:

\[
\begin{array}{c}
D_{2p} \times C_n \\
D_{2p} \times C_{\frac{n}{q}} \\
C_p \times C_{\frac{n}{q}} \\
(1) \times C_n \\
(1) \times C_{\frac{n}{q}}
\end{array}
\]

It follows that \( \psi_P \) is a generalized character if an only if the sums \( \sum_{l=1}^{6} c_l, \sum_{l=3}^{5} c_l, \sum_{l=1}^{4} c_l \), and \( c_2 + c_4 \) are integers. Keeping in mind that we are assuming \( q^2 \) divides \( n \) and that \( p \) is an odd prime, the check that these sums are integers is elementary.

Hence, \([\chi, \psi_P] \in \mathbb{Z}\) for all \( \chi \in \text{Irr}(G) \), so that \( \psi_P \) is a generalized character, implying \( \mathcal{N} \) splits into \( \mathcal{P} \) and \( \mathcal{N} \setminus \mathcal{P} \). \( \square \)

Although the proof of Theorem 4.14 is complete in the case where the atom splits, we have not split it as finely as it can possibly be. We look at an example of this next.

**Example 4.15:** Consider \( G = D_{2p} \times C_n \) where \( p \) is an odd prime and \( n = 2^\alpha \) for some integer \( \alpha \geq 1 \). Then \( \mathcal{N} = \{ (r, x^\frac{n}{2}) : d \mid n \} \cup \{ (r, 1), (s, x^\frac{n}{2}) : 2d \mid n \} \).

Let \( \sigma_\beta := \psi_P \), where \( \mathcal{P} = \{ H : H \) is a nonabelian subgroup of \( D_{2p} \times C_{2^\beta} \} \). That is, \( \sigma_\beta \) is the Thompson counting function for all nonabelian subgroups of a fixed nonabelian subgroup, \( D_{2p} \times C_{2^\beta} \), of \( G \). The values of \( \sigma_\beta \) are described below.

<table>
<thead>
<tr>
<th>( \sigma_\beta(g) )</th>
<th>( (1) \times C_n )</th>
<th>( (C_p \setminus (1)) \times C_{2^\beta} )</th>
<th>( (D_{2p} \setminus C_p) \times C_{2^\beta} )</th>
<th>( (D_{2p} \setminus (1)) \times (C_n \setminus C_{2^\beta}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>( p2^\beta )</td>
<td>( 2(p-1) \cdot 2^\beta )</td>
<td>( 0 )</td>
<td></td>
</tr>
</tbody>
</table>
We have \( \sigma_\beta = \frac{2^{2\beta(2-p)}}{2^{\alpha+1}} \left( 1_{C_p \times C_{2^\beta}} \right)^G - \frac{2^{2\beta}}{2^\alpha} \left( 1_{1 \times C_{2^\beta}} \right)^G + \frac{2^{2\beta+1}(p-1)}{2^\alpha} \left( 1_{D_{2p} \times C_{2^\beta}} \right)^G \). Now \( \ker(\lambda_j) = D_{2p} \times C_{GCD(n,j)} \), \( \ker(\text{sgn}\lambda_j) = C_p \times C_{GCD(n,j)} \), and as \( GCD(n,p) = 1 \), \( \ker(\lambda_jX_k) = C_p \times C_{GCD(n,j)} \). Thus we see that if \( \beta \) is not “too small,” then \( \sigma_\beta \) is a generalized character. It turns out that the degree 2 characters of \( G \) put the most restriction on what \( \beta \) can be to get a generalized character. Indeed, \( 2 \cdot \frac{2^{2\beta(2-p)}}{2^{\alpha+1}} - 2 \cdot \frac{2^{2\beta}}{2^\alpha} = \frac{2^{2\beta}(p-1)}{2^\alpha} \). If we let \( p - 1 = 2^\gamma \cdot q \), where \( q \) is odd, then we see \( \sigma_\beta \) is a generalized character if and only if \( \beta \geq \frac{\alpha - \gamma}{2} \).

In particular, for \( G = D_{2p} \times C_{2^\alpha} \), the proof of the proposition only shows that \( G \) can be split off from the rest of the nonabelian atom. What we have just shown is that for any \( \alpha - \gamma > 1 \), we can split the set of all proper subgroups of \( G \) even further. In fact, it is quite possibly true that the complement in \( N \) of all subgroups of \( D_{2p} \times C_{2^\beta} \) (for the smallest possible \( \beta \) that gives splitting) can be split into conjugacy classes, where the Thompson counting function associated to each conjugacy class is a generalized character.

We end this chapter noting that the reason we get these extra splittings is not just a rewording of Gagola’s work in [2]. (That is, replacing the assumption that \( P \) be admissible with \( xD(x) \subseteq S_P(g) \) for all \( x \in S_P(g) \) and \( P \) closed under conjugation in Lemma G does not explain these extra splittings.) As an example consider the atom \( \text{cl}(S_3) \cup \text{cl}(\tilde{S}_3) \cup \text{cl}(D_{12}) \) of \( S_5 \). We know this splits, by Proposition 4.13, into \( \text{cl}(S_3) \cup \text{cl}(\tilde{S}_3) \) and \( \text{cl}(D_{12}) \). Now consider \( \mathcal{P} = \text{cl}(D_{12}) \) which, while it is closed under conjugation, is not admissible since it fails condition (2) of Definition 1.4. Now consider \( g = (45) \) and let \( x = (12)(34) \). Observe \( gx = (12)(345) \) and \( \langle g, x \rangle = \langle g, gx \rangle \in \text{cl}(D_{12}) \). Let \( D(x) = C_{S_5}(N) \) where \( N = \langle g^y : y \in \langle x \rangle \rangle \). Then \( N = \langle (45), (35) \rangle \cong S_3 \) and \( D(x) = \langle (12) \rangle \). Therefore \( xD(x) = \{ x, (34) \} \). Notice, however, that \( (34) \notin S_P(g) \) as \( \langle g, (34) \rangle \cong S_3 \notin \text{cl}(D_{12}) = \mathcal{P} \).

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Appendix

**Theorem 1.1:** If $\psi$ is an integer-valued class function of a finite group $G$ such that $\psi(g) = \psi(g')$ whenever $\langle g \rangle = \langle g' \rangle$ and $|C_G(g)| | \psi(g)$ for all $g \in G$, then $\psi$ is in the character ring of $G$.

**Proof:** Let $\chi \in \text{Irr}(G)$ and let $\{g_1, g_2, \cdots, g_k\}$ be a complete set of conjugacy class representatives of $G$. As $\psi$ is integer valued, $\psi(g_i) = \psi(g_i)$ for all $i \in \{1, 2, \cdots, k\}$. Thus, the inner product $[\chi, \psi] = \sum_{i=1}^{k} \frac{\psi(g_i)}{|C_G(g_i)|} \chi(g_i)$. By assumption, $\frac{\psi(g_i)}{|C_G(g_i)|} \in \mathbb{Z}$, and Corollary 3.6 of [4] implies $\chi(g_i)$ is an algebraic integer for all $i$. Therefore $[\chi, \psi]$ is an algebraic integer as it is a sum of products of algebraic integers.

Let $X : G \to GL_n(\mathbb{C})$ be a representation that affords $\chi$, $\varepsilon$ a primitive $|G|^\text{th}$ root of unity, and let $\sigma \in \text{Gal}(\mathbb{Q}(\varepsilon)/\mathbb{Q})$ be given by $\sigma(\varepsilon) = \varepsilon^m$, where $\text{GCD}(m, |G|) = 1$. Then for any $g \in G$,

$$ (\chi(g))^\sigma = (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)^\sigma, $$

where $\varepsilon_i \in \langle \varepsilon \rangle$ for all $i$ (by Lemma 2.15 of [4]). Then

$$ (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)^\sigma = \varepsilon_1^\sigma + \varepsilon_2^\sigma + \cdots + \varepsilon_n^\sigma $$

$$ = \varepsilon_1^m + \varepsilon_2^m + \cdots + \varepsilon_n^m $$

$$ = \text{tr}(X(g)^m) $$

$$ = \text{tr}(X(g^m)) $$

$$ = \chi(g^m). $$

Now that we know how $\sigma$ acts on $\chi$ we are ready to see how $\sigma$ acts on $[\chi, \psi]$. Using our
assumption, we have \( \psi(g^m) = \psi(g) \) as \( \langle g \rangle = \langle g^m \rangle \), and so

\[
[x, \psi]^\sigma = \left( \frac{1}{|G|} \sum_{g \in G} \chi(g) \psi(g) \right)^\sigma
\]

\[
= \frac{1}{|G|} \sum_{g \in G} \chi(g^m) \psi(g)
\]

\[
= \frac{1}{|G|} \sum_{g \in G} \chi(g^m) \psi(g^m)
\]

\[
= \frac{1}{|G|} \sum_{g \in G} \chi(g) \psi(g)
\]

\[
= [x, \psi].
\]

We have just shown that \([x, \psi]\) is fixed by an arbitrary element of \( \text{Gal}(\mathbb{Q}(\varepsilon)/\mathbb{Q}) \), and so \([x, \psi] \in \mathbb{Q} \). Then the Rational Zeros Theorem from elementary algebra implies \([x, \psi] \in \mathbb{Z} \).

Since \( \psi = \sum_{\chi \in \text{Irr}(G)} [x, \psi] \chi \), we have that \( \psi \) is a generalized character. □

**Character Table of \( D_{2n} \) for even \( n > 2 \)**

Here we construct the character table of \( D_{2n} \), where \( n > 2 \) is even. Since \( n \) is even we know \( D_{2n}/D'_{2n} = \{ \langle r^2 \rangle , r \langle r^2 \rangle , s \langle r^2 \rangle , sr \langle r^2 \rangle \} \cong C_2 \times C_2 \) and the character table of \( C_2 \times C_2 \), is known:

<table>
<thead>
<tr>
<th>( C_2 \times C_2 )</th>
<th>( \langle r^2 \rangle )</th>
<th>( r \langle r^2 \rangle )</th>
<th>( s \langle r^2 \rangle )</th>
<th>( sr \langle r^2 \rangle )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( \lambda_3 )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( \lambda_4 )</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

This can be computed using orthogonality relations. From this we can define the four linear characters, \( \lambda_i \) of \( D_{2n} \) by the association \( \lambda_i(g) = \hat{\lambda}_i(g \langle r^2 \rangle) \) and we have:

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For the nonlinear characters we induce from the irreducible characters of \( h \).

The character table of a cyclic group is known to be a Vandermonde matrix:

<table>
<thead>
<tr>
<th>( D_{2n} )</th>
<th>1</th>
<th>( r^\pm 1 )</th>
<th>( r^\pm 2 )</th>
<th>( r^\pm 3 )</th>
<th>\ldots</th>
<th>( r_n )</th>
<th>( s )</th>
<th>( sr )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>\ldots</td>
<td>( 1 )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>\ldots</td>
<td>( 1 )</td>
<td>(-1)</td>
<td>(-1)</td>
</tr>
<tr>
<td>( \lambda_3 )</td>
<td>1</td>
<td>(-1)</td>
<td>1</td>
<td>(-1)</td>
<td>\ldots</td>
<td>( (-1)^{\frac{n}{2}} )</td>
<td>1</td>
<td>(-1)</td>
</tr>
<tr>
<td>( \lambda_4 )</td>
<td>1</td>
<td>(-1)</td>
<td>1</td>
<td>(-1)</td>
<td>\ldots</td>
<td>( (-1)^{\frac{n}{2}} )</td>
<td>(-1)</td>
<td>1</td>
</tr>
</tbody>
</table>

For the nonlinear characters we induce from the irreducible characters of \( \langle r \rangle \). The character table of a cyclic group is known to be a Vandermonde matrix:

| \( \langle r \rangle \) | 1 | \( r \) | \( r^2 \) | \( r^3 \) | \ldots | \( r^{(n-1)} \) |
|-----------------|----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( 1_{(r)} \) | 1 | 1 | 1 | 1 | \ldots | \( 1 \) |
| \( \widehat{\chi}_1 \) | 1 | \( \epsilon \) | \( \epsilon^2 \) | \( \epsilon^3 \) | \ldots | \( \epsilon^{(n-1)} \) |
| \( \widehat{\chi}_2 \) | 1 | \( \epsilon^2 \) | \( \epsilon^{2}\) | \( \epsilon^{2}\) | \ldots | \( \epsilon^{3(n-1)} \) |
| \( \widehat{\chi}_3 \) | 1 | \( \epsilon^3 \) | \( \epsilon^{3}\) | \( \epsilon^{3}\) | \ldots | \( \epsilon^{3(n-1)} \) |
| \( \vdots \) | \[ \vdots \] | \[ \vdots \] | \[ \vdots \] | \[ \vdots \] | \ldots | \[ \vdots \] |
| \( \widehat{\chi}_{(n-1)} \) | 1 | \( \epsilon^{(n-1)} \) | \( \epsilon^{(n-1)} \) | \( \epsilon^{(n-1)} \) | \ldots | \( \epsilon^{(n-1)(n-1)} \) |

where \( \epsilon \) is a primitive \( n^{th} \) root of unity.

For \( 1 \leq k \leq n - 1 \), we define \( \chi_k(g) := \widehat{\chi}_k^{D_{2n}}(g) \) and we see

\[
\chi_k(g) = \frac{1}{|\langle r \rangle|} \sum_{x \in D_{2n}} \widehat{\chi}_k^o(g^x)
= \widehat{\chi}_k^o(g) + \widehat{\chi}_k^o(g^s)
= \widehat{\chi}_k^o(g) + \widehat{\chi}_k^o(g^{-1})
= \begin{cases}
0 & \text{if } g \notin \langle r \rangle \\
\epsilon^{kj} + \epsilon^{-kj} & \text{if } g \in \langle r \rangle \text{ such that } g = r^j.
\end{cases}
\]

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Now $\chi_k(1) = \epsilon^{k0} + \epsilon^{-k0} = 1 + 1 = 2$ and hence the $\chi_k$s are nonlinear characters of $G$. Also note that the $\chi_k$s are distinct for $1 \leq k \leq \frac{n}{2}$ and $\chi_{\frac{n}{2}+l} = \chi_{\frac{n}{2}-l}$ for $\frac{n}{2} < l \leq n - 1$. Therefore we assume $1 \leq k \leq \frac{n}{2}$ and test for irreducibility by checking the inner product:

$$[\chi_k, \chi_k] = \frac{1}{|D_{2n}|} \sum_{x \in D_{2n}} \chi_k(x) \overline{\chi_k(x)}$$

$$= \frac{1}{2n} \sum_{j=0}^{n-1} \chi_k(r^j) \overline{\chi_k(r^j)}$$

$$= \frac{1}{2n} \left( \sum_{j=0}^{n-1} (\epsilon^{kj} + \epsilon^{-kj})^2 \right)$$

$$= \frac{1}{2n} \left( \sum_{j=0}^{n-1} (\epsilon^j)^{2k} + \sum_{j=0}^{n-1} 2 + \sum_{j=0}^{n-1} (\epsilon^j)^{n-2k} \right)$$

$$= \begin{cases} \frac{1}{2n} (n + 2n + n) & \text{if } k = \frac{n}{2} \\ \frac{1}{2n} (0 + 2n - 0) & \text{if } k \neq \frac{n}{2} \end{cases}$$

$$= \begin{cases} 2 & \text{if } k = \frac{n}{2} \\ 1 & \text{if } k \neq \frac{n}{2} \end{cases}.$$

Thus, $\chi_k$ is irreducible for $1 \leq k < \frac{n}{2}$ and we have the character table for $D_{2n}$ such that
$n$ is even:

<table>
<thead>
<tr>
<th>$D_{2n}$</th>
<th>1</th>
<th>$r^{\pm 1}$</th>
<th>$r^{\pm 2}$</th>
<th>$\ldots$</th>
<th>$r_{\frac{n}{2}}$</th>
<th>$s$</th>
<th>$sr$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\ldots$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\ldots$</td>
<td>1</td>
<td>$-1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>1</td>
<td>$-1$</td>
<td>1</td>
<td>$\ldots$</td>
<td>$(-1)^{\frac{n}{2}}$</td>
<td>1</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td>1</td>
<td>$-1$</td>
<td>1</td>
<td>$\ldots$</td>
<td>$(-1)^{\frac{n}{4}}$</td>
<td>$-1$</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>2</td>
<td>$\epsilon + \epsilon^{-1}$</td>
<td>$\epsilon^2 + \epsilon^{-2}$</td>
<td>$\ldots$</td>
<td>$\epsilon^{\frac{n}{2}} + \epsilon^{-\frac{n}{2}}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>2</td>
<td>$\epsilon^2 + \epsilon^{-2}$</td>
<td>$\epsilon^{2\cdot 2} + \epsilon^{-2\cdot 2}$</td>
<td>$\ldots$</td>
<td>$\epsilon^{2\cdot \frac{n}{2}} + \epsilon^{-2\cdot \frac{n}{2}}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\vdots$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\chi_{\frac{n}{2}-1}$</td>
<td>2</td>
<td>$\epsilon^{\frac{n}{2}-1} + \epsilon^{-\left(\frac{n}{2}-1\right)}$</td>
<td>$\epsilon^{\frac{n}{2}-1})^{2} + \epsilon^{-\left(\frac{n}{2}-1\right)^2}$</td>
<td>$\ldots$</td>
<td>$\epsilon^{\left(\frac{n}{2}-1\right)^{\frac{n}{2}}} + \epsilon^{-\left(\frac{n}{2}-1\right)^{\frac{n}{2}}}$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Bibliography


