Prime Character Degree Graphs of Solvable Groups
having Diameter Three

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by

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To my son, Thomas.
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Chapter 1

Introduction

In this paper, we study the character degrees of finite solvable groups by studying the prime character degree graph $\Delta(G)$. We denote the set of irreducible characters of a group $G$ as $\text{Irr}(G)$ and the set of character degrees as $\text{cd}(G) = \{\chi(1) \mid \chi \in \text{Irr}(G)\}$. The set of vertices of the prime character degree graph is the set of all primes $p$ where $p$ divides a character degree. We denote this set of vertices as $\rho(G)$. There is an edge between two primes $p$ and $q$ if the product $pq$ divides $a$ for some $a \in \text{cd}(G)$. The first two questions that arise when studying these graphs are:

1. Which graphs can occur as the prime character degree graph for a solvable group?

2. If a graph occurs as a prime character degree graph, what can be said about the structure of the groups that have this graph as their degree graph?

In this paper we give partial answers to both questions. We show that if $G$ is a solvable group with $\Delta(G)$ having diameter three, then $G$ has Fitting height 3. Further, $G$ has exactly one normal Sylow $p$-subgroup and we specify exactly where that prime $p$ must lie in the graph. Both of these improve the results found in [20]. Further, we significantly restrict the graphs that can occur in a comparable manor to Pálfy’s result
on disconnected graphs. His result says that if the smaller component of a disconnected graph has \( n \) vertices, then the larger component must have at least \( 2^n - 1 \) vertices.

We have restrictions on the prime character degree graph from the literature. A summary of known results can be found in [7]. Pálfy proved in [15], that for a solvable group \( G \), given any three primes in \( \rho(G) \), two of them must be adjacent in \( \Delta(G) \). We refer to this as Pálfy’s Condition. Also, Zhang proved in [18] that the graph with four vertices and diameter three is not the prime character degree graph for any solvable group \( G \). Lewis proved in [10] that the graphs with five vertices and diameter three do not occur as prime character degree graphs for any solvable group \( G \). In [9], Lewis proved that there is a solvable group \( G \) that has a prime character degree graph with six vertices and diameter three, and in [1], Dugan generalized that result to show that a family of groups has a prime character degree graph with diameter three. All of those examples have a normal Sylow \( p \)-subgroup and all of them have Fitting height 3.

First, we prove that the two graphs in Figure 1 do not exist as character degree graphs. In order to generalize the above results, we found it necessary to define a partition on the set of vertices. If we label the vertices \( q_1, q_2, q_3, q_4, q_5, q_6 \), as shown in the figure, we see the following: \( q_1 \) is adjacent to \( q_2 \), and \( q_1 \) is not adjacent to any other prime. The vertex \( q_3 \) is adjacent to \( q_2 \) and all of the other vertices, \( q_4, q_5, \) and \( q_6 \).

Notice that the three vertices \( q_4, q_5, q_6 \) form a complete subgraph and are adjacent to \( q_3 \). If we consider the other graph, we see that \( r_3 \) and \( r_4 \) are adjacent, and each are adjacent to \( r_2, r_5 \) and \( r_6 \). It makes sense to define a partition so that we can refer to an arbitrary vertex in the appropriate position.
Thus, to easily describe the graphs with diameter three that satisfy Pálfy’s Condition, we will partition the vertices into four non-empty and disjoint subsets $\rho_1 \cup \rho_2 \cup \rho_3 \cup \rho_4$. Because the graph has diameter three, we can find vertices $p_1$ and $p_4$ where the distance between them is three. Define $\rho_4$ to be the set of all vertices that are distance three from the vertex $p_1$ including the vertex $p_4$, and $\rho_3$ to be the set of all vertices that are distance two from the vertex $p_1$. Because no vertex in $\rho_3 \cup \rho_4$ is adjacent to the vertex $p_1$, the subgraph formed by the vertices in $\rho_3 \cup \rho_4$ is complete by Pálfy’s Condition. Then define $\rho_2$ to be the set of all vertices that are adjacent to the vertex $p_1$ and some vertex in $\rho_3$. Define $\rho_1$ to consist of $p_1$ and the vertices that are adjacent to $p_1$ that are not adjacent to anything in $\rho_3$. Notice that $\rho_1 \cup \rho_2$ also forms a complete subgraph as no vertex is adjacent to the vertex $p_4$. If necessary we interchange $p_1$ and $p_4$ so that $|\rho_1 \cup \rho_2| \leq |\rho_3 \cup \rho_4|$.

With this partition in mind, we are now able to state our main theorems.

**Theorem 1.** Let $G$ be a solvable group with prime character degree graph $\Delta(G)$ with diameter three. Then $G$ has a normal non-abelian Sylow $p$-subgroup for exactly one prime $p$ and $p \in \rho_3$. 
This theorem gives structural information about a solvable group that can have a prime character degree graph with diameter three. Theorem 1 has consequences that restrict the structure of the solvable group further. The following theorem is one of those consequences.

**Theorem 2.** Let $G$ be a solvable group with $\Delta(G)$ having diameter three. Then $G$ has Fitting height 3.

Theorem 1 has consequences beyond the structure of the group. We were able to prove an inequality that relates the number of vertices in $\rho_3 \cup \rho_4$ to the number of vertices in $\rho_1 \cup \rho_2$ in a similar way to Pálfy’s result for disconnected graphs. While this result does not give structure to the group, it severely limits the list of potential graphs that can occur as a prime character degree graph for a solvable group that has diameter three. Further, it shows that our partition is well defined.

**Theorem 3.** Let $G$ be a solvable group with $\Delta(G)$ having diameter three. If $n = |\rho_1 \cup \rho_2|$, then $|\rho_3 \cup \rho_4| \geq 2^n$. 
Chapter 2

Background

2.1 Group Theory

As character theory is a way to study groups, we start with the definition of a group. The standard reference is [3]. A group $G$ is a set with a binary operation that is associative, contains an identity, and all elements have an inverse. The operation, or product, is not necessarily commutative, but when it is we say the group is abelian. A subgroup $H$ is a subset of $G$ such that $H$ is also a group. The index of a subgroup, $|G : H|$ is $|G|/|H|$. This is nonstandard but appropriate for finite groups.

There are many subgroups of importance, arguably the most important is a normal subgroup. A subgroup $H$ is called normal if it is closed under conjugation, that is $gHg^{-1} = H^g = H$ for all $g \in G$ and we write $H \triangleleft G$. The normalizer of a subgroup $H$ is the set of all elements of $G$ such that $H^g = H$, which we write as $N_G(H)$. The normalizer of $H$ is the largest subgroup of $G$ in which $H$ is a normal subgroup. The centralizer in $G$ of a subset $X$ is the set of all elements of $G$ that commute with all elements of $X$, denoted $C_G(X)$ and the center of a group $Z(G) = C_G(G)$. A maximal subgroup of $G$ is a proper subgroup $M$ such that if $M \subseteq H \subseteq G$, then either $H = M$
or $H = G$. The intersection of all maximal subgroups is called the \textit{Frattini subgroup}, denoted $\Phi(G)$.

A \textit{homomorphism} is a mapping between two groups $\theta : G \rightarrow H$ that preserves products, that is, $\theta(ab) = \theta(a)\theta(b)$ for all $a, b \in G$. An \textit{isomorphism} is a homomorphism that is injective and surjective and an \textit{automorphism} is an isomorphism from $G$ onto itself. A subgroup $H$ is called \textit{characteristic} if it is fixed by all automorphisms.

The group $G$ is called a \textit{direct product} if $G$ has two normal subgroups $H$ and $K$, where $H \cap K = 1$, and every element of $G$ can be written as the product $hk$ where $h \in H$ and $k \in K$. Then we write $G = H \times K$. A group $G$ is called a \textit{semi-direct product} if $N$ is a normal subgroup of $G$, the factor group $G/N$ is isomorphic to a subgroup $H$ of $G$, and $N \cap H = 1$, then we write $G = N \cdot H$. We call $H$ a \textit{complement} of $N$. We define a Frobenius group as given in Huppert [2]. Let $H$ be a proper nontrivial subgroup of $G$ such that $H^g \cap H = 1$ for all $g \notin H$. In particular, $H = N_G(H)$. The subgroup $F = G \setminus \cup_{g \in G} (H - 1)^g$ is a normal subgroup of $G$ which we call the \textit{Frobenius kernel}. Then the semi-direct product $G = F \cdot H$ is a \textit{Frobenius group} and $H$ is a \textit{Frobenius complement}.

An operation that is used frequently is called a \textit{commutator}. For two elements $a$ and $b$ in the group $G$, we define $[a, b] = a^{-1}b^{-1}ab$. If $a$ and $b$ commute, then $[a, b] = 1$. We define the commutator of two subgroups, $[A, B]$ to be the group generated by the set of all commutators $[a, b]$ where $a \in A$ and $b \in B$. The \textit{derived subgroup} is $G'$, where $G' = [G, G]$. Two results we use without reference are the following:

\textbf{Lemma 2.1.} \textit{Let $G$ be a group and $A$ and $B$ subgroups of $G$, then:}
1. The subgroup $A$ centralizes $B$ in the group $G$ if and only if $[A, B] = 1$.

2. The subgroup $A$ is normalized by the subgroup $B$ if and only if $[A, B] \subseteq A$.

Another result that uses commutator subgroups that we use is called the Three Subgroups Lemma:

**Lemma 2.2.** [3, Lemma 8.27] Let $X, Y,$ and $Z$ be subgroups of $G$ and assume that $[[X, Y], Z] = 1$ and $[[Y, Z], X] = 1$. Then $[[Z, X], Y] = 1$.

A subgroup $P$ is called a $p$-group for some prime $p$ if $|P|$ is a power of $p$. An abelian $p$-group $P$ is called *elementary abelian* if every non-identity element of $P$ has order $p$. The center of a $p$-group is always nontrivial by Theorem 5.21 of [3].

A *chief series* is a series of normal subgroups $N_i$ where $1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = G$ where each factor group $N_{i+1}/N_i$ is a nontrivial minimal normal subgroup of $G/N_i$. A group $G$ is *solvable* precisely when the chief factors are elementary abelian by Corollary 8.1 of [3].

Sylow subgroups are very important subgroups in this body of research. A subgroup $S$ is called a *Sylow subgroup* if $S$ is a $p$-group for some prime $p$ and the index $|G : S|$ is not divisible by $p$. The following theorem is known as the Sylow Existence, Development, and Conjugacy theorems:

**Theorem 2.3.**

1. [3, Theorem 5.2] Let $G$ be a finite group of order $p^a m$ where $p$ is a prime and $p$ does not divide $m$. Then $G$ has a subgroup $H$ of order $p^a$. 
2. [3, Theorem 5.6] If $P$ is a $p$-subgroup of $G$ then there exists a Sylow $p$-subgroup $S$ with $P \subseteq S$.

3. [3, Theorem 5.7] The Sylow subgroups are conjugate, that is, if $S$ and $P$ are Sylow subgroups, then there exists an element $x \in G$ where $P^x = S$.

We are especially interested when the Sylow subgroups are normal subgroups of $G$.

**Theorem 2.4.** [3, Corollary 5.10] Let $S$ be a Sylow $p$-subgroup of $G$ where $p$ is a prime. The following are equivalent:

1. $S$ is a normal subgroup of $G$;
2. $S$ is the unique Sylow $p$-subgroup of $G$;
3. every $p$-subgroup of $G$ is contained in $S$;
4. $S$ is characteristic in $G$.

An example of a theorem based on Sylow subgroups is the Frattini argument:

**Lemma 2.5.** [3, Lemma 8.10] Let $N$ be a normal subgroup of $G$, where $N$ is finite, and let $P$ be a Sylow $p$-subgroup of $N$. Then $G = N_G(P)N$.

An analogue of a Sylow subgroup is a Hall subgroup. Let $\pi$ be a set of prime numbers. A group $H$ is called a $\pi$-group if all of the primes that divide $|H|$ lie in $\pi$. A Hall $\pi$-subgroup $H$ of $G$ is a $\pi$-group such that the index $|G : H|$ is not divisible by any prime in $\pi$. A group $H$ is called a $\pi'$-group if all of the primes that divide $|H|$ are not in $\pi$. When the group is solvable, Hall $\pi$-subgroups exist and they are conjugate.
The subgroup $O_{\pi}(G)$ is the largest normal $\pi$-subgroup of $G$ and the subgroup $O^\pi(G)$ is the smallest normal subgroup such that the factor group $G/O^\pi(G)$ is a $\pi$-group. The following theorem is known as the Hall-Higman Lemma 1.2.3.

**Theorem 2.6.** [5, Hall-Higman 1.2.3] Let $G$ be a solvable group and assume that $O_{\pi'}(G) = 1$. Then $C_G(O_{\pi}(G)) \subseteq O_{\pi}(G)$.

### 2.2 Character Theory

The standard reference for character theory is [4]. First, we need some information about matrices and groups of matrices. The set of complex numbers is denoted by $\mathbb{C}$. The *general linear group*, denoted $\text{GL}(n, \mathbb{C})$, is the group of all nonsingular (i.e., invertible) $n \times n$ matrices with entries from $\mathbb{C}$. The *trace* of a matrix $A \in \text{GL}(n, \mathbb{C})$ is the sum of diagonal entries of $A$ denoted $\text{tr}(A)$.

Let $G$ be a group and $\mathbb{C}$ the field of complex numbers. A *representation* $\mathcal{X}$ of $G$ is a homomorphism $\mathcal{X} : G \to \text{GL}(n, \mathbb{C})$ for some integer $n$. The *character* $\chi$ of $G$ afforded by $\mathcal{X}$ is the function given by $\chi(g) = \text{tr}(\mathcal{X}(g))$. Characters are constant on conjugacy classes and so all characters are *class functions*. A representation is *irreducible* when it has no nontrivial proper sub-representations and a character is *irreducible* precisely when the representation is irreducible. We define $\text{Irr}(G)$ to be the set of irreducible characters of the group $G$ and $|\text{Irr}(G)|$ is finite by Corollary 2.7 of [4]. In fact, $\text{Irr}(G)$
forms a basis for the set of class functions. If

\[ \chi = \sum_{i=1}^{k} n_i \chi_i \]

then the \( \chi_i \) with \( n_i > 0 \) are called the irreducible constituents of \( \chi \). The inner product \([\chi, \phi]\), where \( \chi \) and \( \phi \) are class functions, is defined to be

\[ \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\phi(g)} \]

and satisfies all the usual properties which makes the set of class functions a finite dimensional Hilbert space, [4]. The degree of the character is \( \chi(1) \) which is the dimension of \( \mathcal{X} \) and is a positive integer.

Define \( \text{cd}(G) \) to be the set of all character degrees, which is \( \{ \chi(1) \mid \chi \in \text{Irr}(G) \} \). As there is only a finite number of irreducible characters, \(|\text{cd}(G)|\) is also finite. Characters are generally not homomorphisms; however, if \( \lambda \) is a character with degree 1, then \( \lambda \) is a homomorphism and we call \( \lambda \) a linear character. All groups have at least one linear character called the principal character, and so 1 is always a character degree. In an abelian group, all of the characters are linear. If we know the characters, then we know all of the normal subgroups without knowing the group itself. Also, groups can have the same set of characters without being isomorphic. As we have thrown away information studying characters instead of representations, we have thrown away even more information by studying character degrees. Many problems are easier to solve
with less information.

A very important topic in character theory is called Clifford theory. Let \( \chi \in \text{Irr}(G) \) and \( N \) a normal subgroup of \( G \). We can always restrict the character \( \chi \) to the subgroup \( N \), but since \( N \) is normal, we have more information:

**Theorem 2.7.** [4, Theorem 6.2] Let \( N \) be a normal subgroup of \( G \) and let \( \chi \in \text{Irr}(G) \).

Let \( \theta \) be an irreducible constituent of \( \chi_N \) and suppose \( \theta = \theta_1, \theta_2, \ldots, \theta_t \) are the distinct conjugates of \( \theta \) in \( G \). Then

\[
\chi_N = e \sum_{i=1}^{t} \theta_i,
\]

where \( e \) is the inner product \([\chi_N, \theta]\).

In this case, we say that \( \theta \) extends if \( \chi_N = \theta \) and we say that \( \theta \) induces irreducibly if \( \theta^G = \chi \). If the character \( \chi \) is the unique irreducible constituent of \( \theta^G \) and \( \theta \) is invariant in \( G \) we say that \( \chi \) and \( \theta \) are fully ramified. The group \( G \) acts on the set of irreducible characters of a subgroup \( H \) by conjugation, ie \( \theta^g \) where \( \theta \in \text{Irr}(H) \) and \( g \in G \). The stabilizer of the character \( \theta \) is the subgroup \( G_\theta = \{ g \in G \mid \theta^g = \theta \} \). If \( G_\theta \) is \( G \), then we say that \( \theta \) is invariant in \( G \). The following theorem gives a sufficient condition for when \( \theta \) extends to a character \( \chi \in \text{Irr}(G) \).

**Theorem 2.8.** [4, Corollary 6.28] Let \( N \) be normal in \( G \) with \( G/N \) solvable and suppose \( \theta \in \text{Irr}(N) \) is invariant in \( G \). If \((|G : N|, |N|) = 1\), then \( \theta \) has an extension, \( \chi \in \text{Irr}(G) \).

Gallagher’s theorem, Theorem 2.9, gives us all of the distinct constituents of \( \theta^G \) when we know \( \theta \) extends to \( G \):
Theorem 2.9. [4, Corollary 6.17] Let $N$ be normal in $G$ and let $\chi \in \text{Irr}(G)$ be such that $\chi_N = \theta$, where $\theta$ is an irreducible character of $N$. Then the characters $\beta\theta$, where $\beta$ is an irreducible character of $G/N$, are irreducible, distinct for distinct $\beta$, and are all of the irreducible constituents of $\theta^G$.

Itô’s theorems give us information about the character degrees:

Theorem 2.10. [4, Theorem 6.15] Let $A \triangleleft G$ be abelian. Then $\chi(1)$ divides $|G : A|$ for all $\chi \in \text{Irr}(G)$.

Theorem 2.11. [4, Corollary 12.34] Let $G$ be solvable. Then $G$ has a normal abelian Sylow $p$-subgroup if and only if every element of $\text{cd}(G)$ is relatively prime to $p$.

We introduce some notation. Let $N$ be a normal subgroup of $G$ and fix a character $\theta \in \text{Irr}(N)$. We define $\text{Irr}(G \mid \theta)$ to be the set of irreducible constituents of $\theta^G$ and $\text{Irr}(G \mid N)$ to be the union of the sets $\text{cd}(G \mid \theta)$ as $\theta$ runs over the non-principal characters of $\text{Irr}(N)$. Similarly, we define $\text{cd}(G \mid \theta) = \{\chi(1) \mid \chi \in \text{Irr}(G \mid \theta)\}$. We set $\text{cd}(G \mid N)$ to be the union of the sets $\text{cd}(G \mid \theta)$ as $\theta$ runs over the non-principal characters of $\text{Irr}(N)$.

The Fitting subgroup $\mathbb{F}(G)$ is the largest normal nilpotent subgroup of $G$. Define the characteristic subgroups $\mathbb{F}_i(G)$ iteratively by letting $\mathbb{F}_0(G) = 1$ and

$$\frac{\mathbb{F}_{i+1}(G)}{\mathbb{F}_i(G)} = \mathbb{F}(G/\mathbb{F}_i(G))$$

Then, $\mathbb{F}_1(G) = \mathbb{F}(G)$ and the smallest $n$ for which $\mathbb{F}_n(G) = G$ is the Fitting height.
which we denote $n(G)$. The index of the Fitting subgroup will give us information on at least one character degree.

**Theorem 2.12.** [14, Lemma 18.1] Suppose that $G/\mathbb{F}(G)$ is abelian. Then there is an irreducible character $\chi \in \text{Irr}(G)$ such that $\chi(1) = |G : \mathbb{F}(G)|$.

We can use this theorem to give us character degrees even when $G/\mathbb{F}(G)$ is not abelian. Let $F = \mathbb{F}(G)$ and $E = \mathbb{F}_2(G)$, that is, $E/F = \mathbb{F}(G/F)$. Then define $Y/F = \mathbb{Z}(E/F)$. By Theorem III.4.2 of [2], we know that $C_G(\mathbb{F}(G)) \subseteq \mathbb{F}(G)$ and so $Y/F$ is nontrivial. Because the center of a $p$-group is nontrivial and the Fitting subgroup is the direct product of the subgroups $O_p(G)$, every prime that divides $|E/F|$ also divides $|Y/F|$. There exists an $\eta \in \text{Irr}(Y)$ with $\eta(1) = |Y : F|$ by the theorem. If $\tau \in \text{Irr}(E | \eta)$, then $\tau(1)$ is divisible by every prime divisor of $|E/F|$. In particular, if $G = HP$, where $P$ is a normal non-abelian Sylow subgroup of $G$, $\mathbb{F}(G) = P$, and $H$ is nilpotent, then there is a character degree in $\text{cd}(G)$ that is divisible by every prime divisor of $|H|$.

The following three results are quoted regularly in our work.

**Theorem 2.13.** [4, Corollary 11.29] Let $N$ be normal in $G$ and $\chi \in \text{Irr}(G)$. Let $\theta \in \text{Irr}(N)$ be a constituent of $\chi_N$. Then $\chi(1)/\theta(1)$ divides the index $|G : N|$.

**Theorem 2.14.** [14, Theorem 12.9] Suppose that $N \triangleleft G$, $\theta \in \text{Irr}(N)$, and $\chi(1)/\theta(1)$ is a $\pi'$-number for all $\chi \in \text{Irr}(G | \theta)$ and a set $\pi$ of primes. Assume that $G/N$ is solvable. Then $G/N$ has an abelian Hall $\pi$-subgroup.
Let $G$ be a solvable group and $\text{cd}(G)$ the set of character degrees of $G$. Define $\rho(G)$ to be the set of all primes $p$ that divide a character degree $a$ in $\text{cd}(G)$.

**Lemma 2.15.** [10] Suppose that $G$ has a non-abelian normal Sylow $p$-subgroup $P$ for a prime $p$. Then $\rho(G/P') = \rho(G) \backslash \{p\}$. 
Chapter 3

Background on Graphs

3.1 Introduction

Much of the background material on character degree graphs can be found in the expository paper [7]. Background on graphs can be found in any undergraduate graph theory or combinatorics text such as [17]. A graph $G$ is the ordered pair $(V, E)$ where $V$ is the set of vertices and $E$ is the set of edges. The set of vertices is a finite set of objects and the set of edges denotes adjacency between two distinct objects. Two vertices $a$ and $b$ are in the same connected component if there is a finite path of edges between $a$ and $b$. Let $n(G)$ denote the number of connected components. The distance, $d(a, b)$, is defined to be the shortest path between $a$ and $b$ within the same connected component. If $a$ and $b$ lie in different components, then the distance is not defined. The diameter, $diam(G)$, is the largest distance among the connected components. This is not the standard definition of diameter, but is consistent in the literature on the graphs we are studying. A graph is called complete if there is an edge between every pair of distinct vertices.

Let $x$ be an integer and $\pi(x)$ the set of prime divisors of $x$. If $\mathcal{X}$ is a nonempty set
of integers, $\rho(\mathcal{X})$ is the union of $\pi(x)$ for all $x \in \mathcal{X}$. A prime divisor graph $\Delta(\mathcal{X})$ is a graph where the vertices are the set of primes $\rho(\mathcal{X})$ and the edges are the pairs $\{p, q\}$ where the product $pq$ divides an integer $a \in \mathcal{X}$. If $\mathcal{Y}$ is a subset of $\mathcal{X}$, then $\Delta(\mathcal{Y})$ is a subgraph of $\Delta(\mathcal{X})$.

We study the prime character degree graph, where $\mathcal{X} = \text{cd}(G)$ and $G$ is a solvable group. The vertices are $\rho(G)$, where $\rho(G)$ is defined to be the set of all primes $p$ such that $p$ divides a character degree. We write $\Delta(G)$ for the prime character degree graph for a solvable group $G$. Because $G$ is a solvable group, there are many restrictions on $\Delta(G)$. Most notable is referred to as Pálfy’s Condition.

**Theorem 3.1.** [15] For a solvable group $G$, given any three primes in $\rho(G)$, two of them must be adjacent in $\Delta(G)$.

The following corollary follows immediately from Pálfy’s Condition, however, parts 1 and 2 were known previously.

**Corollary 3.2.** Let $G$ be a solvable group and $\Delta(G)$ the prime character degree graph. The following hold.

1. [13] The diameter of $\Delta(G)$ is at most three.

2. [12] The graph $\Delta(G)$ has at most 2 connected components.

3. If $\Delta(G)$ is disconnected, then both components are complete.

The following lemma from [11] was originally used to show that the pentagon is not the prime character degree graph for any solvable group $G$. 
Lemma 3.3. [11, Lemma 3.1] Let $G$ be a solvable group and suppose $p$ is a prime in $\rho(G)$. For every proper normal subgroup $H$ of $G$, suppose that $\Delta(H)$ is a proper subgraph of $\Delta(G)$. Assume that no subgraph $\Gamma$ of $\Delta(G)$ obtained by removing the vertex $p$ and all edges incident to $p$, or by removing one or more of the edges incident to $p$, is $\Delta(H)$ for any normal subgroup $H$ of $G$. Then $G = O^p(G)$.

3.2 Disconnected Graphs

Disconnected graphs have been studied extensively and the groups that have a disconnected prime character degree graph were classified. Disconnected graphs occur often in our work and the notation contained in the following theorem will be consistent in this paper.

Theorem 3.4. [8] Let $G$ be a solvable group where $\Delta(G)$ has two connected components. Then $G$ is one of the following examples:

2.1 $G$ has a normal non-abelian Sylow $p$-subgroup $P$ and an abelian $p$-complement $K$ for some prime $p$. The subgroup $P' \subseteq C_P(K)$ and every non-linear irreducible character of $P$ is fully ramified with respect to $P/C_P(K)$.

2.2 $G$ is the semi-direct product of a subgroup $H$ acting on a subgroup $P$, where $P$ is elementary abelian of order $9$ and $\text{cd}(H) = \{1, 2, 3\}$. Let $Z = C_H(P)$. We have $Z \subseteq Z(H)$, and $H/Z \cong \text{SL}_2(3)$, where the action of $H$ on $P$ is the natural action of $\text{SL}_2(3)$ on $P$.

2.3 $G$ is the semi-direct product of a subgroup $H$ acting on a subgroup $P$, where $P$
is elementary abelian of order $9$ and $cd(H) = \{1,2,3,4\}$. Let $Z = C_H(P)$. We have $Z \subseteq Z(H)$, and $H/Z \cong \text{GL}_2(3)$, where the action of $H$ on $P$ is the natural action of $\text{GL}_2(3)$ on $P$.

2.4 $G$ is the semi-direct product of a subgroup $H$ acting on an elementary abelian $p$-group $V$ for some prime $p$. Let $Z = C_H(V)$ and $K$ the Fitting subgroup of $H$. Write $m = |H : K| > 1$, and $|V| = q^m$, where $q$ is a $p$-power. We have $Z \subseteq Z(H)$, $K/Z$ is abelian, $K$ acts irreducibly on $V$, $m$ and $|K : Z|$ are relatively prime, and $(q^m - 1)/(q - 1)$ divides $|K : Z|$.

2.5 $G$ has a normal non-abelian $2$-subgroup $Q$, and an abelian $2$-complement $K$ with the property that $|G : KQ| = 2$ and $G/Q$ is not abelian. Let $Z = C_K(Q)$, and $C = C_Q(K)$. The subgroup $Q' \subseteq C$, and $Z$ is central in $G$. Every non-linear irreducible character of $Q$ is fully ramified with respect to $Q/C$. Furthermore, $Q/C$ is an elementary abelian $2$-group of order $2^{2a}$ for some positive integer $a$, $Q/C$ is irreducible under the action of $K$, and $K/Z$ is abelian of order $2^{a+1}$.

2.6 $G$ is the semi-direct product of an abelian group $D$ acting coprimely on a group $T$ so that $[T, D]$ is a Frobenius group. The Frobenius kernel is $A = T' = [T, D]'$, $A$ is a non-abelian $p$-group for some prime $p$, and a Frobenius complement is $B$ with $[B, D] \subseteq B$. Every character in $\text{Irr}(T \mid A')$ is invariant under the action of $D$ and $A/A'$ is irreducible under the action of $B$. If $m = |D : C_D(A)|$, then $|A : A'| = q^m$ where $q$ is a $p$-power, and $(q^m - 1)/(q - 1)$ divides $|B|$.

The following lemmas from [8] give specific properties of the examples, specifically
information about the connected components of $\Delta(G)$ and the character degrees.

**Lemma 3.5.** Let $G$ be as in Example 2.1, and write $F$ for the Fitting subgroup of $G$.

Then the following are true:

1. $F = P \times (F \cap K)$ and $F \cap K$ is central in $G$;

2. $\text{cd}(G \mid P') = \text{cd}(P) \setminus \{1\}$;

3. $\text{cd}(G/P')$ consists of $1$, $|G : F|$, and possibly other divisors of $|G : F|$;

4. $\Delta(G)$ has two connected components, $\{p\}$ and $\pi(|G : F|)$.

It is important to note that the smaller component of this graph has exactly one vertex $p$, and the group has a normal non-abelian Sylow $p$-subgroup. Further, the group has an abelian $p$-complement.

**Lemma 3.6.** Let $G$ be as in Example 2.2. Then the following are true:

1. $P \times Z$ is the Fitting subgroup of $G$;

2. $\text{cd}(G) = \{1, 2, 3, 8\}$;

3. $\Delta(G)$ has two connected components, $\{2\}$ and $\{3\}$.

**Lemma 3.7.** Let $G$ be as in Example 2.3. Then the following are true:

1. $P \times Z$ is the Fitting subgroup of $G$;

2. $\text{cd}(G) = \{1, 2, 3, 4, 8, 16\}$;
3. \(\Delta(G)\) has two connected components, \(\{2\}\), and \(\{3\}\).

The groups in Example 2.2 and 2.3 do not have any normal non-abelian Sylow subgroups and both of the components have size one. However, whenever the group we are considering has a disconnected graph \(\Delta(G)\) with two components each of size one, the group \(G\) also has a normal non-abelian Sylow \(p\)-subgroup for exactly one prime \(p \in \rho(G)\). In this case, \(G\) is Example 2.1 from above and Examples 2.2 and 2.3 do not occur.

**Lemma 3.8.** Let \(G\) be as in Example 2.4. Write \(F\) and \(E/F\) for the Fitting subgroups of \(G\) and \(G/F\) respectively. Then the following are true:

1. \(V = [E, F] = E'\) and \(Z = Z(G)\);

2. \(F = V \times Z\) and \(E = K\);

3. \(\text{cd}(G) = \text{cd}(G/Z)\) and \(E/Z\) is a Frobenius group with kernel \(F/Z\);

4. \(G/E\) and \(E/F\) are cyclic groups;

5. \(\text{cd}(G/V)\) consists of divisors of \(m\) including \(1\) and \(m\), and

\[
\text{cd}(G \mid V) = \{|E : F|\};
\]

6. \(\Delta(G)\) has two connected components, \(\pi(m)\) and \(\pi(|E : F|)\).

In our research, whenever we have a group with no normal non-abelian Sylow subgroups and a disconnected prime character degree graph, the group is Example 2.4. If
we know that the group has an abelian Fitting subgroup and at least one component in the graph $\Delta(G)$ is larger than one, then the group is Example 2.4.

**Lemma 3.9.** Let $G$ be as in Example 2.5. Write $F$ and $E/F$ for the Fitting subgroups of $G$ and $G/F$ respectively. Then the following are true:

1. $F = Q \times Z$ and $E = QK$;

2. $E$ satisfies the hypotheses of Example 2.1, and $G/Q'$ satisfies the hypotheses of Example 2.4;

3. $\text{cd} \left( G/[Q,K] \right) = \{ 1, 2 \}$ and $\text{cd} \left( G/Q' \mid [Q,K]/Q' \right) = \{ 2^a + 1 \}$;

4. $\text{cd}(G \mid Q')$ consists of powers of 2 that are divisible by $2^a$;

5. $\Delta(G)$ has two connected components, $\{ 2 \}$ and $\pi(2^a + 1)$.

Example 2.5 does not generally occur in our research. It has no normal non-abelian Sylow subgroups and the prime in the smaller component is specifically 2. While this situation could be possible, which we will address, generally the group in question has a normal non-abelian Sylow subgroup which excludes this case from consideration.

**Lemma 3.10.** Let $G$ be as in Example 2.6. Write $F$ and $E/F$ for the Fitting subgroups of $G$ and $G/F$ respectively. Then the following are true:

1. $G$ has a normal Sylow $p$-subgroup $P$ and a $p$-complement $Q$ so that $[Q,D] \subseteq Q$;

2. $F = P \times Z$ where $Z = C_Q(P) \times C_D(P)$ is central in $G$;
3. $A = [P, Q]$ and $P' = A'$;

4. $F/A'$ is the Fitting subgroup of $G/A'$;

5. $G/A'$ satisfies the hypotheses of Example 2.4;

6. $\operatorname{cd}(G | A')$ consists of degrees that divide $|P||E : F|$ and are divisible by $p|B|$;

7. $\Delta(G)$ has two connected components, $\pi(|E : F|) \cup \{p\}$ and $\pi(m)$.

If the graph we are considering has both components larger than one and it has a normal non-abelian Sylow subgroup, then it has to be Example 2.6. If the group has a normal Sylow $p$-subgroup for some prime $p$ and the prime $p$ occurs in the larger component, then it must be Example 2.6.

The following theorem applies when the group in question is Example 2.6 from Theorem 3.4.


Counting arguments on the sets of vertices are often used to prove results about these graphs. The first one we describe is the Zsigmondy Prime Theorem. Let $q$ and $n$ be positive integers. A prime $p$ is called a Zsigmondy prime divisor for $q^n - 1$ if $p$ divides $q^n - 1$ and $p$ does not divide $q^j - 1$ for $1 \leq j < n$.

**Theorem 3.12.** [17] There is a Zsigmondy prime for $q^n - 1$ unless either $n = 2$ and $q = 2^k - 1$ for some integer $k$, or $n = 6$ and $q = 2$. 
The following lemma shows how we use the Zsigmondy Prime Theorem. In the proof of this lemma in [8], we see that the two exceptions do not occur.

**Lemma 3.13.** [8, Lemma 5.1] Let \( m \) be a positive integer, and let \( q \) be a prime power such that \( (q^m - 1)/(q - 1) \) is relatively prime to \( m \). If \( r \) is the number of distinct prime divisors of \( m \), then the quotient \( (q^m - 1)/(q - 1) \) has at least \( 2^r - 1 \) distinct prime divisors.

The second counting argument we use is a lower bound on the number of primes in the larger component for a disconnected graph. Pálfy proved in [16] that there is a difference in the number of vertices in each component. We call this Pálfy’s Inequality.

**Theorem 3.14.** [16] Let \( G \) be a solvable group and \( \Delta(G) \) the prime character degree graph. Suppose \( \Delta(G) \) is disconnected with two components having size \( a \) and \( b \), where \( a \leq b \). Then \( b \geq 2^a - 1 \).

In Examples 2.4 and 2.6 from Theorem 3.4, let \( F = \mathbb{F}(G) \), \( E/F = \mathbb{F}(G/F) \), and \( m = |G : E| \). The prime divisors of \( m \) are precisely the primes in the smaller component of the character degree graph. Let \( r \) be the number of distinct primes that divide \( m \). We know from Pálfy’s Inequality that we must have at least \( 2^r - 1 \) primes in the larger component. By Theorem 5.4 in [8], we know that if the larger component has exactly \( 2^r - 1 \) primes, then \( F \) is abelian and in particular, we are in Example 2.4. When there are more than \( 2^r - 1 \) primes in the larger component, we may have to distinguish between Examples 2.4 and 2.6. Of course, if the group has a normal non-abelian Sylow \( p \)-subgroup \( P \), then we are in Example 2.6. In this case, \( G/P' \) will satisfy Example 2.4.
by Lemma 3 in [8], and so the larger component in $\Delta(G)$ must have at least $2^r$ primes including $p$.

Whenever $G$ is Example 2.4 or 2.6 from Theorem 3.4, the following theorem applies:

**Theorem 3.15.** [8, Theorem 5.3] Let $G$ be a solvable group, let $F$ be the Fitting subgroup of $G$, and suppose that $\Delta(G)$ has two connected components with sizes $a$ and $b$. If $b \geq a \geq 2$, then $G/F$ is not nilpotent and in particular, the Fitting height of $G$ is 3. Also, $O^p(G) < G$ for every prime $p$ in the smaller component.

### 3.3 Diameter Three Graphs

To easily describe the graphs with diameter three that satisfy Pálfy’s Condition, we will partition the set of vertices into four non-empty and disjoint subsets $\rho_1 \cup \rho_2 \cup \rho_3 \cup \rho_4$. Because the graph has diameter three, we can find vertices $p_1$ and $p_4$ where the distance between them is three.

**Definition 3.16.** Let $\Gamma$ be a diameter three graph that satisfies Pálfy’s condition. Fix vertices $p_1$ and $p_4$ where the distance between $p_1$ and $p_4$ is three. Define $\rho_4$ to be the set of all vertices that are distance three from the vertex $p_1$, including the vertex $p_4$, and $\rho_3$ to be the set of all vertices that are distance two from the vertex $p_1$. Because no vertex in $\rho_3 \cup \rho_4$ is adjacent to the vertex $p_1$, the subgraph formed by the vertices in $\rho_3 \cup \rho_4$ is complete by Pálfy’s Condition. Define $\rho_2$ to be the set of all vertices that are adjacent to $p_1$ and adjacent to some prime in $\rho_3$. Define $\rho_1$ to be the vertices consisting of $p_1$ and those that are adjacent to $p_1$ and not adjacent to anything in $\rho_3$. Notice that $\rho_1 \cup \rho_2$
also forms a complete subgraph as no vertex is adjacent to the vertex $p_4$. Because the set $\rho_1$ is the set of all vertices distance three from $p_4$ and $\rho_2$ is the set of all vertices distance two from $p_4$, we interchange $p_1$ and $p_4$ if necessary so that $|\rho_1 \cup \rho_2| \leq |\rho_3 \cup \rho_4|$.

Note that if $p \in \rho(G)$ and $P$ is a normal Sylow $p$-subgroup of $G$, then $P$ is non-abelian by Itô’s Theorem 2.11. When we have specified that $p \in \rho(G)$ we will not say that $P$ is non-abelian as it is redundant.

To date, not much is known about prime character degree graphs that have diameter three. The research focused on determining which graphs that satisfy Pálfy’s Condition and have diameter three occur as prime character degree graphs. The following theorems characterize those results:

Theorem 3.17. [18] The graph in Figure 2 is not the character degree graph of any solvable group.

Theorem 3.18. [10] The graphs in Figure 3 are not the character degree graphs of any solvable group.
Further, it was shown in [10] that if \( G \) is a solvable group that has a prime character degree graph with 6 vertices, then \( \rho_1 \cup \rho_2 \) has two vertices and \( \rho_3 \cup \rho_4 \) has four vertices. The first example of a solvable group that was found with six vertices satisfies this:

**Theorem 3.19.** [9] There exists a solvable group that has a prime character degree graph as shown in Figure 4.

![Graph](image)

Figure 4: Graph with 6 vertices and diameter 3

In [1], Dugan generalized this result to show that there is a family of groups with the graph shown in Figure 4 as prime character degree graph.

The following result from [10] is fundamental in our paper.

**Theorem 3.20.** Suppose \( G \) is a solvable group with \( \Phi(G) = 1 \). Assume that for all nontrivial normal subgroups \( M \), \( \Delta(G/M) \) has two connected components or the diameter of \( \Delta(G/M) \) is at most 2. Then either \( \Delta(G) \) has two connected components or the diameter of \( \Delta(G) \) is at most 2.

Theorem 3.20 gives us a formula to show that the graphs in Figure 1 are not prime character degree graphs for any solvable group \( G \). First we show that \( G \) has no normal non-abelian Sylow \( p \)-subgroup for any prime \( p \in \rho(G) \), and that \( \rho(G/M) \) is proper in \( \rho(G) \) whenever \( M \) is a proper nontrivial normal subgroup of \( G \). We will conclude with
these two facts that $\Phi(G) = 1$. Then we show that $\Delta(G/M)$ does not have diameter three and so Theorem 3.20 gives us the desired result. This is proved formally in Lemma 5.3.

Recently we have been made aware of a result by Zuccari that bounds the number of possible normal non-abelian Sylow $p$-subgroups for a solvable group $G$ with $\Delta(G)$ having diameter three.

**Lemma 3.21.** [20] Let $G$ be a solvable group with $\Delta(G)$ having diameter three. Then $G$ has at most one normal non-abelian Sylow subgroup.
Chapter 4

Preliminary Results

In this section, we prove many of the lemmas necessary to show that the graphs in Figure 1 do not exist as prime character degree graphs for any solvable group $G$.

**Lemma 4.1.** Let $G$ be a solvable group with $\Delta(G)$ having diameter three. Assume that $\Delta(O^{p^2}(G))$ does not have diameter three. If $\Delta(O^{p^2}(G))$ is disconnected, then $O^{p^2}(G)$ cannot be Example 2.4 from Theorem 3.4. In particular, $O^{p^2}(G)$ must have a normal non-abelian Sylow $p$-subgroup.

**Proof.** Suppose that $G$ is a solvable group, $\Delta(G)$ has diameter three, and $G$ has no normal Sylow $p$-subgroups for $p \in \rho(G)$. Let $K = O^{p^2}(G)$ and suppose $K < G$.

**Claim.** The subgroup $K$ is Example 2.4 from Theorem 3.4.

**Proof.** Because $G/K$ is a $\rho_2$-group, all primes in $\rho(G)\setminus \rho_2$ are contained in $\rho(K)$. Because the primes in $\rho_1$ and $\rho_4$ are contained in $\rho(K)$ and $\Delta(K)$ is a subgraph of $\Delta(G)$, we have $\Delta(K)$ is either disconnected or has diameter three. By the hypothesis, $\Delta(K)$ does not have diameter three, and so it must be disconnected. If $p \in \rho(K)$ for $p \in \rho_2$, then since $|\rho_1| \geq 1$, the smaller component has at least two primes. By Theorem 3.15,
$O^p(K) < K$. This contradicts the definition of $K$, and so $\rho_2$ intersects $\rho(K)$ trivially. Thus, the two components of $\Delta(K)$ are $\rho_1$ and $\rho_3 \cup \rho_4$.

If $K$ has a normal Sylow $p$-subgroup for some prime $p \in \rho(K)$, then so does $G$ as $\rho_2$ intersects $\rho(K)$ trivially. Thus, as $G$ has no normal non-abelian Sylow $p$-subgroups, $K$ also has no normal Sylow $p$-subgroups and $K$ is Example 2.4 from Theorem 3.4. □

Thus $K$ is Example 2.4 from Theorem 3.4 with the larger component being $\rho_3 \cup \rho_4$. By Theorem 5.2 of [8], $\text{cd}(K)$ contains only one degree that is divisible by primes in $\rho_3 \cup \rho_4$. Pick primes $p_2 \in \rho_2$, $p_3 \in \rho_3$, and $p_4 \in \rho_4$ such that $p_2$ and $p_3$ are adjacent in $\Delta(G)$. Then there exists a character $\chi \in \text{Irr}(G)$ such that $p_2p_3$ divides $\chi(1)$. Let $\theta$ be an irreducible constituent of $\chi_K$ in $\text{Irr}(K)$. By Corollary 2.13, we know $\chi(1)/\theta(1)$ divides $|G:K|$, and hence, $p_3$ divides $\theta(1)$. Because $K$ has only one degree divisible by primes in $\rho_3 \cup \rho_4$ and that degree is divisible by $p_3$ and $p_4$, we conclude that $p_4$ also divides $\theta(1)$. Thus, $p_2$ and $p_4$ are adjacent in $\Delta(G)$, which is a contradiction. □

The next two lemmas follow from the same arguments given in Claim 3 and Claim 5 of the Main Theorem in [10].

**Lemma 4.2.** Let $G$ be a solvable group with a normal Sylow $p$-subgroup for some prime $p \in \rho(G)$. Suppose $\rho(G/M) = \rho(G)$ implies that $M = 1$. Then $P'$ is a minimal normal subgroup in $G$ and in particular, $P'$ is a central subgroup in $P$.

**Proof.** Let $G$ be a solvable group with a normal Sylow $p$-subgroup for some prime $p \in \rho(G)$. Assume $\rho(G/M) = \rho(G)$ implies $M = 1$. By Lemma 2.15, $\rho(G/P') = \rho(G) \setminus \{p\}$. Let $X$ be a normal subgroup of $G$ that is contained in $P'$ so that $P'/X$ is a chief factor.
for $G$. Because $\rho(G/X)$ contains $p$ and $\rho(G/P')$, we see that $\rho(G/X) = \rho(G)$. Thus $X = 1$ by our hypothesis and $P'$ is a minimal normal subgroup of $G$. Because $\mathbb{Z}(P)$ is a characteristic subgroup of $P$, it is normal in $G$. It follows that $P' \cap \mathbb{Z}(P)$ is a normal subgroup of $G$. Because $P$ is a nilpotent subgroup of $G$, $P' \cap \mathbb{Z}(P)$ is not a trivial subgroup, and so $P' \subseteq \mathbb{Z}(P)$.

\textbf{Lemma 4.3.} Let $G$ be a solvable group and suppose $G$ has a normal Sylow $p$-subgroup for some prime $p \in \rho(G)$. Suppose there is a normal subgroup $N$ in $G$ such that $p$ does not divide $|N|$. Then $\rho(G/N)$ contains every prime in $\rho(G)$ that is not adjacent to $p$ in $\Delta(G)$.

\textbf{Proof.} Let $G$ be a solvable group with a normal Sylow $p$-subgroup $P$ for some prime $p \in \rho(G)$. Let $N$ be a normal subgroup of $G$ such that $p$ does not divide $|N|$. The subgroup $PN/N$ is a normal non-abelian Sylow $p$-subgroup of $G/N$ and so $p \in \rho(G/N)$.

Let $q$ be a prime in $\rho(G) \setminus \rho(G/N)$, and let $Q$ be a Sylow $q$-subgroup of $G$. Since $q$ divides no degree in $\text{cd}(G/N)$, we use Itô’s theorem, Corollary 2.11, to see that $QN/N$ is abelian and $QN$ is normal in $G$. Thus, the direct product $P \times QN$ is normal in $G$.

Let $\chi \in \text{Irr}(G)$ with $q$ dividing $\chi(1)$ and let $\theta \in \text{Irr}(QN)$ be an irreducible constituent of $\chi_{QN}$. We know that $\chi(1)/\theta(1)$ divides $|G:QN|$ by Corollary 2.13. Since $q$ does not divide $|G:QN|$, we see $q$ must divide $\theta(1)$. Thus, $q \in \rho(QN)$, and so $p$ and $q$ will be adjacent in $\Delta(P \times QN)$. Because $\Delta(P \times QN)$ is a subgraph of $\Delta(G)$, the primes $p$ and $q$ are adjacent in $\Delta(G)$. In particular, $\rho(G/N)$ contains every prime in $\rho(G)$ that is not adjacent to $p$ in $\Delta(G)$.

\hfill \Box
Because we know that a solvable group $G$ has at most one normal non-abelian Sylow $p$-subgroup by Lemma 3.21, if $\Delta(G)$ has diameter three, all factor groups $G/M$ have restrictions on the possible normal Sylow $p$-subgroups. This is the case even if $\Delta(G/M)$ does not have diameter three.

**Lemma 4.4.** Let $G$ be a solvable group where $\Delta(G)$ has diameter three. Let $M$ be a minimal normal subgroup of $G$ such that $\Delta(G/M)$ has diameter three. If $G$ has a normal Sylow $p$-subgroup for $p \in \rho(G)$, then $G/M$ does not have a normal Sylow $q$-subgroup for $q \in \rho(G)\{p\}$.

**Proof.** Let $G$ be a solvable group with $\Delta(G)$ having diameter three. Suppose $G$ has a normal Sylow $p$-subgroup $P$ for $p \in \rho(G)$, and let $M$ be a minimal normal subgroup of $G$ and suppose $\Delta(G/M)$ has diameter three. Suppose that the subgroup $G/M$ has a normal non-abelian Sylow $q$-subgroup $Q/M$ for a prime $q \in \rho(G)\{p\}$. Because the subgroup $PM/M$ is a normal Sylow $p$-subgroup of $G/M$, the Sylow subgroup $PM/M$ must be abelian or we have a contradiction of Lemma 3.21. Thus, $P' \subseteq M$. Let $R$ be a Sylow $q$-subgroup of $G$. We see that $RM$ is normal in $G$ and so $RP$ is a normal subgroup in $G$. Further, $[R,P] \subseteq M$ and so $R$ centralizes $P/P'$ by Corollary 3.28 in [5]. Hence, $P = C_p(R)P' = C_p(R)\Phi(P)$. As $P' \subseteq \Phi(P)$, by Lemma 2.5, $P = C_p(R)$ and so $RP$ is a direct product. Thus $R$ is normal in $RP$, so it is characteristic, and $R$ is normal in $G$. This contradicts Lemma 3.21. □

We will eventually show that if $\Delta(G)$ has diameter three, then $G$ has a normal Sylow $p$-subgroup for some prime $p \in \rho_3$. However, if $\rho_3$ is not large enough, the solvable
group $G$ cannot have a normal Sylow $p_3$-subgroup and a diameter three character degree graph. This is necessary to show that the graphs in Figure 1 do not exist as prime character degree graphs.

**Lemma 4.5.** Let $G$ be a solvable group with $\Delta(G)$ having diameter three. Assume for all proper nontrivial normal subgroups $M$ that $\Delta(G/M)$ and $\Delta(M)$ do not have diameter three, and assume $|\rho_3| < 2^n - 1$ where $n = |\rho_1 \cup \rho_2|$. Then $G$ does not have a normal Sylow $p_3$-subgroup for any prime $p_3 \in \rho_3$.

**Proof.** Suppose $G$ has a normal non-abelian Sylow $p_3$-subgroup $P$ for a prime $p_3 \in \rho_3$. Then, since $\rho(G/P') = \rho(G) \setminus \{p_3\}$ by Lemma 2.15 and the group $G/P'$ is a nontrivial proper factor group of $G$, the graph $\Delta(G/P')$ cannot have diameter three by the hypothesis. Because $\rho(G/P')$ contains primes from $\rho_1$ and $\rho_4$, we have that $\Delta(G/P')$ must be disconnected with components $\rho_1 \cup \rho_2$ and $(\rho_3 \cup \rho_4) \setminus \{p_3\}$. Further, $G/P'$ cannot have any normal non-abelian Sylow subgroups or $G$ would have more than one, and that violates Lemma 3.21. So $G/P'$ is Example 2.4 from Theorem 3.4.

Theorem 3.15 gives us that $O^{p_2}(G/P') < G/P'$, and so $O^{p_2}(G) < G$ for every prime $p_2 \in \rho_2$. Fix the prime $p_2 \in \rho_2$ and let $K = O^{p_2}(G)$. Because $K$ is a proper subgroup of $G$ and $\rho(G) \setminus \{p_2\} \subseteq \rho(K)$, the graph $\Delta(K)$ must be disconnected. As $K$ contains the subgroup $P$, it has a normal non-abelian Sylow $p_3$-subgroup and $p_3$ is in the larger component. So $K$ must satisfy the hypotheses of Example 2.6 from Theorem 3.4.

By Lemma 3.10, $K/P'$ satisfies the hypotheses of Example 2.4. Let $S/P' = \mathbb{F}(K/P')$, and $R/S = \mathbb{F}(K/S)$. Because $G/P'$ is Example 2.4, we let $F/P' = \mathbb{F}(G/P')$.
and $E/F = \mathbb{F}(G/F)$. Because $S$ is a characteristic subgroup of $K$ and $K$ is normal in $G$, $S$ is normal in $G$, and $S \subseteq F \cap K$. Because $F \cap K$ is a normal subgroup of $K$, we have $F \cap K = S$. Further, $R$ is characteristic in $K$ and so $R$ is normal in $G$. By the Diamond Lemma in [5], we obtain $R/S \cong RF/F$ is a normal subgroup of $G/F$.

Because $RF/F$ is normal in $G/F$, we have $RF/F \subseteq E/F$ and $RF \subseteq E$. Since $E/F$ is a Hall $(\rho_3 \cup \rho_4) \setminus \{p\}$-subgroup, and $R/S$ is a Hall $(\rho_3 \cup \rho_4) \setminus \{p_3\}$-subgroup by Lemma 3.8, $RF = E$.

Now, because $G/P'$ satisfies the hypotheses of Example 2.4 in Theorem 3.4, $G/P'$ is the semi-direct product of a subgroup $HP'/P'$ acting on an elementary abelian $p_3$-group $P/P'$, the positive integer $m$ is defined to be the index $|H : \mathbb{F}(H)|$, the order of $P/P'$ is $q^m$ where $q$ is a $p_3$-power, and the quotient $(q^m - 1)/(q - 1)$ divides $|E : F|$.

Since there are $|\rho_1 \cup \rho_2| = n$ primes that divide $m$, there must be $2^n - 1$ distinct primes that divide the quotient $(q^m - 1)/(q - 1)$ by the Zsigmondy Prime Theorem. Because $p_3$ does not divide $|E : F|$, and there are less than $2^n - 1$ primes in $\rho_3$, there must be at least one prime $p_4 \in \rho_4$ that divides $(q^m - 1)/(q - 1)$.

As the graph $\Delta(G)$ is connected, there exist primes $p_2 \in \rho_2$ and $p_3 \in \rho_3$ where $p_2$ and $p_3$ are adjacent. Then there is a character $\chi \in \text{Irr}(G)$ such that the product $p_2p_3$ divides $\chi(1)$. Let $\theta$ be an irreducible constituent of $\chi_K$. By Corollary 2.13, $p_3$ divides $\theta(1)$ and so $P'$ is not contained in $\ker \theta$. The size of the smaller component of $\Delta(K)$ is either $|\rho_1 \cup \rho_2| = n$ or $|\rho_1 \cup \rho_2 \setminus \{p_2\}| = n - 1 = a$.

Because $K$ is Example 2.6, $K$ contains a Frobenius group where $P$ is the Frobenius kernel and we let $B$ be a Frobenius complement. The positive integer $m_1$ is defined to be
the index $|K : R|$. There is a $p_3$-power $q_1$ such that $|P : P'| = q_1^{m_1}$ and $(q_1^{m_1} - 1)/(q_1 - 1)$ divides $|B|$. By Lemma 3.10, the character degrees $\text{cd}(K | P')$ are all divisible by $p|B|$. Because $\theta \in \text{Irr}(K | P')$, the quotient $(q_1^{m_1} - 1)/(q_1 - 1)$ divides $\theta(1)$.

If the smaller component of $\Delta(K)$ is $\rho_1 \cup \rho_2$, then $m = m_1$ and $(q_1^{m_1} - 1)/(q_1 - 1) = (q^{m_1} - 1)/(q - 1)$. Since the prime $p_4$ divides $(q^{m_1} - 1)/(q - 1)$, the prime $p_4$ divides $\theta(1)$. If the smaller component of $\Delta(K)$ is $\rho_1 \cup \rho_2 \setminus \{p_2\}$, then let $m_2 = |G : K|$. Then $m_1 = m/m_2$ and $q_1^{m_1} = q^m = |P : P'|$, so $q_1 = q^{m_2}$. Let $r$ be a Zsigmondy prime divisor of $q^m$. Then $r$ divides $q^m - 1$ and $r$ does not divide $q^s - 1$ for any $s < m$. Because $q_1^{m_1} - 1 = q^m - 1$, and $m_2 < m$, the prime $r$ divides the quotient $(q_1^{m_1} - 1)/(q_1 - 1)$. Thus, as $p_4$ is a Zsigmondy prime divisor of $q^m - 1$, the prime $p_4$ divides $(q_1^{m_1} - 1)/(q_1 - 1)$ and so divides $\theta(1)$.

As $p_4$ divides $\theta(1)$, the prime $p_4$ also divides $\chi(1)$. Then the primes $p_2$ and $p_4$ are adjacent, but this is impossible as $p_2 \in \rho_2$ and $p_4 \in \rho_4$. Thus, $G$ has no normal Sylow $p_3$-subgroup.

The following lemma shows that under suitable hypotheses, if we have a nontrivial factor group $G/M$, then $|\rho(G/M)| < |\rho(G)|$.

**Lemma 4.6.** Let $G$ be a solvable group such that $\Delta(G)$ has diameter three. Assume for all proper nontrivial normal subgroups $N$ that $\Delta(G/N)$ and $\Delta(N)$ do not have diameter three. Further, assume $G$ has no normal Sylow $p_4$-subgroup for $p_4 \in \rho_4$, $O^{p_4}(G) = G$, $|\rho_1 \cup \rho_2| = n$, and $|\rho_3| < 2^n - 1$. Then $\rho(G/M) = \rho(G)$ implies $M = 1$, whenever $M$ is
a normal subgroup of $G$.

**Proof.** Suppose $G$ is a solvable group where $\Delta(G)$ has diameter three, $G$ has no normal Sylow $p_4$-subgroups for any prime $p_4 \in \rho_4$, the subgroup $O^{p_4}(G) = G$, $|\rho_1 \cup \rho_2| = n$, and $|\rho_3 \cup \rho_4| < 2^n - 1$. Assume that if $N$ is a proper nontrivial normal subgroup of $G$ that the graphs $\Delta(N)$ and $\Delta(G/N)$ do not have diameter three. Let $M$ be a nontrivial normal subgroup of $G$, and suppose that $\rho(G/M) = \rho(G)$. Then $\Delta(G/M)$ does not have diameter three, and since $\Delta(G/M)$ is a subgraph of $\Delta(G)$, the graph $\Delta(G/M)$ must be disconnected with components $\rho_1 \cup \rho_2$ and $\rho_3 \cup \rho_4$. The group $G/M$ must be one of the examples from Theorem 3.4. Because both components have more than one vertex, we must be in Example 2.4 or Example 2.6. If $G/M$ is Example 2.4, then it has no normal Sylow $p$-subgroups for any prime $p \in \rho(G/M)$. However, $O^{p_2}(G/M) < G/M$ by Theorem 3.15. Let $H/M = O^{p_2}(G/M)$. Then since $|G/M : H/M|$ is a nontrivial power of $p_2$, the index $|G : H|$ is a nontrivial power of the prime $p_2$. So $O^{p_2}(G) < G$. This contradicts Lemma 4.1, hence $G/M$ is Example 2.6 from 3.4. By Lemma 3.10, $G/M$ has a normal Sylow $p$-subgroup for some prime $p \in \rho_3 \cup \rho_4$. Let $Q/M$ be that Sylow $p$-subgroup of $G/M$ and $P$ a Sylow $p$-subgroup of $G$ that is contained in $Q$.

Consider $K = O^{p_2}(G)$ for a fixed prime $p_2 \in \rho_2$. Because $O^{p_2}(G/M) < G/M$ by Theorem 3.15, we have $K$ is a proper subgroup of $G$, and so $\Delta(K)$ must be disconnected. Also, notice $P$ is contained in $K$.

**Claim.** The group $K$ is Example 2.6 from Theorem 3.4.

**Proof.** Because the only prime that could be missing from $\rho(K)$ is the prime $p_2$,
\( \rho(G) \setminus \{ p_2 \} \subseteq \rho(K) \). Thus, \( K \) cannot be Example 2.2 or 2.3 from Theorem 3.4 as \( \rho(K) \) contains more than two vertices. If \( K \) is Example 2.1 or 2.5, then \( |\rho_1| = |\rho_2| = 1 \), the prime \( p_2 \notin \rho(K) \), and \( K \) has an abelian \( p_1 \)-complement. However, this implies \( O^{p_1}(K) \) is a proper subgroup of \( K \). Then \( O^{p_1}(G) \) is a proper subgroup in \( G \) and this is a contradiction.

Suppose \( K \) is Example 2.4. There exists a prime \( p_3 \in \rho_3 \) that is adjacent to \( p_2 \) and a character \( \chi \in \text{Irr}(G) \) such that \( p_2 p_3 \) divides \( \chi(1) \). Let \( \theta \) be an irreducible constituent of \( \chi_K \). Then by Corollary 2.13, the quotient \( \chi(1)/\theta(1) \) divides the index \( |G : K| \). Because the index \( |G : K| \) is a \( p_2 \)-power, \( p_3 \) divides \( \theta(1) \). Because \( K \) is Example 2.4, there is only one character degree that is divisible by the primes in the larger component, \( \rho_3 \cup \rho_4 \), and so \( \theta(1) \) must be that character degree. Hence, every prime \( p_4 \in \rho_4 \) divides \( \theta(1) \). Thus, \( p_4 \) divides \( \chi(1) \) and \( p_2 \) and \( p_4 \) are adjacent, which is a contradiction. Thus, \( K \) must be Example 2.6 from Theorem 3.4.

Since \( K \) is Example 2.6, \( K \) has a normal non-abelian Sylow \( r \)-subgroup \( R \) for some prime \( r \in \rho_3 \cup \rho_4 \). Because \( R \) is a characteristic subgroup of \( K \), \( R \) is normal in \( G \). Further, as \( |G : K| \) is a power of \( p_2 \), \( R \) is a Sylow subgroup of \( G \). Because \( G/M \) has a normal non-abelian Sylow \( p \)-subgroup, and \( RM/M \) is a normal non-abelian Sylow \( r \)-subgroup of \( G/M \), \( p = r \) and \( R = P \). As \( G \) has no normal non-abelian Sylow \( p \)-subgroups for \( p \in \rho_4 \), the prime \( p \) must be in \( \rho_3 \). But this contradicts Lemma 4.5. Thus \( \rho(G/M) \) does not equal \( \rho(G) \).
Chapter 5

Graphs with Six Vertices

In this section we show that if $G$ is a solvable group then $\Delta(G)$ is not one of the two graphs from Figure 1. We start by showing that if $\Delta(G)$ has diameter three, then $G$ does not have a normal non-abelian Sylow $p$-subgroup for any prime $p \in \rho_1 \cup \rho_2$.

**Lemma 5.1.** Let $G$ be a solvable group with $\Delta(G)$ having diameter three. Assume for all proper nontrivial normal subgroups $M$ that $\Delta(G/M)$ and $\Delta(M)$ do not have diameter three. Further, assume $\rho(G/M) = \rho(G)$ implies $M = 1$. Then $G$ does not have a normal Sylow $p_2$-subgroup for any prime $p_2 \in \rho_2$.

**Proof.** Suppose $G$ has a normal Sylow $p_2$-subgroup $P$ for a fixed prime $p_2 \in \rho_2$ and let $H$ be a $p_2$-complement of $G$. By Lemma 2.15, $\rho(G/P') = \rho(G) \setminus \{p_2\}$. Because $\rho(G/P')$ contains the primes in $\rho_1$ and $\rho_4$, and $\Delta(G/P')$ is a subgraph of $\Delta(G)$, the graph $\Delta(G/P')$ is disconnected. Because $P$ is not central in $G$ and $H$ acts on $P$ nontrivially, we see that $P/P'$ is not central in $G/P'$. From Theorem 5.5 of [8], the Fitting subgroup of $G/P'$ has at most one non-central Sylow subgroup, which is $P/P'$. Let $F = \mathbb{F}(G)$. Since all of the other Sylow subgroups of $\mathbb{F}(G/P') = F/P'$ are central, $F/P'$ is abelian, and thus $G/P'$ is as described in Example 2.4 from Theorem 3.4. This is because
any solvable group having an abelian Fitting subgroup whose graph has two connected components, where at least one connected component has size larger than one, must satisfy the hypotheses of Example 2.4. Further, $G/P'$ has Fitting height 3.

Let $E/F = \mathbb{F}(G/F)$. Then $G/E$ is a $(\rho_1 \cup \rho_2)\{p_2\}$-group, $E/F$ is a cyclic $\rho_3 \cup \rho_4$-group and $E \cap H$ is abelian. Further, $H$ has a normal Sylow $p_4$-subgroup $Q$ for some $p_4 \in \rho_4$. By Lemma 3.8, we have that $E \cap H$ acts irreducibly on $[E, F]/P'$. Also, $\rho(PQ) = \{p_2, p_4\}$ and $\Delta(PQ)$ has two connected components. Because the Fitting height of $PQ$ is 2, by Lemma 4.1 of [8] we know that $PQ$ is Example 2.1 of Theorem 3.4. Let $C = \mathbb{C}_P(Q)$. Because $PQ$ is Example 2.1 from Theorem 3.4, $P' \subseteq C$, every non-linear irreducible character is fully ramified with respect to $P/C$, and $Q$ acts nontrivially on $P$ fixing every non-linear irreducible character of $P$. By Theorem 3.11, we have $P' = [P, Q]'$, and so $P' < [P, Q]$. Because $P \subseteq F$ and $Q \subseteq E$, we have $[P, Q] \subseteq [E, F]$. Since $[E, F]/P'$ is irreducible under the action of $E \cap H$, we have $[E, F] = [P, Q]$.

Fix a prime $p_1 \in \rho_1$. As $p_1$ and $p_2$ are adjacent in $\Delta(G)$, there exists a character $\chi \in \text{Irr}(G)$ such that $p_1 p_2$ divides $\chi(1)$. Let $\theta \in \text{Irr}(P)$ be an irreducible constituent of $\chi_P$. Notice that $\theta$ is invariant in $F$ as $F$ is nilpotent, and in $E$ because no prime in $\rho_3 \cup \rho_4$ divides $\chi(1)$. By Glauberman’s Lemma, 13.8 of [4], there exists an $E \cap H$-invariant irreducible constituent of $\theta_{P'}$. Thus, $\mathbb{C}_{P'}(E \cap H) > 1$. By Lemma 4.2, $P'$ is central in $P$ and so $\mathbb{C}_{P'}(E \cap H)$ is normal in $P$. Also, $H$ normalizes $P'$ and $E \cap H$. So, $H$ and $G = PH$ normalize $\mathbb{C}_{P'}(E \cap H)$. Furthermore, $P'$ is minimal normal in $G$ by
Lemma 4.2, thus, $P' \subseteq C_P(E \cap H)$. By Fitting’s Lemma,

$$P/P' = [P,Q]/P' \times C/P' = [E,F]/P' \times C_P(E \cap H)/P'.$$

Because $C_P(E \cap H) \subseteq C$, we have $C_P(E \cap H) = C$ and so $E$ satisfies the hypotheses of Example 2.1 and has components $\{p_2\}$ and $\rho_3 \cup \rho_4$ by Lemma 3.5. There exists a character $\psi \in \text{Irr}(G)$ such that $p_2p_3$ divides $\psi(1)$ for some $p_3 \in \rho_3$. Let $\gamma \in \text{Irr}(E)$ such that $\gamma$ is an irreducible constituent of $\psi_E$. By Corollary 2.13, the quotient $\psi(1)/\gamma(1)$ divides $|G : E|$, and since neither $p_2$ nor $p_3$ divides $|G : E|$, we have that $p_2$ and $p_3$ divide $\gamma(1)$. This is a contradiction to the fact that $\Delta(E)$ is disconnected. Thus, $G$ has no normal Sylow $p_2$-subgroup.

The hypotheses for the following lemma have the added condition that $O_p(G) = G$ for $p \in \rho_3$ or $p \in \rho_4$.

**Lemma 5.2.** Let $G$ be a solvable group with $\Delta(G)$ having diameter three. Assume for all proper nontrivial normal subgroups $M$ that $\Delta(G/M)$ and $\Delta(M)$ do not have diameter three. Further, suppose that $\rho(G/M) = \rho(G)$ implies $M = 1$ and $O^{\rho_3} = G$ or $O^{\rho_4}(G) = G$. Then $G$ does not have a normal Sylow $p_1$-subgroup for any prime $p_1 \in \rho_1$.

**Proof.** Suppose $G$ has a normal Sylow $p_1$-subgroup $P$ for some $p_1 \in \rho_1$, and let $N = O_{p_1}(G)$. By Lemma 4.3, we have $\{p_1\} \cup \rho_3 \cup \rho_4 \subseteq \rho(G/N)$. By the hypotheses, if $N > 1$, then $|\rho(G/N)| < |\rho(G)|$, and so, by Lemma 4.3, there exists a prime $p_2 \in (\rho_1 \cup \rho_2) \backslash \{p_1\}$ that is not in $\rho(G/N)$. Since $\rho(G/N)$ contains $\{p_1\}$ and $\rho_4$, $\Delta(G/N)$ has two connected
components. By our assumption, \(G/N\) has a normal Sylow \(p_1\)-subgroup, and so \(G/N\) is either Example 2.1 or Example 2.6 from Theorem 3.4. Because \(p_1\) is in the smaller component, \(G/N\) must be Example 2.1, and the components of \(\Delta(G/N)\) are \(\{p_1\}\) and \(\rho_3 \cup \rho_4\). In Example 2.1, \(G\) has an abelian Hall \(p_1\)-complement. Thus for \(p \in \rho_3 \cup \rho_4\), the subgroup \(O^p(G)\) is a proper subgroup of \(G\) that contains \(P\), which is a contradiction to the hypothesis that either \(O_{\rho_3}(G) = G\) or \(O_{\rho_4}(G) = G\). Thus, \(N = 1\).

Because \(O_{\rho_1}(G) = 1\), the Fitting subgroup of \(G\) is \(P\). Let \(H\) be a \(p_1\)-complement for \(G\); \(H\) acts faithfully on \(P\), and by Lemma 2.12 and the discussion following, every prime divisor of \(|H|\) occurs in \(\rho(G)\). Pick a character \(\gamma \in \text{Irr}(P)\) with \(\gamma(1) > 1\) and a character \(\chi \in \text{Irr}(G | \gamma)\). Then \(p_1\) divides \(\chi(1)\) and no prime in \(\rho_3 \cup \rho_4\) divides \(\chi(1)\). Thus, \(G/P \cong H\) has an abelian Hall \(\rho_3 \cup \rho_4\)-subgroup by Theorem 2.14. Let \(L = O_{(\rho_1 \cup \rho_2)\setminus \{p_1\}}(H)\) and \(E/L = \mathbb{F}(H/L)\). Then \(H/L\) has an abelian Hall \(\rho_3 \cup \rho_4\)-subgroup, which must be \(E/L\) by the Hall-Higman Theorem 2.6. If \(E = H\), then \(O_{\rho_4}(H) < H\) for \(p_4 \in \rho_3 \cup \rho_4\), and so \(O_{\rho_4}(G) < G\), which is a contradiction to the hypothesis that either \(O_{\rho_3}(G) = G\) or \(O_{\rho_4}(G) = G\). Thus \(E < H\).

Consider the normal subgroup \(PE\). We know \(\{p_1\} \cup \rho_3 \cup \rho_4 \subseteq \rho(PE)\). Since \(PE < G\), we have that \(\Delta(PE)\) must be disconnected. If \(p_2 \in \rho(PE)\) for any prime \(p_2 \in (\rho_1 \cup \rho_2)\setminus \{p_1\}\), then because \(PE\) has a normal Sylow \(p_1\)-subgroup and both components would have size larger than one, we must be in Example 2.6 of Theorem 3.4. However, \(p_1\) must be in the larger component, which it is not, and so \(PE\) satisfies Example 2.1 of Theorem 3.4. Because \(PE\) is Example 2.1 from Theorem 3.4, the components of \(\Delta(PE)\) are \(\{p_1\}\) and \(\rho_3 \cup \rho_4\), the subgroup \(E\) is an abelian Hall \(\rho_3 \cup \rho_4\)-
subgroup, $L = 1$, and $P' \subseteq \mathbb{Z}(PE)$.

Let $Q$ be the Sylow $p_4$-subgroup of $E$ for $p_4 \in \rho_3 \cup \rho_4$. Because $PE$ is Example 2.1 from Theorem 3.4, every non-linear irreducible character of $P$ is fully ramified with respect to $P/\mathbb{C}_P(E)$, the subgroup $Q$ acts faithfully on $P$, fixing every non-linear irreducible character of $P$. By Theorem 3.11, we have $P' = [P, Q]'$. Let $\lambda \in \text{Irr}([P, Q]/P')$ be non-principal. Because $P'$ is central in $P$ by Lemma 4.2, the stabilizer of $\lambda$ in $G$ is $P \mathbb{C}_H(\lambda)$.

Because $Q$ acts faithfully, $\mathbb{C}_Q(\lambda) < Q$, which implies $p_4$ divides $|Q : \mathbb{C}_Q(\lambda)|$. Thus $p_4$ divides $|H : \mathbb{C}_H(\lambda)|$. Since $p_2$ and $p_4$ are not adjacent in $\Delta(G)$ for $p_2 \in (\rho_1 \cup \rho_2) \setminus \{p_1\}$, we have $\mathbb{C}_H(\lambda)$ contains a Hall $(\rho_1 \cup \rho_2) \setminus \{p_1\}$-subgroup of $H$. Further, $\lambda$ extends to $P \mathbb{C}_H(\lambda)$ and $\text{cd}(P \mathbb{C}_H(\lambda) | \lambda) = \text{cd}(\mathbb{C}_H(\lambda))$. So by Clifford’s theory, no degree in $\text{cd}(\mathbb{C}_H(\lambda))$ is divisible by any prime in $(\rho_1 \cup \rho_2) \setminus \{p_1\}$. By Itô’s Theorem, 2.11, $\mathbb{C}_H(\lambda)$ contains a unique Hall $(\rho_1 \cup \rho_2) \setminus \{p_1\}$-subgroup of $H$, which is abelian. Recall, $E = \mathbb{F}(H)$ is abelian and the index $|H : E|$ is only divisible by the primes in $(\rho_1 \cup \rho_2) \setminus \{p_1\}$. By Itô’s Theorem, 2.10, $\text{cd}(H)$ contains only products of primes in $(\rho_1 \cup \rho_2) \setminus \{p_1\}$ and so $\rho(H) = (\rho_1 \cup \rho_2) \setminus \{p_1\}$. Thus, $\mathbb{C}_H(\lambda)$ is abelian. The stabilizer of $\lambda$ in $[P, Q]H$ is $[P, Q] \mathbb{C}_H(\lambda)$ and $\lambda$ extends to this stabilizer.

Consider the group $[P, Q]H$. We have

$$\text{cd}([P, Q]H) = \text{cd} \left( \frac{[P, Q]H}{P'} \right) \cup \text{cd} ([P, Q] | P')$$

$$= \text{cd} \left( \frac{[P, Q]H}{[P, Q]} \right) \cup \text{cd} \left( \frac{[P, Q]H}{P'} \right) \cup \text{cd} ([P, Q] | P').$$
Observe that
\[
\text{cd} ([P, Q]H \mid \lambda) = \{ |H : C_H(\lambda)|a \mid a \in \text{cd}(C_H(\lambda))\} = \{ |H : C_H(\lambda)| \}.
\]

Thus the primes in \( \rho_3 \cup \rho_4 \) are the only possible prime divisors of degrees in
\[
\text{cd} \left( \frac{[P, Q]H}{P'} \mid \frac{[P, Q]}{P'} \right).
\]

By Lemma 1 of [6], \( [P, Q]/P' \) is irreducible under the action of \( Q \) and so \( P' = C_{[P, Q]}(E) \).

Since \( \Delta(PQ) \) has two connected components, every non-linear irreducible character of \( [P, Q] \) is fully ramified with respect to \( [P, Q]/P' \). Then \( [P, Q]E \) is Example 2.1 from Theorem 3.4 and the two connected components are \( \{ p_1 \} \) and \( \rho_3 \cup \rho_4 \).

By Lemma 3.5, \( \text{cd} ([P, Q]E \mid P') \) consists of powers of the prime \( p_1 \). Because
\[
\]
only divisible by primes in \( (\rho_1 \cup \rho_2) \setminus \{ p_1 \} \), the only primes that divide degrees in \( \text{cd} ([P, Q]H \mid P') \) are the primes in \( \rho_1 \cup \rho_2 \). Because \( Q \) acts faithfully on \( [P, Q] \), the prime \( p_4 \in \rho([P, Q]H) \). As \( p_4 \) was chosen arbitrarily from \( \rho_3 \cup \rho_4 \), the graph \( \Delta([P, Q]H) \) has two connected components: \( \rho_1 \cup \rho_2 \) and \( \rho_3 \cup \rho_4 \). Because \( [P, Q]H \) has a normal non-abelian Sylow \( p_1 \)-subgroup and both components have more than one vertex, \( [P, Q]H \) must satisfy Example 2.6 from Theorem 3.4. In fact, \( p_2 \) divides \( |F(H)| \). But \( E = F(H) \) and is a Hall \( \rho_3 \cup \rho_4 \)-subgroup. This is a contradiction and so \( G \) has no normal Sylow \( p_1 \)-subgroups.

As the following argument is used frequently, we have made a lemma. This lemma
shows how Theorem 3.20 can be applied once the group $G$ has been shown to have no normal non-abelian Sylow $p$-subgroups and satisfies the condition that $|\rho(G/M)| < |\rho(G)|$ for all proper nontrivial normal subgroups $M$.

**Lemma 5.3.** Let $G$ be a solvable group and suppose $G$ has no normal Sylow $p$-subgroups for $p \in \rho(G)$. Suppose for all minimal normal subgroups $M$ that $\rho(G/M) < \rho(G)$ and $\Delta(G/M)$ does not have diameter three. Then $\Delta(G)$ does not have diameter three.

**Proof.** We write $F$ for the Fitting subgroup of $G$ and $\Phi(G)$ for the Frattini subgroup of $G$. By Lemma 2.12 and the discussion following, we know that $\rho(G) = \pi(|G : F|)$. Because $F(G)/\Phi(G) = F(G/\Phi(G))$, we see that $\rho(G/\Phi(G)) = \rho(G)$ and so $\Phi(G) = 1$. Let $M$ be a proper nontrivial normal subgroup of $G$ and consider $\Delta(G/M)$. Since $\Delta(G/M)$ is a subgraph of $\Delta(G)$, we see that it cannot have diameter three by hypothesis. Thus, $\Delta(G/M)$ must either have two connected components or have diameter two. By Theorem 3.20, we see that $\Delta(G)$ must also have two connected components or diameter two. \hfill \Box

We now have all the tools to show that the graphs in Figure 1 are not the prime character degree graphs for any solvable group $G$.

**Theorem 5.4.** Let $\Gamma$ be a graph of diameter three that satisfies Pálfy’s Condition. Suppose the number of vertices is 6, and specifically $|\rho_1 \cup \rho_2| = 2$, $|\rho_3| = 1$ or 2, and $|\rho_4| \geq 2$. Then $\Gamma$ is not the prime character degree graph for any solvable group $G$.

**Proof.** Let $\mathcal{F}$ be the family of graphs with diameter three that satisfy Pálfy’s Condition, $|\rho_1 \cup \rho_2| = 2$, $|\rho_3| = 1$ or 2, $|\rho_4| \geq 2$, and $|\rho(G)| = 6$. Let $G$ be a minimal counter-
example such that $\Delta(G) \in \mathcal{F}$. Working by induction, if $M$ is a proper normal subgroup of $G$, then $\Delta(M)$ cannot be in $\mathcal{F}$ and so $\Delta(M)$ is either disconnected or has diameter at most two. If $G$ has a normal Sylow $p_4$-subgroup $P$ for some prime $p_4 \in \rho_4$, then $\rho(G/P') = \rho(G) \setminus \{p_4\}$. The graph $\Delta(G/P')$ has 5 vertices. Because $\rho_4$ has more than one vertex, $\Delta(G/P')$ either has diameter three or is disconnected. Let $p_2 \in \rho_2$ and $p_3 \in \rho_3$ be adjacent in $\Delta(G)$ such that the product $p_2p_3$ divides $\chi(1)$ for a character $\chi \in \text{Irr}(G)$. Because $p_2$ and $p_4$ are not adjacent, we have $P' \leq \ker \chi$. Thus, $\chi \in \text{Irr}(G/P')$ and so $p_2$ and $p_3$ are adjacent in $\Delta(G/P')$. Because a graph with 5 vertices and diameter three does not occur as a prime character degree graph by Theorem 3.18, $G$ does not have a normal Sylow $p_4$-subgroup. Notice that deleting one or more edges incident to the vertex $p_4$ produces a graph that does not satisfy Pálfy’s Condition, and so $O^{p_4}(G) = G$ by Lemma 3.3.

By Lemma 4.5, $G$ does not have a normal Sylow $p_3$-subgroup for any prime $p_3 \in \rho_3$. Because $|\rho_1 \cup \rho_2| = 2$ and $|\rho_3| < 3$, Lemma 4.6 says that $|\rho(G/M)| < |\rho(G)|$ for all proper nontrivial normal subgroups $M$. We see by Lemma 5.1 and Lemma 5.2 that $G$ does not have a normal Sylow $p$-subgroup for any prime $p \in \rho_1 \cup \rho_2$. Thus $G$ has no normal non-abelian Sylow subgroups and so by Lemma 5.3, the graph $\Delta(G)$ does not have diameter three, a contradiction, and so the graphs in $\mathcal{F}$ are not prime character degree graphs for any solvable groups.

Corollary 5.5. Let $G$ be a solvable group where $\Delta(G)$ has diameter three. Then $|\rho_3| \geq 3$. 

Proof. We prove this using induction on the size of $\rho(G)$. Let $G$ be a minimal counterexample such that $|\rho(G)| > 6$, the graph $\Delta(G)$ has diameter three, and $|\rho_3| = 1$ or $2$. Because $|\rho_1 \cup \rho_2| \leq |\rho_3 \cup \rho_4|$, the subset $\rho_4$ has at least two vertices. Suppose $G$ has a normal Sylow $p_4$-subgroup $P$ for some prime $p_4 \in \rho_4$. Then by Lemma 2.15, the set of vertices for $\rho(G/P')$ is $\rho(G) \setminus \{p_4\}$ and $\Delta(G/P')$ has diameter three. This contradicts the induction hypothesis and so $G$ has no normal Sylow $p_4$-subgroups for any prime $p_4 \in \rho_4$. Furthermore, $O^{p_4}(G) = G$ by Lemma 3.3.

Let $M$ be a proper, nontrivial, normal subgroup of $G$. The graphs $\Delta(M)$ and $\Delta(G/M)$ are subgraphs of $\Delta(G)$. Thus, in particular $\rho(G/M) \cap \rho_3$ and $\rho(M) \cap \rho_3$ have at most 2 vertices. Therefore, the graphs $\Delta(M)$ and $\Delta(G/M)$ cannot have diameter three.

Because $|\rho_3| \leq 2$ and $|\rho_1 \cup \rho_2| = a$ is at least 2, $|\rho_3| < 2^a - 1$. Thus by Lemma 4.5, $G$ does not have any normal Sylow $p_3$-subgroups for any prime $p_3 \in \rho_3$. By Lemma 4.6, if $M$ is a normal subgroup of $G$ and $\rho(G/M) = \rho(G)$, then $M = 1$. We see by Lemma 5.1 and Lemma 5.2 that $G$ does not have a normal Sylow $p$-subgroup for any prime in $\rho_1 \cup \rho_2$. Hence, $G$ has no normal non-abelian Sylow subgroups. Thus, by Lemma 5.3, $\Delta(G)$ does not have diameter three and $|\rho_3| \geq 3$. \qed
Chapter 6

No normal non-abelian Sylow $p_4$-subgroup

When showing that the graphs in Figure 1 were not the prime character degree graphs for any solvable group $G$, we were able to use the fact that the graphs in Figure 3 are not prime character degree graphs for any solvable group $G$. Is it possible that there is a group $G$ that has a prime character degree graph $\Delta(G)$ as in Figure 4 and a normal Sylow $p_4$-subgroup $P$ for the prime $p_4 \in \rho_4$? The graph $\Delta(G/P')$ will have diameter two in this case, and so none of our current tools answer this question. The following Lemma shows that it is not possible for a solvable group $G$ to have a prime character degree graph with diameter three and a normal Sylow $p_4$-subgroup for prime $p_4 \in \rho_4$.

Lemma 6.1. Let $G$ be a solvable group with $\Delta(G)$ having diameter three. Then $G$ does not have a normal Sylow $p_4$-subgroup for any prime $p_4 \in \rho_4$.

Proof. Let $G$ be a minimal counter-example where $\Delta(G)$ has diameter three and $G$ has a normal Sylow $p_4$-subgroup $P$ for a prime $p_4 \in \rho_4$. We can assume that $|\rho_3| \geq 3$ by Corollary 5.5.

Claim 1. $O^{p_3}(G) = G$ for all primes $p_3 \in \rho_3 \cup \rho_4 \setminus \{p_4\}$.
Proof. Suppose $O^{p_3}(G) < G$. Since $\Delta(O^{p_3}(G))$ contains all edges not adjacent to $\{p_3\}$ and the primes except possibly the prime $p_3$, we can find a prime $q \in \rho_3$ other than $p_3$ that is adjacent to some prime $p_2 \in \rho_2$. Thus $\Delta(O^{p_3}(G))$ is connected and has diameter three. This contradicts our assumption that $G$ is a minimal counter-example, as $P \subseteq O^{p_3}(G)$.  

Claim 2. The subgroup $O_{p_4}(G) = 1$ and the Fitting subgroup of $G$ is $P$.

Proof. Suppose that $M$ is a minimal normal subgroup contained in $O_{p_4}(G)$. Then $\rho(G/M)$ contains $\rho_1 \cup \rho_2 \cup \{p_4\}$ by Lemma 4.3, and $\Delta(G/M)$ is disconnected or has diameter three. Since $G/M$ has a normal Sylow $p_4$-subgroup $PM/M$, we know that $\Delta(G/M)$ must be disconnected by our assumption, and further $G/M$ is either Example 2.6 or Example 2.1 from Theorem 3.4. The components are either $\rho_1 \cup \rho_2$ and $\{p_4\}$, or $\rho_1 \cup \rho_2$ and $(\rho_3 \cup \rho_4) \cap \rho(G/M)$, where $|\rho_1 \cup \rho_2| = n$ and $|(\rho_3 \cup \rho_4) \cap \rho(G/M)| \geq 2^n - 1$, by Pálfy’s Inequality.

We assume first that both of the connected components of $\Delta(G/M)$ have size larger than one and so $G/M$ is Example 2.6 in Theorem 3.4. Let $F/M = F(G/M)$ and $E/M/F/M = F(G/M/F/M) = E/F = F(G/F)$. Then by Lemma 3.10, $F/M = PM/M \times Z/M$ where $Z/M$ is a central subgroup of $G/M$. Let $\phi$ be a nonlinear irreducible character of the subgroup $P$ so that $p_4$ divides $\phi(1)$. Then $\phi \times 1_M \in \text{Irr}(P \times M)$ and $M$ is contained in the kernel of $\phi \times 1_M$, ie $(\phi \times 1_M)(a) = 1$ for all $a \in M$. By Lemma 3.10, there exists a subgroup $B$ of $G$ such that $PM/M \cdot B/M$ is a Frobenius group, $PM/M$ is the Frobenius kernel and $B/M$ is a Frobenius complement.
By Theorem 6.34 in [4], \((\phi \times 1_M)^{PB/M}\) is an irreducible character of \(PB/M\), and so \((\phi \times 1_M)^{PB}\) is an irreducible character of \(PB\).

Let \(\chi \in \operatorname{Irr}(G)\) with \(\chi(1)\) divisible by \(p_2p_3\) for primes \(p_2 \in \rho_2\) and \(p_3 \in \rho_3\). Let \(\psi \in \operatorname{Irr}(PZ)\) and \(\gamma \in \operatorname{Irr}(Z)\) where \(\psi\) is an irreducible constituent of \(\chi_{PZ}\) and \(\gamma\) is an irreducible constituent of \(\psi_Z\). Notice that if \(\gamma = 1\), then \(Z \subseteq \ker(\chi)\) and so \(M\) is also in \(\ker(\chi)\). But then \(p_2\) and \(p_3\) are adjacent in \(\Delta(G/M)\), which is not the case, and so \(\gamma\) is not the principal character.

Consider \(\phi \times \gamma\), an irreducible character of \(PZ\), and its stabilizer \(G_{\phi \times \gamma}\), which is the intersection of the stabilizers, \(G_{\phi}\) and \(G_{\gamma}\). Notice, that by Corollary 2.8, \(\phi\) extends to \(G_{\phi}\). If \(\theta \in \operatorname{Irr}(G)\) lies over \(\phi\), then since \(p_4\) divides \(\theta(1)\), we know that no prime in \(\rho_1 \cup \rho_2\) divides \(\theta(1)\). By Theorem 2.14, we see that \(G/P\) contains an abelian Hall \(\rho_1 \cup \rho_2\)-subgroup of \(G/P\). Since \(\phi\) extends to \(G_{\phi}\) we can apply Gallagher’s Theorem, Corollary 2.9. Any character degree \(a \in \operatorname{cd}(G_{\phi}/P)\) can be multiplied by \(\phi(1)\) to get a character degree in \(\operatorname{cd}(G)\), and so no prime in \(\rho_1 \cup \rho_2\) divides any character degree of \(G_{\phi}/P\). Thus, by Itô’s Theorem, Corollary 2.11, \(G_{\phi}/P\) contains a unique Hall \(\rho_1 \cup \rho_2\) subgroup \(AP/P\). By Clifford’s theory, we get that no prime in \(\rho_1 \cup \rho_2\) divides \(|G : G_{\phi}|\).

Further, no prime in \(\rho_1 \cup \rho_2\) divides \(|G : G_{\gamma \times \phi}|\). If a prime \(p_1 \in \rho_1 \cup \rho_2\) did divide \(|G : G_{\gamma \times \phi}|\), then \(p_1\) would divide \(|G : G_{\phi}|\) and \(p_1\) and \(p_4\) would be adjacent, which is not possible. So, since \(G_{\phi \times \gamma}/P\) also contains a Hall \(\rho_1 \cup \rho_2\)-subgroup of \(G\), it must contain \(AP/P\). Thus, \(G_{\gamma}\) contains \(A\).

Consider the character \(\phi^g\) and its stabilizer \(G_{\phi^g}\) for some \(g \in G\). The character \(\phi^g\) also extends to its stabilizer by Corollary 2.8. By Gallagher’s theorem, Corollary 2.9,
any character degree in \( \text{cd}(G_{\phi^g}/P) \) is a multiple of \( \phi^g(1) \), and so no prime in \( \rho_1 \cup \rho_2 \) divides any character degree of \( G_{\phi^g}/P \). So, this factor group contains a unique Hall \( \rho_1 \cup \rho_2 \)-subgroup \( A^gP/P \). Because \( G_{\phi^g \times \gamma}/P \) contains \( A^g/P \), \( G_\gamma \) also contains \( A^g \), and so \( G_\gamma \) contains all conjugates of \( A \). Hence, \( G_\gamma \) contains \( O^{(\rho_1 \cup \rho_2)'(G)} \). However, \( G_\gamma \) contains \( P \) and \( O^{p_3}(G) = G \) for all \( p_3 \in \rho_3 \cup \rho_4 \setminus \{p_4\} \), and so \( G_\gamma = G \).

Because \( 1_P \times \gamma \in \text{Irr}(P \times Z) \), and \( E/F \) is cyclic, we get that \( 1_P \times \gamma \) extends to \( E \) by Corollary 11.22 in [4]. Then, as \( G/E \) is cyclic, \( 1_P \times \gamma \) extends to all of the Sylow subgroups of \( G/E \). By Corollary 11.31 in [4], we get that \( 1_P \times \gamma \) extends to \( G \), and hence \( \gamma \) extends to \( G \). Call the extension of \( \gamma \) to \( G \hat{\gamma} \). By Gallagher’s Theorem, \( \eta \hat{\gamma} \) are all of the irreducible constituents of \( \gamma^G \), where \( \eta \in \text{Irr}(G/Z) \). In particular, \( \chi = \eta \hat{\gamma} \) for some \( \eta \in \text{Irr}(G/Z) \). Since \( M \) is contained in \( Z \), we know that \( \Delta(G/Z) \) is disconnected and any \( p_2 \in \rho_2 \) and \( p_3 \in \rho_3 \) are not adjacent. Since \( p_2p_3 \) divides \( \chi(1) \) and \( p_2 \) does not divide \( \gamma(1) \), we know that \( p_2 \) must divide \( \eta(1) \). Thus, \( p_3 \) cannot divide \( \eta(1) \) as \( p_2 \) and \( p_3 \) are not adjacent in \( \Delta(G/Z) \), and so \( p_3 \) divides \( \gamma(1) \). However, there exists an \( \alpha \in \text{Irr}(G/Z) \) such that \( p_1 \) divides \( \alpha(1) \) for some prime \( p_1 \in \rho_1 \) and \( \alpha \gamma \in \text{Irr}(G) \). This contradicts the fact that \( p_1 \) and \( p_3 \) are not adjacent in \( \Delta(G) \).

Thus, \( (\rho_3 \cup \rho_4) \cap \rho(G/M) = \{p_4\} \) and so \( G/M \) is Example 2.1 from Theorem 3.4. Then, \( G/M \) has an abelian Hall \( p_4 \)-complement, and hence \( O^{p_3}(G) < G \) for some prime \( p_3 \in \rho_3 \cup \rho_4 \setminus \{p_4\} \), which we showed cannot happen. Thus \( O^p(G) = 1 \) and \( F(G) = P \). 

\[ \square \]

**Claim 3.** The subgroup \( P' \) is a minimal normal subgroup of \( G \) and \( P' \) is central in \( P \).
\textit{Proof.} Suppose there exists $1 \neq X \subset P'$ where $P'/X$ is a chief factor. Then since $\rho(G/P') = \rho(G) \setminus \{p_4\}$ and $\Delta(G/P')$ is connected, we have that $\rho(G/X) = \rho(G)$ and in particular $\Delta(G/X)$ has diameter three. However, this contradicts our assumption, and $P'$ is minimal normal. Because $\mathbb{Z}(P)$ is characteristic in $P$ and so normal in $G$, it follows that $P' \cap \mathbb{Z}(P)$ is normal in $G$. Because $P$ is nilpotent, $1 < P' \cap \mathbb{Z}(P)$, and so $P' \subseteq \mathbb{Z}(P)$ as $P'$ is a minimal normal subgroup of $P$. \hfill \Box$

Let $\theta \in \text{Irr}(P)$ be nonlinear and $H$ a $p_4$-complement in $G$. Notice that $H$ acts faithfully on $P$ and every prime divisor of $|H|$ occurs in $\rho(G)$. Let $\chi \in \text{Irr}(G)$ be an irreducible constituent of $\theta^G$. Then, since no prime in $\rho_1 \cup \rho_2$ divides $\chi(1)$, by Theorem 2.14, $G/P \cong H$ has an abelian Hall $\rho_1 \cup \rho_2$-subgroup. Let $L = O_{\rho_3 \cup \rho_4 \setminus \{p_4\}}(H)$ and $E/L = \mathbb{F}(H/L)$. Note that $E/L$ is a $\rho_1 \cup \rho_2$-subgroup. Since $H/L$ has an abelian Hall $\rho_1 \cup \rho_2$-subgroup, $E/L$ is the Hall $\rho_1 \cup \rho_2$-subgroup by the Hall-Higman Theorem 2.6, and since $O_{\rho_3 \cup \rho_4 \setminus \{p_4\}}(G) = G$, $E = H$. In particular, $L$ is a Hall $\rho_3 \cup \rho_4 \setminus \{p_4\}$-subgroup of $H$.

Let $K$ be a $p_2$-complement in $H$ for a fixed prime $p_2 \in \rho_2$. Note that $L \subseteq K$. Since $E/L = H/L$ is abelian, $K$ is normal in $H$. Because $G$ is a minimal counter-example and $PK$ has a normal non-abelian Sylow $p_4$-subgroup, the graph $\Delta(PK)$ is disconnected. The two connected components of $\Delta(PK)$ are $(\rho_1 \cup \rho_2) \setminus \{p_2\}$ and $\rho_3 \cup \rho_4$. As the prime $p_4$ is in a component with size larger than one, the group $PK$ must be Example 2.6 in Theorem 3.4. Thus, $PK = TD$ where $D$ is an abelian group acting coprimely on the group $T$. By Lemma 3.10, $P \subseteq T$, and $T$ has a $p_4$-complement $Q$. So, $PK = PQD$. 

and the primes that divide $Q$ are precisely the primes in $p_3 \cup p_4 \setminus \{p_4\}$. Thus, $Q = L$ and, as $Q$ is abelian, $L$ is abelian.

Because the group $PK = PLD$ is Example 2.6, the subgroup $[PL, D]$ is a Frobenius group where $[P, L]$ is the Frobenius kernel by Lemma 3.6. A $p_4$-complement in $[PL, D]$ is $[L, D]$ and so we call a Frobenius complement $B = [L, D]$, which is contained in $L$. From Lemma 3.10, we have $P' = [P, L]' \subseteq [P, L]$. We see that $[P, L]K$ satisfies the hypotheses of Example 2.6 of Theorem 3.4. So $\Delta([P, L]K)$ is disconnected with components $(\rho_1 \cup \rho_2) \setminus \{p_2\}$ and $\rho_3 \cup \rho_4$. The action of $B$ on $P'$ is a Frobenius action, so $C_{P'}(B) = 1$ and $C_{P}(L) \subseteq C_{P}(B)$. Also, $C_{P}(L)' \subseteq C_{P}(L) \cap P' = 1$. Thus, $C_{P}(L)$ is abelian.

Let the character $\lambda \in \text{Irr}(P')$ be non-principal. Because $P'$ is central in $P$, the stabilizer of $\lambda$ is $P \mathcal{C}_H(\lambda)$. By Theorem 13.28 of [4], we can find a $\mathcal{C}_H(\lambda)$-invariant irreducible constituent $\theta$ of $\lambda P'$. Note that the stabilizer of $\theta$ in $G$ is $P \mathcal{C}_H(\theta)$ and $\mathcal{C}_H(\lambda) \subseteq \mathcal{C}_H(\theta)$. Because $P'$ is central and $\theta_{P'}$ has a unique constituent $\lambda$, we have that $\mathcal{C}_H(\lambda) = \mathcal{C}_H(\theta)$.

As $p_4$ divides every degree in $\text{cd}(G \mid \lambda)$, we have that $\mathcal{C}_H(\lambda)$ contains an abelian Hall $\rho_1 \cup \rho_2$-subgroup of $H$. Further, $\theta$ must extend to $P \mathcal{C}_H(\lambda)$ by Corollary 2.8 and so, by Gallagher’s Theorem, no prime in $\rho_1 \cup \rho_2$ is in $\rho(\mathcal{C}_H(\lambda))$. Since $\lambda$ extends to $P' \mathcal{C}_H(\lambda)$, and $p_4$ divides every degree in $\text{cd}(G \mid \lambda)$, we see that no prime in $\rho_1 \cup \rho_2$ divides any degree in $\text{cd}(P'H \mid P')$. On the other hand, $H$ has a normal abelian Hall $\rho_3 \cup \rho_4 \setminus \{p_4\}$-subgroup $L$, and so no prime in $\rho_3 \cup \rho_4$ divides a degree in $\text{cd}(H)$. If $H$ is nilpotent, then by the discussion after Lemma 2.12, there would be a character
degree that equals $|H|$. Since all of the primes in $\rho(G)\setminus\{p_4\}$ divide $|H|$, this is not the case. Because $L \subseteq \mathbb{F}(H)$, we deduce that $p_1 \in \rho(H)$ for some $p_1 \in \rho_1 \cup \rho_2 \setminus \{p_2\}$.

Recall that $L$ contains $B$, so that $P'B$ is a Frobenius group. It follows that at least one prime in $\rho_3 \cup \rho_4 \setminus \{p_4\}$ is in $\rho(P'H)$ and $\Delta(P'H)$ is disconnected. The components are a nonempty subset of $\rho_1 \cup \rho_2$ and a nonempty subset of $\rho_3 \cup \rho_4 \setminus \{p_4\}$. Because $P'H$ has no normal non-abelian Sylow subgroups, $P'H$ is Example 2.4 from Theorem 3.4.

First, we suppose that $\rho_1 \cup \rho_2 \subseteq \rho(P'H)$. Recall that $PK$ is Example 2.6 from Theorem 3.4 and so by Lemma 3.10, $PK/P'$ is Example 2.4. Because $L$ is a normal Hall subgroup of $H$, we know $L \subseteq \mathbb{F}(K)$. By Lemma 3.10, only the primes in $\rho_3 \cup \rho_4$ can divide $\mathbb{F}(K)$. Thus, $\mathbb{F}(K) \subseteq L$ and $L = \mathbb{F}(K)$. Further, both $L$ and $K/L$ are cyclic groups.

Consider the subgroup $HP'/P'$ acting coprimely on the group $[P,L]/P'$. Define $E/P' = \mathbb{F}(HP'/P') \cong \mathbb{F}(H)$. Certainly, $L \subseteq \mathbb{F}(H)$. Because $p_2$ divides $|H|$ and $K$ is a normal $p_2$-complement of $H$, the Fitting subgroup of $K$ is contained in the Fitting subgroup of $H$. Hence, $\mathbb{F}(H) = L \times \mathbb{O}_{p_2}(H)$. Because the Sylow $p_2$-subgroup of $H$ is not a normal subgroup of $H$, the subgroup $\mathbb{O}_{p_2}(H)$ is a proper subgroup of $H$. Further, the subgroup $PK\mathbb{O}_{p_2}(H)$ is a proper normal subgroup of $G$ and $\rho(PK\mathbb{O}_{p_2}(H)) = \rho(G)$. As the subgroup $PK\mathbb{O}_{p_2}(H)$ has a normal Sylow $p_4$-subgroup, and by the minimality of $G$, the graph $\Delta(PK\mathbb{O}_{p_2}(H))$ must be disconnected. Both components have size larger than one and so $PK\mathbb{O}_{p_2}(H)$ is Example 2.6 from Theorem 3.4. By Lemma 3.10, the primes that divide $|\mathbb{F}(H) : \mathbb{F}(PK\mathbb{O}_{p_2}(H))|$ are precisely the primes in $\rho_3 \cup \rho_4$, and so $p_2$ cannot divide that index, which is a contradiction. Hence, $\mathbb{O}_{p_4}(H) = 1$ and $\mathbb{F}(H) = L$
and $E/P' = LP'/P'$.

Recall that the group $[P, L]K$ is Example 2.6 from Theorem 3.4. By Lemma 3.10, the factor group $[P, L]K/P'$ is Example 2.4 from Theorem 3.4. First, we define $Z_1/P' = C_{KP'/P'}([P, L]/P')$ and $Z_2/P' = C_{HP'/P'}([P, L]/P')$. Since $\mathbb{F}(G/P') = P/P'$ and $C_{G/P'}(P/P') \subseteq P/P'$, we see that $Z_1/P' = Z_2/P' = P'/P'$. The Fitting subgroup of $K$ is $L$ and so the Fitting subgroup of $KP'/P'$ is $LP'/P'$ and it is abelian. Since $L$ acts irreducibly on $[P, L]$, we have $LP'/P'$ acts irreducibly on $[P, L]/P'$. Define $m = |H P'/P' : LP'/P'|$, which is equal to $|H : L|$, and so $(m, |L|) = 1$. Let $m_2 = |H : K|$ be the power of $p_2$ that divides $|H|$.

Let $q_1$ be a power of $p_4$ such that $q_1^{m/m_2} = |P/P'|$. Because $[P, L]K/P'$ is Example 2.4, there is a $p_4$-power $q$ such that $q^{m/m_2} = |[P, L] : P'|$. Clearly, $q \leq q_1$. Let $s$ be a Zsigmondy prime divisor of $q_1^{m/m_2} - 1$. We recall that Zsigmondy prime divisors exist except if $m = 2$ and $p_4 = 2$ or $m = 6$ and $q = 2$. These exceptions do not occur as $[P, L]K/P'$ is Example 2.4. Then $s$ divides $q_1^{m/m_2} - 1$ and does not divide $q_1^i - 1$ for any $i < m/m_2$. Further, as $q_1 - 1$ is a factor of $q_1^{m/m_2} - 1$, the prime $s$ divides $(q_1^{m/m_2} - 1)/(q_1 - 1)$, and as the quotient $(q_1^{m/m_2} - 1)/(q_1 - 1)$ divides $|L|$, so does $s$. Now the subgroup $L$ acts Frobeniusly on $[P, L]/P'$. So $L$ divides $|[P, L]/P'| - 1 = q^{m/m_2} - 1$. Hence, $s$ divides $q^{m/m_2} - 1$, and since $s$ is a Zsigmondy prime divisor, $q^{m/m_2} \geq q_1^{m/m_2}$. Thus $q = q_1$.

Because $P'H$ is also Example 2.4, we know that there is a $p_4$-power $q_2$ such that $q_2^n = |P'|$. Because $L$ acts Frobeniusly on $P'$ we know that $|L|$ divides $q_2^n - 1$. Because $s$ divides $|L|$, we know that $s$ divides $q_2^n - 1$ and so $q^{m/m_2} \leq q_2^n$. Let $r$ be a Zsigmondy
prime divisor of $q^m_2 - 1$. The prime $r$ exists because the exceptions do not occur as $P'H$ is Example 2.4. Then $r$ divides $q^m_2 - 1$ and $r$ also divides $(q^m_2 - 1)/(q_2 - 1)$. Hence, $r$ divides $|L|$. But as $|L|$ divides $q^{m/m_2} - 1$, so does $r$. But then $q^m_2 \leq q^{m/m_2}$ and so $q^m_2 = q^{1/m_2}$. Hence, $q_2 = q^{1/m_2}$. Since $(q^m_2 - 1)/(q_2 - 1)$ divides $|L|$, we know that $(q^{m/m_2} - 1)/(q^{1/m_2} - 1)$ divides $|L|$. Thus $[P, L]H/P'$ satisfies Example 2.4 and so the graph $\Delta([P, L]H/P')$ is disconnected.

By Fitting’s Lemma, $P/P' = C_{P/P'}(L) \times [P, L]/P'$. So

$$PH/P' = P/P' \cdot HP'/P' = (C_{P/P'}(L) \times [P, L]/P') \cdot HP'/P'.$$

If $C_P(L)/P'$ is not central in $HP'/P'$, then there is a character degree divisible by $|H : C_H(\lambda)|\theta(1)$. Because $K$ is a $p_2$-complement, $p_2$ divides $|H : C_H(\lambda)|$, and so $p_2$ and $p_4$ are adjacent. This is a contradiction, and so $\rho_2$ is not contained in $\rho(P'H)$. In particular, the prime $p_2$ is not in $\rho(H)$. So $H$ has an abelian Sylow $p_2$-subgroup $Q$ by Corollary 2.11. Further, $PQ$ is a normal subgroup of $G$.

Now, the graph $\Delta(PQ)$ has two connected components, $\{p_2\}$ and $\{p_4\}$. Observe that $Q$ acts coprimely on $P$, fixing all non-linear characters. From Theorem 3.11, we have $[P, Q]' = P'$ and $[P, Q]$ is not abelian. Consider $\Delta([P, L]H)$ determined by

$$\text{cd}([P, L]H) = \text{cd} \left( \frac{[P, L]H}{[P, L]} \right) \cup \text{cd} \left( [P, L]H \mid [P, L] \right).$$

Because $\rho(H) = (\rho_1 \cup \rho_2)\backslash\{p_2\}$, $p_2$ does not divide a character degree in $\text{cd}(H) = \cd([P, L]H)$. So $\Delta([P, L]H/P')$ is disconnected.
cd ([P, L]H/\{P, L\}). Recall that [P, L]K is Example 2.6 and B is a Frobenius complement. Thus by Lemma 3.10, every degree in cd ([P, L]K | [P, L]) is divisible by \(p_4|B|\) and so \(p_2\) does not divide any of those character degrees. We see that \(p_2\) is not in cd([P, L]H), and so the subgroup [P, L]H is proper in \(G\). The graph \(\Delta([P, L]H)\) must be disconnected. Thus, \(Q\) is a normal subgroup of [P, L]H. In particular, \(Q\) centralizes [P, L] as \(Q\) is normal in \(H\). Because \([[P, L], Q] = 1\) and \([L, Q] = 1\), we have \([[L, Q], P] = 1\). Thus \([[Q, P], L] = 1\) by the Three Subgroup Lemma, Lemma 2.2, and so \([P, Q] \subseteq C_P(L)\). This is a contradiction because \(C_P(L)\) is abelian and \([P, Q]\) is not. \(\square\)
Chapter 7

Main Theorems

Now that we have the tools to show that $G$ does not have a normal Sylow $p$-subgroup for any prime $p \in \rho_1 \cup \rho_2 \cup \rho_4$, we need to show that $G$ must have a normal Sylow $p$-subgroup for $p \in \rho_3$. We have restated the theorems from the Introduction for convenience. Recall that Lemma 3.21 says that $G$ can have at most one normal non-abelian Sylow $p$-subgroup for $p \in \rho(G)$ when $\Delta(G)$ has diameter three.

**Theorem 1.** Let $G$ be a solvable group with prime character degree graph $\Delta(G)$ with diameter three. Then $G$ has a normal Sylow $p$-subgroup for exactly one prime $p$ and $p \in \rho_3$.

**Proof.** Let $G$ be a counter-example with $|G|$ minimal such that $\Delta(G)$ has diameter three and $G$ has no normal Sylow $p_3$-subgroups for $p_3 \in \rho_3$. Because $\Delta(G)$ has diameter three, we know $|\rho_3| \geq 3$ by Corollary 5.5. By Lemma 6.1, we see that $G$ does not have a normal Sylow $p_4$-subgroup for any prime $p_4 \in \rho_4$.

**Claim 1.** Let $p_2$ be a prime in $\rho_2$. If $O^{p_2}(G)$ is a proper subgroup of $G$, then $\Delta(O^{p_2}(G))$ is disconnected.
Proof. Let $K = O^{p_2}(G)$ for some prime $p_2 \in \rho_2$. Because $\rho(K)$ contains $\rho_1 \cup \rho_4$, the graph $\Delta(K)$ either has diameter three or is disconnected. Suppose $\Delta(O^{p_2}(G))$ has diameter three. Then by the hypothesis, $O^{p_2}(G)$ has a normal Sylow $p_3$-subgroup $P$ for some prime $p_3 \in \rho_3$. The subgroup $P$ is characteristic in $O^{p_2}(G)$ and so is normal in $G$, which is a contradiction. Hence, $\Delta(O^{p_2}(G))$ cannot have diameter three. Because $\rho(O^{p_2}(G))$ contains all the primes of $\rho(G)$ with the possible exception of the prime $p_2$, we see $\Delta(O^{p_2}(G))$ must be disconnected. 

Suppose there exists a normal subgroup $N$ of $G$ such that $\Delta(G/N)$ has diameter three. Then we can find a minimal normal subgroup $M$ contained in $N$, where $\Delta(G/M)$ has diameter three. By the minimality of $G$, $G/M$ has a normal Sylow $p_3$-subgroup $P/M$ for $p_3 \in \rho_3$. Because $M$ is an elementary abelian $p$-group for some prime $p$, if $p = p_3$ then $G$ has a normal Sylow $p_3$-subgroup and this is a contradiction.

Because $P' \neq 1$, either $M \subseteq P'$ or $M \cap P' = 1$. Suppose that $M \cap P' = 1$. Then because $[P, M] \subseteq P'$ and $[P, M] \subseteq M$, we see $[P, M] = 1$ and so $M$ is central in $P$. Let $P_3$ be a Sylow $p_3$-subgroup of $G$ such that $P_3 \subseteq P$. Since $M$ normalizes $P_3$, $P_3$ is characteristic in $P$ and so is normal in $G$. This is a contradiction, and so we assume that $M \subseteq P'$. Hence $G/M/(P/M)' = G/M/P'/M \cong G/P'$.

Suppose that $G$ has a normal Sylow $q$-subgroup $Q$ for $q \in \rho_1 \cup \rho_2 \cup \rho_4$. Then we have a contradiction of Lemma 4.4 because $G/M$ has a normal Sylow $p_3$-subgroup. Hence, $G$ has no normal non-abelian Sylow subgroups.

Consider the graph $\Delta(G/M/(P/M)') = \Delta(G/P')$. By our hypothesis, if $\Delta(G/P')$
has diameter three, then it has a normal Sylow \( q \)-subgroup for \( q \in \rho_3 \). Since \( q \neq p_3 \), this contradicts Lemma 3.21 for \( G/M \). Hence, \( G/P' \) does not have diameter three and further, does not have any normal Sylow \( p \)-subgroups for \( p \in \rho(G/P') \). Because \( \rho(G/P') \) contains primes from \( \rho_1 \) and \( \rho_4 \), we know that \( \Delta(G/P') \) must be disconnected and is Example 2.4 from Theorem 3.4.

Let \( p_2 \) be a prime in \( \rho_2 \). By Theorem 3.15, \( O^{p_2}(G/P') < G/P' \) and so \( O^{p_2}(G) < G \). Recall that \( \Delta(O^{p_2}(G)) \) is disconnected. However, \( G \) has no normal non-abelian Sylow subgroups, and this contradicts Lemma 4.1. Thus, \( \Delta(G/N) \) cannot have diameter three for any proper, nontrivial, normal subgroup \( N \).

Suppose that \( M \) is a proper normal subgroup of \( G \) such that \( \Delta(M) \) has diameter three. Then \( M \) has a normal Sylow \( p_3 \)-subgroup \( P \) for a prime \( p_3 \in \rho_3 \). Since \( P \) is characteristic in \( M \), and \( M \) is normal in \( G \), we see \( P \) is normal in \( G \). Further, \( P' \) is normal in \( G \), and \( P' \neq 1 \). Recall that \( \Delta(G/P') \) cannot have diameter three. Because \( \rho(G/P') \) contains \( \rho(G)\backslash \{p_3\} \), the graph has components \( \rho_1 \cup \rho_2 \) and \( \rho_3 \cup \rho_4 \backslash \{p_3\} \), with the possibility of containing \( p_3 \) as well. It must be disconnected with both components at least size 2. By Theorem 3.15, \( O^{p_2}(G/P') < G/P' \) for some prime \( p_2 \in \rho_2 \), and \( O^{p_2}(G) < G \). Further, \( P \subseteq O^{p_2}(G) \). We have shown that \( \Delta(O^{p_2}(G)) \) is disconnected and Lemma 4.1 tells us that \( O^{p_2}(G) \) cannot be Example 2.4 from Theorem 3.4.

Suppose \( |\rho_1 \cup \rho_2| = 2 \) and \( p_2 \notin \rho(O^{p_2}(G)) \). Then it is possible that \( O^{p_2}(G) \) is Example 2.1 or 2.5 from Theorem 3.4 and \( O^{p_2}(G) \) has a normal Sylow \( p_1 \)-subgroup \( R \) for \( p_1 \in \rho_1 \). Since \( \Delta(M) \) has diameter three, \( p_1 \in \rho(M) \) and so \( M \) has a normal Sylow \( p_1 \)-subgroup, which contradicts Corollary 3.21. Thus \( O^{p_2}(G) \) cannot be Example 2.1 or 2.5 from
Theorem 3.4. Since at least one component, if not both, has size at least 2, $O^{p_2}(G)$ is Example 2.6 from Theorem 3.4. Hence $O^{p_2}(G)$ has a normal Sylow $p$-subgroup $Q$ for a prime $p \in \rho_3 \cup \rho_4$. But this is a contradiction because $Q$ is characteristic in $O^{p_2}(G)$ and so $Q$ is a normal Sylow $p$-subgroup of $G$ and $G$ does not have a normal Sylow $p$-subgroup for any prime in $\rho_3 \cup \rho_4$. Thus, $\Delta(M)$ cannot have diameter three whenever $M$ is a proper normal subgroup of $G$.

Suppose there exists a prime $p_3 \in \rho_3$ such that $O^{p_3}(G)$ is proper in $G$. Since $O^{p_3}(G)$ is proper in $G$ we know that $\Delta(O^{p_3}(G))$ cannot have diameter three. Because $\rho(O^{p_3}(G))$ contains all of $\rho(G)$ except perhaps the prime $p_3$, we know that $\Delta(O^{p_3}(G))$ is disconnected. Because $|\rho_3| \geq 3$, there is a prime $q \in \rho_3$, not equal to $p_3$, and a prime $p_2 \in \rho_2$, such that $q$ and $p_2$ are adjacent in $\Delta(G)$. However, as $\Delta(O^{p_3}(G))$ contains all edges not incident to the prime $p_3$, we see that $p_2$ and $q$ are adjacent in $\Delta(O^{p_3}(G))$ and $\Delta(O^{p_3}(G))$ must have diameter three. This is a contradiction, and so $O^{p_3}(G) = G$ for all primes $p_3 \in \rho_3$.

Suppose there exists a nontrivial normal subgroup $M$ of $G$ such that $\rho(G/M) = \rho(G)$. Without loss of generality, $M$ is a minimal normal subgroup of $G$. We know that the graph $\Delta(G/M)$ cannot have diameter three, and because $\rho(G/M)$ contains primes from $\rho_1$ and $\rho_4$, the graph $\Delta(G/M)$ must be disconnected. The components are $\rho_1 \cup \rho_2$ and $\rho_3 \cup \rho_4$. Because $\rho_1 \cup \rho_2$ is the smaller component and has size at least 2, $G/M$ is either Example 2.4 or 2.6 from Theorem 3.4. Suppose $G/M$ is Example 2.6. Then $G/M$ has a normal Sylow $p$-subgroup $P/M$ for a prime $p \in \rho_3 \cup \rho_4$. Let $Q$ be a Sylow $p$-subgroup of $G$ such that $Q$ is contained in $P$. If $M$ is not contained in $P'$,
then since \([P,M] \subseteq P'\) and \([P,M] \subseteq M\), we have \([P,M] = 1\). Thus \(M\) is central in \(P\) and, as \(M\) normalizes \(Q\), we have \(Q\) is characteristic in \(P\) and so \(Q\) is normal in \(G\), which is a contradiction and so \(M \subseteq P'\). Because \(\rho(G/M)/(P/M)' = \rho(G)\{p\}\) and \(G/M/(P/M)' = G/P'\), the graph \(\Delta(G/P')\) is disconnected. The factor group \(G/P'\) is Example 2.4 from 3.4. By Lemma 3.15, \(O^{p_2}(G/P')\) is proper in \(G/P'\) and so \(O^{p_2}(G) < G\). But this contradicts Lemma 4.1, thus, \(G/M\) is not Example 2.6, and so is Example 2.4. If \(G\) has a normal non-abelian Sylow subgroup, then so does \(G/M\). Therefore, \(G\) has no normal non-abelian Sylow subgroups. By Theorem 3.15, \(O^{p_2}(G/M) < G/M\). But then \(O^{p_2}(G) < G\), which contradicts Lemma 4.1. Thus, \(\rho(G/M) = \rho(G)\) implies that \(M = 1\).

We have shown that if \(M\) is a proper, nontrivial, normal subgroup of \(G\), then the graphs \(\Delta(M)\) and \(\Delta(G/M)\) do not have diameter three. Further, \(O^{p_3}(G) = G\) for all primes \(p_3 \in \rho_3\), and so Lemma 5.1 and Lemma 5.2 apply. Because we have also shown that \(G\) has no normal Sylow \(p_4\)-subgroup for any prime \(p_4\) in \(\rho_4\), and we have assumed that \(G\) has no normal Sylow \(p_5\)-subgroup for any prime \(p_5\) in \(\rho_3\), \(G\) has no normal non-abelian Sylow subgroups. Thus, by Lemma 5.3, \(G\) does not have diameter three, which is our final contradiction.

Finally, because \(G\) has a normal Sylow \(p_3\)-subgroup \(P\) when \(\Delta(G)\) has diameter three, we can observe that \(\Delta(G/P')\) must be disconnected, and so, \(G/P'\) is in one of the families of groups from Theorem 3.4.

**Theorem 3.** Let \(G\) be a solvable group with \(\Delta(G)\) having diameter three. If \(|\rho_1 \cup \rho_2| = n\)
then $|\rho_3 \cup \rho_4| \geq 2^n$

**Proof.** By Theorem 1, $G$ has a normal non-abelian Sylow $p_3$-subgroup $P$ for some prime $p_3 \in \rho_3$. By Lemma 2.15, $\rho(G/P') = \rho(G) \setminus \{p_3\}$. Because $\rho_3$ has more than three vertices by Lemma 5.5, either $\Delta(G/P')$ is disconnected or it has diameter three. If $\Delta(G/P')$ has diameter three, then by Theorem 1, $\Delta(G/P')$ has a normal non-abelian Sylow $q$-subgroup $Q/P'$ for some prime $q \in \rho(G) \setminus \{p_3\}$. Let $R$ be a Sylow $q$-subgroup of $G$ contained in $Q$. As $RP'/P'$ is a normal subgroup of $G/P'$, the Sylow subgroup $R$ is a normal subgroup of $G$, which contradicts Lemma 3.21. Hence, $\Delta(G/P')$ is disconnected. By Lemma 3.14, if $|\rho_1 \cup \rho_2| = n$, then $|\rho_3 \cup \rho_4 \setminus \{p_3\}| \geq 2^n - 1$. Hence, $|\rho_3 \cup \rho_3| \geq 2^n$. 

**Theorem 2.** Let $G$ be a solvable group with $\Delta(G)$ having diameter three. Then $G$ has Fitting height 3.

**Proof.** By Theorem 1, $G$ has a normal Sylow $p_3$-subgroup $P$ for $p_3 \in \rho_3$. So $G/P'$ has a normal abelian Sylow $p_3$-subgroup. Notice that $\Delta(G/P')$ is disconnected, both components are larger than 2, and $G/P'$ has no normal Sylow $p$-subgroups for $p \in \rho(G/P')$. So $G/P'$ is Example 2.4 from 3.4 and so has Fitting height 3. Let $H$ be a $p_3$-complement of $G$. Anything in $H$ that centralizes $P/P'$ also centralizes $P$. Let $F = \mathbb{F}(G)$ and $E/P' = \mathbb{F}(G/P')$. We have $F/P'$ is a nilpotent normal subgroup of $G/P'$ and $F \subseteq E$. Conversely, $E = P(E \cap H)$, and $E \cap H$ is a nilpotent subgroup that centralizes $P/P'$. So $E = P \times (E \cap H)$ is nilpotent and $E \subseteq F$. Thus the Fitting height of $G$ is the same as $G/P'$, and so $G$ has Fitting height three.
Bibliography


[9] Lewis, M. L., A Solvable Group whose Character Degree Graph has Diameter 3, 


