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CHAPTER 1

Introduction and Background

If $T$ is a continuous linear operator on a Banach space $B$ and $x \in B$, then the orbit of $x$ under $T$ is the sequence $(x, Tx, T^2x, \ldots)$. We say that $x$ is cyclic if the closed, linear span of the orbit is the whole space; that is, if $\text{span}\{T^jx\}_{j=1}^{\infty} = B$.

The study of orbits and in particular, cyclic vectors has a long history and several classical problems and results in analysis can be viewed as problems and results on orbits of operators. In recent years there has been growing interest to study orbits of operators in a more detailed way than was done before. This direction of research is also direction of our investigation. We will below give results on geometric properties of orbits of some classes of operators. But first we give a background to indicate how our investigation fits in with earlier work.

Given a bounded linear operator on a Banach space, a (closed) subspace $Y$ of $X$ is called an invariant subspace for $T$ if $T(Y) \subset Y$.

The classical Invariant Subspace Problem asks whether every continuous linear operator $T$ on a Banach space $B$ has a closed non-trivial invariant subspace. It is easy to see that this can be reformulated as a question on orbits: Is there an operator $T$ on $B$ such that every $x \neq 0$ is cyclic, i.e., such that the orbit of $x$ under $T$, spans the whole space?

This problem was solved in 1975 by P. Enflo [2] who constructed a Banach space $B$ and an operator $T$ on $B$ such that every $x \neq 0$ is cyclic. Later (1985), in their paper [5] C.Read and A.M. Davie gave examples on operators with only trivial invariant subspaces on some of the classical Banach spaces, like $l_1$ and $c_0$ where $l_1$ and $c_0$ are sequence spaces defined respectively by,

$l_1$ = the vector space of all sequences $a = (a_1, a_2, a_3, \ldots)$ such that
\[ \sum_{k=1}^{\infty} |a_k| < \infty. \]

and

\[ c_0 = \text{the vector space of all sequences } (a_k) \text{ such that } \lim_{k \to \infty} a_k = 0. \]

However the Invariant Subspace Problem is still unsolved for operators on Hilbert space. The more general unsolved version of the Invariant Subspace Problem was given by V.Lomonosov [7]: "Does every operator $T^*$ have a closed, non-trivial invariant subspace?" and it is still unsolved in this general formulation. Here $T^*$ means the adjoint of $T$.

This is one of the long-standing fundamental problems in Analysis.

In the Invariant Subspace Problem one is just interested in whether vectors are cyclic. One does not consider more detailed questions about the orbits, for instance, in what way the orbits are cyclic; if the orbits have stronger properties than cyclicity, etc.

One such property that has been studied extensively is hypercyclicity. We say that a vector $x$ is hypercyclic if \( \{x, Tx, \ldots\} = B \), i.e. if the orbit is dense in the whole space. Notice that, If $T$ has a hypercyclic vector $x$, then obviously $T^n x$ is also hypercyclic for every $n$, so $T$ has a dense set of hypercyclic vectors.

Examples of hypercyclic vectors were given already in 1920’s. The first example of a hypercyclic operator is due to Birkhoff [9] in 1929, who showed the existence of an entire function $f$ whose successive translates by a non-constant $b$ are arbitrarily close to any function in the space of entire functions $H(\mathbb{C})$. To be precise, he showed that there exists an entire function $f$ and $b \in \mathbb{C}$, $b \neq 0$ so that for every $g \in H(\mathbb{C})$, the sequence $f(z+n_jb) \to g(z)$ locally uniformly on $\mathbb{C}$, for some sequence $n_j$ that depends on the function $g$. Thus, showing that the translation operator $T_b : f(z) \to f(z + b)$ has a dense orbit, that is, \( \{f(z), f(z + b), f(z + 2b), f(z + 3b), \ldots\} = H(\mathbb{C}) \).

The second example of a hypercyclic operator is due to MacLane [10] in 1952, who showed
the existence of a universal entire function $f$ whose successive derivatives are dense in $H(\mathbb{C})$. He showed that there is an entire function $f$ such that for every $g \in H(\mathbb{C})$, there exists a sequence $n_j$ increasing to $\infty$ so that $f^{(n_j)}(z) \to g(z)$ locally uniformly on $\mathbb{C}$ and thus the differentation operator $D$ on $H(\mathbb{C})$ is hypercyclic.

In 1969 [13] Rolewicz studied multiples of the backward shift operator $S^*$, 

$$S^* : e_n \to e_{n-1}, \quad e_1 \to 0 \text{ on } l_2.$$ 

Here $l_2$ denotes the space of square-summable sequences, $a = (a_1, a_2, a_3, \ldots)$, $\sum_{k=1}^{\infty} |a_k|^2 < \infty$ and $(e_j)$ its natural basis, i.e. $e_i = (0, 0, \ldots, 0, 1, 0, \ldots)$, 1 on the $i^{th}$ position.

He showed that the multiple $\lambda S^*$ of the backward shift $S^*$ has both eigenvectors and hypercyclic vectors provided $|\lambda| > 1$. Since the set of hypercyclic vectors is dense, every eigenvector is near to a hypercyclic vector. Thus orbits are very sensitive to initial conditions and this provides a connection between Chaos Theory and hypercyclic vectors. However, the vigorous and systematic study of hypercyclic operators has only been undertaken since the mid-eighties, starting with the Ph.D. of Kitai [11] and later with the paper of Gethner and Shapiro [12]. Since then the theory of hypercyclicity has become the focus of research by many authors.

It is still an open problem whether there is an operator on Hilbert space for which all non-zero vectors are hypercyclic. For general Banach spaces there are such examples, [6].

In [1], Beauzamy considers also other properties of orbits than those related to cyclicity. In [1] conditions are given on operators to have orbits $(T^nx)$ with $\|T^nx\| \to \infty$ and orbits $(T^nx)$ with $\|T^nx\| \to 0$.

As a background for our investigation, and getting closer to it, we should finally mention Weierstrass’s classical theorem from 1885 [14] where he proved that every continuous function on the interval $(0, 1)$ can be approximated uniformly by polynomials $p(t)$.

This can be stated as a result a result on orbits of operators. To do that, let $C(0, 1)$ be
the space of continuous real-valued functions on \((0, 1)\) with \(\|f\| = \sup_{t \in [0, 1]} |f(t)|\). Let \(L_2[0, 1]\) denote the space of square-integrable functions on \([0, 1]\) with \(\|f\| = \left( \int_0^1 |f(t)|^2 \, dt \right)^{\frac{1}{2}}\).

Let \(T\) denote multiplication by \(t\) in the space \(C(0, 1)\) (or \(L_2[0, 1]\)), i.e., \((Tf)(t) = tf(t)\). Weierstrass’s theorem then becomes: The constant function \(f(t) \equiv 1\) is cyclic for \(T\) in the space \(C(0, 1)\). From this, it follows fairly easily that it is also cyclic in \(L_2[0, 1]\).

A strengthening of Weierstrass’s theorem was given in 1910 [15,16] by Müntz-Szász who proved that not only is \(f(t) \equiv 1\) cyclic, but also a sufficiently dense subsequence of the orbit of the constant function \(1\) spans the whole space; More precisely, \(\text{span}\{t^{n_j}\}_{j=0}^\infty \cup \{1\} = C[0, 1]\) (or \(L_2[0, 1]\)) if and only if \(\sum_{j=1}^\infty \frac{1}{n_j} = \infty\).

In our investigation we will study hyperful orbits. An orbit \((x, Tx, T^2x, \ldots)\) is hyperful if every subsequence \((T^{n_j}x)_{j=1}^\infty\) spans the whole space, i.e, \(\text{span}\{T^{n_j}x\}_{j=0}^\infty = B\) for every subsequence \((n_j)\).

We will mainly study diagonal operators on \(l_2\), that is, operators where \(Te_j = \lambda_j e_j\). This class of operators contains the class of integral operators on \(L_2[0, 1]\) with symmetric kernel, i.e, operators \(T\) with \((Tf)(t) = \int_0^1 K(t,s)f(s)\, ds\) where \(K(t,s) = K(s,t)\), \(K \in \text{L}_2((0, 1) \times (0, 1))\).

Such integral operators have a countable orthogonal set of eigenvectors which span \(L_2[0, 1]\), so they can be viewed as diagonal operators on \(l_2\). In fact, integral equations play an important role in Physics and Engineering, in particular many problems in Physics lead to Fredholm’s integral equation of the second kind. One place where they arise is in the study of small oscillations of elastic systems. Further, they enjoy some similar nice properties with self-adjoint linear operators on Hilbert spaces.

This dissertation is organized as follows:

Chapter 1 contains introduction and background information for our investigation.

Chapter 2 contains characterization of those diagonal operators on \(l_2\) which have cyclic
vectors and which are the cyclic vectors.

Chapter 3 contains characterization of those diagonal operators on $l_2$ that have cyclic vectors with hyperful orbits. A consequence of this characterization is that for diagonal operators, either all cyclic vectors have hyperful orbits or none of the cyclic vectors has a hyperful orbit.

In Chapter 4 we study diagonal operators on other spaces other than $l_2$. We also study the case with complex scalars, we show that on every separable Banach space there are operators with hyperful orbits.

In Chapter 5 we summarize some of our results and present directions for further research.
CHAPTER 2
Cyclic Vectors and Convergence of orbits

Throughout this dissertation the notation $T = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots)$ will denote the diagonal operator,

$$T = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots \\ 0 & \lambda_2 & 0 & \ldots \\ 0 & 0 & \lambda_3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

relative to a sequence of vectors $(e_i)$. The sequence $(e_i)$ will be the natural basis of $l_2$ throughout this chapter.

In this chapter we give our first results on orbits of diagonal operators. Our first observation is that if a diagonal operator $T = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots)$, has $\lambda_i = \lambda_j$ for some $i \neq j$ then it will not have any cyclic vectors, since if $\lambda_i = \lambda_j$, $x = (x_1, x_2, \ldots, x_i, \ldots, x_j, \ldots)$, then

$$T^n x = (\lambda_1^n x_1, \lambda_2^n x_2, \ldots, \lambda_i^n x_i, \ldots, \lambda_j^n x_j, \ldots)$$

and so the ratio between the $i^{th}$ and $j^{th}$ coordinate of $T^n x$ is $x_i / x_j$ for all $n$.

So, below we will only consider $T$ where $\lambda_i \neq \lambda_j$ if $i \neq j$ and $\lambda_j > 0$ for all $j$.

Our first result shows that on $l_2$ $x = (x_1, x_2, x_3, \ldots)$ is cyclic if and only if $x_i \neq 0$ for all $i$.

\begin{align*}
\textbf{Theorem 1.} \text{ Let } T = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots \\ 0 & \lambda_2 & 0 & \ldots \\ 0 & 0 & \lambda_3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}, \text{ } T : l_2 \to l_2 \text{ be a diagonal operator with distinct positive eigenvalues.} \\
\text{Then } x = (x_1, x_2, x_3, \ldots) \in l_2 \text{ is cyclic if and only if } x_i \neq 0 \text{ for all } i.
\end{align*}
To prove the theorem we will need the following lemmas:

**Lemma 1.** Let \( T : l_2 \to l_2 \) be defined as before and let \( (e_j)_{j=1}^\infty \) be the standard basis for \( l_2 \). If \( ST = TS \), then \( e_j \) is an eigenvector of \( S \).

**Proof.** Since \( T \) is diagonal, we have that \( Te_j = \lambda_j e_j, \ \forall j \).
Since \( ST = TS \), we have that,

\[
TS(e_j) = ST(e_j) \\
= S(T(e_j)) \\
= S(\lambda_j e_j) \\
= \lambda_j S(e_j)
\]

\( \implies S(e_j) \) is an eigenvector of \( T \) with an eigenvalue \( \lambda_j \)
\( \implies S(e_j) = \gamma e_j \) for some scalar \( \gamma \).
Thus, \( e_j \) is an eigenvector of \( S \) \qed

**Definition 1.** If \( S \) is a subspace of a Hilbert space \( H \), then we define the orthogonal complement of \( S \) by \( S^\perp = \{ f \in H : (f, g) = 0, \ \forall g \in S \} \)

**Remarks:**
(a) \( S^\perp \) is a closed subspace of \( H \)
(b) \( S^\perp \cap S = 0 \)
(c) \( H = S \bigoplus S^\perp \) (the direct sum of \( S \) and \( S^\perp \))

*Note:* The next three lemmas are classical results and can be found in any standard functional analysis textbook.
Lemma 2. If $T : H \to H$ is self-adjoint and $F \subset H$ (closed linear subspace of $H$) is invariant under $T$, then $F^\perp$ is also invariant under $T$.

Proof. Suppose that $F$ is invariant under $T$ and $T$ is self-adjoint, we want to show that $T(F^\perp) \subset F^\perp$.

Let $T^*$ be the adjoint of $T$, but then it is enough to show that $F^\perp$ is invariant under $T^*$ since $T = T^*$.

Now, let $y \in F^\perp$, then $\langle x, y \rangle = 0$, $\forall x \in F$.

But $\langle x, T^* y \rangle = \langle Tx, y \rangle = 0$ since $F$ is invariant under $T$.

Thus, $\langle x, T^* y \rangle = 0$.

Thus, $F^\perp$ is invariant under $T^*$.

Hence, $F^\perp$ is invariant under $T$. $\square$

- In this case we say that $F$ reduces $T$ or that $T$ is reduced by $F$.

Lemma 3. Let $P$ be the orthogonal projection on a closed linear subspace $F$ of $H$.

Then $F$ is invariant under an operator $T$ if and only if $TP = PTP$.

Proof. ($\Rightarrow$:) Let $F$ be invariant under $T$. Let $x \in H$ be an arbitrary vector.

If $x \in F$, then $Px \in F$. Since $F$ be invariant under $T$, we have that $TPx \in F$.

Further, since $P$ is an orthogonal projection and $F$ is the range of $P$, we have $TPx \in F$ implies $P(TPx) = TPx$.

So, $PTP = TP$.

($\Leftarrow$) Conversely, suppose that $PTP = TP$.

We show that $F$ is invariant under $T$. 

Let $x \in F$. Since $P$ is an orthogonal projection with range of $F$ and $x \in F$, we have,

\[
Tx = T(Px) = (TP)(x) \\
= (PTP)(x) \\
= P(TPx)
\]

Thus, $P(TPx) = TPx = Tx$

Thus, $Tx \in F$. Hence, $F$ is invariant under $T$. \hfill \Box

**Lemma 4.** Let $P_F$ be the orthogonal projection on the closed subspace $F \subset H$.

Then $F$ reduces $T \iff TP_F = P_F T$ (i.e, $P_F$ commutes with $T$).

**Proof.** ($\Rightarrow$:) By lemma 2 we know that $F$ reduces $T$ if and only if $F$ is invariant under $T$ and $T^*$.

So, by lemma 3 we have, $T^*P_F = P_F T^*P_F$ and $TP_F = P_F TP_F$.

Now, $T^*P_F = P_F T^*P_F$ if and only if

$(T^*P_F)^* = (P_F T^*P_F)^*$ if and only if

$P_F T = P_F TP_F$ ($P_F^2 = P_F$, $T^{**} = T$).

Hence $F$ reduces $T$ if and only if $P_F T = P_F TP_F$ and $TP_F = P_F TP_F$,

that is, if and only if $TP_F = P_F T$.

($\Leftarrow$:) Conversely, suppose that $TP_F = P_F T$.

Multiplying the above identity by $P_F$ on the right and on the left one has,

$P_F TP_F = P_F^2 T$ and $TP_F^2 = P_F TP_F$.

But $P_F$ is a projection, so $P_F^2 = P_F$. Hence, we get that

$P_F TP_F = P_F T$ and $TP_F = P_F TP_F$, and it follows that $F$ reduces $T$ by lemma 3. \hfill \Box

**Proof of the theorem.** The “only if”-part is straight forward, that is, if $x_i = 0$ for some $i_0$, then $T^n x_{i_0} = 0$ for all $n$.

Hence $T^n x$ does not span the whole space, therefore $x$ is not cyclic.
To prove the “if” direction, we start by writing $H$ as $H = F \bigoplus F^\perp$, then we have that, $F$ reduces $T \iff TP_F = P_TF$ (by lemma 4).

Thus, $P_F$ has the same eigenvectors as $T$ (by lemma 1).

But then lemma 2 says that $F^\perp$ is an invariant subspace of $H$.

But we know that $Te_j = \lambda_je_j \ \forall j$ and,

(i) $P_F(e_j) = e_j$ whenever $e_j \in F$

(ii) $P_F(e_j) = 0$ whenever $e_j \in F^\perp$

So, we have that for the orthogonal projection $P_F$, either $\lambda_j = 1$ or $\lambda_j = 0$ for all $j$.

Thus, all eigenvectors of $T$ for which $P_F$ has the eigenvalue 1, they span $F$ and all eigenvectors of $T$ for which $P_F$ has the eigenvalue 0, they span $F^\perp$.

But $x_j \neq 0$ for all $j$, hence $x = (x_1, x_2, x_3, \ldots)$ must be cyclic. \hfill \square

Remarks:

(1) We do not know if Theorem 1 extends to other sequence spaces, in particular $l_1$ and $c_0$. Some aspects of this problem will be considered in chapter 3.

(2) Theorem 1 characterizes the cyclic vectors in $l_2$ but it does not provide any information on how vectors are cyclic, what type of linear combinations of vectors in the orbit should be used to approximate a given vector in the space. To get results in this direction we will study subclasses of diagonal operators. Let us first consider a $T$ where $\lambda_1 > \lambda_2 > \lambda_3 \ldots$ and all $\lambda_i$’s are positive. Let us also consider an $x$ where $x_i \neq 0$ for all $i$.

We start with an observation. Let $\lambda_1 > \lambda_2 > \lambda_3 \ldots$, and let $T = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots)$ be the diagonal operator, and let $x = (x_1, x_2, x_3, \ldots)$ be a vector in $l_2$ where $x_1 \neq 0$.

Then,

$$
\frac{T^n x}{\lambda_1^n x_1} = \frac{1}{\lambda_1^n x_1} (\lambda_1^n x_1, \lambda_2^n x_2, \lambda_3^n x_3, \ldots) = \left(1, \left(\frac{\lambda_2}{\lambda_1}\right)^n \frac{x_2}{x_1}, \left(\frac{\lambda_3}{\lambda_1}\right)^n \frac{x_3}{x_1}, \ldots\right)
$$

Thus, $\frac{T^n x}{\lambda_1^n x_1} \to (1, 0, 0, \ldots)$ as $n \to \infty$;
and for $n$ large enough we have,
\[
\left\| \frac{T^n x}{\lambda_1^n x_1} - (1, 0, 0, \ldots) \right\| \leq 2 \left( \frac{\lambda_2}{\lambda_1} \right)^n \left| \frac{x_2}{x_1} \right|
\]

Further, since
\[
\|T^n x\|^2 = (\lambda_1^n x_1)^2 \left( 1 + \left( \frac{\lambda_2}{\lambda_1} \right)^{2n} \left( \frac{x_2}{x_1} \right)^2 + \left( \frac{\lambda_3}{\lambda_1} \right)^{2n} \left( \frac{x_3}{x_1} \right)^2 + \ldots \right)
\]

We also get,
\[
\left\| \frac{T^n x}{\|T^n x\|} - (1, 0, 0, \ldots) \right\| \leq 2 \left( \frac{\lambda_2}{\lambda_1} \right)^n \left| \frac{x_2}{x_1} \right|
\]
for $n$ large enough. Thus, the orbit converges. In order to study this further we will first study diagonal operators with decreasing eigenvalues.

In the next theorem we will study how much faster we can get convergence to \( x = (1, 0, 0, \ldots) \) if we use larger linear combinations.

**Definition 2.** We put \( d(x, A) = \inf_{y \in A} d(x, y) \)

**Theorem 2.** Let \( T : l_2 \to l_2, \ T = \text{diag} (\lambda_1, \lambda_2, \lambda_3, \ldots) \) be an infinite diagonal operator with distinct eigenvalues decreasing to zero; that is, \( \lambda_1 > \lambda_2 > \lambda_3 \ldots \to 0 \).

Suppose \( x = (x_1, x_2, x_3, \ldots) \in l_2 \) where \( x_i \neq 0 \ \forall \ i \). Then,
\[
d \left( (1, 0, 0, \ldots), \text{span}\{T^n x, T^{(n-1)} x, \ldots, T^{(n-(k-1))} x\} \right) \leq C_{(k,x)} \left( \frac{\lambda_k}{\lambda_1} \right)^n,
\]
where \( C_{(k,x)} \) depends on \( k \).

**Proof.** We will construct a linear combination of the elements \( \{T^j x\}_{j=n}^{n-(k-1)} \) that gives our estimate.

We first give the proofs for \( k = 2 \) and \( k = 3 \) and then pass to the general case.
Case 1: We estimate (1) by using the first two elements of the orbit, $T^nx$ and $T^{n-1}x$.

Notice that,

$$T^nx = \{\lambda_1^n x_1, \lambda_2^n x_2, \lambda_3^n x_3, \ldots\}$$

and

$$T^{n-1}x = \{\lambda_1^{n-1} x_1, \lambda_2^{n-1} x_2, \lambda_3^{n-1} x_3, \ldots\}.$$

Since $\lambda_1 > \lambda_2 > \lambda_3 \ldots \rightarrow 0$, one has that,

$$\|T^nx\| \approx \lambda_1^n x_1 \text{ and } \|T^{n-1}x\| \approx \lambda_1^{n-1} x_1 \text{ for large enough } n,$$

where the symbol $\approx$ means that the ratio between the sides tends to 1 as $n$ tends to $\infty$. Now, let,

$$Z_1 = \frac{T^nx}{\lambda_1^n x_1} = \left(1, \left(\frac{\lambda_2}{\lambda_1}\right)^n x_2, \left(\frac{\lambda_3}{\lambda_1}\right)^n x_3, \ldots\right)$$

and

$$Z_2 = \frac{T^{n-1}x}{\lambda_1^n x_1} = \left(1, \left(\frac{\lambda_2}{\lambda_1}\right)^{n-1} x_2, \left(\frac{\lambda_3}{\lambda_1}\right)^{n-1} x_3, \ldots\right).$$

We show that,

$$d \left((1, 0, 0, \ldots), \text{span}\{T^nx, T^{(n-1)}x\}\right) \leq C_{(3,x)} \left(\frac{\lambda_3}{\lambda_1}\right)^n \quad (2)$$

Indeed, if we take $a_1 = \left(\frac{\lambda_2}{\lambda_1}\right)^n$ and $a_2 = \left(\frac{\lambda_2}{\lambda_1}\right)^{n-1}$, then by elementary computations we get,

$$\frac{Z_2}{a_2} - \frac{Z_1}{a_1} = \left(-1 + \left(\frac{\lambda_2}{\lambda_1}\right)^n x_2, 0, \left(\frac{\lambda_3}{\lambda_1}\right)^n x_3, \ldots\right)$$

$$\quad \implies \left\|\frac{Z_2}{a_2} - \frac{Z_1}{a_1}\right\| \approx \left(\frac{\lambda_2}{\lambda_1} - 1\right) \left(\frac{\lambda_2}{\lambda_1}\right)^n \frac{x_2}{x_1}$$

for large enough $n$.

By letting $Z_{1,2} = \frac{z_2 - z_1}{a_2 - a_1}$ we have that, for a suitable $b$

$$b \frac{Z_{1,2}}{\|Z_{1,2}\|} = \left(1, 0, \left(\frac{\lambda_2}{\lambda_3} - 1\right) \left(\frac{\lambda_3}{\lambda_1}\right)^n x_3, \frac{x_2}{(x_2)^2}, \ldots\right).$$
So,
\[
d \left( (1,0,0,\ldots), \text{span} \{ Z_1, Z_2 \} \right) \approx \frac{\left( \frac{\lambda_2}{\lambda_3} - 1 \right) x_3 x_1}{(\frac{\lambda_1}{\lambda_3} - 1) (x_2)^2} \left( \frac{\lambda_3}{\lambda_1} \right)^n.
\]
Hence,
\[
d \left( (1,0,0,\ldots), \text{span} \{ T^n x, T^{(n-1)} x \} \right) \leq C_{(3,x)} \left( \frac{\lambda_3}{\lambda_1} \right)^n
\]
where
\[
C_{(3,x)} = \frac{\left( \frac{\lambda_2}{\lambda_3} - 1 \right) x_3 x_1}{(\frac{\lambda_1}{\lambda_3} - 1) (x_2)^2}.
\]

Case 2: Now, we estimate (1) by using the first three elements of the orbit, \( T^n x \), \( T^{n-1} x \) and \( T^{n-2} x \).

Let \( Z_1, Z_2, a_1, a_2 \) be defined as in step 1 above, \( a_3 = \left( \frac{\lambda_2}{\lambda_3} \right)^{n-2} \) and choose \( b_1 \) such that
\[
Z_3 = b_1 \frac{T^{n-2} x}{\| T^{n-2} x \|} = \left( 1, \left( \frac{\lambda_2}{\lambda_1} \right)^{n-2} \frac{x_2}{x_1}, \left( \frac{\lambda_3}{\lambda_1} \right)^{n-2} \frac{x_3}{x_1}, \ldots \right).
\]
We show that,
\[
d \left( (1,0,0,\ldots), \text{span} \{ T^n x, T^{(n-1)} x, T^{n-2} x \} \right) \leq C_{(4,x)} \left( \frac{\lambda_4}{\lambda_1} \right)^n
\]
(3)
If we let \( c = \frac{\lambda_2}{\lambda_3} + 1 \), then we have the following;
\[
\frac{Z_3 - cZ_2}{a_3} + \frac{(c-1)Z_1}{a_1} = \left( \frac{\lambda_2 \lambda_3 - \lambda_1 \lambda_2}{\lambda_1 \lambda_3} \right) \left( \frac{\lambda_2}{\lambda_1} - 1 \right) \left( \frac{\lambda_2}{\lambda_2} \right)^n \frac{x_1}{x_2}, 0, 0, \left( \frac{\lambda_2 \lambda_3 - \lambda_1 \lambda_4}{\lambda_4 \lambda_3} \right) \left( \frac{\lambda_2}{\lambda_4} - 1 \right) \left( \frac{\lambda_4}{\lambda_2} \right)^n \frac{x_1}{x_2}, \ldots \right)
\]
By letting
\[
Z_{1,2,3} = \frac{Z_3}{a_3} - \frac{cZ_2}{a_2} + \frac{(c-1)Z_1}{a_1},
\]
one has, for a suitable \( b_2 \),
\[
b_2 \frac{Z_{1,2,3}}{\| Z_{1,2,3} \|} = \left( 1, 0, 0, \left( \frac{\lambda_2 \lambda_3 - \lambda_1 \lambda_4}{\lambda_1 \lambda_3} \right) \left( \frac{\lambda_2}{\lambda_4} - 1 \right) \frac{x_4}{x_1} \left( \frac{\lambda_4}{\lambda_1} \right)^n, \ldots \right)
\]
So,
\[
d \left( (1,0,0,\ldots), \text{span} \{ Z_1, Z_2, Z_3 \} \right) \approx \frac{\left( \frac{\lambda_4}{\lambda_3} - 1 \right) x_4}{\left( \frac{\lambda_4}{\lambda_1} \right)^n}.
\]
Thus,

\[ d\left( (1,0,0,\ldots), \text{span}\{T^n x, T^{(n-1)} x, T^{n-2} x\} \right) \approx \frac{\lambda_2 \lambda_3 - \lambda_2 \lambda_4}{\lambda_4 \lambda_3} \left( \frac{\lambda_2}{\lambda_4} - 1 \right) x_4 \left( \frac{\lambda_4}{\lambda_1} \right)^n. \]

Therefore,

\[ d\left( (1,0,0,\ldots), \text{span}\{T^n x, T^{(n-1)} x, T^{(n-2)} x\} \right) \leq C(4,x) \left( \frac{\lambda_4}{\lambda_1} \right)^n \]

where

\[ C(4,x) = \frac{\lambda_2 \lambda_3 - \lambda_2 \lambda_4}{\lambda_4 \lambda_3} \left( \frac{\lambda_2}{\lambda_4} - 1 \right) x_4 \left( \frac{\lambda_4}{\lambda_1} - 1 \right) x_1. \]

**General Case:** In the view of the cases 1 and 2 above, the proof of the Theorem will be complete if we can take the linear combinations of \( k \) elements of the orbit and produce a vector with first coordinate one, the rest zeros except the \( k^{th} \) position, then estimate the distance between this vector and the basis vector \((1,0,0,\ldots)\).

To do this, we first multiply the normalized elements of the orbit \( Z_1, Z_2, Z_3, \ldots, Z_k \) by the scalars \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k \) to get,

\[ \alpha_1 Z_1 = \alpha_1 \left( 1, \left( \frac{\lambda_2}{\lambda_1} \right)^n x_2, \left( \frac{\lambda_3}{\lambda_1} \right)^n x_3, \ldots \right) \]

\[ \alpha_2 Z_2 = \alpha_2 \left( 1, \left( \frac{\lambda_2}{\lambda_1} \right)^{n-1} x_2, \left( \frac{\lambda_3}{\lambda_1} \right)^{n-1} x_3, \ldots \right) \]

\[ \alpha_3 Z_3 = \alpha_3 \left( 1, \left( \frac{\lambda_2}{\lambda_1} \right)^{n-2} x_2, \left( \frac{\lambda_3}{\lambda_1} \right)^{n-2} x_3, \ldots \right) \]

\[ \vdots \]

\[ \alpha_k Z_k = \alpha_k \left( 1, \left( \frac{\lambda_k}{\lambda_1} \right)^{n-(k-1)} x_2, \left( \frac{\lambda_k}{\lambda_1} \right)^{n-(k-1)} x_3, \ldots \right) \]
But then we need the following to happen:

\[
\begin{align*}
\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_k &= 1 \\
\alpha_1 + \alpha_2 \left( \frac{\lambda_2}{\lambda_1} \right)^{n-1} \frac{x_2}{x_1} + \ldots + \alpha_k \left( \frac{\lambda_k}{\lambda_1} \right)^{n-(k-1)} \frac{x_k}{x_1} &= 0 \\
\alpha_1 \left( \frac{\lambda_3}{\lambda_1} \right)^n \frac{x_2}{x_1} + \alpha_2 \left( \frac{\lambda_3}{\lambda_1} \right)^{n-1} \frac{x_2}{x_1} + \ldots + \alpha_k \left( \frac{\lambda_3}{\lambda_1} \right)^{n-(k-1)} \frac{x_k}{x_1} &= 0 \\
& \quad = : \\
\alpha_1 \left( \frac{\lambda_k}{\lambda_1} \right)^n \frac{x_2}{x_1} + \alpha_2 \left( \frac{\lambda_k}{\lambda_1} \right)^{n-1} \frac{x_2}{x_1} + \ldots + \alpha_k \left( \frac{\lambda_k}{\lambda_1} \right)^{n-(k-1)} \frac{x_k}{x_1} &= 0
\end{align*}
\] (5)

Notice that we have a system of equations, thus it suffices to show that (5) has a solution.

Now, since \( \lambda_j \neq 0 \) for \( j = 2, 3, \ldots, k \), we can divide each equation from the second equation by \( \left( \frac{\lambda_k}{\lambda_1} \right)^{n-(k-2)} \) to get the following equivalent system:

\[
\begin{pmatrix}
1 & 1 & \ldots & 1 & 1 \\
(\frac{\lambda_2}{\lambda_1})^{k-1} \frac{x_2}{x_1} & (\frac{\lambda_2}{\lambda_1})^{k-2} \frac{x_2}{x_1} & \ldots & (\frac{\lambda_2}{\lambda_1})^2 \frac{x_2}{x_1} & (\frac{\lambda_2}{\lambda_1}) \frac{x_2}{x_1} \\
(\frac{\lambda_3}{\lambda_1})^{k-1} \frac{x_3}{x_1} & (\frac{\lambda_3}{\lambda_1})^{k-2} \frac{x_3}{x_1} & \ldots & (\frac{\lambda_3}{\lambda_1})^2 \frac{x_3}{x_1} & (\frac{\lambda_3}{\lambda_1}) \frac{x_3}{x_1} \\
& \vdots & \ddots & \vdots & \vdots \\
(\frac{\lambda_k}{\lambda_1})^{k-1} \frac{x_k}{x_1} & (\frac{\lambda_k}{\lambda_1})^{k-2} \frac{x_k}{x_1} & \ldots & (\frac{\lambda_k}{\lambda_1})^2 \frac{x_k}{x_1} & (\frac{\lambda_k}{\lambda_1}) \frac{x_k}{x_1}
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\vdots \\
\alpha_k
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

To this very end, it remains to show that the determinant of the matrix

\[
A = 
\begin{pmatrix}
1 & 1 & \ldots & 1 & 1 \\
(\frac{\lambda_2}{\lambda_1})^{k-1} \frac{x_2}{x_1} & (\frac{\lambda_2}{\lambda_1})^{k-2} \frac{x_2}{x_1} & \ldots & (\frac{\lambda_2}{\lambda_1})^2 \frac{x_2}{x_1} & (\frac{\lambda_2}{\lambda_1}) \frac{x_2}{x_1} \\
(\frac{\lambda_3}{\lambda_1})^{k-1} \frac{x_3}{x_1} & (\frac{\lambda_3}{\lambda_1})^{k-2} \frac{x_3}{x_1} & \ldots & (\frac{\lambda_3}{\lambda_1})^2 \frac{x_3}{x_1} & (\frac{\lambda_3}{\lambda_1}) \frac{x_3}{x_1} \\
& \vdots & \ddots & \vdots & \vdots \\
(\frac{\lambda_k}{\lambda_1})^{k-1} \frac{x_k}{x_1} & (\frac{\lambda_k}{\lambda_1})^{k-2} \frac{x_k}{x_1} & \ldots & (\frac{\lambda_k}{\lambda_1})^2 \frac{x_k}{x_1} & (\frac{\lambda_k}{\lambda_1}) \frac{x_k}{x_1}
\end{pmatrix}
\]

is different from 0 since we have a non-homogeneous system.
We see that each row of the matrix $A$ is a geometric progression, hence $A$ is a $k \times k$ VanderMonde matrix, hence $|A|$ can be computed from the well known formula for VanderMonde matrix and we get,

$$|A| = \frac{1}{\lambda_1^{k-1} x_1^{k-1}} \prod_{1 \leq i < j \leq k} (\gamma_j - \gamma_i)$$

which is clearly $\neq 0$.

$\square$

Remark: We see that, $\sum_{j=1}^{k} \alpha_j z_j = \left(1,0,0,\ldots,0,C\left(\frac{\lambda_k}{\lambda_1}\right)^n\right)$ where $C$ just depends on $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $x$.

In the next theorems we study the convergence of orbits.

**Theorem 3.** Let $T = \begin{pmatrix} r_1 & 0 & 0 & \ldots \\ 0 & r_2 & 0 & \ldots \\ 0 & 0 & r_3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$, $T : l_2 \to l_2$ be such that $r_i \neq r_j$ for $i \neq j$, and the $r_i$’s are real numbers between 0 and 1.

Let $x = (x_1, x_2, x_3, \ldots) \in l_2$ where $x_i \neq 0$ for all $i$. Suppose further that $\sup_i r_i = 1$ and $\inf_i r_i = 0$.

(a) If $\sup_j r_j = r_{j_0} = 1$ for some $j_0$ then,

$$\lim_{n \to \infty} \frac{T^n x}{\|T^n x\|} = e_{j_0} \text{sign}(x_{j_0})$$

where $e_{j_0} = (0,0,\ldots,1,0,0,\ldots)$, one on the $j_0$-position and zero else where.

(b) If the supremum is not attained then $\frac{T^n x}{\|T^n x\|} \to 0$ weakly.

Proof. (a) Let $x = (x_1, x_2, x_3, x_{j_0}, x_{j_0+1}, x_{j_0+1} \ldots)$, where $x_{j_0} \neq 0$ and $r_{j_0} = 1$

We have

$$T^n x = (r_1^n x_1, r_2^n x_2, r_3^n x_3, \ldots, r_{j_0-1}^n x_{j_0}, r_{j_0}^n x_{j_0}, r_{j_0+1}^n x_{j_0+1}, \ldots)$$

So,

$$\|T^n x - (0,0,0,x_{j_0},0,0,\ldots)\|_{l_2} =$$
\[ \| (r_{n_1}^n x_1, r_{n_2}^n x_2, r_{n_3}^n x_3, \ldots, r_{n_{j_0}-1}^n x_{j_0-1}, r_{n_{j_0}}^n x_{j_0}, r_{n_{j_0}+1}^n x_{j_0+1}, \ldots) - (0, 0, 0, x_{j_0}, 0, 0, 0, \ldots) \|_2 \]

\[ = \| r_{n_1}^n x_1, r_{n_2}^n x_2, r_{n_3}^n x_3, \ldots, r_{n_{j_0}-1}^n x_{j_0-1}, 0, r_{n_{j_0}+1}^n x_{j_0+1}, \ldots \|_2 \]

\[ = \sqrt{r_{n_1}^{2n} x_1^2 + r_{n_2}^{2n} x_2^2 + r_{n_3}^{2n} x_3^2 + \ldots + r_{n_{j_0}-1}^{2n} x_{j_0-1}^2 + 0 + r_{n_{j_0}+1}^{2n} x_{j_0+1}^2 + \ldots} \]

But \( x = (x_1, x_2, x_3, \ldots) \in l_2 \), so \( \exists \ N \in \mathbb{N} \) such that \( \sum_{k=N+1}^\infty |x_k|^2 < \epsilon \) \( \forall \epsilon > 0 \).

Since \( 0 < r_j < 1, \forall j \neq j_0 \), we have that,

\[ \sum_{k=N+1}^\infty |x_k|^2 |r_k^n|^2 < \epsilon \]

Notice that,

\[ \sqrt{r_{n_1}^{2n} x_1^2 + r_{n_2}^{2n} x_2^2 + r_{n_3}^{2n} x_3^2 + \ldots + r_{n_{j_0}-1}^{2n} x_{j_0-1}^2 + 0 + r_{n_{j_0}+1}^{2n} x_{j_0+1}^2 + \ldots} = \]

\[ = \sqrt{\sum_{k=1}^N |x_k|^2 |r_k^n|^2 + \sum_{k=N+1}^\infty |x_k|^2 |r_k^n|^2} \]

\[ = \sqrt{I + II} \]

where \( I = \sum_{k=1}^N |x_k|^2 |r_k^n|^2 \) and \( II = \sum_{k=N+1}^\infty |x_k|^2 |r_k^n|^2 \).

But \( II < \epsilon_1 \) by the arguments above.

Thus, it remains to show that \( I < \epsilon_2 \) too.

Indeed \( I < \epsilon_2 \) since there are only finitely many terms and the \( r_j \)'s are such that \( 0 < r_j < 1, \forall j \neq j_0 \).

Hence, we have shown that \( T^nx \to (0, 0, 0, \ldots, x_{j_0}, 0, 0, 0, \ldots) \).

Thus, \( \frac{T^nx}{\|T^nx\|} \to e_{j_0} \text{sign}(x_{j_0}) \), and that finishes the proof of (a).

(b) Let \( \frac{T^nx}{\|T^nx\|} = (a_1, a_2, a_3, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots) \).
Now, if the supremum is not attained, then for any particular \( j \) there exists \( r_j \) such that
\[ r_j > r_{j_0}, \text{ for some } j > j_0. \]

We have,
\[
\frac{T^n x}{\|T^n x\|} = \left( \cdots, \frac{r^n_{j_0} x_{j_0}}{\|T^n x\|}, \cdots, \frac{r^n_{j_0-1} x_{j_0-1}}{\|T^n x\|}, \frac{r^n_{j_0} x_{j_0}}{\|T^n x\|}, \frac{r^n_{j_0+1} x_{j_0+1}}{\|T^n x\|}, \cdots \right).
\]

But by assumption,
\[
\frac{r^n_{j_0} x_{j_0}}{\|T^n x\|} \to a_j \text{ and } \frac{r^n_{j_0} x_{j_0}}{\|T^n x\|} \to a_{j_0}
\]
\[ \Rightarrow \frac{a_{j_0}}{a_j} = \lim_{j \to \infty} \frac{r^n_{j_0} x_{j_0}}{r^n_{j_1} x_{j_1}} = 0 \text{ since } r_j > r_{j_0}, \text{ for some } j > j_0. \]

Further, we have that \( \| \frac{T^n x}{\|T^n x\|} \| = 1 \), so the sequence \( \left( \frac{T^n x}{\|T^n x\|} \right) \) is bounded.

Thus, we have shown that the coordinates of \( \frac{T^n x}{\|T^n x\|} \to 0 \) pointwise and the sequence is bounded.

Hence, \( \frac{T^n x}{\|T^n x\|} \to 0 \) weakly, which completes the proof of (b).

**Theorem 4.** Let the operator \( T \) and the vector \( x \) be defined as in the previous theorem.

Suppose that \( \sup_i r_i = 1 \) but 1 is not attained.

Then \( \lim_{n \to \infty} T^n x = 0 \), in the norm topology for every \( x \).

**Proof.** We want to show that \( \|T^n x - 0\|_2 = \|T^n x\|_2 \to 0 \) as \( n \to \infty \).

As before, \( x = (x_1, x_2, x_3, \ldots) \in l_2 \), so \( \exists \ N \in \mathbb{N} \) such that
\[
\sum_{k=N+1}^{\infty} |x_k|^2 < \epsilon \quad \forall \epsilon > 0.
\]

Now we have,
\[
\|T^n x\|_2 = \sqrt{r_1^{2n} x_1^2 + r_2^{2n} x_2^2 + r_3^{2n} x_3^2 + \ldots + r_j^{2n} x_j^2 + r_{j+1}^{2n} x_{j+1}^2 + \ldots}
\]
\[
= \sqrt{r_1^{2n}x_1^2 + r_2^{2n}x_2^2 + r_3^{2n}x_3^2 + \ldots + r_j^{2n}x_j^2 + r_{j+1}^{2n}x_{j+1}^2 + \ldots}
\]

\[
= \sqrt{r_1^{2n}x_1^2 + r_2^{2n}x_2^2 + \ldots + r_N^{2n}x_N^2 + r_{N+1}^{2n}x_{N+1}^2 + r_{N+2}^{2n}x_{N+2}^2 + \ldots}
\]

\[
= \sqrt{A + B}
\]

where \( A = r_1^{2n}x_1^2 + r_2^{2n}x_2^2 + \ldots + r_N^{2n}x_N^2 \) and \( B = r_{N+1}^{2n}x_{N+1}^2 + r_{N+2}^{2n}x_{N+2}^2 + \ldots \).

Notice \( A < \epsilon_1 \) by the remark above \( (x \in l_2) \) and the fact that \( 0 < r_j < 1, \forall j \).

But also \( B < \epsilon_2 \) since there are only finitely many terms and the \( r_j \)'s are such that \( 0 < r_j < 1, \forall j \).

\( \implies \sqrt{A + B} < \sqrt{\epsilon_1 + \epsilon_2} < \epsilon_3 \), and the theorem is proved. \( \square \)
CHAPTER 3
Hyperful orbits on $l_2$

In this chapter we give necessary and sufficient conditions for a diagonal operator $T : l_2 \to l_2$ to have hyperful orbit. The main results of this chapter are Theorem 8 and Corollary 2. Extensions to some operators on some classes of Banach space are discussed in the next chapter.

**Theorem 5.** Let $T = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$, $T : l_2 \to l_2$,

be a diagonal operator with distinct eigenvalues such that $\lambda_1 > \lambda_2 > \lambda_3 \ldots \to a \geq 0$.

Suppose that $x$ is any vector in $l_2$ with all nonzero coordinates.

Then every subsequence of $\{T^n x : n \in \mathbb{N}\}$ spans $l_2$.

**Proof.** Let $x = (x_1, x_2, x_3, \ldots) \in l_2$ where $x_i \neq 0 \forall i$.

First, we show how to approximate the first basis vector, $(1, 0, 0, 0, \ldots)$.

We know that $T^n x = \{\lambda^n_1 x_1, \lambda^n_2 x_2, \lambda^n_3 x_3, \ldots\}$ for any natural number $n$.

Notice that if $n$ is large enough then $\lambda^n_1 x_1 \gg \lambda^n_j x_j$ for all $j = 2, 3, \ldots$

$\implies \|T^n x\| \approx \lambda^n_1 x_1$ as $n \to \infty$

**Note:** Here $\approx$ means that the difference between the sides tends to 0. So,

$$\frac{T^n x}{\|T^n x\|} \approx \left( 1, \frac{\lambda_2}{\lambda_1} \frac{x_2}{x_1}, \frac{\lambda_3}{\lambda_1} \frac{x_3}{x_1}, \ldots \right) \to (1, 0, 0, 0, \ldots)$$

as $n \to \infty$

Thus, we have the first basis vector.
Next we show how we can approximate the second basis vector, \((0, 1, 0, 0, \ldots)\).

For any natural numbers \(k\) and \(l\) and some coefficients \(a\) and \(b\) near 1 one has,

\[
a \frac{T^k x}{\|T^k x\|} = \left(1, \left(\frac{\lambda_2}{\lambda_1}\right)^k \frac{x_2}{x_1}, \left(\frac{\lambda_3}{\lambda_1}\right)^k \frac{x_3}{x_1}, \ldots \right)
\]

and

\[
b \frac{T^l x}{\|T^l x\|} = \left(1, \left(\frac{\lambda_2}{\lambda_1}\right)^l \frac{x_2}{x_1}, \left(\frac{\lambda_3}{\lambda_1}\right)^l \frac{x_3}{x_1}, \ldots \right)
\]

as \(k, l \to \infty\).

Now,

\[
a \frac{T^k x}{\|T^k x\|} - b \frac{T^l x}{\|T^l x\|} = \left(1, \left(\frac{\lambda_2}{\lambda_1}\right)^k \frac{x_2}{x_1}, \left(\frac{\lambda_3}{\lambda_1}\right)^k \frac{x_3}{x_1}, \ldots \right) - \left(1, \left(\frac{\lambda_2}{\lambda_1}\right)^l \frac{x_2}{x_1}, \left(\frac{\lambda_3}{\lambda_1}\right)^l \frac{x_3}{x_1}, \ldots \right)
\]

\[
= \left(0, \left(\frac{\lambda_2}{\lambda_1}\right)^k \frac{x_2}{x_1} - \left(\frac{\lambda_2}{\lambda_1}\right)^l \frac{x_2}{x_1}, \left(\frac{\lambda_3}{\lambda_1}\right)^k \frac{x_3}{x_1} - \left(\frac{\lambda_3}{\lambda_1}\right)^l \frac{x_3}{x_1}, \ldots \right)
\]

\[
= \left(0, \left(\frac{\lambda_2}{\lambda_1}\right)^k \frac{x_2}{x_1} \left(1 - \left(\frac{\lambda_2}{\lambda_1}\right)^{k-l}\right), \left(\frac{\lambda_3}{\lambda_1}\right)^k \frac{x_3}{x_1} \left(1 - \left(\frac{\lambda_3}{\lambda_1}\right)^{k-l}\right), \ldots \right)
\]

Put \(k = l + r\),

Then,

\[
a \frac{T^{l+r} x}{\|T^{l+r} x\|} - b \frac{T^l x}{\|T^l x\|} = \left(0, \left(\frac{\lambda_2}{\lambda_1}\right)^l \frac{x_2}{x_1} \left(1 - \left(\frac{\lambda_2}{\lambda_1}\right)^r\right), \left(\frac{\lambda_3}{\lambda_1}\right)^l \frac{x_3}{x_1} \left(1 - \left(\frac{\lambda_3}{\lambda_1}\right)^r\right), \ldots \right)
\]

So that

\[
\left\| a \frac{T^{l+r} x}{\|T^{l+r} x\|} - b \frac{T^l x}{\|T^l x\|} \right\| \approx \left(\frac{\lambda_2}{\lambda_1}\right)^l \frac{x_2}{x_1} \left(1 - \left(\frac{\lambda_2}{\lambda_1}\right)^r\right)
\]

as \(l, r \to \infty\).

Thus,

\[
\left\| \frac{T^k x}{\|T^k x\|} - \frac{T^l x}{\|T^l x\|} \right\| \to (0, 1, 0, 0, \ldots)
\]
as $k, l \to \infty$, and we get the second basis vector.

To approximate the third basis vector, $(0, 0, 1, 0, 0, \ldots)$ we proceed as follows:

We take $c_1 = \left(\frac{\lambda_2}{\lambda_1}\right)^k$ and $c_2 = \left(\frac{\lambda_3}{\lambda_1}\right)^{k-1}$, and let

$$S_1 = \frac{T^k x}{\|T^k x\|} \approx \left(1, \left(\frac{\lambda_2}{\lambda_1}\right)^k \frac{x_2}{x_1}, \left(\frac{\lambda_3}{\lambda_1}\right)^k \frac{x_3}{x_1}, \ldots\right)$$

and

$$S_2 = \frac{T^{k-1} x}{\|T^{k-1} x\|} \approx \left(1, \left(\frac{\lambda_2}{\lambda_1}\right)^{k-1} \frac{x_2}{x_1}, \left(\frac{\lambda_3}{\lambda_1}\right)^{k-1} \frac{x_3}{x_1}, \ldots\right).$$

Then we have,

$$\frac{S_2}{c_2} - \frac{S_1}{c_1} = \left(\frac{\lambda_2}{\lambda_1} - 1\right) \left(\frac{\lambda_2}{\lambda_1}\right)^k \frac{x_2}{x_1}, 0, \left(\frac{\lambda_3}{\lambda_1} - 1\right) \left(\frac{\lambda_3}{\lambda_1}\right)^k \frac{x_3}{x_1}, \ldots\right).$$

So that

$$\left\| \frac{S_2}{c_2} - \frac{S_1}{c_1} \right\| \approx \left(\frac{\lambda_2}{\lambda_1} - 1\right) \left(\frac{\lambda_2}{\lambda_1}\right)^k \frac{x_2}{x_1}$$

for $k$ large enough.

Now, by letting $S_{1,2} = \frac{s_2 - s_1}{s_2 - s_2}$, we get,

$$\frac{S_{1,2}}{\|S_{1,2}\|} \approx \left(1, 0, \left(\frac{\lambda_3}{\lambda_1} - 1\right) \left(\frac{\lambda_3}{\lambda_1}\right)^k \frac{x_3 x_1}{(x_2)^2}, \ldots\right)$$

Now, using this vector and the first basis vector obtained above we have that,

$$\frac{S_{1,2}}{\|S_{1,2}\|} - (1, 0, 0, 0, 0, \ldots) = \left(1, 0, \left(\frac{\lambda_3}{\lambda_1} - 1\right) \left(\frac{\lambda_3}{\lambda_1}\right)^k \frac{x_3 x_1}{(x_2)^2}, \ldots\right) - (1, 0, 0, 0, 0, \ldots)$$

$$= \left(0, 0, \left(\frac{\lambda_3}{\lambda_1} - 1\right) \left(\frac{\lambda_3}{\lambda_1}\right)^k \frac{x_3 x_1}{(x_2)^2}, \ldots\right)$$

which converges to $(0, 0, 1, 0, 0, \ldots)$ as $k \to \infty$.

Continuing on in this manner it is clear that we can approximate any of the standard basis vectors with linear combinations of the $\{T^n x : n \in \mathbb{N}\}$.\qed
Theorem 6. Let $T = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots \\ 0 & \lambda_2 & 0 & \ldots \\ 0 & 0 & \lambda_3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$, $T : l_2 \to l_2$, be a diagonal operator with distinct eigenvalues. Let $x = (x_1, x_2, x_3, \ldots) \in l_2$ be such that every coordinate is different from zero. Suppose that the $\{\lambda_j\}$ run through real numbers between 0 and 1, such that such that $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots \to 1$.

Then not every subsequence of $\{T^nx : n \in \mathbb{N}\}$ spans $l_2$.

We will give 2 proofs for this Theorem. The first proof is a simple consequence of Theorem 3(b) and is very short. However, the concrete technique of the second proof should be useful to give results on the problem how dense a sequence $(n_j)$ needs to be in order that $(T^{n_j})$ spans the whole space.

Proof 1. We use Theorem 3(b). It gives that $\frac{T^n x}{\|T^n x\|} \to 0$ weakly.

Thus, there is a subsequence $\left( \frac{T^{n_j} x}{\|T^{n_j} x\|} \right)$ which is a normalized Schauder basis for its span. Thus, by removing just one element from $\left( \frac{T^{n_j} x}{\|T^{n_j} x\|} \right)$ we get a subsequence with a strictly smaller span, so it can not be all of $l_2$.

Proof 2. This proof is a bit technical, so we will start our proof by first considering a concrete example that indicates how the general case works.

Step 1: Suppose $x = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots)$ and take

$$\lambda_n = 1 - \frac{1}{n+1}, \quad n \geq 1$$

$$= \left( \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \right)$$
We want to show that if we take a subsequence \( \{T^{10^{n(n+1)/2}}x : n \in \mathbb{N}\} \) of the sequence \( \{T^n x : n \in \mathbb{N}\} \), then we will not be able to approximate the first basis vector \((1, 0, 0, 0, 0, \ldots)\) by the span of this subsequence.

As before, 

\[ T^n x = \{\lambda^n_1 x_1, \lambda^n_2 x_2, \lambda^n_3 x_3, \ldots\} \]

Let \( n_j = 10^{j(j+1)/2}, j = 1, 2, 3, \ldots \)

We have,

\[ T^{n_1} x = \left(1, \frac{1}{2} \left(\frac{\lambda_2}{\lambda_1}\right)^{n_1}, \frac{1}{3} \left(\frac{\lambda_3}{\lambda_1}\right)^{n_1}, \ldots\right) \]

\[ T^{n_2} x = \left(1, \frac{1}{2} \left(\frac{\lambda_2}{\lambda_1}\right)^{n_2}, \frac{1}{3} \left(\frac{\lambda_3}{\lambda_1}\right)^{n_2}, \ldots\right) \]

\[ T^{n_3} x = \left(1, \frac{1}{2} \left(\frac{\lambda_2}{\lambda_1}\right)^{n_3}, \frac{1}{3} \left(\frac{\lambda_3}{\lambda_1}\right)^{n_3}, \ldots\right) \]

etc.

Now, assume that \( \sum_{j=1}^{\infty} a_j T_j x \lambda_j^{-n_j} = (1, \epsilon_1, \epsilon_2, \epsilon_3, \ldots) \) where \( \sum_{j=1}^{\infty} \epsilon_j^2 < \epsilon < \frac{1}{100} \), so \( \epsilon_j < \frac{1}{10} \) for each \( j \). Since \( \sum_{j=1}^{\infty} a_j = 1 \), there is a smallest \( j \) such that \( |a_j| > \frac{1}{2j^3} \).

We have that,

\[ |a_j| \frac{T_j x}{\lambda_j^{n_j}} > \frac{1}{2j^2} \frac{T_j x}{\lambda_j^{n_j}} = \frac{1}{2j^2} \left(1, \frac{1}{2} \left(\frac{\lambda_2}{\lambda_1}\right)^{n_j}, \frac{1}{3} \left(\frac{\lambda_3}{\lambda_1}\right)^{n_j}, \ldots\right) \]

\[ = \left(\frac{1}{2j^2}, \frac{1}{4j^2} \left(\frac{\lambda_2}{\lambda_1}\right)^{n_j}, \frac{1}{6j^2} \left(\frac{\lambda_3}{\lambda_1}\right)^{n_j}, \ldots\right) \]

so that if \( m < j \) then \( |a_m| \leq \frac{1}{2m^2} \).

Now, suppose \( j = j_0 \), then we get \( |a_{j_0}| \geq \frac{1}{2j_0^3} \).
so that our first vector \( \frac{T^{n_{j_0}} x}{\lambda_1^{n_{j_0}}} \) becomes

\[
\frac{T^{n_{j_0}} x}{\lambda_1^{n_{j_0}}} = \left( 1, \frac{1}{2} \left( \frac{\lambda_2}{\lambda_1} \right)^{n_{j_0}}, \frac{1}{3} \left( \frac{\lambda_3}{\lambda_1} \right)^{n_{j_0}}, \ldots \right)
\]

\[
= \left( 1, \frac{1}{2} \left( \frac{4}{3} \right)^{n_{j_0}}, \frac{1}{3} \left( \frac{2}{3} \right)^{n_{j_0}}, \ldots \right)
\]

Notice that the second coordinate, \( \frac{1}{2} \left( \frac{4}{3} \right)^{n_{j_0}} \) is bigger than \( 1.1 > 1 + \epsilon_1 \), since \( n_{j_0} \geq 10 \).

Further, \( a_{j_0} \frac{1}{2} \left( \frac{4}{3} \right)^{n_{j_0}} - \sum_{j=1}^{j_0-1} b_j \frac{1}{2} \left( \frac{4}{3} \right)^{n_j} > 1.1 > 1 + \epsilon_1 \), too, since if \( j < j_0 \) then \( |a_j| \leq \frac{1}{2j^2} \).

Thus, the second coordinate of \( \sum_{j=1}^{j_0} a_j \frac{T^{n_j} x}{\lambda_1^{n_j}} \) is greater than \( 1 + \epsilon_1 \).

But then this will imply that the second coordinate of \( \sum_{j=j_0+1}^{\infty} a_j \frac{T^{n_j} x}{\lambda_1^{n_j}} \) is less than \(-1\).

Next, we normalize the vectors \( \{T^{n_{j_0+1}} x, T^{n_{j_0+2}} x, T^{n_{j_0+3}} x, \ldots\} \) from \( j > j_0 \) such that the second coordinate is 1 in all of them.

After normalization we have \( \sum_{j \geq j_0+1}^{\infty} b_j \frac{T^{n_j} x}{\lambda_1^{n_j}} \).

Then as above, we have that \( \sum_{j \geq j_0+1} b_j = 1 \), so that there is a smallest \( j \geq j_0 + 1 \) such that \( |b_j| \geq \frac{1}{2j^2} \).

Now, let \( j = j_0 + 1 \), then we get \( |b_{j_0+1}| \geq \frac{1}{2(j_0+1)^2} \), and the second coordinate of the vector \( \frac{T^{n_{j_0+1}} x}{\lambda_1^{n_{j_0+1}}} \) becomes 1, but the third coordinate of this vector is

\[
\frac{x_3}{x_1} \left( \frac{\lambda_3}{\lambda_2} \right)^{n_{j_0+1}} = \frac{1}{2} \left( \frac{4}{3} \right)^{j_0+1} = \frac{2}{3} \left( \frac{9}{8} \right)^{n_{j_0+1}}
\]

which is bigger than \( 1 + \epsilon_2 \). But also,

\[
b_{j_0+1} \left( \frac{1}{2} \right) \left( \frac{4}{3} \right)^{n_{j_0}} - \sum_{j=1}^{j_0-1} b_j \left( \frac{1}{2} \right) \left( \frac{4}{3} \right)^{n_j} > 1 + \epsilon_2 \text{ too, since if } j < j_0 + 1 \text{ then } |b_j| \leq \frac{1}{2j^2}
\]

Hence, the third coordinate is always bigger than 1.

Continuing on in this manner, we see that if \( |b_j| > \frac{1}{2j^2} \), then for that \( j \), the \( j+1 \) coordinate is bigger than 1. Thus, using this subsequence of the orbit we can not approximate the
basis vector \((1, 0, 0, 0, 0, \ldots)\) by the span of this subsequence.

**Step 2:** Now, we take any vector \(x = (x_1, x_2, x_3, \ldots) \in l_2\) such that every coordinate is different from zero, and let the \(\{\lambda_j\}\) run through real numbers between 0 and 1, such that such that \(\lambda_1 < \lambda_2 < \lambda_3 < \ldots \rightarrow 1\). We want to show that there is a subsequence \(\{T^{n_1}x, T^{n_2}x, T^{n_3}x, \ldots\}\) of \(\{T^nx : n \in \mathbb{N}\}\), such that we will not be able to approximate the first basis vector \((1, 0, 0, 0, 0, \ldots)\) by the span of that subsequence.

We have,
\[
\frac{T^{n_1}x}{\lambda_1^{n_1}} = \left(1, \frac{x_2}{x_1} \left(\frac{\lambda_2}{\lambda_1}\right)^{n_1}, \frac{x_3}{x_1} \left(\frac{\lambda_3}{\lambda_1}\right)^{n_1}, \ldots\right)
\]
\[
\frac{T^{n_2}x}{\lambda_1^{n_2}} = \left(1, \frac{x_2}{x_1} \left(\frac{\lambda_2}{\lambda_1}\right)^{n_2}, \frac{x_3}{x_1} \left(\frac{\lambda_3}{\lambda_1}\right)^{n_2}, \ldots\right)
\]

etc.

Now, assume that \(\sum_{j=1}^{\infty} a_j \frac{T^{n_j}x}{\lambda_1^{n_j}} = (1, \beta_1, \beta_2, \beta_3, \ldots)\) where \(\sum_{j=1}^{\infty} \beta_j^2 < \beta < \frac{1}{100}\), so, \(\beta_j < \frac{1}{\sqrt{100}}\) for each \(j\).

Since \(\sum_{j=1}^{\infty} a_j = 1\), there is a smallest \(j\) such that \(|a_j| > \frac{1}{2j^2}\).

\[
|a_j| \frac{T^{n_j}x}{\lambda_1^{n_j}} > \frac{1}{2j^2} \frac{T^{n_j}x}{\lambda_1^{n_j}} = \left(1, \frac{x_2}{x_1} \left(\frac{\lambda_2}{\lambda_1}\right)^{n_j}, \frac{x_3}{x_1} \left(\frac{\lambda_3}{\lambda_1}\right)^{n_j}, \ldots\right)
\]

\[
= \left(\frac{1}{2j^2}, \frac{x_2}{x_1} \left(\frac{\lambda_2}{\lambda_1}\right)^{n_j}, \frac{x_3}{x_1} \left(\frac{\lambda_3}{\lambda_1}\right)^{n_j}, \ldots\right)
\]

so that if \(k < j\) then \(|a_k| \leq \frac{1}{2k^2}\).

Now, suppose \(j = j_0\), then we get \(|a_{j_0}| \geq \frac{1}{2j_0^2}\).

so that our first vector \(\frac{T^{n_{j_0}}x}{\lambda_1^{n_{j_0}}}\) becomes
\[
\frac{T^{n_{j_0}}x}{\lambda_1^{n_{j_0}}} = \left(1, \frac{x_2}{x_1} \left(\frac{\lambda_2}{\lambda_1}\right)^{n_{j_0}}, \frac{x_3}{x_1} \left(\frac{\lambda_3}{\lambda_1}\right)^{n_{j_0}}, \ldots\right)
\]

Notice that \(\frac{\lambda_2}{\lambda_1} > 1\), by our assumption.

Hence, the second coordinate, \(\frac{x_2}{x_1} \left(\frac{\lambda_2}{\lambda_1}\right)^{n_{j_0}} > 1.1 > 1 + \beta_1\) if \(n_{j_0}\) is large enough.
But also, since \( |a_j| \leq \frac{1}{2j^2} \) for \( j < j_0 \) we have that,
\[
a_{j_0} \frac{x_2}{x_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{n_{j_0}} - \sum_{j=1}^{j_0-1} a_j \frac{x_2}{x_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{n_j} > 1.1 > \beta_1,
\]
if \( n_{j_0} \) is sufficiently large depending on \( n_1, n_2, \ldots, n_{j_0} - 1 \) Therefore we get that the second coordinate is bigger than 1.

Next, we normalize the vectors \( \{T^{n_{j_0+1}}x, T^{n_{j_0+2}}x, T^{n_{j_0+3}}x, \ldots\} \) from \( j > j_0 \) such that the second coordinate is 1 in all of them.

After normalization we have, 
\[
\sum_{j \geq j_0+1} \infty b_j \frac{T^{n_j}x}{2^{n_j_2}}.
\]
But then \( \sum_{j \geq j_0+1} b_j < -1 \), so, that there is a smallest \( j \geq j_0 + 1 \) such that \( |b_j| \geq \frac{1}{2j^2} \).

Now, suppose \( j = j_0 + 1 \), then we get \( |b_{j_0+1}| \geq \frac{1}{2(j_0+1)^2} \), and the second coordinate of the vector \( \frac{T^{n_{j_0+1}}x}{\lambda_1^{j_0+1}} \) becomes 1, but the third coordinate of this vector is
\[
\frac{x_3}{x_1} \left( \frac{\lambda_1}{\lambda_2} \right)^{n_{j_0+1}} \text{ which is bigger than } 1 + \beta_2 \text{ if we take } n_{j_0+1} \text{ large enough and use the fact that } \frac{\lambda_1}{\lambda_2} > 1
\]
Further,
\[
b_{j_0+1} \left( \frac{1}{2} \right) \left( \frac{4}{3} \right)^{n_{j_0}} - \sum_{j=1}^{j_0-1} b_j \left( \frac{1}{2} \right) \left( \frac{4}{3} \right)^{n_j} > 1 + \beta_2,
\]
if \( n_{j_0+1} \) is sufficiently large depending on \( n_1, n_2, \ldots, n_{j_0} \) and the fact that \( |b_j| \leq \frac{1}{2j^2} \) if \( j < j_0 + 1 \).

Hence, the third coordinate is bigger than 1.

Continuing on in this fashion, one has that if \( |b_j| > \frac{1}{2j^2} \), then for that \( j \), the \( j + 1 \) coordinate is bigger than 1. Thus, using the subsequence \( \{T^{n_{j_0+1}}x, T^{n_{j_0+2}}x, T^{n_{j_0+3}}x, \ldots\} \) of the orbit \( \{T^nx : n \in \mathbb{N}\} \) we can not approximate the basis vector \( (1, 0, 0, 0, 0, \ldots) \) by the span of this subsequence. This finishes the proof.

It is easy to see that this argument generalizes to the case when there is some increasing infinite subsequence of the eigenvalues \( 0 < \lambda_{i_0} < \lambda_{i_1} < \lambda_{i_2} < \ldots \). We just repeat the proof.
of Theorem (6) for the coordinates $x_{i_0}, x_{i_1}, x_{i_2}, \ldots$. So, we get the following result;

**Corollary 1.** If there is an increasing subsequence of eigenvalues,

$0 < \lambda_{i_0} < \lambda_{i_1} < \lambda_{i_2} < \ldots$, then for every $x = (x_1, x_2, x_3, \ldots) \in l_2$ there is a subsequence

$\{T^{n_j} x : n_j \in \mathbb{N}\}$ that does not span $l_2$.

In Theorem (5) we stated the *hyperful orbit* result where we assume that the set of eigenvalues $\lambda_j$ is decreasing tending to $a > 0$, in other words we have one accumulation point. Well, our next theorem shows that we can still have the same result even if when the set of eigenvalues is not 'purely' decreasing but can be rearranged into two subsets, each subset having a different accumulation point.

**Theorem 7.** Let $T = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & \frac{1}{3} & 0 & 0 & \ldots \\ 0 & 0 & \frac{3}{4} & 0 & 0 & \ldots \\ 0 & 0 & 0 & \frac{5}{8} & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\end{pmatrix}$, $T : l_2 \rightarrow l_2$,

Suppose that $x$ is any vector in $l_2$ with all nonzero coordinates.

Then every subsequence of $\{T^n x : n \in \mathbb{N}\}$ spans $l_2$.

**Proof.** We first divide the set of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ into odd and even terms; Let $A = \{\lambda_1, \lambda_3, \lambda_5, \ldots\}$ and $B = \{\lambda_2, \lambda_4, \lambda_6, \ldots\}$.

Then, clearly we have the following

$$\lambda_k = \begin{cases} \frac{1}{2} + \frac{1}{2k}, & k = 1, 3, 5, \ldots \\ \frac{1}{k+1}, & k = 2, 4, 6, \ldots \end{cases}$$
Hence,

\[
\lim_{k \to \infty} \lambda_k = \begin{cases} 
\frac{1}{2}, & k \in A \\
0, & k \in B
\end{cases}
\]

Thus, we can apply theorem 5 to the set \(A\) of eigenvalues to get the estimates of the basis vectors, \((1, 0, 0, 0, 0, \ldots), (0, 0, 1, 0, 0, \ldots), (0, 0, 0, 1, 0, \ldots), \) etc.

To get the remaining basis vectors we proceed as follows:

We know that every coordinate of \(x = (x_1, x_2, x_3, \ldots)\) is not zero, so with this vector and the basis vectors obtained above we produce the new vector \(x'\) with zeros in the odd coordinates only.

Indeed, Let \(\{e_k\}_{k=1}^\infty\) be the standard basis of \(l_2\).

Then we have that,

\[
x - x_1e_1 - x_3e_3 - x_5e_5 - \ldots = \\
= (x_1, x_2, x_3, x_4, x_5 \ldots) - x_1(1, 0, 0, 0, 0, \ldots) - x_3(0, 0, 1, 0, 0, \ldots) - x_5(0, 0, 0, 1, 0, \ldots) - \ldots \\
= (0, x_2, 0, x_4, 0, x_6, 0, \ldots)
\]

Letting \(x' = (0, x_2, 0, x_4, 0, x_6, 0, \ldots)\), we apply theorem 5 again to the set \(B\) of eigenvalues together with \(x'\) to get the estimates of the basis vectors, \((0, 1, 0, 0, 0, \ldots), (0, 0, 1, 0, 0, \ldots), (0, 0, 0, 0, 1, 0, \ldots), \) etc.

Therefore, we have generated all the basis vectors and this completes the proof of the theorem.

Now, we combine the results of this chapter to the following theorem that characterizes diagonal operators with hyperful orbit.

**Theorem 8.** If a diagonal operator has distinct eigenvalues \((\lambda_i)\), that is, \(\lambda_i \neq \lambda_j\) if \(i \neq j\),

Then \(x = (x_1, x_2, x_3, x_4, \ldots)\) is cyclic if and only if \(x_i \neq 0\) for all \(i\).

Moreover, if there is an infinite increasing sequence, \(\lambda_{n_1} < \lambda_{n_2} < \lambda_{n_3} < \ldots\) of eigenvalues
\( (\lambda_i) \), then not every subsequence of \( \{T^{n_i}x : n_i \in \mathbb{N}\} \) spans \( l_2 \).

If there is no such an infinite increasing sequence, \( \lambda_{n_1} < \lambda_{n_2} < \lambda_{n_3} < \ldots \) of eigenvalues \((\lambda_i)\), then every subsequence of \( \{T^{n_i}x : n_i \in \mathbb{N}\} \) spans \( l_2 \).

Proof. Suppose there is no an infinite increasing sequence, 
\[ \lambda_{n_1} < \lambda_{n_2} < \lambda_{n_3} < \ldots \] of eigenvalues \((\lambda_j)\).

Consider the set of accumulation points, \((\lambda'_{n_i})\) of the sequence \((\lambda_{n_i})\); then none of these can be approached from below by our assumption. This gives that every non-empty subset of the eigenvalues has a largest element and for every eigenvalue \(\lambda_i\) there is a largest eigenvalue \(\lambda_j\) such that \(\lambda_j < \lambda_i\). To see this, consider \(\sup_{\lambda_j < \lambda_i} \lambda_j\). If this is not attained we have an infinite increasing sequence of \(\lambda_j\)'s which violates our assumption. Thus the supremum is attained. Thus, the set of eigenvalues, arranged in decreasing order is a well ordered set. With this well ordering every eigenvalue corresponds in an obvious way to a countable ordinal number.

Now assume that \(x = (x_1, x_2, x_3, \ldots)\) with \(x_i \neq 0\) for each \(i\). Consider a subsequence \((n_j)\) of the positive integers. We will prove that \(\overline{\text{span}}\{T^{n_j}x\} = l_2\). For this we need to show that every basis vector \(e_k = (0, 0, \ldots, 1, 0, \ldots)\) (one on the \(k^{th}\) position, and zero else where) is in \(\overline{\text{span}}\{T^{n_j}x\}\).

We do this by transfinite induction. We have that each vector \((0, 0, \ldots, 1, 0, \ldots)\) corresponds to an eigenvalue and a countable ordinal number.

Let \(\lambda_{j_0}\) be the largest eigenvalue corresponding to the first ordinal number. Let \(\omega_0\) be this ordinal number. Then \(\frac{T^{n_j}x}{\|T^{n_j}x\|}\) will converge to \((0, 0, \ldots, 1, 0, \ldots)\) where one is on the \(j_0^{th}\) position. So, this basis vector is obviously in the span.

Now, take a basis vector \(e_{m_0} = (0, 0, \ldots, 1, 0, \ldots)\) and a subsequence \(\{T^{m_j}x\}\) and assume that all the basis vector corresponding to smaller ordinal numbers \(\omega_j < \omega_0\), are in \(\overline{\text{span}}\{T^{m_j}x\}\). We should then show that also \(e_{m_0} \in \overline{\text{span}}\{T^{m_j}x\}\).
Indeed, consider a sequence \( \{T^n_j x\} \). From each \( \{T^n_j x\} \) put 0 in the coordinates corresponding to eigenvalues greater than \( \lambda_{j_0} \), i.e., ordinal numbers that give basis vectors before \( e_{m_0} \).

Let’s call this new element \( y_j \). So we have that,

\[
y_j = \{T^n_j x\} - \sum_i (T^n_j x)_i e_i \quad \text{where the } \sum \text{ extends over } \lambda_i > \lambda_{m_0}
\]

But then \( \frac{y_j}{\|y_j\|} \) will converge to \( e_{m_0} \).

\[ \Box \]

**Corollary 2.** For a diagonal operator on \( l^2 \) either A or B holds:

(A) Every cyclic vector has a hyperful orbit.

(B) None of the cyclic vectors has a hyperful orbit.
CHAPTER 4

Cyclic operators on Banach spaces other than $l_2$

In this chapter we discuss the cyclic properties of vectors (or operators) in the Banach spaces $c_0$, $C[0,1]$ and $L_2[0,1]$. However on $L_2[0,1]$ we consider a Multiplication Operator in place of a Diagonal Operator. Furthermore, we obtain some characterizations of cyclicity of operators on a general Banach space $X$.

**Theorem 9.** Let $T : c_0 \to c_0$ be a diagonal operator with distinct eigenvalues.

(a) Then $T^* : c_0^* \to c_0^*$ is also a diagonal operator.

(b) Let $x = (x_1, x_2, x_3, ...)$ be such that $x_i \neq 0$ for all $i$. Then, if $\sum_{i=1}^{\infty} |x_i|^2 < \infty$, then $x$ is also cyclic in $c_0$.

(c) If $x$ is non-cyclic in $c_0$, then $x$ is also non-cyclic in $l_1$.

**Proof.** (a) We have that $T^* : l_1 \to l_1$ since $c_0^* = l_1$.

Now for $f = y = (y_n) \in l_1, x \in c_0$, we have,

$$(T^* f)(x) = f(T x)$$

$$= f(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, ...)$$

$$= \sum_{n=1}^{\infty} (\lambda_n y_n) x_n$$

$$\Rightarrow T^* f = (\lambda_n x_n), n = 1, 2, 3, ...$$

$$\Rightarrow T^* y = (\lambda_n x_n), n = 1, 2, 3, ...$$

$$\Rightarrow T^* \text{ is diagonal.}$$

(b) Suppose that $x$ is cyclic in $l_2$, then there exist positive integers $N_1, N_2, N_3, ...$ such that
∥ \sum_{n=1}^{N_1} (a_i T^i x - e_1) \parallel_{l_2} < \epsilon_1,
∥ \sum_{n=1}^{N_2} (a_i T^i x - e_2) \parallel_{l_2} < \epsilon_2,
∥ \sum_{n=1}^{N_3} (a_i T^i x - e_3) \parallel_{l_2} < \epsilon_3, \text{ etc.}

But we know that
∥ \cdot \parallel_{c_0} \leq ∥ \cdot ∥_{l_2},

Therefore we have that,
∥ \sum_{n=1}^{N_1} (a_i T^i x - e_1) \parallel_{c_0} < \epsilon_1,
∥ \sum_{n=1}^{N_2} (a_i T^i x - e_2) \parallel_{c_0} < \epsilon_2,
∥ \sum_{n=1}^{N_3} (a_i T^i x - e_3) \parallel_{c_0} < \epsilon_3, \text{ etc.}

Thus, \( x \) is cyclic in \( c_0 \).

(c) Suppose that \( x \) is non-cyclic in \( c_0 \).

Then the subspace \( M \) spanned by the closed linear span of the orbit of \( X \) under \( T \) is non-trivial, that is, \( M \) is a non-trivial invariant subspace. We proved in the chapter 2 (Lemma 2) that \( M \) is invariant under \( T \) if and only if the orthogonal complement of \( M \) is invariant under \( T^* \).

Thus we get a non-trivial invariant subspace \( M^* \) under \( T^* \).

Hence \( x \) is non-cyclic in \( c_0 \).

Next, we give a general result on hyperful sequences.

**Definition 3.** A Banach space \( X \) is called reflexive if \( X = X^{**} \). Here \( X^{**} \) is the second conjugate space of \( X \).

**Theorem 10.** Suppose that \( S \) is a countable set in the unit sphere of a reflexive Banach space \( B \), and that every infinite subset of \( S \) spans \( B \). Then \( \overline{S} \) is a compact set.

To prove the theorem we will use the following classical Lemma.
**Definition 4.** Let $A$ be a subset of a metric space $(X, d)$. Let $\epsilon > 0$. A finite set of points \( \{x_1, x_2, \ldots, x_n\} \) of $X$ is said to form a (finite) $\epsilon$-net if $A \subseteq B_\epsilon(x_1) \cup B_\epsilon(x_2) \ldots \cup B_\epsilon(x_n)$; where $B_\epsilon(x_n)$ is an open ball with centre $x_n$ and radius $\epsilon$.

**Lemma 5.** Suppose $K$ is a closed subset of a complete metric space $M$. If for every positive $\epsilon$, there exists finite $\epsilon$-net in $K$, then $K$ is compact.

**Proof of Lemma.** We want to show that for every sequence $\{a_n\}$ in $K$, there exists a subsequence $\{a_{n_k}\}$ that converges in $K$.

Let $\{a_n\} \subset K$, let $\epsilon > 0$. Consider an $\epsilon$-net, $\epsilon = \frac{1}{k}, k = 1, 2, 3, \ldots$.

Now, suppose $\exists$ finite $\epsilon$-net $\forall \epsilon > 0$.

Then taking $\epsilon = 1$, we can cover $K$ with finitely many balls of radius 1; then one of them contains $a_n$s for infinitely many $n$’s; i.e., $\exists$ a ball $B_1$ of radius 1 so that there is a subsequence of $a_n$ whose members all belong to $B_1$. We denote this subsequence by $\{a_{n(1)}\}$ and thus all $a_{n(1)}$ belong to $B_1$.

Similarly by taking $\epsilon = \frac{1}{2}$, we can find a subsequence $\{a_{n(2)}\}$ of $\{a_{n(1)}\}$ and a ball $B_2$ of radius $\frac{1}{2}$ so that all $a_{n(2)}$ belong to $B_2$. Continuing on in this manner we obtain for any $k \geq 2$ a subsequence $\{a_{n(k)}\}$ of $\{a_{n(k-1)}\}$ and a ball $B_k$ of radius $2^{-k}$ so that all $\{a_{n(k)}\}$ belong to $B_k$.

Now consider the sequence $\{a_{n(k)}\}$ which is a subsequence of the original sequence. We show that it is a Cauchy-sequence. Notice that if $m \geq n$, then we have by the triangle inequality,

\[
d(a_{m}, a_{n}) \leq d(a_{m}, a_{m-1}) + \ldots + d(a_{n+1}, a_{n}),
\]

and since $a_{j}$ and $a_{j-1}$ are both in $B_{j-1}$ their mutual distance is $\leq 2 \cdot 2^{1-j}$. Thus we have that for $m > n$,

\[
d(a_{m}, a_{n}) \leq 2^{2-m} + \ldots + 2^{2-(n+1)} \leq 2^{2-n}.
\]

$\Rightarrow \{a_{n}\}$ is a Cauchy sequence. Since $K$ is closed and $l_2$ is complete, we have that $\{a_{n}\}$ converges, so we have found a convergent subsequence of $\{a_n\}$. This completes the proof of
Proof of Theorem. To finish the proof of the above theorem, we proceed as follows:

If $S = \{S_n\}$ is not compact, then there is $\{S_k\}$ which has no convergent subsequence. Let’s call this subsequence $\{S_{nk}\}$.

Now we take an $\epsilon$ such that there is no finite $\epsilon$-net and make an $\epsilon$-net out of the subsequence $\{S_{nk}\}$.

We first pick $\{S_{n1}\}$, then by our assumption $\exists \{S_{n2}\}$ so that $d(S_{n1}, S_{n2}) > \frac{\epsilon}{2}$. But then $\exists \{S_{n3}\}$ so that $d(S_{n1}, S_{n3}) > \frac{\epsilon}{2}$ and $d(S_{n2}, S_{n3}) > \frac{\epsilon}{2}$.

And in general, after $S_{n1}$, $S_{n2}$, ..., $S_{nk}$ have been chosen $\exists S_{nk+1}$ so that $d(S_{ni}, S_{nj}) > \frac{\epsilon}{2}$ for $1 \leq n_i < n_j \leq n_{k+1}$.

Thus we have constructed the subsequence $\{S_{nk}\}$ which is $\frac{\epsilon}{2}$-separated.

Notice further that $\{S_n\}$ is bounded so $\{S_{nk}\}$ converges weakly say to $a$.

Define $g_{ni} = S_{nk} - a$,

$\implies \{g_{ni}\} \to 0$ weakly.

Now since $\{g_{ni}\} \to 0$ weakly and $B$ is reflexive, $\exists$ a subsequence $\{g_{nij}\}$ of $\{g_{ni}\}$ that is a Schauder basis of the closed linear span of $\{g_{nij}\}$. By removing one element from this subsequence we get a strictly smaller span. This can not then be the whole space. Hence, some sequence of our original sequence $\{S_k\}$ with not span $B$. This contradicts our assumption.

In the next theorem we consider complex scalars.

Definition 5. Let $X$ be a Banach space. A sequence $(x_n)$ in $X$ is called a Schauder basis in $X$ if every $x \in X$ admits a unique decomposition as the sum of a convergent series

$$x = \sum_{1}^{\infty} \alpha_n x_n, \quad \alpha_n \in \mathbb{C}, \quad \forall n \in \mathbb{N}.$$ 

Theorem 11. Suppose that $(e_1, e_2, e_3, ...)$ is a Schauder basis for a reflexive Banach space $B$. Then for every natural number $m$ there exists a diagonal operator $T$ on $B$ and an $x$ such
that any subsequence of \( \left( \frac{T^m x}{\|T^m x\|} \right)_{n \in \mathbb{N}} \) spans \( B \), but \( \left( \frac{T^m x}{\|T^m x\|} \right)_{n \in \mathbb{N}} \) is not convergent and the set of accumulation points is \( m \)– dimensional.

**Proof. Case 1:** Let’s consider the 2– dimensional case first;

Let \( x = (x_1, x_2) \) where \( x_1 \neq 0, x_2 \neq 0 \).

Let \( T : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \) be a diagonal operator given by;

\[
T = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i \theta} \end{pmatrix}
\]

where \( \theta \) is irrational.

We show that \( \text{span}\{T^i x, T^j x\} = \mathbb{C}^2 \).

Notice that \( T^k x = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i \theta} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 e^{2\pi i \theta} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 e^{2\pi i \theta} \end{pmatrix} \).

It is therefore enough to show that the set two-dimensional set \( \{T^{k_1} x, T^{k_2} x\} \) is linearly independent. Here \( k_1, k_2 \in \mathbb{Z}^+ \).

Now assume that \( \alpha_1 T^{k_1} x + \alpha_2 T^{k_2} x = 0 \). We show that \( \alpha_1 = \alpha_2 = 0 \).

Indeed, \( \alpha_1 \begin{pmatrix} x_1 \\ x_2 e^{2\pi i k_1 \theta} \end{pmatrix} + \alpha_2 \begin{pmatrix} x_1 \\ x_2 e^{2\pi i k_2 \theta} \end{pmatrix} = 0 \) implies,

\[
\alpha_1 x_1 + \alpha_2 x_1 = 0
\]

and

\[
\alpha_1 x_2 e^{2\pi i k_1 \theta} + \alpha_2 x_2 e^{2\pi i k_2 \theta} = 0
\]

\( \implies \alpha_1 + \alpha_2 = 0, \) and \( \alpha_1 + \alpha_2 e^{2\pi i (k_2 - k_1) \theta} = 0 \) since we can divide by non-zero \( x_1 \) and \( x_2 \).

But then we have, \( \alpha_1 = -\alpha_2, \)

so, we get \( \alpha_1 (1 - e^{2\pi i (k_2 - k_1) \theta}) = 0. \)

But \( \theta \) is irrational, hence \( 1 - e^{2\pi i (k_2 - k_1) \theta} \neq 0. \)

Thus \( \alpha_1 = 0 \) and \( \alpha_2 = 0. \)

Thus \( \text{span}\{T^i x, T^j x\} = \mathbb{C}^2 \). Further is easy to see that the set of accumulation points will be the set \( \{1, e^{2\pi i \phi} : \phi \in \mathbb{R} \} = \{(1, z, 0, 0, \ldots) : z \in \mathbb{C}, |z| = 1 \} \), and this finishes the proof.
Case 2 (The General case): From case (1) above, it is easy to see that if $T$ is an $m$ by $m$ diagonal matrix given by $T = \text{diag}(1, e^{2\pi i k_1 \theta_1}, e^{2\pi i k_2 \theta_2}, \ldots, e^{2\pi i k_{m-1} \theta_{m-1}})$, then 
\[
\text{span}\{T^k_1 x, T^k_2 x, \ldots, T^{k_{n-1}} x\} = C^m
\]
and the set of accumulation points of $\{\frac{T^m x}{\|T^m x\|}\}_{m \in \mathbb{N}}$ is the set 
\[
\{1, e^{2\pi i k_1 \phi_1}, e^{2\pi i k_2 \phi_2}, \ldots, e^{2\pi i k_{m-1} \phi_{m-1}} : k_1 \phi_1 + k_2 \phi_2 + \ldots + k_m \phi_{m-1} = s \in \mathbb{R}, k_1, k_2, \ldots, k_{m-1} \in \mathbb{N}\},
\]
which completes the proof of the General case.

Theorem 12. Let $T : l_2 \to l_2$ be an infinite diagonal matrix given by $T = \text{diag}(1, e^{2\pi i \theta}, 1, 1, \ldots)$ where $\theta$ is real, then every subsequence of $\{\frac{T^m x}{\|T^m x\|}\}_{m \in \mathbb{N}}$ spans the whole space but $\{\frac{T^m x}{\|T^m x\|}\}_{m \in \mathbb{N}}$ does not converge.

Proof. Notice that, we have the decreasing eigenvalues except for the first two eigenvalues. Thus by theorem 5 we get that $\frac{T^m x}{\|T^m x\|}$ is hyperful but by the above theorem this sequence will not converge; in fact the set of all accumulation points will be given by the set 
\[
\{1, e^{2\pi i \phi}, 0, 0, 0, \ldots : \phi \in \mathbb{R}\} = \{(1, z, 0, 0, \ldots) : z \in \mathbb{C}, |z| = 1\}.
\]

We now give a sequence of results which will show that multiplication operators on $C(0, 1)$ and $L_2[0, 1]$ can not have vectors with hyperful orbits.

Theorem 13. Let $T : C(0, 1) \to C(0, 1)$ be a multiplication operator defined on $C(0, 1)$ by $T_f(g(x)) = g(x)f(x)$ for every $g(x) \in C(0, 1)$.

Then $\{T^n f\}$ is cyclic if and only if $f(x_1) \neq f(x_2)$, whenever $x_1 \neq x_2$ and $g(x) \neq 0$ for all $x$.

Proof of the Theorem. For the necessity we have the following;

If $f(x_1) = f(x_2)$ for $x_1 \neq x_2$, then we have,

$T_f(g(x)) = f(x)g(x)$

$T^2_f(g(x)) = f^2(x)g(x)$, ...

$T^n_f(g(x)) = f^n(x)g(x)$

So, there is no cyclic function $g(x)$. Thus, in order to have a cyclic vector the multiplication
operator $T_f$ must have $f$ either strictly increasing or strictly decreasing on $(0, 1)$. We now assume $f$ is strictly increasing. Obviously $f$ strictly decreasing gives an almost identical proof. The sufficiency of having $f$ increasing will follow from the following proposition.

**Proposition 1.** Assume that $0 < f(0) < f(1)$ and $f(x)$ is strictly increasing on $C(0, 1)$. Then $\text{span}\{f, f^2, f^3, \ldots\} = C[0, 1]$.

**Proof of the Proposition.** Let $A = \{f, f^2, f^3, \ldots\}$, then clearly $A$ is an algebra of continuous real-valued functions which separates points of $C(0, 1)$, thus, in order to apply the Stone-Weierstrass theorem, it remains to show that $A$ contains the constant functions.

Indeed, let $0 < a < f(t) < b < 1$ for $t \in [0, 1]$.

Then we find a polynomial $p(t) = \sum_{j=1}^{N} c_j t^j$ such that $1 - \epsilon < p(t) < 1 + \epsilon$ if $t \in [a, b]$.

But then this will imply that $1 - \epsilon < \sum_{j=1}^{N} c_j (f(t))^j < 1 + \epsilon$ for $t \in [0, 1]$.

Thus $A$ contains the constant functions, and this finishes the proof of the proposition.

Now, from this proposition, the proof of the sufficiency part of Theorem 13 follows immediately.

This gives us the following theorem,

**Theorem 14.** If $T_f$ is a multiplication operator defined on $C(0, 1)$, then $T_f$ can not have a hyperful orbit.

**Proof.** Theorem 13 gives that $f$ either strictly increasing or strictly decreasing to have a cyclic vector. It is then any easy consequence of M"{u}ntz’s Theorem that $\text{span}\{f^{n_j}g\} \neq C(0, 1)$ if $\sum 1/n_j < \infty$. To see this, WLOG we can assume that $f$ is an increasing homeomorphism of $[0, 1]$. By then considering $(fof^{-1}) (x) = x, (fof^{-1})^2 (x) = x, \ldots$ we see that, $\text{span}\{f^{n_j}(x)\} = C(0, 1)$ if and only if $\text{span} (fof^{-1})^{n_j} (x) = C(0, 1)$, that is, if $\text{span}\{x^{n_j}\} = C(0, 1)$. 

\qed
Theorem 15. Let $T : L_2[0, 1] \to L_2[0, 1]$ be a multiplication operator defined on $L_2[0, 1]$ by $T_f : h \to f \cdot h$.

If $f$ is strictly increasing non-negative function and bounded on $[0, 1]$, then $\frac{T^n_f h}{\|T^n_f h\|} \to 0$ weakly for every $h$.

Proof. First notice that if $f$ is essentially bounded, then $T_f h$ is bounded, and in fact $\|T_f h\| = \|f\|_\infty$ (the essential supremum of $f$ on $C(0,1)$).

Now we are given that $f$ is bounded, so $\exists$ a constant $K$ so that $0 \leq f(t) \leq K$.

Further, since $\{T_f h\}$ is bounded, the proof will be complete if we show that $\frac{T^n_f h}{\|T^n_f h\|} \to 0$ for every $h$.

This is equivalent to showing that given any $g \in L_2[0,1]$ with $\|g\|_2 = 1$, one has that $\frac{\int_0^1 T^n_f h \cdot g}{\|T^n_f h\|} \to 0$ for every $g$.

Indeed,

$$\frac{\int_0^1 T^n_f h \cdot g}{\|T^n_f h\|} = \frac{\int_0^{1-\epsilon} T^n_f h \cdot g + \int_{1-\epsilon}^1 T^n_f h \cdot g}{\|T^n_f h\|}$$

$$= \frac{\int_0^{1-\epsilon} T^n_f h \cdot g}{\|T^n_f h\|} + \frac{\int_{1-\epsilon}^1 T^n_f h \cdot g}{\|T^n_f h\|}$$

$$= I + II.$$  

Now, if we write $g$ as $g = g_{[0,1-\epsilon]} + g_{[1-\epsilon,1]}$, then applying the Cauchy-Schwartz inequality to both $I$ and $II$ we get,

$$\frac{|\int_{1-\epsilon}^1 T^n_f h \cdot g|}{\|T^n_f h\|} \leq \frac{\|T^n_f h\|_{[1-\epsilon,1]}\|g\|_{[1-\epsilon,1]}}{\|T^n_f h\|} \to 0,$$

as $\epsilon \to 0$, by the absolute continuity of the integral of $g$.

Thus, $II \to 0$. 

To see that $I \to 0$ too, we consider the two cases:

**Case 1:** When $h \neq 0$ near 1;

If $h \neq 0$ near 1, then we have,

$$\left| \int_{0}^{1-\epsilon} T_f^{h} \cdot h \cdot g \right| \leq \frac{\|T_f^{h}\|}{\|T_f^{h}\|} \cdot \epsilon \cdot \|g\|_{[0,1-\epsilon]} \cdot \|T_f^{h}\|_{[0,1-\epsilon]},$$

which clearly go to 0, as we are dividing by the integral over the whole interval $[0, 1]$.

**Case 2:** When $h = 0$ near 1; If this case happens, then we take an $\epsilon$ so that $h$ is supported on $[0, 1-\epsilon]$ and apply case 1 above.

So, either way $I \to 0$, and this completes the proof of the theorem.

The following corollary gives the generalization of this result for any $f > 0$ on $L_\infty[0, 1]$, that satisfies $m\{t|f(t) = a\} = 0$ for every $a \in \mathbb{R}$, $a > 0$. Here $m$ denotes the Lebesgue measure on $\mathbb{R}$.

**Corollary 3.** Let $T : L_2[0, 1] \to L_2[0, 1]$ be a multiplication operator defined on $L_2[0, 1]$ by $T_f : h \to f \cdot h$.

If $f > 0$ and bounded on $[0, 1]$, and $m\{t|f(t) = a\} = 0$ for every $a > 0$ then,

$$\frac{T_f^{h}}{\|T_f^{h}\|} \to 0 \text{ weakly for every } h.$$

**Proof.** We replace $f$ by the increasing rearrangement of $f$, and apply above theorem.

**Corollary 4.** Let $T_f$ is a multiplication operator on $L_2[0, 1]$ and $f \in L_\infty(0, 1)$, then $T_f$ does not have any hyperful orbit.

**Proof.** If $m\{t|f(t) = a\} > 0$ for some $a$, then $T_f$ obviously does not have any cyclic vector. If $m\{t|f(t) = a\} = 0$ for all $a$, then by Corollary 3, $\frac{T_f^{h}}{\|T_f^{h}\|} \to 0 \text{ weakly for every } h$. Thus, some subsequence of $\frac{T_f^{h}}{\|T_f^{h}\|}$ is a Schauder basis for its span, and so by previous argument, it is not hyperful.
Theorem 16. Suppose that \( \overline{\text{span}} \{e_j\} = B \) and \( \| \sum_{j=1}^{N} a_j e_j \| \geq \max_{1 \leq j \leq N} |a_j| \). Let \( d_j > 0 \) for all \( j \). Then the diagonal operator \( T : B \to B \) given by
\[
T = \text{diag}(d_1, d_2, d_3, \ldots)
\]
is continuous provided \( \sum_1^\infty d_j < \infty \).

Proof. It is enough to show that \( T \) is a bounded linear operator.
Indeed,
\[
T(\sum_{j=1}^{N} a_j e_j) = \sum_{j=1}^{N} d_j a_j e_j,
\]
so, clearly \( T \) is linear, thus it remains to show that \( T \) is bounded, that is, we show that there is a positive constant \( C \) so that
\[
\| \sum_{j=1}^{N} d_j a_j e_j \| \leq C \| \sum_{j=1}^{N} a_j e_j \|.
\]

Now, by the triangle inequality we have that
\[
\| \sum_{j=1}^{N} d_j a_j e_j \| \leq \sum_{j=1}^{N} |d_j| \| a_j e_j \|.
\]
\[
\leq \max_{1 < j < N} |a_j| \sum_{j=1}^{N} |d_j| \leq \left( \sum_{j=1}^{\infty} |d_j| \right) \left( \sum_{j=1}^{N} a_j e_j \right)
\]
\[
< \infty,
\]
by our assumptions above.

Theorem 17. Let \( B \) be a Banach Space and \( T : B \to B \) be an infinite diagonal matrix defined on \( B \). Let \( d_j > 0 \) for all \( j \) such that \( \sum_1^\infty d_j < \infty \).

Let \( \{e_j\}_{j=1}^\infty \) be any sequence in \( B \) where \( e_j \neq 0 \) for all \( j \).
Suppose \( \overline{\text{span}} \{e_j\} = B \) and that
\[
\| \sum_{j=1}^{N} a_j e_j \| \geq C \max_j |a_j|
\]

Then: (a) For every \( x \in B \) there is \( x \sim \sum_1^\infty a_j e_j \)
(b) If \( a_j \neq 0 \ \forall j \), then \( x \) is cyclic.
Remarks:

(1) If

$$T = \begin{pmatrix}
  d_1 & 0 & 0 & \ldots \\
  0 & d_2 & 0 & \ldots \\
  0 & 0 & d_3 & 0 & \ldots \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
\end{pmatrix}$$

and

$$\sum_{j=1}^{\infty} |d_j| < \infty$$

then $T$ is continuous by the previous theorem.

(2) We don’t require the Fourier series $\sum_{j=1}^{\infty} a_j e_j$ to be convergent.

Proof. (a) Suppose that $x_n = \sum_{j=1}^{N_n} a_j e_j$, then it is enough to show that if $x_n \to x$ as $n \to \infty$, then $a_{jn} \to a_j$ as $n \to \infty$, for all $j$.

Now, suppose to the contrary that $a_{jn}$ does not converge to $a_j$ as $n \to \infty$, then $\exists \epsilon > 0$ such that $|a_{jn} - a_{jm}| > \epsilon$ for arbitrarily large $n$ and $m$.

But we are given that

$$\| \sum_{j=1}^{N_n} a_j e_j \| \geq C_{max} |a_j|$$

For $N_n \geq N_m$, we have that,

$$\left\| \sum_{j=1}^{\max(N_n,N_m)} (a_{jn} - a_{jm}) e_j \right\| \geq C_{max} |(a_{jn} - a_{jm})|$$

So we have that,

$$\left\| \sum_{j=1}^{\max(N_n,N_m)} (a_{jn} - a_{jm}) e_j \right\| \geq C_{max} |a_{jn} - a_{jm}| \geq C|a_{jn} - a_{jm}| \geq C\epsilon$$
for arbitrarily large $n$ and $m$.
i.e,
\[ \|x_n - x_m\| > C \epsilon \]

$\implies \{x_n\}$ is not cauchy. A contradiction.

Thus, $a_{jn} \to a_j$ as $n \to \infty$, for all $j$.

Notice that we have even more, namely that if
\[ x_n = \sum_{j=1}^{N_n} a_{jn} e_j \text{ and } x \sim \sum_{j=1}^{\infty} a_j e_j \]
$\implies a_j \to 0$ as $j \to \infty$.

Indeed, if $\exists \epsilon > 0$ such that $\forall N_n$ there is $j > N_n$ such that $|a_j| > \epsilon$, then there is an $r > n$ such that $|a_{jr}| > \frac{\epsilon}{2}$.

But then $\|x_r - x_n\| > \frac{\epsilon}{2} C$, a contradiction.

Thus, $a_j \to 0$ as $j \to \infty$.

Remark: The last conclusion above is an important generalization of the Riemann-Lebesgue Lemma which states that,

If $f \in L^1(T)$ then $\lim_{n \to \infty} \hat{f}(n) = 0$ where $\hat{f}(n)$ is the $n^{th}$ Fourier coefficient of $f$.

(b) Suppose that $a_j \neq 0 \ \forall j$, we want to show that $x$ is cyclic, that is, we show that linear span$\{\text{Orb}(T,x)\} = B$.

We have that,
\[ T^i x_n = T^i \left( \sum_{j=1}^{N_n} a_{jn} e_j \right) = \sum_{j=1}^{N_n} d^i_j a_{jn} e_j \]

By (a) above we know that $x_n \to x$, so given $\delta > 0$ and fixed $i$ we take $s$ so big that
\[ \frac{\|T^i x_s\|}{\|T^i x\|} < \delta \]
Further, without loss of generality we can assume that $d_1 > d_2 > d_3 > \ldots$, but then for a fixed $n$, we have that,

$$\|T^i x_n\| \approx d_1^i a_{1n} e_1$$

when $i$ is large enough.

But we also know that $a_j \to 0$ and $a_1 \neq 0$, hence given $\epsilon > 0$ we can first choose $i$ so that

$$\epsilon d_1^i a_1 > \sum_{m \geq 2} d_n^i a_m$$

then we choose $r$ so big that $|a_{1r} - a_1| < \frac{|a_1|}{2}$ and $|a_{nr}| \leq 2$ for all $n \neq r$.

But then this will imply that

$$\frac{T^i x_r}{\|T^i x_r\|} - (1, 0, 0, \ldots) < \epsilon$$

if

$$\epsilon d_1^i a_1 > 2 \sum_{m \geq 2} d_m^i a_m.$$

Then by the triangle inequality one has that,

$$\left\| \frac{T^i x_{s}}{\|T^i x_{s}\|} - \frac{T^i x}{\|T^i x\|} \right\| + \left\| \frac{T^i x_r}{\|T^i x_r\|} - (1, 0, 0, \ldots) \right\| < \delta + \epsilon.$$

Thus we get the first standard basis vector $(1, 0, 0, \ldots)$.

Next, we approximate the second basis vector $(0, 1, 0, 0, \ldots)$ by the linear combinations of the elements of the orbit of $x$.

We want to show that if $i$ and $j$ are fixed, then one has that

$$\|(aT^i x + bT^j x) - (0, 1, 0, 0, \ldots)\| < \epsilon \quad \forall \epsilon > 0$$

for some scalars $a$ and $b$.

Now we take $i$ and $j$ so large that

$$\|T^i x_n\| \approx d_1^i a_{1n} e_1$$

and

$$\|T^j x_n\| \approx d_1^j a_{1n} e_1.$$
But then this will imply that
\[
\frac{T^i x_n}{\|T^i x_n\|} \approx \left(1, \left(\frac{d_2}{d_1}\right)^i \left(\frac{a_{2n}}{a_{1n}}\right) e_2, \left(\frac{d_3}{d_1}\right)^i \left(\frac{a_{3n}}{a_{1n}}\right) e_3, \ldots\right)
\]
and
\[
\frac{T^j x_n}{\|T^j x_n\|} \approx \left(1, \left(\frac{d_2}{d_1}\right)^j \left(\frac{a_{2n}}{a_{1n}}\right) e_2, \left(\frac{d_3}{d_1}\right)^j \left(\frac{a_{3n}}{a_{1n}}\right) e_3, \ldots\right).
\]
So we have that,
\[
\frac{T^i x_n}{\|T^i x_n\|} - \frac{T^j x_n}{\|T^j x_n\|} = \left(1, \left(\frac{d_2}{d_1}\right)^j - \left(\frac{d_2}{d_1}\right)^i \right) \left(\frac{a_{2n}}{a_{1n}}\right) e_2, \left(\frac{d_3}{d_1}\right)^j - \left(\frac{d_3}{d_1}\right)^i \left(\frac{a_{3n}}{a_{1n}}\right) e_3, \ldots\right)
\]
for \(s > \text{Max}\{i, j\}\) for all \(n\). Without loss of generality we can assume that \(j = i + 1\), so by letting
\[
S_{i,j,n} = \frac{T^i x_n}{\|T^i x_n\|} - \frac{T^j x_n}{\|T^j x_n\|}
\]
we have that,
\[
S_{i,j,n} = \left(0, \left(\frac{a_{2n}}{a_{1n}}\right) e_2 \left(\frac{d_2}{d_1}\right)^i \left(\frac{d_2}{d_1} - 1\right), \left(\frac{a_{3n}}{a_{1n}}\right) e_2 \left(\frac{d_3}{d_1}\right)^i \left(\frac{d_3}{d_1} - 1\right), \ldots\right).
\]
But since \(a_j \to 0\) and \(a_2 \neq 0\), thus given \(\epsilon > 0\) we can first choose \(j\) so that
\[
\epsilon d_2^j a_1 > \sum_{m \geq 3} d_m^j a_m.
\]
Then we choose \(t\) so big that \(|a_{2t} - a_2| < \frac{|a_2|}{2}\) and \(|a_{nt}| \leq 2\) for all \(n \neq t\).
But then this will imply that
\[
\frac{S_{i,j,n}}{\|S_{i,j,n}\|} - (0, 1, 0, 0, \ldots) \epsilon d_2^j a_2 > 2 \sum_{m \geq 3} d_m^j a_m.
\]
Thus we will be done if we show that
\[
\left\|\left(\frac{T^i x}{\|T^i x\|} + \frac{T^j x}{\|T^j x\|}\right) - \left(\frac{S_{i,j,n}}{\|S_{i,j,n}\|} - (0, 1, 0, 0, \ldots)\right)\right\| < \delta \quad \forall \delta > 0. \tag{6}
\]
Indeed, let \( A = \left( \frac{a_{2a}}{a_{1a}} \right) e_2 \left( \frac{d_1}{dt} \right)^i \),

and since \( x_n \to x \) we choose \( r \) so large that

\[
\frac{\| T^i x \|}{\| T^i x_r \|} - \frac{T^i x_r}{\| T^i x_r \|} < \delta_1 A \quad \forall \delta_1 > 0 \tag{7}
\]

and

\[
\frac{\| T^j x \|}{\| T^j x_r \|} - \frac{T^j x_r}{\| T^j x_r \|} < \delta_2 A \quad \forall \delta_2 > 0. \tag{8}
\]

Now, using conditions 7, 8 and the triangle inequality, clearly condition 6 < \( \delta_1 + \delta_2 \).

Thus, we get the second basis vector \((0, 1, 0, 0, \ldots)\). Similarly we can approximate the third basis vector \((0, 0, 1, 0, 0, \ldots)\), etc.

This finishes the proof of the Theorem. \( \square \)
CHAPTER 5

Conclusions and Further Research

In this paper our main focus has been diagonal operators on Hilbert space and hyperful orbits. We have given a complete characterization of those diagonal operators which have hyperful orbits and show that either all cyclic vectors have hyperful orbits or no cyclic vector has a hyperful orbit. In this concluding chapter we will present some problems and directions for further research. Some of them have already been mentioned earlier in the paper.

A natural question is what other operators on Hilbert space than diagonal ones can have hyperful orbits. Obviously operators which are similar to diagonal operators, i.e, operators of the form $VTV^{-1}$ where $T$ is diagonal and $V$ is an isomorphism can have hyperful orbits. On Hilbert space there are Schauder bases which are not unconditional. Obviously a diagonal operator with respect to such a basis can have hyperful orbit and such an operator will not be similar to a diagonal operator relative to the standard basis. However, we do not know if an operator with a hyperful orbit must have a countable set of eigenvectors which span the space.

Another direction is to consider cyclic vectors $x$ which do not have hyperful orbits and study how dense the sequence $(n_j)$ needs to be in the set of natural numbers in order for $(T^{n_j}x)$ to span the whole space.

For operators on Hilbert space there is a well developed theory and we have used that for several of our results, in particular Theorem 1. Attempts to generalize that to other classes of Banach space lead us to difficult problems about their geometry. In view of this we feel that even partial generalizations of Theorem 1 are interesting.


