STUDENTS’ METAPHORS FOR MATHEMATICAL PROBLEM SOLVING

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by
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The purpose of this study was to determine the metaphors used by students to describe mathematical problem solving. This study focused on identifying how students interpret and perceive mathematical problem solving via conceptual metaphors (Lakoff & Johnson, 2003). These perceptions and interpretations were coded and analyzed qualitatively and quantitatively in search for a coherent structure embedded in the student’s experiences with problem solving.

The participants for this study were 14 students of honors geometry and both honors geometry teachers at a suburban high school in Ohio. The students were interviewed for 10-30 minutes after completing one of three honors geometry common assessments agreed upon by both teachers. Students were interviewed more than once independent of prior interviews if appropriate to the interview criteria. A total of 22 independent student interviews were collected. Both teachers were interviewed before grading each assessment, totaling 6 teacher interviews.

The design of the study revolved around Interpretative Phenomenological Analysis (IPA) of semi-structured interviews. IPA was applied through Conceptual Metaphor Theory (CMT) to identify the metaphors students used in solving mathematics problems. CMT coded participants’ language by interpreting the conceptual metaphor involved. A conceptual metaphor is a mapping from a target domain to a source
domain. This research qualitatively identified and verified the source domains associated with the target domain of problem solving. The frequency and popularity of each source domain was tallied for numerical analyses. A quantitative analysis verified the significance of the source domains and identified correlations between these domains. Data collection and analysis were validated internally via correlations with the student’s score and T-tests variance between teachers.

The results confirmed that the metaphors used by students were not random or isolated, but coherent. A set of coherent metaphors were identified, which verified the existence of a conceptual metaphorical system for mathematical problem solving. A coherent conceptual metaphorical system is valuable in the teacher’s classroom. CMT analysis educates teachers in being more receptive to the language of students. CMT analysis and the results of this study can educate teachers on how to listen for student understanding.
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CHAPTER I
INTRODUCTION

Problem-solving should be the focus of the mathematics curriculum. (NCTM, 1989)

For the last 60 years, mathematical problem solving has been a prevalent dimension of mathematics education. Within those years mathematics educators have perceived mathematical problem solving as a heuristic process (Pólya, 1945), a logic-based program (Newell & Simon, 1972), a means of inductive and deductive discovery (Lakatos, 1976, 1999), a framework with multiple dimensions (Schoenfeld, 1985), methodologies with multiple variables (Kilpatrick, 2004), a standard (NCTM, 1989), and a model-eliciting activity (R. Lesh, Zawojewski, & Lester, 2007). Indeed, in their historical analysis of problem solving, Stanic and Kilpatrick (1988) state, “The term problem solving has become a slogan encompassing different views of what education is, of what schooling is, of what mathematics is, and of why we should teach mathematics in general and problem solving in particular” (p. 1). Consequently, definitions of problem solving can be nebulous, or downright conflicting (Schoenfeld, 1992). As Shumway (1982) states, “We (researchers of problem solving) don’t know what problem solving is, but if you read our definitions you’ll know what it isn’t” (p. 131).

These varying definitions are difficult for educators to agree upon, but even worse for students, who are told that problem solving is vital for their mathematical education. The goal of my study is to approach problem solving from the perspective of the party that has the most to gain or lose from its interpretation, the student. It is vital to understand how students perceive problem solving if educators are to discuss
mathematical problem solving in a meaningful manner with the student. So why is there limited research into the student’s hermeneutic understanding of problem solving?

**Background and Motivation**

My interest in mathematical problem solving began during an action research study I completed for my Masters in Education. I used a research-based rubric (Evered & Uy, 1999) that measured how well students described their metacognitive difficulty instead of accuracy. These results demonstrated a significant positive correlation between levels of complexity (unistructural, multistructural, relational, and extended abstract) on various problems and the student’s ability to complete and justify problems. To this end, I verified the theoretical structure of this research-based rubric for mathematical problem solving. While this technique was new for mathematical problem solving, there was a tacit argument towards the design of the research. The research model was assumed appropriate for the students because of prior research by Evered and Uy (1999) in which they used the level of complexity as a means to quantitatively interpret students’ responses independent of the student’s correctness. I chose this model prior to listening to student’s perception of problem solving because the literature (Evered & Uy, 1999) suggested it was appropriate for problem solving.

I discovered that numerous earlier studies in mathematical problem solving made similar assumptions. There were those researchers who strived for a grounded theory before assuming a model for mathematical problem solving (Schoenfeld, 1985), yet there were many that did not (Newell & Simon, 1972). An example of the former is the earlier work of Schoenfeld (1985), while examples of the latter can be found in the beginning
studies of cognitive science by comparing the human mind and computers. Schoenfeld’s (1985) foundational work inspired the field by offering aspects of mathematical problem-solving knowledge, instead of a general theory of how students should solve mathematical problems. Specifically, Schoenfeld discovered an epistemological categorization of student’s problem solving process through listening and observing what students drew on when solving problems. These categories were resources, heuristics, control, and belief systems. A more thorough discussion of Schoenfeld’s categories will be given in the literature review.

Schoenfeld (1985) observed students solving problems and focused on the idiosyncratic mistakes rather than the solutions. This allowed him to look at many varying perspectives. For example, he studied the student’s time-management skills as an aspect of self-monitoring, which is itself an aspect of metacognition (Schoenfeld, 1992). Within a 20-minute problem-solving session, high school and collegiate students would spend 5% of the time reading the problem and 95% of the time exploring the problem. Yet mathematicians divided the time more evenly and used other techniques such as analyzing, implementing, and verifying. Thus Schoenfeld asked why there was a difference and interpreted the distinction between how mathematicians solved problems versus how collegiate and high school students solved problems through time management.

However Schoenfeld grounded himself in cognitive science (Schoenfeld, 1987) and there were many pioneers of cognitive science who did not follow this perspective of grounding theory in pedagogical practice. Newell and Simon (1972) emphasized the
model of problem solving as a *process*. The metaphor of *process* originated from Newell’s inspiration with computers. Indeed, thinking of problem solving as a mechanical operation aligns directly with the artificial intelligence aspect of cognitive science. Newell used grandmasters of chess as prototypical examples of problem solving and held their processes to a high degree. Bransford, Brown, and Cocking (1999) also referred to certain chess skills, such as clustering, as skills that separated novice problem solvers from expert problem solvers. Yet why should problem solving, a topic deemed valuable by the National Council of Teachers of Mathematics (NCTM, 1989), assume mathematics and mathematics education is analogously performed in the game of chess? The value of such research is not in question, only its application to mathematics education.

Early cognitive science research followed methodical techniques, such as the scientific method, and are described in chapter 2. Such theories were derived from observing students solving problems, reflecting on the techniques of problem solving and prior research, and creating a process that followed how students solved problems or aspects of problems. Similar to my action research study, such theories were tested and data corroborated their design. However, these theoretical models were consistently constructed by the researcher. The lens of the researcher was the primary lens for the model’s development, not the student’s lens.

Lesh and Doerr (2003) offer a fresh perspective that supports the need to deviate from the notion of a problem-solving *process* to the notion of problem solving as a model-eliciting activity. This allows Lesh and Doerr to explicitly observe the “how and
why” of problem solving by suggesting that the student should model how they solved the problem. Specifically, Lesh and Doerr argue for student-generated models of the problems where the solutions are part of the model, not the end result of a process. Ontologically, Lesh and Doerr deconstruct the traditional dichotic view of understanding (i.e. you either get it or you don’t) and replace it with a cycles of modeling within a model-eliciting activity (Richard Lesh & Harel, 2003). These cycles allow the teacher to observe how the student modifies their model of understanding of a problem so that the teacher may understand how the student perceives the problem.

My research was inspired by Schoenfeld’s (1992) framework and Lesh and Doerr’s (2003) modeling design. To encourage all aspects of problem solving, one must first understand how students model the problem and their solution. I focus on letting students model their perception of problem solving, instead of the researcher, to minimize misinterpretations by the researcher. Yet to observe the student’s model of problem solving, there must be some structure. This study requires a design that will minimize misinterpretation and emphasize the student’s ability to explain their thoughts. For this reason, linguistics plays a vital interdisciplinary role. Specifically, I will use the linguistic structure of metaphors to minimize the researcher’s involvement in student modeling because metaphors allow the student to generate an experiential mapping through conceptual metaphor theory so as to improve their explanation and perception of problem solving.
Statement of the Problem

The student is the primary purpose and future of mathematical problem solving. Polya (1981) identified this over 30 years ago:

What the teachers says in the classroom is not unimportant, but what the students think is a thousand times more important. The ideas should be born in the students’ mind and the teacher should act as only the midwife. (p. 104)

This complex metaphor alludes to certain relationships between birth and the role of the midwife, but how can the relationship be made explicit for the teacher? The inception of new ideas being “born” is complicated epistemologically and has been studied under Plato (1980) through a Socratic dialogue known as Meno’s Paradox (the learner’s paradox): How does one attain new knowledge from old knowledge? This will also be discussed in more depth in the literature review. Currently, the importance to Polya’s focus on the student with the teacher as a supportive participant lends itself to the need for more focus on the student’s perception of problem solving. The teacher has significant influence over the student’s perception of problem solving, but to communicate in a language that is meaningful (Ausubel, 1978; Brownell, 2004) to the student, research must start with the student’s perception of problem solving rather than re-conceptualizing a pre-existing paradigm, such as a process, and validating via an alternative hypotheses. Such post-reconceptualization (Malewski, 2010) must be made meaningful to the students and teachers to improve aspects of mathematics curriculum such as problem solving. Thus we arrive at the question, how do we identify students’
interpretations and perceptions of mathematical problem solving without over-imposing the researcher’s interpretation?

**Purpose of Study**

The purpose of this study is to understand the student’s perception of mathematical problem solving while minimizing the researcher’s misinterpretation. Accomplishing this feat requires a significant focus on the language used by the student, the teacher, and the researcher. All three dialects of problem solving must be clearly understood and expressed by the researcher. When moving between such dialects, shared and similar experiences become a vital linguistic translator.

However, similarities must be clearly identified and perceptions of those experiences can vary. Indeed, if I were to say, “mathematical problem solving is completing a puzzle”, what type of puzzles are being referenced? Are they jigsaw puzzles, puzzles about one’s existence, or just objects of curiosity? If I were to say, “mathematical problem solving is pattern recognition” are the definitions of patterns equivalent for students, teachers, and researchers? If not, how can distinctions be clarified and in which dialect?

As the purpose of this study is to maximize student perception and minimize the researchers misinterpretation, the linguistic study of similarity and comparison theory (Miller, 1993) is relevant. Linguists hold to varying perspectives on similarity and comparison theory, but linguistic research has shown that metaphors play an enormous role in how humans relate experiences through language (Black, 1962; Gibbs, 2008; Lakoff & Johnson, 2003; Ortony, 1993; Presmeg, 1997; Sfard, 1997). As such, linguistic
interdisciplinarity becomes mandatory in a study whose focus is on students’ perceptions of mathematical problem solving. This study will use the linguistic idiom of metaphors to interpret the student’s, teacher’s, and researcher’s perception of mathematical problem solving.

**Research Questions**

- How do students perceive mathematical problem solving?
  - What student experiences are lived through the act of solving mathematical problems?
  - Do these student perceptions align with the teacher’s perception of problem solving?
  - Do these perceptions align with current research in mathematical problem solving?

- How do metaphors influence how students model mathematical problem solving?
  - Is there a set of coherent conceptual metaphors that are frequently used?
  - If existence of a coherent set of conceptual metaphors is satisfied, do literal metaphorical expressions align with conceptual metaphors?

**Rationale for the Study**

The rationale for this doctoral study originated from the pilot study I completed in 2010. The pilot study explored how metaphors were used by high school students while solving a set of three mathematical problems. I had nine voluntary participants ranging from 9th to 11th grade. These students were given three mathematical problems and 30 minutes to attempt to solve and justify as many questions as they could. The students
were video-recorded and after the initial 30 minutes, the students watched the video of themselves solving the problems and were instructed to explain their thinking and problem-solving techniques. Each problem varied in its use of deductive and inductive reasoning. Problems were chosen so that little prior knowledge was necessary. Additionally, the metaphors in the three problems were limited so as to evoke the student’s metaphors. Hermeneutic phenomenology (Willig & Stainton Rogers, 2008) was the methodology employed to analyze the data because I focused on interpreting the students’ metaphors and experience. A more guided explanation of the pilot study is given in the methodology.

Results from the pilot study suggest that the student’s use of metaphors for problem solving were complex yet coherent. Specifically, two results encouraged this current study. First, students were able to model their metacognitive problem solving techniques clearly and concisely. For example, students were aware of their subconscious. Multiple students referred to the need to give it (the subconscious) time to work on the problem. In essence students knew when to be patient with their minds. This encouraged me to trust that high school students were capable of perceiving their problem solving skills, and articulate them appropriately.

Secondly, the use of conceptual metaphor theory (CMT; Danesi, 2007; Lakoff & Nunez, 2000) offered a significantly rich perception of problem solving. Danesi (2007) applied CMT analysis to assist eighth grade teachers in their understanding of word problems. Specifically, abstract metaphorical ideas, such as the belief that numbers lie on a line, were interpreted through the student’s lens using concretization (The act of
making an intangible idea concrete through similarity or comparison). In my pilot study, CMT analysis revealed students perceive problem solving through a variety of experiences yet all nine students consistently referred to problem solving using four conceptual metaphors: mathematical problem solving as a journey, a container, building, or discovery. These conceptual metaphors were coherent due to the fluid transition of the student from one conceptual metaphor to the next within the same mathematical problem. These results will be discussed in the methodology while a more thorough explanation of conceptual metaphors will be described in the literature review.

Finally, a significant rationale for this study originates in the field of cognitive science. While the pioneers of cognitive science emphasized the need for understanding the mind preemptively through computation and processes, they also set a foundation for interdisciplinarity and a need for a scientific method to investigating research questions of cognition (Gardner, 1987). The need for interdisciplinarity with linguistics was vital for my study. Linguistics suggested a means, conceptual metaphors, through which student’s experiences with mathematical problem solving could be modeled by the student with minimal misinterpretation by the researcher. Additionally, while this study does not currently employ the scientific method to directly understand cognition, this study exists due to the spirit from which cognitive science was founded. If one can observe someone thinking, one can understand it. Thus my study bases its approaches squarely on the belief that the observation of students solving mathematical problems will generate credible interpretations genuine to how students solve mathematical problems.
In summarizing the methodology, pre-transcendental Husserlian phenomenology (Giorgi & Giorgi, 2008) was applied in tandem with CMT analysis (Lakoff & Nunez, 2000). This research study consisted of 22 video interviews of 14 suburban, high school, geometry students within two different honors geometry classes with different teachers. These video interviews were 6-20 minutes long and discuss a problem (identical for all participants) chosen by the researcher on a common assessment that the student had recently finished. CMT analysis was applied to these videos to describe the student’s model of mathematical problem solving. Three to four students from each teacher for each assessment were chosen at random: one to two who have performed well according to the teacher’s rubric (67-100%), one to two who have shown some progress (33-66%) according to the teacher’s rubric, and one to two students who performed poorly (0-32%) with little progress. This study spanned over a period of three common assessments and repeated the aforementioned selection method for each of the three common assessments. The study also interviewed teachers prior to assessing the students to understand the teacher’s perspective on how they expected the student to solve the mathematics problem. CMT analysis was applied to the teacher’s perspective of the student’s mathematical problem solving techniques as well. A more thorough explanation is given in the methodology chapter.

**Significance of the Study**

Within mathematics educational research, problem solving has taken many different directions and purposes. Yet the student’s perspective and conceptual understanding of problem solving should be at the center of all interpretations and
applications. Without knowing the student’s perspective of problem solving, one cannot be sure the teacher’s perspective aligns with the student’s perspective on a regular basis. Indeed Silver (1985) writes:

In order to give a richer characterization of mathematical problem solving, we must go beyond process-sequence strings and coded protocols. And we need also to go beyond simple procedure-based computer models of performance. We need to develop new techniques for describing problem-solving behavior not only in terms of the procedures utilized but also in terms of the conceptual systems that influence performance. The successful implementation of rigorous modeling techniques of this type would give us more powerful models of problem-solving performance and might provide more suitable models for learning, since they could account for the phenomena of cognitive restructuring and conceptual reorganization that are observed as people learn in complex domains. (p. 258)

The goal of my study is to achieve what Silver describes, a cohesive conceptual model that will allow for “cognitive restructuring” and “conceptual reorganization” via experiential relationships. Current problem solving models are too static in procedure and don’t incorporate re-conceptualization, which many believe to be at the heart of learning (Green, 2009; Henderson & Kesson, 2004; Malewski, 2010). This will be accomplished through the application of metaphors and conceptual metaphor theory analysis, which has never been studied before with the emphasis on the high school student’s mathematical problem-solving skills.
In essence, this study is significant because it will give students a voice in the research models of mathematical problem solving. It offers researchers a chance to improve mathematics education at its base, the students. It will allow teachers to see mathematical problem solving through the student’s lens rather than hypothesizing how students should perceive problem solving. This will then allow a dialogue between how the teacher perceives problem solving and how the student perceives problem solving. Such perceptions and applications through language will give students and teachers alike the opportunity to improve their problem-solving skills through open dialogue.

**Limitations of the Study**

As there are many factors that are involved with mathematical problem solving, certain factors needed to remain fixed so as to focus on other important factors. This is especially important with interdisciplinary studies because contexts of other disciplines (such as linguistics and CMT) require conditions to occur for proper data collection. The pilot study allowed me to hone those factors that should remain fixed to minimize misinterpretation of the student’s explanation.

As with most studies, generalizability is limited by the sample and population. Currently, this study is interested in students at a suburban high school in an honors geometry course. This study in no way needs to be limited to such a demographic, but was chosen so as to have a common baseline between participants and teachers. Nonetheless, generalizing to other demographics may give different results. The primary concern with this study is opening the door to the student’s model and attempting CMT analysis with student’s perspectives of a mathematical topic fundamental to their
mathematics education. Future studies should follow to encourage a more generalizable model. Currently, the goal is to simply start the process.

The focus of this study engages students at the level of language with metaphors. While this is a primary means of communication, it is not the only one. Indeed, many means of communication are vital when teaching. In fact, other linguistic attributes such as semiotics and metonymies, specifically synecdoche, are shown in research to be just as vital (Presmeg, 1997, 2005). Additionally, Nunez (2008) has emphasized the value of gestures and having a teacher conceptually develop new embodied mathematical ideas using their hands. Due to the focus of this study, such attributes are not initially planned.

Finally, this study does assume the resources and curriculum agreed upon by the honors geometry teacher community at this high school, which uses an investigative approach known as College Preparatory Mathematics (CPM). Their means of designing an investigative curriculum immersed in constructivist approaches may influence student’s perceptions of problem solving. Yet again, choosing such consistencies is a necessary evil of any study whose data, application, and purpose demands empirical classroom involvement.

Summary

In sum, my study’s purpose is to understand mathematical problem solving from the student’s perspective in secondary education. To achieve this I interviewed students and used conceptual metaphor theory (CMT) so as to minimize misinterpretation and maximize student perceptions. The goal is to develop a common base of metaphors from which students draw to discuss mathematical problem solving. This is helpful to the field
as a whole because it will allow teachers to communicate more clearly their perception of problem solving in the student’s dialect. Additionally, it will give research a starting point for how problem solving is presently viewed by students.

**Definitions of Terms**

**Complex Metaphors:** Conceptual metaphors that are composed of multiple primary metaphors that carry correspondences between the source domain and target domain. They are more culturally dependent metaphors (Grady, 2005).

**Conceptual Metaphors:** Conceptual metaphors are mappings. Conceptual metaphors map from a target domain to a source domain for the purposes of succinctly identifying conceptual integration (Lakoff & Johnson, 2003). This is stated in the form TARGET DOMAIN IS SOURCE DOMAIN. For example, the literal metaphor “I don’t know where to start to solve this problem” can be summarized by the conceptual metaphor of PROBLEM SOLVING IS A JOURNEY. The use of the word IS remains unidirectional.

**Conceptual Metaphor Theory (CMT):** This theory is based upon the belief that language is not restricted to communication, but involved in cognition. Specifically, conceptual metaphors are a means for which people cognate. In CMT, the researcher interprets the participant’s conceptual model through the conceptual metaphor (Danesi, 2007).

**Frequency of Source Domain:** Frequency refers to participants’ summative use of the source domain associated with a specific target domain with respect to Conceptual Metaphor Theory.
**Grounded Metaphors:** Conceptual metaphors directly grounded in experiences and ideas of the participant. Grounded metaphors are regularly structural conceptual metaphors (Lakoff & Nunez, 2000).

**Interpretive Phenomenological Analysis (IPA):** The detailed examination of individually lived experience and how individuals make sense of that experience (Eatough & Smith, 2008).

**Linking Metaphors:** Unlike grounded metaphors, linking metaphors are conceptual metaphors made through associations with experiences rather than directly experiences (Lakoff & Nunez, 2000). For example, despite never being in war, one could state that the football game on television was quite a battle.

**Literal Metaphors:** Actual metaphors stated verbatim (Kovecses, 2006).

**Metaphorical System:** A set of conceptual metaphors whose source and target domains and regularly associated with one another (Kovecses, 2006). Additionally, the use of the source domains and target domains align with how people communicate their experiences.

**Ontological Conceptual Metaphors:** Ontological metaphors provide target domains with less structure that allow for the construction of a new reality through their use (Kovecses & Benczes, 2010).

**Orientational Conceptual Metaphors:** Orientational metaphors are a broad concept with only an orientation attached through experience (Kovecses & Benczes, 2010). For example, HEALTHY IS UP.
**Popularity of Source Domain:** Popularity refers to whether a participant used the source domain or not associated with a specific target domain with respect to Conceptual Metaphor Theory. This is not a measurement of how many times it was used.

**Primary Metaphors:** Conceptual metaphors are shared by humans through the most basic experiential relationships with objects. Thus they are more culturally independent metaphors. (Grady, 2005)

**Source Domain:** For a conceptual metaphor, the source domain is the experiential domain that is mapped from the target domain (Lakoff & Johnson, 2003). TARGET DOMAIN $\rightarrow$ SOURCE DOMAIN.

**Structural Conceptual Metaphors:** Structural metaphors describe a complex concept, in terms of a concrete experiential object (Kovecses & Benczes, 2010). For example, the literal metaphor “Stop wasting my time” concretizes via the conceptual metaphor TIME IS A LIMITED RESOURCE.

**Target Domain:** For a conceptual metaphor, the target domain is the abstract domain that is mapped onto the source domain (Lakoff & Johnson, 2003). TARGET DOMAIN $\rightarrow$ SOURCE DOMAIN.
CHAPTER II

LITERATURE REVIEW

*Hermeneutics* (Greek *hermeneuein*): to interpret (Harper, 2012).

**Introduction**

The literature is broken into four sections. First, I will begin by discussing the philosophical axioms of the study, along with the justification and the consequences of such theoretical foundations. Second, I will discuss the history of mathematical problem solving. Third, the influence of cognitive science in mathematical problem solving will be discussed. Finally, the interdisciplinary need for linguistics in this study will be discussed.

**Philosophical Axioms**

When designing a study in mathematical problem solving and conceptual metaphor theory, many decisions reduce to one’s view of knowing and being; the epistemological and ontological postulates respectively. As this study’s value is in the hermeneutic influences of the student, it is vital that the values of the researcher are made clear. Ontologically, this study resonates with the French set theorist turned philosopher, Alain Badiou (2002). Badiou argued reality independent of Hume and Kantian philosophies in that he refused to take up the argument whether reason defines ethics or ethics define reason. Badiou suggested truth (not absolute Truth) was found through being aware and “in the moment” (p. 47). This did not define his work in the realm of relativism, because he wasn’t focusing on relative truth, but rather a means to be genuine to your existence. This provides what he referred to as *ethical fidelity*. In sum, ethical fidelity is the need
for a person to transform a situation into an event. It is not enough for one’s situation to exist, one must exist in that situation (i.e. an event of that person). This resonates with the design of my study because I am searching for the student’s truth about problem solving. Thus I need to make sure I do not deny the student’s truth of mathematical problem solving in this interpretation. My study aligns with this phenomenological philosophy because my research is looking for the essence, the truth, of what mathematical problem solving means to the student. Moreover, this philosophy allows for a flexible view of reality so that the study will not be clouded by precursors. This is not to say that there is no reality, but rather that the purpose of this study should be to identify coherence (Lakoff & Johnson, 2003) in realities of the student, teacher, and researcher. Existence of reality is assumed, but not uniqueness.

Epistemologically, what does it mean to know the solution to a mathematical problem or even the problem itself? A current, working definition (not concrete or assumed appropriate for students) of a mathematical problem and problem solving will be taken from Lesh and Zawojewski (2007):

A task, or goal-directed activity, becomes a problem (or problematic) when the “problem solver” (which may be a collaborating group of specialists) needs to develop a more productive way of thinking about a given situation . . . problem solving is defined to be the process of interpreting a situation mathematically, which usually involves several iterative cycles of expressing, testing, and revising mathematical interpretations and of sorting out, integrating, modifying, revising, or
refining clusters of mathematical concepts from various topics within and beyond mathematics. (p. 782)

Obviously, one can see how the cognitive activity of solving mathematical problems is complex. Notice how Lesh and Zawojewski use the word “interpreting” as the primary verb involved with problem solving. Thus this working definition aligns with the ontological axioms developed above via ethical fidelity, yet how does a student know if the problem has been solved correctly by the teacher’s standards?

Attainment of this knowledge is an important epistemological postulate I will make explicit with the perspective known as embodied cognition (Nunez, Edwards, & Filipe Matos, 1999). Nunez (2008) has suggested that mathematics is an embodied field, that is, mathematics is not mind-free. Often, mathematics is romanticized to be thought of as a subject independent of human beings. It is an international language, it offers such consistent phenomena as the number \( \pi \), and still has many unanswered questions such as the Riemann Hypothesis. Nunez posits that despite these amazing properties, all are dependent upon human cognition and thus human experiences. This radical notion has offered much controversy in the field of mathematics and cognitive science. However, Nunez developed embodied cognition within the field of mathematics education, where it is clear that learning mathematics is embodied and not mind-free. Hence, experientialism and embodied cognition hold as epistemological tenets within mathematics education.

In recent years it has become widely accepted that the learning and practice of mathematics are not purely intellectual activities . . . it has been acknowledged that learning and teaching take place, and have always taken place, within
embedding social contexts that do not just influence, but essentially determine the kinds of knowledge and practices that are constructed. . . Human cognition is bodily grounded, that is, embodied within a shared biological and physical context. (Nunez et al., 1999, pp. 45-46)

Thus the process of learning mathematics is epistemologically embodied and not an autonomous procedure within mathematics education.

**Brief History of Mathematical Problem Solving**

**Process Over Product**

In mathematics education, the study of problem solving has resulted in diverse and interdisciplinary research. The last few decades have shown a burgeoning of ideas illuminating the field as extremely complex (R. Lesh & Zawojewski, 2007; Niss, 2007; Schoenfeld, 1985). This complexity originates from an intrinsic value of the discipline to be inclusive of other fields. While researching a specific aspect of student problem solving, various theoretical perspectives such as cognitive psychology, sociocultural theory, or distributed cognition have all offered insight (Cobb, 2007). Additionally, problem solving requires disciplines such as anthropology, psychology, biology, and physics to explain aspects that education alone has not considered (R. Lesh & Zawojewski, 2007). For example, the understanding of unitization and cardinalization within counting would have been impossible without the help of cognitive psychology. Thus there are many dimensions and disciplines involved in mathematical problem solving.
Problem solving is not only complex in structure, but complex in definition as well. The 1989 National Council of Teachers of Mathematics (NCTM, 2012) *Curriculum Standards and Evaluation for School Mathematics* demands high school students are educated in building new mathematical knowledge through problem solving, apply and adapt a variety of appropriate strategies to solve problems, and reflect on the process of mathematical problem solving. Specifically, NCTM states, “Problem solving means engaging in a task for which the solution is not known in advance” (NCTM, 2012). Mathematical problem solving is hard to define because it is regularly dependent upon the context of the problem.

NCTM’s definition has focused on the operation of problem solving over the product. This agrees with Krulik and Rudnick (1989):

Problem solving is a process. It is the means by which an individual uses previously acquired knowledge, skills, and understanding to satisfy the demands of an unfamiliar situation. The process begins with the initial confrontation and concludes when an answer has been obtained and considered with regard to the initial conditions. The student must synthesize what he or she has learned and apply it to the new and different situation. (p. 5)

The idea of emphasizing the process originated from cognitive psychology and bases its arguments on the epistemological logic that all mathematical knowledge is interpreted through the human mind. Thus there is more interest in the study of human mathematics and less interest in a transcendental Platonic mathematics, if it exists (Lakoff & Núñez, 2000). This framework is meaningful due to the extent to which problem solving is
dependent upon the medium of the human mind. As mathematics educators, this is the most rational and vital instrument to study.

Mayer (1982) describes how cognitive psychology can improve mathematical problem solving in 3 important ways. First, the cognitive psychologist views learning as the acquisition of knowledge instead of the acquisition of new behaviors. Second, cognitive psychology views problem solving as a series of mental operations that transform knowledge representations rather than viewing problem solving as a series of learned behaviors. Third, and most important to the teacher, the cognitive psychologist believes instruction should focus on the “process rather than the product” (p. 3). Hence, while a consistent definition may be lacking in mathematical problem solving, the coherent theme that the process, and not the product, is important to research agrees with Shumway’s (1982) aforementioned claim that there is validity to what is not problem solving.

**Modeling the Process**

As problem solving is a diverse subfield of mathematics education, the approaches to interpreting its process have also varied within the domain of cognitive psychology. These interpretations have taken the shape of various models. A foundational model was researched by George Polya in the middle of the twentieth century. Polya is considered the father of mathematical problem solving (Alexanderson, Pólya, Alexanderson, & Boas, 2000; R. Lesh & Zawojewski, 2007; Schoenfeld, 1994; Shumway, 1982). Polya was a pioneer because he initially began in pure mathematics and changed over his lifetime to education. He founded the four-step system to solving
problems, each step of which has been debated thoroughly over the last 50 years (Pólya, 1945):

1. Understand the problem-What knowledge is required to interpret the problem?
2. Devise a plan-What similar problems or methods do you have available?
3. Carry out the plan-Can we execute our technique orderly?
4. Review and reflect on your solution-What are the limitations of this problem? Could I solve the problem another way? How else can I use these results?

These steps seem natural; that is the beauty of Polya’s approach. He embraced the linearity of the process from which humans deduce. Yet it should be noted that Polya was responsible for many deeper ideas in mathematics education beyond process-oriented structure. For example Polya worked on analogical reasoning, inductive and deductive reasoning, modern heuristics, generalization, and the Inventor’s Paradox (Polya, 1945).

Many problem-solving studies branched out from Polya’s foundational four step system. One such example in mathematics education is Erlwanger’s (2004) study of Benny. This was a qualitative study during the “New Math” of the 1970’s when mastery-learning mathematics known as Individually Prescribed Instruction (IPI) was the dominant form of discourse. A sixth grade student named Benny was asked to explain how he solved certain problems on multiple choice tests. Mastery scores on assessments were required before the student moved on to a new lesson that they studied individually with tutor support from the instructor. Benny scored mastery level on many assignments, and Erlwanger sat down to ask him how he solved problems. Through hermeneutic inquiry and deductive questioning for consistency of mathematical properties, it was
discovered that Benny had a solid problem-solving system in his own mind, but in no way followed the laws of mathematics. In other words, Benny found his own process to solve multiple choice problems (guessing and manipulating distracters) that lacked axiomatic closure and consistent reasoning (½ did not equal 2/4). The immediate question arose, what exactly are we teaching students? Yet the more enduring question over the years has been how can assessment theory compensate for the Benny’s of the world? In terms of Polya’s 1st step, it is vital that students understand a problem before they try to solve it? But how does a student come to understand a problem?

Within Erlwanger’s (2004) study were the tenets of constructivism that led to models such as Kilpatrick’s (2004) study via classification of variables within the process of problem-solving. Kilpatrick’s design distinguished between a research variable and a teaching variable. Within each of these categories Kilpatrick delineated between independent and dependent variables via causality of the problem-solving process. Kilpatrick’s description of variables was helpful to researchers of problem solving in seeing how knowledge is constructed. It outlined a need for causality between aspects of a student’s past understanding and their future understanding. Kilpatrick was aware of the complexities of problem-solving methodologies and thus attempted more of a description rather than a prescription (such as Polya’s model) for the process. However, current research has found that the distinction between independent and dependent variables blends and the diversity in the variables can be a limitation to the model in practice (R. Lesh & Zawojewski, 2007).
In the 1980’s these complexities led Alan Schoenfeld (1985) to create categorization of knowledge and behavior in mathematical problem solving. Schoenfeld studied talented collegiate mathematics students proficient with calculus and how they approached various problems. He found an enormous overlap with the affect domain that illuminated how Polya’s four step process required four categories:

1. Resources: Mathematical knowledge possessed by the individual
2. Heuristics: Strategies and techniques for making progress on new problems
3. Control: Global decisions regarding the selection and implementation of resources and strategies.

These categories became Schoenfeld’s model for problem solving. Rather than mapping out the process, Schoenfeld identified key properties/categories that were inherent in the problem-solving gestalt. Thus, Schoenfeld offered an alternative to help educators interpret students’ work through the problem-solving process. Instead of identifying which step of Polya’s process students were unable to overcome, Schoenfeld suggested a holistic approach and interpreting the student’s categorical limitation rather than sequential limitation.

The lineage of these models reflects their diversity and similarities. Schoenfeld’s (1985) categorization is completely different than Kilpatrick’s (2004) classification, yet their foundations and interests still revolve around the process of problem solving. This has led to the current dominant discourse for the problem-solving process, modeling perspectives (R. Lesh & Zawojewski, 2007; R. A. Lesh & Doerr, 2003; Lester &
National Council of Teachers of Mathematics, 2007). The modeling perspective suggests mathematical problems elicit student modeling instead of the textbook or curriculum eliciting a problem-solving model that students then follow.

Solving applied problems involves making mathematical sense of the problem (by paraphrasing, drawing diagrams, and so on) concurrently with the development of a sensible solution. Understanding is not thought of as being an all-or-nothing situation, and mathematical ideas and problem-solving capabilities co-develop during problem-solving. The constructs, processes, and abilities that are needed to solve “real life” problems (i.e., applied problems) are assumed to be at intermediate stages of development, rather than “mastered” prior to engaging in problem solving (R. A. Lesh & Doerr, 2003). The modeling perspective is not limited to distinguishing between applied problems and theoretical problems, but this does signify a significant change in problem-solving paradigms.

Lesh and Harel’s (2003) research used the modeling perspective to create model-eliciting activities such as Big Foot. In this activity students developed a procedure for police detectives to predict a person’s height from their shoe footprint. The design did not imply linear growth as shoe to height may have a quadratic or exponential correlation. The students had to generate a model and a mathematical approach. This is an example of model elicitation. The process of problem solving is not predesigned, but follows from iterations of the model eliciting paradigm, known as model cycling.

The modeling perspective can then take on two roles: modeling for student classroom problems and modeling in problem-solving research (R. A. Lesh & Doerr,
2003). The latter is considered a paradigm that allows research to look at problem solving by the design of its model. Unless otherwise stated, when discussing models of problem solving, I will be referring to the latter. Different topics in mathematics and different ages require different models. Thus the representation of the research model can then be evaluated relative to the significance of that age group. At the student level, models become relevant through social constructs and application (Powell & Kalina, 2009).

The modeling perspective has at least 3 promising lines of research currently being studied: situated cognition, communities of practice, and representational fluency (R. Lesh & Zawojewski, 2007). Situated cognition refers to learning and solving problems in context. This looks at the models with criteria such as; what is their purpose, who is the intended user, and what content knowledge is being used? Communities of practice emphasize the social context of the problem-solving model. Are groups being used? Are the groups heterogeneous or homogeneous? How will the classroom dynamics affect the models in practice? Representational fluency looks at how problems are presented, the tools involved, and different representations that solutions may take. Representational fluency studies the instrument used while solving problems and their effect on the user to manipulate them.

The last promising line of research, representational fluency, has taken many forms and deals with many topics. One such topic is technology. In-depth research has been done in representational fluency on the significance of electronic representations (e.g. Computer Algebra Systems, CAS) and their ability to help or hurt the student’s
problem-solving techniques (Zbiek, Heid, Blume, & Dick, 2007). In this manner, Zbiek, Heid, Blume, and Dick (2007) have done research into measuring and aligning a representation on the computer (The Geometer’s Sketchpad) to that of traditional mathematics (Euclid’s straight-edge and compass). Are these representations fluent? Do aspects and manipulative tools of both representations align with the problems intent? Such measurements question the mathematical fidelity of a representation.

Yet mathematical fidelity and representational fluency may be applied beyond computer technology, to a method of interpretation. What do we mean when we say fluent? Mathematics itself is evolving (Harel & Sowder, 2007). Should a representation align with the mathematical axioms of the topic at the time? What models retain the mathematical discipline and axiomatic structure inherent in the deductive reasoning (Lakatos, 1976)? Within the problem-solving modeling perspective (R. A. Lesh & Doerr, 2003), representational fluency is novel and still being defined. It deals with the resource categorization of Schoenfeld (1985), and the heuristics of Polya (1945). However, such representations can only be discussed after the use of reflection and monitoring of the problem solver. Reflection and monitoring are foundational aspects needed in modeling metacognition.

**Metacognition**

Research on the modeling of the problem-solving process mandates an understanding of the model being constructed. Metacognition is one technique from which to develop and reflect on that understanding. Metacognition is not a new idea. Piaget referred to it as reflective abstraction (Piaget, 1970). Flavell (1976) originally
referred to the term as “The active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects or data on which they bear, usually in the service of some concrete goal or objective” (p. 232). The abbreviated definition for metacognition is to think about how one thinks. This study’s use of modeling aligns best with that of cognitive science (Gardner, 1987) where its purpose is to better understand the mind’s design in problem solving. Modeling how one solves a mathematical problem is thus inherently metacognitive. Certain aspects of metacognitive research are useful in understanding how to model the problem-solving process.

Metacognition has taken many forms by many authors. Polya (1945) rejuvenated Pappus’ (2000) antiquated definition of heuristics (a branch of study in philosophy/logic whose aim is to study the methods and rules of discovery and invention) to a modern application that questioned where insight in mathematical problems originated (thus his four step process). This idea of heuristics has been expanded to cover problems that still have no answer. The study of the method of discovery is metacognitive in nature because it isn’t a process to study as it is discovery-based. Thus the participant must constantly monitor their process to interpret any discovery.

Metacognition can also be applied pedagogically with metacognitive instruction. Succinctly, metacognitive instruction demonstrates to students how to monitor and question their problem-solving process so as to not go in the wrong direction. Kramarski, Mevarech, and Arami (2002) used metacognitive instruction with a control/experimental group design and concluded that students who were exposed to metacognitive instruction significantly outperformed their counterparts on standardized exams. Skemp (1989) used
the metaphor of mental-schemas-as-maps with goal-directed activities helping explain metacognition. Specifically, two monitoring operators, $\Delta_1$ and $\Delta_2$ were introduced by Skemp to understand how thought affected action. $\Delta_1$ is the operator that took in information, processed it relative to desired goals, and directed the proper action one would take to accomplish those goals while $\Delta_2$’s purpose was to optimize $\Delta_1$’s process. Thus the $\Delta_1$ operator involved physical action while the $\Delta_2$ operator involved mental activities analogous to metacognitive monitoring and reflection for optimizing goal-directed activities (Skemp, 1987).

Schoenfeld’s (1994) research had a different approach to metacognitive instruction. Schoenfeld discovered in his semester-long studies that students eventually became trained in a behaviorist manner to apply metacognitive techniques by the instructor. Schoenfeld’s students did not generate genuine problem-solving processes. Schoenfeld concluded that the use of metacognitive instruction was not meaningful as its application was rote, methodical reflection and not genuine heuristics. Thus students habitually used the metacognitive techniques but did not internalize them.

Metacognitive instruction has mixed and nebulous results. The aforementioned studies corroborate a complex relationship of metacognition and metacognitive instruction. Lesh and Zawojewski (2007) argue that productive, metacognitive strategies need not be explicit, but rather intuitive. Yet, pragmatically (Dewey, 1963), how will students metacognate their model of the problem-solving process and how will educators model the student’s modeling in mathematics education? Will metacognitive modeling help future students make sense of problem solving in mathematics?
To better understand the mathematical problem-solving process and the application of metacognition, one must incorporate the aims in mathematics (Noddings, 2009). What mathematical processes demonstrate a meaningful problem-solving process? How can one convince themselves or others that their process is fruitful? To answer these questions, the epistemological roots of mathematics must be studied. The foundation of mathematics as a discipline, proof, is the demonstrative method of verification that a problem has been solved. Thus it is through proof that a mathematical problem-solving model is evaluated.

**Proof**

In mathematical problem solving George Polya distinguished between problems to find and problems to prove (Pólya, 1945). A problem to find is similar to a chess problem where there is a checkmate to find, or the answer to a riddle. Heuristically, an answer is already known to exist. A problem to prove is an assessment of the problem, or claim the problem states, as either true or false. Kramarski et al. (2002) demonstrated research that was tested via standardized tests primarily interested in problems to find. Schoenfeld’s (1994) research emphasized problems to prove. In both cases objective logic is needed, but only in the latter is explanation necessary. It is in proof that the process of problem solving can be evaluated (Pólya, 1945; Pólya, 1954). Therefore, to observe metacognition and the model from which one solves mathematics problems, a proof is desired.

What defines mathematical proof is the search for truth. Harel and Sowder (2007) state “a proof is what establishes truth for a person or community” (p. 806). Post-
structurally, truth is subjective (Sarup, 1993) and grounded in ontological and epistemological beliefs (Malewski, 2010). Thus there are many facets to mathematical proofs and they are inherent in how humans do mathematics, specifically teenage students.

The subjectivity of mathematical proof, mathematical truth, arises from the person or persons one is trying to convince. Harel and Sowder (2007) refer to this as ascertaining versus persuading. Ascertaining refers to the process an individual (or community) uses to remove doubts of a conjecture. Persuading refers to the process an individual (or community) uses to convince others to remove doubts of a conjecture. Once one, or both, of these processes have been accomplished, the conjecture is referred to as mathematical fact.

The epistemological and ontological reasoning in mathematical proof is different than proof in a courtroom, or in a physics laboratory. Polya (1954) distinguishes between two types of reasoning in proofs, plausible reasoning and demonstrative reasoning. Plausible reasoning refers to empirical and experiential knowledge. Demonstrative reasoning originates in mathematics as demonstrative logic, which is pure, rigid, and demands objective truth as right or wrong. Schoenfeld (1985) impresses the difference between inductive (empirical) logic and deductive logic. Many other researchers have recognized this divide in proofs (Lakatos, 1999; 1976; Lakoff & Núñez, 2000; R. Lesh & Zawojewski, 2007; Prus-Wisniowska, 1995). For purposes of this paper, I will refer to deductive (demonstrative) logic and inductive (plausible) logic.
However, heuristics (Pólya, 1945), has given way to a conflux of deductive and inductive reasoning through discovery. Lakatos (1976) has argued that in mathematical discovery, both forms of proof are used. Specifically, Lakatos argues that all proofs are quasi-empirical when discovery is involved. “Any deductive argument has to start from an inductive basis!” (p. 69). Induction is necessary for contextualizing the deductive logic. Similarly, Polya (1954) claimed that the ebb and flow between the two forms of reasoning is needed. Specifically, without plausible reasoning, demonstrative reasoning is merely aesthetic, and without demonstrative reasoning, plausible reasoning cannot achieve consensus. Therefore determining truth requires both inductive and deductive logic. For the purpose of this study, a model for the process a student uses to solve mathematical problems must include elements from both deductive and inductive logic.

Erlwanger’s (2004) study can be interpreted as a misconception between the inductive and deductive logic of Benny. Healy (2000) addressed this concern with a causal comparative study of 14 and 15 year old students in their understanding of proof in algebra. The formal logic of proof was not the focus of the study as it shouldn’t be with students who are unfamiliar or limited in their understanding of proof. However, the beauty of proof is that it is based on explicit reason and thus gives insight into how students solve problems. Healy’s questionnaire requested the student to justify the truth of a certain algorithm. Healy’s results showed the empirical (inductive) approach to dominate across a majority of students, yet the students were aware of the limitations of their argument. Few progressed to the logical (deductive) approach, and those who did used inductive methods initially. This study lends itself to the need for teacher support
and explanation to one of the core structures of mathematics; deductive logic comes from inductive reasoning and problem solving must address how students make sense, and think of making sense, of empirical previous knowledge.

Tall and Vinner (2004) also noticed this distinction in student’s abilities to solve problems by comparing concept images versus concept definitions. Their study of limits and continuity showed students had a concept image, an intuitive sense, a picture in their minds of the concept, in other words an inductive reasoning of a concept. However, students were unable to justify their solutions or manipulate aspects of similar problems because they lacked a concept definition, a formal list of the properties of the image, how to describe them, and how to use them. Prus-Wisniowska’s (1995) dissertation corroborated Tall and Vinner’s (2004) limits/continuity conclusions using an inquiry-based problem solving model involving derivatives that required deductive logic and problems to prove over problems to find (Pólya, 1945).

**Proof Schemes**

What the students of Tall and Vinner (2004) and Prus-Wisniowska (1995) lacked was a culturally accepted proof scheme. “A person’s (or community’s) proof scheme consists of what constitutes ascertaining and persuading for that person (or community).” (Harel & Sowder, 2007) Thus a proof scheme is what allows a conjecture to become a fact. From a problem-solving perspective, a proof scheme models the problem solving technique. Surprisingly, problem solving and proof schemes are researched separately despite the significant overlap in theoretical framework and evolution as a design science (Cobb, 2007).
Harel and Sowder (2007) created a taxonomy of proof schemes so as to express classes of proofs that are more convincing than others. The three main schemes were: external conviction proof scheme, empirical proof schemes, and deductive proof schemes. External conviction proof schemes rely on outside authorities such as a teacher or a calculator. Empirical proof schemes rely on evidence (more inductive reasoning). Deductive proof schemes are considered the greatest of these and require deductive reasoning as well as generality, operational thought, and logical inference (Battista, 2007; Harel & Sowder, 2007; Tall & Vinner, 2004; van Hiele, 2004).

While Harel and Sowder’s (2007) taxonomy is thorough, it is in no way the only scheme for proofs. As previously described, there are many models for inductive and deductive logic. Harel and Sowder’s model does align well with the field of mathematics education. For example, the clarification of deductive proof schemes works well with Tall and Vinner’s (2004) relation to a formal concept definition while the empirical proof scheme aligns with a concept image. Moreover the five stages of the van Hiele (2004) model for geometry find a strong connection with the taxonomy. The stages do not align directly with the levels of the taxonomy, but stage 4 of van Hiele’s stages of geometric thought is formal deduction and stage 5 uses all the types of the deductive proof schemes (Harel & Sowder, 1998).

The extent to which proof schemes can relate to other models in mathematics education is impressive, but not surprising. The search for truth, the essence of proof, is what has historically perpetuated mathematics. From the time of Euclid’s (2003) “Elements” to Hilbert’s (1971) modern “Foundations of Geometry”, mathematics
education has been searching for a balance; a balance within the inevitable interaction of deductive and inductive logic. Simply put, proof schemes offer an explanation of the limitations and classifications of mathematics problem-solving models.

**Proof Schemes for Pedagogy**

Mathematics is a universal language spoken by every culture (Lakoff & Núñez, 2000). Consequently, mathematical proof, mathematical problem solving, and mathematics education perpetuates in every culture and every country. As many studies have shown, American mathematics is not a dominant model in international education (McDaniels, 1998; National Center for Educational Statistics, 2007; Niss, 2007). Recently, the NCTM standards have deemphasized formal proof (NCTM, 2010). However, public and private educational institutions misinterpreted this result as the opportunity to stop teaching proof techniques (Harel & Sowder, 2007). This originates from the average teacher’s misinterpretation that proofs are only used in geometry and thus a local, rather than global, aspect of mathematics (Battista, 2007). Clearly, the propensity for this stigma to continue for future generations can only be abated by the teacher.

How are current mathematics reforms and standards informing teachers and students for the need of proofs? Sadly, they aren’t (Harel & Sowder, 1998; Harel & Sowder, 2007; Healy, 2000). As Polya (1945) said over 60 years ago, mathematics education is focusing its efforts on problems to solve and not problems to prove. The use of persuading arguments needs to be taught for students by teachers (Harel & Sowder, 2007). On this matter, there is a dearth of research in the problem-solving subfield as
well. How are teachers to explain their problem-solving model without imposing its use on the student?

Polya (1945) struggled with this concept 70 years ago when he observed the teaching of formal and informal proof. Formal proof, like concept definition (Tall & Vinner, 2004), requires attention to detail to ascertain (Harel & Sowder, 2007) a deductive argument. However, a more persuasive proof scheme can be accomplished more succinctly with the use of an informal proof (Pólya, 1945). An informal proof assumes the audience can fill in certain details and focus on the counter-intuitive aspects of a proof; from which the main concepts of the proof follow. This proof scheme does not align with Harel and Sowder’s (2007) taxonomy as it includes properties of the external conviction proof scheme and the deductive proof scheme. The danger arises when the teacher assumes the concept definition is well known when in fact the student only has a concept image available (Tall & Vinner, 2004). In this situation, the proof has lost its value and, may be misconstrued if metacognitive awareness (Schoenfeld, 1985) is not being practiced.

Thus teaching proof informally can be as dangerous as not teaching proof at all (Polya, 1945). This need for a teaching and learning perspective of proof challenges the proof-scheme paradigm to ask practical pedagogical questions. Harel and Sowder (2007) raise challenging questions about proofs relevant to problem-solving in the classroom:

What bearing, if any, does the epistemology of proof in the history of mathematics have on the conceptual development of proof with students? What bearing, if any, does the way mathematicians construct proofs have on
instructional treatments of proof? What bearing, if any, does everyday justification and argumentation have on students’ proving behaviors in mathematical contexts? . . . Comprehensive perspectives on proof are needed in an effort to understand students’ difficulties, the roots of the difficulties, and the type of instructional interventions needed to advance students conceptions of and attitudes toward proof. (p. 807)

I believe the pedagogical repercussions of teaching proof must include problem-solving processes from a model perspective. How students, teachers, and researchers model the problem-solving process requires a better understanding of how these parties define justifying mathematical truths. These truths rely on the party’s epistemological beliefs in their constructed proof scheme.

There has been little research done comparing the underpinnings of mathematical problem-solving and mathematical proof. Moreover, the polarization in proof of deductive and inductive logic needs to be deconstructed for purposes of teaching. A spark of illumination exists in Lakatos’ (1976) approach to proof, discovery, and heuristics via problem solving as he claims, “There are no such things as inductive conjectures” (p. 73). Yet how is this discovery made? How do problem solving models relate to proof schemes? To understand such a relation, let us look into a field that has become significant in mathematical problem-solving and proof due to its emphasis on the human mind and a perspective of thinking that holds to empirical conclusions. Cognitive science offers a cognitive model putting mathematical problem-solving models into practice.
Influence of Cognitive Science

Cognitive Science is a field that has burgeoned so much over the last 50 years that it is difficult to understand what theories, practices, and tenets are prototypical of this field (Dupuy, 2009). Its history originates from many fields, and its growth branches out to many fields one of which is mathematics education, where it has sown new seeds in the area of mathematical problem solving. Many mathematics educators well-versed in mathematical problem solving (Schoenfeld, Newell, Davis, Nunez, etc.) consider cognitive science a foundation for their studies. As cognitive science has outgrown its initial design, it is difficult to trace the discipline’s roots back to the source as so many other areas of study have convoluted the direct lineage. This is understandable as cognitive science is an interdisciplinary field.

Definitions for Cognitive Science

Cognitive science varies in practice and in theory among research. Thus definitions are important to clarify. This study will take the adjective cognitive in cognitive science as referring to issues concerning knowledge, thinking, and learning (Gardner, 1987; Reif, 2008). Additionally, the word science will hold to the Merriam-Webster Definition (2011), “a system of knowledge concerned with the physical world and its phenomena”. More specifically, consider the philosophical spirit from which science begins: If a phenomenon is observed, it can be understood. Hence cognitive science approaches the knowledge of teaching and learning from a research-based design. To accomplish this, many factors are involved and many specifications need to be clarified.
Foundations of Cognitive Science

While cognitive science has many avenues from which it originated, a few are significant to the core assumptions and methodological features discussed below. In the first half of the 20th century, psychology was strongly influenced by behaviorism in its scientific endeavors (Davis, 1984; Gardner, 1987; Ormrod, 2004; Schoenfeld, 1987). This is primarily due to behaviorism emphasizing science could not hypothesize on the “mysterious mentalistic entities” (Gardner, 1987). However, practice and theory continued to question if this was sufficient, especially in education (Ormrod, 2004). Some argue that a convolution of researchers in the 1960’s and 1970’s from various fields converged because of circumstances such as the Turning machine (Schoenfeld, 1987).

Gardner’s book (1987), “How the Mind Works”, is considered an accurate historical development of general cognitive science. Within his book, Gardner pinpoints the Hixon Symposium of 1948 at the California Institute of Technology where a tipping point occurred and a significant number of people conferred with Karl Lashley on the limitations of behaviorism (Gardner, 1987). Lashley argued that behaviorism fell short when explaining various language acquisition skills within the English speakers. Others believe the change away from behaviorism was due to the birth of cybernetics in the Macy conferences of the early 1950’s (Dupuy, 2009). In these New York conferences, computer science and mathematics education took center stage as cybernetics developed a following and two credos:

1. Thinking is a form of computation.
2. Physical laws can explain why and how nature has meaning, finality, directionality, and intentionality. In essence, humans think similar to how machines compute. This model has become popular amongst cognitive scientists, but has also admitted to being flawed. Thus already many roots are seen to be taking hold as to the developments of cognitive science.

These approaches and techniques varied in their geography as well. On the west coast, Lashley argued against behaviorism via linguistics (Gardner, 1987). On the east coast, cybernetics and artificial intelligence began to take shape as an alternative way to describe how humans think. In 1977, these two theories gave birth to the Cognitive Science Journal. Yet, where did these ideas come from?

Interdisciplinarity: The Six Fundamental Disciplines

Six disciplines united from 1950 to 1970 to form the foundations of cognitive science. In this manner, it is interdisciplinary as it originates from these fields and as cognitive science develops, theories are regularly shared among these six disciplines. Those disciplines are Psychology, Philosophy, Linguistics, Artificial Intelligence, Neuroscience, and Anthropology (Gardner, 1987). I have constructed a visual representation (Figure 1) to demonstrate the foundations of cognitive science.
Figure 1. The initial six disciplines whose overlap developed cognitive science. This is a Venn diagram representing the overlap of the initial disciplines as cognitive science.

Each discipline has a rich and historically relevant influence to the fruition of cognitive science as a whole. However, the current purpose is to narrow in on cognitive science’s contributions and development in mathematical problem solving, each of the disciplines is only briefly described below.

Psychology

Psychology’s initial contribution was the break from behaviorism towards cognitive psychology. Fundamental to this change was the Gestaltists, and the studies done by Wertheimer to describe the Phi Phenomenon (Ormrod, 2004). This is the phenomenon where the parts cannot define the whole. Wertheimer’s specific example
was how a single flashing light doesn’t demonstrate motion, however, lining up multiple lights and having them flash at sequential times, demonstrates to the human brain a sense of motion as if the light was moving through space. In this manner, one cannot say the singular light makes up the gestalt, but rather it is their holistic display which cannot be dissected. Other significant developments include Piaget’s constructivism and his interpretation of the mind.

**Philosophy**

One of the strongest foundations of cognitive science originates from the German Philosopher Immanuel Kant (Davis, 1984; Gardner, 1987; Schoenfeld, 1987). Although Kant published “Critique of Pure Reason” in 1781, he moved to combine the rationalist and empiricist view of knowledge so that the mind was truly seen as an organ that yields knowledge from experience, and has raw knowledge independent of observational experience. This intrinsic knowledge was the beginning of the schema. The schema allowed researchers to *map* how the brain models thought. It was a process, not a product in its own. These foundations led to the development of cognitive science over a century later because of interpretations on that process. Later developments, such as Richard Skemp’s (1989) metacognitive model via the $\Delta_1$ and $\Delta_2$ operators, paved the way for the schema to take on many roles in how we model learning and performance.

**Linguistics**

Linguistics emphasizes how one says something, not just the purpose of the words, but the process under which the words are used, such as education. Thus this line of thought greatly influences my current research. Reif (2008) argues that cognitive
science is observed by a change of states, from a pre-learned state to a post-learned state. This change may occur with or without learning being involved. In sum, teaching is not necessary, nor sufficient, for learning to occur. It is clear that teaching can occur without learning, as is demonstrated far too often in American schools. However, how does learning occur without teaching? This is a difficult question because teaching may occur from many sources. However, linguistics offers valuable insight because of how babies acquire their first language. Teaching cannot be directly vocalized as the child has no interpretation of what the adult is saying, yet languages are still learned (Reif, 2008). Such conclusions and techniques are the draw for this study’s purpose to use linguistics to interpret mathematical problem solving of students.

**Artificial Intelligence (A.I.)**

While cybernetics was influential, the development and enduring research of artificial intelligence can be traced from many people. Allan Turing and the Turing Machine (Gardner, 1987) made a huge change in how computers could be used. Moreover, Allen Newell and Herbert Simon’s creation lead to a landslide of opportunities with computers (Newell & Simon, 1972). However, this will be emphasized as a significant drive to the integration of mathematics education within cognitive science and thus will be discussed later.

**Neuroscience**

Neuroscience’s initial contribution to cognitive science was already discussed above with respect to Lashley’s deviation from behaviorism. It was his charismatic skepticism that allowed for researchers to start really questioning the brain (Gardner,
1987; Newell & Simon, 1972). Specifically, Lashley argued against the classic localization position that various regions of the brain control certain behavioral patterns. This was justified in those whose brains had suffered trauma and were able to rebuild using the remaining brain tissue. In essence, this provoked the question how do neurons function?

**Anthropology**

Anthropology’s roots can be aptly described by Edward Tylor’s application of the gestalt theory of Wertheimer to the methodology of isolating of test subjects in the current approaches to understanding the mind. This has enormous influence, as discussed with the theory of mathematics being embodied, on current mathematics education views of problem solving. In sum, Tylor emphasized that the complexities of a single human being include many affective and societal influences and thus cannot be isolated from one’s environment (Ratnapalan, 2008). Culture, as agreed upon by Dewey (1963), cannot be separated from the individual. Yet Tylor did believe that one could make sense of a situation if one knew such cultural issues as beliefs, art, morals, laws, customs, etc (Ratnapalan, 2008). Thus Anthropology has encouraged cognitive science to look to the affective domain for explanations at a cultural level.

**Core Assumptions**

While cognitive science has absorbed various aspects of the six aforementioned disciplines, there are core assumptions that give cognitive science its purpose and direction:
1. Representations—While this has changed over time, the belief is that symbols, rules, and images are joined, transformed, or contrasted with one another in a way that can be represented (Gardner, 1987).

2. Computers—A man-made machine’s ability to reason and analyze information indicates that humans should be able to do the same (Gardner, 1987).

3. Complexity—A more recent development in cognitive science is that research has indicated mental structures and process are more complex than originally thought within humans (Reif, 2008; Schoenfeld, 1994).

**Methodological Features**

Finally, Gardner (1987) points to current methodological features that are inherent to cognitive science.

1. Historically, cognitive science has de-emphasized the affective domain including context, culture, and history. This is not to say it does not exist, but rather that it is not the focus of cognitive science research. However, this also is changing as discussed above and as will be discussed below.

2. Interdisciplinarity is fundamental to cognitive science. It is vital to use multiple disciplines to interpret data.

3. Cognitive science is still rooted in classical philosophical problems about the mind. From Descartes (Gardner, 1987) to the current computational paradox (Schoenfeld, 1987), methodologies are founded in philosophical questions.
**Application Within Mathematical Problem Solving**

From these six roots, cognitive science developed in many different directions. To continue with the roots that led to mathematical problem solving, certain historical developments will be described. These developments will narrow the focus from general cognitive science, to applied cognitive science, to education, to mathematics, and then finally problem solving as demonstrated in Figure 2.

![Venn Diagram](image)

*Figure 2.* Visual representation of problem solving embedded within cognitive science. This is a Venn Diagram of problem solving within cognitive science demonstrating how problem solving is contained in mathematics, contained in education, contained in applied cognitive science contained in cognitive science.
**Applied Cognitive Science**

While the theoretical underpinnings of cognitive science have been discussed, the pragmatic influence of research and improvement of society has not. Within cognitive science lies the application of the theory put into practice, applied cognitive science. In carrying with the spirit of the foundations, applied cognitive science focuses on the process over the product, but uses knowledge about the product to discuss the process (Davis, 1984).

Many fields emphasize applied cognitive science. Cognitive Load Theory (CLT) is an application that looks to the limitations of the human mind and how it subsumes large amounts of processed data (Ayres, 2006). Cybernetics is a good example of a field that applies all six of the original disciplines concomitantly to understand how one’s image of the body explains one’s image of the mind (Dupuy, 2009; 2000). However, the application that rooted education within the cognitive-science community was the creation of the General Problem Solving (GPS) Machine (Newell & Simon, 1972).

Newell and Simon (1972) studied logic and its sequential, conditional applications to one’s ability to think. One system of thinking prevalent to their research was chess. Grandmasters of chess took a board’s position and then went down various lines of play until it showed a benefit or a loss. In this manner, Newell and Simon believed a computer program could be written to emulate this method of thinking. Moreover, during this time (1976) the four colored theorem had been proved in mathematics by Appel and Haken via computer exhaustion of all possible mathematical cases (Gonthier, 2005). Thus there was encouragement for Newell and Simon’s method
of thinking. After many tries, Newell and Simon created the GPS machine. The user would encode various axioms and postulates and conditions into GPS, and then the GPS machine would integrate basic properties of logic (converse, inverse, contra positive, etc.) to justify a theorem, similar to how a chess problem requires sequential lines of thought.

This revolutionary breakthrough led to educational means of trying to sequentially develop a curriculum where students were educated to process similar to the GPS machine (Gardner, 1987). Essentially, the GPS machine allowed for a process model of education. Within the realm of mathematics education, Gagne (1965) wrote significantly about this reminding many of the value of sequential thinking similar to the behaviorist’s interpretation of stimuli-response, but at a cognitive level (Schoenfeld, 1987). Overall, this took the theoretical approaches of how one learns and put them squarely in the educational realm with pedagogy from which new curriculums could be developed. Yet the process metaphor dominated this structure. As a result, cognitive science was applied to education training the student mind in processes.

**Education**

Education is one of the most natural applications of cognitive science for research. As cognitive science is a science, it follows from the aforementioned definitions that the results of any cognitive process must be observed. Hence Reif (2008) makes the argument for student performance. Specifically, Reif defines good performance versus poor performance. Reif suggests effectiveness, flexibility, efficiency, and reliability are characteristics of good performance. Additionally, Davis (1984) suggests Ausubel’s (1978) meaningful versus rote understanding is a good method
to judge if the student has taken ownership of the knowledge or if it is simply memorized for the performance. Naturally, this raises questions of knowledge as declarative versus procedural (Reif, 2008).

A study was conducted with 13 year old students within the mathematical domain on how children manage various cognitive loads (hence Cognitive Load Theory, CLT). In this study, Ayers (2006) had groups of students approaching problems by three methods: isolated elements of the problem (Low CLT), integrated holistic approach to the problem (High CLT), and a hybrid of the parts and whole (Varying CLT). The data was analyzed and Ayers found that the higher ability students performed better with the integrated approach, but worse when the problem was broken into isolated parts for them (Low CLT). This is surprising, but suggests that the higher ability student’s performance was low on the low CLT question because the students weren’t engaged in the learning process, it was too easy. This demonstrates that improved performance may not be as easy as simplifying questions.

Yet some educational practitioners regularly try to lower the cognitive load to make complex problems easier for students. Hyde (2007) found that the classic pedagogical method of K-W-L (this is where the students write down for a given problem what they KNOW, what they WANT to know, and afterwards what they have LEARNED) was limiting in getting from what they wanted to know to what they learned. To help improve, he suggested a K-W-C where the first two categories were the same, but the third indicated what CONDITIONS were relevant to the problem at hand. Hyde found student’s median performance improved. Clearly, these are different situations and
performance may be judged differently as may the groupings of student’s abilities. These two examples demonstrate that even within a more systematic field as cognitive science, mathematical problem solving within education is complex as argued above by Schoenfeld (1987).

Education became a field of study within applied cognitive science that was challenging to integrate within school systems. Applying cognitive science emphasizes performances that are observed, judged, and measured. Yet, when applied in the public school systems the research was used for generating an all-inclusive metric for assessment, i.e. standardized tests. While the intentions were good, the standardized tests soon became a broader measuring tool of comparison between school systems rather than within a single school district. Cognitive science is aware of the complexity of problem solving within many fields of education, but standardized tests attempt to use judgment of performance as a value-based system to sterilize that complexity. This faulty attempt at sterility is illuminated well in cognitive science application to mathematics education.

Mathematics Education

As can be seen from Newell and Simon’s GPS Machine (1972) and the Four Color Theorem’s result with computers in 1976 (Hyde, 2007), pure mathematics has always been a significant aspect of cognitive science. Yet mathematics is not considered a foundational discipline because it is a means and result of how the brain interprets objective logic, not an explanation of the processes involved. However, as Piaget’s concept of constructivism began to take root in psychology, cognitive scientists found mathematics education a significant field from which to generate models of how students
As constructivism has its own history independent of cognitive science, it would be complicated to delve into constructivist theory and its theoretical stance, such as radical constructivism (Steffe & Thompson, 2000), with the focus on cognitive science’s influence on mathematics education. The issue will be left with the concept that the roots of constructivism and cognitive science have become intertwined and for the most part run parallel to one another. One such overlap would be within metacognition because how we think about what we think is perceived as constructed (Confrey, 1991). In this manner, cognitive science has helped construct models of how people think about their thinking. One such model has already been discussed as Richard Skemp’s (1989) schema-modifying operators, $\Delta_1$ and $\Delta_2$. Thus there do exist overlaps where cognitive science has helped constructivist thought and vice versa.

The computational paradox is one that still perplexes cognitive scientists and constructivists alike within mathematics education. The GPS machine built by Newell and Simon (1972) was rational in the sense that it would stay focused and work in a logical manner that could be traced in thought. However, as discussed earlier, the problem is that research has shown that humans do not approach mathematics problems the same way and are often irrational, thus a paradox that humans can create a rational machine while being irrational themselves. (Newell & Simon, 1972; Reif, 2008; Schoenfeld, 1985). This has led to the more-current core assumptions of cognitive science that mathematics education is complex (Schoenfeld, 2011).
Various research techniques have been applied to address this problem. Cognitive science within mathematics education is well known for making models to represent student’s thinking process (R. Lesh & Zawojewski, 2007; R. A. Lesh & Doerr, 2003; Schoenfeld, 1987). Indeed, the classic approach to modeling in cognitive science would be to choose a small number of participants, ask them to solve a specific mathematical problem type, and then model how they solved (or did not solve) the problem. Yet, when one looks at the methodology for these approaches, a common theme is that they all revolve in one way or another around dictating a model for student’s mathematical problem-solving skills. This study suggests reversing this model may be more beneficial for research. Specifically, let the students design the model.

**Problem Solving Within Mathematics Education**

From 1964 through the 1970’s, student’s SAT scores continued a regular drop in mathematical performance (Schoenfeld, 1987). Additionally, Newell and Simon’s (1972) emphasis on mathematical problem solving was buttressed by 1977 by the National Council of Supervisors of Mathematics arguing that problem solving is the necessary focus of school mathematics. Moreover, in 1980 National Council of Teachers of Mathematics (NCTM) released “An agenda for action” that argued that problem solving is the most important issue for mathematics education in 1980’s. During this time, American automakers started to see and feel the influence of foreign cars as dominating the automobile market, a market that had begun in America. Certain foreign cars offered better mileage, lasted longer, and cost less than the American equivalent. Thus, a strong
desire and respect was given to mathematics education, one that had problem solving squarely at its center.

There is a significant amount of literature on problem solving through the 1980’s, the big connection with cognitive science truly hit home in the 1989 Standards (the first national mathematical standards for the United States) arguing then, and still today, that fundamental to a productive citizen is the need to become a mathematical problem solver (NCTM, 2010). The 1989 NCTM standards also argued that assessing problem solving needs to rely on more than multiple-choice questions, and that free response was necessary for proper assessment of their understanding. Here is where cognitive science research has surfaced. To observe properly, the constructivist roots and the cognitive science roots argued and have been securely planted in the philosophy that the process is more valuable than the product in mathematics education (Schoenfeld, 1987). Hence, the emphasis in the process has led to a myriad of interpretations of those processes.

Davis (1984) has argued that for meaningful mathematics to take place within problem solving, cognitive scientists should search out representations of students so that the understanding of the material is meaningful and not rote. Davis also encourages researchers to follow Erlwanger’s (2004) approach to representations and that is not to give up or argue a student is wrong when a mistake is made, but rather follow it to its origin to discover the mathematical conception (or misconception) of that child. However, Davis’s (1990) approach was limited by the view that the student’s procedure when solving a problem had short term and long term memory and essentially mimicked
a computer. Today, such analogies have been shown to be false as research has shown such categorizations to not necessarily be true (Zull, 2002).

Lesh (2007) has argued that cognitive science should take this focus of problem solving from a modeling perspective, encouraging modeling at all levels. In this sense, Davis’s representations do not need to be abstracted by a researcher, but has merit in the classroom as done by the student. Specifically, Lesh argues for modeling as a means to improve student’s proof schemes, a structure that convinces yourself or others of the validity of a mathematical claim. Moreover, while the models may vary from student to student, Lesh suggests the need for a meta-model, a model of student’s problem-solving models.

Schoenfeld (1985) approached Polya’s (1945) four-step model (one of the first problems-solving models) differently. Schoenfeld created domains from which problems were solved and interpreted. Those domains overlapped, but had prototypical categorizations that were founded in cognitive science research done by Schoenfeld (1985). The domains he found were: Control, Beliefs, Heuristics, and Resources. Thus when solving a problem Schoenfeld identified factors from these four domains that influence the student’s problem solving abilities.

Other researchers have used cognitive science within problem solving to demonstrate areas of growth; new topics to take root and grow in the development of problem solving within mathematics education. Silver (1987) reminds us of Eisner’s (2002) work in the hidden curriculum within problem solving. Specifically, Silver expands on Schoenfeld’s Belief domain and argues that certain negative beliefs about
mathematics are taught by teacher’s attitudes and beliefs towards teaching mathematics. For example, the claim that mathematics is a list of facts and procedures that reward those who may do them quickly rather than understand their foundation. In this manner, teacher education needs to be critically reflective (Brookfield, 1995) so as to deconstruct student teacher’s prior experience to ensure such hidden curriculums are brought to light to generate their value with ethical fidelity (Badiou, 2002).

Summary

To summarize the result of this historical analysis, the six founding contributors of general cognitive science were contextualized through historical developments and descriptions of various events that led to the birth of cognitive science as its own discipline. Once this was established, a set of events, such as Newell and Simon’s (1972) development of the GPS machine, helped narrow this study down to the influences of cognitive science that currently flourish in my current area of focus, mathematical problem solving within mathematics education. Categorizations and historical events led us from the roots of cognitive science, to applied cognitive science, to education, to mathematics education, and then into the subfield of mathematics education emphasizing problem solving.

Within mathematical problem solving, certain roots of cognitive science have helped develop the field. Those would include Davis’s (1984) work in emphasizing the procedure over the product, Schoenfeld’s (1985) work on problem solving methodology, and Lakoff’s (2000) influence on linguistics as a means to help educators interpret mathematics for various learners. Overall, the greatest cognitive science contribution that
began over a decade ago, is the belief in mathematics education that the process is more valuable to learners than the product. Of the core assumptions and methodologies of cognitive science laid out by Gardner (1987), few have changed. Rather, cognitive science’s application to mathematics education has changed by having mathematics education lay some of its own roots. For example, mathematics education has shown cognitive science that within problem solving, de-emphasizing the affective domain, context, culture, and history may not be the best practical pedagogical approach as mathematics is dominantly seen as embodied (Lakoff & Núñez, 2000). Thus a symbiotic relationship seems a credible analogy between cognitive science and mathematics education. While mathematics and education were not among the six founding philosophies, education is now considered a member of the field (CSS, 2011). As such, mathematics education, specifically problem solving, has a significant overlap with cognitive science. Thus this study parallels certain properties of cognitive science for its purpose. Specifically, this study will emphasize interdisciplinarity with linguistics as a fundamental.

**Linguistics**

**Learning via Acquisition or Participation**

As was discussed above in proof, how one knows something depends on their perception of knowledge and truth. Sfard (1998) discusses the metaphorical importance in one’s perspective of knowledge and how the epistemological difference between perceiving learning through the acquisition lens versus the participatory lens can have a significant influence on one’s development of that knowledge. Essentially, there are pros
and cons to each metaphorical model of knowledge within education. If one perceives knowledge as *attainable*, then it is acquired and one may allow for *transfer* of such ideas. For example, consider the statement, “now take what you just learned and apply it to the next question”. Thus perceiving learning as acquisition allows for transference to be concretized (Danesi, 2007). Sfard (2009) discusses how Piaget’s work finds itself within the frame of learning as acquisition as is demonstrated by concrete terms such as schema, schemata, etc. However, Vygotsky’s work aligns more with the belief that learning is participatory. In this manner, Sfard (2009) argues that becoming more proficient in solving mathematics problems from the participation metaphor means the student is more involved and a competent participant in solving the problem. Sfard (1998) does warn that each perspective has their limitations. While the acquisition metaphor implies the hoarding of knowledge capitalistically, the participation metaphor lacks a means to discuss transfer in a pragmatic method. Sfard (2009) warns, “Our choice of metaphors should depend on the questions we ask” (p. 56). As my perspective on metaphors within problem solving is focusing on a phenomenological methodology emphasizing models of student’s perceptions, the participatory lens will be preferred in this paper. Yet, the concept of knowledge as acquisition may arise if explanation mandates such a lens, for example with transfer.

**The Learning Paradox**

The participation metaphor for learning does offer an interesting perspective to the learning paradox, also referred to as Meno’s Paradox (as mentioned in chapter 1): How does one learn new knowledge from old knowledge? Specifically, Socrates says to
Meno “all learning is but recollection” (Plato & Thompson, 1980). This argument then becomes paradoxical under the perception of knowledge via acquisition (Sfard, 2009). However, with participation, there is an alternative, the metaphor. Metaphors offer people the opportunity to communicate and learn concomitantly while sharing experiences. There is a general base of experiences that people within a conversation share, the physical experiences such as eating, sleeping, and gravity are a few such examples (Lakoff & Johnson, 2003). Comparisons are drawn between such known experiences and similarities to one’s own experiences. In essence, the people conversing are learning by participating with each other’s experiences. Therefore, a solution to the learning paradox is to perceive knowledge and learning through participation, then learning occurs when experiences of others can be related in a new way to one’s own experiences (Petrie & Oshlag, 1993; Sfard, 1997, 1998, 2009). This perspective on knowledge demonstrates why metaphors play such a significant role in mathematics education.

**Analogies**

Analogies reference two concepts already firmly defined in the learners’ mind for purposes of mapping aspects of one concept onto the other (Sfard, 1997). Consider the cognitive difference between stating “\( y \) is like a function” and “\( y \) is a function”. Both are comparative statements; the former is referred to as a simile while the latter is referred to as a metaphor. The former indicates properties of functions and properties of \( y \) may be analogous, while the latter more assertively states they are isomorphic. Metaphors frequently model a new conceptual structure with a pre-existing structure. The
accommodation of known structures into new concepts can define the new concept, and is considered an aspect of conceptualization (Kövecses & Benczes, 2010; Sfard, 1997). Justifiably, Sfard (1997) refers to such conceptual metaphors as implicit analogies and similes as explicit analogies. Hence for mathematical problem solving, application of analogies follows the learner’s understanding of metaphors.

**Metaphors**

Metaphors are not only for poetic representations, but for transfer of ideas. New knowledge is attained by relating (not isomorphically, but analogously) aspects of old knowledge from multiple individually perceived experiences. Indeed, Presmeg (1997) reminds us that the Greek word, *metaphora*, means to transfer or carry over. Traditionally, a metaphor literally denotes one figure of speech as another figure of speech (Merriam-Webster, 2011). Yet the influence of this transfer from a cognitive perspective encouraged Lakoff (2003) to expand on Reddy’s (1993) idea of the conduit metaphor to create the conceptual metaphor. In the conceptual metaphor, the literal relationship between figures of speech is replaced with a conceptual mapping between linguistic expressions (Lakoff, 1993). For example, the literal metaphor “Your theoretical framework lacks a solid foundation” would have the conceptual metaphor, THEORIES ARE BUILDINGS, attached to it. Thus with each metaphor, you have two parts:

- **Literal Metaphor**: The actual literal expression.
- **Conceptual Metaphor**: TARGET DOMAIN IS SOURCE DOMAIN
In all conceptual metaphors, there is a source domain and a target domain. The source domain is the experientially-known domain and the related concept is the target domain. Thus in the metaphorical linguistic expression “The solution escapes me”, the target domain is solutions while the source domain is prey. Hence the conceptual metaphor is SOLUTIONS ARE PREY. It is important to note that despite the use of the being verb, “ARE”, the phrase is unidirectional (TARGET DOMAIN \( \rightarrow \) SOURCE DOMAIN).

Linguists classify these conceptual metaphors into three hierarchal categories: structural, ontological, and orientational (Kövecses & Benczes, 2010; Lakoff & Johnson, 2003). Structural metaphors strive to describe a complex concept, such as time, in terms of a concrete experiential object, such as a limited resource, i.e. “Don’t waste my time”. Ontological metaphors provide target domains with less structure and a new reality in which they may be defined. Personification is regularly ontological; as is the phrase “the solution escapes me”. Orientational metaphors are the most difficult to relate experientially according to linguists. They are broad concepts with a specific direction inherent in human development. The metaphorical linguistic expressions “Things are looking up” and “He fell ill” are examples of the conceptual metaphor “HEALTHY IS UP”. To this end, conceptual metaphors are used to map how the cognitive domains of the person are related (Lakoff & Johnson, 2003).

The language that the students and teachers use to express mathematics involves metaphors both literal and conceptual as well. For example, Lakoff (2000) demonstrates how the understanding of limit is complex and embedded within one of two conceptual metaphors. If I were to write the definition attached to the symbol:
\[
\lim_{x \to c} f(x) = L
\]

“as \( x \) approaches \( c \), the function, \( f(x) \), will approach \( L \).”

Notice that this literal metaphor involves approaching which experientially humans construct through movement. Thus the conceptual metaphor for this definition is LIMITS ARE MOTION.

Karl Weierstrass viewed this limit symbol from a different metaphor which ultimately changed the entire field of analysis in mathematics. Weierstrass’s definition worded the limit by saying,

\[
\lim_{x \to c} f(x) = L
\]

”if \( x \) is close enough to \( c \), then \( f(x) \) will be close enough to \( L \).”

The literal metaphor refers to distance and proximity as a justification for this definition. Thus the conceptual metaphor would be LIMITS ARE PROXIMITY. From this definition, Weierstrass was able to encourage the language of epsilon-delta proofs that have been now widely accepted in mathematical analysis. Both conceptual metaphors are valuable, and distinct in their techniques of proof. In this manner, mathematics is embodied (Lakoff & Núñez, 2000) because it relies on the experiences and language of the people involved. One’s knowledge of mathematics is dependent upon one’s perspective and experiential learning of that knowledge. However, the logic necessary to solve limit-based problems is analogous in both metaphors. Thus an elusive bond exists between problem solving and proofs in which metaphors are squarely centered.
Metaphors are a means to relate experiences through language, thought, and action. The relationship between the experiences of the teacher and the student are vital to mathematics education. Specifically, teachers and students share an experiential set: solving mathematics problems. However, the student’s and teacher’s perspective of what constitutes mathematical problems and/or solutions are complex in structure. (Lakatos, 1976; Lesh & Zawojewski, 2007; Pólya, 1945; Schoenfeld, 1992). Metaphors are culturally designed to articulate these implicit perspectives. Moreover, they have been found to encourage and incite cognition (Lakoff & Núñez, 2000; Sfard, 1997).

**Conceptual Metaphor Theory**

My current variant of cognitive science within mathematical problem solving emphasizes Lakoff and Nunez’s (2000) linguistic approach to mathematics via Conceptual Metaphor Theory (CMT). CMT was introduced by Lakoff and Johnson in 1980 and has burgeoned in many subfields of cognitive science such as language, psychology, education, and neuroscience (Lakoff, 2008). In CMT, the researcher interprets the student model through the mapping of the conceptual metaphor described above. As before, consider a literal metaphor by a student, “5 is larger than 3”. The source domain is focusing on size while the target domain focuses on numbers. Metonymy comes into play here as “5” represents five similar objects and “3” represents three similar objects. Presmeg’s (2005) work on semiotics supports the importance of such metonymic conclusions within mathematics education. There is an added level of thought and symbol embedded behind using “5” and “3” as nouns rather than adjectives. Hence, there exists a perception by the student that NUMBERS ARE COLLECTIONS
OF SAME SIZE OBJECTS. This is an example of CMT analysis in mathematics education.

CMT has been used by many researchers in mathematics education and other fields. For example, Prus-Wisniowska (1995) used this method qualitatively in her dissertation to interpret six undergraduate students use of metaphors with the concept of limits. Bazzini’s (2001) dissertation used CMT to recognize how technology is influencing source domains and how teachers need to be aware of how technology influences student perception. Chapman (1997) used CMT analysis in an interpretive qualitative design to study three teacher’s perspective of mathematical problem solving. These teachers varied in grade-level (elementary to high school) and experience (student teachers to 25 years of teaching). Chapman studied their perspectives on problem solving and how it directly influenced their teaching of problem solving. The three fundamental source domains for problem solving characterized by the three teachers were:

PROBLEM SOLVING IS COMMUNITY

PROBLEM SOLVING IS ADVENTURE

PROBLEM SOLVING IS OPEN ENDED

Chapman foresees the need for more research in this area:

Thus, as the outcome of this study, it should be viewed as a contribution to a needed series of development toward an adequate conception of teachers’ perspective of teaching problem solving. This contribution can also be associated with viewing the study of metaphor as a promising basis for enhancing teacher
education and a promising avenue for future research on teaching mathematical problem solving. (p. 225)

Whereas Chapman focused on the teacher’s perception of mathematical problem solving, my study focuses on the student’s perception. I believe that both studies are valuable and the teacher’s perception of problem solving is tantamount to the student’s perception, yet the student’s perception also needs to be taken into consideration.

Danesi (1995, 2003, 2007) has done many studies with CMT analysis as well. While her earlier work was focused on teaching and learning foreign language (1995), she has since converted to the application of CMT to mathematics education. Danesi (2007) perceives CMT analysis as a means to help in understanding concretization, putting intangible terms into experiential terms:

An abstract concept in CMT is defined simply as a “mapping” or an “extension” of one domain onto the other. This theory suggests that abstract concepts are formed systematically through such extensions and that specific metaphors are traces to the target and source domains. (p. 228)

This mapping of concepts emphasizes the epistemic view of learning as participation because it focuses on the action of mapping, i.e. the conduit (Reddy, 1993). Danesi (2007) used CMT analysis on eighth grader’s perspectives of mathematical word problems and found that their ability to concretize time according to the source domain of a line, hence the time line and the conceptual metaphor of TIME IS A LINE was helpful with basic algebra if diagrammed.
As an extension of Danesi’s research, consider a student who is struggling with improper fractions and their size. The teacher is trying to suggest how to compare numbers and so the teacher says to the student, “14/5 is greater than 2, right?” The student is struggling and the teacher doesn’t know why. The student then proclaims, “How do you have 14 pizzas broken into fifths?” What CMT analysis suggests is that the mathematical experiences have created metaphorical conflicts between your images of the situation. After probing into the pizza analogy, the teacher finds that your use of the word “greater” isn’t making sense in this context to the student. If research interprets mathematical problem solving as post-modern, claiming relativism/subjectivism, no development could take place to discuss this impasse. However, CMT analysis shows that the teacher is claiming use of something that has a less than and greater than value, the number line. The student does not see fractions as points on the number line, but rather contextualized pizzas. Now that this discrepancy is identified, the teacher may now scaffold appropriately so that student can begin to internalize the teacher’s model, or so that the student’s pizza analogy may be expanded to make sense of the fraction. Essentially, coherence can be found between the student’s perception and the teacher’s perception.

The student and the teacher were at an impasse. However, through CMT analysis the teacher was more capable of deconstructing their interpretation of the problem and willing to interpret the student’s schema. Thus current use of cognitive science for mathematics education’s emphasis on problem solving, founded by Lakoff (1985), can truly be reduced to admitting that mathematics is embodied. Lakoff’s (2000) discovery
uses many of the foundational fields in its development; anthropology, neuroscience, linguistics, and philosophy. More important to mathematics education, CMT analysis is applicable to practitioners and researchers alike.

**Coherence and Entailments**

A significant result of CMT analysis aligns with the epistemological view that mathematics is embodied. It should be clarified that this framework assumes coherence and not abstraction from the experientialist perspective (Lakoff & Johnson, 2003). Specifically, CMT does not search for absolutes, but rather coherence between participants. This coherence is perceived only by both parties involved. Let us consider the following perspective on time. If one is to interpret the use of time via CMT analysis, one may find that there exists an orientational metaphor with time. We regularly perceive TIME IS LINEAR. As such we orient ourselves so that we place ourselves on the time line. Consider the phrases:

What happened *last* week?

What happened the day *before*?

Thus the past is behind us and the future is ahead of us. Yet, direct conflict seems to occur with this orientational metaphor when it is said:

The *following* week, something happened to Johnny.

The word “following” implies behind us, yet the sentence is referring to future time. How does this make sense? First, notice the context of the situations. The last example is a narration. As such, the story is already completed and one is looking back on the situation. Second, from the perspective of the narrator, the story is complete and we are
not directly on the metaphorical time line, but rather perceiving the time line from outside the situation. As we read through a story, the time line of the story has future times written following the current moment in the story. Thus there is no conflict in the perception of time as linear, but rather the context and perspective of the people involved. This is what is meant by having coherence (Kövecses & Benczes, 2010; Lakoff & Johnson, 2003). This will happen regularly in a conversation, but misinterpretation regularly comes from misperception of such metaphorical systems.

Additionally, as discussed earlier, conceptual metaphors are unidirectional. That is to say not all aspects of the source domain carry over to the target domain. If such concepts did, then Black (1962) refers to such metaphors as dead metaphors because they have become so ingrained conceptually, that the participant is completely unaware of the mapping and considers such ideas tautological. Thus, entailment occurs when additional information of the source domain is carried to the target domain. Kövecses and Benczes (2010) gives the example in the following conversation:

TEACHER: You look like a healthy apple.

AUTHOR: I hope it’s not rotten inside.

TEACHER: I hope, too, that it will last a long time. (p. 123)

The conceptual metaphor of PEOPLE ARE FRUITS is given. Yet what defines healthy fruits in this context (longevity or good throughout) is not agreed upon by the participants. CMT analysis regularly shows that aspects of one source domain is not enough for a complex target domain (Kövecses & Benczes, 2010; Lakoff & Johnson, 2003; Lakoff & Nunez, 2000). When this occurs, misinterpretations and overlaps in
mappings can lead to such metaphorical entailments. Imagine if a young child who doesn’t perceive numbers as same-sized objects as discussed above. If you were to say “11 is bigger than 10”. The size and shape of the numbers as symbols do not hold to this property as the zero can be interpreted as “larger” than a 1 in shape. Or perhaps, when written out in English one could argue the words for numbers may be confusing, “Nine” is smaller than “Seven”. Thus one must be vigilant when performing CMT analysis to be aware of such misunderstandings. This is done by reviewing literal metaphors used by participants to identify coherent properties and entailments involved with the conceptual metaphor.

**Conceptual Blending, Complex and Primary Metaphors**

Regular misunderstandings as well as regular understandings of entailments lead to Fauconnier and Turner’s (2002) idea of conceptual blending. Conceptual blending develops when two concepts (for example source and target domains) are associated closely together in language and action. As a result, the concepts can become blended. For example, Grady (2005) identifies that humans associate their faces to the sun, but then blend certain aspects of both, such as a sunny disposition associating brightness with smiling. In this manner specific aspects of both the source and target domains are associated similar to entailments. However, one can distinguish aspects of a specific conceptual metaphor from broader metaphors by looking to conceptual metaphors that are less varying between cultures. Conceptual metaphors that are shared by humans through the most basic experiential relationships with objects are known as primary metaphors (Grady, 2005). Thus primary metaphors are fundamentally experiential but
more importantly, less (if not completely) culturally dependent. For example, MORE IS UP is understood through gestures and the concept of “piles” in most cultures. On the other hand, when conceptual metaphors have multiple aspects that could be blended and are more culturally dependent, we refer to these as complex metaphors (Grady, 2005). Essentially, complex metaphors are composed of primary metaphors. This study will classify participant’s metaphors as primary or complex, dependent on the aspects of domains shared. This is done to verify that the source domains are mapped appropriately onto the given target domains.

**Grounded and Linking Metaphors**

Lakoff and Nunez (2000) also found that there is a categorization of experiential metaphors within mathematical ideas, grounding and linking metaphors. Grounding metaphors give basic, directly grounded ideas and are regularly structural because these metaphors are more directly grounded in experience. For example, one’s perception of addition usually develops from adding objects to a collection. Linking metaphors connect concepts within mathematics itself that may or may not be based on physical experiences. Some examples of this in mathematics are numbers on a line, or understanding inequalities and absolute values properties within an epsilon-delta proof of limit. Mathematical problem solving requires both metaphors to be used because the linking of mathematical concepts requires that students create metaphors (metaphors for linking) and mathematical problem solving will be described through experiences with solving mathematical problems (metaphors for grounding).
Summary

In conclusion, the approach I have taken towards perceiving mathematical problem solving focuses on students’ modeling (R. Lesh & Zawojewski, 2007). Problem solving is a complex topic in mathematics education, primarily because its history and definitions are convoluted (Kilpatrick, 2004; Schoenfeld, 1985, 1992; Shumway, 1982; Silver, 1985). To help clarify and justify future research studies in mathematical problem solving, the student’s perception must be the purpose. This paper’s theoretical framework assumes mathematics education is embodied and not mind-free (Lakoff & Nunez, 2000). Hence, the purpose of this study is to interpret student’s models of problem solving for the purposes of understanding how and whether their perception varies from the teacher’s perception.

This study will focus on the student’s perceptions. To assist in minimizing the researcher’s misinterpretation so as to minimize the threat to internal reliability of the study, cognitive science inspired the use of interdisciplinarity (Gardner, 1987). Specifically, cognitive science’s foundations suggest that many fields are useful to study together in mathematics education. However, the emphasis was on linguistics and the focus of interpretations collected via written and oral responses. The ideal instrument for developing models of student’s perceptions of mathematical problem solving is CMT analysis because the model is based on the student’s experience, and the researcher’s purpose is to stay true to this experience (Lakoff & Nunez, 2000). Other studies have used CMT analysis, but none have chosen to apply it to high school student’s perception of mathematical problem solving. While using CMT analysis, the researcher must
remain vigilant and clarify discrepancies between source and target domains. Yet, this technique offers a means through which to succinctly express students’ models of mathematical problem solving without generating a problem-solving model prior to the study.
CHAPTER III

METHODOLOGY

*Heuristics (Greek Heureskein): serving to discover (Harper, 2012).*

**Pilot Study**

Before discussing the methodology of this study, a synopsis of the pilot study will be described so as to justify choices made in the current study’s methodology.

This pilot study’s purpose was to question how metaphors are used in mathematical problem solving, specifically:

- Q1. How do metaphors help in understanding a problem?
- Q2. How do metaphors guide the process of solving a problem?
- Q3. What metaphors are used to describe mathematical problem solving?
- Q4. How do metaphors connect problem-solving and proof?
- Q5. How can knowledge of conceptual metaphors help teachers improve student learning?

**Design of Pilot Study**

The pilot study used naturalistic inquiry (Donmoyer, 2001) to study how metaphors influenced students’ problem solving. There is limited research within mathematics education that uses the linguistic instrument of metaphors within mathematical problem solving to allow the student’s voice to be heard. Moreover, limiting the use of previously designed models offers the opportunity to naturally discover such models within the study. In tandem, phenomenological inquiry (Short, 1991) was used to search for the essence of the student’s mathematical problem-solving
techniques. Specifically, Interpretative Phenomenological Analysis (IPA) was the methodology used because of the interest in the student’s lived experiences, their methods of communication, and how both played into their perspective on mathematical problem solving (Eatough & Smith, 2008). This hermeneutic technique applied to a small homogeneous sample with semi-structured interviews emphasizing dialogical and idiographic interviews. Thus IPA was the ideal design for the pilot study.

The students were given the three mathematical problems shown below:

P1. Imagine you had a piece of string. How would you bend this string to make a triangle bounded by the string with the greatest area?

P2. Humans have classified numbers on the number line into two categories, rational and irrational. Rational numbers are those that can be written as fractions, irrational numbers cannot be written as a fraction. Suppose I have an irrational number. If I add one to that number will it be rational or irrational?

P3. How could you cut a cylindrical birthday cake so that you have 8 slices using only 3 straight cuts with a knife?

The techniques and justification for each problem varied mathematically to identify differences or similarities in student’s problem-solving techniques and metaphorical conceptualization. Moreover, the problems were specifically designed to be metaphorically sterile so as to evoke conceptual metaphors from the students without bias. The problems could be done in any order and manipulatives were available to help
the students, including cork board, dry erase markers/board, string, pencils, paper, calculators, thumb tacks, pipe cleaners, and straight edges for graphing.

As Steffe (1983) poignantly noted in studies involving children solving problems, there are multiple interpretive mediums involved. The experience of the student is expressed by the student, interpreted by the researcher, related via the researcher’s experience, and then expressed by the researcher. To minimize the amount of interpretation of the researcher and to maximize the metacognitive expressions of the student, Reynolds’ (1993) design was applied. In this design, the student would attempt to solve the above problems (primary video) and then immediately watch themselves solving the problems with explicit instructions to explain their thought process (secondary video). Thus students worked with the researcher on the above problems for 30 minutes and then watched and commented on the video of their problem-solving process with the researcher for 30 minutes.

**Participants of Pilot Study**

The homogeneous sample was composed of students chosen according to a list of criteria that indicated the student had a propensity towards mathematics and expressing their thoughts. The following is a clarification of the criteria:

1. Talented in mathematics due to work ethic and sincere interest in mathematics beyond grades.
2. Participants are less likely to be influenced by the researcher and the environment and more by the content of the problem.
3. Participants are willing to solve mathematics problems for fun rather than work.

4. Participants who can clearly communicate how and what they are thinking without fear of the researcher’s judgment of that communication.

5. Participants who are willing to make mistakes and continue despite disappointment.

Participants were neither included or excluded based on ethnicity, sex, age, health or association with a special class. The population of interest was high school students who were planning on going to college who also had a propensity for mathematics. Twenty students were asked to participate in the study and nine students volunteered.

Nine students at a suburban high school in Ohio participated: three freshman, two sophomores, and four juniors. There were three females and six males with English as their native language. Each student met with the researcher individually after school for an hour. The names of the students were coded according to the first nine letters of the Greek alphabet.

**Results of Pilot Study**

Using mixed methods, two analyses were completed. The first analysis was qualitative and interpreted the student’s conceptual metaphors of problem solving. The second analysis was quantitative and ascertained two results; linguistic hierarchy of conceptual metaphors is viable within mathematical problem solving, and the student’s use of the word *like* has a negative correlation with their performance on Q1, Q2, and Q3.
First Analysis: Qualitative

The initial analysis involved multiple observations of all the videos using a phenomenological design and recording significant metaphors, problem-solving techniques, and all justifications. The first analysis revealed that the students evoked metaphors rich in context and culture. For example, when working with P2, Beta stated “Adding rational and irrational numbers is kind of like mixing oil and water”. Initially, this analogy helped the student conceive of the question, but then raised complicated issues when deductive reasoning was needed.

Table 1 details the conceptual metaphors (mainly structural) that the students related to their problem-solving strategies through metaphorical linguistic expression in the first analysis:

The percentages show the percentage of students who used the conceptual metaphor at least once in the interview. Many conclusions can be drawn from this data. For example, despite the emphasis in current mathematics education on cognitive understanding of heuristics as a mathematical process involved in solving problems, students only claimed it to be a process 56% of the time. Perhaps other more student-relevant metaphors such as JOURNEYS should be studied by mathematics educators and researchers. Additionally, notice how problem solving is never personified as a creature with its own intelligence. Yet, the problem-solving process is always action-based. For example, no one suggested PROBLEM-SOLVING IS THINKING. These results demonstrate a metaphorical system (Kövecses & Benczes, 2010) with certain coherent (Lakoff & Johnson, 2003) traits.
Table 1

*Pilot Study Conceptual Metaphors for Problem Solving and Percentage*

<table>
<thead>
<tr>
<th>TARGET DOMAIN</th>
<th>SOURCE DOMAIN</th>
<th>% of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>PROBLEM SOLVING IS</td>
<td>BUILDING</td>
<td>100%</td>
</tr>
<tr>
<td></td>
<td>A JOURNEY</td>
<td>100%</td>
</tr>
<tr>
<td></td>
<td>DISCOVERING</td>
<td>67%</td>
</tr>
<tr>
<td></td>
<td>EXPERIMENTING</td>
<td>56%</td>
</tr>
<tr>
<td></td>
<td>A MACHINE</td>
<td>56%</td>
</tr>
<tr>
<td></td>
<td>PLAYING</td>
<td>44%</td>
</tr>
<tr>
<td></td>
<td>SEARCHING</td>
<td>44%</td>
</tr>
<tr>
<td></td>
<td>TRICKS</td>
<td>33%</td>
</tr>
<tr>
<td></td>
<td>STRATEGIES</td>
<td>22%</td>
</tr>
<tr>
<td></td>
<td>A PRODUCT</td>
<td>22%</td>
</tr>
<tr>
<td></td>
<td>A DESTINATION</td>
<td>22%</td>
</tr>
<tr>
<td></td>
<td>A GOAL</td>
<td>22%</td>
</tr>
<tr>
<td></td>
<td>ILLUMINATING</td>
<td>11%</td>
</tr>
<tr>
<td></td>
<td>CHOOSING</td>
<td>11%</td>
</tr>
<tr>
<td></td>
<td>TRAVELING</td>
<td>11%</td>
</tr>
<tr>
<td></td>
<td>APPROXIMATING</td>
<td>11%</td>
</tr>
<tr>
<td></td>
<td>A HUNT</td>
<td>11%</td>
</tr>
</tbody>
</table>

... Another unexpected result of the first analysis was the abundant use of the word *like*. Students used the word *like* frequently demonstrating examples or counterexamples to guide their intuition. When trying to understand the problem, students used the word *like* for inductive reasoning rather than deductive...
reasoning. For example, Iota said “Consider a vector space like three dimensions”. One can replace Iota’s *like* with the phrase *for example*. This use is referenced in the Merriam-Webster Dictionary (2011) as a possible adjective definition, yet research in linguistics and mathematics education does not mention Iota’s use of *like*, despite its frequent use. Iota’s use of *like* is different because Iota is emphasizing a specific example (three dimensions) in reference to the broad concept (vector space). Thus Iota narrows or focuses her understanding of the problem to a specific example so as to concretely work with the mathematical problem within a familiar context. In essence (phenomenologically), Iota’s *like* transfers from global to local understanding.

There was significant evidence that a classification of metaphors (structural, ontological, orientational) was applicable to problem solving. A plethora of structural metaphors were used as shown in Table 1. For example, Delta’s phrase, “I don’t know where to start” aligned with the conceptual metaphor PROBLEM SOLVING IS A JOURNEY. Students were consistently able to discuss their problem-solving techniques as if their brain was a separate entity. In the secondary video, students evoked ontological metaphors personifying their mind as an entity from which they were analyzing. For example, Epsilon changed from one question to another because she had to let her “subconscious work on it for a while”. Zeta stated “my mind plays games on me.” Thus students demonstrated the ontological metaphor of MIND IS A SEPARATE ENTITY. The number of orientational metaphors were scarce as will be shown in the quantitative analysis.
Eta stated that his process had to be “written out” while Theta similarly argued that the calculations needed to be “worked out”. Both of these reference the container metaphor orienting CALCULATIONS ARE EXTERNAL. These three conceptual metaphor structures were then analyzed quantitatively.

**Second Analysis: Quantitative**

The second analysis (quantitative) had two parts. The first attempted to verify Kövecses and Benczes (2010) research in linguistics, that there is a hierarchy between structural, ontological, and orientational metaphors in mathematical problem solving. The second attempted to verify that the frequency of the word *like* was related to the student’s performance. It is important to note this voluntary pilot study included only nine participants and thus nine degrees of freedom which limited the study (*N*<30).

The first part of the quantitative analysis was calculated by counting the number of times each conceptual metaphor was used during the primary and secondary videos. The descriptive statistics are shown in Table 2.

**Table 2**

*Pilot Study Descriptive Statistics for Conceptual Metaphor Hierarchy*

<table>
<thead>
<tr>
<th>Descriptive Statistics</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>STRUCTURAL</td>
<td>32.22</td>
<td>10.462</td>
<td>9</td>
</tr>
<tr>
<td>ONTOLOGICAL</td>
<td>19.78</td>
<td>6.438</td>
<td>9</td>
</tr>
<tr>
<td>ORIENTATIONAL</td>
<td>14.78</td>
<td>7.870</td>
<td>9</td>
</tr>
</tbody>
</table>
MANOVA was performed on the data, and the Wilk’s Lambda showed a strong significant variability between the conceptual metaphors ($F(3,9)=14.292, p=.003$ with \(\text{alpha}=.05\)). Additionally, the structural metaphors were most frequent (\(\mu =32.22\)) followed by ontological metaphors (\(\mu =19.78\)) and then orientational metaphors (\(\mu =14.78\)) as was expected according to cognitive linguists.

The second part of the second analysis demonstrated that the total number of times a student used the word *like* was related to their overall score. For each problem the student solved, but could not justify, the student was given a score of \(\frac{1}{2}\). For each problem the student solved and could justify, the student was given a score of “1”. Students could receive an overall score between 0 and 3. There was a nearly-significant moderate negative correlation between the overall score and the number of times a student used the word *like* ($r=-.634, p=.067$). There was a strong negative correlation to their score on P3 to the number of times a student used the word *like* in P3 ($r=-.937, p<0.01$). Both of these results demonstrate that the more the student used the word *like*, the worse their solution and justification of the problem. All of this quantitative data can be found in Appendix C. While the results of the quantitative data with the word *like* are novel and have potential for a great number of studies, this was a tangent result of my pilot study and will not be discussed further here. However, it offers some exciting lines of inquiry for stemming from this research study.

In sum, the pilot study offered many intriguing avenues from which to study metaphors within problem solving. The direction I have chosen for this dissertation holds to the main conclusion of the pilot study: high school students can model their
mathematical thinking and experiences with metaphors. Thus this study revolves around the pilot studies conclusion that student’s literal metaphors for mathematical problem solving can be understood via conceptual metaphors.

Design of Study

The pilot study suggested that there were coherent metaphors involved in the student’s perceptions of mathematical problem solving. Moreover, the source domains involved with the target domain of mathematical problem solving suggest some greater systematic perspective by high school students. The pilot study suggests the need to expand the study’s participants beyond students with a propensity towards mathematics. Moreover, the pilot study allowed the researcher to gain experience in identifying conceptual metaphors given a specific mathematical problem. Thus a broader study with less devotion to interpretative phenomenological analysis (as IPA methodology requires) would be more appealing for purposes of generalizability. However, phenomenology would still be the candidate for CMT analysis due to the need to verify the results of the pilot study with a different population. To this end, pre-transcendental Husserlian phenomenology aligns best to the goal of this study (Giorgi & Giorgi, 2008). Pre-transcendental perspectives do not perceive consciousness as a universal that can transcend experience:

For Husserl, to assume a transcendental perspective means to view what is given from a non-human perspective . . . While the object or event being experienced is considered merely as something present to the experience, the subjectivity or consciousness that is the base of experience is understood to be existing and
related to the world. Transcendental perspective does not posit a real, existing subjectivity as the source of its acts. (p.171)

As the purpose of the study is to embrace the subjectivity of student’s perspectives on problem solving so as to find coherence and not absolutes, the need for pre-transcendental Husserlian phenomenology is evident. While this does not directly imply any results to the methodology it gives a framework from which to hold to a basic phenomenological study:

1. Researcher obtains concrete descriptions of the phenomenon as lived through the person.
2. Once a whole has been established, meaning units (literal metaphors) are discerned as partial meaning of the whole.
3. The meaning units are then conveyed directly through the researcher’s interpretation (conceptual metaphors).
4. The meaning units offer a structure of experience to articulate the phenomenon, specifically problem solving.

This method aligns with CMT analysis directly because it focuses on the student’s experience through which meaning units (conceptual metaphors) were constructed and interpreted with minimal research influence. Specifically, the conceptual metaphors were analyzed so that any metaphorical entailments are clarified. Hence source domains will demonstrate coherence or lack thereof (Kövecses & Benczes, 2010).

Moreover, a quantitative analysis on the structure and frequency of the classifications of conceptual metaphors will be useful to the mathematics education and
linguistics community. This study is useful to mathematics education because it will inform the field as to whether conceptual metaphors can or cannot be thought of in terms of the current linguistic approach. Additionally, this study is helpful to linguists as they will have a tangible example or counter-example to the applications of conceptual metaphor theory in education. Essentially, all results will lead to improved and open interdisciplinarity.

There are a significant number of participants to warrant this quantitative study using T-tests, Correlation Matrices, and Descriptive Statistics for the frequency of the types of metaphors used, the three linguistic categories of cognitive function (structural, ontological, orientational), and the two experiential categories (linking and grounding metaphors). This was lacking in the pilot study as there were only nine participants. Thus this study’s overall design involves qualitative and quantitative techniques implying a mixed methods approach. Fundamentally, the reason behind mixed methods is to be as thorough as possible in analyzing the given data using conceptual metaphors.

Participants of Study

The participants for the study were high school students and teachers of honors geometry mathematics courses who volunteer to be part of this research study. The course of honors geometry was chosen due to the inherent value of proofs to problem solving as discussed in chapter 2. Specifically, proofs encourage students to justify their conclusions and thus lead to perceptions of the student about problem solving. Honors geometry students were chosen because there is a greater likelihood that honors geometry students will desire to learn mathematics. Participants were students in a suburban high
school between the ages of 14 and 17. This is broader than the pilot study’s population and more applicable to other school districts. No students were filtered to volunteer for the study beyond the criteria of being in the honors geometry class to be entered into the study. A total of 22 distinct students volunteered for the study. Of those students who volunteered, only certain students were chosen to participate. The students were chosen to participate in a systematic way that aligned with their performance on the common assessment. The participants for this study were three to four students per honors geometry teacher per common assessment for three assessments. A total of 14 students were interviewed, but this was not a one-to-one correspondence with the number of interviews. A few students interviewed repeatedly for different assessments, but as each interview was based on a separate mathematical problem, each interview was considered an independent event. Thus the total number of independent interviews with students was 22 over 3 separate assessments (7 interviews for assessment 1, 7 interviews for assessment 2, and 8 interviews for assessment 3). Despite not being part of the participant selection process, gender was evenly distributed with 11 student interviews being female and 11 student interviews being male. All students spoke English and only one student in one interview had English as a secondary language. Participants were coded by a Greek letter and a number to identify which assessment they took. For example, Zeta1 is student Zeta’s interview on common assessment 1, while Zeta 3 was Zeta’s interview on common assessment 3. How these students are chosen is described in the procedure section.
One high school was chosen so as to keep with uniformity of the curriculum to limit the number of varying factors. Within the chosen high school, two teachers were responsible for teaching all of the honors geometry classes. These two teachers form the honors geometry team, known as the honors geometry professional learning community or honors geometry PLC. Each teacher taught between two to four classes of honors geometry. One teacher was male and the other was female. Both teachers were voluntary participants in this study. Each teacher was interviewed before each common assessment. The teacher’s were coded as Alpha and Beta with a number to identify which assessment they took. For example, Beta2 was the interview with teacher Beta discussing how students would solve a problem form common assessment 2. Table 3 clarifies the student and teacher participants for each interview.

Informed Consent

All parties involved were given full explanation of the study. The principal of the high school was given full disclosure of the study and asked to sign a consent form agreeing to allow the study to take place within the high school. Both honors geometry teachers signed a consent form with full disclosure of the study’s purpose and questions to be posed within the interview. At the beginning of the course, all students of both teachers were asked to volunteer for this study and were given consent forms to sign again with full disclosure as to the purpose of the study. Again, it is not true that all students who volunteer participated and only those who sign the consent forms were allowed to participate. As video
interviews are involved, there was a video consent form for both the teachers and students as well. Moreover, as there were minors within the

Table 3

*Participants by Common Assessment*

<table>
<thead>
<tr>
<th>Interview</th>
<th>Alpha Teacher’s Participants</th>
<th>Beta Teacher’s Participants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Common Assessment1</td>
<td>Psi1</td>
<td>Eta1</td>
</tr>
<tr>
<td></td>
<td>Phi1</td>
<td>Delta1</td>
</tr>
<tr>
<td></td>
<td>Rho1</td>
<td>Zeta1</td>
</tr>
<tr>
<td></td>
<td>Omega1</td>
<td></td>
</tr>
<tr>
<td>Common Assessment2</td>
<td>Upsilon2</td>
<td>Eta2</td>
</tr>
<tr>
<td></td>
<td>Psi2</td>
<td>Epsilon2</td>
</tr>
<tr>
<td></td>
<td>Kappa2</td>
<td>Theta2</td>
</tr>
<tr>
<td></td>
<td>Lambda2</td>
<td></td>
</tr>
<tr>
<td>Common Assessment3</td>
<td>Kappa3</td>
<td>Zeta3</td>
</tr>
<tr>
<td></td>
<td>Upsilon3</td>
<td>Eta3</td>
</tr>
<tr>
<td></td>
<td>Nu3</td>
<td>Theta3</td>
</tr>
<tr>
<td></td>
<td>Omicron3</td>
<td>Epsilon3</td>
</tr>
</tbody>
</table>

study great effort has been taken so that guardian approval was also required. The guardians were given the full consent form. All consent forms can be found in Appendices M through R. The Institution Review Board (IRB) has approved this study and all consent forms.
Privacy and Confidentiality

All data collected was coded to protect the participant’s identity. The only key to the coded information is kept by the researcher in a secure locked location. All video data, written data, and consent forms is stored in this location. The video consent form signed by all students and teachers gives students the option of allowing the researcher to share the video interviews at presentations. If the student or teacher declined, then the video and audio data was not used in any form to link the participant to the study. Coded transcriptions were used omitting any references to the involved participant.

Instruments of Study

This study was concerned with two forms of data: video data and written data. All written assessments and videos were coded thoroughly for procedural decisions. The student’s written assessment may have been used during that specific student’s video session so as to remind the student of their initial intents when solving the problem. Videos and transcriptions of the videos was coded and collected as data.

Video Interview

The video interviews with the student focused on the student’s response to only one question of the chosen assessment as described in the procedure. These videos were recorded using a digital video camcorder. These videos are kept in digital format in a secure location. The teacher’s videos were recorded in the same fashion.

Transcription

Transcriptions of the conversations with teachers and students were coded, created, and held securely in digital format. The transcriptions were referred to for
purposes of both the qualitative (pre-transcendental Husserlian phenomenological study) and quantitative (T-tests, correlations, and descriptive data) analysis so that literal metaphors and conceptual metaphors could be identified.

**CMT Analysis**

CMT analysis was done to identify and classify conceptual metaphors. Specifically, this study’s primary focus was to determine what source domains that students associate with the target domain of mathematical problem solving. In other words, the primary goal was to determine how student’s literal metaphors complete the phrase PROBLEM SOLVING IS . . . However, the pilot study discovered that asking students to broadly complete this phrase is different than when they put it into practice. Thus all literal and conceptual metaphors involved in the interview were recorded. To make sure that the participant stays true to the purpose, the researcher posed questions in the interview for clarification. These questions can be found in the procedure section. The collected conceptual metaphors were recorded as were each associated literal metaphor. The aspects of the source and target domain were recorded for trustworthiness and posterity. Each literal/conceptual metaphor pair was categorized by cognitive function (structural, ontological, or orientational), experiential function (grounding or linking), and complexity (complex or primary conceptual metaphors according to the described aspects of source and target domain) each time they are used by the student (Kövecses, 2006). Entailments were recorded to describe any dissonance given by the student or interpreted by the researcher. Such dissonance describes whether the study
lends itself to a coherent set of source domains for mathematical problem solving through the student’s perspective.

**Procedures of Study**

**Data Collection**

The data was collected from honors geometry classes at the beginning of the 2011-2012 school year. It is the high school’s standard procedure for both participating honors geometry teachers to agree upon a common assessment at the end of each chapter of their textbook, *Geometry Connections* (Dietiker, 2007). After agreeing upon the common assessment, the researcher chose one problem on the common assessment that requires problem solving according to Lesh and Zawojewski’s (2007) definition referenced in the literature review. This process was done for the first three chapters of the honors geometry textbook. The chosen questions are stated in Table 4.

The teachers were interviewed individually with the researcher via video prior to giving the test so as to explain what problem solving techniques they expect the students to use. The teacher asked the following six questions in a semi-structured interview:

1. How will students perceive the mathematical problem?
2. Why do you think students will perceive the problem the way you have described it?
3. How will students solve the problem?
4. Why will students solve the problem that way?
5. Could students solve the problem differently?
<table>
<thead>
<tr>
<th>Common Assessment</th>
<th>Question #</th>
<th>Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>A regular hexagon has rotational symmetry about its center. What is the minimum number of degrees of rotation necessary to show rotational symmetry? Explain your reasoning using math vocabulary. (No picture included)</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>Find the <strong>area</strong> and <strong>perimeter</strong> of this figure.</td>
</tr>
<tr>
<td>3</td>
<td>1a and 1c</td>
<td>Determine whether or not each pair of triangles is similar.</td>
</tr>
</tbody>
</table>

Justify each by using a flowchart.
6. What possible problem solving techniques do you see them using to solve this problem?

The interviews with the teachers were between 6 and 20 minutes long. The interviews were transcribed and literal metaphors corresponding to conceptual metaphors were recorded. Again, the focus remained centered on the teacher’s perspective of the source domains mapped onto the target domain of mathematical problem solving. The teacher gave the researcher a copy of their rubric for the specific problem for each assessment. Each teacher had their own rubric, which was not agreed upon between teachers, yet the rubrics were very similar in point breakdown.

The students then took the common assessment. After the teacher completed grading the common assessments, students who volunteered to be part of the study were separated by the teacher and their tests were given to the researcher. The researcher then randomized the assessments of each teacher. The researcher began looking at how the student performed relative to the teacher’s rubric on only the chosen question. A student whose solution to the chosen problem scored 67%-100% of the given points for that specific problem according to the teacher’s rubric was considered to have aligned well to the teacher’s expectations. A student whose solution to the chosen problem scored 33%-66% of the given points for that specific problem according to the teacher’s rubric was considered to have aligned moderately to the teacher’s expectations. A student whose solution to the chosen problem scored 0%-32% of the given points for that specific problem according to the teacher’s rubric was considered to have aligned poorly to the teacher’s expectations. One to three tests from each category of student-teacher
alignment were chosen randomly per assessment. Some categories could not be met due to the limited number of volunteers that could fit into each category according to the teacher’s rubric. Moreover, there were two mortalities in the data collection due to students who were unable to make the interview meetings. These mortalities affected the percentage distribution, but could not have been prevented. Table 5 below summarizes the participant choices.

The limitation to the number of students who aligned poorly with the teacher’s rubric were less due to multiple factors that were not controllable. First, the mortality rate took a few of the students from that category. Secondly, the study was voluntary and a natural sample bias may exist between students who volunteered and those who performed well. However, the students volunteered prior to taking any of the tests, so this could not have been anticipated or prevented. Thirdly, tests that omitted any attempt at the chosen problem were omitted. The students must have shown some degree of work to offer some assurance that the interview would be beneficial.

Tests of the chosen students were then photocopied so that the teacher may return the original to the student. However, the student’s identification was coded so that the student artifact can be used confidentially. The students whose tests were chosen randomly by the researcher were then interviewed by video for 6-20 minutes. Manipulatives allowed during the test (calculator, ruler, pencil, and paper) were allowed during the video interview. The student was also given a copy of his/her test so that the student could refer to it during the video.
### Table 5

*Student Participants by Performance*

<table>
<thead>
<tr>
<th>Alignment with Teacher’s Expectations via Teacher’s Rubric</th>
<th>Number of Students Chosen from Each Honors Geometry Teacher</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Assessment 1</strong></td>
<td></td>
</tr>
<tr>
<td>Aligned Well to Teacher’s Rubric (67-100%)</td>
<td>1 (ALPHA) 2 (BETA)</td>
</tr>
<tr>
<td>Aligned Moderately Teacher’s Rubric (34-67%)</td>
<td>2 (ALPHA) 1 (BETA)</td>
</tr>
<tr>
<td>Aligned Poorly Teacher’s Rubric (0-34%)</td>
<td>1 (ALPHA) 0 (BETA)</td>
</tr>
<tr>
<td><strong>Assessment 2</strong></td>
<td></td>
</tr>
<tr>
<td>Aligned Well to Teacher’s Rubric (67-100%)</td>
<td>3 (ALPHA) 3 (BETA)</td>
</tr>
<tr>
<td>Aligned Moderately Teacher’s Rubric (34-67%)</td>
<td>1 (ALPHA) 0 (BETA)</td>
</tr>
<tr>
<td>Aligned Poorly Teacher’s Rubric (0-34%)</td>
<td>0 (ALPHA) 0 (BETA)</td>
</tr>
<tr>
<td><strong>Assessment 3</strong></td>
<td></td>
</tr>
<tr>
<td>Aligned Well to Teacher’s Rubric (67-100%)</td>
<td>1 (ALPHA) 3 (BETA)</td>
</tr>
<tr>
<td>Aligned Moderately Teacher’s Rubric (34-67%)</td>
<td>3 (ALPHA) 1 (BETA)</td>
</tr>
<tr>
<td>Aligned Poorly Teacher’s Rubric (0-34%)</td>
<td>0 (ALPHA) 0 (BETA)</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>12 (ALPHA) 10 (BETA)</td>
</tr>
</tbody>
</table>
As stated in the phenomenological design, the interview was semi-structured asking five primary questions of the students:

1. How did you perceive the mathematical problem?
2. What does it mean to solve the problem?
3. How did you solve the problem?
4. Why did you solve the problem that way?
5. Could you have solved the problem differently?

Follow-up questions and clarification were requested by the researcher. This entire process was repeated for three questions from three distinct common assessments. These three questions came from chapters one, two, and three of their textbook, *Geometry Connections* (Dietiker, 2007). While the same student may have been interviewed more than once as this involves multiple assessments, this was not considered a longitudinal study. As the assessment questions are completely independent, if the same student is interviewed again for a different assessment question, the student’s video interview was considered an independent event as was his/her responses and CMT analysis.

**Data Analysis**

As described in the design of the study, each video was transcribed and then analyzed using mixed methods. First, the data was analyzed using pre-transcendental Husserlian phenomenology with CMT analysis. Each literal and conceptual metaphor was identified and categorized via cognitive function (structural, ontological, or orientational), experiential function (grounding or linking), and complexity (complex or primary). The complexity of the conceptual metaphors used for problem solving was
more subjective and was not the focus of the interviews. Thus the classification of conceptual metaphors as primary or complex is more closely related to the described parts of the source and target domain for each conceptual metaphor for justification of the described conceptual metaphor. In this manner, the categorization need not be quantitatively analyzed.

Additionally, conceptual and literal metaphors relating to topics involved in mathematical problem solving were recorded as well. For example if a student maps solutions or problems to a specific source domain, those were recorded as well. The resulting conceptual metaphors were analyzed for coherence and entailments.

Second, the quantitative analysis counts the frequency of conceptual metaphors used by students that are associated with problem solving or aspects of problem solving. Specifically, the frequency of each conceptual metaphor associated with problem solving was tallied for T-test significance, correlative data, and descriptive data. Moreover, a study of the frequency of various source domains associated with mathematical problem solving was studied to determine which source domains dominate students’ perception of mathematical problem solving. For example, if students regularly perceive PROBLEM SOLVING IS A JOURNEY through literal metaphors such as “I don’t know where to go with this problem”, then it would be recorded and described through descriptive statistics and other appropriate quantitative analyses.

Together, these two analyses determined if a metaphorical system (Kövecses & Benczes, 2010) was developed. If no such coherent system exists, then an analysis of why there was a lacking of coherent metaphors for mathematical problem solving offers
quite a fresh perspective on how students perceive mathematical problem solving and how relevant it is to the standards of mathematics education. Models of teachers’ perceptions of student problem solving using conceptual metaphors was compared to the models of students’ perceptions of mathematical problem solving to identify similarities and differences. However, the focus remained squarely on the student’s perception.
CHAPTER IV

ANALYSIS AND RESULTS

Epistemology (Greek epistasthai): know how to do, understand (Harper, 2012).

To identify student’s interpretations and perceptions of mathematical problem solving without over-imposing the teacher’s interpretation, there were two parts to the mixed methods analysis of the data. First, CMT Analysis was performed that followed pre-transcendental Husserlian phenomenology that coded the literal and conceptual metaphors. In doing so, data was analyzed for coherence, entailments, and complexity so as to minimize misinterpretation in the metaphorical coding. Specific examples and clarification are described in the qualitative analysis section. Second, a quantitative analysis compared frequency of conceptual metaphors used, student scores from common assessments, and their distribution amongst the interviewed students using T-tests, correlation matrices, and descriptive statistics.

The two teachers were coded via the Greek letters Alpha and Beta. The number following their identity refers to the interview and assessment that took place. For example, Beta2 was the second interview discussing the Chapter 2 test (Common Assessment2), question 7 with Beta as described on Table 4. All students were coded by other Greek letters and numerically coded according to the assessment as well. Notice, the number following the student’s code describes which assessment and question the student was answering. For example, Eta3 was Eta’s response to the Chapter 3 test (Common Assessment 3), questions 1a and 1c as described on Table 4.
Qualitative Analysis

CMT Analysis

There were a surprising number of diverse conceptual metaphors used by students and teachers for mathematical problem solving. Below are examples of conceptual metaphors used frequently by students and teachers. Verification is given through description with the literal metaphors and aspects of target and source domains.

**Problem solving is experimenting.** Using problem solving as the target domain, the structural source domain of experimenting was evident when describing the inductive technique for solving the problem without being able to deductively justify (i.e. prove). Eta3 stated, “I could have to test out each one of those (possible solutions).” Theta3 stated, “For that you just had to guess and check many times.” The “guess and check” method stems from students’ experimental design because students will cognitively experiment. With Common Assessment 3 part 1c, Eta3 and Theta3 were referring to calculating the triangle ratios with every possible combination of side lengths to verify that the two triangles were not proportional. In this manner, Eta3 and Theta3 were deductively solving 1c) via exhaustion, but required the experimental (albeit formulaic) data verifying that no combination demonstrated proportionality. Sociomathematical norms (Cobb & Yackel, 1996) in an honors geometry course do not value “guess and check” techniques at the high school level. Deductive reasoning is given more credit and definitely partial credit as demonstrated by Beta3 when he states "I certainly wouldn’t take a blanket number of points off if they didn’t (use guess and check) and if they have their correct thinking.” The demonstration of their cognitive abilities is valued more by
the teacher than a correct answer, yet the guess and check technique is still used. Indeed, Theta3 continues to not align to these expectations when stating on the Chapter 3 test that “Since there is only some space on the paper I tested it out on my calculator prior to writing it down with this one.” This is another example of experimenting with the calculator independent of demonstrating on an assessment deductive reasoning.

**Problem solving is a process.** Students regularly referred to problem solving as a process.

Psi1: “I had a different thought process and I’m kind of thinking that these can’t equal 360”

Rho1: “I do Kumon, and that lets me have an automatic brain . . . Find the process then find the solution, and once you got the solution you just um, you just write down your process. Teachers always want the process.”

Omega1: “it is the way that my brain processes it that it just flips or doesn’t make sense.”

Eta2: “taking apart things and I guess that’s how my mind works.”

Upsilon3: “I used the highest number and lowest number method. I just drew a circle around the lowest one and I underlined the second lowest one and then I drew a circle around the highest number.”

The common theme for this source domain is the linearity and sequential connotation that originates from the process metaphor. More specifically, the process metaphor emphasizes the “how” perspective of solving problems. Eta2’s view of the mind is one of a machine performing an operation. Upsilon3’s view of method demonstrates the
“how” clearly. Rho1’s reference to the “automatic brain” is a wonderful illustration of the utility of the process metaphor.

Teachers also referred to problem solving as a process. Alpha3 pointed to the student’s cognitive use of problem solving as a process when she stated, “So it’s also them understanding what notation is there but what notation they need to use to explain their thought process.” Similarly Beta3 refers to a specific problem-solving process when he states, “I think that they’re a little bit more comfortable with the flow chart process and they know to get right into the geometry part”. Again, both teachers use of “process” reify the “how” of problem solving. Both teachers demonstrate how this reification manifests itself in the form of the source domain of a procedural process.

Problem solving is discovery. A word that both teachers used in every interview I had with them was the word “realize”. Here are some prototypical examples.

Alpha1: “To really make them realize that difference between I’m talking about that central angle that we have kind of added to the shape versus the angles that some of them already have.”

Alpha2: “They need to realize that this is not going to be part of the perimeter”.

Alpha3: “If they don’t realize that—then you will get things like there is not enough information if they don’t realize having parallel lines gives you other information. “

Beta1: “Once they realize that that’s six sides then they’ll see that we’re looking for rotational symmetry.”

Beta2: “I think they’re going to realize that this is an abnormal shape”
Beta3: “I think the first thing that students have to realize about part (a) is some will look at that and will not really see that there are two triangles there”

In all of the examples, the word “realize” represents an action by the students that will make them aware of important information. The ideal word that aligns with this definition is discovery. While in some cases the teacher expects inductive discovery, deductive discovery is discussed in other cases, yet both require the Merriam-Webster Dictionary (2011) definition of discovery: “To make known or visible”. Additionally, a few students used the word “realize” which aligns with the metaphor of discovery.

Zeta1: “when I was using my other method I kind of realized that that was completely off because it was over 90 degrees.”

Kapp3: “I knew that was going to be important because when two lines were parallel you can realize what angles are similar and stuff.”

Thus throughout many interviews discovery became an important aspect of problem solving when the word “realized” was used. However, two other phrases impacted student problem solving as much as the word “realized”.

Students regularly solved a problem using the phrases, “figure out” and “find out”. Consider the data below.

Zeta3: “You’re supposed to figure out whether each triangle is similar or not or use a flow chart again and you had use what you knew from chapter three.”

Theta3: “For all of them you actually have to find out if they’re similar”

Eta2: “I knew how to solve it and I could have done it if I looked at it I could have reapplied it and figured out the mistake.”
Omega3: “I take the basic thing of total area and total degree of shape and then figure out how that is going to be split up to create that symmetry.”

Upsilon3: “I saw two triangles that they didn’t show which sides were um, corresponding to each other. We had to find out which sides were corresponding to each other”

One may think that the use of “find” defined the use of the word to align with the conceptual metaphor of PROBLEM SOLVING IS SEARCHING or the use of the word “figure” as PROBLEM SOLVING IS CALCULATING. However, when looking at the context surrounding the uses of “figure out” and “find out”, the context did not align with SEARCHING or CALCULATING alone. Omega3 referenced the action, to “figure out” as a separate operation distinct from the “splitting” metaphor. Upsilon3 isn’t searching alone as the sides of the triangle were already identified. Similarly, Zeta3’s use of “figure out” indicates a need for identification, not calculation. Cognitively, all of the uses are of “figure out” and “find out” reference a need to determine through understanding and observation. The word “out” made a significant difference in the meaning the student was trying to convey when using the word “figure” or “find”.

Notice how so many of the deductive and inductive steps refer to the more semantically correct use of discover, to make known or visible. This supports why Merriam-Webster (2011) has the word “discover” as synonymous with “find out”, “figure out”, and “realize”. Thus the appropriate source domain to associate with these phrases is PROBLEM SOLVING IS DISCOVERY.
While it could be argued that these are ontological conceptual metaphors with the use of “out”, there was little direct validation with students representing the inverse operator to describe what was “in”. Nonetheless, if the conceptual metaphor for “figure out” and “find out” was perceived as ontological, the concept would be that PROBLEMS ARE CONTAINERS with the solutions being outside and techniques of problem solving constituting the inside. These conceptual metaphors were tallied as ontological conceptual metaphors. However, they directly related to problem solving semantically as DISCOVERY in context of the problems and thus were recorded as such for CMT analysis of problem solving.

**Problem solving is building.** Both students and teachers referred to the building metaphor when solving mathematics problems. Such uses regularly aligned with constructivist approaches of learning. For example, Omega1 referred to the cognitive aspect of problem solving when saying, “I put all those components together to create one big complex thought.” The teachers also used the building metaphor in discussing how the students learned through inductive reasoning. Beta1 stated, “We tried to build up the pattern showing them what’s the degree of rotation of an equilateral triangle”. Alpha3 also expressed the building metaphor when stating, “They know they need to set up a flow chart” because setting up references how to build a foundation. In this manner, one can see the constructivist approach to learning being expressed through the teacher’s perception of understanding. However, constructivism was only one of many approaches described by the students. The building metaphor was not the singular dominant
discourse in the students’ or teachers’ conceptual metaphor. However, it was a prevalent conceptual metaphor as the following examples demonstrate.

Alpha2: “dividing into parts that you know putting the full parts together to get the total amount”

Epsilon2: “You probably could have made it like a trapezoid I guess.”

Omicron3: “it made it a lot more organized by stating facts randomly all over.”

Omega1: “if I want to take that triangle and then rotate it creating that um hexagon”

Eta2: “I know that if I add them together it would equal the same amount of different figures put together because it’s just them separated.”

Zeta3: “I just did make like little mistakes.”

As is clear, the action of “making” was a regular reoccurrence of solving problems (or lack thereof) through building.

**Problem solving is visualization.** Due to the course being a geometry course, visualization was an important factor in problem solving. This is not to say that visualization (or other metaphors) was completely dependent upon the content, but rather to suggest that content affected how the metaphors were used. There were examples of directly addressing visualization, described by Omega1, “It is physically not possible for me to just visualize it. It gets very confusing and flustering.” However, the source domain of visualization regularly had multiple facets in the literal metaphors. There were examples of external visualization (written on paper) and internal visualization (mental,
visual operations). Additionally, both the internal and external visualizations were described with static and dynamic perceptions. Four such examples are in Table 6.

Table 6

Variations on the Source Domain of Visualization

<table>
<thead>
<tr>
<th>Name</th>
<th>Visualization</th>
<th>Literal Metaphor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eta2</td>
<td>Internal, Static</td>
<td>“I can’t see all the rules for the one shape in my head.”</td>
</tr>
<tr>
<td>Rho1</td>
<td>External, Static</td>
<td>“I saw six equilateral triangles (in the regular hexagon)”</td>
</tr>
<tr>
<td>Kappa3</td>
<td>Internal, Dynamic</td>
<td>“I saw that the triangle was turned, so I tried to turn it in my mind to make it face the same way so I could figure out which lines or which sides are corresponding”</td>
</tr>
<tr>
<td>Nu3</td>
<td>External, Dynamic</td>
<td>“So I flip this one over and longest so I can visualize it better.”</td>
</tr>
<tr>
<td>Beta1</td>
<td>External, Dynamic</td>
<td>“So I think that they will still try to visualize what’s actually happening as that hexagon is rotating”</td>
</tr>
</tbody>
</table>

When mapping from the target domain of problem solving, visualization also was entailed with justification. For example when Zeta3 states, “Because you can’t really tell just by looking at it, like it could be completely differently from what it looks like, same
with c.” when referring to a picture on the common assessment 3. This originates from the beliefs and orientations (Schoenfeld, 2011) that teachers impose on students not to trust their eyes alone when solving problems. Indeed, Beta3 demonstrates the need for students to not trust their eyes alone and rather the conditional information, “once again they’re going to have to look at what they’re given”. Additionally, teachers associated visualization with problem solving when referring to analogous problems. Alpha3 stated, “They have done numerous problems that look like these”. These may be static or dynamic, but the significance is in the external and experiential reference of similar problems. Did the problems truly look like these or did the solutions?

**Problem solving is partitioning.** The source domain of partitioning occurred most frequently with students in the second common assessment question (see Table 4), but not exclusively. This was primarily due to the geometric nature of question 7 on common assessment 2 where students attempted to manipulate the given problem into multiple similar problems. However, on all three assessments students manipulated the given problem into similar problems and many attempted to partition the problem as demonstrated by the examples.

Rho1: “I separated the hexagon into triangles”

Kappa2: “Since there are two parts to the question you have to make sure you do the two parts separately”

Epsilon2: “Well I like to break it up into two smaller parts and then just add things together.”
Eta2: “I solve them individually it’s a lot easier than having to solve it all at once.”

Theta2: “you could just break it up into the basic shapes that we learned.”

Eta3: “you have to take them apart, that’s what I did, I took them apart in my head.”

One can see the common words of “break” and “apart” for purposes of simplifying the problem into multiple problems the students are more comfortable in approaching.

Similarly, the teachers mapped to the source domain of partitioning when describing how they would expect students to solve the mathematics problems, but only with the second common assessment question. Clearly this was due to the content of second common assessment, but it is surprising that this conceptual metaphor was not used by either teacher on either of the other two assessments. Alpha2 claimed that to solve the problems students “need to break it down into separate parts and taking the problem and making it into simpler problems.” Similarly, Beta2 also recognized that students are “going to have to break this shape into figures that are more accessible figures that are more recognizable.” Again, the words “break” and “parts” arise in the teachers’ choice of words. One can see the similar language between teachers Alpha and Beta, but more importantly, one can see the direct qualitative relationship to the language used above by their students. This is a strong indicator that the language used by the teachers influences the language used by the students in this context.

Problem solving as searching. This was a dominant form of metaphorical discourse, but difficult to code. This was primarily due to the distinction between
“finding” the answer, and being able to “find out” something about the problem. Again, in context when the student referred to “finding out” it referenced a notion of discovery through surrounding context. Consider Epsilon3’s quote:

To find out if they’re similar or not and by using angle, angle similarity or side, side, side similarity or side angle side similarity. So I had to either match up the sides or match up the angles to see if they were corresponding.

Notice that the source domain of searching is not relevant in this context despite using the word “find”. This is clear because Epsilon3 has already reduced the problem to three possible techniques (angle-angle, side-side-side, or side-angle-side), hence the searching aspect was already complete. Epsilon3 now is attempting to use experimentation and deduction to discover which technique is appropriate. Thus this is an example when searching was not the indicative conceptual metaphor despite the use of the word “find”.

Nonetheless, the following statements from teachers and students do indicate that searching is a dominant source domain.

Kappa2: “since there was a triangle there I knew that if I found it, the full length of it, I could do the Pythagorean Theorem.”

Lambda2: “Well I found the area which was 240 squared by finding the area of the two separate triangles and the rectangle.”

Epsilon2: “There is probably like a missing side or missing angle or something.”

Omicron3: “I looked at the angles and found that the measure of angle YAK, is equal to ARM”

Theta3: “to solve it for me it meant that I had to find it somehow.”
Beta1: “What were we doing to find that degree of rotation?”

Alpha2: “I think they’re going to run into problems in terms of finding information that they would need.”

Beta2: “use the Pythagorean Theorem for the missing side.”

Beta3: “congruent angles somewhere you have to go searching for them.”

Thus, the conceptual metaphor of searching was frequented through the perception of “finding” the “missing” solution. This structural metaphor is clearly grounded in one’s experiences searching for objects. Additionally, other source domains were regularly entailed with the searching source domain because one’s experience in searching develops with other concrete experiences, such as location and journey. These entailments are discussed in the Entailments and Coherence section.

**Problem solving is a journey.** This was by far the most abundant metaphor for problem solving. Inherently, everyone travels and thus it was described frequently as a grounded metaphor. The most common mapping to journey occurred when student’s referred to “the way” to solve a problem. In this manner, the experiential “way” is short for passage way. Eta3 stated, “I couldn’t really think of a way to solve it and if I did find a different way to solve it wouldn’t have been as easy”. Kappa2 stated “I’m not sure if you could find a different way but I’m sure there are other ways”. Rho1 stated “I think there is a way using that but I haven’t been able to figure it out yet.” Even the teachers (both Alpha and Beta) used the word “way” in this manner. Alpha3 stated, “I don’t have any issues with them doing it that way as long as they still have all needed stuff that’s there.” As a primary conceptual metaphor, “way” is being used as a path or trail, but as a
complex metaphor, the aspects of this path or trail emphasize the journey the student is on to solve the problem.

To see these connections more clearly, notice the following other aspects of journey as described by the students and teachers. Teacher Beta3 mentions that as the students solve the problem they will have to “use the short cut conjectures that we’ve talked about.” The aspect of the target domain of problem solving of interest is the use of the short cut, which is clearly an aspect of the journey source domain. The source domain of journey occurred more subtly in some student’s interviews. Psi1 used journey when referring to the reflective properties of problem solving by saying, “when I look back at what I did”. Notice how looking back is embedded in traveling and thus the journey metaphor. Moreover, look at Omicron3’s statement, “I don’t know, that wouldn’t have gotten me too far”. Again, Omicron3 is referring to distance from what? The final location aligning to the solution, indicating that the theme involved is traveling to the solution, i.e. the journey. What is enlightening about the dominant use of the source domain of journey is that in many ways teachers encourage students to believe that the journey is equally credible in solving math problems. In this manner, it is ideal for teachers to use metaphors related to journey when encouraging students to study and in describing an efficient “way” to solve the problem.

Teacher’s Perspective of Problem Solving

While the students and teachers used multiple source domains for problem solving, the teachers described how they perceived how students would perceive the problem. Within the language they used to describe such problem-solving techniques,
the teachers revealed experiences that they had shared with their class, and perspectives on problem solving that were at times more descriptive than the students. For example, Beta2 and Beta3 revealed that PROBLEM SOLVING IS A RACE with quotes such as “The biggest hurdle that I think that they might run into is” (Beta2) and “That is a tough hurdle for a lot of the kids.” (Beta3) In the second interview, Beta uses the word “run”, this could indicate a fast-paced journey such as a race. Hurdles are iconic of races, thus corroborating the experiential association with races. Coherence and verification of PROBLEM SOLVING IS A RACE is found in Beta3’s perspective of race when he states, “It’s one thing that I think might trip up a few students or might at least at the beginning.” When one is tripped up at the beginning of a race, they may or may not be able to continue. However, this metaphor references a shared experience of the source and target domain, “the beginning”. When Beta3 states “at least at the beginning” he is implying that the beginning is not the end of the problem-solving technique, thus implying that the hurdle, or the object that trips up the student, impedes the student but does not terminate their efforts. Moreover, by referencing the experience of a race, the teacher is indicating that speed is an important factor in solving problems. While the teacher may be only partially aware that they are using such metaphors, these metaphors carry significant consequences for students who presume that solving math problems is a race. Specifically, students may identify that the speed at which one solves a problem is more significant than the quality or understanding of the problem. Or perhaps the metaphor of race may over emphasize the need for competition, to finish the test in a speedy fashion as a measure of intelligence. Fundamentally, there seem to be significant
sociomathematical norms (Cobb & Yackel, 1996) derived from the choice of metaphors of the teacher. These norms give rise to an implicit curricula (Eisner, 2002) inherently defined by the teacher’s discourse.

Alpha1 also demonstrated a surprising orientational conceptual metaphor in her interview with the phrase, “I mean that’s really the only way to end up getting the correct answer in this one.” Later on in the same interview, Alpha1 also said “Some of them will come up with 360 divided by 6”. Notice there is an entailment in the first metaphor with the use of the word “way” indicated a path and the journey source domain. This leads to a better understanding of the phrase “come up” and “end up”. In fact, if these are looked at from a complex conceptual metaphor perspective, a clear relationship exists with a direction for problem solving, SOLVING IS UP. Naturally, such a grounded metaphor may have arisen (notice its use in this sentence with the word “arisen”) from the experiences with objects that are higher are easier to find. Or perhaps that when dealing with objects lost in liquid buoyancy has them “end up” or “come up”. Either way, this is an intriguing demonstration in Alpha1 of a phrase that humans in American culture commonly use and understand but struggle in finding its tangible experience that has clearly been shared. Moreover, Alpha1 only used it for common assessment 1, yet it allowed for an orientation for problem solving.

Another surprising conceptual metaphor used by Alpha1, Beta1, Alpha3, and Beta3 was how speaking and writing were related in problem solving by students as well as by themselves. Consider the phrases stated by participating teachers below:
Alpha1: “We’re really looking for them (the students) saying somewhere that the central angle is 360 degrees.”

Beta1: “I would expect them to say, the rule is 360 divided by N.”

Alpha3: “So they do need to state that the lines are parallel interpreting what this means and then either of the angle pairs so the corresponding angles.”

Beta3: “when we first look at this problem there is nothing telling me what corresponding side corresponds with what”

All four of these exemplars demonstrate that the solution read by the teacher is being expressed verbally by the student. The ontological conceptual metaphor is thus WRITTEN SOLUTIONS ARE SPOKEN SOLUTIONS. This is exciting due to the ontological ramifications. Both teachers frequently demonstrated this conceptual metaphor in their interviews. The implication is that what students write is what they speak and consequently what they think. In other words, when teachers Alpha and Beta grade their students’ exams, they perceive the writing as a student directly talking to them. A dangerous consequence of this perception is that humans frequently hear more than is spoken. In this manner, the teachers may be making assumptions about the student’s understanding and the teacher may essentially read between the lines when assessing. Moreover, this ontological perspective suggests that teachers presume that students open a dialogue with the given problem as demonstrated above by Alpha3’s use of the word “interpret”. This is an enormous result in perceptions of assessment theory and of great insight into how teachers perceive assessment and problem solving.
Entailments and Coherence

Coherence between conceptual metaphors was seen in many participants and lack of coherence was demonstrated as well. The latter was shown by teacher Beta2’s use of the word dissect:

They’re used to and they’ve been taking these shapes and breaking them up and dissecting them into various more familiar shapes. I think the first two that would be the most recognizable to them would be the rectangle and the triangle. Yeah, I mean specifically because the triangle ends up being the right triangle, they’ll look at that and dissect into those two . . .

At this point, I asked Beta2 to describe why students would use this idea of dissecting to solve the problem, to which Beta2 responded:

We’ve talked about dissecting the problem. I think looking at this I think they will see this as being easier to dissect than it will be so surround the shape with a rectangle and subtract the triangle and rectangle that wouldn’t be involved . . .

The biggest hurdle that I think that they might run into is realizing when they do dissect this, that missing piece there is the full height of their triangle.

Two things are interesting about this use of dissect. Occasionally it is mixed with other literal metaphors, such as “surround” or “hurdle” or “break apart”. Also, there is no other experiential reference to dissection. For example, Beta2 never refers to cutting inside or slicing or any surgical operation language. This is a demonstration of what Max Black (1962) referred to as dead metaphors due to their lack of interaction with other aspects of the current experience and the rote use by the participant. Moreover, no student of Beta
(or Alpha) ever used the word dissection in any other interview. Finally, notice that the word dissection is isolated in the sense that other words could replace it and not disturb the meaning of his statements, such as “partition”. Thus, there lacked coherence in the use of the word “dissection” in Beta2’s interview.

Nonetheless, Beta3’s interview still showed that coherent metaphorical systems were consistent in his language as described in the previous section with his multiple examples of PROBLEM SOLVING IS A RACE, that were reinforced regularly in that interview. Students also demonstrated coherent conceptual metaphorical systems on a regular basis when entailments indicated how conceptual metaphors overlapped. Consider Theta2’s comment, “I knew that it was the same as just adding them all together and uh finding the answer that way.” Notice the use of the PROBLEM SOLVING IS SEARCHING and PROBLEM SOLVING IS A JOURNEY near the end of his statement via the use of the words “finding” and “way”. However, searching for a something is usually done in a certain manner, or direction, using a means one describes as a path. Thus while the two source domains are entailed, it is natural that searching requires a direction in which one searches. To this end, the metaphors create a coherent metaphorical system between searching and journey. Other students (Psi2, Upsilon2, and Omega1) regularly entailed these two metaphors with the common phrase, “I need to find another way”.

More often than not, entailments were natural and coherence was smooth enough so that the mixing of the source domains was not disturbing. Consider Zeta3 below:
For all of them you actually have to find out if they’re similar but for 1a) in particular to solve the problem to see if the smaller triangle host inside of the larger triangle is different just a smaller size of the same triangle with the same angle measurements. For 1c) to solve it for me it meant that I had to find if somehow, some way, which side lengths corresponded.

Initially, Zeta3 uses “find out” as to discover, but then moves into visualization with the word “see”, then moves into searching with the word “find” alone, and then the phrase “some way” indicating a journey or path to take. It is important to note that while multiple metaphors were used in this system to describe Zeta’s approach to questions 1a) and 1c) on the third common assessment, they worked cohesively via similar experiences in this culture. When someone needs to discover something, they need to be looking for it via visualization. Additionally, when one searches for something, sometimes one needs to be on the right path to find it. Thus there was cohesion to Zeta3’s use of the multiple metaphors relating to her experience in problem solving. It would have been more disturbing if Zeta3 had said, “I knew I was going in the wrong direction, that’s how I found the solution. I was able to see the solution because I was on the wrong path.”

The conceptual metaphors that Zeta3 described tell a story of experiences that are grounded and shared amongst most when searching, seeing, and travelling.

**Complex and Primary Metaphors**

In the previous example with Zeta3, the three conceptual metaphors for mathematical problem solving (SEARCHING, VISUALIZATION, JOURNEY) were all primary conceptual metaphors as each independently relies on basic human experiences.
However, when they are all used together to describe problem solving, while remaining grounded the concepts became blended through their relationship to mathematical problem solving. Thus when used to blend conceptually with mathematical problem solving, they become more complex. It is important to note that the coherence was dependent on the given cultural norms due to the complexity of the primary metaphors working together as a complex metaphorical system. Nonetheless, coherence existed as seen through the language in this culture of problem solving.

One student, Theta3, offered a complex metaphorical system that diverged from current cultural experiences when discussing distinctions between proving and solving mathematical problems. In the following dialogue, the student has recently learned how to use a flow chart as a means of writing a proof, similar to a 2 column proof, but with ovals and arrows instead of statements and reasons.

Researcher: What is the difference between solving and justifying I guess. How would you see these two ideas come together or not come together in part 1a) and then we’ll do 1c).

Theta3: I think for 1a) or in general solving and justifying, solving is just you actually prove why you say your answer. For justifying it is just you state what you think the answer is.

Researcher: Sorry say that again, justifying is you state what the answer is?
Theta3: Like you show your work to see how you got to your answer but for solving you—they’re both similar, the two terms are similar but I think that
solving is more deeply proving why you have that. Why is it true and why is it like that?

Researcher: You think that’s under solving, solving is proof?

Theta3: Yes, and uh, I’m sorry what was the other question?

Researcher: Just the difference between justifying and solving you’re thinking solving is more proving why and you were saying that justifying is not.

Theta3: Like you can justify by showing uh, that, like for 1a) it says justify each by using a flow chart. Well you use the flow chart and you show the flow chart and you draw it out and you say which ones are uh, you say which angles are similar which are equal and you then you show your conclusion and say why you use it. Those reasons, but if you solve it then you could actually show how uh, how possibly you could take a protractor if you need one and show that each angle is the same like angle a and angle a are the same and others are corresponding and uh, for 1c) you could do the same except using a ruler instead of a protractor.

One can see some counter-intuitive results from this conversation. First, the student does not see flow charts as a means of justifying his thoughts. Instead, he sees it as a manner of following a process demonstrated by the teacher to show your conclusion. What is most surprising is that the flow chart design was introduced because it was specifically designed to better help students justify their thoughts. However, this student sees problem solving as the genuine way in which he justifies his thoughts because when you solve a problem, that is when you need to “show how” (PROBLEM SOLVING IS
Visualization). The flow chart is not a practical means of demonstrating knowledge to Theta3.

Thus this complex metaphor is broken down into the primary conceptual metaphors inherent in his experiences with flow charts and problem solving. However, the complex metaphor that arises becomes PROBLEM SOLVING IS PROVING to this student with proving defined by demonstrating why your solution is correct via constructivist practices. To Theta3, JUSTIFICATION IS A PROCESS, specifically the flow chart. At the primary metaphorical level, Theta3’s experience with justification in the mathematics classroom demanded a procedure known as the flow chart. Similarly, Theta3’s experience with problem solving required constructions with tools such as the protractor that directly demonstrated how the solution was determined. Thus to Theta3, PROVING IS SHOWING HOW. Surprisingly, the current mathematics education culture attempts to teach the opposite; solving a problem is not the same as proving a problem. These cultural complex metaphors demonstrated by Theta3’s experience are in direct conflict with the sociomathematical norms (Cobb & Yackel, 1996) that are the aims of the teachers as demonstrated in the teacher interviews, despite the primary conceptual metaphors aligning with the teacher’s perspectives of what defines good problem solving.

Unorthodox Metaphors

There were a few metaphors that were difficult to map, but deserve to be mentioned for future study. A few conceptual metaphors were only used by one or two students to describe problem solving, yet deserve some recognition due to how the
student used them. Additionally, during the qualitative analysis any literal metaphors that became difficult to map were recorded for posterity. This section describes a few unique conceptual metaphors and literal metaphors that were difficult to map.

A conceptual metaphor came from Epsilon2 that was not regularly used by other students, but offered insight into how Epsilon2 thought of problem solving. Epsilon2 stated “I mean we did a lot of them in class and in our homework so I didn’t really find it all that challenging to some.” For Epsilon2, a problem needs to be challenging. Specifically, if a majority of people (i.e. the class) could do them, then it was not a challenge. Epsilon2 is describing the conceptual metaphor of PROBLEM SOLVING IS A CONTEST as other participants are involved. At the same time, he also argues that it was similar to homework problems and thus he didn’t “find” (PROBLEM SOLVING IS SEARCHING) the problem that “challenging”. While isolated as a conceptual metaphor, Epsilon2 is demonstrating that a test should be different from previous activities in the homework and in class. In this manner, Epsilon2 is distinguishing an important difference between problems and exercises. For Epsilon2 problems that are not challenging are exercises.

Returning to the notion of “realize” both teachers Alpha1 and Beta1 used unique metaphors to describe how students were to recognize/discover specific information about the question on the first common assessment. First, Alpha1 states, “That hopefully one of those things that they have physically done will spark their memory”. Beta1 states, “In my opinion that will be the first term that that jumps out at them.” Both teachers are referring to recognizing/discovering information about the given hexagon.
While one is referring to sparks and fire, the other is referring to “jumping out”. Both experiences of ignition of fire and jumping are actions that are perceived as sudden. Both actions are a changing of states, jumping is moving from static into motion, ignition is moving from not burning to burning quickly. Specifically, both actions do not indicate how the change of states was achieved, but is rather difficult to understand such a change due to the speed at which they occur. This leads back to the word “realize” as, again, the notion of how a student is to be aware of this transformation (spark or jump) is purposefully not described by the teachers. Thus the only coherent aspect of problem solving that can be drawn from such complex, linking metaphors is that PROBLEM SOLVING IS A CHANGE OF STATES.

When Eta3 was asked to describe the difference between solving a mathematics problem and justifying a mathematics problem, he said the following:

That’s a very good question. Justifying is like proving this is right or that is right or like proving your thoughts. Like justifying this like, you say whatever you say you stand by what you say and you try to find out what it is, I’m not too sure.

Two parts of this quote are of interest and difficult to map conceptually. First is Eta3’s reference to “proving your thoughts” and the second is the phrase “stand by what you say”. Both are for convincing, but the difficulty arises in who Eta3 is attempting to convince. Eta3 is bringing to the forefront what Harel and Sowder (2007) refer to as the distinction between ascertaining with proof and persuading with proof. Eta3 references proving one’s thoughts in referring to proving it to himself as his thoughts are the subject, not their expression (ascertaining). Eta3 references standing by what you say, indicating
the experience has shifted to demonstrating to others via convictions rather than deduction (persuading). The difficulty here is that there is no coherence between proving one’s thoughts and standing by one’s words because the topics address different subjects. Moreover, “stand by” is a sign of support usually reserved for conflict-based metaphors. Nonetheless, this cognitive dissonance in Eta3 demonstrates a reflective moment of disequilibrium. This may suggest that when a student is aware of the lack of coherence in the metaphors being used, there is an opportunity to cause disequilibrium. However, more research needs to be collected before arguing this conclusion.

Upsilon3 generated a phrase that was used twice by her and was a means of determining truth, “if these two lines were equal and were, yeah, I think that would be enough to determine two sides.” To generate validity (ascertaining, not persuading) for Upsilon3, she argues that there needs to be “enough” convincing data. This post-positivistic thinking of the episteme leads to a nebulous understanding of deductive reasoning. Upsilon3 is stating that KNOWLEDGE IS A PHYSICAL OBJECT as Upsilon3 is measuring the amount of knowledge from the problem to determine missing information. Unlike formal, deductive, mathematical reasoning where it is assumed a conditional statement is true or false independent of the amount of evidence shown, Upsilon3 is arguing that there is a subjectivity of “enough” dependent on inductive reasoning. While a conceptual metaphor for problem solving is not clear from this statement, the interplay between inductive and deductive reasoning is evident.
Quantitative Analysis

After using CMT analysis to classify and categorize all conceptual and literal metaphors pertaining to problem solving of the students and teachers, quantitative analyses were done with the frequency and popularity of the conceptual metaphors used by the students. As there were only six teacher interviews, quantitative analysis of the teacher’s conceptual metaphors was limited to descriptive statistics along with graphical representations of the teacher’s frequency of conceptual metaphors.

Variables

SPSS software was used to identify interactions and main effects of independent variables to the dependent variables listed below.

Independent Variables

- Gender
- Grade Level
- Teacher (manipulated independent variable)
- Common Assessment One, Two, or Three (manipulated independent variable)

Dependent Variables

- Score according to teacher’s assessment
- Total Number of Conceptual Metaphors Used for Problem Solving
- Popularity of Each Conceptual Metaphors Used for Problem Solving
- Frequency of Each Conceptual Metaphor Used for Problem Solving
- Categorization of Each Conceptual Metaphor Used for Problem Solving (Grounding/Linking, Structural/Ontological/Orientational, Primary/Complex)
**Descriptive Statistics**

Below is a list of descriptive statics of the student’s frequency and use of conceptual metaphors associated with mathematical problem solving. While similar descriptive statistics were available for teachers, as there were only six interviews with the teachers, such results did not warrant a quantitive summary. The students’ descriptive statistics were taken from raw quantitative data found in Appendix D. Descriptive statistics are also included for the student’s Score (according to their teacher’s rubric), Structural Conceptual Metaphors, Ontological Conceptual Metaphors, Orientational Conceptual Metaphors, Grounded Conceptual Metaphors, and Linking Conceptual Metaphors, Number of Metaphors Used, and Different Metaphors Used. For each student the Number of Metaphors Used refers to the total number of conceptual metaphors used by the student when describing mathematical problem solving. Different Metaphors Used refers to the categorical number of different metaphors. For example, a student who only demonstrated PROBLEM SOLVING IS A JOURNEY and PROBLEM SOLVING IS A DISCOVERY would have two Different Metaphors Used. If that same student used the JOURNEY conceptual metaphor four times and the DISCOVERY conceptual metaphor three times in the interview, then his Number of Metaphors Used would be nine. All conceptual metaphors used by the students to describe mathematical problem solving were recorded in Table 7.
### Table 7

**Descriptive Statistics of Students’ Conceptual Metaphors for Problem Solving**

<table>
<thead>
<tr>
<th>Conceptual Metaphor</th>
<th>N</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Mean</th>
<th>Standard Deviation</th>
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<td>1</td>
<td>0.05</td>
<td>0.213</td>
</tr>
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<td>0.05</td>
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<td>0</td>
<td>5</td>
<td>0.41</td>
<td>1.141</td>
</tr>
<tr>
<td>IMAGINING</td>
<td>22</td>
<td>0</td>
<td>1</td>
<td>0.09</td>
<td>0.294</td>
</tr>
<tr>
<td>JOURNEY</td>
<td>22</td>
<td>0</td>
<td>17</td>
<td>4.91</td>
<td>4.116</td>
</tr>
<tr>
<td>PARTITIONING</td>
<td>22</td>
<td>0</td>
<td>6</td>
<td>1.45</td>
<td>1.896</td>
</tr>
<tr>
<td>PROCESS</td>
<td>22</td>
<td>0</td>
<td>5</td>
<td>1.64</td>
<td>1.529</td>
</tr>
<tr>
<td>PROVING</td>
<td>22</td>
<td>0</td>
<td>2</td>
<td>0.09</td>
<td>0.426</td>
</tr>
</tbody>
</table>

*(table continues)*
Table 7 (continued)

*Descriptive Statistics of Students’ Conceptual Metaphors for Problem Solving*

<table>
<thead>
<tr>
<th>Metaphor</th>
<th>N</th>
<th>N1</th>
<th>N2</th>
<th>Score 1</th>
<th>Score 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>REVIEWING</td>
<td>22</td>
<td>0</td>
<td>1</td>
<td>0.14</td>
<td>0.351</td>
</tr>
<tr>
<td>SEARCHING</td>
<td>22</td>
<td>0</td>
<td>13</td>
<td>3.41</td>
<td>3.5</td>
</tr>
<tr>
<td>THINKING</td>
<td>22</td>
<td>0</td>
<td>2</td>
<td>0.09</td>
<td>0.426</td>
</tr>
<tr>
<td>VISUALIZATION</td>
<td>22</td>
<td>0</td>
<td>10</td>
<td>2.5</td>
<td>2.405</td>
</tr>
<tr>
<td>VOCALIZATION</td>
<td>22</td>
<td>0</td>
<td>1</td>
<td>0.05</td>
<td>0.213</td>
</tr>
<tr>
<td>WAR</td>
<td>22</td>
<td>0</td>
<td>1</td>
<td>0.05</td>
<td>0.213</td>
</tr>
<tr>
<td>Score</td>
<td>22</td>
<td>50%</td>
<td>100%</td>
<td>76.2%</td>
<td>18.5%</td>
</tr>
<tr>
<td>Structural Metaphors</td>
<td>22</td>
<td>8</td>
<td>45</td>
<td>17.55</td>
<td>9.101</td>
</tr>
<tr>
<td>Ontological Metaphors</td>
<td>22</td>
<td>0</td>
<td>18</td>
<td>5.91</td>
<td>5.154</td>
</tr>
<tr>
<td>Orientational Metaphors</td>
<td>22</td>
<td>0</td>
<td>3</td>
<td>0.59</td>
<td>1.008</td>
</tr>
<tr>
<td>Grounded Metaphors</td>
<td>22</td>
<td>7</td>
<td>52</td>
<td>18.45</td>
<td>10.423</td>
</tr>
<tr>
<td>Linking Metaphors</td>
<td>22</td>
<td>1</td>
<td>13</td>
<td>5.73</td>
<td>3.93</td>
</tr>
<tr>
<td>Number of Metaphors Used</td>
<td>22</td>
<td>5</td>
<td>53</td>
<td>19.14</td>
<td>10.723</td>
</tr>
<tr>
<td>Different Metaphors Used</td>
<td>22</td>
<td>3</td>
<td>9</td>
<td>6.27</td>
<td>1.518</td>
</tr>
</tbody>
</table>

From Table 7, there are some relevant results to mention. First, the average score of the questions chosen for this study (assessed by the respective teacher) was 76.2% with a standard deviation of 18.5%. This aligns with a grade letter of C which is what the teachers strove to have as the average.
Second, the average number of conceptual metaphors categorized by cognitive function was consistent with the linguistic expectations. On average each student used 17.55 structural metaphors, 5.91 ontological metaphors, and 0.59 orientational metaphors. This agrees with the linguistic hierarchical claim that structural metaphors are the most common while orientational are the least frequented.

Third, the grounded metaphors occurred much more frequently than the linking metaphors. On average, students used 18.45 grounded metaphors during an interview while only using 5.73 linking metaphors. This supports the claim that direct experiences are more relevant to students’ descriptions of how they solve mathematical problems than associated experiences. Additionally, this buttresses the argument that mathematical problem solving is cognitively embodied.

Fourth, on average students used 19.14 metaphors in each interview with 6.27 different conceptual metaphors associated with mathematical problem solving. It is important to note that the standard deviation for metaphors used was quite high at 10.723 while the standard deviation for different conceptual metaphors was only 1.518. This variation in the number of metaphors used stems from two students Delta1 and Phi1 whose primary language was not English. Thus Delta1 and Phi1 used very few conceptual metaphors in their English descriptions. However, these were the only two outliers of the study and they did offer valuable qualitative and quantitative data and were thus still included in the analysis.
To better understand the descriptive statistics of student source domains relative to the target domain of problem solving, Figure 3 was developed to demonstrate how many times each metaphor was used. This chart demonstrates the total number of uses of each source domain by all students together. When referring to the total number of uses of each source domain, it will succinctly be referenced as the frequency of a source domain. However, the total number of times a source domain was used may not demonstrate how pervasive it was amongst all students. For example, it may have been the case that Eta3 used PROBLEM SOLVING IS A JOURNEY enough times in his interview to compensate for another student not using the same conceptual metaphor at all. Thus Figure 4 depicts the source domain by use. When referring to the number of students who used the metaphor, it will be succinctly referenced as the popularity of the source domain. If any student used PROBLEM SOLVING IS A JOURNEY at all during his interview, it was only recorded as “used”. In this manner, Figure 4 demonstrates the number of students who used the metaphor (its popularity), not how many times the students used the metaphor (its frequency). Figure 4 follows after Figure 3.
Figure 3. Students’ conceptual metaphors for problem solving by frequency. This pie chart demonstrates the frequency of each source domain mapped from the students’ perception of mathematical problem solving. This chart sums all students who participated in the study. All source domains less than 1% were not shown.
Figure 4. Students’ conceptual metaphors for problem solving by popularity. This chart represents the percentage of students (22 total) that used each source domain. If a student used it more than once, it was only recorded as “used”.

The results demonstrated in Figure 3 show that JOURNEY (26%), SEARCHING (18%), VISUALIZATION (13%), DISCOVERY (12%), PROCESS (9%), BUILDING (8%), and PARTITIONING (8%) were the most frequently used source domains for mapping from mathematical problem solving for students. For example, of all the conceptual metaphors for problem solving, PROBLEM SOLVING IS A JOURNEY was used 26% of the time. To verify that this was not a few students who used these source domains repetitively, Figure 4 represents the number of students who used the metaphors.
The results demonstrated in Figure 4 show that JOURNEY (95%), VISUALIZATION (86%), SEARCHING (82%), PROCESS (73%), BUILDING (68%), DISCOVERY (68%), and PARTITIONING (55%) were the most popular source domains. For example, 95% of the 22 students used the conceptual metaphor PROBLEM SOLVING IS A JOURNEY at least once in the interview.

These analyses demonstrate two things. First, there are clearly a set of source domains students attach to problem solving that were used in the interview more often than others (Figure 3). Specifically, there were seven source domains (JOURNEY, SEARCHING, VISUALIZATION, DISCOVERY, PROCESS, BUILDING, and PARTITIONING) that were used regularly according to Figure 3 as stated above. Secondly, Figure 4 reveals that these source domains were not frequented by only a few students, but rather all of the students. When comparing the frequency to the popularity of the source domains, Figure 3 and Figure 4 revealed that the source domains used with over 50% of the students were the exact same seven (JOURNEY, VISUALIZATION, SEARCHING, PROCESS, BUILDING, DISCOVERY, and PARTITIONING). Notice that the order varied slightly between the frequency and popularity of the top seven source domains, but their validity relative to the remaining source domains was significant (significance greater than 5%).

This analysis was done on the teacher’s conceptual metaphors as well to look for commonalities between the student’s and teacher’s source domains. Figure 5 below is the equivalent of Figure 3 above using teachers instead of students. Notice that teachers
used a different set of source domains. Similarly, Figure 6 is the equivalent of Figure 4 using teachers instead of students.

*Figure 5.* Teachers’ conceptual metaphors for problem solving by frequency. This pie chart demonstrates the frequency of each source domain mapped from the teachers’ perception of mathematical problem solving. This chart sums all teachers who participated in the study. All source domains less than 2% were not shown.
Figure 6. Teachers’ conceptual metaphors for problem solving by popularity. This chart represents the percentage of teachers (6 teacher interviews total) that used each source domain. If a teacher used it more than once, it was only recorded as “used”.

The results demonstrated in Figure 5 show that JOURNEY (29%), DISCOVERY (17%), BUILDING (11%), VISUALIZATION (10%), PARTITIONING (7%),
SEARCHING (7%), PROCESS (6%), and were the most frequently used source domains for mapping from mathematical problem solving for teachers. For example, of all the conceptual metaphors for problem solving, PROBLEM SOLVING IS A JOURNEY was used 29% of the time. To verify that this was not a few teacher interviews which used these source domains repetitively, Figure 6 represents the popularity of the source domains. The results demonstrated in Figure 6 show that JOURNEY (100%), DISCOVERY (100%), SEARCHING (83%), BUILDING (83%), VISUALIZATION (67%), PROCESS (50%), PARITIONING (33%), RACE (33%), and SETS OF SKILLS (33%) were the popular source domains in a majority of teacher interviews. For example, 100% of the six teacher interviews used the conceptual metaphor PROBLEM SOLVING IS A JOURNEY at least once in the interview.

These analyses demonstrate three things. First, there are clearly a set of source domains teachers attach to problem solving that were used in the interview more often than others (Figure 5). Specifically, there were seven source domains (JOURNEY, DISCOVERY, BUILDING, VISUALIZATION, PARITIONING, SEARCHING, and PROCESS) that were used regularly according to Figure 5 as stated above. Secondly, Figure 6 reveals that these source domains were not frequented in only a few teacher interviews, but rather all of the teacher interviews. When comparing the frequency to the popularity of the source domains, Figure 6 revealed that the source domains used with over 33% of the students included these seven source domains (JOURNEY, DISCOVERY, SEARCHING, BUILDING, VISUALIZATION, PROCESS, PARITIONING, RACE, and SETS OF SKILLS). Notice that the order varied slightly
between the frequency and popularity of the top seven source domains, but their validity relative to the remaining source domains was significant (> 0.05%).

Thirdly, when comparing the teachers’ dominant source domains to the student’s dominant source domains for the target domain of mathematical problem solving, there are not only a few overlaps, but almost complete uniformity. Table 8 demonstrates this significant relationship.

**T-Test Results**

As the teacher was an independent manipulated variable, it is important to verify that there was no sample bias relative to students chosen from teachers Alpha and Beta for purposes of internal validity. For example, the researcher found that both Alpha and Beta used the source domain of partitioning (p. 108), as they both described students solving problems by breaking the given problem apart. However, if the teacher used a different conceptual metaphor (such as dissection) regularly in their interview with the researcher, did this influence the researcher’s interpretation of the student’s conceptual metaphors? Did the researcher’s interpretation of the students’ conceptual metaphors vary because the teacher’s perception influenced the researcher’s interpretation? To verify that the teacher was not an influence on the source domains, an ANOVA T-test was run to determine if the following null hypothesis was false.
Notice, that despite there being a few more popular source domains from the teachers, the top seven source domains remained the same in popularity and frequency for both teacher and student. These results demonstrate a coherent system of conceptual metaphors within mathematical problem solving in high school mathematics classrooms.
$H_0 =$ There is no difference between Alpha’s students and Beta’s students with respect to the means of their frequency with any source domain, their student’s scores, Number of Metaphors Used by each student, Different Metaphors Used by each student, grounded metaphors used by students, linking metaphors used students, structural metaphors used by students, ontological metaphors used by students, or orientational metaphors used by students.

The null hypothesis was tested using T-Tests with alpha-levels of 5% and 10%. T-test results can be found in Appendix I. The null hypothesis was not rejected with both 90% and 95% confidence (alpha-levels of 5% and 10% with 2-tailed significance). In sum, the means of Alpha’s students and Beta’s students did not vary significantly with respect to any source domain associated with problem solving, the student’s scores on each common assessment question, Number of Metaphors Used by each student, Different Metaphors Used by each student, grounded metaphors used by students, linking metaphors used students, structural metaphors used by students, ontological metaphors used by students, or orientational metaphors used by students. For example, the variance between the use of the JOURNEY source domain of Alpha’s students and Beta’s students was not significant ($N=20, p=0.834$). Thus, teacher Alpha and teacher Beta did not differ in influencing students’ frequency of source-domain descriptions of problem solving, their students’ scores on the interviewed questions, the cognitive functions (structural, ontological, orientational), nor their experiential functions (grounded, learning). This supports the internal validity of the study because the teacher’s bias did not affect the researcher’s interpretation of the student’s conceptual metaphors. It is
important to note that there were 12 interviews with students of teacher Alpha and 10 interviews with students of teacher Beta.

An ANOVA T-Test was also run on gender to see if any variance in frequency was significant between males and females. This analysis was not to support internal validity of the research design, but to identify any gender results that may have arisen. The students were grouped according to gender ($1=$male, $0=$female). There were 11 female and 11 male students. Descriptive statistics of this grouping were collected and can be found in Appendix J. An ANOVA T-test was also run to determine if the following null hypothesis is false.

$$H_0 = \text{There is no difference between males and females with respect to the means of their frequency with any source domain, student's scores, Number of Metaphors Used by each student, Different Metaphors Used by each student, grounded metaphors used by students, linking metaphors used students, structural metaphors used by students, ontological metaphors used by students, or orientational metaphors used by students.}$$

The null hypothesis was tested using T-Tests with an alpha-level of 5%. T-test results can be found in Appendix K. The frequency of all categories did not vary significantly except for the source domain of JOURNEY. As shown in Appendix K, males and females varied in their frequency of use of the source domain JOURNEY ($N=20$, $p=0.046$). As shown in Appendix J, the females used the source domain of JOURNEY on average 3.18 times ($M=3.18$, $SD=2.523$) while males used the source domain of JOURNEY nearly twice as often on average 6.64 times ($M=6.64$, $SD=4.760$). Thus, the
null hypothesis was rejected with an alpha-level of 5% with 2-tailed significance in only one category, the source domain of JOURNEY.

In sum, the means of males and females did not vary with respect to any of the aforementioned categories associated with problem solving other than JOURNEY. Reason for the variance in the JOURNEY source domain between genders is unclear. The variation itself may imply an experiential distinction between males and females within the JOURNEY source domain. Perhaps males perceive problem solving more as a journey because they have had more experiences with journeys than females. Another feasible possibility is that male conceptual understanding of problem solving is more involved with the JOURNEY source domain than females. Despite such gender-based variance, the JOURNEY source domain was a prevalent source domain in frequency and popularity with both genders. A study more focused on distinguishing gender characteristics is recommended as this research study focused on identifying coherent conceptual metaphors.

**Correlations**

Upon completing the initial quantitative analysis demonstrated in Table 7, Table 8, and Figure 3 through Figure 6, a correlation matrix was analyzed to look for relationships between many of the factors. These factors included Number of Metaphors Used, Different Metaphors Used, Score, and frequency of all source domains mapped from problem solving. A Pearson Correlation Coefficient ($r$) and 2-Tailed Significance ($p$) were recorded for each correlation with the correlation coefficient being a function of the degrees of freedom. For example, there was no significant relationship between the
number of times a student used the BUILDING source domain and the JOURNEY source domain ($r(22)=0.347, p=0.114$). This means that for JOURNEY and BUILDING, there was a correlation coefficient of 0.347 with 22 degrees of freedom with significance of 11.4% and the significance-level was set at 5% for all correlations. The entire correlation matrix can be found in Appendix L.

Before getting into the individual relationships of all of the compared variables, a decision was made to omit source domains that were used only by one student. This was done because such data could not be argued generalizable if no other student used such metaphors. While the data was coded with frequency zero for any student who did not use a specific source domain, it would be improper to use that frequency in the correlation data as it was an isolated source domain. To this end, 11 source domains were used by at least two people: JOURNEY, SEARCHING, CALCULATING, VISUALIZATION, REVIEWING, PROCESS, EXPERIMENTING, IMAGINING, PARTITIONING, BUILDING, and DISCOVERY. Hence these will be the only source domains used when referencing the correlation matrix.

**Score.** There was no relationship between the score students received by the teacher on the specific question of the chosen common assessment and all other metaphorical properties measured. That is to say, there was no correlation between the student’s score on the specific question and the Number of Metaphors Used, the student’s score on the specific question and Different Metaphors Used, or the student’s score on the specific question and any source domain for the target domain of mathematical
problem solving. This means that there is no relationship between a student using one metaphor and a better score on their assessments for the chosen problems.

**Number of metaphors used.** The Number of Metaphors Used by students showed a strong positive correlation with the number of Different Metaphors Used ($r(22)=0.726, p<0.01$), the BUILDING source domain ($r(22)=0.603, p<0.01$), the DISCOVERY source domain ($r(22)=0.739, p<0.01$), the JOURNEY source domain ($r(22)=0.823, p<0.01$), the PARTITIONING source domain ($r(22)=0.505, p=0.016$), and the SEARCHING source domain ($r(22)=0.581, p<0.01$). It is important to note the strong positive correlation between the number of metaphors used by a student and the different metaphors used by a student ($r(22)=0.726, p<0.01$). This indicates only correlation, and not causation. This could mean the more often the student uses metaphors for problem solving, the more different types metaphors for problem solving the student will use, or vice versa. As causality is not clear in this situation, the relationship to the indicated source domains does little to validate any research questions posed.

**Different metaphors used.** The number of Different Metaphors Used by a student showed a strong positive correlation with the Number of Metaphors Used ($r(22)=0.726, p<0.01$) by a student, the DISCOVERY source domain ($r(22)=0.636, p<0.01$), the JOURNEY source domain ($r(22)=0.515, p=0.014$), and the PROCESS source domain ($r(22)=0.578, p<0.01$). The relationship to the Number of Metaphors Used is described in the previous analysis. As indicated above, causality between the
Different Metaphors Used and specific source domains cannot be verified, only correlated. Thus such correlation data is not helpful in the research questions.

**Source domain correlations.** The correlations between source domains will be analyzed. A correlation between two source domains indicates that there is a relationship between how frequently a student uses one source domain to describe problem solving, and how often they use another source domain to describe problem solving. However, causality is still not definitive. For example, there is a strong positive correlation between DISCOVERY and BUILDING \((r(22)=0.554, p<0.01)\). This means either the more a student uses the DISCOVERY source domain the more likely they are to use the BUILDING source domain, or the more a student uses the BUILDING source domain the more likely they are to use the DISCOVERY source domain.

As one cannot discern causality, it may seem futile to continue. However, if the purpose of the analysis is to understand cognitively which source domains are associated with each other, causality is not a concern. Specifically, if conceptual metaphors are maps between target and source domains, then correlations can be mappings between the source domains. These mappings between source domains help explain entailments and coherence of the given metaphorical system. Figure 7 below shows all possible mappings between the 11 source domains.
Figure 7. Possible correlations between the 11 viable source domains for problem solving. Each line segment (55 total) represents a possible correlation in frequency.

If each of the 11 source domains is considered a vertex of a regular 11-gon, one sees the possible ways in which the source domains could be correlated. There are a total of 55 possible correlations between the source domains. The correlation matrix in Appendix L filters these possible correlations and reveals a fixed number of correlations between only a few of the source domains. These strong correlations are represented in Figure 8.
Figure 8. Actual correlations between the 11 viable source domains for problem solving. Only the 6 line segments represent strong correlations. Correlation data is included in figure.

Figure 8 reveals a lot of information about the cognitive associations students use when describing mathematical problem solving. First, notice how all of the correlations are strongly positive. This means that there is a positive relationship between the source
domains. For example, the strong positive correlation between EXPERIMENTING and VISUALIZATION means one or both of the following.

1. If a student frequently associates problem solving with EXPERIMENTING, then they will also frequently associate problem solving with VISUALIZATION.

2. If a student frequently associates problem solving with VISUALIZATION, then they will also frequently associate problem solving with EXPERIMENTING.

There were no negative correlations indicating that no source domain had an adverse effect on a student using another source domain.

Second, the relationship between JOURNEY and SEARCHING was already mentioned in the qualitative analysis of entailments. However, in the qualitative analysis of entailments, it was suggested through the study of Zeta3 that perhaps JOURNEY, SEARCHING, and VISUALIZATION were closely related source domains (Section 4.1.3). The advantage of the quantitative analysis is that it allowed the entailments via correlations to not be defined by exemplars, but rather through a larger lens of all 22 students involved in the study. In this manner, the quantitative analysis showed that while one student may associate multiple source domains in an example, the sample as a whole demonstrated generalizable results relating specific source domains with others. This quantitative data only buttresses the argument that such entailments can be beneficial to understanding coherence between conceptual metaphors.
Third, notice the structure between PARTITIONING, BUILDING, DISCOVERY, JOURNEY, and SEARCHING. There are exactly two correlations for each source domain. For example, JOURNEY is related only with DISCOVERY \(r(22)=0.590, p<0.01\) and SEARCHING \(r(22)=0.459, p=0.032\), not BUILDING or PARTITIONING. The fact that JOURNEY or other source domains had at most two strong positive correlations is surprising because one would expect multiple correlations between the most popular or frequented source domains. Moreover, one would not expect each source domain to have the same number of correlations and create a closed circuit as demonstrated with PARTITIONING, BUILDING, DISCOVERY, JOURNEY, and SEARCHING. Instead, it seems likely that at least one word would be related to other words outside of the circuit or another word inside the circuit. This may suggest a path for constructing students’ understanding of problem solving through metaphors.

**Summarizing Analysis and Methodology**

Figure 8 offers evidence that a coherent conceptual metaphorical system exists between the students’ experiences. There were only certain conceptual metaphors students demonstrated qualitatively with the CMT analysis. Through the qualitative study of those conceptual metaphors and their source domains associated with mathematical problem solving, this study identified the important source domains that may be entailed. The descriptive statistics then identified those source domains that were most prevalent in frequency and popularity. These results were described in Table 8. Moreover, the correlation matrix identified which viable source domains were related by frequency and which were not. These results were expressed in Figure 8. The
reoccurrence of the same source domains in Table 8 and Figure 8 lend credence to a coherent system (Kövecses & Benczes, 2010) of conceptual metaphors that students use when solving mathematical problems.
CHAPTER V
CONCLUSIONS

*Curriculum (Latin Currere): To run the course (Harper, 2012).*

**Overview of Conclusions**

This study began with the research question, how do we identify students’ interpretations and perceptions of mathematical problem solving without over-imposing the researcher’s interpretation? To answer this question, Conceptual Metaphor Theory (CMT) was applied through quantitative and qualitative analyses to determine if the linguistic tool of metaphor could be useful in identifying the interpretations and perceptions of students through experiential understanding. Did CMT analysis accomplish this task? Yes.

Using CMT analysis, this study qualitatively identified and verified the source domains associated with the target domain of mathematical problem solving. That is CMT analysis determined what perceptions and interpretations students described using to solve mathematical problems. CMT analysis was then extended to determine how the students most frequently described problem solving as demonstrated in Figure 3. The use of CMT analysis confirmed that these were not an isolated set of students who used these source domains to describe mathematical problem solving, but rather all students used the most-frequented source domains as demonstrated in Figure 4. Moreover, as shown on Figure 5 and Figure 6, CMT analysis was used to compare the teachers’ perception of problem solving through the teacher’s choices of source domains associated with problem solving. To this end, it was verified that the students and the teachers shared a
specific set of source domains mapped from mathematical problem solving as shown in Table 8.

Furthermore, the coherence between the given source domains was validated to not be influenced directly by the teacher involved as the null hypothesis held true in with every involved factor. This was verified using an Analysis of Variance with T-tests demonstrating that the variance with score, metaphor frequency, metaphor popularity, cognitive function, experiential function, and every single source domain was not significant between grouping the students by teachers Alpha and Beta. Again, this supports the internal validity of the study because it demonstrates that the choice of teacher was not a significant factor in the conceptual metaphors chosen by the students.

Finally, by looking for correlations between the frequencies of source domains used by students, a specific set of source domains were significantly related as shown in Figure 8. Moreover, the significant correlations gave a dynamic structure to the source domains as demonstrated in Figure 8. For example, if a student frequented the source domain BUILDING, then they may also frequent the source domain PARTITIONING or DISCOVERY, but not SEARCHING or JOURNEY. These are ideal results as they demonstrate specific interpretations students have of mathematical problem solving and how they perceive those interpretations through the specific connections of specific source domains. The rest of this chapter elaborates on these conclusions and their implications.
Responses to Research Questions

The following research questions were asked at the beginning of this study. The results of this study illuminated responses to those questions and are described below.

How do students perceive mathematical problem solving? This study used CMT analysis and identified 22 source domains mapped from problem solving for students (See Appendix D). Of these 22 source domains, seven proved prevalent among the students in frequency and popularity. Thus to a majority of students, PROBLEM SOLVING is

A JOURNEY
SEARCHING
VISUALIZATION
A PROCESS
DISCOVERY
A BUILDING
PARTITIONING

This was determined by a thorough qualitative analysis with CMT analysis and IPA methodology. This was then validated quantitatively using Descriptive Statistics, T-tests, and a Correlation Matrix.

What student experiences are lived through the act of solving mathematical problems? The experiences that most closely relate to problem solving are those grounded metaphors of the seven primary metaphors described above. For example, as stated in the qualitative results, Psi1 used journey when referring to the reflective
properties of problem solving by saying, “When I look back at what I did”. Additionally, Lambda2 perceived problem solving as searching when saying, “Well I found the area which was 240 squared by finding the area of the two separate triangles and the rectangle.” Of course many examples have been discussed, but the result that is valuable for generalizability is the specific influence of the seven stated above. These are the genuine student experiences described through student language.

**Do these student perceptions align with the teacher’s perception of problem solving?** Yes, CMT analysis and IPA methodology was also performed on the teacher’s source domains mapped from mathematical problem solving. Despite only having two teachers (Alpha and Beta) through three interviews each, these teachers generated 21 source domains. Of those 21 source domains, only seven were frequented and popular by both teachers through all six interviews. Those seven source domains were:

- JOURNEY
- DISCOVERY
- BUILDING
- VISUALIZATION
- PARTITIONING
- SEARCHING
- PROCESS

Surprisingly, these are the same seven source domains dominated in the student’s discourse. While this may seem coincidental, Table 8 demonstrates that this was determined among all four hierarchies (frequency/popularity of student/teacher). Thus,
there seems to be a more significant result beyond coincidental data. These results demonstrate a high degree of reliability between the teacher and student in their perceptions and experiences with mathematical problem solving. While the complete list of source domains used by the students (22) and the complete list of source domains used by the teachers (21) differed, the most prevalent source domains were the same for both parties.

**Do these perceptions align with current research in mathematical problem solving?** As this is novel interdisciplinary research, this question is difficult to answer. Schoenfeld’s (1985) four categories are difficult to apply holistically to the CMT methodology. Each conceptual metaphor carries with it properties of the target and source domains that intrinsically have resources and beliefs involved. Additionally, the act of choosing the target and source domain (the act of generating the mapping from target to source domain) mandates the involvement of Schoenfeld’s heuristic and control categories. More specifically, the choice and construction of the conceptual metaphor requires metacognition.

Recall in the qualitative analysis when Zeta3 states, “For 1c) to solve it for me it meant that I had to find if somehow, some way, which side lengths corresponded.” Zeta3 used two distinct source domains to describe how she solved the problem: SEARCHING and JOURNEY. The statement demonstrates Zeta3’s belief that a direction and journey would lead her to end her search. Moreover, the active role of searching indicates the involvement with the problem rather than treating it as an exercise. If it were perceived as an exercise, the path would either be known or not known and no search would be
necessary for the solution. Zeta3 is drawing on the resource of mathematically searching to determine the path. Thus, resources and beliefs are embedded in the use of the source domains. The choice of blending the concepts, blending the source domains, demands control and heuristic understanding of the situation involved. As other students have shown, there are many other source domains that are used to describe problem solving.

The choice to use the JOURNEY and SEARCHING source domains concomitantly is the degree of control available to the student. Moreover, metacognitively Zeta3 is not sure if this conceptual blend is a viable means to a solution. That is why she begins by saying, “To solve it for me meant”. Thus illustrating that this is her approach and how meaning has developed for her. This is her heuristic means to discover a solution. Hence Schoenfeld’s work is a different lens from which to perceive the conceptual metaphor and parallels certain results of this study within mathematical problem solving.

Finally, it is important to note that the linguistic research into conceptual metaphors held true in this study with mathematics education. The cognitive functions demonstrated a frequency-based hierarchy (Lakoff & Johnson, 2003) in the descriptive statistics on Table 7. As mentioned in the quantitative analysis, each student used an average of 17.55 structural metaphors, 5.91 ontological metaphors, and 0.59 orientational metaphors. Additionally, the theoretical framework of an embodied cognition was appropriate as supported via the experiential functions descriptive statistics in the quantitative analysis. On average, students used 18.45 grounded metaphors during an interview while only using 5.73 linking metaphors. While both are meaningful and
valuable to students, the 3:1 ratio between grounded and linking metaphors does support the theoretical stance that problem solving is not mind-free (Nunez, et al., 1999).

**How do metaphors influence how students model mathematical problem solving?** This study validates Lesh and Doerr’s (2003) belief that the solution is only one aspect of the problem solving model. Indeed, students regularly discussed the solution as the end or goal and the problem as a starting point when using the JOURNEY metaphor, but regularly, students went beyond the solution. In the previous section, Psi1 “looked back” when reflecting, keeping with the JOURNEY metaphor he had used frequently throughout his interview. The source domains described by the students align directly with Lesh and Doerr’s description of what is necessary for significant learning.

Consequently, in cases where the conceptual systems that students develop are mathematically significant sense-making systems, the constructs that are extended, revised, or refined may involve situated versions of some of the most powerful elementary—but deep—constructs that provide the foundations for elementary mathematical reasoning. (p. 5)

While Lesh and Doerr focused on elementary education and this study emphasized secondary education curriculum, the idea of Lesh and Doerr’s conceptual system is analogous to a conceptual metaphorical system. Thus the metaphorical system discovered and described in Figure 8 offer specific ways that metaphors influence mathematical problem solving. More specifically, the constructs that Lesh and Doerr argue to be elementary, powerful, and deep are the ideal experiential structures that the conceptual metaphors offer. As metaphors are a means to describe ones experiences in
terms of another person’s experiences, naturally they are a means of sense making for individual students.

**Is there a set of coherent conceptual metaphors that are frequently used?**

Yes. There is evidence from all analyses that there is a strong connection between specific source domains for the target domain of problem solving. Seven source domains dominated the interviews as demonstrated in Table 8 amongst teachers and students in popularity and frequency. Additionally, Figure 8 showed that five of the seven showed distinct strong correlations with one another in frequency.

There are two different conclusions about the coherent conceptual metaphors. First, one could conclude from Table 8 that there are seven ideal source domains as they were shared amongst all parties in popularity and frequency (JOURNEY, DISCOVERY, BUILDING, VISUALIZATION, PARTITIONING, SEARCHING, and PROCESS). However, as each list in Table 8 varied slightly in the order of the seven source domains, relationships among the seven source domains via frequency may be a better determinant. To this end, Figure 8 suggests limiting those seven to only five (DISCOVERY, JOURNEY, SEARCHING, PARTITIONING, and BUILDING) as the other two (PROCESS and VISUALIZATION) aren’t significantly correlated to the other five which each have two significantly strong correlations via frequency of use.

As frequency of use is the only metric used in determining the correlations significance, limiting the seven source domains to only five could limit future research as well as generalizability of this study. Thus for initial purposes of development for meaningful understanding of the students’ perceptions of mathematical problem solving,
all seven source domains will be included in the set of conceptual metaphors that this study finds prevalent in student’s perceptions of mathematical problem solving. The correlative result is valuable, and will continue to be discussed, but will not be a determining factor in defining the set of coherent conceptual metaphors. Instead, it will be used to better understand the relationships between those source domains involved.

**If existence of a coherent set of conceptual metaphors is satisfied, do literal metaphorical expressions align with conceptual metaphors?** The coherent, conceptual metaphor system described in Table 8 and Figure 8 are validated by most literal metaphors as described in the CMT analysis. There were examples of teacher’s and student’s blending concepts and thus involving entailments, but as discussed in the qualitative analysis, such blends were reasonable and a majority aligned with Figure 8 correlations. Zeta3’s example of overlapping the JOURNEY and SEARCHING metaphor described above is the staple exemplar of such entailments. There were a few examples of Black’s (1962) concept of dead metaphor as described in teacher Beta3’s interview in the qualitative analysis. However, this only encourages the need for cognition when dealing with conceptual metaphors not the lack of alignment between literal and conceptual metaphors. There were a few literal metaphors that were difficult to place, but regularly only one per interview and well under five percent.

**Limitations**

There are some significant limitations worth discussing in this study. First, this study focused on the student metaphors at a single high school, with a single course (honors geometry). These limitations were necessary to limit the influence of multiple
variables and to first conclude the existence of a conceptual metaphorical system that students use when solving mathematical problems. A larger more diverse sample group with other mathematics courses would be valuable to better discuss uniqueness of the metaphorical system. For example, this study may now be extended to other mathematics classes (honors or regular), other grade levels, and other schools.

The sample size was valid for the chosen T-tests, yet a slightly larger sample ($N>25$) could offer more support for the results as one could then assume normal distribution. The mortality rate from volunteers to participants is where the difficulty arose. Additionally, a larger sample size of teachers would be more valuable so that a teacher’s system could be established with more quantitatively significant data on the teachers. This will be discussed further below in the Future Studies section.

When dealing with separating and choosing participants by their score, it was unforeseeable that there would be so few students performing below the teacher’s expectations. Nor was it perceivable that the low-performing students would not show up to participate. The lack of participation of the students who performed below the teacher’s expectations could have been for many reasons. The lack of low-performing participants may be due to the course being an honors geometry course and thus may limit the number of lower performing students. Another reason may be that the students did not participate because they were embarrassed about their performance on the assessment. However, the interview focused on one question from the assessment and that question did not reflect the student’s overall grade. The question chosen from the assessment was one that required the use of problem-solving techniques so as to interpret
the students’ perceptions of mathematical problem solving. The student did not know prior to the interview which question would be discussed in the interview. Thus it is possible that the student chose not to participate because they perceived their performance as poor, but cannot be confirmed. To curb this possibility for the future, the volunteer forms should suggest that performance and adherence to teacher expectations is not relevant to the student’s participation. These limitations should be taken into effect so as to have contingencies in place for future studies.

Finally, the limitation of the researcher’s interpretation was a challenge. The researcher’s purpose was to limit misinterpretations of the students’ conceptual metaphors. This was done by having the researcher go over each interview at least twice, and verifying the conceptual metaphors and source domains remained true to the student’s perceptions by looking for collaborative and coherent language used by the student. Specifically, the researcher listened for how the metaphor was included so as to support the conversion from literal to conceptual metaphor. For example, “find out” and “figure out” were discussed in the qualitative analysis as they were used by Zeta3, Theta3, Eta2, Omega3, and Upsilon3. To this end, after long deductive and responsive analyses, the researcher concluded that the students were genuinely referencing DISCOVERY more often than they were referencing SEARCHING. This was an important delineation that could only occur from thorough, vigilant dedication to the student’s voice. Methodologically, encouraging future research to hold to this degree of listening is vital and yet difficult to outline. It was the challenging need for the researcher to limit their misinterpretations that allowed these results to be illuminated.
This novel line of research, CMT analysis, currently limits the possibility for triangulation as few other researchers are available to confirm the researcher’s conclusions. Naturally, this is the case with most new lines of inquiry. However, the researcher took every available opportunity and ran every foreseeable analysis to check for reliability and validity of the results. First, the researcher used qualitative and quantitative analyses so that all results were transparent. This use of mixed methods demonstrated all deductive reasoning that led to student interpretations. Second, validity of interpretation was incorporated into the methodology as the interview was semi-structured so that the researcher could ask the participant directly to validate linguistic conclusions. For example, during the interview with Rho1, the researcher directly asked if Rho1 was suggesting that problem solving was a process because Rho1’s choice of words was suggesting that metaphor, but not conclusively. Third, the researcher reviewed all data at least twice in identifying all conceptual metaphors involved. Fourth, T-Tests were run to determine if variance between teachers was significant. The resulting T-Tests analysis of variance demonstrated that no variance between Alpha’s students and Beta’s students existed with any linguistic category that was quantitatively analyzed. Fifth, correlations were run to verify that no source domain correlated with the student’s score as determined by the teacher. The existence of such a correlation might suggest the researcher or teacher had a bias towards a specific metaphor that affected the student’s score. As demonstrated, no such correlation existed. Sixth, coherence in the students’ and teachers’ use of conceptual metaphors was interpreted via frequency and popularity to corroborate results of a coherent metaphorical system. Moreover, the
source domains associated with problem solving were cross-referenced between teacher and student as well as frequency and popularity as demonstrated in Table 8. Correlations between the frequency of source domains within the coherent conceptual metaphorical system were even analyzed (Figure 8) to verify how the students related certain source domains.

This method of research is new within mathematics education. Future studies will require a similar vigilance that is difficult to formulaically describe for future researchers. Nonetheless, the usefulness of this study is just as important as its generalizability. The strength within this study for the practitioner is how research can improve teachers’ ability to listen as will be described in Implications for Practice.

**Discussion**

**Conceptual Metaphorical System**

In Kovecses and Benczes (2010) work, they discovered two conceptual metaphorical systems that helped them understand how people perceive and categorize things and events, specifically nouns and verbs. In describing how things (nouns) are perceived they identified the GREAT CHAIN OF BEING where the following conceptual map exists:
In Koveces and Benczes work, these four source domains defined how humans understand abstract complex systems. Moreover, they argued that a specific hierarchy exists between the source domains relative to their use in language.

When beginning this study, there existed very little substantial evidence to know whether the students and teachers had similar perspectives on the source domains associated with the target domain of PROBLEM SOLVING. The pilot study did suggest that there were reoccurring themes in the use of metaphors, but the quantitative data was too limited. Thus this study was not guaranteed to demonstrate the results of a coherent metaphorical system. The results have the potential to be immensely valuable to the field of mathematics education as existence of a conceptual metaphorical system for mathematical problem solving has been demonstrated in Table 8. As discussed above, in answering the initially-posed research questions Figure 8 lends insight into understanding this metaphorical system, but Table 8 is a strong descriptor for the system.
A hierarchy equivalent to Kovecse and Bencsés (2010) work seems to exist (as seen on Table 8) but varies relative to frequency or popularity as well as student or teacher. This hierarchy did show certain trends but beyond these common trends the data is too restrictive due to the varying significance between the teacher and students as well as the limited number of teacher interviews ($N=6$). Hence this study discusses only certain trends for illustrating a possible hierarchy for problem solving. As discussed in the Limitations section, larger studies may indicate a greater understanding, now that a conceptual metaphorical system has been established.

One trend among students and teachers was that JOURNEY was the most frequent and popular source domain associated with PROBLEM SOLVING. In all four metrics of Table 8, JOURNEY was the dominant discourse. Other source domains varied between teacher and student. VISUALIZATION and SEARCHING were popular and frequented by students, while their use was less frequent and less popular among teachers. Conversely, DISCOVERY and BUILDING were highly frequented and
popular to teachers, but less valued by the students. PROCESS was in the middle of the pack among students, but in the lower half for teachers. However, among both teachers and students, PARTITIONING was least frequent and popular.

While these are the most apparent source domains, certain trends did exist between them demographically. These trends varied between teacher and student, a specific hierarchy requires further research. Yet two trends were agreed upon by all metrics. JOURNEY is the most common and popular source domain for PROBLEM SOLVING and PARTITIONING was the least popular and frequent among these seven source domains.

While these trends offer insight into the metaphorical system involved, Figure 8 demonstrates that perhaps a hierarchy is too restrictive for a few of the seven source domains. For example, despite VISUALIZATION and SEARCHING staying close together with respect to their demographic in popularity and frequency found in Table 8, the correlative data was not significant \( r(22)=-0.53 \ p=0.107 \) as shown in Appendix L. However, DISCOVERY and BUILDING did demonstrate a strongly significant correlative data \( r(22)=-0.554 \ p<0.01 \). For these reasons, the quantitative data limits conclusions being made about a directed hierarchy for the given conceptual metaphorical system defined within mathematical problem solving.

**Additional Questions That Arose**

The CMT qualitative and quantitative analyses led to some surprising results as described in Chapter 4. A few questions arose from those results that are worth mentioning.
Why was the students’ scores not directly correlated with the frequency of any source domain? One of the surprising results was the lack of relationships between score and source domains. The students’ scores were not significantly related to any of the conceptual metaphor’s source domains. If there had been a significant correlation, one would be able to argue that the correlated source domain was perhaps related to a higher score on the appropriate assessment. This in turn would lead to questioning whether a specific source domain is a more correct way to think about mathematical problem solving. As this did not occur, one can only conclude that there is no ideal metaphor to improve the score for mathematical problem solving. Initially, this was surprising, but upon further review it should be that no source domain dominates over another. The conceptual metaphor is a mapping from difficult notions to experiences (grounded or linking). Ideally, solving mathematical problems should not be accomplished from one experience alone. Otherwise, the philosophy and study of mathematics would be much less interesting. Hence, while no singular metaphorical experience can improve a student’s problem solving ability, the more conceptual metaphors and experiences the student have with mathematics will definitely offer the student more methods to approach mathematics problems. While the research question goes unanswered, one finds solace in the lack of correlation because it validates the perspective that there isn’t a single experiential description for mathematical problem solving.

Why were the teachers’ metaphors different from the students’ metaphors?

Naturally, teachers and students see mathematics differently and thus they will see
mathematical problem solving differently. As shown in Figure 4 and Figure 6, while the students and teachers agreed on the top seven popular and frequent source domains, there were other perceptions of mathematical problem solving that differed between the two. For example, teachers used other less frequent source domains such as TOOLBOX, RULES, HABITS, FAMILIARITY, DOING BUSINESS, and CONFLICT. Contrasting this list, students used a different set of source domains such as APPROXIMATING, CALCULATING, A CONTEST, CONVINCING, and THINKING. Despite not being the most frequent or popular metaphors, seeing these differences can indicate the distinctions between teachers and students when solving problems. Moreover, these distinctions may indicate a different meaning or understanding of problem solving between teachers and students. The use of a TOOLBOX or DOING BUSINESS is not well understood or experienced by teenagers who have not had such life experiences. However, the life experience of a CONTEST or CONVINCING is quite standard for teenagers when dealing with sports or parental arguments. While a complete understanding may never exist, listening to these distinctions can be helpful in constructing scenarios for teacher educator courses.

**Did teacher’s metaphors influence students?** Yes and No. There were examples in the study where certain conceptual metaphors showed no value in the students’ interviews. For example, Beta3’s use of “dissection” as described in the qualitative analysis was not demonstrated by any students. However, there were examples where the teacher’s source domain and use of that source domain was prevalent in their students’ use as well. For example, the discussion in the qualitative analysis with
PROBLEM SOLVING IS DISCOVERY demonstrated nearly all teacher interviews involved the word “realize” as did many of their students interviews in a similar fashion. This does not dictate that the teacher directly planted that metaphor into the students’ minds, but does offer a possibility of this source domain allowing the student to internalize the mathematical reasoning involved in the given problem.

**Does listening to student metaphors improve the quality of teachers?** This is an important question and will be discussed in the Implications for Practice section. This study was only able to conclude the existence of a coherent conceptual metaphorical system, to verify the value in applying its existence is discussed in the Implications for Practice section.

**Do preservice teachers, and teachers in general, share their metaphorical usage?** Currently, little research has been done to indicate that mathematics teachers discuss the metaphors with other mathematics teachers. However, such sharing may occur less formally and is an excellent topic that will be discussed in the Future Studies section.

**Implications**

**Implications for Research**

The goal of this study was to determine the conceptual metaphors students associate with mathematical problem solving. More specifically, the goal was to determine the source domains students mapped from the target domain of problem solving. Historically, cognitive science and the 1989 NCTM Standards suggested that problem solving was a process. This concept of process was one of the source domains
students used when mapping from problem solving, but it was not the only one or the
most dominant discourse used by students. In fact, PROCESS wasn’t the most dominant
discourse used by teachers either. This study’s results demonstrated that JOURNEY was
the most popular and frequented source domain for problem solving from both students
and teachers. More importantly, there were seven source domains that were all
significant to students in frequency and popularity that should be considered.

These seven are significant because they have given the students a consistently
strong voice in how they solve problems. Moreover, it will give research a framework
from which to describe problem solving in a meaningful way that is significant to
students. While there may not be a specific definition given to problem solving that the
research community will agree upon, in American culture these seven source domains
will give researchers a set of fundamental experiences inherent to students’ investigations
into mathematics problems. Naturally, other countries and other cultures may or may not
differ in their fundamental experiences as language has evolved so much from those
experiences. Indeed, a fresh perspective can now be offered to Meno’s Paradox (Plato,
1980), one that describes how students learn to solve new problems from old problems
using the coherent conceptual metaphorical system. It is plausible that students attempt
to solve new mathematical problems by embodying the problem with the previous
mathematical problem-solving experiences they have cognitively mapped onto from the
seven source domains.

Referring back to the foundations of this study, CMT analysis provided
mathematics education with a model (Lesh & Zawojewski, 2007) to interpret the
student’s perceptions and interpretations of problem solving. This model’s purpose is to offer a flexible methodology under which experiences are shared and summarized (similar to IPA) using conceptual metaphors. This methodology has expansive possibilities for research, but it should be made clear that CMT analysis needs to be limited to certain topics. Mathematical Problem Solving was ideal because it has been known to have nebulous definitions yet it is founded in experience. Marcel Danesi (2008) says it best:

The dilemma in education, it would seem, has always lied specifically in the teaching of abstract concepts (not concrete ones). It is precisely in attempting to resolve this dilemma that CMT presents itself as a potentially useful framework because it suggests that abstract concepts can be concretized in exactly the same way as concrete ones such as positive and negative numbers. (p.143)

This concretization mandates a sense of embodiment from which research has developed. However, such concretization is only useful with topics of less tangible understanding. Furthermore, something must be gained practically through the concretization. The mathematics education community finds value in this coherent conceptual metaphorical system because it incorporates current theories to an experiential basis to work with the practitioner.

**Implications for Practice**

The validity of the coherent metaphorical system has many applications for the practitioner. First, this system offers teachers a specific set of experiences to try out with students who are struggling when solving mathematics problems. If a student is
reviewing a past test or quiz with a teacher to prepare for the next large assessment, many times the teacher dictates how to solve the problem instead of explaining how they knew how to solve the problem. Many times, the teacher is at a disadvantage because they are unable to explain their development of the solution. Instead, teachers regularly try to indicate what hints or memorized nuances a student may take to identify similar problems. Even worse, teachers could say, “I just see it”. This is by far the worst response because it reinforces the elitism of mathematical reasoning that is so pervasive in the educational system.

The seven source domains identified above give the teacher seven dialects to explain to the student, through shared experiences, how the teacher knew how to solve a mathematics problem. Moreover, as Table 8 offers a suggested order in which to use these seven source domains. For example, a teacher may start with the JOURNEY source domain by saying,

When I read the problem, I was just as lost as you were, but to help me find my way, I slowed down and tried to get my bearings by determining what would solve the problem, where I needed to go.

In this manner, the imagery (Reynolds, 1993) and experiences of JOURNEY are shared and can, if nothing else, offer a language from which the student is familiar.

Fundamentally, at the core of applying this metaphorical system is improving the teacher’s ability to listen. Listening is at the heart of CMT analysis, yet listening needs to be learned. In this form, training teachers to use CMT analysis in real time will help them go beyond the seven source domains to narrow in on student learning. Specifically,
if a teacher is aware of the conceptual metaphors being used by the students, such as PROBLEM SOLVING IS A RACE, they can use the situation by stating, “Jessica seems to be ahead of Johnny in understanding the problem. What could Jessica say to help Johnny catch up?” or diffuse the situation by bluntly saying, “Jessica, I don’t want to think of solving problems as a race, while you may have to move fast on tests, speed does not indicate your ability to be thorough or that you understand the material.” To understand the student’s perception of problem solving, the teacher needs to actively listen. CMT analysis is an excellent means to train a teacher to listen to the student’s model of problem solving, rather than imposing their own.

**Future Studies**

There are three directions one can take with the results of this study. First, as mentioned in previous sections, a large study can be applied using secondary education students while varying a few parameters. This study’s results can be bolstered by applying CMT Analysis with problem solving to multiple schools, multiple grade levels, or multiple mathematics classrooms. If multiple schools were used, grouping students by school may show cultural relationships in the source domains used to describe the target domain of mathematical problem solving. Additionally, with a larger sample, grouping students by gender may offer insight into how the two genders perceive problem solving similarly and differently. If multiple grade levels were used (middle school and high school), it may be better to remain with the same school district so as to limit the changes in curriculum designs and assessment metrics. If multiple classrooms were used (within a certain course such as Algebra 2) the teachers’ source domains may be more
comparable as the students may be influenced by the teachers’ metaphors. A common rubric or at least a common assessment would be a suggested frame of reference with multiple classrooms. A common assessment is valuable because it has the teachers agree upon a language and the topic of importance from which they are assessing. As was done in this study, it allows the research to look for similarities or differences in the teacher’s teaching style. To this end, it is recommended that the researcher also interview the teachers and interpret their conceptual metaphor system. In all of these cases, it would better to have a sample size for each group of students above 25 ($N>25$). To this end, it is recommended that future research initially aim for 35 student interviews so that the research compensates for the mortality rate.

Secondly, future studies can change the focus to teachers. This includes current teachers in the classroom, but addressing pre-service teachers is valuable in improving their skill at listening to students so that when they begin their career, the student’s voice is not lost within the course’s overwhelming curriculum. To this end, modifying future studies so that the participants are current and pre-service teachers would be incredibly valuable and is the goal of the researcher at this time. As this study showed, the teacher’s conceptual metaphors do influence the student’s conceptual metaphors with respect to mathematical problem solving. Thus it is important to understand what conceptual metaphors teachers are bringing into the classroom which describe their perception of mathematics and mathematical problem solving. A future study could involve teachers (current and preservice) solving mathematics problems and describing their perceptions to the researcher, or the teachers could describe a problem they plan on giving students
for assessment and have the teacher describe how they would expect the student to solve the problem. In a similar vein, future studies could identify metaphors teachers use to solve mathematics problems and metaphors they use as a communicative tool to teach mathematics problems. Perhaps a significant difference exists between the metaphors to teach problem solving and the metaphors used to solve problems from the teacher’s perspective. Either way, this study’s results could gain more traction with a better understanding of the teacher’s conceptual metaphors.

Third, in a more horizontal approach, CMT analysis could be used to better understand student’s and teacher’s experiences with other vague mathematics educational concepts. Specifically, CMT analysis could help to determine if a metaphorical system existed for target domains other than problem solving. Five popular topics in mathematics education that are of significant value to experientially define would be:

- GENERALIZATION
- ABSTRACTION
- REFLECTION
- PROOF
- RIGOR

Currently, mathematics education uses these words as a means to describe important aspects of mathematics that are important for students to value. However, their definitions vary significantly between researchers, teachers, and students. To develop an experiential basis, as set of source domains for each word that may lead to a conceptual
metaphor system, could be beneficial to the professionalism of the field of mathematics education.

Finally, this study limited the interviews to being isolated so that the individual student’s voice was clearly heard. After more studies have been completed using CMT analysis, it may be possible to apply the technique to a more social setting where multiple students interact and conceptual metaphors are formed by groups rather than individuals. These conceptual metaphors may originate from an individual but manifest themselves differently once the group redefines their purpose. However, this social mediation would require a significant modification to the theoretical framework as the conceptual metaphors are more socially constructed in real time.

**Summary**

This study began by searching for the student’s hermeneutic understanding of problem solving. To persevere in this endeavor, the diversity of problem-solving research was difficult to sort through to find the student’s voice. A cognitive model (Lesh & Harel, 2003) was desired that would succinctly interpret students’ experiences with problem solving so as to determine how educators can relate to those experiences. Linguistics Conceptual Metaphor Theory (CMT) offered such a cognitive model, but its methodology was unclear. To this end, Danesi’s (2007), Lakoff’s and Nunez’s (2000), and Kovecses’ and Bencses’ (2010) work encouraged an application of CMT analysis that could generate a system of conceptual metaphors that would succinctly describe how students share their problem solving experiences.
Methodologically, there was a need for a mixed methods study with qualitative and quantitative analysis. The qualitative analysis required the application of CMT analysis with a semi-structured interview implementing Interpretative Phenomenological Analysis (IPA). Through IPA, the conceptual metaphors of both students and teachers were identified along with their cognitive function, experiential function, and specific details that corroborated the chosen source and target domains. Frequency and popularity of the source domains associated with problem solving were then counted so as to run quantitative analyses. The result of these quantitative studies was a list of source domains mapped from the target domain of mathematical problem solving that were relevant to students and teachers. Surprisingly, the highest seven source domains were in accord with students and teachers in frequency and popularity and are presented on Table 8. Additionally, a quantitative correlation matrix was run to determine how certain source domains related to each other in terms of frequency. To this end, the qualitative and quantitative data gave us empirical evidence that a coherent, conceptual metaphorical system exists.

The Conceptual Metaphor System Conjecture

This study’s results offer an inductive argument to the existence of a culturally-accepted, coherent, conceptual metaphorical system as relevant to teacher education. As existence of the coherent conceptual metaphorical system has been shown, a deductive argument for its practical application may now be given:

If a coherent system of conceptual metaphors exists in American culture for mathematical problem solving, then it is beneficial to teachers pedagogically and
mathematically to understand coherent systems of conceptual metaphors rather than accepting the literal and idiosyncratic metaphors within their classroom.

**The Deductive Argument**

The conjecture will be deductively argued by contradiction. The condition of the conjecture is that a coherent conceptual system exists. Assume the conjecture’s conclusion is false. Assume that the acceptance of a diverse group of literal and conceptual metaphors (including idiosyncratic) for mathematical problem solving is as beneficial to teachers as a coherent conceptual metaphor system. By definition, a coherent system for the individual is a system under which their experience, as demonstrated through language, holds meaning and is in equilibrium (Piaget, 1970) with the individual’s conceptual understanding.

If it is assumed that the teacher is to teach with the coherent and idiosyncratic metaphors embodied in the mathematics curriculum to students, then the student will interpret each metaphor and acquire only those that are coherent to their system. Any metaphors that cause disequilibrium (Piaget, 1970) may be dealt with by the student on the individual level intrinsically. However, as the teacher is demonstrating all such metaphors relevant to the topic at hand (coherent and idiosyncratic), the teacher pedagogically lacks the time or purpose to demonstrate the overlap and influence of one metaphor over another because it was assumed the conjecture was false (it is not more beneficial to teach a coherent conceptual metaphor system). Hence the development of the student's coherent system (which exists by assumption) is done individually and is not directly understood by the teacher. To this end the teacher is unaware if the student's
problem solving strategy aligns with a culturally coherent system. At this point, there are two arguments that yield a contradiction. One is logically weaker than another.

**Weaker Argument of Falsity**

The purpose of the teacher is to express, demonstrate, and challenge the cultural metaphors (although not necessarily conceptual, hence the weaker argument) for the purpose of student learning. If these metaphors are not coherent for the classroom as a whole, the teacher is not able to assess their use without the application of his/her own coherent conceptual metaphorical system. However, the teacher’s conceptual metaphorical system is not more valuable than the idiosyncratic and literal conceptual metaphors by assumption, so it cannot be used for assessment. Contradiction.

**Stronger Argument of Falsity**

As mathematics is axiomatic and deductive by nature, future mathematical ideas are sequentially developed using logic and problem-solving strategies. If the metaphors for mathematical problem solving are not socially or culturally coherent, the order in which such mathematical ideas are developed are subject to the whims of the individual who is developing them. Such whims may not constitute consistent conclusions to mathematical problems. Two contradictions hold at this point. First, from a practical standpoint, it is impossible for a teacher to teach every (let alone two) sequential structure for mathematical ideas that develop from each other in different orders due to realistic time limits. Second, even if the teacher could teach all idiosyncratic mathematical problem-solving techniques, their judicial assessment (which also would be incredibly long) is still based upon their coherent system and is biased. Contradiction.
This conjecture indicates the value of teachers learning how to identify coherent conceptual metaphors. Both arguments of falsity are resolved through the necessity of teacher assessment. Thus, due to the need for assessment from the teacher, understanding coherent systems of conceptual metaphors is valuable to teachers.

Research

In truth, the latter contradiction regularly arises as is demonstrated by Stanley Erlwanger's (2004) Benny. In Erlwanger’s infamous study, a student named Benny is interviewed in 1973 in an Individually Prescribed Instruction (IPI) Plan as part of the “Back to Basics” movement. The design of this instruction program is to have the students take individual assessments at their chosen pace after they studied for mastery. The students took a multiple choice test and were able to move on to the next section when they scored a certain percentage (mastery). Benny had shown mastery on many assessments, yet when Erlwanger interviewed him, his mathematical understanding of fractions was inconsistent and mathematically incorrect. Benny had developed his own perception of mathematics from the assessments. He had created his own method from his experiences with reviewing the test and making corrections. For example, Benny stated:

\[ 0.3 + 0.4 = 0.07 \text{ and that } \frac{3}{10} + \frac{4}{10} = 0.7 \]

Benny never saw a conflict because his method worked with the specific multiple choice questions from the assessments. As Benny states, “It’s like a wild goose chase” (p. 53). Benny constructed an entire rule-based system for fractions, along with exceptions to the rules, as well as how to manipulate them. In this manner, Benny had taken the teacher’s
assessment and used it to create rules that defined mathematics for himself. Benny created his own coherent mathematical problem-solving system.

Teachers regularly assess relative to their own coherent mathematical problem-solving system (which is laden with their own conceptual metaphors of what defines mathematical problems solving). This causes distress within a teacher when a student, such as Benny, derives mathematical ideas that are not in the coherent system of the teacher. This research on students’ conceptual metaphors for mathematical problem solving has demonstrated the conditions necessary for the Conceptual Metaphor System Conjecture, the existence of coherent conceptual metaphorical systems. Hence the results of the conjecture hold. It is beneficial to teachers to understand what coherent system is being used in their classroom so as to improve the students’ understanding of how and why humans solve mathematical problems.

Finally, this research demonstrates existence of a coherent metaphorical system for mathematical problem solving so as to improve the field of mathematics education. It is vital to the educational community that the theorem’s condition is only existence of conceptual metaphor systems. For this group of honors geometry students, the research demonstrated the existence of a coherent metaphorical system, not uniqueness. Regularly in mathematics, once existence has been confirmed, uniqueness (given additional conditions) is pursued. However, in mathematics education, one cannot apply a positivist perspective to search for a unique system. Each student and each teacher has their own system defined by their experiences. This research is not to be used to achieve an ideal metaphorical system for problem solving. Instead this research is looking for coherence.
In Joseph Schwab’s (1970) classic, “The practical: A language for curriculum”, Schwab distinguishes between theoretic inquiry and practical inquiry. Theoretic inquiry’s purpose is comprehension while practical inquiry’s purpose is making a decision in a particular situation. Despite their differing purposes, the nexus of their overlapping existence in curriculum is what Lakoff and Johnson (2003) identified as coherence. Rather than looking for an absolute metaphorical system or excepting the post-modern perspective that no two systems are identical, coherence desires a system which will be close enough theoretically to be practically beneficial. To this end, this research study should not be referenced to demonstrate the uniqueness of a metaphorical system (an absolute), but rather an existence. The Conceptual Metaphor System Theorem then states that teachers will find value in identifying coherent conceptual metaphor systems, not a singular coherent conceptual metaphor system. The system identified by this research study is only a starting point, a system from which to orient one’s current comprehension of student metaphors, not an absolute. This research will improve mathematics education not by identifying an absolute set of experiences, but rather a flexible coherent metaphor system to be modified by teachers in their classrooms.

In its most basic pedagogical form, conceptual metaphor theory educates teachers on how to listen to experiences. These experiences are the practical foundations of multiple curriculum theories; from radical constructivism (Glasersfeld, 1991) to goal-oriented decision making (Schoenfeld, 2011). So much research depends on the validity of the researcher’s interpretation of the data. Conceptual metaphor theory offers a methodology to help researchers and practitioners navigate the interpretive medium
derived from student experiences, while educating researchers and practitioners on the value of listening.
APPENDICES
APPENDIX A

PILOT STUDY RAW DATA
## Appendix A

### Pilot Study Raw Data

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PILOT STUDY DESCRIPTIVE STATISTICS AND MANOVA
## Appendix B

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PILOT STUDY CORRELATION DATA
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**Pilot Study Correlation Data**

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APPENDIX E

DISSERTATION TEACHER RAW DATA
## Appendix E

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APPENDIX F

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APPENDIX H

DISSERTATION TEACHER-GROUPED DESCRIPTIVE STATISTICS
### Appendix H

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APPENDIX I

DISSERTATION TEACHER-GROUPED ANOVA T-TEST
## Appendix I

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For each learner’s test, the table shows the t-statistic (t), degrees of freedom (df), and the lower and upper 95% confidence interval for the difference in means. The t-test for equality of means is used to compare the means of two independent groups. The test statistic (t) and the corresponding p-value are reported, along with the 95% confidence interval for the difference in means. The null hypothesis is that the two groups have equal means, and the alternative hypothesis is that they differ. A significant p-value (less than 0.05) suggests that the null hypothesis can be rejected, indicating a significant difference in means between the groups.
APPENDIX J

DISSERTATION GENDER-GROUPED DESCRIPTIVE STATISTICS
## Appendix J

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APPENDIX K

DISSERTATION GENDER GROUPED ANOVA T-TEST
### Appendix K

#### Dissertation Gender Grouped Anova T-Test

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APPENDIX L

DISSERTATION CORRELATION MATRIX
Appendix L

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Pearson Correlation

Sig. (2-tailed)
APPENDIX M

PRINCIPAL CONSENT FOR RESEARCH STUDY
Appendix M

Principal Consent for Research Study

**Study Title:** Student’s Metaphors for Mathematical Problem Solving

**Principal Investigator:** Sean Yee

You are being asked to allow a research study to occur inside your school. This consent form will provide you with information on the research project and the associated risks and benefits of the research. Your participation is voluntary. Please read this form carefully. It is important that you ask questions and fully understand the research in order to make an informed decision. You will receive a copy of this document to take with you.

**Purpose**
The purpose of this study is to understand the student’s perception of mathematical problem solving while minimizing the researcher’s interpretation. Accomplishing this feat requires a significant focus on the language used by the student, the teacher, and the researcher. All three dialects of problem solving must be clearly understood and expressed by the researcher. As such, linguistic interdisciplinarity becomes mandatory in a study whose focus is on students’ perceptions of mathematical problem solving. This study will use the linguistic idiom of metaphors to interpret the student’s, teacher’s and researcher’s perception of mathematical problem solving.

**Procedures**
This study will be interviewing teachers and students by video outside of class. The participants will be the two honors geometry teachers and their students who volunteer for the study. Before a common assessment, teachers will be interviewed by video individually with the researcher for 15 minutes. From the common assessment, one question will be studied that requires the student to express their problem-solving techniques. The researcher will be asking the following questions and focusing on metaphors used by the teachers. With respect to the one common assessment question, the teacher will be asked:

1. How will students perceive the mathematical problem?
2. Why do you think students will perceive the problem the way you have described it?
3. How will students solve the problem?
4. Why will students solve the problem that way?
5. Could students solve the problem differently?
6. What possible problem-solving techniques do you see them using to solve this problem?

The teacher will give the assessment, grade the assessment, and filter out the students who did not volunteer for the study. Of the remaining students who did volunteer for the study, the researcher will randomize and choose three to six tests.
These students will then be interviewed individually by the researcher outside of their classroom via video for 15 minutes. The researcher will be asking the following questions and focusing on metaphors used by the students to describe their problem-solving techniques:

1. How did you perceive the mathematical problem?
2. What does it mean to solve the problem?
3. How did you solve the problem?
4. Why did you solve the problem that way?
5. Could you have solved the problem differently?

The data will then be transcribed and analyzed by the researcher using conceptual metaphor theory (CMT). Patterns and similar metaphors and images will be recorded and analyzed using CMT. This process will be repeated three times using different assessments so that three questions from three different assessments are studied with a total number of students involved is between 18 and 36. There will be no follow-up interviews unless the same students are chosen randomly for a second test question. As the question is different, this will be considered an independent interview. This process will take place from August until November.

**Audio and Video Recording and Photography**

Students and teachers will be recorded using a hand-held digital video recorder. Students and teachers will give permission to be video recorded when volunteering to be part of the study. The videos will be kept confidentially by the researcher, although permission of the student and teacher to use the videos anonymously in conference presentations will be requested.

**Benefits**

This study will most likely be beneficial to the student participants because they will have the opportunity to reflect and describe their problem-solving techniques which will benefit them in future assessments and courses. Similarly, this has the same potential to benefit teachers because they will be given the opportunity to describe multiple ways in which students could solve the problem and thus be open to better solutions and more ways in which the test could be rewritten.

In essence, this study is significant because it will give students a voice in the research models of mathematical problem solving. It offers researchers a chance to improve mathematics education at its base, the student. It will allow teachers to see mathematical problem solving through the student’s lens rather than hypothesizing how students should perceive problem solving. This will then allow a dialogue between how the teacher perceives problem solving and how the student perceives problem solving. Such perceptions and application through language will give students and teachers alike the opportunity to improve their problem-solving skills through open dialogue.

**Risks and Discomforts**

There are no anticipated risks beyond those encountered in everyday life.
Privacy and Confidentiality
The data from this study will be kept confidential. Both the students and the teachers will be given a form similar to this describing the entire process. Moreover, their consent indicates they understand that the researcher will do everything possible to keep all data confidential. The digital files of the student’s and teacher’s interviews along with all transcribed data will be coded and kept confidentially in a secure location. No identifying information will be collected. The student’s and teacher’s signed consent form will be kept separate from their study data, and responses will not be linked back to them. Identifying information will not be included in the data that you provide.

The study related information will be kept confidential within the limits of the law. Any identifying information will be kept in a secure location and only the researchers will have access to the data. Research participants will not be identified in any publication or presentation of research results; only aggregate data will be used. Participants research information may, in certain circumstances, be disclosed to the Institutional Review Board (IRB), which oversees research at Kent State University, or to certain federal agencies. Confidentiality may not be maintained if they indicate that they may do harm to yourself or others.

Compensation
No compensation will be given as this is a voluntary study.

Voluntary Participation
Taking part in this research study is entirely voluntary. Teachers and students may choose not to participate or may discontinue participation at any time without penalty or loss of benefits to which they are otherwise entitled. Teachers and students will be informed of any new, relevant information that may affect your health, welfare, or willingness to continue study participation.

Contact Information
If you have any questions or concerns about this research, you may contact Sean Yee at 330-672-7031. This project has been approved by the Kent State University Institutional Review Board. If you have any questions about your rights as a research participant or complaints about the research, you may call the IRB at 330.672.2704.

Consent Statement and Signature
I have read this consent form and have had the opportunity to have my questions answered to my satisfaction. I voluntarily agree to participate in this study and allow the study to occur at Solon High School. I understand that a copy of this consent will be provided to me for future reference.

________________________________  _____________________  
Participant Signature     Date
APPENDIX N

TEACHER CONSENT DOCUMENT FOR RESEARCH STUDY
Appendix N

Teacher Consent Document for Research Study

**Study Title:** Student’s Metaphors for Mathematical Problem Solving  
**Principal Investigator:** Sean Yee

You are being asked to participate in a research study. This consent form will provide you with information on the research project and the associated risks and benefits of the research. Your participation is voluntary. Please read this form carefully. It is important that you ask questions and fully understand the research in order to make an informed decision. You will receive a copy of this document to take with you.

**Purpose**  
The purpose of this study is to understand the student’s perception of mathematical problem solving while minimizing the researcher’s interpretation. Accomplishing this feat requires a significant focus on the language used by the student, the teacher, and the researcher. All three dialects of problem solving must be clearly understood and expressed by the researcher. As such, linguistic interdisciplinarity becomes mandatory in a study whose focus is on students’ perceptions of mathematical problem solving. This study will use the linguistic idiom of metaphors to interpret the student’s, teacher’s, and researcher’s perception of mathematical problem solving.

**Procedures**  
This study will be interviewing teachers and students by video outside of class. The participants will be the two honors geometry teachers and their students who volunteer for the study. Before a common assessment, teachers will be interviewed by video individually with the researcher for 15 minutes. From the common assessment, one question will be studied that requires the student to express their problem-solving techniques. The researcher will be asking the following questions and focusing on metaphors used by the teachers. With respect to the one common assessment question, the teacher will be asked:

1. How will students perceive the mathematical problem?  
2. Why do you think students will perceive the problem the way you have described it?  
3. How will students solve the problem?  
4. Why will students solve the problem that way?  
5. Could students solve the problem differently?  
6. What possible problem-solving techniques do you see them using to solve this problem?

The teacher will give the assessment, grade the assessment, and filter out the students who did not volunteer for the study. Of the remaining students who did volunteer for the study, the researcher will randomize and choose three to six tests.
These students will then be interviewed individually by the researcher outside of their classroom via video for 15 minutes. The researcher will be asking the following questions and focusing on metaphors used by the students to describe their problem-solving techniques:

1. How did you perceive the mathematical problem?
2. What does it mean to solve the problem?
3. How did you solve the problem?
4. Why did you solve the problem that way?
5. Could you have solved the problem differently?

The data will then be transcribed and analyzed by the researcher using conceptual metaphor theory (CMT). Patterns and similar metaphors and images will be recorded and analyzed using CMT. This process will be repeated three times using different assessments so that three questions from three different assessments are studied with a total number of students involved is between 18 and 36. There will be no follow-up interviews unless the same students are chosen randomly for a second test question. As the question is different, this will be considered an independent interview. This process will take place from August until November.

**Audio and Video Recording and Photography**

Students and teachers will be recorded using a hand-held digital video recorder. Students and teachers will give permission to be video recorded when volunteering to be part of the study. The videos will be kept confidentially by the researcher, although permission of the student and teacher to use the videos anonymously in conference presentations will be requested.

**Benefits**

This study will most likely be beneficial to the student participants because they will have the opportunity to reflect and describe their problem-solving techniques which will benefit them in future assessments and courses. Similarly, this will benefit teachers because they will be given the opportunity to describe multiple ways in which students could solve the problem and thus be open to better solutions and more ways in which the test could be rewritten.

In essence, this study is significant because it will give students a voice in the research models of mathematical problem solving. It offers researchers a chance to improve mathematics education at its base, the student. It will allow teachers to see mathematical problem solving through the student’s lens rather than hypothesizing how students should perceive problem solving. This will then allow a dialogue between how the teacher perceives problem solving and how the student perceives problem solving. Such perceptions and application through language will give students and teachers alike the opportunity to improve their problem-solving skills through open dialogue.

**Risks and Discomforts**

There are no anticipated risks beyond those encountered in everyday life.
Privacy and Confidentiality
The data from this study will be kept confidential. The digital files of the student’s and teacher’s interviews along with all transcribed data will be coded kept confidentially in a secure location. No identifying information will be collected. The student’s and teacher’s signed consent form will be kept securely separate from the study’s data, and responses will not be linked back to them. Identifying information will not be included in the data that you provide.

The study related information will be kept confidential within the limits of the law. Any identifying information will be kept in a secure location and only the researchers will have access to the data. Research participants will not be identified in any publication or presentation of research results; only aggregate data will be used. No video or audio information will be shared unless the participants give permission to use any video or audio recordings in research presentations (via the video consent form). Even with permission, the participants will be referred to only by their coded name. Participants research information may, in certain circumstances, be disclosed to the Institutional Review Board (IRB), which oversees research at Kent State University, or to certain federal agencies. Confidentiality may not be maintained if they indicate that they may do harm to yourself or others.

Compensation
Currently no compensation will be given to teachers as this is a voluntary study. However, with the approval of grants that are still being processed, teachers may be financially compensated at their hourly rate of the school district for the amount of time they volunteer to the study.

Voluntary Participation
Taking part in this research study is entirely voluntary. Teachers and students may choose not to participate or may discontinue participation at any time without penalty or loss of benefits to which they are otherwise entitled. Teachers and students will be informed of any new, relevant information that may affect your health, welfare, or willingness to continue study participation.

Contact Information
If you have any questions or concerns about this research, you may contact Sean Yee at 330-672-7031. This project has been approved by the Kent State University Institutional Review Board. If you have any questions about your rights as a research participant or complaints about the research, you may call the IRB at 330.672.2704.

Consent Statement and Signature
I have read this consent form and have had the opportunity to have my questions answered to my satisfaction. I voluntarily agree to participate in this study and allow the
study to occur at Solon High School. I understand that a copy of this consent will be provided to me for future reference.

Participant Signature     Date
APPENDIX O

STUDENT CONSENT FORM
Appendix O

Student Consent Form

Study Title: Student’s Metaphors for Mathematical Problem Solving
Principal Investigator: Sean Yee

You are being invited to participate in a research study. This consent form will provide you with information on the research project, what you will need to do, and the associated risks and benefits of the research. Your participation is voluntary. Please read this form carefully. It is important that you ask questions and fully understand the research in order to make an informed decision. You will receive a copy of this document to take with you.

Purpose:
The purpose of this study is to understand the student’s perception of mathematical problem solving.

Procedures:
This study will be interviewing you, the student, by video outside of class. All students in honors geometry were offered this opportunity. After one of your common assessments in honors geometry (tests from each chapter), the researcher will choose one question from the test that reveals certain problem solving techniques. The researcher will then ask you a few questions about the chosen problem and how you solved it. This will take no more than 10-15 minutes and will be done outside of your classroom. You will have already received your test back so that you will know how you performed relative to your teacher’s expectations. In no way do you need to prepare for these interviews. You will have a copy of your test at the interview as well as other tools you had during the test such as a calculator.

It is important to know that even though you volunteer, you may not be chosen to participate. The researcher will be choosing students randomly so there is no guarantee that you will be in this study. If you are chosen to participate in the study, you will find out in your honors geometry class after the assessment (chapter test). The researcher will meet with you when it is most convenient (most likely before or after school). Again, the researcher will be using metaphors to help understand your problem-solving techniques to help teachers improve how problem solving is taught. This study will involve three common assessments and will only run at the beginning of the school year. There will be no follow-up interviews, but you may be chosen randomly for the second or third common assessment. As each common assessment is different, a second interview will be considered an independent interview. This process will take place from August until November. All data will be coded and will be kept confidential as described below.
Audio and Video Recording and Photography
The interview will take place using a hand-held digital video recorder. A video consent form will be given if you are chosen to participate. You will be given the opportunity to sign the video consent form before or during the interview to give permission for the video to be used in the research study. The videos will be kept confidentially by the researcher and your identity will be coded within the research so that your real name and personal information is not used.

Benefits
The potential benefits of participating in this study include the opportunity to reflect and describe their problem-solving techniques which will benefit them in future assessments and courses. Specifically, by reviewing the chosen problem on the test and reviewing their problem-solving techniques, it is likely that you will discover how and why your mind chose to solve the problem that way. Moreover, you may be able to define categories of mathematical problem solving that may help you for future mathematical problems.

In essence, this study is significant because it will give students a voice in the research models of mathematical problem solving. It offers researchers a chance to improve mathematics education at its base, the student. It will allow teachers to see mathematical problem solving through the student’s lens rather than hypothesizing how students should perceive problem solving. This will then allow a dialogue between how the teacher perceives problem solving and how the student perceives problem solving. Such perceptions and application through language will give students and teachers alike the opportunity to improve their problem-solving skills through open dialogue.

Risks and Discomforts
There are no anticipated risks beyond those encountered in everyday life.

Privacy and Confidentiality
The data from this study will be kept confidential. The digital files of the student’s and teacher’s interviews along with all transcribed data will be coded kept confidentially in a secure location. No identifying information will be collected. The student’s and teacher’s signed consent form will be kept securely separate from the study’s data, and responses will not be linked back to them. Identifying information will not be included in the data that you provide.

The study related information will be kept confidential within the limits of the law. Any identifying information will be kept in a secure location and only the researchers will have access to the data. Research participants will not be identified in any publication or presentation of research results; only aggregate data will be used. No video or audio information will be shared unless the participants give permission to use any video or audio recordings in research presentations (via the video consent form). Even with permission, the participants will be referred to only by their coded name. Participants research information may, in certain circumstances, be disclosed to the
Institutional Review Board (IRB), which oversees research at Kent State University, or to certain federal agencies. Confidentiality may not be maintained if they indicate that they may do harm to yourself or others.

**Compensation**
There is no financial or grade compensation.

**Voluntary Participation**
Taking part in this research study is entirely up to you. You may choose not to participate or you may discontinue your participation at any time without penalty or loss of benefits to which you are otherwise entitled. You will be informed of any new, relevant information that may affect your health, welfare, or willingness to continue your study participation.

**Contact Information**
If you have any questions or concerns about this research, you may contact Sean Yee at 330-672-7031. This project has been approved by the Kent State University Institutional Review Board. If you have any questions about your rights as a research participant or complaints about the research, you may call the IRB at 330.672.2704.

**Consent Statement and Signature**
I have read this consent form and have had the opportunity to have my questions answered to my satisfaction. I voluntarily agree to participate in this study. I understand that a copy of this consent will be provided to me for future reference.

________________________________  _____________________
Participant Signature     Date

**Consent Statement and Signature of Guardian**
I am the parent/legal guardian of this child and I have read the consent form. I believe that the participants listed above have been fully informed, understand the project and what they will have to do, and have voluntarily agreed to participate. Moreover, I give my permission for:

___________________________________________ to participate in this study.

Name of Participant

________________________________  _____________________
Parent or Guardian’s Signature     Date
APPENDIX P

STATEMENT TO READ BEFORE INTERVIEW FOR DISSERTATION
Appendix P

Statement to Read Before Interview for Dissertation

Thank you for being a willing and voluntary participant to my study. There are 4 important points before we get started.

1. This is a completely voluntary study. In no way will your performance be judged. In no way will this affect any relationship you have with any teacher in any classroom. No teacher, other than the researcher, will be privy to the data.

2. This process involves a video interview over a single problem from one of your recent honors geometry assessments. I will begin by giving you a few minutes to remind yourself of the problem. I will then turn on the camera, read the problem out loud, and ask you a few questions about the problem. It is very important that you are not afraid to communicate your thoughts. You do not need to have a complete answer, nor do you need to have a plan. Please verbalize all of your thoughts to the best of your ability as we work together. You may use all manipulatives available from the test. This interview should last between 10 and 15 minutes but can be cut shorter or extended if necessary. I may pose questions or ask for clarification. However, this is not to suggest how to solve the problem, but rather better understand how you think.

3. It is vital to this study that you do not discuss, share, or work on the questions with anyone as this study will take a month to complete and relies on genuine responses. You will sign this form at the end of our session agreeing to not share questions discussed in this interview with other students.

4. You may be called upon again if you are randomly selected to participate again. The format of that interview will be very similar to this one.

If you accept all of these conditions, we will continue. Otherwise, you may quit the study now with no consequences. DO YOU WISH TO CONTINUE?

___Yes  ___No     Signature ___________________________ Date _________________
APPENDIX Q

STUDENT AUDIOTAPE/VIDEO CONSENT FORM
Appendix Q

Student Audiotape/Video Consent Form

STUDENT’S METPAHORS FOR MATHEMATICAL PROBLEM SOLVING
SEAN YEE

I agree to participate in an audio-taped/video taped interview about mathematical problem solving as part of this project and for the purposes of data analysis. I agree that Sean Yee may audio-tape/video tape this interview. The date, time and place of the interview will be mutually agreed upon.

__________________________________________
Signature Date

I have been told that I have the right to listen to the recording of the interview before it is used. I have decided that I:

_____ want to listen to the recording    _____ do not want to listen to the recording

Sign now below if you do not want to listen to the recording. If you want to listen to the recording, you will be asked to sign after listening to them.

Sean Yee may / may not (circle one) use the audio-tapes/video tapes made of me. The original tapes or copies may be used for:

_____ this research project _____ publication _____ presentation at professional meetings

__________________________________________
Signature Date

Address:
Appendix R

Teacher Audiotape/Video Consent Form

STUDENT’S METAPHORS FOR MATHEMATICAL PROBLEM SOLVING
SEAN YEE

I agree to participate in an audio-taped/video taped interview about mathematical problem solving as part of this project and for the purposes of data analysis. I agree that Sean Yee may audio-tape/video tape this interview. The date, time and place of the interview will be mutually agreed upon.

____________________________________________________________________________
Signature      Date

I have been told that I have the right to listen to the recording of the interview before it is used. I have decided that I:

_____ want to listen to the recording
_____ do not want to listen to the recording

Sign now below if you do not want to listen to the recording. If you want to listen to the recording, you will be asked to sign after listening to them.

Sean Yee may / may not (circle one) use the audio-tapes/video tapes made of me. The original tapes or copies may be used for:

_____this research project _____publication _____presentation at professional meetings

____________________________________________________________________________
Signature      Date

Address:
REFERENCES
REFERENCES


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http://cognitivesciencesociety.org/index.html


Cambridge University Press.


Erlbaum Associates.


