DETERMINING GROUP STRUCTURE FROM THE SETS OF CHARACTER DEGREES

A dissertation submitted to Kent State University in partial fulfillment of the requirements for the degree of Doctor of Philosophy

by

Kamal Azizihers

May 2011
To the Loving Memory of My Father,

Mohammad Aziziheris
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ACKNOWLEDGEMENTS

This dissertation is due to many whom I owe a huge debt of gratitude. I would especially like to thank the following individuals for their support, encouragement, and inspiration along this long and often difficult journey.

First and foremost, I offer thanks to my advisor Prof. Mark Lewis. This dissertation would not have been possible without your countless hours of advice and support. Thank you for modeling the actions and behaviors of an accomplished mathematician, an excellent teacher, and a remarkable person. I have learned so much from you during my career at Kent State and strive to become such a successful mathematician and person. Thank you for your continued support. The support of the “algebra group” at Kent State has also been amazing. I would like to state a special thank you to Prof. Steven Gagola and Donald White for their advice and encouragement. I further offer my gratitude to the Department of Mathematical Sciences at Kent State University.

The supportive environment that it offers to its graduate students is remarkable. I would especially like to thank Misty Sommers-Tackett and Virginia Wright for always being available to answer questions and help in any way. I also extend my thanks to Prof. Andrew Tonge, Prof. Artem Zvavitch, Prof. Austin Melton, Dr. Mahbobeh Vezvaei, and Dr. Beverly Reed for their support and encouragement. I must also thank Dr. Thomas Wakefield, Dr. Carrie Dugan, Terence Hanchin, and Nabil Mlaiki for assisting me through the early years, for giving me the confidence that I could survive the later years, and for continuing to be such wonderful friends. I have been inspired and motivated by many incredible professors during my college career at the University of Tabriz. In particular, I would like to thank Professor A. A. Mehrvarz, who introduced me to the beauty of
mathematics and still challenges me to always do my best. I could not have begun such journey without the limitless love and support of my family.

Unfortunately, during my second year of my graduate studies at Kent State University, my father was diagnosed with a terminal illness. I returned home to take care of my father and share his last months. After one year, my father passed away and left me alone. Dad, I really miss you, I love you “Ajan”, God bless you, and I hope to have lived up to your expectations. Dad and Mom, thank you for your support, encouragement, and love. Grandpa and Grandma, God bless you, thank you for the lessons on how to live and love; and for believing in me through all my challenges. My sisters and brothers, Sakineh, Fahimeh, Jalal, and Ata, thanks for believing in me, and your continued reminder that I am not alone. I could always count on you for a good laugh. My uncle, Ali “Dayi”, thank you for teaching me to have a big heart. Vahid Sharifi, Mohammad Sobhani, Mohammad Assar and Mahbobeh Vezvaei, my close friends in Cleveland, thank you for your help and advice. This dissertation would never have been completed without the support of such an amazing family.

Finally, I thank my wonderful wife Zohreh who tolerated me through the most difficult parts of this journey. Our path through graduate school has been long and winding, but it is almost complete! Your limitless patience, support, and willingness to sacrifice have made all the difference.
INTRODUCTION

Throughout this dissertation, $G$ will denote a finite group. A group $G$ is solvable if there exist normal subgroups $N_i$ such that

$$1 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_r = G,$$

where $N_i/N_{i-1}$ is abelian for $1 \leq i \leq r$.

The commutator subgroup (or derived subgroup) of an arbitrary group $G$ is the subgroup $G'$ generated by all commutators $[x, y] = x^{-1}y^{-1}xy$, with $x, y \in G$. The subgroup $(G')'$ is usually denoted $G''$, and its derived subgroup, is $G'''$. This notation rapidly gets unwieldy, however, and so we write $G^{(n)}$ for the $n$th derived subgroup. In other words, $G^{(0)} = G$ and for $n > 0$, we define $G^{(n)} = (G^{(n-1)})'$. These subgroups constitute the derived series of $G$.

Note that $G$ is solvable if and only if $G^{(n)} = 1$ for some integer $n$. We mention that the derived length of a solvable group $G$ is the smallest integer $n$ such that $G^{(n)} = 1$.

Let $p$ be a prime divisor of $|G|$ and let $\pi$ be a set of prime divisors of $|G|$. A subgroup $P$ of $G$ is said to be a Sylow $p$-subgroup of $G$ if $|P|$ is a power of $p$ and the index $|G : P|$ is not divisible by $p$. Similarly, a subgroup $H$ of $G$ is said to be a Hall $\pi$-subgroup of $G$ if $|H|$ is divisible by only primes in $\pi$ and the index $|G : H|$ is not divisible by any prime in $\pi$. It is well-known that $G$ has a Sylow $p$-subgroup for all prime divisors $p$ of $|G|$ even if $G$ is not solvable. It can happen that $G$ does not have any Hall $\pi$-subgroup. But if $G$ is solvable, then $G$ has a Hall $\pi$-subgroup for all sets of prime divisors $\pi$ of $|G|$. Also, a subgroup $R$ of $G$ is said to be a $p$-complement of $G$ if $|R|$ is not divisible by $p$ and the index $|G : R|$ is a power of $p$. Similarly, a subgroup $S$ of $G$ is said to be a $\pi$-complement of $G$ if $|S|$ is not divisible by any prime in $\pi$ and the index $|G : S|$ is divisible by only primes in $\pi$. Hence, a subgroup $U$ of $G$ is a $p$-complement of $G$ if and only if $U$ is Hall $p'$-subgroup of $G$, where
$p'$ is the set of all prime divisors of $|G|$ except $p$, and a subgroup $V$ of $G$ is a $\pi$–complement of $G$ if and only if $V$ is Hall $\pi'$–subgroup of $G$, where $\pi'$ is the set of all prime divisors of $|G|$ except primes in $\pi$.

We introduce the notation $\text{Syl}_p(G)$ and $\text{Hall}_\pi(G)$ to denote the set of all Sylow $p$–subgroups of $G$ and the set of all Hall $\pi$–subgroups of $G$, respectively. The intersection $\bigcap \text{Syl}_p(G)$ of all Sylow $p$–subgroups of $G$ is denoted by $\text{O}_p(G)$, and this is the largest normal $p$–subgroup of $G$. The intersection $\bigcap \text{Hall}_\pi(G)$ of all Hall $\pi$–subgroups of $G$ is denoted by $\text{O}_\pi(G)$, and this is the largest normal $\pi$–subgroup of $G$.

It is easy to show that the finite group $G$ has a (necessarily unique) normal subgroup $N$ such that $G/N$ is a $\pi$–group and $M \supseteq N$ whenever $M$ is normal in $G$ and $G/M$ is a $\pi$–group. This subgroup $N$ is denoted $\text{O}^\pi(G)$. If $\pi$ is the singleton $\{p\}$, then we write $\text{O}^p(G)$. Thus, $\text{O}^\pi(G)$ is the smallest normal subgroup of $G$ such that the quotient group $G/\text{O}^\pi(G)$ is a $\pi$–group.

We define the Fitting subgroup of $G$, denoted $\text{F}(G)$, to be the product of the subgroups $\text{O}_p(G)$ as $p$ runs over the prime divisors of $G$. We say that the group $G$ is nilpotent if $G = \text{F}(G)$. It is well-known that all nilpotent groups are solvable groups, and the group $G$ is nilpotent if and only if all Sylow subgroups of $G$ are normal in $G$. Also, the Fitting subgroup $\text{F}(G)$ of $G$ is the largest normal nilpotent subgroup of $G$. Inductively, we define $F_0 = 1$ and $F_{i+1}/F_i = \text{F}(G/F_i)$ for integers $i \geq 0$. When $G$ is solvable, it is clear that there is some integer $i$ so that $F_i = G$. We define the Fitting height of $G$ to be the smallest integer $i$ so that $F_i = G$.

Let $A$ and $B$ be normal subgroups of $G$. We say $G$ is the direct product of $A$ and $B$ and we denote this by $G = A \times B$ if and only if $G = AB$ and $A \cap B = 1$. If $G = A \times B$ is the direct product of $A$ and $B$, then we can get all information about the structure of $G$ from the structures of $A$ and $B$. For example, if $G = A \times B$, then $\text{Z}(G) = \text{Z}(A) \times \text{Z}(B)$, $\text{F}(G) = \text{F}(A) \times \text{F}(B)$, $G' = A' \times B'$, $\text{O}_\pi(G) = \text{O}_\pi(A) \times \text{O}_\pi(B)$ for all set of primes $\pi$, and
Syl\(_p(G) = \{ P_1 \times P_2 \mid P_1 \in \text{Syl}_p(A), P_2 \in \text{Syl}_p(B) \}\) for all primes \(p\). Therefore, being a direct product is not only completely useful to find information about the structure of the group but also makes it easy to work on the group.

Let \(\mathbb{C}\) be the field of complex numbers. A \(\mathbb{C}\)-representation of \(G\) is a homomorphism \(T\) from \(G\) into \(\text{GL}_n(\mathbb{C})\), the group of \(n \times n\) invertible matrices over the field \(\mathbb{C}\). The \(\mathbb{C}\)-character associated with \(T\) is the function \(\chi : G \rightarrow \mathbb{C}\) given by \(\chi(g) = \text{trace}(T(g))\). The character \(\chi\) is said to be irreducible if \(\chi\) cannot be written as the sum of two or more characters. We will denote the set of complex irreducible characters of \(G\) by \(\text{Irr}(G)\). The degree of the character \(\chi\), computed as \(\chi(1) = n\), is the rank of the matrix in the representation. We will let \(\text{cd}(G)\) denote the set of complex character degrees of \(G\), i.e., \(\text{cd}(G) = \{\chi(1) \mid \chi \in \text{Irr}(G)\}\).

There is a number of papers that indicate there exists a connection between the character degree set of a finite group and its structure. It is not difficult to see that the character degree set of a finite group does not completely determine the structure of the group, and in fact, it often gives us little information about the structure of the group. As far as we know, there have been few papers in which the structure of a finite group with a particular character degree set has been investigated.

In this dissertation, we look at a particular character degree set and we obtain strong structural information about the group more than what we expected. This gives more evidence to the existence of a deep connection between the structure of a group and its character degree set if this set is a reasonable particular set.

As we said, in general, the structure of the character degree set of \(G\) does not completely determine the structure of \(G\), but it gives us some information about the structure of \(G\). For example, Thompson’s theorem (Corollary (12.2) of [10]) implies that if a prime number \(p\) divides \(\chi(1)\) for every nonlinear \(\chi \in \text{Irr}(G)\), then \(G\) has a normal \(p\)-complement. Also, by Itô-Michler’s theorem (Corollary (12.34) of [10]), a group \(G\) has a normal abelian Sylow \(p\)-subgroup if and only if every element of \(\text{cd}(G)\) is relatively prime to \(p\).
Let $I$ and $J$ be finite sets of positive integers. We define $I \times J = \{ij \mid i \in I, j \in J\}$, which is again a subset of positive integers. It is well-known that if $G = A \times B$ is the direct product of normal subgroups $A$ and $B$, then $\text{Irr}(G) = \{\theta \times \varphi \mid \theta \in \text{Irr}(A), \varphi \in \text{Irr}(B)\}$, and so $\text{cd}(G) = \text{cd}(A) \times \text{cd}(B)$. It seems reasonable to ask that if $G$ is group such that $\text{cd}(G) = I \times J$ for subsets $I$ and $J$ of positive integers both including 1, then must $G = A \times B$ for some normal subgroups $A$ and $B$ with $\text{cd}(A) = I$ and $\text{cd}(B) = J$?

In Theorem A of [19], Lewis assumed that $I_1 = \{1, p\}$, $J_1 = \{1, q, r\}$, and $\text{cd}(G) = I_1 \times J_1$, where $p, q, r$ are distinct primes. Then, he was able to answer question above positively. In other words, he proved that $G$ is the direct product of two non-abelian groups $A$ and $B$ such that $\text{cd}(A) = I_1 = \{1, p\}$ and $\text{cd}(B) = J_1 = \{1, q, r\}$. Also, in Theorem B of [19], he answered question above positively if $\text{cd}(G) = I_2 \times J_2$, where $I_2 = \{1, p, q\}$ and $J_2 = \{1, r, s\}$ for distinct primes $p, q, r,$ and $s$.

To prove these theorems, Lewis reduced the problems to the solvable case by [7], which is dependent upon the classification of finite simple groups. Also, in [19], he proposed an open question about loosening the primeness hypothesis of theorems above.

**Question.** Is it possible to relax the primeness hypothesis in Theorems A and B of [19]? By this he mean the following: suppose that $a, b, c,$ and $d$ are pairwise relatively prime positive integers and assume that $I_1 = \{1, a\}$, $I_2 = \{1, a, b\}$, $J_1 = \{1, b, c\}$, and $J_2 = \{1, c, d\}$. If $G$ is a group with $\text{cd}(G) = I_1 \times J_1$, must $G = A \times B$, where $\text{cd}(A) = I_1$ and $\text{cd}(B) = J_1$? Similarly, if $\text{cd}(G) = I_2 \times J_2$, must $G = A \times B$, where $\text{cd}(A) = I_2$ and $\text{cd}(B) = J_2$?

Lewis’ methods of proof of Theorems A and B of [19] requires verifying the following two steps.

1. Let $p, q, r$ be distinct primes and write $K = O^p(G)$. Assume that $\text{cd}(K) = $
\{1, q, r\} \subseteq \text{cd}(G)$. Then \(G = C_K(N) \times N[K, N]\) and \(\text{cd}(K) = \text{cd}(C_K(N))\).

2. Let \(p, q,\) and \(r\) be distinct primes and let \(G\) be a finite group. If \(\text{cd}(G) = \{1, p, q, r, pq, pr\}\), then \(\text{cd}(O^p(G)) = \{1, q, r\}\).

Our goal is to answer Lewis’ question by generalizing Lewis’ theorems. We begin by generalizing Steps (1) and (2). Suppose \(p\) is a prime and \(m > 1\) is an integer. We define the ordered pair \((p, m)\) to be a \textbf{strongly coprime pair} if \(m\) is not divisible by \(p\) and also \(p\) does not divide \(u - 1\), where \(1 < u < m\) is any proper prime power divisor of \(m\). For example, the pair \((3, 22)\) is a strongly coprime pair while the pair \((3, 16)\) is not. Also, for any two distinct primes \(p\) and \(q\), the pair \((p, q)\) is a strongly coprime pair. We extend Step (1) of Lewis’ method as follows. The following theorem is the main theorem of chapter 2.

**Theorem A.** Let \(\pi\) be a set of primes, and let \(m\) and \(n\) be coprime integers such that for every \(p \in \pi\) the pairs \((p, m)\) and \((p, n)\) are strongly coprime pairs. Assume that \(G\) is a group such that \(\text{cd}(K) = \{1, m, n\} \subseteq \text{cd}(G)\), where \(K = O^\pi(G)\). Also, suppose that \(U\) is a Hall \(\pi\)–subgroup of \(K\). Then \(G\) splits over \(K\) and there is a complement \(N\) for \(K\) such that one of the following holds:

1. \([K, N] \subseteq Z(K), N \subseteq C_G(U), K = C_K(N) \times [K, N], G = C_K(N) \times N[K, N], \) and \(\text{cd}(K) = \text{cd}(C_K(N))\).

2. There is a prime \(t\) such that \(G\) has a normal Sylow \(t\)–subgroup \(T\) such that \(\text{cd}(T) = \{1, t^l\}\) for some integer \(l \geq 2, t^l \in \{n, m\}\), \(K/T\) is abelian, and the Fitting height of \(K\) is 2.

This result is a key step in generalizing Theorems A and B of [19]. Notice that Theorem A above is a generalization of Step (1) of Lewis’ argument (Theorem (5.2) of [19]). Also, we present an example, Example (6.1), of a group that implies that Conclusion (b) of
Theorem A above does occur.

By using Theorem A above, we obtain the following corollary. This corollary has a fundamental importance in proving the main results of this dissertation.

**Corollary B.** Let \( a, b, \) and \( c \) be pairwise coprime integers greater than 1 such that the pairs \((p, b)\) and \((p, c)\) are strongly coprime pairs for each prime divisor \( p \) of \( a \). Suppose that \( \text{cd}(G) = \{1, a, b, c, ab, ac\} \) and \( \text{cd}(K) = \{1, b, c\} \), where \( K = \text{O}^\pi(G) \) and \( \pi = \pi(a) \). Then one of the following holds:

1. \( G = A \times B \), where \( \text{cd}(A) = \{1, b, c\} \) and \( \text{cd}(B) = \{1, a\} \).

2. There is a prime \( t \) such that \( G \) has a normal Sylow \( t \)–subgroup \( T \) such that \( \text{cd}(T) = \{1, t^l\} \) for some integer \( l \geq 2 \), \( t^l \in \{b, c\} \), \( K/T \) is abelian, and the Fitting height of \( K \) is 2.

To generalize Step (2) of Lewis’ argument, let \( m \) and \( n \) be coprime integers greater than 1 and let \( p \) be a prime such that the pairs \((p, m)\) and \((p, n)\) are strongly coprime pairs. We suppose \( G \) is a solvable group such that \( \text{cd}(G) = \{1, p, n, m, pm, pm\} \). Then, in the following theorem, we show that \( \text{cd}(\text{O}^p(G)) = \{1, n, m\} \) if \( G \) has either a Frobenius factor group with \( pm \) or \( pm \) as the index of the kernel or a non-abelian factor \( p \)–group. This theorem, which will be the main theorem of chapter 3, extends Step (2) of Lewis’ argument because if \( m \) and \( n \) are primes, then the pairs \((p, m)\) and \((p, n)\) are automatically strongly coprime pairs as \( p, m, \) and \( n \) are distinct primes.

**Theorem C.** Let \( p \) be a prime number and let \( m \) and \( n \) be coprime integers such that the pairs \((p, m)\) and \((p, n)\) are strongly coprime pairs. Assume that \( G \) is a solvable group such
that \( \text{cd}(G) = \{1, p, m, n, pm, pn\} \). If \( G \) has either a Frobenius factor group with \( pn \) or \( pm \) as the index of the kernel or a non-abelian factor \( p \)-group, then \( \text{cd}(\mathcal{O}^p(G)) = \{1, m, n\} \).

By using Theorems A and C above, we obtain the following generalization of Theorem A of [19], which is **Theorem D** of this dissertation.

**Theorem D.** Let \( m \) and \( n \) be coprime integers greater than 1 and let \( p \) be a prime such that the pairs \((p, m)\) and \((p, n)\) are strongly coprime pairs. If \( G \) is a solvable group such that \( \text{cd}(G) = \{1, p, n, m, pm, pn\} \), then \( \text{cd}(\mathcal{O}^p(G)) = \{1, n, m\} \) and one of the following holds:

1. \( G = A \times B \), where \( \text{cd}(A) = \{1, p\} \) and \( \text{cd}(B) = \{1, m, n\} \).

2. There is a prime \( t \) such that \( G \) has a normal Sylow \( t \)-subgroup \( T \) with \( \text{cd}(T) = \{1, t^l\} \) for some integer \( l \geq 2 \), \( t^l \in \{n, m\} \), \( \mathcal{O}^p(G)/T \) is abelian, the Fitting height of \( \mathcal{O}^p(G) \) is 2, and so the Fitting height of \( G \) is at most 3.

As we said, Lewis was able to prove that if

\[
\text{cd}(G) = \{1, p, q, r, s, pr, ps, qr, qs\},
\]

where \( p, q, r, \) and \( s \) are distinct primes, then \( G = A \times B \), where \( \text{cd}(A) = \{1, p, q\} \) and \( \text{cd}(B) = \{1, r, s\} \), and he again proposed the question of whether it is possible to relax the primeness hypothesis in this theorem. In **Theorem E** of this dissertation, we consider how we can drop the primeness hypothesis of \( \text{cd}(G) \) and yet still obtain a similar conclusion. Hence, the following theorem extends Theorem B of [19].

**Theorem E.** Let \( p \) and \( q \) be distinct primes and let \( m \) and \( n \) be coprime integers greater than 1 such that the pairs \((p, m), (p, n), (q, m), \) and \((q, n)\) are strongly coprime pairs. If \( G \) is
a solvable group such that \( cd(G) = \{1, p, q, n, m, pn, pm, qn, qm\} \), then one of the following holds:

1. \( G = A \times B \), where \( cd(A) = \{1, p, q\} \) and \( cd(B) = \{1, n, m\} \), and either \( cd(O^{\{p,q\}}(G)) = \{1, n, m\} \) or \( cd(O^{\{\pi,\sigma\}}(G)) = \{1, p, q\} \), where \( \pi = \pi(m) \) and \( \sigma = \pi(n) \) are the sets of all prime divisors of \( m \) and \( n \), respectively.

2. \( cd(O^{\{p,q\}}(G)) = \{1, n, m\} \) and there is a prime \( t \) such that \( G \) has a normal Sylow \( t \)-subgroup \( T \) with \( cd(T) = \{1, t^l\} \) for some integer \( l \geq 2 \), \( t^l \in \{m, n\} \), \( O^{\{p,q\}}(G)/T \) is abelian, the Fitting height of \( O^{\{p,q\}}(G) \) is 2, and the Fitting height of \( G \) is at most 4.

We do not have an example of a group that meets the conclusion (2) of Theorem E, so it is an open question whether such a group exists. Also, our strategy to prove Theorem E shows why we need the primeness of \( p \) and \( q \) in Theorem E.

In chapter 6, we present two examples of groups. The first example, Example (6.1), implies that the conclusion (2) of Theorem D does occur. The second example, Example (6.2), shows that in Theorem D the condition that the pairs \( (p, m) \) and \( (p, n) \) are strongly coprime pairs cannot be removed. Thus, we conclude that if \( G \) is a solvable group with \( cd(G) = \{1, a, b, c, ab, ac\} \) for some pairwise coprime integers \( a, b, \) and \( c \), then \( G \) need not be a direct product. Hence, we are interested to study the group structure of solvable groups that are not direct products and whose character degree sets are of the form \( \{1, a, b, c, ab, ac\} \) for some pairwise coprime integers \( a, b, \) and \( c \). In Theorem F of this dissertation, we prove that if \( G \) is a solvable group such that \( cd(G) = \{1, a, b, c, ab, ac\} \), where \( a, b, \) and \( c \) are pairwise coprime integers, then \( dl(G) \leq 4 \) and 4 is the best bound.

**Theorem F.** Let \( G \) be a finite solvable group with \( \{1, a, b, c, ab, ac\} \) as the character degree set, where \( a, b, \) and \( c \) are pairwise coprime integers greater than 1. Then the derived length
of $G$ is at most $4$.

As we mentioned, we do not have an example of a group that is not a direct product and has character degree set of the form $\{1, a, b, c, d, ac, ad, bc, bd\}$ for some pairwise coprime integers $a, b, c,$ and $d$. As a corollary of Theorem F, we obtain that if $G$ is a solvable group such that $\text{cd}(G) = \{1, a, b, c, d, ac, ad, bc, bd\}$, where $a, b, c,$ and $d$ are pairwise coprime integers greater than $1$, then we can bound the derived length of $G$. This corollary is Theorem G of this dissertation.

**Theorem G.** Let $a, b, c,$ and $d$ be pairwise coprime integers greater than $1$. If $G$ is a solvable group such that $\text{cd}(G) = \{1, a, b, c, d, ac, ad, bc, bd\}$, then $\text{dl}(G) \leq 5$.

We devote chapter 1 to background results necessary to proceed through the three steps of Lewis’ argument. We include many basic results from the Clifford Theory.

In chapter 2, we work through the step 1 in Lewis’ argument to answer Lewis’ question. The main result of chapter 2 is Theorem A above. For doing this, we consider a group $G$ which has a normal subgroup $K$, where $\text{cd}(K) = \{1, m, n\}$ for coprime integers $m$ and $n$. We also have a set of primes $\pi$ with the property that $O^\pi(K)$, the smallest normal subgroup of $K$ whose quotient is a $\pi$–group, is $K$ itself and the pairs $(p, m)$ and $(p, n)$ are strongly coprime pairs for every $p \in \pi$. Since $\text{cd}(K) = \{1, m, n\}$, it follows that the Fitting height of $G$ is either $2$ or $3$. We consider two cases depending on the Fitting height of $K$ to prove that if $P \in \text{Hall}_\pi(G)$, then $PK$ is a direct product or there is a prime $t$ such that $PK$ has a normal Sylow $t$–subgroup $T$ such that $\text{cd}(T) = \{1, t^\beta\}$ for some integer $\beta \geq 2$, $t^\beta \in \{n, m\}$, $K/T$ is abelian, and the Fitting height of $K$ is $2$.

In chapter 3, we work to extend the step 2 of Lewis’ argument. The main result of chapter 3 is Theorem C above. To do this, let $m$ and $n$ be coprime integers greater than
1 and let $p$ be a prime such that the pairs $(p, m)$ and $(p, n)$ are strongly coprime pairs. We suppose $G$ is a solvable group such that $\text{cd}(G) = \{1, p, m, pm, pn\}$. Then we show that $\text{cd}(O^p(G)) = \{1, n, m\}$. Our strategy depends on the following. We take $K$ to be a subgroup of $G$ that is maximal with the property that $K$ is normal in $G$ and $G/K$ is non-abelian. We know that $\text{cd}(G/K) = \{1, f\}$ for some character degree $f$ and $G/K$ is either an $s$–group for some prime $s$ or $G/K$ is a Frobenius group (this is Lemma (1.3) of chapter 1). We break up our proof into different cases depending on the value of $f$. Since $\text{cd}(G)$ is symmetric in $m$ and $n$, there are three different cases: $f = p$, $f \in \{pn, pm\}$, and $f \in \{n, m\}$. In the cases $f = p$ and $f \in \{pn, pm\}$, we use Lewis' characterization of solvable groups $G$ whose degree graphs $\Delta(G)$ have two connected components (This is the main Theorem of [21] and Lemma (1.26) of chapter 1), where the degree graph $\Delta(G)$ has vertex set $\rho(G)$ that consists of the primes that divide the character degrees of $G$ and there is an edge between $p$ and $q$ if $pq$ divides some degree $a \in \text{cd}(G)$. When $f \in \{n, m\}$, we apply Lemma (1.25) of chapter 1.

We devote chapter 4 to proving Theorems D and E. In chapter 5, we will prove Theorems F and G. Finally, in chapter 6, we construct examples of groups related to Theorems D, E, and F. We present three examples of groups. The first example, Example (6.1), implies that the conclusion (2) of Theorem D does occur. In fact, we present an example of a solvable group $G$ such that $\text{cd}(G) = \{1, 3, 2, 25, 6, 75\}$ and $G$ is not a direct product, and hence $G$ satisfy the conclusion (2) of Theorem D. The second example, Example (6.2), shows that in Theorem D the condition that the pairs $(p, m)$ and $(p, n)$ are strongly coprime pairs cannot be removed. Eventually, in Example (6.3), we construct a family of groups whose derived lengths are 4 and character degree sets are in the form $\{1, p, b, pb, q^p, pq^p\}$, where $p$ is a prime number, $q$ is a power of another prime $q_0$ which is odd, and $b > 1$ is an integer not divisible by $p$ or $q_0$. Hence, the bound 4 in Theorem F is the best bound that can be hoped for the derived length of solvable groups whose character degree sets are in
the form \( \{1, a, b, c, ab, ac\} \) for pairwise coprime integers \( a, b, \) and \( c. \)
CHAPTER 1

BACKGROUND RESULTS AND FACTS

We will require several lemmas to carry out the answer of Lewis’ Question. We begin with the following results from the Clifford Theory. These are presented as stated in Theorem 6.2, Corollary 6.17, and Theorem 6.11 in [10]. Suppose that $N \trianglelefteq G$ and $\chi \in \text{Irr}(G)$. Lemma 1.1(a) states that $\chi_N$ is a sum of conjugate irreducible characters of $N$ with the same multiplicity. Lemma 1.1(b) is often referred to as Gallagher’s Theorem and shows that if $\chi_N$ is an irreducible character of $N$, then $\chi\theta \in \text{Irr}(G)$ for every $\theta \in \text{Irr}(G/N)$. Also, Lemma 1.1(b) is a useful tool to produce irreducible character degrees.

**Lemma 1.1.** Let $N \trianglelefteq G$ and let $\chi \in \text{Irr}(G)$. If $\theta$ is an irreducible constituent of $\chi_N$, then

(a) $\chi_N = e(\theta_1 + \cdots + \theta_k)$, where $\theta = \theta_1, \theta_2, \ldots, \theta_k \in \text{Irr}(N)$ are the distinct conjugates of $\theta$ in $G$, and $e = [\chi_N, \theta]$.

(b) If $\chi_N \in \text{Irr}(N)$, then $\chi\theta \in \text{Irr}(G)$ for every $\theta \in \text{Irr}(G/N)$.

Assume that $N$ is a normal subgroup of $G$ and $\theta \in \text{Irr}(N)$. Then

$$\text{Stab}_G(\theta) = \{g \in G | g^\theta = \theta\}$$

is the stabilizer of $\theta$ in the action of $G$ on $\text{Irr}(N)$. Lemma 1.2 is often referred to as the Clifford correspondence and is of fundamental importance in the character theory of normal subgroups.

**Lemma 1.2.** Suppose $N \trianglelefteq G$ and $\theta \in \text{Irr}(N)$. Let $T = \text{Stab}_G(\theta)$ denote the stabilizer subgroup of $\theta$ in $G$. Let $A = \{\psi \in \text{Irr}(T) \mid [\psi_N, \theta] \neq 0\}$ and $B = \{\chi \in \text{Irr}(G) \mid [\chi_N, \theta] \neq 0\}$. The following hold.
(a) If $\psi \in A$, then $\psi^G \in B$.

(b) The map $\psi \mapsto \psi^G$ is a bijection of $A$ onto $B$.

(c) If $\psi^G = \chi$, with $\psi \in A$, then $\psi$ is the unique irreducible constituent of $\chi_T$ which lies in $A$.

(c) If $\psi^G = \chi$, with $\psi \in A$, then $[\psi_N, \theta] = [\chi_N, \theta]$.

The following lemma is very useful for inductive proofs of theorems giving information about $G$ when $cd(G)$ is known.

Given non-abelian group $G$, let $K \leq G$ be maximal such that $G/K$ is non-abelian. Then $(G/K)'$ is the unique minimal normal subgroup of $G/K$. Thus $G/K$ satisfies the hypotheses of the following lemma if $G/K$ is solvable.

We will also need the following result, stated as Lemma 12.3 and Lemma 12.4 in [10].

**Lemma 1.3.** Let $K$ be maximal such that $G/K$ is non-abelian solvable. Then two cases can occur.

(a) $G/K$ is a $p$-group for some prime $p$. Hence there exists $\psi \in \text{Irr}(G/K)$ such that $\psi(1) = p^b > 1$. If $\chi \in \text{Irr}(G)$ and $p \nmid \chi(1)$, then $\chi \tau \in \text{Irr}(G)$ for all $\tau \in \text{Irr}(G/K)$.

(b) $G/K$ is a Frobenius group with an elementary abelian Frobenius kernel $N/K$. Thus $|G : N| \in \text{cd}(G)$ while $|N : K| = p^a$ for some prime $p$ and $N/K$ is an irreducible module for the cyclic group $G/N$, hence $a$ is the smallest integer such that $p^a - 1 \equiv 0 \pmod{|G/F|}$. If $\psi \in \text{Irr}(N)$, then either $|G : N| \psi(1) \in \text{cd}(G)$ or $|N : K|$ divides $\psi(1)^2$. In the latter case, $p$ divides $\psi(1)$. If no proper multiple of $|G : N|$ is a character degree of $G$, then $\chi(1)$ divides $|G : N|$ for all $\chi \in \text{Irr}(G)$ such that $p \nmid \chi(1)$.

The following lemma is Corollary 11.29 in [10].
Lemma 1.4. Let $N \trianglelefteq G$ and $\chi \in \text{Irr}(G)$. Let $\theta \in \text{Irr}(N)$ be a constituent of $\chi_N$. Then $\chi(1)/\theta(1)$ divides $|G : N|$.

The following lemmas (Lemmas (1.5), (1.6), and (1.7)) appear as Corollary 8.16, Theorem 6.26, and Corollary 11.22 in [10], respectively. In these lemmas, we find when an irreducible character of a normal subgroup is extendible to the whole group.

Lemma 1.5. Let $N \trianglelefteq G$ and $\theta \in \text{Irr}(N)$ with $\theta$ invariant in $G$. Suppose $(|G : N|, |N|) = 1$. Then $\theta$ is extendible to $G$.

Lemma 1.6. Let $N \trianglelefteq G$ and suppose $\lambda$ is a linear character of $N$ which is invariant in $G$. For each prime $p | o(\lambda)$, choose $H_p \subseteq G$ with $H_p/N \in \text{Syl}_p(G/N)$, and assume that $\lambda$ is extendible to $H_p$. Then $\lambda$ is extendible to $G$.

Lemma 1.7. Let $N \trianglelefteq G$ with $G/N$ cyclic and let $\theta \in \text{Irr}(N)$ be invariant in $G$. Then $\theta$ is extendible to $G$.

The next lemma indicates that if a group has only three irreducible character degrees, then $G$ is solvable and $G''' = 1$. This lemma appears as Theorem 12.15 in [10].

Lemma 1.8. Let $|\text{cd}(G)| = 3$. Then $G''' = 1$, that is, $G$ is solvable of derived length $\leq 3$.

Suppose that $A$ is a normal abelian subgroup of $G$. The following lemma shows every irreducible character degree of $G$ divides the index of $A$ in $G$. This lemma is Corollary 6.15 in [10].

Lemma 1.9. Let $A$ be a normal abelian subgroup of $G$. Then $\chi(1)$ divides $|G : A|$ for all $\chi \in \text{Irr}(G)$.

The next lemma is a character theoretic equivalent for having normal abelian Sylow subgroup. This lemma appears as Corollary 12.34 in [10].
**Lemma 1.10.** Let $G$ be solvable. Then $G$ has a normal abelian Sylow $p$-subgroup if and only if every element of $cd(G)$ is relatively prime to $p$.

Let $N \trianglelefteq G$ so that $G$ permutes $\text{Irr}(N)$. It is also true that $G$ permutes the set of conjugacy classes of $N$ and it is natural to consider the relationship between these two actions. It is not the case that they are necessarily permutation isomorphic although they are closely related. The following lemma appears as Corollary 6.32 in [10].

**Lemma 1.11.** Let $A$ be a group which acts on $\text{Irr}(G)$ and on the set of conjugacy classes of $G$. Assume that $\chi(g) = \chi^a(g^a)$ for all $\chi \in \text{Irr}(G)$, $a \in A$ and $g \in G$, where $g^a$ is an element of $\text{Cl}(g)^a$. Then for each $a \in A$, the number of fixed irreducible characters of $G$ is equal to the number of fixed classes.

The following lemma appears as Corollary 8.28 in [12]. This lemma is often referred to as Three Subgroups Lemma.

**Lemma 1.12.** Let $X, Y$, and $Z$ be subgroups of $G$. Suppose $N$ is a normal subgroup of $G$ such that $[X,Y,Z] \subseteq N$ and $[Y,Z,X] \subseteq N$. Then $[Z,X,Y] \subseteq N$.

In the next lemmas, suppose that the groups $S$ and $G$ both act on a set $\Omega$ and also $S$ acts coprimely on $G$ such that $(\alpha.g).s = (\alpha.s).g^s$ for all $\alpha \in \Omega$, $g \in G$, and $s \in S$, and $G$ is transitive on $\Omega$. If one of $S$ or $G$ is solvable, then $S$ fixes a point of $\Omega$ and also the set of $S$–fixed points of $\Omega$ is an orbit under the action of $C_G(S)$. These lemmas appear as Lemma 13.8 and Corollary 13.9 in [10]. Lemma 1.8 and Lemma 1.9 are often referred to as Glauberman’s lemma.

**Lemma 1.13.** Let $S$ act on $G$ with $(|S|,|G|) = 1$. Assume that one of $S$ or $G$ is solvable. Let $S$ and $G$ both act on a set $\Omega$ such that

1. $(\alpha.g).s = (\alpha.s).g^s$ for all $\alpha \in \Omega$, $g \in G$, and $s \in S$. 
2. $G$ is transitive on $\Omega$.

Then $S$ fixes a point of $\Omega$.

**Lemma 1.14.** In the situation of previous lemma, the set of $S$–fixed points of $\Omega$ is an orbit under the action of $C_G(S)$.

The following lemma appears as Lemma 12.4 in [23].

**Lemma 1.15.** Suppose $G/N$ is abelian, $\varphi \in \text{Irr}(N)$, and $\chi \in \text{Irr}(G|\varphi)$ is fully ramified with respect to $G/N$. Suppose $S$ acts on $G$ fixing $N$, $\varphi$, and (hence) $\chi$. Assume $(|S|, |G/N|) = 1$. Set $C/N = C_{G/N}(S)$ and $D/N = [G/N, S]$. Then $\chi$ is fully ramified with respect to both $G/C$ and $G/D$, and $\varphi$ is fully ramified with respect to both $C/N$ and $D/N$.

Let $N$ be a normal subgroup of a finite group $G$. Consider a character $\theta \in \text{Irr}(N)$, and in the standard notation define $\text{Irr}(G|\theta)$ to be the set of characters in $\text{Irr}(G)$ that are constituents of $\theta^G$. Following this notation, we define $\text{cd}(G|\theta) = \{\chi(1) \mid \chi \in \text{Irr}(G|\theta)\}$. In the spirit of [14], $\text{Irr}(G|N)$ is the union of the sets $\text{Irr}(G|\theta)$ where $\theta$ runs through all the non-principal characters in $\text{Irr}(N)$. Our interest is with the set $\text{cd}(G|N) = \{\chi(1) \mid \chi \in \text{Irr}(G|N)\}$. In proving Theorem F of this dissertation, we rely on the results of Isaacs’ and Knutson’s investigation in [14] of the relationship between $\text{cd}(G|N)$ and the structure of $N$. They proved that $d_l(N) \leq |\text{cd}(G|N)|$ when $|\text{cd}(G|N)| \leq 3$ (see Theorems B and C of [14]).

**Lemma 1.16.** Let $N$ be normal subgroup of a finite group $G$. Then $d_l(N) \leq |\text{cd}(G|N)|$ when $|\text{cd}(G|N)| \leq 3$.

Let $H$ be a subgroup of $G$. We use the notation $\text{cd}_H(G|a)$ to represent the union of all the sets $\text{cd}(G|\theta)$ over all $\theta \in \text{Irr}(H)$, where $\theta(1) = a$. The next lemma is an easy consequence of Lemma 1.1(b) and Lemma 1.4 that is Lemma 6.1 in [19].

**Lemma 1.17.** Let $G$ be a finite group, and suppose that $K$ is a normal subgroup of $G$. Assume that the character degrees $a$, $b$, and $f$ lie in $\text{cd}(K)$, $\text{cd}(G)$, and $\text{cd}(G/K)$, respectively.
If \( a \in \text{cd}_K(G|a) \), then \( af \in \text{cd}_K(G|a) \). Furthermore if \( (b, |G : K|) = 1 \), then \( b \in \text{cd}(K) \) and \( bf \in \text{cd}_K(G|b) \).

The following lemma can be obtained by using Lemma 1.3(b) that appears as Lemma 6.2 in [19].

**Lemma 1.18.** Let \( K \) be a normal subgroup of \( G \) such that \( G/K \) is a Frobenius group with kernel \( N/K \) an elementary abelian \( p \)-group for some prime \( p \). Suppose that \( a \in \text{cd}(G) \) is relatively prime to \( |G : N| \). If \( a|G : N| \notin \text{cd}(G) \), then \( p \) divides \( a \).

In [26], it has been proved that if \( G \) is a finite group so that \( \text{cd}(G) = \{1, m, n, mn\} \), where \( m \) and \( n \) are relatively prime integers, then \( G \) is solvable and the derived length of \( G \) is at most 3 unless \( \text{cd}(G) = \{1, 3, 13, 39\} \).

**Lemma 1.19.** If a finite group \( G \) possesses precisely four irreducible character degrees \( 1, m, n, mn \), where \( m \) and \( n \) are relatively prime integers, then \( G \) is solvable. Furthermore, the derived length of \( G \) is at most 3 unless the four irreducible character degrees are \( 1, 3, 13, 39 \).

The following lemma gives us some helpful information about the structure of non-nilpotent groups that have only two character degrees. This lemma is Lemma 1.6 in [25].

**Lemma 1.20.** Suppose \( \text{cd}(G) = \{1, m\} \) and \( G \) is not nilpotent. If all Sylow subgroups of \( G \) are abelian, then \( F(G) = G' \times Z(G) \) and \( G/Z(G) \) is a Frobenius group with kernel \( F(G)/Z(G) \), where \( F(G) \) is the Fitting subgroup of \( G \).

Suppose that \( V < N < G \) are normal subgroups of \( G \) such that \( G/N \) respectively \( N/V \) is cyclic of order \( a \) respectively \( b \). Moreover let \( V \) be elementary abelian and suppose that both \( G/V \) and \( N \) are Frobenius groups with kernel \( N/V \) respectively. \( V \). Then the next lemma shows that we can compute the character degree set of \( G \). This lemma appears as Lemma 1.10 in [25].
Lemma 1.21. Let $V < N < G$ be normal subgroups of $G$ such that $G/N$ respectively $N/V$ is cyclic of order $a$ respectively $b$. Moreover let $V$ be elementary abelian and suppose that both $G/V$ and $N$ are Frobenius groups with kernel $N/V$ respectively $V$. Then $\text{cd}(G) \cup \{ab\} = \{1, a\} \cup \{ib \mid i \text{ divides } a\}$.

The following lemma, which appears as Lemma 4.1 in [19], gives some useful structural information on groups $G$ with $\text{cd}(G) = \{1, m, n\}$, where $m$ and $n$ are relatively prime integers.

Lemma 1.22. Let $G$ be a group with $\text{cd}(G) = \{1, m, n\}$, where $m$ and $n$ are relatively prime integers. Write $F = \text{F}(G)$ (the Fitting subgroup of $G$).

(a) Assume $G$ has Fitting height 3. Write $E/F = \text{F}(G/F)$ and $P = [E, F]$. Then the following are true:

1. $F = P \times \text{Z}(G)$ and $\text{cd}(G) = \text{cd}(G/\text{Z}(G)) = \{1, |G : E|, |E : F|\}$.
2. $P$ is a minimal normal subgroup of $G$ and $P = E'$.
3. $\text{cd}(E) = \{1, |E : F|\}$ and $F$ is abelian.
4. $E/\text{Z}(G)$ is a Frobenius group with kernel $F/\text{Z}(G)$.
5. $|G : E|$ is a prime number and $E/F$ is a cyclic group.
6. $|P| = p^{\alpha(G:E)}$ for some prime $p$ and some positive integer $\alpha$.
7. $\frac{|E:F|}{(|E:F|, p^\alpha - 1)} = \frac{p^{\alpha(G:E)} - 1}{p^\alpha - 1}$.

(b) Assume $G$ has Fitting height 2. Then the following are true:

1. $F = P \times \text{Z}$, where $P$ is Sylow $p$–subgroup of $G$ for some prime $p$ such that $P$ has nilpotence class 2.
2. $Z \subseteq \mathbb{Z}(G)$ and $\text{cd}(G) = \text{cd}(G/Z)$.

3. $|G:F| \in \text{cd}(G)$ and $\text{cd}(F) = \text{cd}(G) \setminus \{|G:F|\}$.

4. $G/F$ is cyclic.

5. Let $R$ be a $p$–complement for $G$, and write $C = C_p(R)$. Then:
   
   (i) $G' = [P,R]$ and $P' \subseteq C$.

   (ii) $P/P' = C/P' \times G'/P'$ and $G/P' = C/P' \times G'R/P'$.

   (iii) If $\delta \in \text{Irr}(P)$ is a nonlinear character, then $\delta$ is fully ramified with respect to $P/C$.

The next lemma indicates that if $S$ acts coprimely on $G$ such that $[G,S] \subseteq \mathbb{Z}(G)$, then $G = [G,S] \times C_G(S)$. This lemma, which is Lemma 3.2 in [19], has fundamental importance to produce direct products in proving Theorem D and Theorem E of this dissertation.

**Lemma 1.23.** Let $S$ act on $G$ such that $(|S|,|G|) = 1$. If $[G,S] \subseteq \mathbb{Z}(G)$, then $G = [G,S] \times C_G(S)$.

We will use the following lemma in proving the main theorem of chapter 2. This lemma appears as Lemma 5.1 in [19].

**Lemma 1.24.** Let $A$ be a cyclic normal subgroup of $G$ and write $C = C_G(A)$. Then $G' \subseteq C$. Furthermore, if $\alpha$ is a faithful irreducible character of $A$, then $C$ is the stabilizer of $\alpha$ in $G$. Finally, if $B$ is a normal subgroup in $G$ with $C \cap B \subseteq A$, then $[G,B] \subseteq A$.

Suppose that $G$ is a solvable group and $K$ is a normal subgroup of $G$ such that $G/K$ is a $p'$–group with $\text{cd}(G/K') = \{1,m,n\}$ and $\text{cd}(K) = \{1,p\}$, where $p$ is a prime number that does not divide $mn$, and $m$ and $n$ are coprime integers. Also, assume that $\text{cd}(G) \subseteq \{1,p,m,n,pm,pn\}$. Then the next lemma shows that we can obtain the character degree
set of $O^p(G)$. This lemma is Theorem 7.3 in [19] and we will use this lemma in proving Theorem D of this dissertation.

**Lemma 1.25.** Let $p$ be a prime, and let $m$ and $n$ be coprime integers that are greater than 1 and not divisible by $p$. Suppose that $G$ is a solvable group and $K$ is a normal subgroup of $G$ such that $G/K$ is a $p'$-group with $\text{cd}(G/K') = \{1, m, n\}$ and $\text{cd}(K) = \{1, p\}$. Assume that $\text{cd}(G) \subseteq \{1, p, m, n, pm, pn\}$. If $p \in \text{cd}(G)$, then $\text{cd}(O^p(G)) = \{1, m, n\}$.

In [21], Lewis found a characterization of solvable groups $G$ whose degree graphs $\Delta(G)$ have two connected components, where the degree graph $\Delta(G)$ has vertex set $\rho(G)$ that consists of the primes that divide the character degrees of $G$ and there is an edge between $p$ and $q$ if $pq$ divides some degree $a \in \text{cd}(G)$. This graph was introduced in [23]. In particular, in [23] it is proved that if $G$ is solvable then $\Delta(G)$ has at most two connected components. The following lemma is a result of the main theorem of [21]:

**Lemma 1.26.** Let $G$ be a solvable group such that $\Delta(G)$ has two connected components. Then one of the followings holds:

1. $G$ has a normal non-abelian Sylow $p$-subgroup $P$ and an abelian $p$-complement $K$ for some prime $p$. Also, if $F$ is the Fitting subgroup of $G$, we have $F = P \times (F \cap K)$, $G/F$ is abelian, $F \cap K$ is central in $G$, $\text{cd}(G|P') = \text{cd}(P) - \{1\}$, $\text{cd}(G/P')$ consists of 1, $|G:F|$, and possibly other divisors of $|G:F|$, and $\Delta(G)$ has two connected components $\{p\}$ and $\pi(|G:F|)$.

2. $\Delta(G)$ has two vertices.

3. $\Delta(G)$ has two connected components $\pi(|G:E|)$ and $\pi(|E:F|)$, where $F$ is the Fitting subgroup of $G$, and $E/F$ is the Fitting subgroup of $G/F$ such that $E/F$ and $G/E$ are both cyclic and all divisors of $|G:E|$ including 1 and $|G:E|$ are character degrees of $G$. Also, $F = E' \times \mathbf{Z}(G)$, $\text{cd}(G) = \{u \mid u \text{ divides } |G:E|\} \cup \{|E:F|\}$, and $E/\mathbf{Z}(G)$ is
a Frobenius group with kernel $F/\mathbb{Z}(G)$.

4. $\Delta(G)$ has two connected components $\{2\}$ and $\pi(2^a + 1)$, where $a$ is a positive integer such that $2^a + 1$ is a character degree of $G$ and also $cd(G)$ consists of 2, all powers of 2 that are divisible by $2^a$, and $2^a + 1$.

5. There is a normal subgroup $A$ of $G$ such that $G/A'$ satisfies the hypothesis of the case (3) and $F/A'$ is the Fitting subgroup of $G/A'$, where $F$ is the Fitting subgroup of $G$. Also, $\Delta(G)$ has two connected components, $\pi(|G : E|)$ and $\pi(|E : F|) \cup \{t\}$, where $t$ is a prime such that $G$ has a normal Sylow $t$-subgroup $T$, the Fitting subgroup $F$ of $G$ is $T \times Z$ for some central subgroup $Z$ of $G$, and $E/F$ is the Fitting subgroup of $G/F$ such that $G/E$ and $E/F$ are cyclic, and $cd(G|A')$ consists of degrees that divide $|T||E : F|$ and are divisible by $t|B|$, where $|B|$ is some divisor of $|E : F|$. 
CHAPTER 2

DIRECT PRODUCTS WHEN \( \text{cd}(O^r(G)) = \{1, n, m\} \)

In this chapter, we prove Theorem A and Corollary B of this dissertation and so we generalize Step 1 of Lewis’ argument which he used to prove Theorems A and B of [19]. Recall that Step 1 of Lewis’ argument is the following theorem, which is Theorem (5.2) of [19].

**Theorem 2.1.** Let \( p, q, \) and \( r \) be distinct primes. Assume that \( G \) is a group such that \( \text{cd}(K) = \{1, q, r\} \subseteq \text{cd}(G) \), where \( K = O^p(G) \). Then \( G \) splits over \( K \) and there is a complement \( N \) for \( K \) such that \( [K, N] \subseteq Z(K) \), \( Z(K) \) is a \( p' \)-group, \( K = C_K(N) \times [K, N] \), \( \text{cd}(K) = \text{cd}(C_K(N)) \), and \( G = C_K(N) \times N[K, N] \).

To generalize Theorem (2.1), we consider a group \( G \), where \( \text{cd}(G) = \{1, m, n\} \) for coprime integers \( m \) and \( n \). The case where \( m \) and \( n \) are distinct primes was originally studied by Isaacs and Passman in their paper [13]. A paper by Noritzsch, [25], has taken their results and expanded them to the general case. These groups fall into two categories based on their Fitting heights. Using standard notation we define the **Fitting subgroup** \( F(G) \) of \( G \) to be the largest normal nilpotent subgroup of \( G \). Inductively, we define \( F_0 = 1 \) and \( F_{i+1}/F_i = F(G/F_i) \) for integers \( i \geq 0 \). When \( G \) is solvable, it is clear that there is some integer \( i \) so that \( F_i = G \). We define the Fitting height of \( G \) to be the smallest integer \( i \) so that \( F_i = G \). From a theorem of Garrison (Corollary (12.21) of [10]), we know that the Fitting height of \( G \) is less than or equal to \( |\text{cd}(G)| \). When \( \text{cd}(G) = \{1, m, n\} \), it follows that the Fitting height of \( G \) is at most 3. Since \( mn \notin \text{cd}(G) \), we can see that \( G \) is not nilpotent.
and $G$ has Fitting height at least 2. The groups of interest can be categorized as those having Fitting height 3 and those having Fitting height 2. More detail of these groups is the content of Lemma (1.22). We will use this lemma in this chapter and also in proving Theorems D and E (Chapter 4).

Now, we consider a group $G$ which has a normal subgroup $K$, where $\text{cd}(K) = \{1, m, n\}$ for coprime integers $m$ and $n$. We also have a set of primes $\pi$ with the property that $O^\pi(K)$, the smallest normal subgroup of $K$ whose quotient is a $\pi$–group, is $K$ itself and the pairs $(p, m)$ and $(p, n)$ are strongly coprime pairs for every $p \in \pi$. If $P \in \text{Hall}_\pi(G)$, then we prove that $PK$ is a direct product or there is a prime $t$ such that $PK$ has a normal Sylow $t$–subgroup $H$ such that $\text{cd}(H) = \{1, t^l\}$ for some integer $l \geq 2$, $t^l \in \{m, n\}$, $K/H$ is abelian, and the Fitting height of $K$ is 2.

This result is a key step in proving Theorems D and E. Also, notice that the following Theorem (2.5) is a generalization of Theorem (2.1) because if we replace $\pi$ by the singleton $\{p\}$ and $m$ and $n$ are primes, then the conclusion (2) of Theorem (2.5) does not occur.

To obtain the main result of this section, we need the following lemmas.

**Lemma 2.2.** Let $K$ be a normal $\pi$–complement of a finite group $G$ and let $N$ be a Hall $\pi$–subgroup of $G$, where $\pi$ is a set of primes. If $[K, N] \subseteq Z(K)$, then $K = C_K(N) \times [K, N]$, $G = C_K(N) \times N[K, N]$, and $\text{cd}(K) = \text{cd}(C_K(N))$.

**Proof.** Since $K$ is a normal subgroup of $G$, this implies that $N$ acts coprimely on $K$. From general facts about coprime actions, we know that $K = [K, N]C_K(N)$ and $[K, N, N] = [K, N]$. It follows that $C_K(N)$ and $[K, N]$ are both normal subgroups of $K$ as $[K, N] \subseteq Z(K)$. Since $[K, N]$ is abelian, we deduce that $[K, N] = [K, N, N] \times C_{[K, N]}(N)$ by applying Fitting’s lemma. We obtain that $C_{[K, N]}(N) = 1$ as $[K, N] = [K, N, N]$. We conclude that $[K, N] \cap C_K(N) = 1$, and so $K = C_K(N) \times [K, N]$. Also, since $[K, N]$ is abelian, we have
that \( \text{cd}(K) = \text{cd}(C_K(N)) \).

On the other hand, we have \( G = KN \). We determine that \( G = N[K,N]C_K(N) \). Note that \( C_K(N) \) centralizes both \([K,N]\) and \( N \), hence \( C_K(N) \) and \( N[K,N] \) are both normal subgroups of \( G \). It is easy to see that \( C_K(N) \cap N[K,N] = 1 \). Thus, we deduce that \( G = C_K(N) \times N[K,N] \). □

**Lemma 2.3.** Let \( \pi \) be a set of primes, and let \( m \) and \( n \) be coprime integers such that \( p \) does not divide \( mn \) for every \( p \in \pi \). Assume that \( G \) is a group and suppose that \( \text{cd}(K) = \{1, m, n\} \subseteq \text{cd}(G) \), where \( K = O^\pi(G) \). Then \( G \) splits over \( K \) and there is a complement \( N \) for \( K \) such that the following hold:

1. \( K \) has a normal abelian Hall \( \pi \)-subgroup \( U \) such that \( P = UN \) is a Hall \( \pi \)-subgroup of \( G \), and \( Z(K) \) is a \( \pi' \)-group.

2. Suppose that the Fitting height of \( K \) is 3 and let \( F = F(K) \) be the Fitting subgroup of \( K \) such that \([K,N] \subseteq F\). If \( F/Z(K) \) is a \( \pi' \)-group, then \([K,N] \subseteq Z(K)\).

**Proof.** Since every prime \( t \in \pi \) divides no character degree of \( K \), we know that \( K \) has a normal abelian Hall \( \pi \)-subgroup \( U \) (see Lemma (1.10) and Lemma (1.8)). Letting \( R \) be a \( \pi \)-complement of \( K \), we obtain \( K = RU \). Observe that \([U,R]R \) is a normal subgroup of \( K \) having index that is a \( \pi \)-number. Since \( K = O^\pi(K) \), we conclude that \( K = [U,R]R \) and \( U = [U,R] \). By Fitting’s lemma, this implies that \( C_U(R) = 1 \), and thus \( N_K(R) = R \) and \( Z(K) \cap U = 1 \). In particular, \( Z(K) \) is a \( \pi' \)-group (this is the second part of (1)). By a Frattini argument, we have \( G = KN_G(R) \). It follows that \( R \) is a normal \( \pi \)-complement of \( N_G(R) \). By choosing a Hall \( \pi \)-subgroup \( N \) of \( N_G(R) \), we have \( K \cap N = 1 \) and \( G = KN \).

We observe that \( P = UN \) is a Hall \( \pi \)-subgroup of \( G \). This completes the proof of (1).

To prove (2), we assume that \( E/F = F(K/F) \) is the Fitting subgroup of \( K/F \). From Lemma (1.22)(a), we know that \( F = S \times Z(K) \), where \( S = [E,F] \) is a minimal normal
subgroup of $K$. Then $S$ is an elementary abelian $s$–group for some prime $s$ that does not divide $|E : F|$. Also, Lemma (1.22)(a) yields the fact that $\text{cd}(K) = \{1, |K : E|, |E : F|\} = \{1, m, n\}$, and $K/E$ and $E/F$ are both cyclic groups. We assume that $|K : E| = n$ and $|E : F| = m$.

By hypothesis of (2), we have that $F/\mathbb{Z}(K)$ is a $\pi'$–group, and so $s \notin \pi$. This implies that $t$ does not divide $|S|$ for all primes $t \in \pi$. Recall that $F = S \times \mathbb{Z}(K)$. Each prime $t$ in the set $\pi$ divides neither $|\mathbb{Z}(K)|$ nor $|K : F|$. Thus, $t$ does not divide $|F|$ nor does it divide $|K|$ for all primes $t \in \pi$. It follows that $U = 1$, $P = N$, and $R = K$. Now, $N$ acts coprimely on $K$ and $[K, N] = [K, N, N] \subseteq [F, N] \subseteq [K, N]$ as $[K, N] \subseteq F$ by the hypothesis. Therefore, we have $[K, N] = [F, N]$ and $[K, N] \mathbb{Z}(K)$ is a normal subgroup of $K$. Since $F/\mathbb{Z}(K) \cong S$ is a chief factor for $K$, either $F = [K, N] \mathbb{Z}(K)$ or $[K, N] \subseteq \mathbb{Z}(K)$. Consider a character $\chi \in \text{Irr}(G)$ with $\chi(1) = m$, so by Lemma (1.4), $\chi_E \in \text{Irr}(E)$. Because $\chi(1) > 1$, we obtain $S = E' \not\subseteq \ker(\chi_E)$. When we take characters $\sigma \in \text{Irr}(S)$ and $\zeta \in \text{Irr}(\mathbb{Z}(K))$ where $\sigma \times \zeta$ is an irreducible constituent of $\chi_F$, then $\sigma \neq 1_S$.

Since $\chi_E$ is $N$–invariant, $N$ acts on the irreducible constituents of $\chi_F$. We also know that $E/F$ acts transitively on the irreducible constituents of $\chi_F$. Because the action of $N$ on $E/F$ is coprime and central, we conclude that all of the irreducible constituents of $\chi_F$ are $N$–invariant (this is Lemma (1.14)). Also, let $\mu \in \text{Irr}(F)$ be an irreducible constituent of $\chi_F$. Then $\mu$ is $N$-invariant. There exists $\tau \in \text{Irr}(S)$ and $\zeta \in \text{Irr}(\mathbb{Z}(K))$ such that $\mu = \tau \times \zeta$. Note that $S = [E, F]$, where both $E$ and $F$ are characteristic in $K$. It follows that $S$ must be characteristic in $K$, and $N$ acts on $S$. Because $\tau \times \zeta$ is $N$–invariant, $\tau$, and thus $\tau \times 1_{\mathbb{Z}(K)}$, must themselves be $N$–invariant. We now know that $[F, N] \mathbb{Z}(K) \subseteq \ker(\tau \times 1) < F$. We conclude that $[K, N] = [F, N] \subseteq \mathbb{Z}(K)$. \qed

In the next theorem, we state a weak type of generalization of Theorem (2.1). Recall that the hypotheses of Theorem (2.1) indicate that $\text{cd}(K) = \{1, q, r\}$, where $K = \mathbb{O}^p(G)$. 


In the following theorem, we assume that \( \text{cd}(\mathbf{O}^{\pi}(G)) = \{1, q, r\} \), where \( \pi \) is a set of primes not including \( q \) and \( r \). In fact, we replace the singleton \( \{p\} \) by a set of primes \( \pi \) and we get the same result. We should mention that the proofs of Subcase 1 and Subcase 2 of the following theorem are similar to the proofs of Case 1 and Case 2 of Theorem (5.2) of [19].

**Theorem 2.4.** Let \( q \) and \( r \) be distinct primes, and let \( \pi \) be a set of primes including neither \( q \) nor \( r \). Assume that \( G \) is a group such that \( \text{cd}(K) = \{1, q, r\} \subseteq \text{cd}(G) \), where \( K = \mathbf{O}^{\pi}(G) \).

Also, suppose that \( U \) is a Hall \( \pi \)-subgroup of \( K \). Then \( G \) splits over \( K \) and there is a complement \( N \) for \( K \) such that \( [K, N] \subseteq \mathbf{Z}(K) \), \( N \subseteq C_G(U) \), \( K = C_K(N) \times [K, N] \), \( G = C_K(N) \times N[K, N] \), and \( \text{cd}(K) = \text{cd}(C_K(N)) \).

**Proof.** By applying Lemma (2.3), we obtain that \( U \) is a normal abelian Hall \( \pi \)-subgroup of \( K \), \( G \) splits over \( K \), and there is a complement \( N \) for \( K \) such that \( P = UN \) is a Hall \( \pi \)-subgroup of \( G \). Letting \( R \) be a \( \pi \)-complement of \( K \), we have \( K = RU \). We consider two cases: either \( N \) is a cyclic \( p \)-group for some prime \( p \in \pi \) or \( N \) is not a cyclic \( p \)-group for any prime \( p \in \pi \).

**Case 1:** \( N \) is a cyclic \( p \)-group for some prime \( p \in \pi \).

Let \( F \) denote \( \mathbf{F}(K) \). From the earlier discussion, we know that \( K \) has Fitting height 2 or 3. The proof now splits into two subcases based on the Fitting height of \( K \).

**Subcase 1:** \( K \) has Fitting height 3.

Write \( E/F = \mathbf{F}(K/F) \). From Lemma (1.22)(a), we know that \( F = S \times \mathbf{Z}(K) \), where \( S = [E, F] \) is a minimal normal subgroup of \( K \). Then \( S \) is an elementary abelian \( s \)-group for some prime \( s \) that does not divide \( |E : F| \). Also, Lemma (1.22)(a) yields the fact that \( \text{cd}(K) = \text{cd}(K/\mathbf{Z}(K)) = \{1, |K : E|, |E : F|\} = \{1, q, r\} \), and \( K/E \) and \( E/F \) are both cyclic groups. We assume that \( |K : E| = q \) and \( |E : F| = r \). Also, Lemma (1.22)(a) yields that \( E/\mathbf{Z}(K) \) is a Frobenius group with Frobenius kernel \( F/\mathbf{Z}(K) \), which
is isomorphic to $S$. Since $|E : F| = r$, we obtain that $s \neq r$. We know that $K/F$ is not nilpotent and $E/F$ is cyclic. Let $C/F = C_{G/F}(E/F)$. Then $C/F \cap K/F = C_{K/F}(E/F)$ and observe that $E/F \subseteq C_{K/F}(E/F) \subseteq K/F$. Since $E/F = F(K/F)$, it is well-known that $C_{K/F}(E/F) \subseteq E/F$. Hence, $E = C \cap K$. By applying Lemma (1.24) to $G/F$ and $E/F$, we conclude that $[G/F, K/F] \subseteq E/F$. Hence $K/E$ is central in $G/E$, and $K/E$ being a $p$–complement for $G/E$ yields the consequence that $PE$ is a normal subgroup of $G$ as $PE/E = NE/E \in Syl_p(G/E)$.

Consider a character $\chi \in \text{Irr}(G)$ such that $\chi(1) = q$. In view of (1.4), we see that $\chi_K \in \text{Irr}(K)$. From Lemma (1.22)(a), we know that $\text{cd}(E) = \{1, r\}$. If $\theta$ is an irreducible constituent of $\chi_E$, then $\theta(1)$ lies in $\text{cd}(E)$ and $\theta(1)$ divides $q$. The only way that these two statements can both be true is if $\theta(1) = 1$. By Frobenius reciprocity, $\chi_K$ is a constituent of $\theta^K$, and because $\chi(1) = q = |K : E| = \theta^K(1)$, we deduce that $\chi_K = \theta^K$. From Lemma (1.22)(a), $S = E'$, and since $\theta(1) = 1$, this implies that $S \subseteq \ker(\theta)$ and $\theta_S = 1_S$. Now we have $\theta_F = 1_S \times \zeta$ for some character $\zeta \in \text{Irr}(Z(K))$. Clearly, this character is $K$–invariant, and as $E/F$ is cyclic, it extends to $E$. It is not difficult to prove that $1_S \times \zeta$ has a unique $K$–invariant extension $\hat{\zeta} \in \text{Irr}(E)$. By Gallagher’s theorem (Lemma (1.1)), there exists a character $\lambda \in \text{Irr}(E/F)$ so that $\theta = \lambda \hat{\zeta}$. Since $K/E$ is cyclic and $\hat{\zeta}$ is $K$–invariant, it follows that $\hat{\zeta}$ extends to $\tilde{\zeta} \in \text{Irr}(K)$. Thus, we see that $\chi_K = \theta^K = (\lambda \tilde{\zeta})^K = \lambda^K \tilde{\zeta}$, and hence $\lambda^K \in \text{Irr}(K/F)$. We conclude that $\lambda \neq 1$ and so $\ker(\lambda) < E$. Because $|E : F| = r$, we know that $\lambda$ is a faithful character of $E/F$. Then by Lemma (1.24), we determine that $C$ is the stabilizer of $\lambda$ in $G$, where $C = C_G(E/F)$.

Write $T$ for the stabilizer of $\theta$ in $G$ so that $T \cap K = E$ by Lemma (1.24). Since $\chi_K$ is $G$–invariant, we use a Frattini argument to decide that $G = TK$, and it follows that $|G : T| = |K : T \cap K| = |K : E| = q$. Observe that any element that stabilizes $\theta$ must stabilize $\theta_F = 1_S \times \zeta$. By the uniqueness of $\tilde{\zeta}$, this element must stabilize $\tilde{\zeta}$. Thus, $T$ is
contained in the stabilizer of $\hat{\zeta}$. Consider an element \( t \in T \), so that we have

\[
\lambda \hat{\zeta} = \theta = \theta^t = \lambda^t \hat{\zeta}^t = \lambda^t \hat{\zeta},
\]

and applying Gallagher’s theorem (Lemma (1.1)), we determine that \( t \) stabilizes \( \lambda \). Thus, we conclude that \( T \subseteq C \), since \( C \) is the stabilizer of \( \lambda \). We have

\[
q = |G : T| \geq |G : C| \geq |CK : C| = |K : E| = q,
\]

which implies that \( G = CK \) and \( T = C \) since there is equality throughout this statement. Because \( C \) is normal in \( G \) and \( |G : C| = q \), we determine that \( P \subseteq C \), and so, \( PE \subseteq C \). Therefore, \( E/F \) is central in \( PE/F \). Observe that \( E/F \) is a \( p \)-complement in \( PE/F \) which implies that \( PF \) is normal in \( PE \). Because \( PF/F \) is a Sylow \( p \)-subgroup of \( G/F \) and \( PE \) is normal in \( G \), it follows that \( PF \) is normal in \( G \). Now we see that \([P, K] \subseteq [PF, K] \subseteq PF \cap K = F \) and \([N, R] \subseteq [P, K] \cap R \subseteq F \cap R \). We must deal with two possibilities: either \( s \in \pi \) or \( s \notin \pi \).

First, assume that \( s \notin \pi \). This implies that \( F/Z(K) \) is a \( \pi' \)-group. By applying Lemma (2.3), we obtain that \( Z(K) \) is a \( \pi' \)-group and \([N, K] \subseteq Z(K)\), as we know \([K, N] \subseteq [K, P] \subseteq F \). Thus, \( K \) is a \( \pi' \)-group. This implies that \( U = 1 \), and so \( N \subseteq C_G(U) \). Since \( N \) acts coprimely on \( K \) and \([N, K] \subseteq Z(K)\), it follows from Lemma (2.2) that \( K = C_K(N) \times [N, K] \), \( cd(C_K(N)) = cd(K) \), and \( G = C_K(N) \times N[N, K] \). Hence, we get the desired conclusion.

Now, we consider the case where \( s \in \pi \). We deal with two possibilities: either \( s = p \) or \( s \neq p \).

First, we assume that \( s = p \). By the second part of Lemma (2.3)(1), we know that \( Z(K) \) is a \( \pi' \)-group. Also, we have that \( K/F \) is a \( \pi' \)-group. It follows that \( S \) is a Hall \( \pi \)-subgroup of \( K \), and so \( S = U \). This implies that \( P \in \text{Syl}_p(G) \). Recall that \( PF \) is a normal subgroup of \( G \). This implies that \( R \) normalizes \( PF \), so \( R \) acts on the quotient \( PF/Z(K) \). Also, we see that \( PF = PSZ(K) = PZ(K) \) and that \( PF/Z(K) \cong P \), which
implies that $PF/Z(K)$ is a $p$–group. It follows that $Z(P)Z(K)/Z(K) = Z(PZ(K)/Z(K))$, and $Z(P)Z(K)$ is normalized by $R$. Observe that $F/Z(K)$ is a non-trivial normal subgroup of the $p$–group $PZ(K)/Z(K)$, and so $F/Z(K) \cap Z(PZ(K)/Z(K))$ is non-trivial. Then, since $Z(P)Z(K)/Z(K) = Z(PZ(K)/Z(K))$, we deduce $Z(K) < F \cap Z(P)Z(K)$ and $F \cap Z(P)Z(K)$ is normal in $K$. Since $F/Z(K)$ is a chief factor for $K$, we have $U \times Z(K) = F \subseteq Z(P)Z(K)$.

Because $U \subseteq P$, we conclude that $U \subseteq Z(P)$. In particular, $N \subseteq C_G(U)$, so $[K,N] = [UR,N] = [R,N]$ and $C_K(N) = UC_R(N)$.

Observe that $Z(K)$ is a $\pi$–complement for $F$. This implies that $Z(K) = F \cap R \subseteq R$, and so $Z(K) = F \cap R \subseteq Z(R)$. Recall that $[R,N] \subseteq F \cap R = Z(K)$. This implies that $[R,N] \subseteq F \cap R = Z(K) \subseteq Z(R)$. Hence, $N$ acts coprimely on $R$ such that $[R,N] \subseteq Z(R)$.

By applying Lemma (2.2), we obtain that $R = C_R(N) \times [N,R]$. Note that $K = RU = [N,R]UC_R(N) = [K,N]C_K(N)$, and it is easy to see that $C_K(N) \cap [K,N] = 1$. Since $[K,N] \subseteq Z(K)$, we conclude that $K = C_K(N) \times [N,K]$, and as $[N,K]$ is abelian, we obtain $cd(C_K(N)) = cd(K)$. Also, we determine that $G = N[N,K]C_K(N)$. Since $C_K(N)$ centralizes both $[K,N]$ and $N$, $C_K(N)$ and $N[N,K]$ are both normal in $G$. It is easy to show $N[K,N] \cap C_K(N) = 1$, and this yields $G = C_K(N) \times N[N,K]$. This completes the proof in this case.

Finally, we assume $s \neq p$. This forces that $K$ is a $p'$–group, and hence $K = O^p(G)$ and also $N \in Syl_p(G)$. Replacing $\pi$ by the singleton $\{p\}$ and using Lemma (2.3)(1), we deduce that $G$ splits over $K$ and there is a complement $W$ for $K$ which is a Sylow $p$–subgroup of $G$, as $K$ is a $p'$–group and $G/K$ is a $p$–group. We have $N,W \in Syl_p(G)$, and so $N = W^g$ for some $g \in G$. Recall that $[N,K] \subseteq [P,K] \subseteq F$. This implies $[W,K] = [N^{g^{-1}},K] = [N,K]^{g^{-1}} \subseteq F^{g^{-1}} = F$. Since $F/Z(K)$ is a $p'$–group and the Fitting height of $K$ is 3, it follows from Lemma (2.3)(2) that $[K,W] \subseteq Z(K)$ as $[W,K] \subseteq F$. We conclude that $[K,N] = [K,W^g] = [K,W]^g \subseteq Z(K)^g = Z(K)$. We have $[U,N] \subseteq [K,N] \subseteq Z(K)$, and hence $[U,N]$ is a $\pi'$–group. On the other hand, $U$ is a normal Hall $\pi$–subgroup of $K$. This
implies that $U$ is a normal subgroup of $G$, so $[U, N]$ is a subgroup of $U$. Thus, $[U, N]$ is a $\pi$–group, and hence $[U, N] = 1$. We obtain that $N \subseteq C_G(U)$.

Observe that $N$ acts coprimely on $K$ such that $[K, N] \subseteq Z(K)$. By applying Lemma (2.2), we get the desired conclusion.

**Subcase 2:** $K$ has Fitting height 2.

By Lemma (1.22)(b), we know that $|K : F| \in \text{cd}(K)$. Since $\text{cd}(K)$ is symmetric in $q$ and $r$, without loss of generality, we may assume that $|K : F| = q$. We obtain from Lemma (1.22)(b) that $\text{cd}(F) = \{1, r\}$, $F = H \times Z$ where $H \in \text{Syl}_t(K)$ (for some prime divisor $t$ of $|K|$) has nilpotence class 2 (i.e., $H' \subseteq Z(H)$), and $Z \subseteq Z(K)$. Hence, $t = r$ and $H$ is a Sylow $r$–subgroup of $K$. Also, we have $\text{cd}(H) = \text{cd}(F) = \{1, r\}$ and $K' \subseteq H$. Because $h$ does not divide either $|H|$ or $|K : K'|$ for all $h \in \pi$, we conclude that $h$ does not divide $|K|$ for all $h \in \pi$. Thus, we deduce that $K = R$, $U = 1$, and $N = P$. It follows that $N$ centralizes $U$. By Glauberman’s lemma (Lemma (1.13)), we may choose an $r$–complement $V$ for $K$ so that $[V, N] \subseteq V$. Write $C = C_H(V)$. From Lemma (1.22)(b), we know that $K' = [H, V]$, $H/H' = K'/H' \times C/H'$ and $K/H' = C/H' \times K'/V/H'$. Note that $V \cap F = Z$ and $K = FV$. As $H' \subseteq C$, we see that $C$ is normal in $H$.

Consider a character $\chi \in \text{Irr}(G)$ so that $\chi(1) = q$. Observe that $\chi_K \in \text{Irr}(K)$ (Lemma (1.4)), and write $\theta = \chi_K$. Note that every irreducible constituent of $\chi_H$ is linear and $H' \subseteq \ker(\theta)$. This implies that $\theta = \alpha \times \beta$ for characters $\alpha \in \text{Irr}(C/H')$ and $\beta \in \text{Irr}([H, V]/V/H')$. Since $C/H'$ is abelian, we have $\alpha(1) = 1$, and hence, $\beta(1) = \theta(1) = q$. Because $\theta$ is $N$–invariant, it follows that $1_C \times \beta$ is $N$–invariant. Thus, $N$ acts on the set $\Omega$ of irreducible constituents of $(1_C \times \beta)_H$ and $V$ acts transitively on $\Omega$. By Glauberman’s lemma (Lemma (1.13)), there is an $N$–invariant irreducible constituent $\gamma \in \text{Irr}(H/C)$ of $(1_C \times \beta)_H$. Observing that $(1_C \times \beta)_H = 1_C \times \beta_{[H, V]}$, we determine that $\gamma_{[H, V]}$ is irreducible and is a constituent of $\beta_{[H, V]}$. Use $\beta(1) \neq 1$ to see that $[H, V] = K' \not\subseteq \ker(\beta)$. This implies that $\gamma_{[H, V]} \neq 1_{[H, V]}$,
and hence, $\gamma \neq 1_H$. On the other hand, it is easy to show that $[H, N] \subseteq \ker(\gamma)$. Considering $C \subseteq \ker(\gamma)$, we conclude that $[H, N]C < H$.

Fix an irreducible character $\psi \in \Irr(G)$ with $\psi(1) = r$. By Lemma (1.4), we know that $\psi_K \in \Irr(K)$. Since $r$ does not divide $|K : H|$, we deduce that $\delta = \psi_H \in \Irr(H)$. Clearly, $\delta$ is $N$–invariant, and with $N$ centralizing $H/[H, N]C$, it follows that every irreducible constituent of $\delta|_{[H, N]C}$ is $N$–invariant (see Lemma (1.14)). Because $\delta \in \Irr(H)$ is nonlinear, $\delta$ must be $V$–invariant. From Lemma (1.22)(b), we see that $\delta$ is fully ramified with respect to $H/C$. As $\delta(1) = r$, we determine that $|H : C| = r^2$, and using Lemma (1.15), we have that $\delta$ is fully ramified with respect to $H/[H, N]C$. When we recall that $C \subseteq [H, N]C < H$ and $|H : C| = r^2$, the previous statement yields $[H, N]C = C$ and $[H, N] \subseteq C$. Thus, we have $[H, V, N] \subseteq [H, N] \subseteq C$ and $[N, H, V] \subseteq [C, V] = 1$. Applying Lemma (1.12), we obtain $[[V, N], H] \subseteq C$ and $[[V, N], H] \subseteq [V, H]$ since $V$ was chosen with $[V, N] \subseteq V$. It follows that $[[V, N], H] \subseteq C \cap [V, H] = H' < [V, H]$ and $[V, N] < V$ since if $[V, N] = V$, then $[[V, N], H] = [V, H]$. We now utilize $|V : V \cap Z(K)| = q$ to say that either $[V, N] \subseteq Z(K)$ or $V = [V, N](V \cap Z(K))$. If $V = [V, N](V \cap Z(K))$, then $[V, H] = [[V, N], H]$, which we know does not happen. Thus, we determine that $[V, N] \subseteq Z(K)$.

On the other hand, we claim that $[H, N] \subseteq Z(H)$. Suppose that $\varphi \in \Irr(H)$ is an arbitrary nonlinear irreducible character of $H$. Again, it follows from Lemma (1.22)(b) that $\varphi$ is fully ramified with respect to $H/C$. This implies that $r^2 = |H : C|$ and $\varphi_C = r\lambda$ for some linear character $\lambda$ of $C$. We obtain that $C \subseteq Z(\varphi)$, and so $C \subseteq Z(H)$. We conclude that $[H, N] \subseteq Z(H)$ as $[H, N] \subseteq C$.

Now, we show that $Z(H) \subseteq Z(K)$. We know that $r^2 = |H : C|$ and $C \subseteq Z(H)$. Also, since $H$ is an $r$–group, we have that $r^2$ divides $|H : Z(H)|$. We deduce that $C = Z(H)$, and so $H \subseteq C_K(C)$. Recall that $C = C_H(V)$, and hence $V \subseteq C_K(C)$. We determine that $K = HV \subseteq C_K(C)$, so $C \subseteq Z(K)$. We conclude that $Z(H) \subseteq Z(K)$.

Finally, we deduce that $[K, N] = [HV, N] \subseteq [H, N][V, N] \subseteq Z(H)Z(K) \subseteq Z(K)$, and
this yields the desired conclusion by using Lemma (2.2).

**Case 2:** $N$ is not a cyclic $p$–group for all $p \in \pi$.

Since $N$ is not a cyclic $p$–group for all $p \in \pi$, $G/K$ is not a cyclic $p$–group as $G/K$ is isomorphic to $N$. Then $N = \langle N_1, N_2, \ldots, N_r \rangle$, where $N_i$ is cyclic $p_i$–subgroup of $N$ with $p_i \in \pi$ for $1 \leq i \leq r$. Since $G/K$ is not a cyclic $p$–group for all $p \in \pi$, we deduce that $KN_i < G$. By induction, we have $[N_i, K] \subseteq Z(K)$ and $N_i \subseteq C_G(U)$. Thus, we obtain that $[N_i, K] \subseteq Z(K)$ and $N_i \subseteq C_G(U)$ for all $i$. We deduce that $N = \langle N_1, N_2, \ldots, N_r \rangle \subseteq C_G(K/Z(K))$ and also $N \subseteq C_G(U)$. This implies that $[N, K] \subseteq Z(K)$. We have $[K, N] = [RU, N] = [R, N]$ as $[N, U] = 1$. It follows that $[R, N] \subseteq Z(K) \subseteq R$ as $R$ is a Hall $\pi'$–subgroup of $K$ and $Z(K)$ is a normal $\pi'$–subgroup of $K$.

We determine that $[R, N] \subseteq Z(R)$. By applying Lemma (2.2), we conclude that $R = C_R(N) \times [R, N]$. It follows that $K = RU = UC_R(N) \times [K, N]$ as $UC_R(N) \cap [R, N] = 1$ and $[R, N] = [K, N] \subseteq Z(K)$. Since $N$ centralizes $U$ and $K = RU$, we have $UC_R(N) = C_K(N)$. Thus, we obtain that $K = C_K(N) \times [K, N]$. Since $[N, K]$ is abelian, we deduce $cd(C_K(N)) = cd(K)$. Observe that $G = KN$ and $N$ centralizes $C_K(N)$. We conclude that $G = C_K(N) \times N[N, K]$, and hence we get the desired conclusion. □

Now, we are ready to prove the main result of this chapter which is Theorem A of this dissertation.

**Theorem 2.5.** Let $\pi$ be a set of primes, and let $m$ and $n$ be coprime integers such that for every $p \in \pi$ the pairs $(p, m)$ and $(p, n)$ are strongly coprime pairs. Assume that $G$ is a group such that $cd(K) = \{1, m, n\} \subseteq cd(G)$, where $K = O^\pi(G)$. Also, suppose that $U$ is a Hall $\pi$–subgroup of $K$. Then $G$ splits over $K$ and there is a complement $N$ for $K$ such that one of the following holds:

1. $[K, N] \subseteq Z(K)$, $N \subseteq C_G(U)$, $K = C_K(N) \times [K, N]$, $G = C_K(N) \times N[K, N]$, and
\[ \text{cd}(K) = \text{cd}(C_K(N)). \]

2. There is a prime \( t \) such that \( G \) has a normal Sylow \( t \)-subgroup \( H \) such that \( \text{cd}(H) = \{1, t^l\} \) for some integer \( l \geq 2, t^l \in \{n, m\} \), \( K/H \) is abelian, and the Fitting height of \( K \) is 2.

**Proof.** By applying Lemma (2.3), we obtain that \( U \) is a normal abelian Hall \( \pi \)-subgroup of \( K \), \( G \) splits over \( K \), and there is a complement \( N \) for \( K \) such that \( K = UN \) is a Hall \( \pi \)-subgroup of \( G \). Letting \( R \) be a \( \pi \)-complement of \( K \), we have \( K = RU \). We consider two cases: either \( N \) is a cyclic \( p \)-group for some prime \( p \in \pi \) or \( N \) is not a cyclic \( p \)-group for all primes \( p \in \pi \).

**Case 1:** \( N \) is a cyclic \( p \)-group for some prime \( p \in \pi \).

Let \( F \) denote \( F(K) \). From the earlier discussion, we know that \( K \) has Fitting height 2 or 3. The proof now splits into two subcases based on the Fitting height of \( K \).

**Subcase 1:** \( K \) has Fitting height 3.

Write \( E/F = F(K/F) \). From Lemma (1.22)(a), we know that \( F = S \times Z(K) \), where \( S = [E,F] \) is a minimal normal subgroup of \( K \). Then \( S \) is an elementary abelian \( s \)-group for some prime \( s \) that does not divide \( |E:F| \). Also, Lemma (1.22)(a) yields the fact that \( \text{cd}(K) = \text{cd}(K/Z(K)) = \{1, |K:E|, |E:F|\} = \{1, m, n\} \), \( |K:E| \) is prime, and \( K/E \) and \( E/F \) are both cyclic groups. We assume that \( |K:E| = n \) is a prime and \( |E:F| = m \). Also, Lemma (1.22)(a) yields that \( E/Z(K) \) is a Frobenius group with Frobenius kernel \( F/Z(K) \), which is isomorphic to \( S \). Since \( |E:F| = m \), we obtain that \( s \) does not divide \( m \). We know that \( K/F \) is not nilpotent and \( E/F \) is cyclic. Let \( C/F = C_{G/F}(E/F) \). Then \( C/F \cap K/F = C_{K/F}(E/F) \) and observe that \( E/F \subseteq C_{K/F}(E/F) \subseteq K/F \). Since \( E/F = F(K/F) \), it is well-known that \( C_{K/F}(E/F) \subseteq E/F \). Hence, \( E = C \cap K \).

We consider two possibilities: \( m \) is prime and \( m \) is not prime. If \( m \) is prime, then since \( n \) is also prime, we can apply Case (1) of Theorem (2.4) to deduce \( [N,K] \subseteq Z(K) \).
$N \subseteq C_G(U)$, $K = C_K(N) \times [K, N]$, $G = C_K(N) \times N[K, N]$, and $\text{cd}(K) = \text{cd}(C_K(N))$. Thus, we get the conclusion (1).

Assume $m$ is not prime and let $m = q_1^{e_1}q_2^{e_2} \cdots q_j^{e_j}$ be the decomposition of $m$ into prime powers. Observe that $E/F$ is a cyclic group of order $m$, and hence $\text{Aut}(E/F)$ is an abelian group of order $q_1^{e_1-1}(q_1-1)q_2^{e_2-1}(q_2-1) \cdots q_j^{e_j-1}(q_j-1)$. By hypothesis, we know the pairs $(p_i, m)$ are strongly coprime pairs for every $p_i \in \pi$. It follows that $p_i$ does not divide $q_i - 1$ for every $p_i \in \pi$ and for every $1 \leq i \leq j$. This implies that $\text{Aut}(E/F)$ is a $\pi'$-group. Since $N$ is a $p$-group with $p \in \pi$, we deduce that $N$ acts trivially on $E/F$, and so $N \subseteq C$. Hence, $NE \subseteq C$. But since $(NE)K = G$, it follows that $C = NE(C \cap K) = NEE = NE$ by using Dedekind’s lemma. Therefore, $NE$ is a normal subgroup of $G$, as $C$ is a normal subgroup of $G$. Now, we have $(NF)E = NE$, $NF \cap E = F$, and $NF/F$ is a Sylow $p$-subgroup of $NE/F$. As $N$ centralizes $E/F$, $NE/F = NF/F \times E/F$ is a direct product. This implies that $NF$ is a normal subgroup of $G$ as $NF/F$ is a characteristic subgroup of $NE/F$.

Observe that $G/F = (K/F)(NF/F)$, and $NF/F$ and $PF/F$ are both Hall $\pi'$-subgroups of $G/F$ with $NF \subseteq PF$ as $N \subseteq P$. We deduce that $NF = PF$, and so $PF$ is a normal subgroup of $G$. Now, we see that $[P, K] \subseteq [PF, K] \subseteq PF \cap K = F$. We conclude that $[N, R] \subseteq [P, K] \cap R \subseteq F \cap R$. We must deal with two possibilities: either $s \in \pi$ or $s \notin \pi$.

First, assume that $s \notin \pi$. This implies that $F/\mathbf{Z}(K)$ is a $\pi'$-group. By applying Lemma (2.3), we obtain that $\mathbf{Z}(K)$ is a $\pi'$-group and $[N, K] \subseteq \mathbf{Z}(K)$, as we know $[K, N] \subseteq [K, P] \subseteq F$. Thus, $K$ is a $\pi'$-group. This implies $U = 1$, and so $N \subseteq C_G(U)$. Since $N$ acts coprimely on $K$ and $[N, K] \subseteq \mathbf{Z}(K)$, it follows from Lemma (2.2) that $K = C_K(N) \times [N, K]$, $\text{cd}(C_K(N)) = \text{cd}(K)$, and $G = C_K(N) \times N[N, K]$. Hence, we get the conclusion (1).

Now, we consider the case where $s \in \pi$. We deal with two possibilities: either $s = p$ or $s \neq p$.

First, we assume that $s = p$. By the second part of Lemma (2.3)(1), we know that $\mathbf{Z}(K)$ is a $\pi'$-group. Also, we have that $K/F$ is a $\pi'$-group. It follows that $S$ is a Hall
π–subgroup of $K$, and so $S = U$. This implies that $P \in \text{Syl}_{p}(G)$. Recall that $PF$ is a normal subgroup of $G$. This implies that $R$ normalizes $PF$, so $R$ acts on the quotient $PF/\mathcal{Z}(K)$. Also, we see that $PF = PS\mathcal{Z}(K) = P\mathcal{Z}(K)$ and that $PF/\mathcal{Z}(K) \cong P$, which implies that $PF/\mathcal{Z}(K)$ is a $p$–group. It follows that $\mathcal{Z}(P)\mathcal{Z}(K)/\mathcal{Z}(K) = \mathcal{Z}(P\mathcal{Z}(K)/\mathcal{Z}(K))$, and $\mathcal{Z}(P)\mathcal{Z}(K)$ is normalized by $R$. Observe that $F/\mathcal{Z}(K)$ is a non-trivial normal subgroup of the $p$–group $P\mathcal{Z}(K)/\mathcal{Z}(K)$, and so $F/\mathcal{Z}(K) \cap \mathcal{Z}(P\mathcal{Z}(K)/\mathcal{Z}(K))$ is non-trivial. Then, since $\mathcal{Z}(P)\mathcal{Z}(K)/\mathcal{Z}(K) = \mathcal{Z}(P\mathcal{Z}(K)/\mathcal{Z}(K))$, we deduce $\mathcal{Z}(K) < F \cap \mathcal{Z}(P)\mathcal{Z}(K)$ and $F \cap \mathcal{Z}(P)\mathcal{Z}(K)$ is normal in $K$. Since $F/\mathcal{Z}(K)$ is a chief factor for $K$, we have $U \times \mathcal{Z}(K) = F \subseteq \mathcal{Z}(P)\mathcal{Z}(K)$.

Because $U \subseteq P$, we conclude that $U \subseteq \mathcal{Z}(P)$. In particular, $N \subseteq C_{G}(U)$, so $[K,N] = [UR,N] = [R,N]$ and $C_{K}(N) = U\mathcal{C}_{R}(N)$.

Observe that $\mathcal{Z}(K)$ is a π–complement for $F$. This implies that $\mathcal{Z}(K) = F \cap R \subseteq R$, and so $\mathcal{Z}(K) = F \cap R \subseteq \mathcal{Z}(R)$. Recall that $[R,N] \subseteq F \cap R = \mathcal{Z}(K)$. This implies that $[R,N] \subseteq F \cap R$. Hence, $N$ acts coprimely on $R$ such that $[R,N] \subseteq \mathcal{Z}(R)$. By applying Lemma (2.2), we obtain $R = C_{R}(N) \times [N,R]$. Note that $K = RU = [N,R]U\mathcal{C}_{R}(N) = [K,N]C_{K}(N)$, and it is easy to see that $C_{K}(N) \cap [K,N] = 1$. Since $[K,N] \subseteq \mathcal{Z}(K)$, we conclude that $K = C_{K}(N) \times [N,K]$, and as $[N,K]$ is abelian, we obtain $\text{cd}(C_{K}(N)) = \text{cd}(K)$. Also, we determine that $G = N[N,K]C_{K}(N)$. Since $C_{K}(N)$ centralizes both $[K,N]$ and $N$, so $C_{K}(N)$ and $N[N,K]$ are both normal in $G$. It is easy to show $N[K,N] \cap C_{K}(N) = 1$, and this yields $G = C_{K}(N) \times N[N,K]$. This completes the proof of the conclusion (1) in this case.

Finally, we assume $s \neq p$. This forces that $K$ is a $p'$–group, and hence $K = O^{p}(G)$ and also $N \in \text{Syl}_{p}(G)$. Replacing $\pi$ by the singleton \{p\} and using Lemma (2.3)(1), we deduce that $G$ splits over $K$ and there is a complement $W$ for $K$, which is a Sylow $p$–subgroup of $G$ as $K$ is a $p'$–group and $G/K$ is a $p$–group. We have $N,W \in \text{Syl}_{p}(G)$, and so $N = W^{g}$ for some $g \in G$. Recall that $[N,K] \subseteq [P,K] \subseteq F$. This implies that $[W,K] = [N^{g^{-1}},K] = [N,K]^{g^{-1}} \subseteq F^{g^{-1}} = F$. Since $F/\mathcal{Z}(K)$ is a $p'$–group and the Fitting height of $K$ is 3,
it follows from Lemma (2.3)(2) that \([K,W] \subseteq Z(K)\) as \([W,K] \subseteq F\). We conclude that 
\([K,N] = [K,W]^g = Z(K) = \{K,W\} \subseteq Z(K)\). We have \([U,N] \subseteq [K,N] \subseteq Z(K)\), and hence \([U,N] \subseteq Z(K)\) is a \(\pi'\)-group. On the other hand, \(U\) is a normal Hall \(\pi\)-subgroup of \(K\). This implies that \(U\) is a normal subgroup of \(G\), so \([U,N] \subseteq Z(K)\) is a \(\pi\)-group, and hence \([U,N] = 1\). We obtain that \(N \subseteq C_G(U)\).

Observe that \(N\) acts coprimely on \(K\) such that \([K,N] \subseteq Z(K)\). By applying Lemma (2.2), we get the conclusion (1).

**Subcase 2:** \(K\) has Fitting height 2.

By Lemma (1.22)(b), we know that \(|K:F| \in \text{cd}(K)\). Since \(\text{cd}(K)\) is symmetric in \(m\) and \(n\), without loss of generality, we may assume that \(|K:F| = n\).

We obtain from Lemma (1.22)(b) that \(\text{cd}(F) = \{1,m\}\), \(F = H \times Z\), where \(H \in \text{Syl}_t(K)\) (for some prime divisor \(t\) of \(|K|\)) has nilpotence class 2 (i.e., \(H' \subseteq Z(H)\)), and \(Z \subseteq Z(K)\). Hence, \(m = t^l\) for some integer \(l \geq 1\), and \(\text{cd}(F) = \text{cd}(H) = \{1,m\}\). Also, observe that \(H \in \text{Syl}_t(G)\) is normal in \(G\), and since by Lemma (1.22)(b) we know that \(K/F\) is cyclic, we obtain that \(K/H\) is abelian. If \(m\) is not prime, then \(l \geq 2\), and hence we get the desired conclusion of (2).

Thus, we assume \(m\) is prime. Because \(h\) does not divide either \(|H|\) or \(|K : K'|\) for every \(h \in \pi\), we conclude that \(h\) does not divide \(|K|\) for every \(h \in \pi\). Thus, we deduce that \(K = R, U = 1, N = P\). This implies that \(N \subseteq C_G(U)\). By Glauberman’s lemma (Lemma (1.13)), we may choose a \(t\)-complement \(V\) in \(K\) so that \([V,N] \subseteq V\).

By Case (2) of Theorem (2.4), we may assume \(n\) is not a prime number. Let \(n = r_1^{f_1} r_2^{f_2} \cdots r_a^{f_a}\) be the decomposition of \(n\) into prime powers. Let \(K_1/F\) be a subgroup of \(K/F\) of order \(r_1\). Recall that \(K/F\) is cyclic, and hence \(K_1\) is a normal subgroup of \(G\). Then we claim that \(\text{cd}(K_1) = \{1,r_1,m\}\). To see this, let \(\nu \in \text{Irr}(K_1)\) be a nonlinear irreducible character of \(K_1\) and let \(\theta \in \text{Irr}(K)\) be an irreducible constituent of \(\nu^K\). By Frobenius
Reciprocity, $\nu \in \text{Irr}(K_1)$ is an irreducible constituent of $\theta_{K_1}$. Recall that $\text{cd}(K) = \{1, m, n\}$. If $\theta(1) = m$, then Lemma (1.4) implies that $\theta_{K_1}$ is irreducible because $(m, |K/K_1|) = 1$. It follows that $\theta_{K_1} = \nu$, and so $m \in \text{cd}(K_1)$. If $\theta(1) = n$, let $\omega$ be an irreducible constituent of $\nu_F$. Then $\omega$ is an irreducible constituent of $\theta_F$. Hence, $\omega(1) \in \text{cd}(F) = \{1, m\}$ and also $\omega(1)$ divides $\theta(1) = n$. Since $(m, n) = 1$, we deduce $\omega(1) = 1$ and so $\omega^K(1) = |K : F| = n = \theta(1)$. It follows that $\omega^K$ is irreducible, and so $\omega^{K_1}$ is irreducible. But $\nu$ is an irreducible constituent of $\omega^{K_1}$. This implies $\nu = \omega^{K_1}$, and hence $\nu(1) = \omega^{K_1}(1) = |K_1 : F| = r_1$. We conclude that $\text{cd}(K_1) = \{1, r_1, m\}$. Also, we have that $F = F(K_1), \ |K_1 : F| = r_1$ is a prime number, $K_1$ has Fitting height 2, and $K_1 = O^\pi(NK_1)$ because $K_1$ is a normal $\pi'$–subgroup of $NK_1$ whose index in $NK_1$ is a $\pi$–number (in fact, the index of $K_1$ in $NK_1$ is a $p$–power). We set $V_1 = V \cap K_1$. Note that $V_1$ is an $N$–invariant $t$–complement in $K_1$.

Since $N$ is a $p$–group, by applying Case (2) of Theorem (2.4) to the group $NK_1$, we deduce that $[V_1, N] \subseteq \mathbf{Z}(K_1)$.

On the other hand, we claim that $[H, N] \subseteq \mathbf{Z}(H)$. Write $C = C_H(V)$. From Lemma (1.22)(b), we know that $K' = [H, V], H/H' = K'/H' \times C/H'$ and $K/H' = C/H' \times K'/V/H'$. Note that $V \cap F = Z$ and $K = FV$. As $H' \subseteq C$, we see that $C$ is normal in $H$.

Consider a character $\chi \in \text{Irr}(G)$ so that $\chi(1) = n$. It follows from Lemma (1.4) that $\vartheta = \chi_K \in \text{Irr}(K)$. Note that every irreducible constituent of $\chi_H$ is linear and $H' \subseteq \ker(\vartheta)$. This implies that $\vartheta = \alpha \times \beta$ for characters $\alpha \in \text{Irr}(C/H')$ and $\beta \in \text{Irr}([H, V]/V/H')$. Since $C/H'$ is abelian, we have $\alpha(1) = 1$, and hence, $\beta(1) = \vartheta(1) = n$. Because $\vartheta$ is $N$–invariant, it follows that $1_C \times \beta$ is $N$–invariant. Thus, $N$ acts on the set $\Omega$ of irreducible constituents of $(1_C \times \beta)_H$ and $V$ acts transitively on $\Omega$. By Glauberman’s lemma (Lemma (1.13)), there is an $N$–invariant irreducible constituent $\gamma \in \text{Irr}(H/C)$ of $(1_C \times \beta)_H$. Observing that $(1_C \times \beta)_H = 1_C \times \beta_{[H, V]}$, we obtain that $\gamma_{[H, V]}$ is irreducible and is a constituent of $\beta_{[H, V]}$. Since $\beta$ is non-linear, we see that $[H, V] = K' \not\subseteq \ker(\beta)$. This implies that $\gamma_{[H, V]} \neq 1_{[H, V]}$, and hence $\gamma \neq 1_H$. Also, we have that $[H, N] \subseteq \ker(\gamma)$ as $\gamma$ is $N$–invariant. Considering
\[ C \subseteq \ker(\gamma), \text{ we conclude that } [H, N]C < H. \]

Fix an irreducible character \( \psi \in \text{Irr}(G) \) with \( \psi(1) = m. \) By Lemma (1.4), we know that \( \psi_K \in \text{Irr}(K) \). Recall that \( m \) is a prime. Since \( m \) does not divide \( |K : H| \), we deduce that \( \delta = \psi_H \in \text{Irr}(H) \). Clearly, \( \delta \) is \( N \)-invariant, and with \( N \) centralizing \( H/[H, N]C \), it follows from Lemma (1.14) that every irreducible constituent of \( \delta|_{H, N}C \) is \( N \)-invariant.

From Lemma (1.22)(b), we see that \( \delta \) is fully ramified with respect to \( H/C \). As \( \delta(1) = m \), we determine that \( |H : C| = m^2 \), and using Lemma (1.15), we have that \( \delta \) is fully ramified with respect to \( H/[H, N]C \). When we recall that \( C \subseteq [H, N]C < H \) and \( |H : C| = m^2 \), the previous statement yields \( [H, N]C = C \) and \( [H, N] \subseteq C \).

Suppose that \( \varphi \in \text{Irr}(H) \) is an arbitrary non-linear irreducible character of \( H \). Again, it follows from Lemma (1.22)(b) that \( \varphi \) is fully ramified with respect to \( H/C \). This implies that \( m^2 = |H : C| \) and \( \varphi_C = m\lambda \) for some linear character \( \lambda \) of \( C \). We obtain that \( C \subseteq Z(\varphi) \), and so \( C \subseteq Z(H) \). We conclude that \( [H, N] \subseteq Z(H) \), as \( [H, N] \subseteq C \).

Now, we show that \( Z(H) \subseteq Z(K) \). We know that \( m^2 = |H : C| \) and \( C \subseteq Z(H) \). Also, since \( H \) is an \( m \)-group, we have that \( m^2 \) divides \( |H : Z(H)| \). We deduce that \( C = Z(H) \), and so \( H \subseteq C_K(C) \). Recall that \( C = C_H(V) \), and hence \( V \subseteq C_K(C) \). We determine that \( K = HV \subseteq C_K(C) \), so \( C \subseteq Z(K) \). We conclude that \( Z(H) \subseteq Z(K) \).

Recall that \( F = H \times Z \) is the Fitting subgroup of \( K \), where \( Z \subseteq Z(K) \). Also, since \( F \subseteq K_1 \), it follows that \( F \) is the Fitting subgroup of \( K_1 \) as \( |K_1 : F| = r_1 \). This implies that \( Z \subseteq Z(K_1) \), and so \( Z(K_1) \subseteq Z(F) = Z(H) \times Z \subseteq Z(K) \), as \( Z(H) \subseteq Z(K) \).

Observe that \( K/K_1 \) is a cyclic group of order \( n/r_1 \), and hence \( \text{Aut}(K/K_1) \) is an abelian group of order \( r_1^{l_1-2}(r_1 - 1)r_2^{l_2-1}(r_2 - 1) \cdots r_u^{l_u-1}(r_u - 1) \). By hypothesis, we know that the pair \( (p, n) \) is a strongly coprime pair. It follows that \( p \) does not divide \( r_i - 1 \) for every \( 1 \leq i \leq u \). This implies that \( \text{Aut}(K/K_1) \) is a \( p' \)-group. Since \( NK_1/K_1 \) acts on \( K/K_1 \) and \( NK_1/K_1 \) is a \( p \)-group, we deduce that \( NK_1/K_1 \) centralizes \( K/K_1 \), and so \( NK_1 \) is a normal subgroup of \( G \), as \( G = NK \). This implies \( [V, N] \subseteq V \cap NK_1 = V_1 \). Since \( N \) acts on \( V \)}
coprimely, we have

\[ [V, N] = [V, N, N] \subseteq [V_1, N] \subseteq \mathbb{Z}(K_1) \subseteq \mathbb{Z}(K). \]

Finally, we deduce that \([K, N] = [HV, N] \subseteq [H, N][V, N] \subseteq \mathbb{Z}(H)\mathbb{Z}(K) \subseteq \mathbb{Z}(K)\), and this yields the desired conclusion by using Lemma (2.2).

**Case 2:** \(N\) is not a cyclic \(p\)-group for all \(p \in \pi\).

Since \(N\) is not a cyclic \(p\)-group for all \(p \in \pi\), this implies that \(G/K\) is not a cyclic \(p\)-group as \(G/K\) is isomorphic to \(N\). Then \(N = \langle N_1, N_2, \ldots, N_r \rangle\), where \(N_i\) is cyclic \(p_i\)-subgroup of \(N\) with \(p_i \in \pi\) for \(1 \leq i \leq r\). Since \(G/K\) is not a cyclic \(p\)-group for all \(p \in \pi\), we deduce that \(KN_i < G\). By induction, either \(KN_i\) has a normal Sylow \(t\)-subgroup for some prime \(t\) with the properties of (2) or we have \([N_i, K] \subseteq \mathbb{Z}(K)\) and \(N_i \subseteq C_G(U)\).

But if \(KN_i\) has a normal Sylow \(t\)-subgroup for some prime \(t\) with the properties of (2), then \(K\) has a normal Sylow \(t\)-subgroup for some prime \(t\) with the properties of (2), and so \(G\) has a normal Sylow \(t\)-subgroup for some prime \(t\) with the properties of (2). Thus, we assume that \([N_i, K] \subseteq \mathbb{Z}(K)\) and \(N_i \subseteq C_G(K/Z(K))\) for all \(i\). In other words, \(N_i \subseteq C_G(K/Z(K))\) for all \(i\). We deduce that \(N = \langle N_1, N_2, \ldots, N_r \rangle \subseteq C_G(K/Z(K))\) and also \(N \subseteq C_G(U)\).

This implies that \([N, K] \subseteq \mathbb{Z}(K)\). We have \([K, N] = [RU, N] = [R, N]\), as \([N, U] = 1\).

It follows that \([R, N] \subseteq \mathbb{Z}(K) \subseteq R\), as \(R\) is a Hall \(\pi'\)-subgroup of \(K\) and \(\mathbb{Z}(K)\) is a normal \(\pi'\)-subgroup of \(K\). We determine that \([R, N] \subseteq \mathbb{Z}(R)\). By applying Lemma (2.2), we conclude that \(R = C_R(N) \times [R, N]\). It follows that \(K = RU = UC_R(N) \times [K, N]\), as \(UC_R(N) \cap [R, N] = 1\) and \([R, N] = [K, N] \subseteq \mathbb{Z}(K)\). Since \(N\) centralizes \(U\) and \(K = RU\), we have \(UC_R(N) = C_K(N)\). Thus, we obtain that \(K = C_K(N) \times [K, N]\). Since \([N, K]\) is abelian, we deduce \(cd(C_K(N)) = cd(K)\). Observe that \(G = KN\) and \(N\) centralizes \(C_K(N)\).

We conclude that \(G = C_K(N) \times N[N, K]\), and hence (1) holds.

\[\square\]

In the following corollary, we suppose that \(p\) and \(q\) are distinct primes and \(cd(O^\pi(G)) =\]
\{1, p, q\} \subseteq \text{cd}(G), \text{ where } \pi \text{ is a set of primes not including } p \text{ and } q. \text{ Then it is easy to see that the conclusion (2) of Theorem (2.5) does not occur.}

**Corollary 2.6.** Let $\pi$ be a set of primes not including the primes $p$ and $q$. Assume that $G$ is a group so that $\text{cd}(K) = \{1, p, q\} \subseteq \text{cd}(G)$, where $K = O^\pi(G)$. Then $G$ splits over $K$ and there is a complement $N$ such that $G = C_K(N) \times N[K, N]$ and $\text{cd}(K) = \text{cd}(C_K(N))$.

**Proof.** Since $\text{cd}(O^\pi(G)) = \{1, p, q\} \subseteq \text{cd}(G)$, and $p$ and $q$ are primes, it follows that Conclusion (2) of Theorem (2.5) does not occur. By Conclusion (1) of Theorem (2.5), we obtain that $G$ splits over $K$ and there is a complement $N$ such that $G = C_K(N) \times N[K, N]$ and $\text{cd}(K) = \text{cd}(C_K(N))$. \(\square\)

We now obtain another corollary which is Corollary B of this dissertation. In this corollary, we assume that we have a group $G$ so that $\text{cd}(G) = \{1, a, b, c, ab, ac\}$, where $a, b,$ and $c$ are pairwise coprime integers greater than 1 such that the pairs $(p, b)$ and $(p, c)$ are strongly coprime pairs for each prime divisor $p$ of $a$. We also assume that $\text{cd}(O^\pi(G)) = \{1, b, c\}$, where $\pi = \pi(a)$ is the set of all prime divisors of $a$. Under this additional hypothesis, we prove the conclusion of Theorem D.

**Corollary 2.7.** Let $a, b,$ and $c$ be pairwise coprime integers greater than 1 such that the pairs $(p, b)$ and $(p, c)$ are strongly coprime pairs for each prime divisor $p$ of $a$. Suppose that $\text{cd}(G) = \{1, a, b, c, ab, ac\}$ and $\text{cd}(K) = \{1, b, c\}$, where $K = O^\pi(G)$ and $\pi = \pi(a)$. Then one of the following holds:

1. $G = A \times B$, where $\text{cd}(A) = \{1, a\}$ and $\text{cd}(B) = \{1, b, c\}$.

2. There is a prime $t$ such that $G$ has a normal Sylow $t$-subgroup $T$ with $\text{cd}(T) = \{1, t^l\}$.
for some integer \( l \geq 2 \), \( t^l \in \{b, c\} \), \( K/T \) is abelian, and the Fitting height of \( K \) is 2.

**Proof.** We assume that the conclusion (2) is not true and we prove the conclusion (1). By Theorem (2.5), there is a complement \( N \) for \( K \) in \( G \) so that \( G = C_K(N) \times [K, N]N \) and \( cd(K) = cd(C_K(N)) \). Take \( A = [K, N]N \) and \( B = C_K(N) \). Then \( cd(B) = cd(K) = \{1, b, c\} \). It is easy to see that \( cd(A) = \{1, a\} \), and we have the result. \( \square \)
CHAPTER 3

OBTAINING THE CHARACTER DEGREES OF $O^\pi(G)$

In this chapter, we prove Theorem C of this dissertation and hence we generalize Step 2 of Lewis’ argument which he used to prove Theorems A and B of [19].

Recall that Step 2 asserts that if $p, q,$ and $r$ are distinct primes and $G$ is a finite group with $\text{cd}(G) = \{1, p, q, r, pq, pr\}$, then $\text{cd}(O^p(G)) = \{1, q, r\}$.

In [21], Lewis found a characterization of solvable groups $G$ whose degree graphs $\Delta(G)$ have two connected components, where the degree graph $\Delta(G)$ has vertex set $\rho(G)$ that consists of the primes that divide the character degrees of $G$ and there is an edge between $p$ and $q$ if $pq$ divides some degree $a \in \text{cd}(G)$. This graph was introduced in [23]. In particular, in [23] it is proved that if $G$ is solvable then $\Delta(G)$ has at most two connected components.

In fact, in section (2) of [21], Lewis presented six examples (Examples (2.1)-(2.6) of [21]) of solvable groups that have degree graphs with two connected components, and then in the main theorem of [21], he proved that if $G$ is a solvable group, then $\Delta(G)$ has two connected components if and only if $G$ is one of the groups included in Examples (2.1)-(2.6) of [21]. Also, in section (3) of that paper, he found the connected components of each example included in Examples (2.1)-(2.6) (Lemmas (3.1)-(3.6) of [21]). We briefly stated these lemmas as Lemma (1.26), which we will use in the following.

**Lemma 3.1.** Let $a, b,$ and $c$ be pairwise coprime integers greater than 1. Suppose that $G$ is a solvable group such that $\text{cd}(G) = \{1, a, b, bu_1, bu_2, \ldots, bu_i\}$, where $u_1, u_2, \ldots, u_i$ are some non-trivial divisors of $c$. Also, suppose that $G$ does not satisfy the hypotheses of Case
(5) of Lemma (1.26). Then \( \text{cd}(\text{O}^\pi(G)) = \{1, a, b\} \), where \( \pi = \pi(c) \).

**Proof.** Observe that \( \Delta(G) \) has two connected components, \( \pi(a) \) and \( \pi(b) \cup \pi(u_1 u_2 \cdots u_i) \). Hence, \( G \) satisfies the hypotheses of one of Cases (1)-(5) of Lemma (1.26). Since \( G \) does not satisfy the hypotheses of Case (5) of Lemma (1.26), to complete this proof it suffices to show that \( G \) does not satisfy the hypotheses of Cases (2)-(4) and if \( G \) satisfies the hypotheses of Case (1), then we have the desired result.

If \( G \) satisfies the hypotheses of Case (1) of Lemma (1.26), then Lemma (1.26)(1) implies that \( a = s^n \) for some prime \( s \) and some positive integer \( n \), and \( G \) has a normal non-abelian Sylow \( s \)-subgroup \( S \) such that \( G/S \) is abelian. Let \( C/S \) be the Hall \( \pi' \)-subgroup of \( G/S \). Since \( C \) is a normal subgroup of \( G \) whose order is a \( \pi' \)-number and whose index is a \( \pi \)-number, we deduce that \( C = \text{O}^\pi(G) \). We claim that \( \text{cd}(C) = \{1, a, b\} \). To see this, since \( a, b \in \text{cd}(G) \) are \( \pi' \)-numbers, Lemma (1.4) implies that \( a, b \in \text{cd}(C) \). Also, if \( \theta \in \text{Irr}(C) \) and \( \chi \in \text{Irr}(G \mid \theta) \), then \( \chi(1) = et\theta(1) \), where \( e \) and \( t \) divide \( |G:C| \), and so are \( \pi \)-numbers. Also, \( \theta(1) \) is a \( \pi' \)-number as \( C \) is a \( \pi' \)-group. Hence, \( \theta(1) \) is the \( \pi' \)-part of some character degree of \( G \). From the list of character degrees, the \( \pi' \)-parts of character degrees of \( G \) are \( 1, a, b \). We conclude that \( \text{cd}(C) = \{1, a, b\} \). Hence, we are done in this case.

Since \( \Delta(G) \) has more than two vertices, Case (2) of Lemma (1.26) does not occur. If \( G \) satisfies the hypotheses of Case (3) of Lemma (1.26), then by applying Lemma (1.26)(3) we conclude \( \Delta(G) \) has two connected components \( \pi(|G:E|) \) and \( \pi(|E:F|) \), where \( F \) is the Fitting subgroup of \( G \), \( E/F \) is the Fitting subgroup of \( G/F \) such that \( G/E \) and \( E/F \) are cyclic. Also, we know \( \text{cd}(G) = \{d \mid d \text{ divides } |G:E| \} \cup \{|E:F|\} \) by Lemma (1.26)(3) and Theorem (3.2) of [24]. Since none of \( u_1, u_2, \ldots, u_i \) is a character degree of \( G \), this implies that \( bu_j \) does not divide \( |G:E| \) for each \( 1 \leq j \leq i \), and so \( bu_1 = bu_2 = \cdots = bu_i = |E:F| \) as \( bu_1, bu_2, \ldots, bu_i \in \text{cd}(G) \). But \( \Delta(G) \) has two connected components, \( \pi(|G:E|) \) and
\[ \pi(|E : F|) = \pi(bu_1), \text{ and } |E : F| = bu_1 \text{ is the only degree in } \text{cd}(G) \text{ divisible by primes in } \pi(|E : F|) = \pi(bu_1) \text{ as } \text{cd}(G) = \{d | d \text{ divides } |G : E|\} \cup \{|E : F|\}. \] On the other hand, \( b \) is also a degree in \( \text{cd}(G) \) which is divisible by primes in \( \pi(bu_1) \). This forces that \( b = bu_1 \), and hence \( u_1 = 1 \), which is a contradiction.

If \( G \) satisfies the hypotheses of Case (4), then Lemma (1.26)(4) shows that \( \{2\} \) and \( \pi(2^l + 1) \), where \( l \) is a positive integer such that \( 2^l + 1 \) is a character degree of \( G \), are two connected components of \( \Delta(G) \). Also, by Lemma (1.26)(4), we know that \( \text{cd}(G) \) consists of 1, powers of 2, and \( 2^l + 1 \). Since \( \text{cd}(G) = \{1, a, b, bu_1, bu_2, \ldots, bu_i\} \), this is a contradiction as \( a, b, \) and \( u_j \) are pairwise coprime for each \( 1 \leq j \leq i \).

In the following lemma, we let \( G \) be a solvable group such that the character degree set of \( G \) is \( \text{cd}(G) = \{1, a, b, bu_1, bu_2, \ldots, bu_i\} \), where \( a, b, c \) are pairwise coprime integers, and \( u_1, u_2, \ldots, u_i \) are some non-trivial divisors of \( c \). We suppose that \( F \) and \( E/F \) are the Fitting subgroups of \( G \) and \( G/F \), respectively, and we assume that \( G \) satisfies the hypotheses of Case (5) of Lemma (1.26). Then we prove that \( a \) is a prime number, \( |G : E| = a \), and \( |E : F| = b \).

**Lemma 3.2.** Let \( a, b, \) and \( c \) be pairwise coprime integers greater than 1. Suppose that \( G \) is a solvable group such that \( \text{cd}(G) = \{1, a, b, bu_1, bu_2, \ldots, bu_i\} \), where \( u_1, u_2, \ldots, u_i \) are some non-trivial divisors of \( c \). Also, suppose that \( G \) satisfies the hypotheses of Case (5) of Lemma (1.26). Then \( a \) is a prime number, \( |G : E| = a \), and \( |E : F| = b \), where \( F \) and \( E/F \) are the Fitting subgroups of \( G \) and \( G/F \), respectively.

**Proof.** Since \( G \) satisfies the hypotheses of Case (5) of Lemma (1.26), Lemma (1.26)(5) implies that \( \Delta(G) \) has two connected components \( \pi(|G : E|) \) and \( \pi(|E : F|) \cup \{p\} \), where \( p \) is some prime such that \( G \) has a normal Sylow \( p \)-subgroup \( P \), \( F \) is the Fitting subgroup
of $G$, and $E/F$ is the Fitting subgroup of $G/F$. Also, there is a normal subgroup $A$ of $G$ such that $A' \subseteq F$ and $G/A'$ satisfies the hypotheses of Case (3) of Lemma (1.26), and so by applying Lemma (1.26)(3) and Theorem (3.2) of [24], we have that $\text{cd}(G/A') = \{d \mid d \text{ divides } |G : E| \} \cup \{|E : F|\}$. We note that $p$ does not divide $|E : F|$, as $G$ has a normal Sylow $p$–subgroup $P$ which is contained in $F$. We observe that $\text{cd}(G)$ contains degrees divisible by $p$ and we know $|E : F| \in \text{cd}(G)$ is divisible by only primes in $\pi(|E : F|)$ and not by $p$. Also, since $a$ is the only degree in $\text{cd}(G)$ divisible by primes in $\pi(a)$, which is one of two connected components of $\Delta(G)$, we conclude that $|G : E| = a$, and so all divisors of $a$ are character degrees of $G$, as $\Omega \subseteq \text{cd}(G)$, where $\Omega$ is the set of all divisors of $|G : E|$. But we know that $a$ is the only degree in $\text{cd}(G)$ divisible by primes in $\pi(a)$. We obtain that $a$ must be a prime.

Therefore, since $|G : E| = a$ is a prime number, we deduce that

$$\text{cd}(G/A') = \{1, a, |E : F|\}.$$ 

If $|E : F| = bu_j$ for some $1 \leq j \leq i$, then $p$ does not divide $bu_j$ and $\text{cd}(G/A') = \{1, a, bu_j\}$. This implies that $p$ does not divide $b$ and $\{b, bu_l \mid l \neq j\} \subseteq \text{cd}(G/A')$, where $\text{cd}(G/A') = \{\theta(1) \mid \theta \in \text{Irr}(G/A')\}$ and $\text{Irr}(G/A')$ is the set of irreducible characters of $G$ whose kernels do not contain $A'$. On the other hand, we know from Lemma (1.26)(5) that $p$ divides all members of $\text{cd}(G/A')$. In particular, $p$ divides $b$, which is a contradiction. We conclude that $|E : F| = b$. \qed

In the next lemma, we assume that $G$ is a solvable group and $M$ is a normal subgroup of $G$ such that $G/M$ is a Frobenius group with Frobenius kernel $N/M$ an elementary abelian $t$–group for some prime $t$. Also, we assume that $|G : N| = m$ for some integer $m > 1$ such that the numbers 1 and $m$ are the only $\pi$-parts of degrees in $\text{cd}(G)$, where $\pi = \pi(m)$. Then, under some conditions on $\text{cd}(N)$, we show $\text{cd}(O^*(G)) = \{1, a, b\}$. 

Lemma 3.3. Let $a, b,$ and $m$ be pairwise coprime integers greater than $1$ such that $m$ does not divide $u - 1$ when $u$ is a proper prime divisor of $b$. Suppose that $G$ is a solvable group and $M$ is a normal subgroup of $G$ such that $G/M$ is a Frobenius group with Frobenius kernel $N/M$ an elementary abelian $t$–group for some prime divisor $t$ of $b$. Also, suppose that $|G : N| = m$, the numbers $1$ and $m$ are the only $\pi$–parts of degrees in $cd(G)$, where $\pi = \pi(m)$, and $cd(N) = \{1, a, b, bu_1, bu_2, \ldots, bu_i\}$, where $u_1, u_2, \ldots, u_i$ are some non-trivial divisors of $m$. Then $cd(O^\pi(G)) = \{1, a, b\}$.

Proof. Observe that $\Delta(N)$ has two connected components $\pi(a)$ and $\pi(b) \cup \pi(u_1u_2 \cdots u_i)$.

We have two possibilities: either $N$ does not satisfy the hypotheses of Case (5) of Lemma (1.26) or $N$ satisfies the hypotheses of Case (5):

If $N$ does not satisfy the hypotheses of Case (5) of Lemma (1.26), then Lemma (3.1) implies that $cd(O^\pi(N)) = \{1, a, b\}$, and so $cd(O^\pi(G)) = \{1, a, b\}$, as $|G : N| = m$ and $\pi = \pi(m)$.

Assume $N$ satisfies the hypotheses of Case (5) of Lemma (1.26). Lemma (1.26)(5) implies that $\Delta(N)$ has two connected components, $\pi(|N : E|)$ and $\pi(|E : F|) \cup \{p\}$, where $p$ is some prime such that $N$ has a normal Sylow $p$–subgroup $P$, $F = P \times Z$ is the Fitting subgroup of $N$ for some central subgroup $Z$ of $N$, and $E/F$ is the Fitting subgroup of $N/F$ such that $N/E$ and $E/F$ are cyclic. By applying previous lemma, we have that $|N : E| = a$ and $|E : F| = b$. This implies that $u_1, u_2, \ldots, u_i$ are $p$–powers, as $a, b,$ and $m$ are pairwise coprime integers.

We claim that $F \subseteq M$. To see this, we know that $p \in \pi$ as $u_1, u_2, \ldots, u_i$ are $p$–powers. Also, since $t$ divides $b$ by hypothesis, we have that $p \neq t$. Hence, $P \subseteq M$ as $M$ is a normal subgroup of $N$ of $t$–power index. On the other hand, if all irreducible characters of $Z$ are $G$–invariant, then $\chi_Z = f\lambda$ for some integer $f$ and $\lambda \in \text{Irr}(Z)$, where $\chi \in \text{Irr}(G)$ is an
arbitrary irreducible character of $G$. Thus, $Z \subseteq Z(\chi)$, and so $Z \subseteq Z(G)$. Suppose that $x \in Z$ is an arbitrary element of $Z$. Since $x \in Z \subseteq Z(G)$, we must have $C_{G/M}(xM) = G/M$. But we know that $G/M$ is a Frobenius group. We conclude that $x \in M$, and hence $Z \subseteq M$. We obtain that $F = P \times Z \subseteq M$ as claimed.

Since $G/M$ is a Frobenius group with kernel $N/M$, this implies that $|G : N| = m$ divides $|N/M| - 1$. We have $(|N : M|, |N : E|) = 1$, so $N = ME$. Observe that $E/E \cap M$ is cyclic because $F \subseteq E \cap M$ and $E/F$ is cyclic. Hence, $N/M$ is both an elementary abelian $t$–group and cyclic. This forces that $|N : M| = t$, and hence $|N : M|$ is a prime divisor of $|E : F|$. Since $m$ does not divide $u - 1$ when $u$ is a proper prime divisor of $b$, we obtain that $|N : M| = |E : F|$ and $F = E \cap M$. Thus, $N/F = E/F \times M/F$ is abelian which is a contradiction.

We conclude that $Z$ has an irreducible character $\lambda$ that is not $G$–invariant. It follows that $N \subseteq \text{Stab}_G(\lambda) < G$. Since $|G : N| = m$, this implies that $|G : \text{Stab}_G(\lambda)|$ is a divisor of $m$. Observe that $\lambda$ extends to $1_P \times \lambda \in \text{Irr}(F)$ and $1_P \times \lambda$ is invariant in $\text{Stab}_G(\lambda)$. Since all Sylow subgroups of $G/F$ are cyclic, we can use Lemma (1.7) and Lemma (1.6) to see that $\lambda$ extends to $\text{Stab}_G(\lambda)$. We deduce that $|G : \text{Stab}_G(\lambda)|$ is a character degree in $G$. Since it is a $\pi$–degree, it must be $m$.

We know that $u_1, u_2, \ldots, u_i$ are all character degrees of $P$. Now, let $\alpha \in \text{Irr}(P)$ with degree $u_1$. Then $\text{Stab}_G(\alpha \times \lambda) \subseteq \text{Stab}_G(\lambda)$. By the Clifford correspondence, $m u_1$ divides an irreducible character degree of $G$. This is a contradiction as $1$ and $m$ are the only $\pi$-parts of degrees in $\text{cd}(G)$.

In light of Corollary (2.7), to prove Theorem D of this dissertation it suffices to show that $\text{cd}(O^p(G)) = \{1, n, m\}$. Our strategy depends on the following. We take $K$ to be a subgroup of $G$ that is maximal with the property that $K$ is normal in $G$ and $G/K$ is non-abelian. We know that $\text{cd}(G/K) = \{1, f\}$ for some character degree $f$ and $G/K$ is either an
s–group for some prime s or G/K is a Frobenius group (this is Lemma (1.3)). We break up our proof into different cases depending on the value of f. There are three different cases: 
f = p, f ∈ \{pm, pm\}, and f ∈ \{n, m\}. It is obvious that when f ∈ \{pm, pm\}, G/K cannot be an s–group for any prime s. When f ∈ \{n, m\}, we will apply Lemma (1.25).

In the next lemma, which is part (1) of Theorem C of this dissertation, we prove the result when f ∈ \{pm, pm\}, and in fact, we prove a more general result. We present the more general context. Suppose a, b, and m are pairwise coprime integers greater than 1. Also, assume that G is a group with cd(G) = \{1, m, a, b, ma, mb\} and there is a normal subgroup K of G such that G/K is a Frobenius group with kernel N/K whose index is ma. If m does not divide u − 1 for all proper prime power divisors u of b, then cd(O^\pi(G)) = \{1, a, b\}, where \pi = \pi(m).

**Lemma 3.4.** Let a, b, and m be pairwise coprime integers greater than 1 such that m does not divide u − 1 for all proper prime power divisors u of b. Suppose that G is a solvable group with cd(G) = \{1, m, a, b, ma, mb\}, and that K is a normal subgroup of G so that G/K is a Frobenius group with kernel N/K. If |G : N| = ma, then cd(O^\pi(G)) = \{1, a, b\}, where \pi = \pi(m).

**Proof.** We may assume that N/K is a chief factor of G because if K ⊆ K_1 ⊆ N, where N/K_1 is a chief factor of G, then G/K_1 is still a Frobenius group with kernel N/K_1, and notation can be changed so that K = K_1.

By Lemma (1.3)(b), we deduce that N/K is an elementary abelian q–group for some prime divisor q of b because b ∈ cd(N) and b|G : N| is not a character degree of G. Suppose that R/N is a normal subgroup of G/N of index m and order a. We know that for all non-trivial divisors u and v of m and a, respectively, by applying Lemma(1.3)(b), u, v, and uv are not character degrees of N. It follows from Lemma (1.4) that if u is any non-trivial
divisor of $m$ and $v$ is any non-trivial proper divisor of $a$, then $u, v,$ and $uv$ are not character degrees of $R$ as $|R : N| = a$ and $|G : R| = m$. Therefore,

$$\{1, a, b\} \subseteq \text{cd}(R) \subseteq \{1, a, b, bu_1, bu_2, \ldots, bu_i\},$$

where $u_1, u_2, \ldots, u_i$ are some divisors of $m$. If $\{1, a, b\} = \text{cd}(R)$, then we are done since $O^\pi(G) \subseteq R$ and $\pi = \pi(m)$.

Thus, we assume that $\text{cd}(R) = \{1, a, b, bu_1, bu_2, \ldots, bu_i\}$, where $u_1, u_2, \ldots, u_i$ are some non-trivial divisors of $m$, and hence $\text{cd}(N) = \{1, b, bu_1, bu_2, \ldots, bu_i\}$ as we said in the previous paragraph. Observe that $\Delta(R)$ has two connected components $\pi(a)$ and $\pi(b) \cup \pi(u_1u_2\cdots u_i)$. By using Lemma (1.26) and Lemma (3.1), we can consider two possibilities: either $R$ does not satisfy the hypotheses of Case (5) of Lemma (1.26) or $R$ satisfies the hypotheses of Case (5) of Lemma (1.26).

If $R$ does not satisfy the hypotheses of Case (5) of Lemma (1.26), then Lemma (3.1) implies that $\text{cd}(O^\pi(R)) = \{1, a, b\}$, and hence $\text{cd}(O^\pi(G)) = \{1, a, b\}$ as $O^\pi(G) = O^\pi(R)$.

If $R$ satisfies the hypotheses of Case (5) of Lemma (1.26), then Case (5) of Lemma (1.26) implies that $\Delta(R)$ has two connected components $\pi(|R : E|)$ and $\pi(|E : F|) \cup \{t\}$, where $t$ is some prime such that $R$ has a normal Sylow $t$–subgroup $T$, $F = T \times Z$ is the Fitting subgroup of $R$ for some central subgroup $Z$ of $R$, and $E/F$ is the Fitting subgroup of $R/F$. Also, by applying Lemma (1.26)(5), $R/T'$ satisfies the hypotheses of Case (3) of Lemma (1.26), and so it follows from Case (3) of Lemma (1.26) that $F/T' = E'/T' \times Y/T'$, and $E/Y$ is a Frobenius group with kernel $F/Y$, where $Y/T'' = Z(R/T')$. We know that $|R : E| = a$, $|E : F| = b$, and $a$ is a prime number by using Lemma (3.2). This implies that $u_1, u_2, \ldots, u_i$ are $t$–powers as $a, b$, and $m$ are pairwise coprime integers.

On the other hand, we have that $R/K$ is a Frobenius group with complement of order $a$ and elementary abelian $q$–group kernel $N/K$. Since $FK/K$ is a normal nilpotent subgroup of $R/K$ and $R/K$ is a Frobenius group with kernel $N/K$, we obtain that $F \subseteq N$. We know...
that $|R : E| = |R : N| = a$ is a prime number, and so $E = N$. Since $N/K$ is a chief factor of $G$, we deduce that $KY$ is either $K$ or $N$. Recall that $N/Y$ is a Frobenius group with kernel $F/Y$ whose index is $b$, and so $\text{cd}(N/Y) = \{1, b\}$. Hence, if $KY = N$, then $N/Y$ is isomorphic to $K/K \cap Y$, and so $\text{cd}(K/K \cap Y) = \{1, b\}$. We conclude that $b \in \text{cd}(K)$. Since $\text{cd}(R/K) = \{1, a\}$, it follows from Gallagher’s theorem (Lemma (1.1)(b)) that $ab \in \text{cd}(R)$, which is a contradiction.

Thus, we conclude that $KY = K$, and so $Y \subseteq K$. Since $N/K$ is abelian, it follows that $N' \subseteq K$. But we know that $F/T' = N'/T' \times Y/T'$, as $N = E$. This forces that $F \subseteq K$. If $F \neq K$, then $|N : K|$ is a proper prime power divisor of $b$ as $|N : F| = b$. Since $G/K$ is a Frobenius group with kernel $N/K$, we have that $ma = |G : N|$ divides $|N : K| - 1$, and so $m$ divides $|N : K| - 1$. This contradicts the hypothesis because we know that $|N : K|$ is a proper prime power divisor of $b$.

We conclude that $F = K$. We know that $N/Y$ is a Frobenius group with kernel $F/Y$. On the other hand, $G/F$ is a Frobenius group with kernel $N/F$, and so $G/N$ is cyclic of order $am$. We may apply Lemma (1.21) to see that

$$\text{cd}(G/Y) \cup \{abm\} = \{1, am\} \cup \{ib \mid i \text{ divides } am\}.$$

This contradicts the fact that $ab \notin \text{cd}(G)$. \hfill \Box

Next, we show that $\text{cd}(O^p(G)) = \{1, n, m\}$ when $f = p$, which is part (2) of Theorem C of this dissertation. In fact, we prove that if a solvable group $G$ has a normal subgroup $K$ such that $\text{cd}(G/K) = \{1, p\}$, where $p$ is a prime number, and $\text{cd}(G) = \{1, p, n, m, pm, pm\}$, where $m$ and $n$ are coprime integers greater than 1 such that the pairs $(p, m)$ and $(p, n)$ are strongly coprime pairs, then $\text{cd}(O^p(G)) = \{1, n, m\}$.

**Theorem 3.5.** Let $m$ and $n$ be coprime integers greater than 1 and let $p$ be a prime
such that the pairs \((p,m)\) and \((p,n)\) are strongly coprime pairs. Suppose that \(G\) is a solvable group with \(cd(G) = \{1, p, n, m, pn, pm\}\). If \(K\) is a normal subgroup of \(G\) so that \(cd(G/K) = \{1, p\}\), then \(cd(O^p(G)) = \{1, n, m\}\).

**Proof.** Let \(M\) be a subgroup of \(G\) containing \(K\) maximal with respect to normality in \(G\) and \(G/M\) not abelian. Since \(cd(G/M) \subseteq cd(G/K)\), we have \(cd(G/M) = cd(G/K) = \{1, p\}\). From Lemma (1.3), we know that \(G/M\) is either a \(p\)-group or \(G/M\) is a Frobenius group. Suppose first that \(G/M\) is a \(p\)-group. Then \(O^p(G) \subseteq M\), and \(n, m \in cd(M)\) (this is Lemma (1.17)). Consider character degrees \(a \in cd(M)\) and \(b \in cd_M(G|a)\). We use Lemma (1.4) to show that \(b/a\) divides \(|G : M|\) and is thus a power of \(p\). If \(b > a\), then \(p\) divides \(b\) and \(b \in \{p, pm, pm\}\). It follows that \(a \in \{1, n, m\}\). If \(b = a\), then by Lemma (1.17), we have \(pa \in cd(G)\). Again, we obtain \(pa \in \{p, pm, pm\}\), and we deduce that \(a \in \{1, n, m\}\). We conclude that \(cd(M) = \{1, n, m\}\), and since \(|M : O^p(G)|\) is a power of \(p\), we have proved the result in this case.

We now assume that \(G/M\) is a Frobenius group with kernel \(N/M\). By Lemma (1.3)(b), we know that \(p = |G : N|\) and that \(N/M\) is an elementary abelian \(s\)-group for some prime \(s \neq p\). Observe that \(n\) and \(m\) both lie in \(cd(N)\) (again, this is Lemma (1.17)) and that \(O^p(G) \subseteq N\). Consider a character degree \(a \in cd(N)\). By Lemma (1.3)(b), we know that either \(pa \in cd(G)\) or \(s\) divides \(a\). If \(pa \in cd(G)\), then \(pa \in \{p, pm, pm\}\) and \(a \in \{1, n, m\}\). If \(s \notin \pi(nm)\), then \(s\) divides no character degree of \(G\), and hence it divides no character degree of \(N\). Therefore, we determine that \(cd(N) = \{1, n, m\}\) to prove the result.

Hence, we may assume that \(s \in \pi(nm)\). Since \(cd(G)\) is symmetric in \(m\) and \(n\), we assume that \(s \in \pi(m)\), and so \(s \notin \pi(n)\) as \((m, n) = 1\). Consider character degrees \(a \in cd(N)\) and \(b \in cd_N(G|a)\) so that \(s\) divides \(a\). It follows that \(s\) divides \(b\), and \(b \in \{m, pm\}\). Hence, we see that \(a\) is either \(m\) or \(pm\), and we have shown that \(\{1, n, m\} \subseteq cd(N) \subseteq \{1, n, pm, m\}\). If \(cd(N) = \{1, n, m\}\), we are done. Therefore, we assume that \(cd(N) = \{1, n, pm, m\}\). Then
by using Lemma (3.3), we get the result. □
CHAPTER 4

PROOFS OF THEOREMS D AND E

In this chapter, we prove Theorems D and E.

Proof of Theorem D. In view of Corollary (2.7), it suffices to show that \( \text{cd}(O^p(G)) = \{1, n, m\} \). Let \( K \) be maximal in \( G \) so that \( K \) is normal in \( G \) and \( G/K \) is not abelian. By Lemma (1.3), we know that \( \text{cd}(G/K) = \{1, f\} \) for some integer \( f \in \text{cd}(G) \). If \( f = p \), then we are done by Theorem (3.5). If \( f \in \{pm, pn\} \), then by Lemma (1.3), we know that \( G/K \) is a Frobenius group, and we are done by Lemma (3.4). The remaining possibility is that \( f \in \{n, m\} \). Since \( \text{cd}(G) \) is symmetric in \( m \) and \( n \), without loss of generality, we assume that \( f = n \).

Observe that if \( G/K \) were a \( q \)-group for some prime \( q \), then \( \text{cd}(G/K) = \{1, n\} \) would imply that \( n \) is a \( q \)-power, and so \( (m, |G : K|) = 1 \). It would follow that \( m \in \text{cd}(K) \) by Lemma (1.4), and hence we would conclude that \( mn \in \text{cd}(G) \) by Gallagher’s theorem (Lemma (1.1)(b)), which is a contradiction.

Hence, \( G/K \) is a Frobenius group with kernel \( N/K \), where \( |G : N| = f = n \). By Lemma (1.3)(b), we know that \( N/K \) is an \( r \)-group for some prime divisor \( r \) of \( m \). Suppose that \( \pi = \pi(m) \) is the set of all prime divisors of \( m \). Take \( H = O^p(G) \) and \( M = O^\pi(N) \). Then \( M \subseteq K \) as \( N/K \) is an \( r \)-group for some prime divisor \( r \) of \( m \). Since \( G/H \) is a \( p \)-group, it follows from Lemma (1.4) that \( \{1, n, m\} \subseteq \text{cd}(H) \subseteq \{1, p, n, m, pm, pn\} \). If \( p \notin \text{cd}(H) \), then the restriction of a character of degree \( p \) of \( G \) to \( H \) is a sum of linear characters of \( H \). This implies that \( p \in \text{cd}(G/H') \). On the other hand, by applying Itô’s theorem (Lemma (1.9)), we know that every degree in \( \text{cd}(G/H') \) must divide \( |G : H| \) as \( H/H' \) is a normal abelian subgroup of \( G/H' \). We deduce that \( \text{cd}(G/H') = \{1, p\} \). Let \( J/H' \) be maximal in \( G/H' \)
so that $J/H'$ is normal in $G/H'$ and $G/J$ is not abelian. By Lemma (1.3), we know that $\text{cd}(G/J) = \{1, p\}$, and so we are done by Theorem (3.5).

Thus, we assume that $p \in \text{cd}(H)$ and we seek a contradiction. Since $|G : H|, |G : K| = 1$, this implies that $G = HK$ and $G/K$ isomorphic to $H/H \cap K$, and so $H/H \cap K$ is a Frobenius group with kernel $(H \cap N)/(H \cap K)$. Take $U = H \cap N$ and $V = H \cap K$. Since $H/V$ is a Frobenius group with kernel $U/V$, which is an elementary abelian $r$–group for some prime divisor $r$ of $m$, and $|H : U| = n$, by Lemma (1.3)(b), we conclude that $h, ph \notin \text{cd}(U)$ for all non-trivial divisors $h$ of $n$. Therefore, $\text{cd}(U)$ consists of $1, p, m,$ and maybe $pm$ depending on whether $pm \in \text{cd}(H)$ or $pm \notin \text{cd}(H)$.

Recall that $\pi = \pi(m)$ is the set of all prime divisors of $m$. Let $W = O^\pi(U)$. It follows from Lemma (1.4) that all character degrees of $W$ are in the form $p^iv$, where $i = 0$ or $1$ and $v$ can be any divisor of $m$, as $U/W$ is a $\pi$–group and $\{1, p, m\} \subseteq \text{cd}(U) \subseteq \{1, p, m, pm\}$. Let $T$ be maximal in $W$ so that $T$ is normal in $W$ and $W/T$ is non-abelian. By Lemma (1.3), we know that $\text{cd}(W/T) = \{1, g\}$ for some integer $g \in \text{cd}(W)$ and either $W/T$ is a $t$–group for some prime $t$ or $W/T$ is a Frobenius group with abelian complement of order $g$. If $g = v$ or $pw$, where $v$ and $w$ are some divisors of $m$, then, in both cases, we will have a proper subgroup of $W$ with $\pi$–index and normal in $U$, which is a contradiction because $W = O^\pi(U)$. We conclude that $g = p$, and either $W/T$ is a $p$–group or $W/T$ is a Frobenius group with abelian complement of order $p$. By Lemma (1.3), in both cases, we obtain that $O^p(W) < W$. Hence, we choose $D$ normal in $H$ so that $W/D$ is a chief factor for $H$ and $W/D$ is a $p$–group. Since $W/D$ is normal abelian subgroup of $U/D$ whose index is a $\pi$–number, by Itô’s theorem (Lemma (1.9)), we determine that $\text{cd}(U/D) \subseteq \{1, m\}$. We have two possibilities: either $U/D$ is not nilpotent or $U/D$ is nilpotent.

First, assume that $U/D$ is not nilpotent, and so $\text{cd}(U/D) = \{1, m\}$. It follows that $\text{cd}(H/D) = \{1, n, m\}$ and $H/D$ has Fitting height 3. Suppose $F/D$ and $E/F$ are the Fitting subgroups of $H/D$ and $H/F$, respectively. Let $Q/D$ be the Fitting subgroup of
This implies that $Q/D$ is a normal nilpotent subgroup of $G/D$ and $U/Q$ is also nilpotent as $\text{cd}(U/D) = \{1, m\}$. Since $Q/D$ is a normal and nilpotent subgroup of $H/D$ and $W/D$ is normal abelian subgroup of $H/D$, we deduce that $W \subseteq Q \subseteq F$. Also, $UF/F$ is a normal nilpotent subgroup of $H/F$ as it is isomorphic to a factor group of $U/Q$ which is nilpotent. Hence, $UF \subseteq E$ since $E/F$ is the Fitting subgroup of $H/F$. We obtain that $U \subseteq E$. Since $|H : U| = n$ and $|H : E| \in \text{cd}(H)$ by Lemma (1.22)(a), this implies that $U = E$, and hence $U/F$ is the Fitting subgroup of $H/F$ whose index is $n$. Also, by Lemma (1.22)(a), we know that $U/F$ has order $m$, $U/Z$ is a Frobenius group with kernel $F/Z$, where $Z/D = Z(H/D)$, and $F/Z$ is an elementary abelian $s$–group of order $s^a$, where $s$ is a prime that does not divide $m$ and $a$ is some integer.

We claim that $\text{cd}(F) = \{1, p\}$. Since $U/Z$ is a Frobenius group with kernel $F/Z$, Lemma (1.3)(b) implies that $u \notin \text{cd}(F)$ for all divisors $u$ of $m$, and so all character degrees of $F$ are $1, p$, and perhaps $pu$, where $u$ is some divisor of $m$ as $|U : F| = m$. If $s \neq p$, then again, by using Lemma (1.3)(b), we have that $pu \notin \text{cd}(F)$ for all divisors $u$ of $m$. Thus, we have $\text{cd}(F) = \{1, p\}$ in this case as claimed. We assume that $s = p$. If $pu \in \text{cd}(F)$ for some divisor $u$ of $m$, then since $|U : F|pu \notin \text{cd}(F)$, by Lemma (1.3)(b), we conclude that $|F/Z| = p^an$ divides $p^2u^2$. Hence, $a = 1$ and $n = 2$. Also, by Lemma (1.22)(a), we know that $p + 1$ divides $m$. Since $n = 2$, the prime $p$ is odd, and so $p + 1$ is even. It follows that $m$ is even, but this is a contradiction because $m$ and $n$ are coprime.

Therefore, we must have $\text{cd}(F) = \{1, p\}$. As $F/W$ is a $\pi$–group, we deduce that $\text{cd}(W) = \{1, p\}$, and hence $\{1, n, m\} \subseteq \text{cd}(H/W')$. On the other hand, $W/W'$ is a normal abelian subgroup of $H/W'$ whose index is divisible only by primes in $\pi(n) \cup \pi(m)$. It follows from Itô’s theorem (Lemma (1.9)) that $\text{cd}(H/W') = \{1, n, m\}$. By applying Lemma (1.25), we obtain $\text{cd}(O^p(H)) = \{1, n, m\}$, and so $O^p(H) < H$ since $p \in \text{cd}(H)$. As $H = O^p(G)$, this implies that $O^p(H) < H$ is a contradiction.

Now, assume that $U/D$ is nilpotent. Take $R/D$ to be the normal Hall $\pi$-subgroup of
$U/D$. It follows that $|H/R| = p^l n$, where $l$ is some integer, and so $U/R$ is the normal abelian Sylow $p$–subgroup of $H/R$. By Itô’s theorem (Lemma (1.9)), $\text{cd}(H/R) \subseteq \{1,n\}$, and so $H/R$ is abelian or $H/R$ has a factor group which is a Frobenius group with kernel of index $n$ and $p$–power order. If $H/R$ is abelian, then $O_p(H) < H$, which is a contradiction as $H = O_p(G)$. Thus, $H/R$ has a factor group which is a Frobenius group with kernel of index $n$ and $p$–power order. By using Lemma (1.3)(b), we deduce that $nm \in \text{cd}(H)$, which is a contradiction. □

**Proof of Theorem E.** Let $K$ be maximal in $G$ with respect to the properties that $K$ is normal in $G$ and $G/K$ is not abelian. There is an integer $f > 1$ so that $\text{cd}(G/K) = \{1,f\}$. From Lemma (1.3), we know that $G/K$ is either a $t$–group for some prime $t$ or $G/K$ is a Frobenius group. Suppose that $G/K$ is a $t$–group for some prime $t$. It follows that $f$ is a power of $t$ and a character degree of $G$. This implies that $f \in \{p,q,n,m\}$ and $t \in \{p,q,r,s\}$ for some prime divisors $r$ of $n$ and $s$ of $m$. If $f$ is $p$ or $q$, then $pq$ is a degree of $G$, and if $f$ is $n$ or $m$, then $nm$ is a degree by Lemma (1.17). In all cases, this is a contradiction, so $G/K$ cannot be a $t$–group for any prime $t$.

Thus, $G/K$ must be a Frobenius group with elementary abelian $t$–group kernel $L/K$ for some prime $t$ and cyclic complement (see Lemma (1.3)(b)). We also utilize that result to see that $|G : L| = f$. Suppose first that $f \in \{pn,pm,qn,qm\}$. By symmetry, we assume that $f = pn$. Then we may apply Lemma (1.18) to see that $t = q$ and $t$ divides $m$, which is a contradiction as $(q,m) = 1$.

Therefore, we are left with $f \in \{p,q,n,m\}$. So, since $\text{cd}(G)$ is symmetric in $p$ and $q$ and $\text{cd}(G)$ is symmetric in $n$ and $m$, we have two cases to consider. The case where $f = p$ and the case where $f = n$.

**Case 1:** $f = p$. In light of Lemma (1.17), we obtain $\{1,q,n,m,qn,qm\} \subseteq \text{cd}(L)$. Using
Lemma (1.18), we see that $L/K$ is a $q$–group. Suppose that we have degrees $a \in \text{cd}(L)$ and $b \in \text{cd}_L(G|a)$. Then by Lemma (1.3)(b), either $pa \in \text{cd}(G)$ or $q$ divides $a$. If $pa \in \text{cd}(G)$, it follows that $pa \in \{p, pn, pm\}$, and hence $a \in \{1, n, m\}$. If $q$ divides $a$, then $q$ divides $b$ so that $b \in \{q, qn, qm\}$. It follows that $a \in \{q, qn, qm\}$ because by Lemma (1.4) we know that $b/a$ divides $p = |G : L|$. This implies that $\text{cd}(L) = \{1, q, n, m, qn, qm\}$. Because of Lemma (1.17), we determine that $\{1, n, m\} \subseteq \text{cd}(K)$. Given characters $\theta \in \text{Irr}(K)$ with $\theta(1) \in \{q, qn, qm\}$ and $\chi \in \text{Irr}(G|\theta)$, then since $\chi(1)/\theta(1)$ divides $|G : K| = pq^u$ by Lemma (1.4), where $u$ is some positive integer, we must have that $\chi_K = \theta$. By Gallagher’s theorem (Lemma (1.1)(b)), $p\chi(1) \in \text{cd}(G)$, which is a contradiction. Therefore, we conclude that $\text{cd}(K) = \{1, n, m\}$. Let $M = O^{[p, q]}(G)$ so that $M \subseteq K$ and $\text{cd}(M) = \{1, n, m\}$. We obtain the result by applying Theorem (2.5). Notice that Theorem A implies that the Fitting height of $L$ is at most 3 as $\text{cd}(L) = \{1, q, n, m, qn, qm\}$, and hence the Fitting height of $G$ is at most 4 since $G/L$ is cyclic.

**Case 2:** $f = n$. In light of Lemma (1.17), we obtain $\{1, p, q, m, pm, qm\} \subseteq \text{cd}(L)$. Using Lemma (1.18), we see that $L/K$ is a $t'$–group, where $t'$ is some prime divisor of $m$. Suppose that we have degrees $a' \in \text{cd}(L)$ and $b' \in \text{cd}_L(G|a')$. Then by Lemma (1.3)(b), either $na' \in \text{cd}(G)$ or $t'$ divides $a'$. If $na' \in \text{cd}(G)$, it follows that $na' \in \{n, pn, qn\}$, and hence $a' \in \{1, p, q\}$. If $t'$ divides $a'$, then $t'$ divides $b'$ so that $b' \in \{m, pm, qm\}$. It follows that $a' \in \{m, pm, qm\}$ because by Lemma (1.4) we know that $b'/a'$ divides $n = |G : L|$. This implies that $\text{cd}(L) = \{1, p, q, m, pm, qm\}$.

If $\text{cd}(O^{\pi \cup \sigma}(G)) = \{1, p, q\}$, where $\pi$ and $\sigma$ are the sets of all prime divisors of $m$ and $n$ respectively, then we apply Theorem (2.5). Since $p$ and $q$ are primes, Conclusion (2) of Theorem (2.5) does not occur. We conclude that Conclusion (1) of it does occur, and so $G = A \times B$, where $\text{cd}(A) = \{1, p, q\}$ and $\text{cd}(B) = \{1, m, n\}$. Hence, to get Conclusion (1) of Theorem B, it suffices to prove $\text{cd}(O^{\pi \cup \sigma}(G)) = \{1, p, q\}$. 

Let $R$ be maximal in $L$ with respect to the properties that $R$ is normal in $L$ and $L/R$ is not abelian. There is an integer $g > 1$ so that $\cd(L/R) = \{1, g\}$. From Lemma (1.3), we know that $L/R$ is either an $s'$–group for some prime $s'$ or $L/R$ is a Frobenius group. Since $pq \not\in \cd(L)$, it follows that $L$ cannot have a non-abelian quotient that is a $p$–group or a $q$–group. Thus, we can consider the following subcases:

**Subcase 1:** $g = m$ and $L/R$ is a Frobenius group with kernel $U/R$ which is an elementary abelian $l$–group for some prime $l$ that does not divide $m$. If neither $l = p$ nor $l = q$, then by using Lemma (1.3)(b), we deduce that $\cd(U) = \{1, p, q\}$. Since $|G : U| = |G : L||L : U| = nm$, we have that $O^{\pi_{\lambda \sigma}}(G) \subseteq U$, and so $\cd(O^{\pi_{\lambda \sigma}}(G)) = \{1, p, q\}$ by applying Lemma (1.17).

By symmetry, we may assume that $l = p$, hence $l \neq q$. Lemma (1.3)(b) and Lemma (1.17) imply that $\cd(U)$ consists of $1, p, q$, and probably $pd$, where $d$ can be any divisor of $m$. By Lemma (3.3), we obtain $\cd(O^\pi(L)) = \{1, p, q\}$, and so $\cd(O^\pi_{\lambda \sigma}(G)) = \{1, p, q\}$ as $O^{\pi_{\lambda \sigma}}(G) \subseteq O^\pi(L)$.

**Subcase 2:** $g = mp$ and so $L/R$ must be a Frobenius group with kernel $V/R$, which is an elementary abelian $q$–group, by Lemma (1.3)(b). It follows from Lemma (3.4) that $\cd(O^\pi(L)) = \{1, p, q\}$, and so $\cd(O^\pi_{\lambda \sigma}(G)) = \{1, p, q\}$.

**Subcase 3:** $g = m$ and $L/R$ is a $t''$–group for some prime $t''$. Since $L/R$ is a $t''$–group whose character degree set is $\{1, m\}$, $m = (t'')^i$ for some positive integer $i$. Recall that $L/K$ is a $t$–group, where $t$ is a prime divisor of $m$. This implies that $t'' = t$. Let $N = O^t(L)$. It follows that $N \subseteq R$, and so $L/N$ is a non-abelian $t$–group as $\cd(L/R) = \{1, m\}$. By
applying Lemma (1.4), it is easy to see that
\[
\{1, p, q\} \subseteq \text{cd}(N) \subseteq \{1, p, q\} \cup \{u_1, u_2p, u_3q \mid u_1, u_2, u_3 \in \Omega\},
\]
where Ω is the set of all divisors of m.

Let T be maximal in N with respect to the properties that T is normal in N and N/T is not abelian. There is an integer \(g' > 1\) so that \(\text{cd}(N/T) = \{1, g'\}\) and we know that N/T is either an \(s''\)-group for some prime \(s''\) or N/T is a Frobenius group. Since \(N = O^t(L)\), we obtain that N has not a quotient group with \(t\)-power order. Observe that \(pq \notin \text{cd}(L)\) and \(\text{cd}(L)\) is symmetric in \(p\) and \(q\). We deduce that we can assume that N/T is a Frobenius group with kernel J/T whose index is \(p\) and order is \(q^j\) for some positive integer \(j\). Recall that Ω is the set of all divisors of \(m\). By Lemma (1.3)(b) and Lemma (1.4), we conclude that
\[
\{1, q\} \subseteq \text{cd}(J) \subseteq \{1, q\} \cup \{v, v'q \mid v, v' \in \Omega\}.
\]
Let \(W = O^p(N)\), so \(W \subseteq J\). Hence, \(\{1, q\} \subseteq \text{cd}(W) \subseteq \{1, q\} \cup \{w, w'q \mid w, w' \in \Omega\}\). In particular, \(p \notin \text{cd}(W)\). Notice that since \(L\) is a normal subgroup of \(G\), we have that \(N\) is a normal subgroup of \(G\), and thus, \(W\) is a normal subgroup of \(G\).

If \(N/W\) is not abelian, then \(\text{cd}(N/W) = \{1, p\}\), and so by Lemma (1.17), we conclude that \(pq \in \text{cd}(N)\) as \(|N : W|\) and \(q \in \text{cd}(N)\) are coprime. This is a contradiction. It follows that \(N/W\) is abelian. Recall that \(G/K\) is a Frobenius group with kernel \(L/K\), which is a \(t\)-group, and so \(N \subseteq K\). Since \(\text{cd}(G/K) = \{1, n\}\) and \(\text{cd}(L/N) = \{1, m\}\), we obtain that \(m, n \in \text{cd}(G/W)\). Itô’s theorem (Lemma (1.9)) implies that \(\text{cd}(G/W) = \{1, n, m\}\) as \(N/W\) is a normal abelian subgroup of \(G/W\). Let \(F/W\) be the Fitting subgroup of \(G/W\). We have two possibilities: either \(G/W\) has Fitting height 2 or 3.

First, suppose that \(G/W\) has Fitting height 2. Observe that \(W \subseteq F \cap K\), so \(F/F \cap K\) is a quotient group of \(F/W\). This implies that \(FK/K\) is nilpotent. But we know that the Fitting subgroup of \(G/K\) is \(L/K\) as \(G/K\) is a Frobenius group with kernel \(L/K\). We
deduce that \( F \subseteq L \), and hence \( |G : L| = n \) divides \( |G : F| \). But by Lemma (1.22)(b), we know that \( |G : F| \in \text{cd}(G/W) = \{1, m, n\} \). Since \((m, n) = 1\), it follows that \( |G : F| = n \), and so \( F = L \). By Lemma (1.22)(b), we have that \( L/W = Y/W \times N/W \), where \( Y/W \) is a Sylow \( t \)-subgroup of \( G/W \), and so \( N/W \) is a central subgroup of \( G/W \) by applying Lemma (1.22)(b). Observe that \( N/W \in \text{Syl}_p(G/W) \), hence \( G/W \) has a normal \( p \)-complement \( R/W \) since \( N/W \) is a central subgroup of \( G/W \). It follows that \( R \) is a normal subgroup of \( G \) whose index is a \( p \)-power as \( G/R \) is isomorphic to \( N/W \).

If \( p \notin \text{cd}(R) \), then \( p \in \text{cd}(G/R') \). As \( R/R' \) is normal abelian subgroup of \( G/R' \) whose index is a \( p \)-power, Itô’s theorem (Lemma (1.9)) implies that \( \text{cd}(G/R') = \{1, p\} \). Suppose \( X/R' \) is maximal in \( G/R' \) with respect to the properties that \( X/R' \) is normal in \( G/R' \) and \( G/X \) is not abelian. Then \( \text{cd}(G/X) = \{1, p\} \), and so we are done by Case (1). Thus, we assume that \( p \in \text{cd}(R) \). Since \( R/W \) is isomorphic to \( G/N \), this implies that \( W \) is a normal subgroup of \( R \) whose index is a \( p' \)-number. We deduce from Lemma (1.4) that \( p \in \text{cd}(W) \), which is a contradiction as we know \( p \notin \text{cd}(W) \).

Therefore, we can assume that \( G/W \) has Fitting height 3. Let \( E/F \) be the Fitting subgroup of \( G/F \). By Lemma (1.22)(a), we have \( F/W = S/W \times Z/W \), where \( S/W \) is a chief factor of \( G \) and \( Z/W \) is the center subgroup of \( G/W \). Also, by the same lemma, \( E/F \) is cyclic and \( E/Z \) is a Frobenius group with kernel \( F/Z \). We claim that all Sylow subgroups of \( E/W \) are abelian. To see this, let \( y \) be a prime divisor of \( |E/W| \). If \( y \) does not divide \( |E : F| \), then a Sylow \( y \)-subgroup of \( E/W \) is contained in \( F/W \), and so is abelian. If \( y \) does divide \( |E : F| \), then the intersection of a Sylow \( y \)-subgroup of \( E/W \) with \( F/W \) is contained in \( Z/W \) because \( y \) does not divide \( F/Z \). This implies that a Sylow \( y \)-subgroup of \( E/W \) is cyclic-by-central which implies it is abelian. Observe that \( LF/F \) is a normal nilpotent subgroup of \( G/F \). It follows that \( L \subseteq E \) as \( E/F \) is the Fitting subgroup of \( G/F \).

We conclude that all Sylow subgroups of \( L/W \) are abelian. But we know that a Sylow \( t \)-subgroup of \( L/W \) is isomorphic to \( L/N \), which is non-abelian. This is a contradiction,
and hence we are done in this subcase.

**Subcase 4:** $g = p$. Since $pq \notin \mathfrak{c}(L)$, $L/R$ cannot be a $p$-group. Hence, $L/R$ is a Frobenius group with kernel $Q/R$, which is an elementary abelian $q$-group by using Lemma (1.3)(b). Suppose that $P = \mathfrak{S}^{p,q}(L)$, and so $P \subseteq R$. Since $L/P$ is not abelian, we have that $G/P$ is not abelian whose order is divisible by only primes in $\pi(n) \cup \{p, q\}$. Let $D/P$ be maximal in $G/P$ with respect to the properties that $D/P$ is normal in $G/P$ and $G/D$ is not abelian. There is an integer $d' > 1$ so that $\mathfrak{c}(G/D) = \{1, d'\}$. From Lemma (1.3), we know that $G/D$ is either an $r'$-group for some prime $r'$ or $G/D$ is a Frobenius group. As we proved at the start of the proof of Theorem B, $G/D$ cannot be an $h$-group for any prime $h$. Thus, $G/D$ is a Frobenius group with kernel $C/D$, which is an elementary abelian $l'$-group for some prime $l'$. If $|G : C| = n$, then $C/D$ is an elementary abelian $p$-group or $q$-group since the order of $G/D$ is divisible by only primes in $\pi(n) \cup \{p, q\}$. Lemma (1.3)(b) implies that $nm \in \mathfrak{c}(G)$, which is a contradiction. Thus, $|G : C| = p$ or $q$, and so we can get the desired result by Case (1) above. □
CHAPTER 5

PROOFS OF THEOREMS F AND G

In this chapter, we prove Theorems F and G. The following lemma will be used to prove Theorem F. It is stated as Lemma (2.1) in [2]. We will use this lemma just in the last case of the proof of Theorem F. The lemma implies that if $G$ is a solvable group such that the $p$-parts of all members of $\text{cd}(G)$ lie in $\{1, p^a\}$, then we can get some useful information about the $p$-structure of $G$.

**Lemma 5.1.** If $G$ is a solvable group and the $p$-parts of all members of $\text{cd}(G)$ lie in $\{1, p^a\}$, then either $G$ is the direct product of an abelian group and a $p$-group or else $G$ has a non-abelian factor group with an abelian Sylow $p$-subgroup.

**Proof.** Let $N = \mathbf{O}^{p'}(G)$ and $M = \mathbf{O}^p(N)$, and let $D/M = (N/M)'$. If $M = 1$ then $N$ is $p$-group with $p'$-index. It follows that $N$ is a normal Sylow $p$-subgroup of $G$ and $D = N'$. If $G/D$ is non-abelian, then since $N \in \text{Syl}_p(G)$ and $D = N'$, we have that $N/D \in \text{Syl}_p(G/D)$ is abelian. In this case, $G$ has a non-abelian factor group $G/D$ with an abelian Sylow $p$-subgroup $N/D$, and so we are done in this case. Therefore, we assume that $G/D$ is abelian.

Now, a $p$-complement $K$ for $G$ is abelian because $K$ is isomorphic to $G/N$, which is a factor group of the abelian group $G/D$. We have that

$$[K, N] \subseteq G' \subseteq D = N' \subseteq \Phi(N),$$

and $N = C_N(K)[K, N]$ since $N$ is a $p$-group and $K$ is a $p'$-group. We conclude that $N = C_N(K)\Phi(N)$ and so $N = C_N(K)$. Hence, $K$ centralizes $N$ and hence $G$ is the direct product of an abelian group and a $p$-group in this case. This proves the result when $M = 1$. 

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Thus, we can assume that $M > 1$. Choose a normal subgroup $L$ of $G$ so that $M/L$ is a chief factor of $G$, which must be an elementary abelian $q$–group for some prime $q$. Let $\lambda \in \text{Irr}(M/L)$ be arbitrary. Now, if $\chi \in \text{Irr}(N|\lambda)$ and $\psi \in \text{Irr}(G|\chi)$, then $\frac{\psi(1)}{\chi(1)}$ divides $|G : N|$, which is a $p'$–number, and $\frac{\chi(1)}{\lambda(1)} = \chi(1)$ divides $|N : M|$, which is a $p$–power. We conclude that $\chi(1)$ is a $p$–power and $\chi(1)p = \psi(1)p \in \{1, p^a\}$. It follows that $\text{cd}(N|\lambda) \subseteq \{1, p^a\}$, and hence, the stabilizer $T$ of $\lambda$ in $N$ satisfies $|N : T| \leq p^a$. If $\theta \in \text{Irr}(N)$ is a constituent of $(1_T)^N$ then $1_T$ is a constituent of $\theta_T$. We deduce that $1_M$ is a constituent of $\theta_M$, and so $\theta(1) \in \text{cd}(N|1_M) \subseteq \{1, p^a\}$. Since $(1_T)^N(1) = |N : T| \leq p^a$ and $1_N$ is a constituent of $(1_T)^N$, we conclude that $\theta(1) = 1$, and so all irreducible constituents of $(1_T)^N$ are linear. Hence, if $\mu \in \text{Irr}(N|1_T)$ then, since $\mu$ is linear, we have $\mu_T = 1_T$, and so $T \subseteq \ker(\mu)$. It is clear that core$_N(T) = \ker(1_T)^N \subseteq T$. On the other hand, we know that $\ker(1_T)^N = \cap_{\mu \in \text{Irr}(N|1_T)} \ker(\mu)$. We deduce that $T = \cap_{\mu \in \text{Irr}(N|1_T)} \ker(\mu)$, and so $D \subseteq T$. Since $M/L$ is abelian and $\lambda \in \text{Irr}(M/L)$ is arbitrary, it follows that $M/L$ is central in $D/L$.

Since $M = \text{O}_p^0(N)$ (and so $D/M$ is a $p$–group), $L < M$, and $M/L$ is a $q$–group, hence $q \neq p$, and so $D/L = M/L \times S/L$, where $S/L \in \text{Syl}_p(D/L)$. We claim that $G/S$ is a non-abelian group and has a Sylow $p$–subgroup isomorphic to $N/D$, which is abelian, and so we will be done. If $G/S$ is abelian, since

$$|G/S| = |G : N||N : D||D : S| = p'tq^su,$$

where $u = |G/N|$ is a $p'$–number and $t, s$ are some nonnegative integers, if we take a Sylow $p$–subgroup $F/S$ of $G/S$, then $F$ is normal in $G$ and $|G/F| = q^su$, which is $p'$–number. This contradicts the fact that $N = \text{O}^p(G)$ because $|G/N| = u$. Therefore, $G/S$ must be non-abelian. Finally, $N/D$ is isomorphic to $\frac{N/S}{\text{Syl}_p(D/L)}$ which is a factor group of $G/S$. Also, $|G/S|_p = |N/D|$. We deduce that $G/S$ has a Sylow $p$–subgroup isomorphic to $N/D$, which is abelian. □
**Proof of Theorem F.** Let $K$ be maximal in $G$ so that $K$ is normal in $G$ and $G/K$ is not abelian. By Lemma (1.3), we know that $\text{cd}(G/K) = \{1, f\}$ for some integer $f \in \text{cd}(G)$. Since $\text{cd}(G) = \{1, a, b, c, ab, ac\}$ and $\text{cd}(G)$ is symmetric in $b, c$, we consider the following three cases:

**Case 1:** $f = b$. In this case, $G/K$ cannot be an $r$–group for some prime $r$ because if $G/K$ is an $r$–group for some prime $r$, then since $c \in \text{cd}(K)$, we will deduce that $bc \in \text{cd}(G)$ by using the Gallagher’s theorem (Lemma (1.1)(b)), which is a contradiction. Hence, $G/K$ is a Frobenius group with Frobenius kernel $N/K$ for some normal subgroup $N$ of $G$. We know that $N/K$ is an elementary abelian $q$–group for some prime $q$, $|G : N| = b$, and $q$ does not divide $b$. Since $|G : N| = b$ is coprime to $a, c$, and $ac$, we see that irreducible characters of $G$ of these degrees restrict irreducibly to $N$, and thus $\{1, a, c, ac\} \subseteq \text{cd}(N)$. If $q$ does not divide $c$, then since $N$ has an irreducible character of degree $c$, by Lemma (1.3)(b), it would follow that $bc \in \text{cd}(G)$, which is not the case.

We argue now that all irreducible characters of $G$ of degree $b$ restrict to $N$ as a sum of linear characters, and that all irreducible characters of $G$ of degree $ab$ restrict to $N$ as a sum of irreducible characters of degree $a$. To see this, let $u$ be the degree of an irreducible character of $N$ that lies under an irreducible character of $G$ with degree $b$ or $ab$. Since $q$ divides $c$, it does not divide $u$, so $\text{cd}(G)$ contains $ub$, and thus $u = 1$ or $u = a$, as wanted. We now have $\text{cd}(N) = \{1, a, c, ac\}$, and thus Lemma (1.19) implies that either $N$ has derived length at most 3 or $\text{cd}(N) = \{1, a, c, ac\} = \{1, 3, 13, 39\}$. If $\text{dl}(N) \leq 3$, then $\text{dl}(G) \leq 4$ since $G/N$ is cyclic.

It follows that we can assume $\text{cd}(N) = \{1, a, c, ac\} = \{1, 3, 13, 39\}$. Recall that $G/K$ is a Frobenius group with Frobenius kernel $N/K$, which is an elementary abelian $c$–group. Notice that $a$ and $c$ are prime numbers. Since $(a, |N : K|) = 1$ and $a \in \text{cd}(N)$, Lemma (1.4)
implies that $a \in \text{cd}(K)$. On the other hand, given an irreducible character $\varphi \in \text{Irr}(K)$ with degree $\varphi(1) \in \{c, ac\}$, since $(|G : N|, ac) = 1$, we obtain that $\varphi$ is extendible to $\chi \in \text{Irr}(G)$ by looking at the possible values in $\text{cd}(G)$. Also, we know that $\text{cd}(G/K) = \{1, b\}$. By Gallagher’s theorem (Lemma (1.1)(b)), we deduce that $b\varphi(1) \in \text{cd}(G)$. This is a contradiction because $b\varphi(1)$ is divisible by $bc$ but no character degree of $G$ is divisible by $bc$. Therefore, we can conclude that $\text{cd}(K) = \{1, a\}$, and so $\text{dl}(K) \leq 2$. On the other hand, we know that $\text{cd}(G/K) = \{1, b\}$, and hence $\text{dl}(G/K) \leq 2$. We obtain that $\text{dl}(G) \leq 4$, and so we are done in this case.

**Case 2:** $f = ab$. In this case, $G/K$ is a Frobenius group with Frobenius kernel $N/K$ for some normal subgroup $N$ of $G$. We know that $N/K$ is an elementary abelian $q$–group for some prime $q$, $|G : N| = ab$, and $q$ does not divide $ab$. As before, $N$ must have irreducible characters of degree $c$ and thus, since $\text{cd}(G)$ does not contain $abc$, it follows that $q$ divides $c$. We argue now that all irreducible characters of $G$ of degree different from $c$ or $ac$ restrict to $N$ as a sum of linear characters. To see this, let $u$ be the degree of an irreducible constituent of such a restriction. Then $u$ divides $ab$, so it is not divisible by $q$. Then $uab \in \text{cd}(G)$, and thus $u = 1$, as claimed. Thus $\text{cd}(G|N')| \leq 2$. Hence, Lemma (1.16) implies that $\text{dl}(N') \leq 2$. Since $G/N$ is cyclic, we obtain $\text{dl}(G) \leq 4$.

**Case 3:** $f = a$. In this case we have two subcases:

**Subcase 1:** $G/K$ is a Frobenius group with Frobenius kernel $N/K$.

In this subcase, by Lemma (1.3)(b), we know that $N/K$ is an elementary abelian $q$–group for some prime $q$, $|G : N| = a$, and $q$ does not divide $a$. Also, since $|G : N| = a$, we have that $\{1, b, c\} \subseteq \text{cd}(N)$. By Lemma (1.3)(b), we know that if $1 < v \in \text{cd}(N)$ and $q$ does not divide $v$, then $av \in \text{cd}(G)$, and so $v \in \{1, b, c\}$. Hence, $h$ is not a character degree of $N$
for all divisors $h$ of $a$ and also, if $q$ does not divide $bc$, then none of $hb$ and $dc$ are character
degrees of $N$ for all divisors $h$ and $d$ of $a$. Thus, $\text{cd}(N) = \{1, b, c\}$, and so, by Lemma (1.8),
$\text{dl}(N) \leq 3$. Therefore, $\text{dl}(G) \leq 4$, and so we are done. Hence, we assume that $q$ divides $c$.
It follows that $q$ does not divide $b$. Now, $hb$ is not a character degree of $N$ for any divisor $h$ of $a$. Possibly $dc$ is a character degree of $N$ for some divisor $d$ of $a$. Thus, we have

$$\{1, b, c\} \subseteq \text{cd}(N) \subseteq \{1, b, c, d_1c, d_2c, \ldots, d_n c\},$$

where $\{d_1, d_2, \ldots, d_n\}$ is the set of all divisors of $a$. If $\{1, b, c\} = \text{cd}(N)$, then as we men-
tioned, we are done. Therefore, we may assume that $\text{cd}(N) = \{1, b, c, d_1c, d_2c, \ldots, d_i c\}$,
where $d_1, d_2, \ldots, d_i$ are some divisors of $a$.

Observe that $\Delta(N)$ has at least three vertices and two connected components $\pi(b)$ and
$\pi(c) \cup \pi(d_1d_2 \cdots d_i)$. Since the character degree graphs of Case (2) of Lemma (1.26) have
only two vertices, by Lemma (1.26), we have four possibilities: $N$ satisfies the hypotheses
of Case (1), (3), (4), or (5) of Lemma (1.26).

If $N$ satisfies the hypotheses of Case (1) of Lemma (1.26), then $N$ has a normal non-
abelian Sylow $p$-subgroup $P$ and an abelian $p$-complement $K$ for some prime $p$. Also,
$\text{cd}(P) \subseteq \text{cd}(N)$ and $\Delta(N)$ has two connected components $\{p\}$ and $\pi(|N : F|)$, where $F$
is the Fitting subgroup of $N$. Since $\pi(b)$ and $\pi(c) \cup \pi(d_1d_2 \cdots d_i)$ are the two connected
components of $\Delta(N)$, we deduce that $b$ is a $p$–power, and so $\text{dl}(P) \leq 2$ because of the
structure of $\text{cd}(N)$. Since $N/P$ is abelian, we conclude that $\text{dl}(N) \leq 3$, and so $\text{dl}(G) \leq 4$.

If $N$ satisfies the hypotheses of Case (3) of Lemma (1.26), then $\Delta(N)$ has two connected
components $\pi(|N : E|)$ and $\pi(|E : F|)$, where $F$ is the Fitting subgroup of $N$, $E/F$ is
the Fitting subgroup of $N/F$ such that all divisors of $|N : E|$ including 1 and $|N : E|$ are character degrees of $N$. Also, $\text{cd}(N) = \{u|u \text{ divides } n\} \cup \{|E : F|\}$. Since $\text{cd}(N) = \{1, b, c, d_1c, d_2c, \ldots, d_i c\}$, it follows that $b = n$ is a prime and so $|\text{cd}(N)| \leq 3$, which is a
contradiction. Hence, this case does not occur.
If $N$ satisfies the hypotheses of Case (4) of Lemma (1.26), $\Delta(N)$ has two connected components $\{2\}$ and $\pi(2^a + 1)$, where $a$ is a positive integer such that $2^a + 1$ is a character degree of $G$ and also $\text{cd}(G)$ consists of 2, all powers of 2 that are divisible by $2^a$ and divide $|G|$, and $2^a + 1$. By the structure of $\text{cd}(N)$, we deduce that $b$ is a power of 2, and so $|\text{cd}(N)| \leq 3$, which is a contradiction. This implies that this case does not occur, either.

Finally, we suppose that $N$ satisfies the hypotheses of Case (5) of Lemma (1.26). Then $\Delta(N)$ has two connected components $\pi(|N:E|)$ and $\pi(|E:F|) \cup \{t\}$, where $t$ is some prime, $F = \mathbf{F}(N)$ is the Fitting subgroup of $N$, and $E/F = \mathbf{F}(N/F)$ is the Fitting subgroup of $N/F$. Also, $|N:E|, |E:F| \in \text{cd}(N)$. Since $\text{cd}(N) = \{1, b, c, d, \ldots, d_i\}$, we must have $|N:E| = b$, $|E:F| = c$, and $d_1, d_2, \ldots, d_i$ are powers of $t$. Also, in this case $N$ has a normal Sylow $t$-subgroup $P$ such that $F = P \times Z$, where $Z$ is a central subgroup of $N$. All divisors of $|N:E| = b$ are character degrees of $N$. This implies that $b$ is a prime. Recall that $G/K$ is a Frobenius group with Frobenius kernel $N/K$, which is an elementary abelian $q$-group for some prime $q$ dividing $c$. Since $(|N:K|, t) = 1$ and $K$ is a normal subgroup of $N$, it follows that $P$ is contained in $K$. We claim that $F$ is contained in $K$.

For this, it suffices to show that $Z$ is contained in $K$. If $x \in Z \setminus K$, then $C_G(x)K/K \subseteq C_{G/K}(xK) \subseteq N/K$ because $G/K$ is a Frobenius group with Frobenius kernel $N/K$. Hence, $C_G(x)$ is contained in $N$, but since $Z$ is central in $N$, we have that $C_G(x) = N$. Then, by Brauer’s Permutation theorem, Lemma (1.11), we conclude that there is $\lambda \in \text{Irr}(Z)$ such that $\text{Stab}_G(\lambda)$ is contained in $N$. Let $\mu \in \text{Irr}(P)$ with $\mu(1) = d_j$, where $d_j = \max\{d_1, d_2, \ldots, d_i\}$. Then $\text{Stab}_G(\mu \times \lambda)$ is contained in $N$, and so by the Clifford Correspondence, Lemma 1.2, $ad_j$ divides some irreducible character degree of $G$, which is a contradiction.

Therefore, we deduce that $F$ is contained in $K$. Since $E/F$ is a normal abelian subgroup of $N/F$, by Itô’s theorem, Lemma 1.9, $\text{cd}(N/F) = \{1, b\}$. Since $N/F$ is not nilpotent and all Sylow subgroups of $N/F$ are abelian, by Lemma (1.20), we deduce that $N/X$ is a Frobenius
group with Frobenius kernel $E/X$, where $X/F$ is the center subgroup of $N/F$. Let $B$ be a subgroup of $E$ containing $X$ such that $|E : B|$ is a prime. Then $N/B$ is a Frobenius group with Frobenius kernel $E/B$. We claim that $a, b$, and $ab$ are not elements of $\text{cd}(G|E')$. For this, suppose $\chi \in \text{Irr}(G|E')$ is arbitrary of degree $a, b$, or $ab$. Then there exists $\theta \in \text{Irr}(E)$ such that $\theta$ is a constituent of $\chi_E$ and $\theta(1) > 1$. This implies that $\theta(1)$ divides $\chi(1)$, and so $\theta(1)$ and $|E : B|$ are coprime. Then by Lemma (1.3)(b), we deduce that $\theta(1)b \in \text{cd}(N)$, which is a contradiction. It follows that $|\text{cd}(G|E')| \leq 2$, and so $\text{dl}(E) \leq 3$ by Lemma (1.16).

Also, $G/C$ is a subgroup of $\text{Aut}(E/B)$, which is abelian, where $C/B$ is the centralizer of $E/B$ in $G$. Hence, $G' \subseteq C$, and so $G' \subseteq C \cap N = E$. We conclude that $G' \subseteq E$, and so $\text{dl}(G) \leq 4$. Thus, the proof is complete.

**Subcase 2: $G/K$ is a $p$–group.**

Since $\text{cd}(G/K) = \{1, a\}$, we deduce that $a$ is a $p$–power. All $p$–parts of degrees are 1 and $a$, so Lemma (5.1) applies. By Lemma (5.1), we conclude that $G$ has a non-abelian factor group $G/H$ with an abelian Sylow $p$–subgroup. We can choose $K_0$ to be maximal in $G$ so that $K_0$ is normal in $G$, $G/K_0$ is non-abelian, and $H \subseteq K_0$. Therefore, $G/K_0$ has an abelian Sylow $p$–subgroup. If $G/K_0$ does not satisfy the cases (1), (2), or the subcase (1) of case (3), then $G/K_0$ is a $p$–group. But then $G/K_0$ has to be abelian because it has an abelian Sylow $p$–subgroup, which is a contradiction. Hence, we deduce that $G/K_0$ satisfies one of the cases (1), (2), or the subcase (1) of case (3), and so we are done.

**Proof of Theorem G.** Let $K$ be maximal in $G$ with respect to the properties that $K$ is normal in $G$ and $G/K$ is not abelian. Then there is an integer $f > 1$ so that $\text{cd}(G/K) = \{1, f\}$. From Lemma (1.3), we know that $G/K$ is either a $p$-group for some prime $p$ or $G/K$ is a Frobenius group. Suppose that $G/K$ is a $p$-group for some prime $p$. It follows that $f$ is a power of $p$. Since $f$ is a character degree of $G$, $f \in \{a, b, c, d\}$. If $f$ is $a$ or $b$, then by...
Gallagher’s theorem (Lemma (1.1)(b)), $ab \in \text{cd}(G)$, and if $f$ is $c$ or $d$, then by Gallagher’s theorem, $cd \in \text{cd}(G)$, which is a contradiction.

Hence, $G/K$ cannot be a $p$-group for some prime $p$, and so $G/K$ must be a Frobenius group with elementary abelian $p$-group kernel $L/K$ for some prime $p$ and cyclic complement (see Lemma (1.3)(b)). We also utilize that result to see that $|G : L| = f$. Suppose first that $f \in \{ac, ad, bc, bd\}$. By symmetry, we may assume that $f = ac$. Then since $b, d \in \text{cd}(G)$ are relatively prime to $|G : L|$, and $b|G : L|$ and $d|G : L|$ are not character degrees of $G$, we deduce that $p$ divides $b$ and $d$, which is a contradiction.

Therefore, we have $f \in \{a, b, c, d\}$. By symmetry, we may assume that $f = a$. Then by using Lemma (1.4), we conclude that $\{1, b, c, d, bc, bd\} \subseteq \text{cd}(L)$. Also, since $b \in \text{cd}(G)$ is relatively prime to $|G : L|$, and $b|G : L|$ is not a character degree of $G$, we deduce that $p$ divides $b$. Suppose we have $\theta \in \text{Irr}(L)$ with $\theta(1) = t$, where $t$ is some integer, and $\chi \in \text{Irr}(G/\theta)$ and let $\chi(1) = s$. Then, by Lemma (1.3)(b), either $at \in \text{cd}(G)$ or $p$ divides $t$. If $at \in \text{cd}(G)$, then $at \in \{a, ac, ad\}$, and hence, $t \in \{1, c, d\}$. If $p$ divides $t$, then $p$ divides $s$, so that $s \in \{b, bc, bd\}$. It follows that $t \in \{b, bc, bd\}$ because, by Lemma (1.4), $s/t$ divides $|G : L| = a$. This implies that $\text{cd}(L) = \{1, b, c, d, bc, bd\}$. By using Theorem F, we conclude that $\text{dl}(L) \leq 4$, and so $\text{dl}(G) \leq 5$. \qed
In this chapter, we present three examples. In the first example, we show that Case (2) of Theorem D does occur. In fact, we present an example of a solvable group $G$ such that $\text{cd}(G) = \{1, 3, 2, 25, 6, 75\}$, and $G$ is not a direct product. Note that the pairs $(3, 2)$ and $(3, 25)$ are strongly coprime pairs.

**Example 6.1.** Let $L$ be an extra special group of order $5^3$ of exponent 5, and let $E_1$ and $E_2$ be copies of $L$. Then the central direct product of $E_1$ and $E_2$ is $E$, which is an extra special $5$-group of order $5^5$. Take $\sigma \in \text{Aut}(E_1)$, $\alpha \in \text{Aut}(E_2)$ such that $[\sigma, Z(E_1)] = [\alpha, Z(E_2)] = 1$ and $o(\sigma) = 2$ and $o(\alpha) = 3$. Let $A$ be an elementary abelian group of order $5^2$. Since $A$ has an automorphism of order 3, we can consider $\sigma$ as an automorphism of $A$ of order 2 that acts trivially on $A$ and $\alpha$ as an automorphism of $A$ of order 3. Let $G = \langle (\sigma, \alpha) \rangle \ltimes (E \times A)$. Then $G$ is not a direct product and $\text{cd}(G) = \{1, 3, 2, 25, 6, 75\}$.

**Proof.** First, suppose that $G = U \times V$ and we prove that $U$ or $V$ is trivial. Since a Sylow 3-subgroup of $G$ has order 3 and $U$ and $V$ are normal subgroups of $G$, we may assume that $U$ contains all Sylow 3-subgroups of $G$. In particular, $U$ contains $\alpha$. Now $K = E \times A$ is a normal Sylow 5-subgroup of $G$. Notice that $E_2/E' = [E, \alpha]E'/E'$, and so $E_2 = [E, \alpha]E' = [E, \alpha]\Phi(E_2) = [E, \alpha]$ and we know that $[A, \alpha] = A$. It follows that $[K, \alpha] = E_2 \times A$. Since $\alpha$ centralizes $V$, we have that $[G, \alpha] = [UV, \alpha] = [U, \alpha] \subseteq U$. Hence, $E_2A \subseteq U$. In particular, $Z(E) \subseteq U$, and so $Z(K) \subseteq U$. This implies that $Z(K) \cap V = 1$. Since $K \cap V$ is a normal subgroup of the 5-group $K$, this implies that $K \cap V = 1$. But $K$
is the normal Sylow 5–subgroup of $G$, so 5 does not divide $|V|$. This implies that $U$ must contain $K$. Since $\sigma$ does not centralize $K$, we cannot have $\sigma \in V$. Since one of $U$ or $V$ must contain all the Sylow 2–subgroups of $G$, this forces that $\sigma \in U$. Observe that $G$ is generated by $K$, $\alpha$, and $\sigma$, so we conclude that $G = U$.

Now, we show $\text{cd}(G) = \{1, 3, 2, 25, 6, 75\}$. Take $S = \langle E_1 / E', \sigma \rangle$ and $T = \langle (E_2 / E') \times A, \alpha \rangle$. Notice that $(E_2 / E') \times A$ is elementary abelian of order $5^4$, $\langle \alpha \rangle$ has order 3, and $T$ is the semi-direct product of $\langle \alpha \rangle$ acting on $(E_2 / E') \times A$. Similarly, $E_1 / E'$ is elementary abelian of order $5^2$, $\langle \sigma \rangle$ has order 2, and $S$ is the semi-direct product of $\langle \sigma \rangle$ acting on $E_1 / E'$. We know that $\sigma$ centralizes $(E_2 / E') \times A$ and $\alpha$ centralizes $E_1 / E'$. Thus, $S$ and $T$ centralize each other. Since $G / E' = ST$, we conclude that $G / E' = S \times T$, and so $\text{cd}(G / E') = \{1, 2, 3, 6\}$.

Suppose $\theta \in \text{Irr}(E)$ has degree 25. Then, $\theta$ extends to $\theta \times 1$ on $K$. Since $\theta$ is fully ramified with respect $E / \mathbb{Z}(E)$, this implies that $\theta$ is $G$–invariant, so $\theta \times 1$ is $G$–invariant. Since $(|K|, |G : K|) = 1$, this implies that $\theta \times 1$ extends to $G$, and hence $\theta$ extends to $G$. This yields that $25 \in \text{cd}(G)$. Also, $G / E$ is isomorphic to the semi-direct product of $(\langle \sigma, \alpha \rangle)$ acting on $A$, so $\text{cd}(G / E) = \{1, 3\}$. Applying Gallagher’s theorem (Lemma (1.1)(b)), we conclude that $\text{cd}(G \mid \theta) = \{25, 75\}$, and hence, $\text{cd}(G)$ is as claimed. □

In the next example, we show that we cannot remove the condition that the pairs $(p, m)$ and $(p, n)$ are strongly coprime pairs in Theorem D.

**Example 6.2.** Let $q$ and $r$ be distinct primes and let $p$ be a prime such that $r$ does not divide $p^q - 1$, $q$ does not divide $p^r - 1$, and $\frac{p^r - 1}{p - 1}$ is not a prime. Suppose that $d > 1$ is a proper divisor of $\frac{p^r - 1}{p - 1}$ and either $rq \neq 6$ or $p \neq 2$. Assume that $F$ is the additive group of the field of order $p^{qr}$, and $C$ is the cyclic subgroup of the multiplicative group of the field of order $p^{qr}$ with order $\frac{d(p^r - 1)}{p - 1}$. Also, assume that $\Lambda$ is the Galois group of the field of order $p^{qr}$. Take $G = (F \rtimes C) \rtimes \Lambda$. Then $G$ is not a direct product, $\text{cd}(G) = \{1, r, q, \frac{d(p^r - 1)}{p - 1}, rq, \frac{rd(p^r - 1)}{p - 1}\}$. 

and the pair \((r, \frac{d(pqr-1)}{p^r-1})\) is not a strongly coprime pair.

**Proof.** First, suppose that \(G = A \times B\) and we prove that \(A\) or \(B\) is trivial. Observe that \(F\) is the normal Sylow \(p\)-subgroup of \(G\) that is also the Fitting subgroup of \(G\), and \(C\) is a Hall \(\pi\)-subgroup of \(G\), where \(\pi = \pi(|C|)\) is the set of all prime divisors of \(|C|\). Let \(l \in \pi\) be a prime divisor of \(|C|\). Since \(|G| = |A||B|\), this implies \(l\) divides either \(|A|\) or \(|B|\). We may assume that \(l\) divides \(|A|\), and hence \(A \cap C > 1\) because \(A\) is normal in \(G\) and \(C\) is a Hall \(\pi\)-subgroup of \(G\). If \(p\) divides \(|B|\), then \(F \cap B > 1\) as \(F\) is the normal Sylow \(p\)-subgroup of \(G\), and so \(A \cap C\) centralizes \(F \cap B\). This is a contradiction as \(C\) acts Frobeniusly on \(F\).

We deduce that \(p\) does not divide \(|B|\), and hence \(F \subseteq A\) as \(A\) is normal in \(G\) and \(F\) is the normal Sylow \(p\)-subgroup of \(G\). But since \(F\) is the Fitting subgroup of \(G\), it is well-known that \(C_G(F) \subseteq F\). We obtain that

\[
B \subseteq C_G(A) \subseteq C_G(F) \subseteq F \subseteq A,
\]

and so \(B \subseteq A \cap B = 1\).

Now, we show \(cd(G) = \{1, r, q, \frac{d(pqr-1)}{p^r-1}, rq, \frac{rd(pqr-1)}{p^r-1}\}\). We know that \(G/F\) is isomorphic to the semi-direct product of \(A\) acting on \(C\). Recall that \(T\) and \(S\) are cyclic subgroups of \(A\) of orders \(q\) and \(r\), respectively. Since \((p^q - 1, |C|) > 1\) and \((p^r - 1, |C|) > 1\), it follows that there are some non-trivial points \(x\) and \(y\) in \(C\) such that \(T \subseteq \text{Stab}_A(x), T \nsubseteq \text{Stab}_A(y), S \subseteq \text{Stab}_A(y), \) and \(S \nsubseteq \text{Stab}_A(x)\). Also, we know that there is \(c \in C\) so that \(\text{Stab}_A(c) = 1\).

We conclude that \(cd(G/F) = \{1, r, q, rq\}\). On the other hand, we know that there is a non-principal irreducible character \(\lambda\) of \(F\) such that \(\Lambda \subseteq \text{Stab}_G(\lambda)\). Thus, \(FA \subseteq \text{Stab}_G(\lambda)\).

Since \(C\) acts Frobeniusly on \(F\), we observe that \(\text{Stab}_G(\lambda) \cap C = 1\). Since \(G = (FA)C\), \(FA \subseteq \text{Stab}_G(\lambda)\), and \(\text{Stab}_G(\lambda) \cap C = 1\), it follows that \(\text{Stab}_G(\lambda) = FA\). Since \(FA/F\) is cyclic, \(\lambda\) extends to \(\mu\) on \(FA\). By the Clifford Correspondence (Lemma (1.2)), this implies that \(\mu^G \in \text{Irr}(G)\), and so \(|C| \in cd(G)\).
We claim that $C$ acts irreducibly on $F$. To see this, observe that if $U$ is a $C$-invariant subgroup of $F$, then $U$ is a union of $C$-orbits. This implies that $|U| = 1 + a|C|$ for some integer $a$ as $C$ acts Frobeniusly on $F$. Hence, $|U|$ is a $p$-power with the property that $|C|$ divides $|U| - 1$. Assume $|U| > 1$. We know that $(p^{rq} - 1)/(p^r - 1)$ divides $|C|$. Since $rq ≠ 6$ or $p ≠ 2$, we know that $(p^{rq} - 1)/(p^r - 1)$ is divisible by a Zsigmondy prime $s$. By the definition of Zsigmondy primes, $p^{rq}$ is the smallest non-trivial power of $p$ so that $s$ divides $p^{rq} - 1$. Since $s$ divides $|U| - 1$, this implies that $|U| = p^{rq} = |F|$, and hence $F$ is irreducible.

Take $H = (F ∅ C) ∅ T$. Then $H$ is a normal subgroup of $G$ whose index is $r$ and $H$ satisfies the hypotheses of Case (3) of Lemma (1.26) since $H$ is the semi-direct product of $C ∅ T$ acting on elementary abelian $p$-group $F$, the Fitting subgroup of $C ∅ T$ is $C$ whose index is $q$, $|F| = p^{rq}$, $(q, |C|) = 1$, $(p^{rq} - 1)/(p^r - 1)$ divides $|C|$, and also $C$ acts irreducibly on $F$. Therefore, it follows from Lemma (1.26)(3) that $cd(H) = \{1, q, |C|\}$, and hence $cd(G) = \{1, r, q, |C|, rq\}$ or $cd(G) = \{1, r, q, |C|, rq, r|C|\}$ as $|G : H| = r$.

But if $cd(G) = \{1, r, q, |C|, rq\}$, then the character degree graph $Δ(G)$ of $G$ has two connected components $\{r, q\}$ and $\pi(|C|)$. By using Lemma (1.26), we observe that $G$ does not satisfy the hypotheses of Cases (1)-(5) of Lemma (1.26). This forces that $cd(G) = \{1, r, q, |C|, rq\}$ is not possible. We conclude that $cd(G) = \{1, r, q, |C|, rq, r|C|\}$.

Finally, we show the pair $(r, |C|)$ is not a strongly coprime pair. Recall that $S$ is the subgroup of $Λ$ of order $r$. Since $S$ acts non-trivially on $C$, this implies $r$ divides $|\text{Aut}(C)|$, and so $r$ divides $v - 1$, where $v$ is some prime divisor of $|C|$ as $C$ is a cyclic group of order not divisible by $r$. We conclude that the pair $(r, |C|)$ is not a strongly coprime pair. □

Finally, the following example shows that the bound 4 in Theorem F is the best possible.

**Example 6.3.** Let $C_b = \langle σ_1 \rangle$ be cyclic group of order $b$ and let $p$ be a prime for which the cyclic group $C_p = \langle a \rangle$ acts Frobeniusly on $C_b = \langle σ_1 \rangle$. Assume that $σ_i^a = σ_i^1$, where $i > 1$.
is some integer. Choose any odd prime \( q_0 \) not equal \( p \) and suppose \( q \) is any power of \( q_0 \) for which \( b \) divides \( q - 1 \) or \( q + 1 \). Then \( C_b \) acts on an extra special group \( E \) of order \( q_0 q^2 \), \( C_b \) centralizes \( Z(E) \), and \( C_b \times E \) is Frobenius on \( E/Z(E) \). Assume that \( H = C_p \rtimes ((C_b E) \times (C_b E) \times \cdots \times (C_b E)) = (C_b E) \rtimes C_p \) is the wreath product of \( C_b E \) with \( C_p \), and suppose that \( W = \{(z_1, z_2, \ldots, z_p) \in Z(E \times E \times \cdots \times E) \mid \prod z_i = 1 \} \). Let \( E_1 = (E \times 1 \times \cdots \times 1) W/W \), \( E_2 = (1 \times E \times \cdots \times 1) W/W \), \( \ldots \), and \( E_p = (1 \times 1 \times \cdots \times 1) E W/W \). Then

\[
\sigma_2 = \sigma^{a_1} \in \text{Aut}(E_2), \quad \sigma_3 = \sigma^{a_1^2} \in \text{Aut}(E_3), \ldots, \quad \text{and} \quad \sigma_p = \sigma^{a_1^{p-1}} \in \text{Aut}(E_p). \]

Thus, \( E_j^{a_1} = E_{j+1} \) for all \( 1 \leq j \leq p - 1 \) and \( E_p^{a_1} = E_1 \). Take the subgroup \( G \leq H/W \) to be the subgroup of \( H/W \) generated by \( \alpha, \sigma \), and \( E_1 E_2 \cdots E_p \), where

\[
\sigma = \sigma_1 \sigma_2^{p-1} \sigma_3^{p-2} \cdots \sigma_p^1.
\]

Then \( \sigma^p = \sigma \), and \( G = \langle \alpha \rangle \langle \sigma \rangle \Xi \), where \( \Xi = E_1 E_2 \cdots E_p \) is an extra special group of order \( q^{1+2p} \). Then \( \text{dl}(G) = 4 \) and \( \text{cd}(G) = \{1, p, b, pb, q^p, pq^p\} \).

**Proof.** First, we show that the derived length of \( G \) is 4. To see this, it suffices to prove that \( G'' = \Xi \). Observe that \( E_1/\Xi' = [\Xi, \sigma_2 \sigma_3 \cdots \sigma_p] \Xi'/\Xi' \), and so \( E_1 = [\Xi, \sigma_2 \sigma_3 \cdots \sigma_p] \Xi' = [\Xi, \sigma_2 \sigma_3 \cdots \sigma_p] \Phi(E_1) = [\Xi, \sigma_2 \sigma_3 \cdots \sigma_p] \Xi' \). This implies that \( [\Xi, \sigma_2 \sigma_3 \cdots \sigma_p] = E_1 \). Similarly, \( [\Xi, \sigma_1 \sigma_3 \cdots \sigma_p] = E_2, \ldots, \) and \( [\Xi, \sigma_1 \sigma_2 \cdots \sigma_{p-1}] = E_p \). Hence, \( [\Xi, \sigma] = \Xi \), and so \( ([\Xi, \sigma], [\sigma, \alpha]) = \Xi \). We obtain that \( \Xi \subseteq G'' \). On the other hand, it is easy to see that \( \langle \sigma, \alpha \rangle \) is a Frobenius group with cyclic complement of order \( p \) and cyclic kernel of order \( b \). This implies that \( G'' \subseteq \Xi \) because \( \langle \sigma, \alpha \rangle \) is isomorphic to \( G/\Xi \). We conclude that \( G'' = \Xi \), and hence \( \text{dl}(G) = 4 \).

Now, we show \( \text{cd}(G) = \{1, p, b, pb, q^p, pq^p\} \). Take \( S_1 = E_1/\Xi', \quad S_2 = E_2/\Xi', \ldots, \) and \( S_p = E_p/\Xi' \). Then \( S_1, S_2, \ldots, \) and \( S_p \) are elementary abelian \( q_0 \)-groups, and \( \Xi/\mathbb{Z}(\Xi) \) is \( S_1 \times S_2 \times \cdots \times S_p \). Suppose \( \lambda \) is a non-principal irreducible character of \( S_1 \). We observe that \( \text{Stab}_G(\lambda, \lambda, \ldots, \lambda) = \langle \Xi, \alpha \rangle \) and \( \text{Stab}_G(\lambda, 1, \ldots, 1) = \Xi \). By the Clifford Correspondence
(Lemma (1.2)), we conclude that \( \text{cd}(G/Z(\Xi)) = \{1, p, b, pb\} \).

On the other hand, suppose \( \theta \in \text{Irr}(\Xi) \) has degree \( q^p \). Since \( \theta \) is fully ramified with respect to \( \Xi/Z(\Xi) \), this implies that \( \theta \) is \( G \)-invariant, and hence \( \theta \) extends to \( G \) because \( (|E|, [G : E]) = 1 \). This yields that \( q^p \in \text{cd}(G) \). Also, \( G/\Xi \) is a Frobenius group with cyclic complement of order \( p \) and cyclic kernel of order \( b \), so \( \text{cd}(G/\Xi) = \{1, p\} \). Applying Gallagher’s theorem (Lemma (1.1)(b)), we conclude that \( \text{cd}(G | \theta) = \{q^p, pq^p\} \), and hence, \( \text{cd}(G) \) is as claimed. \( \square \)
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