ASPECTS OF NON-PERTURBATIVE QCD FOR MESON PHYSICS

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Chapter 1

Introduction

1.1 Hadrons and Hadronic Physics

Hadrons are particles made from quarks and/or gluons and bound together by their strong interactions. They have no net strong charge (or color charge) but they do have residual strong interactions due to their color-charged substructure. Many types of hadrons were discovered in the 50’s and 60’s and the idea of quarks was first proposed to explain the many observed hadrons. There are two classes of hadrons: baryons and mesons.

Baryons have a basic structure of three quarks (and anti-baryons three antiquarks) and are fermions. Protons and neutrons, generically referred to as nucleons, are the most common baryons. The proton is the only stable baryon in isolation, with a basic structure of two up quarks and one down quark. Neutrons, with a basic structure of two down quarks and one up quark, are stable in certain nuclei but not in isolation. More massive baryons may be made from any set of three quarks. Baryons containing more massive quarks are all unstable because these quarks decay via weak interactions. There are also more massive baryons that have the same quark content as a proton or a neutron but have additional angular momentum. These are typically very short-lived because they can decay to a proton or a neutron and a meson via residual strong interactions.

Mesons are color-neutral particles made up of one quark and one antiquark, and are thus bosons. There are no stable mesons in nature. The most common mesons, pions and K mesons (kaons), are the only types of mesons which are long-lived enough to be seen directly by their tracks in a detector. More massive mesons with the same quark content but higher angular momentum, as well as others containing one or more of the more massive quark types, are all very short lived and are identified by their decay products.

Theoretically, there are additional types of hadrons that have a basic structure that is made purely
from gluons. These are called glueballs [1, 2]. As such particles contain virtual quark-antiquark pair fluctuations, it is quite difficult to find an experimental signature that can distinguish such a particle from an ordinary meson. There are, however, some observed particles that are thought to be chiefly glueball-like.

The quarks inside a meson or baryon are continually interacting with one another via the strong force field. At any instant in time, they may contain many virtual particles: gluons and additional quark-antiquark pairs. The picture of a proton as made of three quarks is thus a gross simplification; for example, it is known from measurements that in a high-momentum proton only about half of the momentum is carried by the quarks and antiquarks, the rest being carried by gluons.

Hadronic physics involves the study of strongly interacting matter and its underlying theory, Quantum Chromodynamics (QCD). The field had its beginnings in the 60’s, when hadrons were discovered in ever increasing numbers. At present it encompasses topics like the quark-gluon structure of hadrons at varying scales, the quark-gluon plasma and hadronic matter at extreme temperature and density; it also underpins nuclear physics and has significant impact on particle physics, astrophysics, and cosmology. Among the goals of hadronic physics are to determine the parameters of QCD, understand the origin and characteristics of confinement, understand the dynamics and consequences of dynamical chiral symmetry breaking, explore the role of quarks and gluons in nuclei and in matter under extreme conditions and understand the quark and gluon structure of hadrons.

There are two approaches to hadronic structure that predate QCD: the quark model (see review in, e.g., [3]) and current algebra (see review in, e.g., [4, 5]). The quark model provides a simple (and very successful) scheme based on the idea that hadrons can be understood as bound states of non-relativistic constituent quarks. Current algebra is based on the (approximate) $SU(2)_L \times SU(2)_R$ chiral symmetry of the strong interaction. The fact that this symmetry is not apparent in the hadronic spectrum led to the important concept that chiral symmetry is spontaneously broken in the ground state. Also, since the “current” quark masses appearing as symmetry breaking terms in the effective chiral Lagrangian are small, it became clear that the constituent quarks of the non-relativistic quark model have to be effective, composite, objects. With the advent of QCD, it was realized that current algebra is a rigorous consequence of the (approximate) chiral invariance of the QCD Lagrangian. It
was also clear that quark confinement and chiral symmetry breaking are consistent with QCD.

1.2 Quantum Chromodynamics

Quantum Chromodynamics (QCD) is the gauge field theory which describes the strong interactions of colored quarks and gluons. Excellent reviews can be found in [6, 7]. A quark of specific flavor (such as an up or down quark) comes in three colors; gluons come in eight colors; hadrons are color-singlet combinations of quarks, antiquarks, and gluons. The Lagrangian describing the interactions of quarks and gluons is (up to gauge-fixing terms)

\[
L_{QCD} = -\frac{1}{4} F^{(a)\mu\nu} F^{(a)\mu\nu} + \sum_q \bar{\psi}_q^i \left[ \gamma^\mu (i D^\mu)_{(ij)} - m_q \right] \psi_q^j,
\]

\[
F^{(a)\mu\nu} = \partial_\mu A^{a\mu}_\nu - \partial_\nu A^{a\mu}_\mu - g_s f_{abc} A^{b\mu}_\mu A^{c\nu}_\nu,
\]

\[
(D^\mu)_{ij} = \delta_{ij} \partial_\mu + i g_s \sum_a \frac{\lambda_a^{ij}}{2} A^a_\mu,
\]

where \(g_s\) is the QCD coupling constant, and both the \(\lambda^a\) and the \(f_{abc}\) are the hermitian generators and the structure constants respectively of the SU(3) algebra. The \(\psi_q^i\) are the 4-component Dirac spinors associated with each quark field of (3) color \(i\), flavor \(q\) and mass \(m_q\), and the \(A^{a\mu}_\mu(x)\) are the (8) Yang-Mills (gluon) fields. Thus \(L_{QCD}\) describes not only quark-gluon interactions but also gluodynamics, the specific gluon self-interactions which have no analog in Quantum Electrodynamics, QED. The quark-gluon strong coupling is commonly expressed as

\[
\alpha_s = \frac{g_s^2}{4\pi},
\]

in analogy with the e.m. coupling \(\alpha_{em} = e^2/4\pi\).

QCD yields two qualitatively different pictures of quark-gluon interactions (Fig. 1.1):

1) at high momentum-transfers (large \(Q\)), corresponding to short distances, perturbative expansions in \(\alpha_s\) are applicable in terms of Feynman diagrams involving quark and gluon propagators and vertices. In this region the scale-dependence (running) of the coupling and quark masses need to be properly taken into account. This is the region of perturbative QCD (pQCD) [8, 9].

2) at low energy (momentum) scales, corresponding to long distances, perturbative techniques (perturbation theory) are no longer viable, since quarks and gluons interact strongly and form
hadrons. This is the domain where non-perturbative techniques are necessary for both the structure and low-energy processes involving hadrons.

![Graph](image.png)

Fig. 1.1: Schematic representation of the behavior of the strong coupling, \( \alpha_s \), at different scales.

Accordingly, QCD is being developed on two different fronts. The first deals with short-distance physics accessible at high energies. One studies specific processes/observables calculable (at least partly) in form of a perturbative expansion in \( \alpha_s \). A typical short-distance process is jet production in high energy hadron-hadron collisions. An excellent review is contained in [8].

The second direction deals with non-perturbative quark-gluon interactions at long distances and with hadron dynamics. A complete analytical evaluation of hadronic masses and other parameters directly from \( L_{QCD} \) is not yet accessible; instead the powerful numerical method of simulating QCD on the space-time lattice (Lattice QCD [10]) has been developed for non-perturbative studies directly from the QCD Lagrangian. There are also various continuum field-theoretical approaches and diverse models currently employed to understand low-energy (strong) QCD.
1.2.1 Perturbative QCD

In both QED and QCD the coupling “constant” is not constant at all but evolves with $Q^2$ and can be calculated perturbatively [9]:

$$\alpha_{\text{QED}} = \frac{\alpha}{1 - \alpha/3\pi \ln(Q^2/m_e^2)}, \quad \alpha \simeq \frac{1}{137},$$  \hspace{1cm} (1.5)

$$\alpha_{S} = \frac{12\pi}{(11N_c - 2N_f) \ln \left(\frac{Q^2/\Lambda_{\text{QCD}}^2}{\Lambda^2} \right)},$$  \hspace{1cm} (1.6)

where the “$11N_c$”, $N_c = 3$ is the number of colours, appears because the gluons are self-interacting. $N_f$ is the number of fermion flavours. As depicted in Fig. 1.2, $\alpha_S(Q)$ decreases with increasing $Q$, which is opposite to the behaviour of $\alpha_{\text{QED}}(Q)$ quoted in Eq. (1.5). The rate of the evolution is also very different, being determined by vastly different mass scales: $\Lambda_{\text{QCD}}/m_e \sim 450$. The reduction in $\alpha_S$ with increasing $Q$ is called “asymptotic freedom”. It is due to gluon self-interactions and entails that perturbation theory is valid at large $Q^2$ in QCD; i.e., at short distances, and not at long distances as is the case in QED.

Perturbation theory is the most widely used, systematic tool in physics and in QCD it is quite accurate in various high energy reactions, and this is thus the domain where QCD has been most
precisely tested. In particular it has been widely applied successfully to describe processes in deep inelastic scattering (DIS) and ultrarelativistic heavy ion collisions [8]. The general framework for utilizing perturbation theory in QCD is as follows: to describe processes involving hadrons one separates the long distance (nonperturbative) parts from the short distance (perturbative) parts, a procedure known as factorization. The perturbative part (hard scattering) is calculable order by order in pQCD. The nonperturbative parts (for example the parton distribution functions (PDFs) and fragmentation functions) are currently determined most economically from experiments. This framework is general enough not only for describing hadron-hadron collisions but also nucleus-nucleus collisions.

1.2.2 Nonperturbative Effects in QCD

However, there are domains and problems in QCD that perturbation theory simply cannot describe. This is very obvious in Fig. 1.2 where the coupling is seen to increase with the inter-particle separation: QCD becomes a strong coupling theory for $Q < 2 \text{GeV}$ [$x > 0.1 \text{ fm}$]. Confinement, dynamical chiral symmetry breaking (DCSB), bound state structure and phase transitions are intrinsically non-perturbative phenomena and are at the core of hadron physics. Hence non-perturbative methods are essential in understanding low-energy QCD. The remarkable effects in QCD can be explained in terms of the properties of dressed-quark and -gluon propagators. They describe the “in-medium” propagation characteristics of QCD’s elementary quanta, with the “medium” being the nontrivial ground state (vacuum) of QCD. A photon propagating through a dense $e^-$-gas provides a familiar example of the effect a medium has on the propagation of an elementary particle. Due to particle-hole excitations the propagation of the photon is modified:

$$\frac{1}{Q^2} \rightarrow \frac{1}{Q^2 + m_D^2};$$ (1.7)

i.e., the photon acquires an effective mass. The “Debye” mass, $m_D \propto k_F$, the Fermi momentum, and it screens the interaction so that in the dense $e^-$-gas the Coulomb interaction has a finite range: $r \propto 1/m_D$. Quark and gluon propagators are modified in a similar way. They acquire momentum-dependent effective masses, which have observable effects on hadron properties.
Various techniques, both rigorously field-theoretical and phenomenological (QCD-derived), are currently applied to understand the non-perturbative regime of QCD [12, 13, 14, 15]. The most fundamental of these approaches is Lattice QCD, where non-perturbative studies are carried out by solving QCD numerically on a lattice. The computational efforts and resources required increase dramatically with lattice size, however, and thus going to the continuum limit is problematic. Therefore continuum field-theoretical approaches like the Dyson-Schwinger Equations (DSEs) are particularly useful in this regard.

Continuum field-theoretic models of QCD in a fixed gauge, such as the light-cone or the Coulomb gauge [15], and Dyson-Schwinger Bethe-Salpeter models [13] provide an important approach towards understanding hadron substructure. They address all of the relevant features of QCD such as the structure of the vacuum, chiral-symmetry breaking, confinement and strong decays, as well as hadronic interactions and the connection to the parton model. In particular, Dyson-Schwinger equation studies have made important advances, for example they provide a microscopic understanding of the dual nature of the pion as a Goldstone boson and a $q\bar{q}$ bound state. Contemporary analyses and modeling of continuum Coulomb gauge QCD can also address the dual nature of the pion and have supported the glueball spectrum obtained in lattice QCD simulations. We use the Dyson-Schwinger Equations (DSEs) primarily in our studies; exhaustive reviews of DSEs and their application to hadronic phenomenology can be found in [13, 16].

1.3 Dyson-Schwinger Equations

The Dyson-Schwinger Equations (DSEs) [13, 16, 17, 18, 19] are an infinite tower of coupled integral equations, with the equation for a particular $n$-point function involving at least one $m > n$-point function; e.g., the quark DSE involved the dressed-gluon propagator, a 2-point function, and the dressed-quark-gluon vertex, a 3-point function. The collection of DSEs provide a Poincaré invariant, continuum approach to solving quantum field theories. However, as an infinite collection of coupled equations, a tractable problem is only obtained if one truncates the system.

The simplest truncation scheme for the DSEs is the weak-coupling expansion, which shows that
they contain perturbation theory; i.e., for any given theory the weak-coupling expansion generates all the diagrams of perturbation theory. However, the most important feature of the DSEs is the antithesis of this weak-coupling expansion: the DSEs are intrinsically nonperturbative and their solution contains information that is not present in perturbation theory. Here we give a brief review of the four lowest $n$-point functions important for hadronic phenomenology. These are the gluon and quark propagators (2-point functions), the vertex function or Bethe-Salpeter amplitudes (3-point functions) and the scattering kernel which is a 4-point function. The material presented in this section is essentially from [13, 19], and further details can be found in these reviews and the references therein.

1.3.1 Gluon Propagator

In Landau gauge the dressed-gluon propagator has the form

$$g^2 D_{\mu\nu}(k) = \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{\mathcal{G}(k^2)}{k^2}, \quad \mathcal{G}(k^2) := \frac{g^2}{[1 + \Pi(k^2)]},$$

(1.8)

where $\Pi(k^2)$ is the vacuum polarisation, which contains all the dynamical information about gluon propagation. It satisfies the DSE [a nonlinear integral equation] depicted in Fig. 1.3. A weak coupling expansion of the equation reproduces perturbation theory and shows directly that in the one-loop expression for the running coupling constant, Eq. (1.6), the “$11N_c$” comes from the charge-antiscreening gluon loop and the “$2N_f$” from the charge-screening fermion loop. This illustrates how the non-Abelian structure of QCD is responsible for asymptotic freedom and suggests that confinement is related to the importance of gluon self-interactions.

Numerous studies of the gluon DSE [13] have shown that if the ghost-loop is unimportant, then the charge-antiscreening 3-gluon vertex dominates and, relative to the free gauge boson propagator, the dressed gluon propagator is significantly enhanced in the vicinity of $k^2 = 0$. The enhancement persists to $k^2 \sim 1\,\text{GeV}^2$, where a perturbative analysis becomes quantitatively reliable. In the neighbourhood of $k^2 = 0$ the enhancement can be represented as a regularisation of $1/k^4$ as a distribution. A dressed-gluon propagator of this type generates confinement and DCSB without fine-tuning.
1.3.2 Quark Propagator

In a covariant gauge the dressed-quark propagator can be written as [13]

\[
S(p) := \frac{1}{i\gamma \cdot p + \Sigma(p)} = \frac{-i\gamma \cdot pA(p^2) + B(p^2)}{p^2A^2(p^2) + B^2(p^2)},
\]

where \(\Sigma(p)\) is the self-energy, expressed in terms of the scalar functions, \(A\) and \(B\), which are \(p^2\)-dependent because the interaction is momentum-dependent. A widely-used equivalent form is given by:

\[
S(p) = -i\gamma \cdot p \sigma V(p^2) + \sigma S(p^2) = \frac{Z(p^2)}{i\gamma \cdot p + M(p^2)}.
\]

The Dyson-Schwinger equation (DSE) for the quark self-energy [the QCD “gap equation”] is [13]

\[
i\gamma \cdot p A(p^2) + B(p^2) = Z_2 i\gamma \cdot p + Z_4 m + Z_1 \int_\Lambda \frac{d^4\ell}{(2\pi)^4} g^2 D_{\mu\nu}(p - \ell) \gamma_\mu \frac{\lambda^a}{2} i\gamma \cdot \ell A(\ell^2) + B(\ell^2) \Gamma^a_\nu(\ell, p),
\]

and depicted in Fig. 1.4.
Fig. 1.4: DSE for the dressed-quark self-energy. The kernel of this equation is constructed from the dressed-gluon propagator ($D$ - spring) and the dressed-quark-gluon vertex ($\Gamma$ - open circle). One of the vertices is bare (labelled by $\gamma$) as required to avoid over-counting.

The quark DSE is a nonlinear integral equation for $A$ and $B$ and its nonlinearity is what makes it possible to generate nonperturbative effects. The kernel of the equation is composed of the dressed-gluon propagator:

$$g^2 D_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \frac{G(k^2)}{k^2}, \quad G(k^2) := \frac{g^2}{1 + \Pi(k^2)},$$

where $\Pi(k^2)$ is the vacuum polarisation, which contains all the dynamical information about gluon propagation, and the dressed-quark-gluon vertex: $\Gamma^a_{\mu}(k, p)$. The bare (undressed) vertex is

$$\Gamma^a_{\mu}(k, p)_{\text{bare}} = \gamma_{\mu} \frac{\lambda^a}{2}. \quad (1.13)$$

Once $D_{\mu\nu}$ and $\Gamma^a_{\mu}$ are known, Eq. (1.11) is straightforward to solve by iteration. One chooses an initial seed for the solution functions: $0A$ and $0B$, and evaluates the integral on the right-hand-side (r.h.s.). The bare propagator values: $0A = 1$ and $0B = m$ are often adequate. This first iteration yields new functions: $1A$ and $1B$, which are reintroduced on the r.h.s. to yield $2A$ and $2B$, etc. The procedure is repeated until $nA = n_{A+1}$ and $nB = n_{B+1}$ to the desired accuracy.

As depicted in Fig. 1.5, solving the quark DSE, Eq. (1.11), using a dressed-gluon propagator of the type described above, one obtains a quark mass-function, $M(p^2)$, that mirrors the enhancement of the dressed-gluon propagator. The results in the figure were obtained with current-quark masses corresponding to

$$m_{u/d}^{1 \text{GeV}} \quad m_{s}^{1 \text{GeV}} \quad m_{c}^{1 \text{GeV}} \quad m_{b}^{1 \text{GeV}}$$

$$6.6 \text{ MeV} \quad 140 \text{ MeV} \quad 1.0 \text{ GeV} \quad 3.4 \text{ GeV}.$$
The quark DSE was also solved in the chiral limit, which in QCD is obtained by setting the Lagrangian current-quark bare mass to zero [20]. It is observed immediately that the mass-function is nonzero even in this case. This is a manifestation of dynamical chiral symmetry breaking (DCSB): a momentum-dependent quark mass, generated dynamically, in the absence of any term in the action that breaks chiral symmetry explicitly. This entails a nonzero value for the quark condensate in the chiral limit. That $M(p^2) \neq 0$ in the chiral limit is independent of the details of the infrared enhancement in the dressed-gluon propagator.

Figure 1.5 illustrates that for light quarks ($u$, $d$ and $s$) there are two distinct domains: perturbative and non-perturbative. In the perturbative domain the magnitude of $M(p^2)$ is governed by the the current-quark mass. For $p^2 < 1 \text{GeV}^2$ the mass-function rises sharply. This is the non-perturbative domain where the magnitude of $M(p^2)$ is determined by the DCSB mechanism; i.e., the enhancement in the dressed-gluon propagator. This emphasises that DCSB is more than just a nonzero value of the quark condensate in the chiral limit.

Fig. 1.5: Dressed-quark mass-function obtained in solving the quark DSE.
1.3.3 Hadrons as Bound States

The properties of hadrons can be understood in terms of their substructure by studying covariant bound state equations: the Bethe-Salpeter equation [BSE] for mesons and the covariant Fadde’ev equation for baryons. The mesons have been studied most extensively and their internal structure is described by a Bethe-Salpeter amplitude obtained as a solution of

\[ [\Gamma_H(k; P)]_{ta} = \int_{\Lambda} \frac{d^4q}{(2\pi)^4} [\chi_H(q; P)]_{sr} K_{ta}^{sr}(q, k; P), \tag{1.15} \]

where \( \chi_H(q; P) := S(q_+)\Gamma_H(q; P)S(q_-); S(q) = \text{diag}(S_u(q), S_d(q), S_s(q), \ldots); q_+ = q + \eta P, \)
\( q_- = q - (1 - \eta P) P, \) with \( P \) the total momentum of the bound state; and \( r, \ldots, u \) represent colour-, Dirac- and flavour-matrix indices. The amplitude for a pseudoscalar bound state has the form

\[
\Gamma_H(k; P) = T^H \gamma_5 \left[ iE_H(k; P) + \gamma \cdot P F_H(k; P) \right. \\
+ \gamma \cdot k - k \cdot P G_H(k; P) + \sigma_{\mu\nu} k_\mu P_\nu H_H(k; P) \left. \right], \tag{1.16}
\]

where \( T^H \) is a flavour matrix that determines the channel under consideration; e.g., \( T^{K^+} = (1/2) (\lambda^4 + i\lambda^5), \) with \( \{\lambda^j, j = 1 \ldots 8\} \) the Gell-Mann matrices.

In Eq. (1.15), \( K \) is the renormalised, fully-amputated, quark-antiquark scattering kernel and important in the successful application of DSEs is that it has a systematic skeleton expansion in terms of the elementary, dressed-particle Schwinger functions; e.g., the dressed-quark and -gluon propagators. The expansion introduced in [21] provides a means of constructing a kernel that, order-by-order in the number of vertices, ensures the preservation of vector and axial-vector Ward-Takahashi identities; i.e., current conservation.

In any study of meson properties, one chooses a truncation for \( K \). The BSE is then fully specified and straightforward to solve, yielding the bound state mass and amplitude. The “ladder” truncation of \( K \) combined with the “rainbow” truncation of the quark DSE [\( \Gamma_\mu \rightarrow \gamma_\mu \) in Eq. (1.11)] is the simplest and most often used. The expansion of Fig. 1.6 provides the explanation [21] for why this Ward-Takahashi identity preserving truncation is accurate for flavour-nonsinglet pseudoscalar and vector mesons: there are cancellations between the higher-order diagrams. It also shows why it
provides a poor approximation in the study of scalar mesons, where the higher-order terms do not cancel, and for flavour-singlet mesons, where it omits timelike gluon exchange diagrams.

1.3.4 Confinement

Confinement is the absence of quark and gluon production thresholds in colour-singlet-to-singlet S-matrix amplitudes. That is ensured if the dressed-quark and -gluon propagators do not have a Lehmann representation.

To illustrate a Lehmann representation, let us consider the 2-point free-scalar propagator: $\Delta(k^2) = 1/(k^2 + m^2)$. One can write

$$\Delta(z) = \int_0^\infty d\sigma \frac{\rho(\sigma)}{z + \sigma},$$

where in this case the spectral density is

$$\rho(x) := \frac{1}{2\pi i} \lim_{\epsilon \to 0} [\Delta(-x - i\epsilon) - \Delta(-x + i\epsilon)] = \delta(m^2 - x),$$

which is non-negative. This is a Lehmann representation: each scalar function necessary to specify the $n$-point function completely has a spectral decomposition with non-negative spectral densities. Only those functions whose poles or branch points lie at timelike, real-$k^2$ have a Lehmann representation.
The existence of a Lehmann representation for a dressed-particle propagator is necessary if the construction of asymptotic “in” and “out” states for the associated quanta is to proceed; i.e., it is necessary if these quanta are to propagate to a “detector”. In its absence there are no asymptotic states with the quantum numbers of the field whose propagation characteristics are described by the propagator. Structurally, the nonexistence of a Lehmann representation for the dressed-propagators of elementary fields ensures the absence of pinch singularities in loops and hence the absence of quark and gluon production thresholds.

The mechanism can be generalised and applied to coloured bound states, such as colour-antitriplet quark-quark composites (diquarks). A study [21] of the quark-quark scattering matrix shows that it does not have a spectral decomposition with non-negative spectral densities and hence there are no diquark bound states. The same argument that demonstrates the absence of diquarks in the spectrum of $SU(N_c = 3)$ also proves that in $SU(N_c = 2)$ the “baryons”, which are necessarily diquarks in this theory, are degenerate with the mesons.

Dressed-gluon propagators with the infrared enhancement described above do not have a Lehmann representation and using forms like this in the kernel of the quark DSE yields a dressed-quark propagator that also does not have a Lehmann representation.

1.4 Condensates, Chiral Symmetry Breaking, and Correlation Functions

1.4.1 Vacuum Condensates

QCD at short distances does not essentially help in understanding the long-distance dynamics of quarks and gluons. From the short-distance side it is known that the running coupling $\alpha_s(Q)$ increases at low-momentum scales and eventually diverges at $Q \sim \Lambda_{QCD}$ (see Fig. 1.1). The question of whether the growth of $\alpha_s$ is the only dynamical reason for confinement is not answerable within the framework of perturbative QCD, because the language of Feynman diagrams involving propagators and vertices is not applicable already at $\alpha_s \sim 1$. QCD in the non-perturbative regime is currently being studied using other methods, most notably lattice simulations. From these studies there is growing evidence that long-distance dynamics is closely related to the nontrivial properties
of the physical vacuum in QCD.

For a given dynamical system, the vacuum is the state with the minimum possible energy. Evidently, in QCD, the vacuum state contains no hadrons, since the creation of any hadron involves a certain amount of energy. Though the vacuum contains no hadrons, quantum fluctuations of quark and gluon fields with nonvanishing densities are present. The existence of vacuum fields is manifested, for example, by instantons, the special solutions of QCD equations of motion having a form of localized dense gluonic fields [22]. Lattice QCD also provides an independent evidence for quark/gluon fields in the vacuum. Thus properties of hadrons are influenced by the existence of quark and gluon vacuum fluctuations with nonvanishing average densities, the so-called vacuum condensates [14].

Formally, in the presence of vacuum fields, the matrix elements of quark and gluon field operators between the initial \( |0\rangle \) and final \( \langle 0| \) states are different from zero. The combinations of fields have to obey Lorentz-invariance, colour gauge symmetry and flavour conservation, so that the simplest allowed composite operators are

\[
O_3 = \bar{\psi}_q i \psi_q^j, \quad O_4 = G_{\mu \nu}^a G^{a \mu \nu}, \quad O_5 = \bar{\psi}_q \Gamma^a \lambda^i \psi^j, \quad O_6 = \left[ \bar{\psi}_q (\Gamma^a)^i_k \psi^j \right] \left[ \bar{\psi}_q (\Gamma^a)^j_l \psi^l \right],
\]

where \( \sigma_{\mu \nu} = (i/2)[\gamma_\mu, \gamma_\nu] \) and \( \Gamma^a \) are various combinations of Lorentz- and colour matrices. The indices at \( O_d \) reflect their dimension \( d \) in GeV units. Furthermore, if the operators are taken at different 4-points, care should be taken of the local gauge invariance. For instance, the quark-antiquark nonlocal matrix element has the following form:

\[
\langle 0 | \bar{\psi}_q(x) [x, 0] \psi_q(0) | 0 \rangle,
\]

where \([x, 0] = \exp \left[ i g_s \int_0^1 dv x^\mu A^a_\mu (vx) (\lambda^a / 2) \right] \) is the so-called gauge factor. Only the matrix elements with the light quarks \( q = u, d, s \) are relevant for the non-perturbative long-distance dynamics, since a pair of heavy c (b) quarks can be created in vacuum only at short distances/times of \( O(1/2m_c) \) (\( O(1/2m_b) \)), i.e., perturbatively.

Without fully solving QCD, very little can be said about vacuum fields, in particular about their fluctuations at long distances which have typical “wavelengths” of \( O(1/\Lambda_{QCD}) \). Thus matrix
elements like (1.20) cannot be calculated explicitly as a function of $x$. It is still possible though to investigate the vacuum phenomena in QCD by applying certain approximations, of which one possibility is to study the average local densities. The vacuum average of the product of quark and antiquark fields,

$$\langle 0 | \bar{\psi}_q k \psi_q^k | 0 \rangle \equiv \langle \bar{q}q \rangle,$$

(1.21)
corresponds to the $x \to 0$ limit of the matrix element (1.20). The simplest vacuum density of gluon fields is

$$\langle 0 | G^a_{\mu\nu} G^{a\mu\nu} | 0 \rangle \equiv \langle GG \rangle.$$

(1.22)

Due to translational invariance, both $\langle \bar{q}q \rangle$ and $\langle GG \rangle$ are independent of the 4-coordinate. These universal parameters are usually called the densities of quark and gluon condensates, respectively. In particular the nonvanishing quark condensate drastically influences the symmetry properties of QCD.

1.4.2 Quark Condensate and Chiral Symmetry Breaking

Consider the isospin symmetry limit of the QCD Lagrangian:

$$L_{QCD}^{(u=d)} = \bar{\Psi}(iD_\mu \gamma^\mu - \tilde{m})\Psi + L_{glue} + \ldots.$$  (1.23)

Since $\tilde{m} \simeq m_u \simeq m_d \ll \Lambda_{QCD}$, a reasonable approximation is to put $\tilde{m} \to 0$, so that $u$- and $d$-quark components of the $\Psi$-doublet become massless.

Each Dirac spinor can be decomposed into the left- and right-handed components:

$$\psi_q = \frac{1 + \gamma_5}{2} \psi_q^L + \frac{1 - \gamma_5}{2} \psi_q^R \equiv \psi_q^L + \psi_q^R,$$

(1.24)

where, by definition, the left-handed (right-handed) quark has an antiparallel (parallel) spin projection on its 3-momentum. Similarly, for the conjugated fields one has:

$$\bar{\psi}_q = \bar{\psi}_q^L \frac{1 - \gamma_5}{2} + \bar{\psi}_q^R \frac{1 + \gamma_5}{2} \equiv \bar{\psi}_q^R + \bar{\psi}_q^L.$$  (1.25)
Rewriting $\Psi$ in terms of the left- and right-handed components we obtain the following decomposition of the Lagrangian (1.23) in the massless limit:

$$L_{QCD}^{(u=d)} = \overline{\Psi}_R i D_\mu \gamma^\mu \Psi_R + \overline{\Psi}_L i D_\mu \gamma^\mu \Psi_L + L_{\text{glue}} + \ldots .$$  \hspace{1cm} (1.26)$$

The quark-gluon interaction term in $L_{QCD}^{(u=d)}$ is now split into two parts, $g_s \overline{\Psi}_R \gamma_\mu A_\mu^{a} (\lambda^{a}/2) \Psi_R$ and $g_s \overline{\Psi}_L \gamma_\mu A_\mu^{a} (\lambda^{a}/2) \Psi_L$, so that quarks conserve their chirality (left- or right-handedness) after emitting/absorbing an arbitrary number of gluons. In the massless (chiral) limit, quarks of left- and right chiralities propagate and interact independently of each other. In fact, it is possible to introduce two independent isospin $SU(2)$ transformations, separately for $L$ and $R$ fields:

$$\Psi_L \rightarrow \Psi_L' = \exp \left( -i \frac{\sigma^a}{2} \omega_L \right) \Psi_L ,$$

$$\Psi_R \rightarrow \Psi_R' = \exp \left( -i \frac{\sigma^a}{2} \omega_R \right) \Psi_R .$$ \hspace{1cm} (1.27)$$

Restoring the mass in $L_{QCD}^{(u=d)}$ leads to the violation of chiral symmetry; the Lagrangian mass term can be represented as an effective transition between left- and right-handed quarks:

$$\tilde{m}\overline{\Psi}\Psi = \tilde{m} \left( \overline{\Psi}_L \Psi_R + \overline{\Psi}_R \Psi_L \right) .$$ \hspace{1cm} (1.28)$$

Exact chiral symmetry entails that degenerate parity multiplets must be present in the spectrum of the theory. For many reasons, the masses of the $u$- and $d$-quarks are expected to be very small; i.e., $m_u \sim m_d \ll \Lambda_{QCD}$. Therefore chiral symmetry should only be weakly broken, with the strong interaction spectrum exhibiting nearly degenerate parity partners. In reality, the symmetry is violated quite substantially. The experimental comparison is presented in Eq. (1.29):

| $N(\frac{1}{2}^+, 938)$ | $\pi(0^-, 140)$ | $\rho(1^-, 770)$ |
| $N(\frac{1}{2}^-, 1535)$ | $a_0(0^+, 980)$ | $a_1(1^+, 1260)$ |

(1.29)$$

Clearly the expectation is very badly violated, with the splitting much too large to be described by the small current-quark masses.

An additional source of the chiral symmetry violation is provided by the quark condensate. Decomposing the quark and antiquark fields in (1.21) in the left-handed and right-handed components,

$$\langle 0 | \bar{\psi}_q \gamma_i \psi_q | 0 \rangle = \langle 0 | (\bar{\psi}_q R + \bar{\psi}_q L) (\psi_q R + \psi_q L) | 0 \rangle = \langle 0 | \bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R | 0 \rangle \neq 0 ,$$ \hspace{1cm} (1.30)$$
we realize that condensate causes vacuum transitions between quarks of different chiralities. Hence in QCD one encounters a spontaneously broken chiral symmetry, a specific situation where the interaction (in this case $L_{QCD}$) obeys the symmetry (up to the small $O(m_{u,d})$ corrections), whereas the lowest-energy state (QCD vacuum) violates it. The conclusion is thus that in order to correctly reproduce the properties of hadrons and hadronic amplitudes (e.g., correlation functions), one has to take into account the vacuum fields, in particular, the quark condensate.

This is reflected in the well known Gell-Mann-Oakes-Renner relation 

$$- (m_u + m_d)(\langle \bar{u}u \rangle + \langle \bar{d}d \rangle) + O(m_{u,d}^2) = f_\pi^2 m_\pi^2. \tag{1.31}$$

This relation manifests the special nature of pion in QCD. The anomalously small pion mass is not accidental and is closely related to the spontaneous chiral symmetry breaking via condensate. If a symmetry in quantum field theory is broken spontaneously, there should be massless states (Nambu-Goldstone particles), one per each degree of freedom of broken symmetry. The three pions, $\pi^+, \pi^-, \pi^0$ play a role of massless Nambu-Goldstone particles in QCD. In other words, due to the specific structure of QCD vacuum fields, the amount of energy needed to produce a pion state tends to zero. The fact that pions still have small nonvanishing masses is due to the explicit violation of chiral symmetry via $u, d$ quark masses.

1.4.3 Correlation Functions

In a relativistic field theory, correlation functions of gauge-invariant local operators are the proper tools to study the spectrum of the theory. The correlation functions can be calculated either from physical states (mesons, baryons, glueballs) or in terms of the fundamental fields (quarks and gluons). In the latter case, there is a variety of available techniques, ranging from perturbative QCD, the operator product expansion (OPE), to models of QCD and lattice simulations. For this reason, correlation functions provide a bridge between hadronic phenomenology on the one side and the underlying structure of the QCD vacuum on the other. Loosely speaking, hadronic correlation functions play the same role for understanding the forces between quarks as the NN scattering phase shifts did in the case of nuclear forces. In the case of quarks, however, confinement implies that we
cannot define scattering amplitudes in the usual way. Instead, one has to focus on the behavior of
gauge-invariant correlation functions at short and intermediate distance scales. A detailed review of
both the theoretical and phenomenological aspects of hadronic correlation functions is contained in
[22, 24], and the material in this section follows essentially that treatment.

Consider hadronic point-to-point correlation functions

$$\Pi_h(x) = \langle 0 | j_h(x) j_h^\dagger(0) | 0 \rangle, \quad (1.32)$$

where $j_h(x)$ is a local operator with the quantum numbers of a hadronic state $h$. Mesonic and
baryonic currents can be defined by

$$j_{\text{mes}}(x) = \delta^{ab}\bar{\psi}_a(x)\Gamma\psi_b(x), \quad (1.33)$$
$$j_{\text{bar}}(x) = \epsilon^{abc}(\bar{\psi}_aT(x)C\Gamma\psi_b(x))\Gamma'\psi_c(x). \quad (1.34)$$

Here, $a, b, c$ are color indices and $\Gamma, \Gamma'$ are isospin and Dirac matrices. At zero temperature it is
sufficient to focus exclusively on correlators for spacelike (or Euclidean) separation $\tau = \sqrt{-x^2}$.
The reason is that spacelike correlators are exponentially suppressed rather than oscillatory at large
distances. In addition, most theoretical approaches, like the Operator Product Expansion (OPE)
[25], lattice calculations [10] or the instanton model [22] deal with Euclidean correlators.

Hadronic correlation functions are completely determined by the spectrum (and the coupling
constants) of the physical excitations with the quantum numbers of the current $j_h$. For a scalar
correlation function, the standard dispersion relation is of the form

$$\Pi(Q^2) = \frac{(-Q^2)^n}{\pi} \int ds \frac{\text{Im}\Pi(s)}{s^n(s + Q^2)} + a_0 + a_1Q^2 + \ldots, \quad (1.35)$$

where $Q^2 = -q^2$ is the Euclidean momentum transfer and $a_i$ are possible subtraction constants.
The spectral function $\rho(s) = \frac{1}{\pi} \text{Im}\Pi(s)$ can be expressed in terms of physical states,

$$\rho(s = -q^2) = (2\pi)^3 \sum_n \delta^4(q - q_n) \langle 0 | j_h(0) | n \rangle \langle n | j_h^\dagger(0) | 0 \rangle, \quad (1.36)$$

where $|n\rangle$ is a complete set of hadronic states. Correlation functions with non-zero spin can be
decomposed into Lorentz covariant tensors and scalar functions. Fourier transforming the relation
(1.35) gives a spectral representation of the coordinate space correlation function

\[ \Pi(\tau) = \int ds \, \rho(s) D(\sqrt{s}, \tau). \quad (1.37) \]

Here, \( D(m, \tau) \) is the Euclidean propagator of a scalar particle with mass \( m \),

\[ D(m, \tau) = \frac{m}{4\pi^2} K_1(m\tau), \quad (1.38) \]

where \( K_1 \) is the 1st-order modified Bessel function. It should be noted that except for possible contact terms, subtraction constants do not affect the coordinate space correlator. For large arguments, the correlation function decays exponentially, \( \Pi(\tau) \sim \exp(-m\tau) \), where the decay is governed by the lowest pole in the spectral function. This is the basis of hadronic spectroscopy on the lattice.

Correlation functions of currents built from quarks fields only (like the meson and baryon currents introduced above) can be expressed in terms of the full quark propagator. For an isovector meson current \( j_{I=1} = \bar{u}\Gamma d \) (where \( \Gamma \) is only a Dirac matrix), the correlator is given by the “one-loop” term

\[ \Pi_{I=1}(x) = \langle \text{Tr} \left[ S^{ab}(0, x) \Gamma S^{ba}(x, 0) \Gamma \right] \rangle. \quad (1.39) \]

The averaging is performed over all gauge configurations. It is pertinent to note that the quark propagator is not translation invariant before the vacuum average is performed, so the propagator depends on both arguments. Also, although (1.39) has the appearance of a one-loop (perturbative) graph, it includes arbitrarily complicated, multi-loop, gluon exchanges as well as non-perturbative effects. All of these effects are hidden in the vacuum average. Correlators of isosinglet meson currents \( j_{I=0} = \frac{1}{\sqrt{2}}(\bar{u}\Gamma u + \bar{d}\Gamma d) \) receive an additional two-loop, or disconnected, contribution

\[ \Pi_{I=0}(x) = \left\langle \text{Tr} \left[ S^{ab}(0, x) \Gamma S^{ba}(x, 0) \Gamma \right] \right\rangle - 2 \left\langle \text{Tr} \left[ S^{aa}(0, 0) \Gamma \right] \text{Tr} \left[ S^{bb}(x, x) \Gamma \right] \right\rangle. \quad (1.40) \]

In an analogous fashion, baryon correlators can be expressed as vacuum averages of three quark propagators.

At short distance, asymptotic freedom implies that the correlation functions are determined by free quark propagation. The free quark propagator is given by

\[ S_0(x) = \frac{i}{2\pi^2} \frac{\gamma \cdot x}{x^4}. \quad (1.41) \]
This means that mesonic and baryonic correlation functions at short distance behave as $\Pi_{\text{mes}} \sim 1/x^6$ and $\Pi_{\text{bar}} \sim 1/x^9$, respectively. Deviations from asymptotic freedom at intermediate distances can be studied using the operator product expansion (OPE). The basic idea [26, 27] is to expand the product of currents in (1.32) into a series of coefficient functions $c_n(x)$ multiplied by local operators $O_n(0)$

$$\Pi(x) = \sum_n c_n(x) \langle O_n(0) \rangle. \quad (1.42)$$

From dimensional considerations it is clear that the most singular contributions correspond to operators of the lowest possible dimension. Ordinary perturbative contributions are contained in the coefficient of the unit operator. The leading non-perturbative corrections are controlled by the quark and gluon condensates of dimension three and four.

Ultimately, the best source of information about hadronic correlation functions is the lattice, though at present most lattice calculations use complicated non-local sources. There are also other sources of phenomenological information about correlation functions [24]; ideally, the spectral function is determined from an experimentally measured cross section using the optical theorem. This is the case, for example, in the vector-isovector (rho meson) channel, where the necessary input is provided by the ratio

$$R(s) = \frac{\sigma(e^+e^- \to (I = 1 \text{ hadrons}))}{\sigma(e^+e^- \to \mu^+\mu^-)}, \quad (1.43)$$

where $s$ is the invariant mass of the lepton pair. Similarly, in the axial-vector ($a_1$ meson channel) the spectral function below the $\tau$ mass can be determined from the hadronic decay width of the $\tau$ lepton $\Gamma(\tau \to \nu_\tau + \text{ hadrons})$. In some cases, the coupling constants of a few resonances can be extracted indirectly, for example using low energy theorems. In this way, the approximate shape of the pseudo-scalar $\pi, K, \eta, \eta'$ and some glueball correlators can be determined. In general, given the fundamental nature of hadronic correlators, all models of hadronic structure or the QCD vacuum should be tested against the available information on the correlators.
1.4.4 Gluon Condensate

The gluon condensate density is another important characteristics of non-perturbative QCD. This parameter cannot be easily estimated from correlation functions with light quarks, because the latter are dominated by quark condensates. A very useful object, sensitive to the gluon condensate is the correlation function of $c$-quark currents:

$$\Pi_{\mu\nu}^c = i \int d^4x e^{iqx} \langle 0 | T\{ j^c_\mu(x) j^c_\nu(0) \} | 0 \rangle = (-g_{\mu\nu} q^2 + q_\mu q_\nu) \Pi^c(q^2),$$

(1.44)

where $j^c_\mu = \bar{c} \gamma_\mu c$ is the $c$-quark part of the quark e.m. current $j^e_\mu$. Following the same derivation as in section 1.4.3, we write down the dispersion relation for $\Pi^c(q^2)$ relating $\text{Im}\Pi^c(s)$ with the ratio $R_c(s)$ defined as:

$$R_c = \frac{\sigma(e^+e^- \rightarrow \text{charm})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)},$$

(1.45)

where the cross section $\sigma(e^+e^- \rightarrow \text{charm})$ includes hadronic states with $\bar{c}c$ content produced in $e^+e^-$: charmonium resonances ($J/\psi, \psi', ...$), pairs of charmed hadrons etc. The value of the gluon condensate density is usually given multiplied by $\alpha_s$ for convenience, as the product is scale-independent.

1.5 Deep Inelastic Scattering and Quark Distributions in Hadrons

Quarks provide a means of understanding much of the regularity in the hadron spectrum. However, they do not form part of that spectrum: quarks carry the colour quantum number and the spectrum consists only of colourless objects. Are they then a mathematical artifice useful only as a means of realising group theory? No, they are observed in inclusive, deep inelastic scattering: $e p \rightarrow e' + \text{"debris"}$, even though the debris never contains an isolated quark.

The cross section for deep inelastic scattering from a proton target is

$$\frac{d^2\sigma}{dQ^2 d\nu} = \frac{\alpha^2}{4E_e^2 \sin^4 \frac{\theta}{2}} \left( 2W_1(Q^2, \nu) \sin^2 \frac{\theta}{2} + W_2(Q^2, \nu) \cos^2 \frac{\theta}{2} \right),$$

(1.46)

where $\theta$ is the scattering angle, $Q^2$ is the spacelike squared momentum transfer, and $\nu = E_{e'} - E_e$ for $\vec{p}_p = 0$. It is characterised by two scalar functions: $W_1, W_2$, which contain a great deal of
information about proton structure. If the proton is a composite particle composed of pointlike constituents, then it follows that for \( Q^2 \to \infty, \nu \to \infty \) with \( \nu \gg Q^2 \)

\[
\begin{align*}
W_1(Q^2, \nu^2) & \to F_1(Q^2/\nu^2), \\
\nu W_2(Q^2, \nu^2) & \to F_2(Q^2/\nu^2);
\end{align*}
\tag{1.47}
\]

i.e., that in inclusive, deep inelastic processes \( W_{1/2} \) are functions only of \( x := Q^2/\nu^2 \) and not of \( Q^2 \) and \( \nu^2 \) independently.

Such experiments were first performed at SLAC in the late sixties and have been extended widely since then. Equation (1.47) is confirmed completely [7], it is correct as \( Q^2 \) varies over four orders-of-magnitude. The presence of pointlike constituents is the only possible explanation of this behaviour, and many other observations made in such experiments. Hence quarks exist but are “confined” in colourless bound states. The presence of “free” pointlike constituents in a highly relativistic hadron is the essence of the Parton Model (see e.g.[28, 29]).

The parton model is applicable, with varying degrees of accuracy, to any hadronic cross section involving a large momentum transfer. It is in essence a generalization of the impulse approximation. We assume that any physically observed hadron, of momentum \( p^\mu \), is made up of constituent particles, its “partons”, which can be identified with quarks and gluons, the QCD degrees of freedom. At high energy the masses of hadrons and partons are neglected compared to the scale \( Q \) of the hard scattering. Furthermore, it is assumed that every relevant parton entering the hard scattering from an initial-state hadron has momentum \( xp^\mu \), with \( 0 \leq x \leq 1 \); here \( p^\mu \) is the momentum of the parent hadron. Parton-model cross sections are calculated from tree graphs (no loops) for partonic scattering, by combining them with probability densities, the parton distribution functions (PDFs). The PDFs are the probability densities of finding a parton of a given type in a hadron, with a momentum fraction \( x \). In the naive parton model, these distributions depend only on \( x \), and are not explicitly dependent on the momentum transferred \( Q^2 \). This is known as scaling [30]. The PDFs are determined from structure functions measured in deep inelastic scattering.

Scaling is only approximate. The structure functions are known to depend on the momentum transferred \( Q^2 \), although rather weakly. Thus the PDFs also depend on \( Q^2 \), a condition referred to as scaling violation. This dependence is described by pQCD. Thus although the PDFs are not
calculable in pQCD, their evolution with $Q^2$ is perturbatively calculable. Perturbative QCD also
gives the general framework in which to calculate higher-order terms in the hard scattering; thus the
parton model can be regarded as the lowest-order (only tree graphs) term in a systematic expansion
under pQCD.

The general picture is thus as follows: to describe processes involving hadrons one separates the
long distance (nonperturbative) parts from the short distance (perturbative) parts, a procedure known
as factorization [25]. The perturbative part (hard scattering) is calculable order by order in pQCD.
The nonperturbative parts (for example the PDFs) are taken from experiments. This framework is
general enough not only for describing hadron-hadron collisions but also nucleus-nucleus collisions.

1.6 Outline of this work

Two studies with rather different themes are presented in this work. The first deals with phenomena
at low energy scales of the order of $\Lambda_{QCD}$. The structure of the QCD vacuum plays an important
role here, and the physics is determined to a large extent by the vacuum condensates, in particular the
quark and gluon condensates. Correlation functions, which are the expectation values of products
of local operators, are the natural tools of choice in elucidating the dynamics involved in this non-
perturbative regime.

Therefore in Chapter 2 we consider current-current correlation functions as a probe of non-
perturbative dynamics. The correlation functions (correlators) considered are the vector and axial-
vector correlators built from vector and axial-vector currents respectively. We investigate the differ-
ence, sum, and ratio of the difference and sum of these correlators. In the chiral limit and at large
momentum scales (large $q^2$) where perturbative techniques are applicable, these correlators have
basically the same structure. Thus perturbatively the difference of these correlators is zero. There-
fore the study of the difference of the vector and axial-vector correlators yield important information
since the difference is mainly sensitive to non-perturbative physics. The sum of these correlators, on
the other hand, has the important feature that it remains close to free-field behavior for distances as
large as 1fm [22]. Therefore the ratio of the difference and sum in coordinate space, gives an indi-
cation of the scale for the onset of dynamical chiral symmetry breaking and QCD non-perturbative dynamics. Results are compared, where possible, with available experimental data, and also with results from lattice QCD and instanton physics.

The second study has roots in the high energy (short-distance) regime where perturbative QCD should be applicable. In high energy collisions involving one or two hadrons in the initial state, for instance, deep inelastic electron-proton collision or the Drell-Yan lepton pair production, the cross sections are expressed in terms of structure functions. The structure functions, in turn, are defined by correlation functions of hadronic currents and encode the information about the structure of the hadrons involved in the collisions. Thus these structure functions are intrinsically non-perturbative in nature, and their determination from “first principles” require non-perturbative techniques. In light of this, structure functions for the most common hadrons are currently determined from deep inelastic hadron-lepton scattering and Drell-Yan lepton-pair production.

Thus Chapter 3 is devoted to the determination of the nonsinglet structure functions in pseudoscalar mesons, namely pions and kaons, using the non-perturbative DSE machinery. These nonsinglet structure functions yield information about the valence quark distributions in these mesons, and thus from the calculated structure functions, we can determine the valence quark distributions for the pseudoscalar mesons. An important aspect is the scale ($Q^2$) evolution of the structure functions and, by implication, the valence quark distributions; this is addressed in this Chapter also. Results at different scales are compared with available experimental data and fits to data.

The common theme underlying these two different studies is the use of inputs from the Dyson-Schwinger Equations (DSEs) to calculate the non-perturbative quantities addressed in these studies. Both studies utilize the 2- and 3- point functions from the DSEs, namely the dressed quark and gluon propagators, and the dressed vertex functions. Thus the DSEs furnish the framework needed to address the issues considered in both studies. Comparisons with experimental results, lattice calculations, and other relevant models of strong QCD help to highlight the strengths and weaknesses of the various approximations applied, in particular the efficacy of the Ladder-Rainbow truncation of the DSEs.

We summarize and conclude our investigations in Chapter 4.
Chapter 2

Current-Current Correlators and Non-perturbative Dynamics

Correlation functions or correlators, which are the vacuum expectation values of the product of gauge-invariant local operators, are useful tools in understanding non-perturbative aspects of Quantum Chromodynamics (QCD) and hadronics physics [31, 32]. They can be considered in terms of the fundamental QCD fields, quarks and gluons, or in terms of physical intermediate states, using the vast hadronic phenomenology of masses, coupling constants, etc [24].

Quark helicity and chirality in QCD are increasingly good quantum numbers at short distances or at momentum scales significantly larger than any mass scale [33]. One manifestation of this is that, for chiral quarks, the correlator of a pair of vector currents is identical to the corresponding correlator of a pair of axial vector currents to all finite orders of pQCD. In the non-perturbative circumstance, the difference of such correlators measures chirality flips, and the leading non-zero contribution in the ultraviolet identifies the leading non-perturbative phenomenon in QCD, the four quark condensate [34]. The popular ladder-rainbow truncation of the DSE for the scattering kernel leads to the representation of the difference of the vector and axial vector correlators as a vacuum polarization integral in momentum space, where the propagators are dressed and the vector and axial vector vertices are generated in a way consistent with the appropriate symmetries. We use the large spacelike momentum dependence to extract the leading coefficient (four-quark condensate); and a Fourier transform identifies the leading non-perturbative distance scale.
2.1 Dressed Quark Propagator and The Quark Condensate

As described in Sec. 1.3.2 the renormalised dressed quark propagator, which is obtained from the quark Dyson-Schwinger equation, has the general form

\[ S^{-1}(p) = i\gamma.pA(p^2) + B(p^2) \equiv (-i\gamma.p\sigma_v(p^2) + \sigma_s(p^2))^{-1}. \]  

(2.1)

Here \( A(p^2) \) and \( B(p^2) \) (or equivalently \( \sigma_v(p^2) \) and \( \sigma_s(p^2) \)) are scalar functions of the momentum \( p^2 \). They represent respectively the coefficient of the vector and scalar components of the propagator. This relation is depicted diagrammatically in Fig. 2.1 where the left-hand side represents the inverse of the fully dressed quark propagator. The first diagram on the right is the inverse of the bare (free)

propagator while the second denotes the quark self energy which satisfies the Dyson-Schwinger equation (the QCD “gap” equation) given in Eq. 1.11.

In the chiral (massless) limit of the non-perturbative domain, the dynamical mass,

\[ M_0(p^2) = \frac{B_0(p^2)}{A_0(p^2)}, \]  

(2.2)

is nonzero, leading to chirality flips. Here the subscript 0 denotes quantities in the chiral limit.

The Dyson-Schwinger equations, in general, have to be solved numerically at discrete values of their arguments, and these computations are nontrivial. For calculations requiring repeated use of the solutions as inputs, it is thus advantageous and economical to fit these numerical solutions to suitable analytic continuous functions. The dressed quark propagator can be efficiently represented analytically as a sum of \( N \) pairs of complex conjugate mass poles,

\[ S(p) = \sum_{i=1}^{N} \left\{ \frac{z_i}{i\gamma.p + m_i} + \frac{z_i^*}{i\gamma.p + m_i^*} \right\}, \]  

(2.3)

where \( m_i \) are complex-valued mass scales and \( z_i \) are complex coefficients. In this representation,
the scalar amplitude of the quark propagator, $\sigma_s(P^2)$, is given by
\[
\sigma_s(P^2) = \sum_{i=1}^{N} \left\{ \frac{z_i m_i}{P^2 + m_i^2} + \frac{z_i^* m_i^*}{P^2 + m_i^{*2}} \right\},
\] (2.4)
and the vector amplitude, $\sigma_v(P^2)$, by
\[
\sigma_v(P^2) = \sum_{i=1}^{N} \left\{ \frac{z_i}{P^2 + m_i^2} + \frac{z_i^*}{P^2 + m_i^{*2}} \right\},
\] (2.5)
with $*$ denoting complex conjugation.

At very large momenta, the full quark propagator is essentially just the bare (free) propagator. The function $A(P^2)$ and the dynamical mass $M(P^2)$ at large momenta are related to the parameters of the analytic representation as:
\[
A^{-1}(\infty) = \sum_{i=1}^{N} 2 \text{Real}(z_i),
\] (2.6)
\[
M(\infty) = \sum_{i=1}^{N} 2 \text{Real}(z_i m_i),
\] (2.7)
and
\[
\sigma_s(P^2)|_{P^2 \to \infty} \to \frac{C}{P^4},
\] (2.8)
where
\[
C = \sum_{i=1}^{N} (-2 \text{Real}(z_i m_i^3)).
\] (2.9)

In this study we fit the numerical solutions of the DSE for the quark propagator at the renormalization scale $\mu = 19 GeV$. The DSE solutions for the quark propagator in both the chiral limit and for finite quark masses is well represented by 3 pairs of complex conjugate mass poles, shown in Table. 2.1. The quality of the fits of both $\sigma_s(P^2)$ and $\sigma_v(P^2)$ to the numerical solutions of the DSE for the quark propagator is presented in Fig. 2.2.

The quark condensate can be related to the quark propagator in the chiral limit. It is formally defined, at a renormalization scale $\mu$, as [13, 35]
\[
[- < \bar{q}q >^0_\mu] = \lim_{\Lambda \to \infty} Z_4(\mu^2, \Lambda^2) \text{Tr} S(x = 0, m = 0)
\] (2.10)
\[
= \lim_{\Lambda \to \infty} Z_4(\mu^2, \Lambda^2) \frac{3}{4\pi^2} \int_0^{\Lambda^2} ds s \sigma^0_s(s).
\] (2.11)
Fig. 2.2: Fits to the solutions of the quark Dyson-Schwinger equation in the chiral limit using the analytic parametrization (three pairs of complex-conjugate mass poles) discussed in the text.
Table 2.1: Parameters of the analytic parametrization (3 pairs of complex-conjugate mass poles fit) of the numerical solutions of the quark Dyson-Schwinger equation for the chiral, u/d and s quarks.

<table>
<thead>
<tr>
<th>quark</th>
<th>n</th>
<th>$m_r$(GeV)</th>
<th>$m_i$(GeV)</th>
<th>$z_r$</th>
<th>$z_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>chiral</td>
<td>1</td>
<td>0.60258</td>
<td>0.30380</td>
<td>0.21957</td>
<td>0.48272</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-1.37664</td>
<td>0.63271</td>
<td>0.14513</td>
<td>0.01362</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.42373</td>
<td>0.6566</td>
<td>0.1353</td>
<td>-0.04587</td>
</tr>
<tr>
<td>u/d</td>
<td>1</td>
<td>0.5434</td>
<td>0.3239</td>
<td>0.1999</td>
<td>0.4579</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-1.2129</td>
<td>0.559</td>
<td>0.1402</td>
<td>0.03714</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.427</td>
<td>0.716</td>
<td>0.1599</td>
<td>-0.01215</td>
</tr>
<tr>
<td>s</td>
<td>1</td>
<td>0.7901</td>
<td>0.4881</td>
<td>0.1831</td>
<td>0.4631</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-1.4958</td>
<td>0.0063</td>
<td>0.1315</td>
<td>5.2021</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2.6360</td>
<td>-0.1058</td>
<td>0.1946</td>
<td>-1.5621</td>
</tr>
</tbody>
</table>

Here $Z_4(\mu^2, \Lambda^2)$ is the wavefunction renormalization at $\mu$ and cut-off scale $\Lambda$. $\sigma^0_s(s)$ denotes the scalar function of the quark propagator in the chiral limit. To one loop in perturbation theory, the wavefunction renormalization is given by

$$Z_4(\mu^2, \Lambda^2)|_{one\ loop} = \frac{\ln(\frac{\mu^2}{\Lambda_{QCD}^2})}{\ln(\frac{\Lambda^2}{\Lambda_{QCD}^2})}. \quad (2.12)$$

Therefore the quark condensate becomes

$$[-\langle \bar{q} q \rangle^0_\mu] = \frac{3}{4\pi^2} C \ln(\frac{\mu^2}{\Lambda_{QCD}^2}), \quad (2.13)$$

where $C$ is as defined in Eq.2.9.

2.2 Mesonic Correlation Functions

Mesonic correlation functions (correlators) are correlation functions built from currents having the quantum numbers of the particular meson under consideration [36]. We focus attention here on
the vector and axial-vector correlators which are derived from the vector and axial-vector currents describing these mesons respectively.

In the chiral limit of QCD and in a color singlet and flavor non-singlet channel, the difference of the vector and axial-vector correlators is zero to all orders in perturbation theory. As an example of the efficiency of this so-called V-A correlator in probing nonperturbative phenomena, its leading ultraviolet term is proportional to the scalar 4-quark condensate, $\langle \bar{q}q\bar{q}q \rangle$. This condensate is a key ingredient in QCD sum rule analyses of hadronic properties [14, 37] and, in the absence of independent information, it is often assumed that vacuum saturation (vacuum dominance) [14] holds:

$$\langle \bar{q}q\bar{q}q \rangle \approx (\bar{q}q)^2$$  \hspace{1cm} (2.14)

In the present study we carry out an independent estimate of this condensate based upon direct evaluation of the current-current correlators within a ladder-rainbow truncated model of the DSEs of QCD. This enables us thereby to test the validity of the vacuum saturation hypothesis.

### 2.2.1 Vector and Axial-Vector Correlation Functions

Let us consider the momentum space vector current-current correlator in the chiral limit (the momentum space corelator is the Fourier transform of the coordinate space correlator defined in Eq. 1.32):

$$\Pi_{\mu\nu}^V(P) = \int d^4x \ e^{iP \cdot x} <0|T j_\mu(x) j_\nu^+(0)|0>.$$ \hspace{1cm} (2.15)

Here $j_\mu$ is a vector current, defined generically as

$$j_\mu(x) = \bar{\psi}(x) \Gamma_\mu \psi(x),$$ \hspace{1cm} (2.16)

where $\Gamma_\mu^V$ is the vector vertex. Conservation of vector current ensures that the vector correlator is purely transverse:

$$\Pi_{\mu\nu}^V(P) = (P^2 \delta_{\mu\nu} - P_\mu P_\nu) \Pi_T^V(P^2).$$ \hspace{1cm} (2.17)

Thus

$$P^\mu \Pi_{\mu\nu}^V(P) = 0.$$ \hspace{1cm} (2.18)
This vector correlator in Euclidean metric (Appendix A) can be defined as a “quark loop” (vacuum polarization) as depicted in Fig. 2.3:

$$\Pi_{\mu\nu}^V(P) = - \int \frac{d^4q}{(2\pi)^4} \text{Tr}\{\gamma_\mu S(q_+)\Gamma_\nu^V(q, P) S(q_-)\}. \quad (2.19)$$

Here $S(q_\pm)$ are the dressed quark propagators, with $q_\pm = q \pm \frac{P}{2}$. As mentioned in Sec.2.1, we use

The 3- complex-conjugate-poles fit to the numerical solutions of the Dyson-Schwinger equation for $S(q)$ employing the Maris-Tandy ladder-rainbow kernel [38]

$\Gamma_\nu^V(q, P)$, the dressed of quark-antiquark vector vertex, can be decomposed into a longitudinal component and a transverse part consisting of eight transverse tensors. The longitudinal part satisfies the vector Ward-Takahashi identity (WTI) [39]:

$$iP_\nu \Gamma_\nu = S_{+1}^{-1} - S_{-1}^{-1}. \quad (2.20)$$

Thus the vector WTI imposes stringent conditions on the form of the longitudinal part of the vector vertex in terms of the propagator. We do not solve the DSE for the vector vertex in the present study; instead we use a popular vertex Ansatz satisfying Eq. 2.20, the vector Ball-Chiu vertex [19]:

$$\Gamma_{\nu}^{BC}(q, P) = \gamma_\nu \frac{A(q_+^2) + A(q_-^2)}{2} + (q_+ + q_-)_\nu \left\{ \gamma_{\nu} \frac{A(q_+^2) - A(q_-^2)}{2} \frac{q_+^2 - q_-^2}{q_+^2 - q_-^2} - i \frac{B(q_+^2) - B(q_-^2)}{q_+^2 - q_-^2} \right\}. \quad (2.21)$$

It should be noted that the vector Ward-Takahashi identity constrains only the longitudinal component of the vector vertex, leaving the transverse part essentially undetermined.
The integral in Eq. 2.19 needs to be regularized in order to be well defined. There are various regularization schemes, and the choice boils down to which symmetries are to be conserved by the regularization. Here we use the Pauli-Villars regularization \([40]\) to regularize the integral. This involves subtracting from the integrand a function which has the same asymptotic behaviour, in order for the resulting integrand to fall off fast enough with increasing \(q\). Introducing a set of auxiliary masses \(M_i\) and constants \(C_i\), the vector correlator is replaced as follows:

\[
\Pi^V_\mu\nu(P) = -\int \frac{d^4q}{(2\pi)^4} \left\{ Tr\{\gamma_\mu S(q+)\Gamma^V_\nu(q, P)S(q^-)\} - \sum_{i=1}^{2} C_i Tr\{\gamma_i \gamma_\mu q_+ + M_i \gamma_\nu q_- + M_i\} \right\}. \tag{2.22}
\]

The constants \(M_i, C_i\) are determined such that the regularized vector correlator is given by a convergent integral. The original correlator is recovered in the limit of infinitely heavy masses \(M_i\).

At large momentum \(q\) the dressed vertex and propagators become bare (free). We denote the free or bare propagator by the lower case letter \(s\). Therefore

\[
Tr\{\gamma_\mu s(q+)\gamma_\nu s(q_-)\} = \frac{4\delta_{\mu\nu}(q^2 - \frac{P^2}{4} + m^2) - 8q_\mu q_\nu + 2P_\mu P_\nu}{((q + \frac{P}{2})^2 + m^2)((q - \frac{P}{2})^2 + m^2)}
\]

\[
= \frac{q^2\{4\delta_{\mu\nu} - 8\hat{q}_\mu \hat{q}_\nu + \frac{1}{q^2}[\hat{P}^2 \delta_{\mu\nu} + 4m^2 \delta_{\mu\nu} + 2P_\mu P_\nu]\}}{q^4\{1 + \frac{1}{q^2}\frac{P^2}{2} + 2m^2 - (\hat{q}.P)^2 + \frac{1}{q^4}\frac{P^2}{16} + \frac{m^2P^2}{2} + m^4\}}
\]

\[
\equiv n_{\mu\nu} + \frac{1}{q^2}\frac{m_{\mu\nu}}{\hat{q}^2(1 + \frac{d}{4} + \frac{e}{q^4})}, \tag{2.23}
\]

where

\[
n_{\mu\nu} = 4\delta_{\mu\nu} - 8\hat{q}_\mu \hat{q}_\nu, \quad m_{\mu\nu} = (4m^2 - P^2)\delta_{\mu\nu} + 2P_\mu P_\nu,
\]

\[
d = \frac{P^2}{2} + 2m^2 - (\hat{q}.P)^2, \quad e = \frac{P^2}{16} + \frac{m^2P^2}{2} + m^4, \tag{2.24}
\]

and \(\hat{q}_\mu = q_\mu/|q|\) is a unit four-vector. Thus for very large momentum \(q\) we have that

\[
Tr\{\gamma_\mu s(q+)\gamma_\nu s(q_-)\}|_{q \to \infty} = \frac{1}{q^2}\left\{ n_{\mu\nu} + \frac{m_{\mu\nu}}{q^2}\right\}[1 - (\frac{d}{q^2} + \frac{e}{q^4}) + (\frac{d}{q^2} + \frac{e}{q^4})^2 + \cdots]
\]

\[
= \frac{n_{\mu\nu}}{q^2} + \frac{1}{q^2}\left[m_{\mu\nu} - dn_{\mu\nu}\right] + \frac{1}{q^4}\left[-n_{\mu\nu}(e - d^2) - dm_{\mu\nu}\right] + \cdots \tag{2.25}
\]
Including the Pauli-Villars subtraction terms gives

\[
\text{Tr}\{\gamma_\mu s(q_+)\gamma_\nu s(q_-)|_{q\to\infty} - \sum_i C_i \gamma_\mu s^{M_i}(q_+)\gamma_\nu s^{M_i}(q_-)|_{q\to\infty}\} = \frac{1}{q^2} n_{\mu\nu} [1 - \sum_i C_i] + \frac{1}{q^4} [-4\delta_{\mu\nu} + 16q_\mu q_\nu][m^2 - \sum_i C_i M_i] + O\left(\frac{1}{q^6}\right). \tag{2.26}
\]

Since we want the regularized integral to be convergent, the Pauli-Villars terms should cancel the large \(q\) part of the free correlator. This is achieved when Eq. 2.26 is zero, and hence the Pauli-Villars regularization conditions are

\[
C_1 + C_2 = 1,
\]

\[
C_1 M_1^2 + C_2 M_2^2 = m^2. \tag{2.27}
\]

To minimize the unspecified parameters \((M_1, M_2)\) we adopt the choice \([40]\)

\[
M_1^2 = m^2 + 2\Lambda^2, \quad M_2^2 = m^2 + \Lambda^2. \tag{2.28}
\]

where \(\Lambda\) is a cut-off parameter. In the chiral limit \(m = 0\), we therefore have

\[
M_1^2 = 2\Lambda^2, \quad M_2^2 = \Lambda^2. \tag{2.29}
\]

Therefore Eq. 2.27 and Eq. 2.29 yield

\[
C_1 = -1, \quad C_2 = 2. \tag{2.30}
\]

In complete analogy with the vector case, the momentum space axial-vector current-current correlator is defined as

\[
\Pi_{\mu\nu}^A(P) = \int d^4x \ e^{ip\cdot x} < 0|T_j^5(\mu)(x)j^5_{\nu}(0)|0 >. \tag{2.31}
\]

Here \(j^5_{\mu}\) is an axial-vector current, defined generically as

\[
j^5_{\mu}(x) = \bar{\psi}(x)\Gamma^5_{\mu}\psi(x), \tag{2.32}
\]

where \(\Gamma^5_{\mu}\) is the axial-vector vertex. This can be depicted diagrammatically as in Fig. 2.4: and
written as:

$$\Pi_{\mu\nu}^A(P) = -\int \frac{d^4q}{(2\pi)^4} Tr\{\gamma_5\gamma_\mu S(q+)\Gamma_\nu^5(q, P)S(q-}\}. \quad (2.33)$$

A remark about axial-vector correlators is in order. With exact chiral symmetry (all quark masses zero), the non-singlet axial-vector currents are conserved. But unlike the vector case, the axial-vector correlators are not purely transverse, because the massless Goldstone mode, the pion, is coupled to the axial-vector current and contributes to the longitudinal term:

$$\Pi_{\mu\nu}^A(P) = (P^2\delta_{\mu\nu} - P_\mu P_\nu)\Pi_T^A(P^2) + f_\pi^2\frac{P_\mu P_\nu}{P^2}. \quad (2.34)$$

This term does not spoil current conservation; simply convoluting indices cancels the pole. Also multiplying by $P_\mu$ leads to neither propagator nor singularity; in the $x$-space it is just a contact term, which can be ignored or removed. Thus at small $P$ the correlator is dominated by the Goldstone pole:

$$\Pi_{\mu\nu}^A(P) = (\delta_{\mu\nu} - \frac{P_\mu P_\nu}{P^2}) [f_\pi^2 + O(P^2)]. \quad (2.35)$$

In the chiral limit of zero current quark mass, the axial-vector Ward-Takahashi identity also constrains the form of the axial-vector vertex, and thus leads to the equivalent axial-vector Ball-Chiu Ansatz for the axial-vector vertex [19]:

$$\Gamma_{5\mu\nu}^{BC}(q, P) = \gamma_5\{\gamma_\nu A(q^2) + A(q^2) + 2q_\nu\gamma.q\frac{A(q^2) - A(q^2)}{q_+^2 - q_-^2} + i\frac{P_\nu}{P^2}[B(q^2) + B(q^2)]\}. \quad (2.36)$$
It should be noted that the axial-vector vertex contains the pion pole. As in the case of the vector correlator, the integral defining the axial-vector correlator, Eq. 2.33, needs to be regularized. Since both the vector and axial-vector correlators are identical in chiral limit of pQCD, therefore the Pauli-Villars regularization employed in the vector correlator is also used for the axial-vector correlator. The regularized correlator now becomes

\[
\Pi_{\mu\nu}^A(P) = -\int \frac{d^4q}{(2\pi)^4} \left\{ Tr \left\{ \gamma_5 \gamma_\mu S(q^+) \Gamma^5_{\nu}(q, P) S(q^-) \right\} \right. \\
\left. - \sum_{i=1}^2 C_i Tr \left\{ \frac{1}{i\gamma\cdot q^+ + M_i} \gamma_\nu \frac{1}{i\gamma\cdot q^- + M_i} \right\} \right\},
\]  

(2.37)

where \( M_i \) and \( C_i \) are as in Eq. 2.29 and Eq. 2.30.

2.2.2 The Difference of Vector and Axial-Vector Correlators (V-A Correlator)

We now consider the difference of the vector and axial-vector correlator, generally referred to as the V-A correlator. This combination is sensitive to chiral symmetry breaking, while perturbative diagrams, as well as gluonic operators cancel. Thus the correlator is nonzero only due to the effects of chiral symmetry breaking, as evident if the currents are expressed in terms left- and right-handed currents. The correlator is then manifestly nondiagonal in chirality, leading to chirality flips. Thus the V-A correlator is sensitive to non-perturbative dynamics since it is zero in the chiral limit to any finite order of perturbation theory. It is convenient to decompose the vector and axial vector correlators into transverse and longitudinal components, through the use of projection operators.

As stated earlier, conservation of vector current ensures that the vector correlator is purely transverse:

\[
\Pi_{\mu\nu}^V(P) = (P^2 \delta_{\mu\nu} - P_\mu P_\nu) \Pi_T^V(P^2).
\]  

(2.38)

The axial-vector correlator, on the other hand, has both transverse and longitudinal components:

\[
\Pi_{\mu\nu}^A(P) = (P^2 \delta_{\mu\nu} - P_\mu P_\nu) \Pi_T^A(P^2) + P_\mu P_\nu \Pi_L^A(P^2).
\]  

(2.39)

It should be noted that the pion pole in the axial-vector correlator occurs in the longitudinal part.
The scalar transverse coefficients of the correlators are extracted through the use of the transverse projector $\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}$. Therefore

$$P^2 \Pi^V_T(P^2) = \frac{1}{3} (\delta_{\mu\nu} - \frac{P_\mu P_\nu}{P^2}) \Pi^V_{\mu\nu}(P),$$  \hspace{1cm} (2.40)

and

$$P^2 \Pi^A_T(P^2) = \frac{1}{3} (\delta_{\mu\nu} - \frac{P_\mu P_\nu}{P^2}) \Pi^A_{\mu\nu}(P).$$  \hspace{1cm} (2.41)

Putting Eq.2.22 (Eq.2.37) into Eq.2.40 (Eq.2.41) respectively and taking the trace of the gamma matrices (Appendix A), we have

$$P^2 \Pi^V_T(P^2) = -\frac{4}{3} \int \frac{d^4q}{(2\pi)^4} \left\{ \left[ (q^2 + 2\frac{(q.P)^2}{P^2} - 3\frac{P^2}{4})\sigma_v(q^2_+)\sigma_v(q^2_-) + 3\sigma_s(q^2_+)\sigma_s(q^2_-) \right] F_1 \right\} +$$

$$2\frac{(q.P)^2}{P^2} - q^2 \right\} \left[ \left[ (q^2 + \frac{P^2}{4})\sigma_v(q^2_+)\sigma_v(q^2_-) - \sigma_s(q^2_+)\sigma_s(q^2_-) \right] F_2 +$$

$$\left[ \sigma_v(q^2_+)\sigma_s(q^2_-) + \sigma_v(q^2_-)\sigma_s(q^2_+) \right] F_3 \right\} - \sum_{i=1}^2 C_i \left[ \frac{q^2 - 3\frac{P^2}{4} + 2\frac{(q.P)^2}{P^2} + 3M_i}{(q^2_+ + M_i^2)(q^2_- + M_i^2)} \right],$$

(2.42)

and

$$P^2 \Pi^A_T(P^2) = -\frac{4}{3} \int \frac{d^4q}{(2\pi)^4} \left\{ \left[ (q^2 + 2\frac{(q.P)^2}{P^2} - 3\frac{P^2}{4})\sigma_v(q^2_+)\sigma_v(q^2_-) - 3\sigma_s(q^2_+)\sigma_s(q^2_-) \right] F_1 \right\} +$$

$$2\frac{(q.P)^2}{P^2} - q^2 \right\} \left[ \left[ (q^2 + \frac{P^2}{4})\sigma_v(q^2_+)\sigma_v(q^2_-) + \sigma_s(q^2_+)\sigma_s(q^2_-) \right] F_2 +$$

$$-\sum_{i=1}^2 C_i \left[ \frac{q^2 - 3\frac{P^2}{4} + 2\frac{(q.P)^2}{P^2} + 3M_i}{(q^2_+ + M_i^2)(q^2_- + M_i^2)} \right],$$

(2.43)

where

$$F_1 = A(q^2_+) + A(q^2_-),$$

$$F_2 = \frac{A(q^2_+) - A(q^2_-)}{q^2_+ - q^2_-},$$

$$F_3 = \frac{B(q^2_+) - B(q^2_-)}{q^2_+ - q^2_-}. \hspace{1cm} (2.44)$$

Here we shall concentrate on the difference of the transverse coefficients of the vector and axial vector correlators in the chiral limit:

$$P^2 \Pi^V_{T-A}(P^2) \equiv P^2 \Pi^V_T(P^2) - P^2 \Pi^A_T(P^2).$$  \hspace{1cm} (2.45)
This correlator does not acquire any perturbative contributions; it is thus sensitive entirely to the
effect of dynamical chiral symmetry breaking (DCSB). It is also independent of the regularized
masses. Thus using Eq. 2.42 and Eq. 2.43 the transverse component of the V-A correlator becomes
\[
P^2 \Pi_{T}^{V-A}(P^2) = -\frac{4}{3} \int \frac{d^4q}{(2\pi)^4} \left\{ [3 F_1 - 4 F_2 \left( \frac{(q.P)^2}{P^2} - q^2 \right)] \sigma_s(q_+^2)\sigma_s(q_-^2) + 2 F_3 \left( \frac{(q.P)^2}{P^2} - q^2 \right) \left[ \sigma_v(q_+^2)\sigma_s(q_-^2) + \sigma_v(q_-^2)\sigma_s(q_+^2) \right] \right\}.
\]

Notice that the integrand in Eq. 2.46 is function of \( P^2, q^2 \) and \( q.P \). Therefore using the expression
for the Euclidean space four-dimensional volume element (Appendix A), Eq. 2.46 takes the form
\[
P^2 \Pi_{T}^{V-A}(P^2) = -\frac{4(4\pi)^3}{3(2\pi)^4} \int_0^\infty q^3 dq \int_{-1}^1 \sqrt{1 - z^2} dz \left\{ [3 F_1 - 4 F_2 (q^2 z^2 - q^2)] \sigma_s(q_+^2)\sigma_s(q_-^2) + 2 F_3 (q^2 z^2 - q^2) \left[ \sigma_v(q_+^2)\sigma_s(q_-^2) + \sigma_v(q_-^2)\sigma_s(q_+^2) \right] \right\}.
\]

The numerical evaluation of Eq. 2.47 is presented in Fig. 2.5.

2.2.3 Results from V-A Correlator: Sum Rules and Extraction of The 4-quark Condensate

It is appropriate at this stage to test the efficacy of our calculated V-A correlator. Direct and integral
tests can be done by using sum rules relations (in the chiral limit) to compare to experimental results.
Another important result from our V-A calculation is the extraction of the four-quark condensate
and comparison with the squared of the quark condensate, as assumed by the vacuum saturation
(vacuum dominance) hypothesis.

Let us now compare our calculations with results from QCD sum rules in the chiral limit: The
first Weinberg sum rule (WSRI) \([41, 42]\) establishes a relation between the V-A correlator at very
low momenta \((P^2 \to 0)\) and the pion decay constant:
\[
[P^2 \Pi_{T}^{V-A}(P^2)]_{P^2 \to 0} = -f_\pi^2.
\]

We evaluate our correlator at \( P^2 = 0 \), with the result
\[
[P^2 \Pi_{T}^{V-A}(P^2)]_{P^2 = 0} = -0.0052845 GeV^2 = -(72.8 MeV)^2,
\]
Fig. 2.5: The difference of the transverse coefficients of the vector and axial-vector correlators, $P^2 \Pi_{T}^{V-A}(P^2)$ and $P^6 \Pi_{T}^{V-A}(P^2)$. The four-quark condensate, the leading non-perturbative phenomenon at large momenta, is given by the large-momentum component of $P^6 \Pi_{T}^{V-A}(P^2)$. 
which is \( \sim 1.27 \) times smaller than the experimental value of the pion decay constant \( f_\pi = 92.4 \pm 0.3 \, MeV \) [43] obtained from the decays \( \pi^- \to \mu^- \nu_\mu \) and \( \pi^- \to \mu^- \bar{\nu}_\mu \gamma \). This is due to the fact that the numerically-evaluated vector correlator is not exactly zero at \( P^2 = 0 \). Actually, the pion decay constant is given by the transverse part of the axial vector correlator at small \( P^2 \), i.e. \( P^2 \approx 0 \). The vector correlator at \( P^2 = 0 \) should be zero or negligibly small, so that the V-A correlator at \( P^2 = 0 \) gives the negative of the squared of the pion decay constant. Evaluating our axial vector correlator at \( P^2 = 0 \) gives

\[
[P^2 \Pi_T^A(P^2)]_{P^2=0} = 0.00809515 \, GeV^2 = (90 \, MeV)^2,
\]

which implies that \( f_\pi = 90 \, MeV \), in quite reasonable agreement with the experimental value. The lower value given by the V-A correlator is an indication of the severity of the contamination of the non-zero value of the vector correlator at \( P^2 = 0 \).

The second Weinberg sum rule (WSRII) [41, 42] requires that:

\[
P^2 [P^2 \Pi_T^{V-A}(P^2)]|_{P^2 \to \infty} = 0.
\]  

(2.51)

The result of our calculation is in perfect agreement with this sum rule.

The Das-Guranik-Mathur-Low-Yuong (DGMLY) sum rule relates the integral of the V-A correlator to the electromagnetic component of the pion mass difference [42, 44]:

\[
\int_0^\infty dP^2 [P^2 \Pi_T^{V-A}(P^2)] = -\frac{4\pi f_\pi^2}{3\alpha} [m_{\pi^\pm} - m_{\pi^0}].
\]  

(2.52)

Here \( \alpha \approx 1/137 \) is the fine structure constant.

Our calculation yields

\[
\int_0^\infty dP^2 [P^2 \Pi_T^{V-A}(P^2)] = -0.00676886 \, GeV^4.
\]  

(2.53)

From Eq. 2.53 and Eq. 2.52 and using the experimental value of the pion decay constant \( f_\pi = 0.00854 \, GeV^2 \) we estimate the electromagnetic pion mass difference to be

\[
m_{\pi^\pm} - m_{\pi^0} = 4.86 \, MeV;
\]  

(2.54)

which agrees nicely with the experimental value (after subtracting the \((m_d - m_u)\) effect) [45]

\[
m_{\pi^\pm} - m_{\pi^0} = 4.43 \pm 0.03 \, MeV.
\]  

(2.55)
The four-quark condensate is an important element in the QCD sum rules analyses of various hadronic properties. It is the vacuum expectation value of a local composite operator, the product of four quark field operators, and denoted by $\langle \bar{q}q\bar{q}q \rangle$. Shifman et al [14] proposed the factorization hypothesis of vacuum saturation (vacuum dominance) in order to evaluate the vacuum expectation values (VEVs) of complicated composite operators. Vacuum saturation assumes that if one can split an operator into two parts, so that the vacuum expectation value of each exists, it should be the dominant one. Thus the four-quark condensate is assumed to be of the form:

$$\langle \bar{q}q\bar{q}q \rangle \approx \langle \bar{q}q \rangle^2.$$  \hspace{1cm} (2.56)

This hypothesis is not always accurate, especially in the case of gluonic operators. Here we evaluate the four-quark condensate from our V-A correlator and then compare with the result assuming vacuum saturation.

Fig. 2.5 shows that our numerical calculation of $P^6 \Pi_T^{V-A}(P^2)$ identifies a leading ultraviolet ($P^2 > 10^5$ GeV$^2$) constant reasonably well, viz

$$P^6 \Pi_T^{V-A}(P^2)\big|_{P^2 \to \infty} \to \text{const},$$  \hspace{1cm} (2.57)

where

$$\text{const} \equiv \langle O_6 \rangle = 0.000261.$$  \hspace{1cm} (2.58)

Non-perturbative contributions to the two-point correlation function, starting with dimension $d = 6$ and involving the four-quark condensate, is given by [46, 47]

$$\Pi_T^{V-A}(P^2) = -\frac{32\pi}{9} \frac{\alpha_s}{P^6} \langle \bar{q}q\bar{q}q \rangle \left(1 + \frac{\alpha_s(P^2)}{4\pi} \left[\frac{247}{12} + \ln\left(\frac{\mu^2}{P^2}\right)\right]\right) + O\left(\frac{1}{P^8}\right).$$  \hspace{1cm} (2.59)

We use Eq. 2.59 and $\alpha_s(100 \text{ GeV}) = 0.12$ to extract the four-quark condensate:

$$\langle \bar{q}q\bar{q}q \rangle = \frac{9 \times \langle O_6 \rangle}{32 \pi \alpha_s} \left(1 + \frac{\alpha_s(P^2)}{4\pi} \left[\frac{247}{12} + \ln\left(\frac{\mu^2}{P^2}\right)\right]\right) = (0.235 \text{ GeV})^6.$$  \hspace{1cm} (2.60)

In general, we can write

$$\langle \bar{q}q\bar{q}q \rangle = \kappa \langle \bar{q}q \rangle^2,$$  \hspace{1cm} (2.61)
where the parameter $\kappa$ is a quantitative measure of the deviation from the vacuum saturation value ($\kappa = 1$).

The ratio of the four-quark condensate to the square of the two-quark condensate (from Eq. 2.13) at the renormalization scale $\mu = 19$ GeV is

$$R(\mu) = \kappa = \frac{\langle \bar{q}q\bar{q}q \rangle_{\mu}}{\langle \bar{q}q \rangle^2} = 1.65.$$  \hspace{1cm} (2.62)

This ratio is significantly greater than unity, with the attendant implication that the four-quark condensate is significantly greater than the value assumed by vacuum saturation, by a factor of 1.65.

This value of $\kappa$ can be compared to other values of $\kappa$ determined from the QCD sum rule approaches: $\kappa = 1$ \cite{48}, $\kappa = 2.36$ \cite{49}, and $\kappa = 6$ \cite{50}.

As a final test, we present, in Fig. 2.6, the calculation of $\frac{P^2 \Pi_{\perp}^{V-A}(P^2)}{f_\pi^2}$, where $f_\pi^2$ is the value from Eq. 2.48. It agrees nicely with the result from a nonlocal chiral quark model presented in Fig.12 in \cite{42}.

### 2.2.4 The Sum of Vector and Axial-Vector Correlators (V+A Correlator)

We now consider the V+A correlation function, that is, the sum of the vector and axial-vector correlators. The unique feature of this function is the fact that the correlator remains close to free-field behavior for distances as large as 1fm.

To preserve the masslessness of the photon, a subtraction has to be made to the new transverse coefficient of the vector correlator, called $P^2 \tilde{\Pi}_T^V(P^2)$ to satisfy $[P^2 \tilde{\Pi}_T^V(P^2)]_{p^2=0} = 0$ as follows:

$$P^2 \tilde{\Pi}_T^V(P^2) = P^2 \Pi_T^V(P^2) - [P^2 \Pi_T^V(P^2)]_{p^2=0}. \hspace{1cm} (2.63)$$

The transverse coefficient of the axial vector correlator at $P^2 = 0$ is the pion decay constant square,

$$P^2 \Pi_T^A(P^2)|_{p^2=0} = f_\pi^2. \hspace{1cm} (2.64)$$

The sum of the transverse coefficient of vector and axial vector correlation function is thus

$$P^2 \Pi_T^{V+A}(P^2) \equiv P^2 \tilde{\Pi}_T^V(P^2) + P^2 \Pi_T^A(P^2), \hspace{1cm} (2.65)$$
Fig. 2.6: The ratio of the difference of the transverse coefficients of the vector and axial vector correlators (V-A correlator) to the pion decay constant, $f_\pi^2$. Note that $f_\pi^2 \equiv \frac{[P^2 \Pi_T^{V-A}(P^2)]_{P^2=0}}{f_\pi^2}$. 
and
\[ P^2 \Pi_T^{V+A}(P^2)|_{P^2=0} = f^2_\pi. \] (2.66)

The numerical calculation of the sum of the correlators is presented in Fig. 2.7.

### 2.2.5 Ratios of Sum and Difference of Correlators

To examine the momentum or distance scale for the onset of the leading non-perturbative phenomena in QCD, we investigate the ratio of the difference (V-A) and sum (V+A) of vector and axial vector correlators.

The ratio of the difference and sum of the transverse coefficients of the correlators in momentum space is defined as
\[ R(P^2) = \frac{\Pi_T^{V-A}(P^2)}{\Pi_T^{V+A}(P^2)}. \] (2.67)

The numerical evaluation of the momentum space ratio (Eq. 2.67) is shown in Fig. 2.8.

It is also of great interest to calculate the ratio in coordinate space, especially for comparison with lattice results and results from other models, for example, the instanton models. To accomplish this we Fourier transform the momentum space V-A and V+A correlators respectively to yield the corresponding x space V-A and V+A correlators. The ratio in the coordinate space (or x space) is then given by
\[ R(x) = \frac{\int d^4 P e^{-iP.x} P^2\Pi_T^{V-A}(P^2)}{\int d^4 P e^{-iP.x} P^2\Pi_T^{V+A}(P^2)}. \] (2.68)

In general, the Fourier transform of any arbitrary correlator, \( \Pi(P^2) \), can be written as
\[ \Pi(x) = \int \frac{d^4 P}{(2\pi)^4} e^{-iP.x} \Pi(P^2) = \frac{2\pi^2}{(2\pi)^4} \int dP^2 P^2 \frac{J_1(\sqrt{P^2}x)}{\sqrt{P^2}x} \Pi(P^2), \] (2.69)

where \( J_1 \) is the Bessel function of the first kind.

Since the correlators are very strongly x-dependent, it is convenient to plot them normalized to free ones. Such ratios yield direct information about interquark interaction. In addition, all nonessential normalization factors drop out. Thus the difference of the correlators in coordinate space, normalized to free behavior (see Appendix A), is presented in Fig. 2.9.
Fig. 2.7: The sum of the vector and axial-vector correlators, $P^2 \Pi^{V+A}_T(P^2)$. 
Fig. 2.8: The ratio of the transverse component of the V-A correlator, $\Pi_{T}^{V-A}(P^2)$, to the transverse component of the V+A correlator, $\Pi_{T}^{V+A}(P^2)$ in momentum space. This ratio gives the onset of non-perturbative dynamics in momentum space.
Fig. 2.9: The transverse V-A correlator, $\Pi^{V-A}_T(x)$, normalized to the free (perturbative) correlator.
Due to the slowly convergent integral of the Fourier transform of the V+A correlator, we fit the momentum space V+A correlator by a sum of Gaussians. This circumvents the convergence problem of the transform since the Fourier transform of a Gaussian is also a Gaussian:

$$\int \frac{d^4P}{(2\pi)^4} e^{-iP.x} e^{-a^2P^2} = \frac{1}{16\pi^2a^4} e^{-\frac{x^2}{a^2}}. \quad (2.70)$$

The ratio in $x$-space (Eq. 2.68) is shown in Fig. 2.10.

2.3 Result from Ratios: Correlators as a Probe of Non-perturbative Dynamics

Fig. 2.8 presents the ratio of the difference and sum of the vector and axial vector correlation functions in momentum space (Eq. 2.67). As $P^2$ increases the V-A correlator tends to zero while the V+A correlator remains finite. Thus the ratio decreases to zero. On the other hand, the transition from the perturbative QCD (pQCD) regime (large $P^2$) to the non-perturbative regime (low $P^2$) causes the ratio of the correlators to increase dramatically. This increase manifestly demonstrates the influence of dynamical chiral symmetry breaking, DCSB. The scale for the onset of DCSB, from Fig. 2.8, is $P \sim 0.3 \, GeV$ corresponding to $x \sim 0.6 \, fm$.

We now turn to the coordinate space. Fig. 2.10 presents the ratio in the $x$ (coordinate) space, $R(x)$, as given by Eq. 2.68. As $x$ gradually increases from zero, $R(x)$ increases slowly until around $x \sim 0.6 \, fm$ where the increase is manifestly very rapid. This is exactly the behavior observed in the momentum space ratio, with the same scale for the onset of DCSB. This is to be expected, since the V-A correlator vanishes in the perturbative region which corresponds to very small $x$. In the non-perturbative region ($x > 0$), the V-A correlator is finite and thus probes the underlying non-perturbative dynamics. As previously discussed, in the chiral limit, vector and axial-vector current-current correlators are identical to any finite order of perturbation theory in QCD. Thus quark chirality is preserved. In the non-perturbative regime, dynamical chiral symmetry breaking generates a mass scale, leading to chirality flips (chirality is discussed in Sec.1.4.2).

We now compare our result to other existing studies. The ALEPH $\tau$ lepton decay data [51] has been analysed in [31], where analytical expressions are given for both V-A and V+A correlators normalized to their free forms. We use these analytical expressions to evaluate $R(x)$, and the com-
Fig. 2.10: The chirality-flip ratio $R(x)$. Finiteness (non-zero value) of the ratio manifests dynamical chiral symmetry breaking (DCSB), a characteristic signature of non-perturbative dynamics. The scale for the onset of DCSB is $\sim 0.6$ fm ($\sim 3$ GeV$^{-1}$).
parison of our result to the evaluated $R(x)$ is shown in Fig. 2.11. It can be seen that the shapes have the same form, although our result shows a steeper rise with $x$ at a somewhat smaller value of $x$ than that of the evaluated $R(x)$. In [52] the result of a quenched lattice evaluation of $R(x)$ is analysed using the Instanton Liquid Model (ILM) [53, 54, 55]. Our result compares favorably with the ILM analysis, as shown in Fig. 2.12. Thus we observe that DCSB and instantons generate the same scale for the onset of non-perturbative QCD dynamics.
Fig. 2.11: Comparison of our result for $R(x)$ with the fit to the ALEPH $\tau$ lepton decay data.
Fig. 2.12: Comparison of our result for $R(x)$ with the Lattice, One Instanton, Random Instanton Liquid Model (RILM) [52], and the Constituent Quark Model (CQM).
CHAPTER 3

VALENCE QUARK DISTRIBUTIONS IN PSEUDOSCALAR MESONS

3.1 Deep Inelastic Scattering

Deep inelastic scattering (DIS) is the archetype for hard processes in QCD: a high energy lepton (electron, muon, or neutrino) scatters off a target hadron (proton or a nucleus), transferring large quantities of both energy and invariant squared-four-momentum. For charged leptons the dominant reaction mechanism is electromagnetism and one photon exchange is a good approximation. For neutrinos either $W^\pm$ (charged current) or $Z^0$ (neutral current) exchange may occur. The weak interactions of electrons may also be studied either by means of small parity violating asymmetries originating in $\gamma-Z^0$ interference, or by means of the charged current reaction $e^- \to \nu_e$. This section is a general review of DIS in Minkowski space. An excellent review of the subject is contained in [56, 57] and additional details can be found in the references therein. The major part of the material in this section follows essentially the treatment in these two references.

3.1.1 Cross Section and Structure Functions

For simplicity consider charged lepton scattering by one photon exchange, with the kinematics as shown in fig. (3.1). The initial lepton with momentum $k$ and energy $E$ exchanges a photon of momentum $q$ with a the target with momentum $P$. Only the outgoing electron with momentum $k'$ and energy $E'$ is detected. Two invariants can be defined

$$q^2 \equiv (k-k')^2 = q_0^2 - q^2 = -4EE' \sin^2(\theta/2) = -Q^2 < 0 \quad (3.1)$$

$$M\nu \equiv P \cdot q = (E - E') \quad (3.2)$$

where the lepton mass has been neglected. The meaning of the scattering angle $\theta$ is clear from fig. (3.1). The quantities $E, E', \theta$ and $q^0 \equiv E - E'$ refer to the target rest frame. The deep inelastic,
Fig. 3.1: Inclusive deep inelastic lepton-nucleon scattering. The virtual photon momentum is $q$. The final hadronic state is not measured, and is denoted by $X$.

or Bjorken limit is where $Q^2$ and $\nu$ both go to infinity with the ratio, $x \equiv Q^2/2M\nu$ fixed. The variable $x$ is known as the Bjorken (scaling) variable.

Since the invariant mass of the hadronic final state is larger than or equal to the mass of the target, $(P+q)^2 \geq M^2$, one has $0 < x \leq 1$. It is convenient also to measure the energy loss using a dimensionless variable,

$$0 \leq y \equiv \frac{\nu}{E} \leq 1.$$

The differential cross-section for inclusive scattering ($eP \to e'X$) is given by:

$$d\sigma = \frac{1}{J} \frac{d^3k'}{2E'(2\pi)^3} \sum_X \frac{n_X}{4\pi} \int \frac{d^3p_i}{(2\pi)^32p_i0} |A|^2 (2\pi)^4 \delta^4(P + q - \sum_i p_i).$$

The flux factor for the incoming nucleon and electron is denoted by $J = 4P \cdot k$, which is equal to $J = 4ME$ in the rest frame of the nucleon. The sum runs over all hadronic final states $X$ which are not observed. Each hadronic final state consists of $n_X$ particles with momenta $p_i$ ($\sum_{i=1}^{n_X} p_i \equiv p_X$).

The squared amplitude $|A|^2$ can be separated into a leptonic ($l^{\mu\nu}$) and a hadronic ($W^{\mu\nu}$) tensor (see fig. 3.2):

$$\frac{|A|^2}{4\pi} = \frac{\alpha^2}{Q^4} l^{\mu\nu} W_{\mu\nu},$$

where $\alpha \sim 1/137$ is the electromagnetic fine structure constant. The leptonic tensor $l^{\mu\nu}$ is given...
Fig. 3.2: The squared amplitude $|A|^2$ for electron-hadron scattering is separable into two parts: a leptonic tensor $l^{\mu\nu}$ and a hadronic tensor $W^{\mu\nu}$.

by the square of the elementary spin $1/2$ current (summed over final spins):

$$l^{\mu\nu} = \sum_{s'} \bar{u}(k, s)\gamma^\mu u(k', s')\bar{u}(k', s')\gamma^\nu u(k, s)$$

$$= 2(k'^\mu k^\nu + k'^\nu k^\mu) - 2g^{\mu\nu}k \cdot k' + 2ie^{\mu\nu\lambda\sigma}q_\lambda s_\sigma,$$

and consists of parts symmetric and antisymmetric in $\mu$ and $\nu$. The antisymmetric part is linear in the spin vector $s$, which is normalized to $s^2 = -m^2$. While the leptonic tensor is known completely, $W^{\mu\nu}$, which describes the internal structure of the nucleon, depends on non-perturbative strong interaction dynamics. It is expressed in terms of the current $J^\mu$ as:

$$4\pi W^{\mu\nu} = \sum_X \langle PS|J^\mu|X\rangle\langle X|J^\nu|PS\rangle(2\pi)^4\delta(P + q - p_X)$$

$$= \int d^4\xi e^{iq \cdot \xi}\langle PS|[J^\mu(\xi), J^\nu(0)]|PS\rangle.$$

The states are covariantly normalized to:

$$\langle P|P'\rangle = 2E(2\pi)^3\delta^3(P - P').$$

Using Lorentz covariance, gauge invariance, parity conservation in electromagnetism and standard discrete symmetries of the strong interactions, $W^{\mu\nu}$ can be parametrized in terms of four
scalar dimensionless structure functions $F_1(x, Q^2)$, $F_2(x, Q^2)$, $g_1(x, Q^2)$ and $g_2(x, Q^2)$. They depend only on the two invariants $Q^2$ and $\nu$, or alternatively on $Q^2$ and the dimensionless Bjorken variable $x$. Splitting $W^{\mu \nu}$ into symmetric and anti-symmetric parts [56] we have,

$$W^{\mu \nu} = W^{(\mu \nu)} + W^{[\mu \nu]},$$

with

$$W^{(\mu \nu)} = \left(-g^{\mu \nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1 + \left[\left(P^\mu - \frac{P \cdot q}{q^2} q^\mu\right) \left(P^\nu - \frac{P \cdot q}{q^2} q^\nu\right)\right] \frac{F_2}{P \cdot q},$$

$$W^{[\mu \nu]} = -ie^{\mu \nu \lambda \sigma} q_\lambda \left(\frac{S_\sigma}{P \cdot q} (g_1 + g_2) - \frac{q \cdot S P_\sigma}{(P \cdot q)^2} g_2\right),$$

where $S^\sigma$ is the polarization vector of the nucleon ($S^2 = -M^2$), $P \cdot S = 0$. $S^\sigma$ is a pseudovector. Since $W^{[\mu \nu]}$ is a normal tensor, parity demands that the $S^\mu$ appear with another pseudotensor, and the only one available is the $e^{\mu \nu \sigma \lambda}$. Lorentz invariance demands that $W^{\mu \nu}$, defined in eq. (3.7) be linear in the initial and final nucleon spinors, $U(P, S)$ and $\bar{U}(P, S)$. Tensors constructed from these are either spin independent ($\bar{U}(P, S) \gamma^\mu U(P, S) = 2P^\mu$) or linear in $S^\mu$ ($(\bar{U}(P, S) \gamma^\mu \gamma_5 U(P, S) = 2S^\mu)$).

### 3.1.2 Bjorken Scaling

The variable $x$ was first introduced by Bjorken, and is crucial to understanding deep inelastic scattering. This is because QCD predicts that structure functions are functions of $x$ and do not explicitly depend on $Q^2$ to leading order, a property known as scaling.

In the Bjorken limit, i.e. at large momentum and energy transfers,

$$Q^2 = -q^2 \rightarrow \infty, \quad P \cdot q \rightarrow \infty,$$

but fixed ratio $Q^2 / P \cdot q$, the unpolarized structure functions

$$F_1(x, Q^2) \xrightarrow{Q^2 \rightarrow \infty} F_1(x),$$

$$F_2(x, Q^2) \xrightarrow{Q^2 \rightarrow \infty} F_2(x).$$
are observed to depend to a good approximation only on the dimensionless Bjorken scaling variable $x$. Variations of the structure functions with $Q^2$ at fixed $x$ turn out to be small. They are discussed further in Sec. 3.3.

A similar scaling behavior is expected for the spin-dependent structure functions $g_1$ and $g_2$ which likewise reduce to functions of $x$ only when the limit $Q^2 \to \infty$ is taken.

### 3.1.3 Virtual Compton Scattering

As stated earlier, the hadronic tensor, $W^\mu\nu$, can be expressed as the Fourier transform of a correlation function of currents;

$$4\pi W^\mu\nu = \int d^4\xi e^{iq\cdot\xi} \langle PS|[(J^\mu(\xi), J^\nu(0))]|PS\rangle_c.$$  (3.15)

Here the subscript $c$ denotes connected diagrams. It is related to the imaginary part of the forward virtual Compton scattering amplitude, $T$:

$$T^\mu\nu = i \int d^4\xi e^{iq\cdot\xi} \langle PS|T(J^\mu(\xi)J^\nu(0))|PS\rangle,$$  (3.16)

via the optical theorem:

$$2\pi W^\mu\nu = \text{Im} ~ T^\mu\nu.$$  (3.17)
This relationship is depicted graphically in fig. (3.3). Thus it is sufficient to study the physics of the forward virtual Compton scattering in order to elucidate the important features of the hadronic tensor part of the DIS cross section.

3.1.4 Light-Cone Dominance

The four-momenta $P^\mu$ and $q^\mu$ can be used to define a frame and a spatial direction [56]. Without loss of generality one can choose a frame such that $P^\mu$ and $q^\mu$ have components only in the time and $\hat{e}_3$ directions. It is useful to introduce the light-like vectors

$$p^\mu = \frac{2\omega}{M} (1, 0, 0, 1),$$

$$n^\mu = \frac{M}{2\omega} (1, 0, 0, -1)$$

with $n^2 = p^2 = 0$ and $n \cdot p = 2$. Up to the scale factor $\omega$, the vectors $p^\mu$ and $n^\mu$ function as unit vectors along opposite tangents to the light-cone. They may be used to expand $P^\mu$ and $q^\mu$,

$$q^\mu = \frac{1}{2} \left( \nu - \sqrt{\nu^2 + Q^2} \right) p^\mu + \frac{1}{2} \left( \nu + \sqrt{\nu^2 + Q^2} \right) n^\mu,$$

$$P^\mu = \frac{M}{2} (p^\mu + n^\mu)$$

In the Bjorken limit $q^\mu$ simplifies to

$$\lim_{\mathcal{B}} q^\mu \sim \left( \nu + \frac{M}{2} x \right) n^\mu - \frac{M}{2} x p^\mu + O \left( \frac{1}{\nu} \right).$$

$\omega$ selects a specific frame. For example $\omega = M/2$ yields the target rest frame, while $\omega \to \infty$ selects the infinite momentum frame. The decomposition along $p^\mu$ and $n^\mu$ is equivalent to the use of light-cone coordinates (Appendix A). The transformation to light-cone components can be written as an expansion in the basis vectors $p^\mu$ and $n^\mu$,

$$a^\mu = \left( \frac{a^-}{2\omega} \right) p^\mu + \left( \frac{a^+ \omega}{2} \right) n^\mu + a^\perp \mu.$$  

The light-cone dominant diagram in the Bjorken limit, the “handbag” diagram, is depicted in Fig. (3.4).

The general form $W^{\mu\nu}$:
Fig. 3.4: Leading diagram in deep inelastic scattering (DIS). The quark propagator between the two currents carries the large momentum $q^\mu$ and leads to a $1/\zeta^3$ behavior at small distances.

$$W^{(\mu\nu)} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1 + \left[(P^\mu - \frac{P.q}{q^2} q^\mu) \left(P^\nu - \frac{P.q}{q^2} q^\nu\right)\right] \frac{F_2}{P.q}, \quad (3.23)$$

which is symmetric under the interchange $\mu \leftrightarrow \nu$ and satisfies current conservation

$$P.\nu W^{(\mu\nu)} P.\nu = 0, \quad (3.24)$$

is expanded as powers in $\nu = \frac{P.q}{M}$

$$W^{(\mu\nu)}(\nu) = \nu \left(\frac{n^\mu n^\nu}{2Mx}\right) \left[\frac{F_2}{2x} - F_1\right] +
\left[\frac{F_2}{2x} - F_1\right] \frac{1}{4} \left[2n^\mu n^\nu - n^\mu p^\nu - p^\mu n^\nu\right] + g^{\mu\nu} F_1 + \frac{F_2}{2x} \left(\frac{n^\mu P^\nu}{M} + \frac{P^\mu n^\nu}{M}\right) +
+ O\left(\frac{1}{\nu}\right). \quad (3.25)$$

In the Bjorken limit ($\nu \to \infty$), the condition that $W^{(\mu\nu)}$ is finite, it is required that

$$F_2(x) = 2x F_1(x), \quad (3.26)$$

which is the Callan-Gross relation.

So $W^{(\mu\nu)}$ simplifies to general form in Bjorken limit

$$\lim_{Bj} W^{(\mu\nu)}(\nu) = \left[-g^{\mu\nu} + \left(\frac{n^\mu P^\nu}{M} + \frac{P^\mu n^\nu}{M}\right)\right] F_1 + O\left(\frac{1}{\nu}\right). \quad (3.27)$$
3.2 Field Theory Derivation of Quark Distributions

We now give a field theoretical derivation of structure functions in DIS \[57\]. Here we will work in
the target frame and choose the negative z-axis to lie along the virtual photon four-momentum \(q\):

\[ q = (\nu, 0, 0, -\sqrt{\nu^2 + Q^2}). \]  

(3.28)

In the Bjorken limit, as \(Q^2 \to \infty\) with \(x\) fixed, \(Q^2/\nu^2 \to 0\) so

\[ q = (\nu, 0, 0, -\nu - M_T x_T), \]  

(3.29)

where \(M_T\) and \(x_T\) are the mass and Bjorken \(x\) of the target respectively. Introducing the “light-
cone” coordinates (Appendix A)

\[ q^\pm \equiv (q_0 \pm q_3), \]  

(3.30)

in the Bjorken limit \(q^- \to \infty\) but \(q^+ \to -M_T x_T\), i.e., \(q^+\) remains finite. The space-time separation, \(\hat{z}\), between the points at which the currents \(J_\mu\) and \(J_\nu\) act, is

\[ \hat{z} = \left(\frac{z^-}{2}, 0, 0, -\frac{z^-}{2}\right), \]

\[ \hat{z}^2 = 0, \quad z^+ = z_\perp = 0. \]  

(3.31)

In the target rest frame define \(F_2\) as

\[ F_2(x) = x \sum_a Q_a^2 (f_a(x) + \bar{f}_a(x)), \]  

(3.32)

where the sum is over all quark flavors \(a\) and \(Q_a\) is the electromagnetic charge of a quark of flavor \(a\). The functions \(f_a(x)\) and \(\bar{f}_a(x)\) are given by:

\[ f_a(x) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dz^- e^{\frac{i}{2} q^+ z^-} < P | \bar{\psi}_a(z) \gamma^+ \psi_a(0) | P > e, \]  

(3.33)

\[ \bar{f}_a(x) = -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} dz^- e^{\frac{i}{2} q^+ z^-} < P | \bar{\psi}_a(0) \gamma^+ \psi_a(z) | P > e = -f_a(-x). \]  

(3.34)

We define a projection operator, \(\mathcal{P}^\pm\), by

\[ \mathcal{P}^\pm = \frac{1}{2} \gamma^0 \gamma^\pm \]

\[ \gamma^\pm = (\gamma^0 \pm \gamma^3), \]  

(3.35)
with the properties:

\[ P^- P^+ = P^+ P^- = 0 \]
\[ P^{±2} = P^± \]
\[ P^- + P^+ = 1. \] (3.36)

If the “light-cone projections” of the Dirac field are defined by

\[ \psi_{±} ≡ P_{±} \psi, \] (3.37)

then Eq. (3.33) may be rewritten as

\[
\begin{align*}
  f_a(x) &= \frac{1}{4\pi} \int_{-\infty}^{\infty} dz^- e^{\frac{i}{2}q^- z^-} < P |\psi_{a+}^\dagger (\hat{z}) \psi_{a+} (0) | P >_c, \\
  f_{\bar{a}}(x) &= \frac{1}{4\pi} \int_{-\infty}^{\infty} dz^- e^{\frac{i}{2}q^- z^-} < P |\psi_{a+} (\hat{z}) \psi_{a+}^\dagger (0) | P >_c .
\end{align*}
\] (3.38)

Here we have made use of

\[
< P |\psi_{a+} (\hat{z}) \psi_{a+} (0) | P >_c =< P | : \psi_{a+} (\hat{z}) \psi_{a+} (0) : | P >
= -< P |\psi_{a+} (0) \psi_{a+}^\dagger (\hat{z}) | P >_c ,
\] (3.39)

to interchange the quark fields in \( f_{\bar{a}}(x) \). Inserting a complete set of states (\( \sum_n |n > < n| = 1 \)) between quark fields, using

\[ \psi(\hat{z}) = e^{\frac{1}{2}iP^+ \hat{z}} \psi(0)e^{-\frac{1}{2}iP^+ \hat{z}}, \] (3.40)

and integrating over \( z^- \), we get

\[
\begin{align*}
  f_a(x) &= \sum_n \delta[(1 - x)P^+ - P^+_n] |< n |\psi_{a+} (0) | P > |^2 \\
  f_{\bar{a}}(x) &= \sum_n \delta[(1 - x)P^+ - P^+_n] |< n |\psi_{a+}^\dagger (0) | P > |^2.
\end{align*}
\] (3.41)

Eq. (3.41) enables us to give a concrete probability interpretation to the functions \( f_a(x) \) and \( f_{\bar{a}}(x) \). \( f_a(x) \) is the probability to remove from the target a quark of flavor \( a \) with \( P^+ \)-fraction \( x \) leaving behind a physical state (|\( n > \)) with \( P^+_n = (1 - x)P^+ \). Similarly, \( f_{\bar{a}}(x) \) is the probability to remove from the target an antiquark with \( P^+ \)-fraction \( x \) leaving behind a physical state with \( P^+_n = \)
Fig. 3.5: (a): The quark distribution function $f_a(x)$; (b): The antiquark distribution function $f_{\bar{a}}(x)$.

$(1 - x)P^+$. $f_a(x)$ and $f_{\bar{a}}(x)$ are shown graphically in Fig. 3.5. These functions are referred to as the distribution functions for quarks (antiquarks) of flavor $a$ respectively. Integrating these distributions over all $x$ leads to the sum rule:

$$
\int_{-\infty}^{\infty} dx f_a(x) = \int_{0}^{1} dx (f_a(x) - f_{\bar{a}}(x)) = N_a - N_{\bar{a}} = N^V_a.
$$

(3.42)

Here $N_a$ is the number of quarks of flavor $a$ and $N_{\bar{a}}$ the number of antiquarks of flavor $a$ respectively. The difference, $N^V_a$, is thus the number of valence quarks of flavor $a$.

We now relate $N^V_a$ to the physical normalization of the hadronic state $|P>$:

$$
N^V_a = \int_{-\infty}^{\infty} dx f_a(x) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dz^{-} \int_{-\infty}^{\infty} dx e^{-i\frac{Mz}{2}z^{-}} < P|\bar{\psi_a}(\hat{z})\gamma^+\gamma^0|P >
$$

$$
= \frac{1}{2P^+} \int_{-\infty}^{\infty} dz^{-} \delta(z^-) < P|\bar{\psi_a}(\hat{z})\gamma^+\psi_a(0)|P >
$$

$$
= \frac{1}{2P^+} < P|J^+_a(0)|P >, \quad (3.43)
$$

where $M \equiv P^+$.

Let us now use the pion to illustrate the normalization procedure. The pion Bethe-Salpeter normalization in the usual Ladder-Rainbow truncation in Minkowski space is

$$
2P^\mu = (-i) \frac{\partial}{\partial P^\mu} \int \frac{d^4k}{(2\pi)^4} tr_{c,s} \left[ \tilde{\Gamma}_\pi(k; -R)S(k + P) \Gamma_\pi(k; R)S(k) \right] |_{R^2 = P^2}, \quad (3.44)
$$
where

\[ S(p) = \frac{1}{\gamma.pA(p^2) - B(p^2)}. \tag{3.45} \]

The differential of the propagator is given by

\[ \frac{\partial}{\partial P_\mu} S(k + P) = -S(k + P) \left[ \frac{\partial}{\partial P_\mu} S^{-1}(k + P) \right] S(k + P) = -S(k + P) \Gamma^\mu S(k + P), \tag{3.46} \]

where \( \Gamma^\mu \rightarrow_{UV} \gamma^\mu + \cdots \).

![Diagram](image-url)

Fig. 3.6: The triangle diagram involved in the normalization of the pion Bethe-Salpeter amplitude.

The light-cone momentum \( P^+ \) is given by the integral of the triangle diagram depicted in Fig. 3.6:

\[ 2P^+ = i \int \frac{d^4k}{(2\pi)^4} \text{tr}_{c,s} \left[ \Gamma^+(k; -P)S(k) \Gamma^+ S(k) \Gamma^+(k; P)S(k - P) \right], \tag{3.47} \]

where \( \Gamma^+ \), the light-cone dressed vertex, is given by \( \Gamma^+ = 2 \frac{\partial}{\partial k^-} S^{-1}(k) \). This is Ladder-Rainbow truncation of

\[ 1 = \frac{1}{2P^+} < P|\bar{\psi}_a(0)\gamma^+\psi_a(0)|P > \epsilon = \frac{F_\pi(Q^2 = 0)}{Q_\pi}, \tag{3.48} \]

where \( F_\pi \) and \( Q_\pi \) are the pion form factor and electromagnetic charge respectively. Thus Eq. (3.43) and Eq. (3.48) indicate that the number of valence quarks of a given flavor is unity, i.e. \( N^V_a = 1 \).
3.3 Evolution of Distribution Functions

Structure functions systematically exhibit a weak $Q^2$-dependence, even at large $Q^2$. These scaling violations can be described within the framework of perturbative QCD (pQCD) which incorporates the interaction between quarks and gluons in the hadron in a perturbative way (see e.g. [9, 58]). The scale at which this interaction is resolved is determined by the momentum transfer. The $Q^2$-dependence of quark distributions, e.g.

$$F_2(x, Q^2) = \sum_a Q_a^2 x \left[ f_a(x, Q^2) + f_{\bar{a}}(x, Q^2) \right],$$

(3.49)
is described by the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations [59, 60, 61]. These equations are different for flavor non-singlet and singlet distribution functions (structure functions). Typical examples of non-singlet combinations are the difference of quark and antiquark distribution functions (valence quark distributions), or the difference of up and down quark distributions. The difference of the proton and neutron structure function, $F_2^p - F_2^n$, also behaves as a flavor non-singlet, whereas the deuteron structure function $F_2^d = F_2^p + F_2^n$ is an almost pure flavor singlet combination. We shall deal exclusively with the non-singlet evolution here, since we are interested mainly in the valence quark distributions in pseudoscalar mesons. We only mention the singlet evolution for completeness.

3.3.1 Non-singlet Evolution

The DGLAP evolution equations for the nonsinglet quark distributions (valence quark distributions), $q^{NS}(x, Q^2)$, can be written as

$$\frac{\partial q^{NS}(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{dy}{y} P_{qq}(x/y) q^{NS}(y, Q^2).$$

(3.50)

Here $\alpha_s(Q^2)$ is the running QCD coupling strength. The splitting function $P_{qq}(x/y)$ determines the probability for a quark to radiate a gluon such that the quark momentum is reduced by a fraction $x/y$. 
The splitting function is of the form

\[ P_{qq}(z) = \frac{4}{3} \left[ \frac{1 + z^2}{(1 - z)_+} + \frac{3}{2} \delta(1 - z) \right], \]  

(3.51)

where the \(+\) distribution is defined as

\[ \int_x^1 \frac{f(z)}{(1 - z)_+} dz = \int_x^1 \frac{dz}{1 - z} [f(z) - f(1)] + f(1) \ln(1 - x). \]  

(3.52)

Since

\[ \int_0^1 dz P_{qq}(z) = 0, \]  

(3.53)

the total number of valence quarks is independent of the resolution:

\[ \frac{\partial}{\partial \ln Q^2} \int_0^1 dx q_{NS}^2(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \int_0^1 dy q_{NS}^2(y, Q^2) \int_0^1 dz P_{qq}(z) = 0, \]  

(3.54)

with change of variables \( z = x/y \).

The \( n^{th} \) moment of the valence quark distribution is defined as

\[ M^n(Q^2) = \int_0^1 dx x^{n-1} q_{NS}^2(x, Q^2). \]  

(3.55)

Differentiating this with respect to \( \ln(Q^2) \), inserting the DGLAP equation, Eq. (3.50), and using Eq. (1.6) results in

\[ \frac{\partial M^n(Q^2)}{\partial \ln Q^2} = -\frac{1}{\ln(Q^2/\Lambda_{QCD})} d^{(0)}_{NS}(n) M^n(Q^2), \]  

(3.56)

where the so-called (leading order) anomalous dimension is defined as

\[ d^{(0)}_{NS}(n) = -\frac{6}{33 - 2N_f} \int_0^1 dz z^{n-1} P_{qq}(z). \]  

(3.57)

We may solve Eq. (3.56) to obtain the \( n^{th} \) moment at arbitrary \( Q^2 \) in terms of it value at some initial point, \( Q_0^2 \) :

\[ M^n(Q^2) = M^n(Q_0^2) \left( \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)} \right)^{d^{(0)}_{NS}(n)}. \]  

(3.58)

Substituting Eq. (3.51) in Eq. (3.50) gives

\[ \frac{\partial q_{NS}^2(x, Q^2)}{\partial \ln Q^2} = \frac{2\alpha_s(Q^2)}{3\pi} \int_x^1 dz \frac{dz}{z} q_{NS}^2(x/z, Q^2) \left[ \frac{1 + z^2}{(1 - z)_+} + \frac{3}{2} \delta(1 - z) \right]. \]  

(3.59)
The first term is integrated using Eq. (3.52) with \( f(z) = \frac{1+z^2}{z} q^{\text{NS}}(x/z, Q^2) \), to give the expression
\[
\frac{\partial q^{\text{NS}}(x, Q^2)}{\partial \ln Q^2} = \frac{2\alpha_s(Q^2)}{3\pi} \left\{ \left[ \frac{3}{2} + 2\ln(1-x) \right] q^{\text{NS}}(x, Q^2) + \int_x^1 \frac{dz}{1-z} \left[ \frac{1+z^2}{z} q^{\text{NS}}(x/z, Q^2) - 2q^{\text{NS}}(x, Q^2) \right] \right\}. \tag{3.60}
\]
This equation takes an almost identical form for the non-singlet contribution to \( F_2, F_2^{\text{NS}} \equiv x q^{\text{NS}} \):
\[
\frac{\partial F_2^{\text{NS}}(x, Q^2)}{\partial \ln Q^2} = \frac{2\alpha_s(Q^2)}{3\pi} \left\{ \left[ \frac{3}{2} + 2\ln(1-x) \right] F_2^{\text{NS}}(x, Q^2) + \int_x^1 \frac{dz}{1-z} \left[ (1+z^2) F_2^{\text{NS}}(x/z, Q^2) - 2F_2^{\text{NS}}(x, Q^2) \right] \right\}. \tag{3.61}
\]

### 3.3.2 Singlet Evolution

The evolution of singlet and gluon distributions are both more complicated and more interesting as they couple:
\[
\frac{\partial}{\partial \ln Q^2} \left( \begin{array}{l} q^S(x, Q^2) \\ g(x, Q^2) \end{array} \right) = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{dy}{y} \left( \begin{array}{cc} P_{qq}(\frac{x}{y}) & P_{qg}(\frac{x}{y}) \\ P_{gq}(\frac{x}{y}) & P_{gg}(\frac{x}{y}) \end{array} \right) \left( \begin{array}{c} q^S(y, Q^2) \\ g(y, Q^2) \end{array} \right). \tag{3.62}
\]
Thus the evolution of singlet quark distributions involves both quark and gluon distributions, unlike the case of non-singlet quark distributions. In a reciprocal relation, the evolution of the gluon distribution too depends on both quark and gluon distributions. This coupled nature of the equations makes them harder to solve than the non-singlet evolution. Further details can be found in [9].

### 3.4 Valence Distributions in Pseudoscalar Mesons

The last three sections have been devoted to the general treatment of DIS and quark distributions in hadrons. We now specialize this general framework to the case where the hadrons under consideration are the light pseudoscalar mesons, namely pions and kaons, and shall limit ourselves to a detailed treatment of the valence quark distributions in these mesons. The treatment presented here follows closely the treatment in [62]. The major difference between the present study and the earlier investigation by Hecht et al [62] is primarily in the treatment of the input elements needed for the
Fig. 3.7: “Handbag” contributions to the virtual photon-pseudoscalar meson forward Compton scattering amplitude, which are the only impulse approximation diagrams that survive in the deep inelastic Bjorken limit. meson, double-line; \(\gamma\), wavy-line; internal solid-line: dressed-quark propagator represented by pairs of 3-complex-conjugate-poles fit to DSE solution. The filled circles represent the pseudoscalar meson’s Bethe-Salpeter amplitude, \(\Gamma_\zeta\), and the quark-photon vertex, \(\gamma_\mu\), depending on which external line they begin/end.

calculations: the dressed quark propagators and the pseudoscalar meson’s dressed vertices. These elements are the solutions of the quark Dyson-Schwinger equation and the Bethe-Salpeter equation respectively. In our work we use the actual solutions of these two equations for the input elements whereas in \([62]\) the authors employ algebraic parametrisations \([63, 64, 65, 66, 67, 68, 69]\) for the dressed-quark propagators and the dressed vertices.

3.4.1 DIS on Pseudoscalar Mesons

In the Bjorken limit, according to Sec. 3.1.4, we can study deep inelastic scattering from a pseudoscalar meson (denoted generically as \(\zeta\): pion or kaon) target via the diagrams in Fig. 3.7. These diagrams describe virtual photon-meson forward Compton scattering in the impulse approximation \([70]\), and the relationship between DIS and virtual Compton scattering has been elucidated in Sec.3.1.3.

Two diagrams contribute to the virtual Compton scattering amplitude, \(T\), and they are related
by crossing. The left diagram represents the renormalised matrix element in Minkowski space

\[
T_{\mu\nu}^a(q, P) = -i \text{tr} \int_M \frac{d^4k}{(2\pi)^4} \tau_{-\gamma_5} \tilde{\Gamma}_\xi(k_\Gamma; -P) S(k) eQ\gamma_{\mu} \\
\times S(k + q) eQ\gamma_{\nu} S(k) \tau_{+\gamma_5} \Gamma_\xi(k_\Gamma; P) S(k_s),
\]

(3.63)

We now work in Euclidean space (Appendix A) and use the Euclidean metric convention in which 
\( a \cdot b := \sum_{i=1}^4 a_i b_i \), so that a spacelike vector, \( Q_\mu \), has \( Q^2 > 0 \) and an on-shell physical hadron state is described by \( P^2 = -M^2 \). The Dirac matrices are Hermitian and are defined by the algebra \( \{ \gamma_\mu, \gamma_\nu \} = 2 \delta_{\mu\nu} \).

The left diagram represents the renormalised matrix element in Euclidean space

\[
T_{\mu\nu}^a(q, P) = \text{tr} \int_E \frac{d^4k}{(2\pi)^4} \tau_{-\Gamma_\xi}(k_\Gamma; -P) S(k) ieQ\gamma_{\mu} \\
\times S(k + q) ieQ\gamma_{\nu} S(k) \tau_{+\Gamma_\xi}(k_\Gamma; P) S(k_s),
\]

(3.64)

where \( \Gamma_\xi(\ell; P) \) is the light pseudoscalar meson’s Bethe-Salpeter amplitude and

\[
\Gamma_\xi(\ell; -P) = C^\dagger \Gamma_\xi(-\ell; -P)^T C,
\]

(3.65)

with \( \tau_{\pm} = \frac{1}{2}(\tau_1 \pm i\tau_2) \), \( C = \gamma_2\gamma_4 \), the charge conjugation matrix, and \( (\cdot)^T \) denoting matrix transpose; \( S(\ell) \) is the dressed-quark propagator:

\[
S^{-1}(\ell) = i\gamma.\ell A(\ell^2) + B(\ell^2) \equiv (-i\gamma.\ell \sigma_v(\ell^2) + \sigma_s(\ell^2))^{-1},
\]

(3.66)

with \( Q = \text{diag}(2/3, -1/3) \) the quark-charge matrix; \( k_\Gamma = k - P/2 \), \( k_s = k - P \); and the trace is over colour, flavour and Dirac indices. The matrix element represented by the right diagram is the crossing partner of Eq. (3.64) and is obvious by analogy.

From Sec.3.1.3, the hadronic tensor relevant to inclusive deep inelastic lepton-meson scattering can be obtained from the forward Compton process via the optical theorem:

\[
W_{\mu\nu}(q; P) = \frac{1}{2\pi} \text{Im} \left[ T_{\mu\nu}^a(q; P) + T_{\mu\nu}^d(q; P) \right],
\]

(3.67)

Due to current conservation, the hadronic tensor in Euclidean space can be expressed in terms of only two invariant structure functions:

\[
W_{\mu\nu}(q; P) = F_1(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) - \left[ (P^\mu - \frac{P.q}{q^2} q^\mu) \left( P^\nu - \frac{P.q}{q^2} q^\nu \right) \right] \frac{F_2}{P.q},
\]

(3.68)
Taking the trace over color and flavor indices, Eq. (3.64) assumes the form

\[ T_{\mu\nu}^{\alpha}(q, P) = e_u^2 N_c \int \frac{d^4k}{(2\pi)^4} \mathrm{tr}_D \left[ \bar{\Gamma}_\zeta(k_G; -P) S(k) i\gamma_\mu S(k + q)i\gamma_\nu S(k) \Gamma_\zeta(k_G; P) S(k_s) \right], \tag{3.69} \]

where \( \mathrm{tr}_D \) denotes trace over Dirac indices.

Introducing the integration variable transformations

\[ k = \kappa + \beta q \frac{M_c}{\nu} + \alpha P, \quad \kappa \cdot q = 0 = \kappa \cdot P, \tag{3.70} \]

with Jacobian \( J = i P \) in the Bjorken limit, we find

\[ (k + q)^2 = 2P \cdot q (\alpha - x) + O((P \cdot q)^0), \tag{3.71} \]

and

\[ -i\gamma \cdot (k + q) = P \cdot q \frac{-i\gamma^+}{M_\zeta} + O((P \cdot q)^0), \tag{3.72} \]

where \( \gamma^+ \) in Euclidean space is defined as

\[ \gamma^+ = -i\gamma_4 + \gamma_3. \tag{3.73} \]

Thus on the domain relevant to inclusive deep inelastic scattering, the dressed connected-quark propagator becomes, in the Bjorken limit

\[ S(k + q) = \frac{-i\gamma \cdot (k + q)}{(k + q)^2 + i\epsilon} \to \frac{-i\gamma^+}{2M_\zeta} \frac{1}{\alpha - x + i\epsilon}. \tag{3.74} \]

Note that with \( \alpha = \frac{k^+}{M_\zeta}; k^+ = k^0 + k^3 = -ik_4 + k_3. \) We see that

\[ \frac{1}{2i} \lim_{\epsilon \to 0} [T_{\mu\nu}(\epsilon) - T_{\mu\nu}(-\epsilon)] = 0, \tag{3.75} \]

if \( k_4 \) has a real domain of integration only. One can only obtain the physical discontinuity from integration along the real \( k^0 \) axis \( ie. \) imaginary \( k_4 \) axis. So we must perform a Wick rotation \((\int_k^E \to (-i) \int_k^M)\) in order for \( k^+ \) to have a real domain.

In the Bjorken limit we have

\[ T_{\mu\nu}^{\alpha}(q, P) = e_u^2 N_c \frac{iP^2}{(2\pi)^4} \int d\alpha d\beta d^2k \frac{-i}{2M_\zeta} \frac{1}{\alpha - x + i\epsilon} \mathrm{tr}_D \left[ \bar{\Gamma}_\zeta(k_G; -P) S(k) i\gamma_\mu \gamma^+ i\gamma_\nu S(k) \Gamma_\zeta(k_G; P) S(k_s) \right], \tag{3.76} \]
and

\[ W_{\mu\nu}^{u}(q, P) = \frac{1}{2\pi} \text{Im} T_{\mu\nu}^{u}(q, P) = \frac{1}{2\pi} \frac{1}{2i} \text{Disc} T_{\mu\nu}^{u}(q, P), \]  

(3.77)

where the discontinuity leads to

\[ \frac{1}{2i} \left[ \frac{1}{\alpha - x + i\epsilon} - \frac{1}{\alpha - x - i\epsilon} \right] \approx -\pi\delta(\alpha - x). \]  

(3.78)

The Lorentz structure of Eq. (3.76) is simplified by the identity

\[ \gamma^{\mu}\gamma^{\rho}\gamma^{\nu} = S_{\mu\rho\nu\alpha}\gamma^{\alpha} + \epsilon_{\mu\rho\nu\alpha}\gamma^{5}, \]

\[ S_{\mu\rho\nu\alpha} \equiv \frac{1}{4} \text{Tr} \gamma_{\mu}\gamma_{\rho}\gamma_{\nu}\gamma_{\alpha} = \delta_{\mu\rho}\delta_{\nu\alpha} + \delta_{\mu\alpha}\delta_{\nu\rho} - \delta_{\mu\nu}\delta_{\alpha\rho}. \]  

(3.79)

The term containing the totally antisymmetric tensor, \( \epsilon_{\mu\rho\nu\alpha} \), does not contribute.

The contribution of \( T_{\mu\nu}^{u} \) to \( W_{\mu\nu}^{u} \) can thus be written as

\[ W_{\mu\nu}^{u}(q, P) = \delta_{\mu\nu} e_{u}^{2} N_{c} \frac{P^{2}}{(2\pi)^{4}} \left( \frac{-1}{4M_{\zeta}} \int d\alpha d\beta d^{2}\kappa \delta(\alpha - x) \right. \]

\[ \left. \text{tr}_{D} \left[ \bar{\Gamma}_{\zeta}(k_{\Gamma}; -P) S(k) \gamma^{+} S(k) \Gamma_{\zeta}(k_{\Gamma}; P) S(k_{s}) \right] + \cdots \right. \]  

(3.80)

\( W_{\mu\nu}^{u}(q; P) \) shows that in the Bjorken limit the struck quark carries a fraction \( x \) of the meson's momentum.

From Equations (3.68) and (3.80) we identify the structure function \( F_{1}^{u}(x) \) as

\[ F_{1}^{u}(x) = -e_{u}^{2} N_{c} \frac{P^{2}}{(2\pi)^{4}} \frac{1}{4M_{\zeta}} \left( \frac{-1}{4M_{\zeta}} \int d\alpha d\beta d^{2}\kappa \delta(\alpha - x) \right. \]

\[ \left. \text{tr}_{D} \left[ \bar{\Gamma}_{\zeta}(k_{\Gamma}; -P) S(k) \gamma^{+} S(k) \Gamma_{\zeta}(k_{\Gamma}; P) S(k_{s}) \right] \right). \]  

(3.81)

Using \( F_{2}(x) = x \sum_{a} Q_{a}^{2}(f_{a}(x) + \bar{f}_{a}(x)) \) and the Callan-Gross relation yield

\[ u(x) = -2N_{c} \frac{P^{2}}{(2\pi)^{4}} \frac{1}{4M_{\zeta}} \left( \frac{-1}{4M_{\zeta}} \int d\alpha d\beta d^{2}\kappa \delta(\alpha - x) \right. \]

\[ \left. \text{tr}_{D} \left[ \bar{\Gamma}_{\zeta}(k_{\Gamma}; -P) S(k) \gamma^{+} S(k) \Gamma_{\zeta}(k_{\Gamma}; P) S(k_{s}) \right] \right). \]  

(3.82)

After transforming the integral in Eq.(3.82) back to four dimensional integral we have

\[ u(x) = \frac{iN_{c}}{2M_{\zeta}} \int \frac{d^{4}k}{(2\pi)^{4}} \delta\left( \frac{k^{+}}{M_{\zeta}} - x \right) \]

\[ \text{tr}_{D} \left[ \bar{\Gamma}_{\zeta}(k_{\Gamma}; -P) S(k) \gamma^{+} S(k) \Gamma_{\zeta}(k_{\Gamma}; P) S(k_{s}) \right] \right). \]  

(3.83)
We can calculate the moments by integrating Eq. (3.83) over \( x \) from 0 → 1:

\[
\langle x^n \rangle = \int_0^1 x^n u(x) \, dx = \frac{iN_c}{2M_\zeta} \int \frac{d^4 k}{(2\pi)^4} \int_0^1 x^n dx \delta\left(\frac{k^+}{M_\zeta} - x\right) \text{tr}_D \left[ \bar{\Gamma}_\zeta(k_\Gamma; -P) S(k) \gamma^+ S(k) \Gamma_\zeta(k_\Gamma; P) S(k_s) \right].
\] (3.84)

Therefore

\[
\langle x^n \rangle = \frac{iN_c}{2M_\zeta} \int \frac{d^4 k}{(2\pi)^4} \left(\frac{k^+}{M_\zeta}\right)^n \text{tr}_D \left[ \bar{\Gamma}_\zeta(k_\Gamma; -P) S(k) \gamma^+ S(k) \Gamma_\zeta(k_\Gamma; P) S(k_s) \right].
\] (3.85)

This novel approach to calculate the moments is still in progress.

To simply our calculation, we can express the dressed spectator-quark propagator as

\[
S(k) = \Delta(k^2)N(k),
\] (3.86)

\[
\Delta(k^2) = 1/[k^2 + M^2(k^2)],
\] (3.87)

\[
N(k) = Z(k^2) \left[ -i\gamma \cdot k + M(k^2) \right].
\] (3.88)

Here \( Z(k^2) \) is the dressed-quark wave-function renormalisation and \( M(k^2) \) is the dressed-quark mass function.

The integrand in Eq. (3.81) can now be simplified

\[
F_1^u(x) = -\frac{P^2}{4M_\zeta^2} c_u^2 N_c \frac{1}{(2\pi)^4} \int d\beta d^2\kappa \left[ \Delta(k_s^2) T^u(k, q, P) \right] |_{\alpha=x},
\] (3.89)

where

\[
T^u = \text{tr}_D \left[ \bar{\Gamma}_\zeta(k_\Gamma; -P) S(k) \gamma^+ S(k) \Gamma_\zeta(k_\Gamma; P) N(k_s) \right].
\] (3.90)

Using Eqs. (3.70) to express

\[
s := k_s^2 = 2P^2(x - 1) \left[ \beta - \frac{1-x}{2} + \frac{\kappa^2}{2P^2(x - 1)} \right],
\] (3.91)

then

\[
\Delta(s) = -\frac{1}{2P^2(1 - x)} \frac{1}{\beta - \frac{1-x}{2} - \frac{\kappa^2 + M^2(s)}{2P^2(1-x)}}.
\] (3.92)
The integral over $\beta$ of Eq. 3.89 is given by
\[
\int d\beta \frac{f(\beta)}{\beta - \beta_0 - i\epsilon} = \int d\beta f(\beta) [\text{PV} \left( \frac{1}{\beta - \beta_0} \right) + i\pi \delta(\beta - \beta_0)]
\] (3.93)
where $\beta_0 = \frac{1-x}{2} + \frac{\kappa^2 + \tilde{M}^2}{2P^2(1-x)}$, the valence quark mass $\tilde{M}$ is the constituent mass and PV stands for the principal value. The imaginary part of the $\beta$-integral is needed in order for the structure function to be real.

\[
F^u_1(x) = \frac{1}{2} \frac{1}{\Lambda_c^2} e_u^2 N_c \frac{\pi}{(2\pi)^4} \int d^2\kappa \frac{i}{4} \left[ T^u(k, q, P) \right]_{\beta = \beta_0}^{\alpha = x}
\] (3.94)

The momentum of the remaining dressed-quark line can also be written as
\[
\mu := k^2 = \kappa^2 + P^2 x (x + 2\beta).
\] (3.95)

This illustrates that the integrand depends only on $\kappa^2$ as an integration variable. Thus another shift of integration variables can be performed:
\[
\int d^2\kappa = \pi \int d\kappa^2,
\]
where
\[
\kappa^2 = (1 - x)(\mu - \mu_{\text{min}})
\] (3.96)
and
\[
\mu_{\text{min}} = x \left[ P^2 + \tilde{M}^2/(1-x) \right]
\] (3.97)

Therefore
\[
F^u_1(q, P) = \frac{1}{2} e_u^2 N_c \frac{1}{(4\pi)^2} \int_{\mu_{\text{min}}}^{\infty} d\mu \frac{i}{4M_c^2} \left[ T^u(k, q, P) \right]_{\beta = \beta_0}^{\alpha = x}
\] (3.98)
with $\beta_0 = \frac{1}{2P^2} \left[ \mu - xP^2 + \mu_{\text{min}}(1-x)/x \right]$.

and the Callan-Gross relation:
\[
F^u_2(x) = 2x F^u_1(x).
\] (3.99)

Due to the contraction of the integration domain ($\mu_{\text{min}} \to \infty$ as $x \to 1$), we have that
\[
F^u_{1,2}(x) \to 0 \text{ as } x \to 1.
\] (3.100)

The analysis of $W^d_{\mu\nu}$ follows similarly with the same result. Thus $F_{1,2} = F^u_{1,2} + F^d_{1,2}$.

Equations (3.67), (3.80), (3.90), (3.98) and (3.99) provide a model-independent starting point for calculating the valence-quark distributions in light pseudoscalar mesons. In practice, model inputs
are needed for the internal elements represented in Fig. 3.7; i.e., the dressed-quark propagator and the pseudoscalar meson Bethe-Salpeter amplitude.

The formalism presented so far determines the valence-quark distributions. This is because, although sea-quarks are implicitly contained in the dressing of the propagators and calculation of the dressed vertices, the “handbag” impulse approximation diagrams in Fig. 3.7 only admit a coupling of the photon to the propagator of the dressed-quark constituent. The internal structure of the dressed-quark is not resolved and therefore the calculation yields the distribution at a scale $q_0$ characteristic of the resolution: $q_0$ is an a priori undetermined parameter in the formalism, although it is anticipated that $0.3 \lesssim q_0 \lesssim 1.0$ GeV, with the lower bound set by the Euclidean constituent-quark mass and the upper by the onset of the perturbative domain. A sea-quark distribution is generated via the renormalisation group (evolution) equations when the valence distribution is evolved to that $q^2$-scale appropriate to a given experiment. The explicit generation of the sea-quark contributions at the scale $q_0^2$ requires going beyond the impulse (handbag) approximation; e.g., incorporating photon couplings to the intermediate-state quark-meson-loops that can appear as a dressing of the quark propagator: $\pi^+ = u \bar{d} \rightarrow (u\bar{s}s) \bar{d} = (K^+s) \bar{d} \rightarrow u \bar{d} = \pi^+$, with $\gamma K^+ s \rightarrow K^+ s \gamma$, etc. Such intermediate states arise as vertex corrections in the quark-DSE.

### 3.4.2 Valence Quark Distributions in Pions

We now address the case where the pseudoscalar meson under consideration is the pion ($\zeta \equiv \pi^+$). The pion has two constituent quarks: $u$ and $\bar{d}$. Thus from Eq 3.32 the structure functions can be expressed as:

$$
F^u_2(x; q_0) = \frac{4}{9} x u_v(x; q_0) = \frac{4}{9} x [u(x; q_0) - \bar{u}(x; q_0)],
$$

$$
F^\bar{d}_2(x; q_0) = \frac{1}{9} x \bar{d}_v(x; q_0) = \frac{1}{9} x [\bar{d}(x; q_0) - d(x; q_0)].
$$

(3.101)

It can be demonstrated algebraically that

$$
\bar{d}^{\pi^+}_v (x; q_0) = u^{\pi^+}_v (x; q_0) = d^{\pi^-}_v (x; q_0).
$$

(3.102)
The calculations are subject to the constraint
\[
\int_0^1 dx \, u_\pi(x; q_0) = 1 = \int_0^1 dx \, \bar{d}_\pi(x; q_0) .
\] (3.103)
This ensures that the \( \pi^+ \) contains only one \( u \) valence quark and one \( \bar{d} \) valence quark.

In the \( G \)-parity symmetric limit \( u_\pi^+(x) = \bar{d}_\pi^+(x) \). However, in this limit \( \omega \to \pi\pi \) is forbidden and hence the scale of \( G \)-parity symmetry violation in nature is characterised by the ratio \( \Gamma_{\omega \to \pi\pi}/\Gamma_{\rho \to \pi\pi} = 0.1\% \). This bound on any difference between the pion’s quark distribution functions is consistent with the model estimate in Ref. [72]. Thus we focus attention on just the valence \( u \) quark. From Eqs. (3.98), (3.99), and (3.101) we have
\[
\Gamma_\pi^u(x) = \frac{N_c}{(4\pi)^2} \int_{\mu_{\min}}^\infty d\mu \frac{i}{4M_\pi} \left[ T^u(k, q, P) \right]_{\alpha=x}^{\beta=\bar{\beta}_0}(3.104)
\]
where
\[
T^u = \text{tr}_D \left[ \Gamma_\pi(k\Gamma; -P) S(k) \gamma^+ S(k) \Gamma_\pi(k\Gamma; P) N(k_s) \right] .
\] (3.105)
The basic inputs needed for the calculations are the dressed-quark propagators and the pion’s dressed vertices. For the dressed-quark propagators we employ the complex-conjugate mass poles parametrization of the solution of the quark DSE, as in Chapter 2. The 3 pairs of complex conjugate mass poles (for \( u/d \) quarks) are presented in the second block of Table 2.1. The general form of the \( \pi \)-meson Bethe-Salpeter amplitude (dressed vertex) is
\[
\Gamma_\pi(k; Q) = \gamma_5 \left[ iE_\pi(k; Q) + \gamma \cdot Q F_\pi(k; Q) + \gamma \cdot k \cdot Q G_\pi(k; Q) + \sigma_{\mu\nu} k_\mu Q_\nu H_\pi(k; Q) \right] ,
\] (3.106)
where the behaviour of the invariant functions is constrained to a large extent by the axial-vector Ward-Takahashi identity [73, 74]. These vertices are solutions of the inhomogeneous Bethe-Salpeter equations. The equations are solved numerically at discrete values of the arguments, and an efficient interpolation routine is employed to evaluate the solutions at other values of the arguments. An important aspect in the numerical solution of these equations is the expansion of the angular dependence of the amplitudes in terms of the Chebyshev polynomials of the second kind. With
\( k \cdot Q = kQu \), the expansion of an arbitrary invariant function \( A(k; Q) \) in the Bethe-Salpeter amplitude is

\[
A(k; Q) = \sum_{m=0}^{\infty} (kQ)^m U_m(u) A_m(k^2, Q^2),
\]

(3.107)

where \( U_m(u) \) denotes the Chebyshev polynomial of order \( m \). The pion amplitudes \( E_\pi, F_\pi, G_\pi, \) and \( H_\pi \) are even functions of \( k \cdot Q \) and thus need only the even Chebyshev moments. An often-used approximation is to limit the expansion to just the first Chebyshev polynomial (no angle dependence); the resulting amplitude is comparable to the input dressed vertex in [62].

In order to preserve the normalization of the distribution we replace \( \gamma^+ \) by (see section 3.2 for detail)

\[
\Gamma^+(x, k) = 2i \frac{\partial}{\partial k} S^{-1}(k).
\]

(3.108)

If no approximations are employed for the spectator propagator, then \( \int_0^1 dx \ u_\pi(x; q_0) = 1 \). Additionally, the calculation of the one-dimensional integral that yields \( u_\pi(x) \) via Eqs. (3.104) and (3.105) requires a determination of the valence-quark mass, \( \tilde{M} \). Since the parametrisation of the dressed-quark propagator is confining, it does not admit a solution of \( \tilde{M} = M(-\tilde{M}^2) \). Therefore, in our calculation of the pion’s valence-quark distribution we use a constituent mass given by

\[
\tilde{M} = 0.40 \text{ GeV}.
\]

(3.109)

Using \( \gamma^+ \) in Eq. (3.105) the integral over \( x \) of the valence u-quark distribution gives

\[
\int_0^1 dx \ u_\pi(x; q_0) = 0.98,
\]

(3.110)

whereas replacing \( \gamma^+ \) by \( \Gamma^+ \) the integral is 1.44, a 47% increase. We present the distributions for both cases after normalizing to 1 in Fig. 3.8. The shapes of these distributions are slightly different, and the calculated moments are within ~ 3%, but after evolving these distributions to the experimental scale, the difference is barely distinguishable.

Our calculation of the pion’s valence u-quark distribution, \( u_\pi(x; q_0) \), is presented in Fig. 3.9. In accordance with the kinematic constraint expressed in Eq. (3.97) it vanishes at \( x = 1 \), and has a finite value of \( F_1(x = 0) \), which is a signal of the absence of sea-quark contamination. In addition
Fig. 3.8: The pion’s valence $u$-quark distribution. The solid curve denotes the distribution using $\gamma^+$ while the dashed curve depicts the distribution using $\Gamma^+$ (Eq.(3.108) in the text).
\( u_\pi(x = 0; q_0) \neq 0 \), which is acceptable as there is no constraint that requires it to vanish at this point. The shape of the calculated distribution is characteristic of a strongly bound system \[75\]: cf. for a weakly bound system \( u_v(x) \approx \delta(x - \frac{1}{2}) \). This is not surprising for a light bound state of heavy constituents.

The average momentum-fraction carried by the valence-quarks at this resolving scale is

\[
\int_0^1 dx \left[ u_\pi(x; q_0) + \bar{d_\pi}(x; q_0) \right] = 0.74, \tag{3.111}
\]

with the remainder carried by the gluons that effect the binding of the pion bound state, which are invisible to the electromagnetic probe. The second and third moments of the distribution are

\[
\langle x^2 \rangle_{q_0} = 0.19, \quad \langle x^3 \rangle_{q_0} = 0.12. \tag{3.112}
\]

As mentioned before, \( q_0 \), the resolving scale, is not given by the formalism. To determine this scale we employ the 4-flavour value of \( \Lambda_{QCD} \), \( \Lambda_{QCD}^{n_f=4} = 0.234 \text{ GeV} \), and the leading-order, nonsinglet renormalisation group (evolution) equations (see Section 3.3 for details) to evolve the distribution in Fig. 3.9 up to \( q = 2 \text{ GeV} \), and require agreement between the first and second moments of our evolved distribution and those calculated from the phenomenological fits of Ref. \[76\]. With

\[
q_0 = 0.57 \text{ GeV} = 1/(0.35 \text{ fm}), \tag{3.113}
\]

we obtain the moments of pion’s valence-quark distribution at \( q = 2 \text{ GeV} \), \( q = 4.05 \text{ GeV} \), and \( q = 5.2 \text{ GeV} \) respectively. The moments are presented in Table. 3.1, in conjunction with the moments from the phenomenological fit in \[76\] (Fit1), a lattice QCD calculation \[77\], and the fit in \[78\] (Fit2). Our results reproduce the data rather well for all considered moments. We also use the result obtained by Hecht \textit{et al} \[62\] to calculate the moments at \( q = 4.05 \text{ GeV} \) and \( q = 5.2 \text{ GeV} \). Our result at \( q = 5.2 \text{ GeV} \) agrees nicely with the fit by Wijesooriya \textit{et al} \[78\] while the Hecht’s result is smaller. For example, for the calculated second moment, our result and Hecht’s result are 2% and 9% smaller respectively than the fit by Wijesooriya \textit{et al}. The difference is even more dramatic when the third moments are considered: our result and Hecht’s result are within 7% and 18% respectively of the fit.
Fig. 3.9: The pion’s valence $u$-quark distribution $u_\pi(x;q)$. The solid line depicts the distribution at the resolving scale $q_0$, $u_\pi(x;q_0)$, calculated using the DSE elements and the full pion’s Bethe-Salpeter amplitude. The valence-quark mass is $\bar{M} = 0.40 \text{ GeV}$ and the resolving scale $q_0 = 0.57 \text{ GeV} = 1/(0.35 \text{ fm})$ is fixed as described in connection with Eq. (3.113). The dashed line represents $u_\pi(x;q_0)$ calculated using the first Chebyshev moment of the pion’s Bethe-Salpeter amplitudes.
Table 3.1: The moments of the pion valence-quark distribution at $q = 2$ GeV, $q = 4.05$ GeV, and $q = 5.2$ GeV compared to Hecht et al [62], Fit1 [76], Lattice [77], and Fit2 [78].

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\langle x \rangle_q$</th>
<th>$\langle x^2 \rangle_q$</th>
<th>$\langle x^3 \rangle_q$</th>
</tr>
</thead>
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<tr>
<td>2 (GeV)</td>
<td>Calc. 0.256</td>
<td>0.11</td>
<td>0.057</td>
</tr>
<tr>
<td></td>
<td>Hecht [62] 0.24</td>
<td>0.098</td>
<td>0.049</td>
</tr>
<tr>
<td></td>
<td>Fit[76] 0.24 ± 0.01</td>
<td>0.10 ± 0.01</td>
<td>0.058 ± 0.004</td>
</tr>
<tr>
<td></td>
<td>Latt.[77] 0.27 ± 0.01</td>
<td>0.11 ± 0.3</td>
<td>0.048 ± 0.020</td>
</tr>
<tr>
<td>4.05 (GeV)</td>
<td>Calc. 0.227</td>
<td>0.09</td>
<td>0.045</td>
</tr>
<tr>
<td></td>
<td>Hecht [62] 0.22</td>
<td>0.083</td>
<td>0.040</td>
</tr>
<tr>
<td>5.2 (GeV)</td>
<td>Calc. 0.219</td>
<td>0.085</td>
<td>0.042</td>
</tr>
<tr>
<td></td>
<td>Hecht [62] 0.212</td>
<td>0.079</td>
<td>0.037</td>
</tr>
<tr>
<td></td>
<td>Fit [78] 0.217 ± 0.011</td>
<td>0.087 ± 0.005</td>
<td>0.045 ± 0.003</td>
</tr>
</tbody>
</table>

The valence-quarks at $q = 2$ GeV carry an average momentum-fraction of 0.51. Also at the resolving scale $q_0$, $\alpha_s/(2\pi) = 0.13$, which is the combination appearing in the evolution equation.

The natural combination occurring in the structure function $F_2$ is $x$ times the distribution function, i.e. $x u_\pi(x, q)$. This “momentum-fraction” distribution (generally referred to as momentum distribution) at the resolving scale $q_0$ is presented in Fig. 3.10. Also presented are the evolved distributions at $q = 2$ GeV, with the fit to the E615 Drell-Yan experimental data at $q = 2$ GeV of Ref [76]. The effects of evolution on the distributions are readily apparent from the figure: a depletion at higher $x$ and an enhancement at lower $x$, in line with the structure of the splitting function governing the non-singlet evolution equation.

In Fig. 3.11 we display the evolved distribution (at $q = 4.05$ GeV) using the full Dyson-Schwinger Equation-Bethe-Salpeter machinery. We also show the distribution in which the Bethe-Salpeter amplitudes involve only the first Chebyshev moment, the result of an earlier calculation carried out by Hecht et al [62], and the E615 $\pi N$ Drell-Yan experimental data from Ref. [79].
Fig. 3.10: Evolution of the pion’s valence quark distribution. Dashed line depicts distribution at resolving scale, $xu_\pi(x; q_0)$. The solid line represents the evolved distribution, $xu_\pi(x; q = 2 \text{ GeV})$, evolved with a 4-flavour value of $\Lambda_{\text{QCD}} = 0.234 \text{ GeV}$. The dot-dashed line denotes the phenomenological fit of Ref [76] at 2 GeV.
three distributions are similar at low $x$, $x < 0.2$. The distribution involving the first Chebyshev moment is similar to that of [62] except in the region $0.44 < x < 0.8$, where it is systematically higher. Both are higher than the full DSE-BSE distribution for $0.16 < x < 0.48$ and lower for $x > 0.6$. All three underpredict the data at high $x$, with the full DSE-BSE distribution showing the least deviation.

Let us focus attention on the full DSE-BSE distribution, and compare it with the Drell-Yan data. As can be seen from Fig. 3.10 the agreement is reasonable for $x < 0.7$ and systematically underpredicts the data at high $x$, i.e. $x > 0.7$, with the severity of underprediction significantly appreciable for $0.76 < x < 1$. This high $x$ region is expected to be influenced by perturbative QCD (pQCD) which dictates that the functional dependence in this region is of the form $(1 - x)^\delta$, with $\delta$ determined from fit to the high $x$ data. The non-perturbative calculations in [62, 63, 64, 65], pQCD calculations in [80, 81, 82], and the parton model calculations in [83] all indicate that $\delta \gtrsim 2$.

To study the high $x$ behavior we fit the high $x$ region of the evolved valence quark distribution of the pion at $q = 4.05\, GeV$ using a functional fit of the form:

$$xu_\pi(x; q) = Ax(1 - x)^{\alpha(x)},$$

$$\alpha(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2. \quad (3.114)$$

It should be noted that the exponent in our fit is quadratic in $x$, instead of a constant. This ensures that the functional form is adequately flexible enough to give a good parametrization. The result of the fit is as shown in Fig. 3.12, where we display the distributions for both the full DSE-BSE calculation and the calculation in [62] and their respective fits for $x > 0.7$. For the calculation reported in [62], the fit is very identical to the actual calculated distribution. In the case of the full DSE-BSE calculated distribution, the fit is identical only for $x > 0.85$, with the deviation increasing as $x$ decreases, as can be clearly seen from the fits to the calculation in Fig. 3.13. This is due to our Bethe-Salpeter amplitude: the pQCD behavior sets in at a larger scale than in the Bethe-Salpeter amplitude parametrisation employed in [62] and as shown in Fig. 3.14.

The parameters of the fits are presented in Table. 3.2. Thus our calculation gives a high-$x$ dependence of the form $(1 - x)^{2.41}$ whereas Ref. [62] yields $(1 - x)^{2.32}$, which is as much as
Fig. 3.11: The pion’s valence distributions at $q = 4.05$ GeV. The solid line depicts the evolved distribution using the full DSE-BSE machinery, the dashed line is the evolved distribution using the DSE-first Chebyshev moment of the pion’s Bethe-Salpeter equation, while the dot-dashed line denotes the result by Hecht et al., Ref. [62]. The data, from Ref. [79], have been obtained for invariant $\mu^+ \mu^-$-mass $> 4.05$ GeV and inferred from the differential pion-nucleon Drell-Yan cross section using simple distribution parametrisations.
Table 3.2: Fit parameters of the pion’s valence quark distributions at $q = 4.05$ GeV from (a) the full DSE-BSE calculation and (b) the Hecht et al. calculation [62] respectively, using Eq. (3.114).

<table>
<thead>
<tr>
<th>Fit</th>
<th>A</th>
<th>$\alpha_0$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DSE-BSE</td>
<td>3.29</td>
<td>1.396</td>
<td>0.22</td>
<td>0.79</td>
<td>2.41</td>
</tr>
<tr>
<td>Hecht et al.</td>
<td>1.83</td>
<td>0.84</td>
<td>1.48</td>
<td>0.00005</td>
<td>2.32</td>
</tr>
</tbody>
</table>

expected from the DSE model of the pion and pQCD. This result is presented in Fig. 3.15.

In order to see the improvement of our calculation relative to the earlier calculation by Hecht et al [62], we evolved their result from $q = 4.05$ GeV to $q = 5.2$ GeV. We then compare both results at $q = 5.2$ GeV to the fit by Wijesooriya et al. [78] at $q = 5.2$ GeV, where the pion’s valence quark distribution is parametrized as:

$$x u_v^{\text{mom}}(x; q) = A_v [x^{\eta_1} (1 - x)^{\eta_2} (1 - \epsilon \sqrt{x} + \nu x) + \gamma \frac{2x^2}{9m^2\gamma^2}].$$  \hspace{1cm} (3.115)

Our calculation, the result by Hecht et al., and the fit by Wijesooriya et al (all at $q = 5.2$ GeV) are presented in Fig. 3.16. As apparent from the figure, the fit gives a good overall (global) representation of the data, slightly higher than the data between $0.2 < x < 0.5$ and a slight undershoot for $x > 0.7$. In both regions the severity of the overshoot and undershoot is less in our calculation than that of Hecht et al.. Thus overall, our model is in better agreement with the fit [78] than the result by Hecht et al.. This is even more apparent when we compare the higher moments of the pion’s valence quark distribution, as shown in Table. 3.3 which includes the percentage deviations of our results and Hecht’s from the fit by Wijesooriya et al. Thus, using as inputs the analytical fit to the numerical solution of the quark DSE and the direct numerical solution of the pion Bethe-Salpeter equation, our model gives a better prediction of the pion’s valence quark distribution.
Fig. 3.12: High-\(x\) behavior of pion’s valence quark distribution. The solid line depicts our calculation using the full DSE-BSE machinery, the dot-dashed line is the fit to the DSE-BSE calculation, the dashed line represents the result obtained by Hecht et al. [62] while the dot-double-dashed curve denotes the fit to the Hecht et al.’s result. The data are from Ref. [79]
Fig. 3.13: Subtraction of the pion valence-quark distribution from its fit. Solid line denotes the DSE-BSE calculation, while the dashed line is that of Hecht et al. [62].
Fig. 3.14: The pion amplitude \( E_\pi \). Solid line denotes the numerical solution of BSE with no angle dependence, long-dashed line denotes the chiral limit of \( E \) obtained from 3-complex-conjugate-poles parametrization. The short-dashed line denotes the chiral limit of \( E \) as obtained from the algebraic parametrisation employed in [62].
Fig. 3.15: High-$x$ fit exponent ($\alpha(x)$) from fits to DSE-BSE calculation (solid line) and Hecht et al. [62] (dashed line) respectively.
Fig. 3.16: The pion’s valence-quark distribution at $q = 5.2$ GeV. Solid line depicts the full DSE-BSE, dashed line the fit from Wijesooriya et al. [78], and the dot-dashed line the result from Hecht et al. [62]; The data are from Ref. [79]
Table 3.3: The 4th to 9th moments of the pion valence-quark distribution at $q = 5.2$ GeV from our calculation, result by Hecht et al [62], and the parametrization by Wijesooriya et al. [78] respectively. The last two rows show the percentage deviations of our calculation and that of Hecht et al. respectively from the fit by Wijesooriya et al.

<table>
<thead>
<tr>
<th></th>
<th>$\langle x^4 \rangle_q$</th>
<th>$\langle x^5 \rangle_q$</th>
<th>$\langle x^6 \rangle_q$</th>
<th>$\langle x^7 \rangle_q$</th>
<th>$\langle x^8 \rangle_q$</th>
<th>$\langle x^9 \rangle_q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calc.</td>
<td>0.024</td>
<td>0.015</td>
<td>0.0095</td>
<td>0.0066</td>
<td>0.0047</td>
<td>0.0034</td>
</tr>
<tr>
<td>Hecht [62]</td>
<td>0.020</td>
<td>0.012</td>
<td>0.0075</td>
<td>0.005</td>
<td>0.0035</td>
<td>0.0025</td>
</tr>
<tr>
<td>Fit [78]</td>
<td>0.026</td>
<td>0.017</td>
<td>0.012</td>
<td>0.0086</td>
<td>0.0065</td>
<td>0.005</td>
</tr>
<tr>
<td>$\Delta_{\text{calc.}}$($%$)</td>
<td>11</td>
<td>15</td>
<td>20</td>
<td>24</td>
<td>28</td>
<td>31</td>
</tr>
<tr>
<td>$\Delta_{\text{Hecht}}$($%$)</td>
<td>25</td>
<td>31</td>
<td>37</td>
<td>42</td>
<td>47</td>
<td>50</td>
</tr>
</tbody>
</table>

3.4.3 Valence Quark Distributions in Kaons

We now turn our attention to the valence quark distributions in the kaon ($\zeta \equiv K^+$). The treatment parallels that of the pion, with the only essential difference being that the kaon consists of the $u$ and $\bar{s}$ quarks, as opposed to the $u$ and $\bar{d}$ in the $\pi^+$. Though the $s$-quark is generally considered as being light, it is still heavier than both the $u$ and the $d$ quarks. Also the $s$ quark has a nonzero strangeness quantum number ($S = 1$), whereas both $u$ and $d$ quarks have zero strangeness number ($S = 0$).

While the heavier mass of the $s$ quark has a more direct influence on the calculation of the valence distributions in the kaon, the nonzero strangeness primarily determines its decay properties.

As stated above the kaon has two constituent quarks: $u$ and $\bar{s}$. Thus from Eq 3.32 the $F_2$ structure functions can be expressed as:

$$F_2^u(x; q_0) = \frac{4}{9} x u(x; q_0) = \frac{4}{9} x \left[ u(x; q_0) - \bar{u}(x; q_0) \right],$$

$$F_2^s(x; q_0) = \frac{1}{9} x \bar{s}(x; q_0) = \frac{1}{9} x \left[ \bar{s}(x; q_0) - s(x; q_0) \right].$$

(3.116)
It can also be demonstrated algebraically that

\[
\bar{s}_u^{k^+}(x; q_0) = s_u^{k^-}(x; q_0), \quad (3.117)
\]

\[
u_u^{k^+}(x; q_0) = \bar{u}_u^{k^-}(x; q_0). \quad (3.118)
\]

As in the case of the pion, the calculations are subject to the constraint

\[
\int_0^1 dx u_K(x; q_0) = 1 = \int_0^1 dx \bar{s}_K(x; q_0). \quad (3.119)
\]

This ensures that the \( k^+ \) contains only one \( u \) valence quark and one \( \bar{s} \) valence quark.

The needed inputs for the calculations are the dressed-quark propagators and the kaon’s dressed vertices. For the dressed-quark propagators we use the 3 pairs of complex conjugate mass poles (for \( u/d \) quarks and \( s \) quark) as presented in the second and third blocks of Table. 2.1. The general form of the kaon Bethe-Salpeter amplitude (dressed vertex) can be written as

\[
\Gamma_k(k; Q) = \gamma_5 \left[ i E_k(k; Q) + \gamma \cdot Q F_k(k; Q) + \gamma \cdot k G_k(k; Q) + \sigma_{\mu\nu} k_\mu Q_\nu H_k(k; Q) \right], \quad (3.120)
\]

with the behaviour of the invariant functions constrained to a large extent by the axial-vector Ward-Takahashi identity (WTI). Unlike the pion, the kaon amplitudes are not charge parity eigenstates, therefore \( E_k, F_k, G_k, \) and \( H_k \) have both even and odd Chebyshev moments.

The average momentum-fraction carried by the kaon’s valence quarks at the resolving scale \((q_0 = 0.57 \text{ GeV})\) is

\[
\int_0^1 dx x \left[ u_K(x; q_0) + \bar{s}_K(x; q_0) \right] = 0.76. \quad (3.121)
\]

Since this is less than unity, the remainder is obviously carried by the gluons that are responsible for the binding of the kaon bound state. The moments of the distribution for the valence \( u \) and \( s \) quark are presented in Table. 3.4. Our results indicate that the heavy \( s \)-quark carries a larger fraction of kaon momentum than the light \( u \)-quark, as expected.

In Fig. 3.17 we present the result of our calculation of the kaon’s valence \( u \)-quark distribution, \( xu_K(x; q_0) \), at the resolving scale \( q_0 = 0.57 \text{ GeV} \). We also show the distribution evolved to \( q = 4.05 \text{ GeV} \), as in the case of the pion. This will facilitate a direct comparison of both the pion and kaon \( u_v \) distributions. Fig. 3.17 shows the usual trend of non-singlet evolution: an enhancement at
Table 3.4: The moments of the kaon’s valence-quark distribution at $q_0 = 0.57$ GeV and $q = 4.05$ GeV respectively.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\langle x \rangle_q$</th>
<th>$\langle x^2 \rangle_q$</th>
<th>$\langle x^3 \rangle_q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.57$ (GeV)</td>
<td>$u_K(x; q_0)$</td>
<td>0.365</td>
<td>0.18</td>
</tr>
<tr>
<td></td>
<td>$s_K(x; q_0)$</td>
<td>0.395</td>
<td>0.21</td>
</tr>
<tr>
<td>$4.05$ (GeV)</td>
<td>$u_K(x; q)$</td>
<td>0.222</td>
<td>0.085</td>
</tr>
<tr>
<td></td>
<td>$s_K(x; q)$</td>
<td>0.24</td>
<td>0.097</td>
</tr>
</tbody>
</table>

low $x$ at the expense of a depletion at high $x$. It should be noted that the constituent valence $s$-quark mass is $\tilde{M} = 0.55$ GeV, compared with $\tilde{M} = 0.40$ GeV for the valence $u$ quark.

It is interesting compare the evolved valence $u$ quark distribution in the kaon to that of the pion. In the pion, the spectator quark is a $\bar{d}$, which has the same mass as the $u$ quark. In the kaon, the spectator is a $\bar{s}$ which is heavier than both the $u$ and the $\bar{d}$ quarks. Thus one expects some differences in the valence $u$ quark distributions. This comparison is shown in Fig. 3.18, with the experimental Drell-Yan data for the pion. The kaon distribution is practically the same as the pion for $x < 0.12$, slightly greater for $0.12 < x < 0.48$, and for $x > 0.5$ the $u$-valence distribution in the kaon decreases faster than that of the pion. The pion $u$-quark carries more momentum fraction than the kaon $u$-quark, as can be seen by considering the higher moments, which are presented in Table. 3.5.

In fact, the kaon $u$-quark distribution bears a striking resemblance to the pion $u$-valence distribution employing just the first Chebyshev polynomial in the Bethe-Salpeter amplitude (see Fig. 3.19). This difference with the pion’s valence $u$-quark distribution is probably due to the influence of the larger constituent mass of the spectator $s$ quark. In Fig. 3.20 we present the ratio of the kaon to pion valence $u$-quark distributions $u_K/u_\pi$ at $q = 5.2$ GeV with the existing experimental data in [84]. The authors measured the ratio of $u_K(x)/u_\pi(x)$ using the Drell-Yan process in the scale interval $4.1 \leq q \leq 8.5$ GeV. Our result, as can be seen in Fig. 3.20, is in reasonable agreement with the experimental data.
Fig. 3.17: The kaon’s valence u-quark distribution ($x u_K(x; q_0)$) calculated using the DSE elements and the full kaon’s Bethe-Salpeter amplitude (solid line). The valence s-quark mass is $\tilde{M} = 0.55$ GeV and the resolving scale is $q_0 = 0.57$ GeV = 1/(0.35 fm). The dashed line denotes the evolved distribution, $x u_{K}(x; q = 4.05 \text{ GeV})$. 
Fig. 3.18: Valence u-quark distributions for pions and kaons. The solid line is the pion’s evolved distribution, $xu_\pi(x; q = 4.05\text{ GeV})$, while the dashed line denotes the kaon’s evolved distribution, $xu_K(x; q = 4.05\text{ GeV})$. The experimental data are from Ref. [79].
Fig. 3.19: Comparison of the evolved kaon’s valence u-quark distribution \( xu_K(x; q = 4.05 \text{ GeV}) \) (solid line) to the evolved pion’s valence u-quark distribution using just the DSE-first Chebyshev moment of the Bethe-Salpeter equation (dashed line).
Fig. 3.20: The ratio of kaon to pion valence u-quark distribution $u_K / u_\pi$ at $q = 5.2$ GeV. Data taken from Ref. [84].
Table 3.5: The moments of the kaon and pion valence u-quark distributions at $q = 4.05$ GeV. The last row shows the percentage deviations of kaon u-quark distribution relative to that of the pion.

<table>
<thead>
<tr>
<th></th>
<th>$\langle x \rangle_q$</th>
<th>$\langle x^2 \rangle_q$</th>
<th>$\langle x^3 \rangle_q$</th>
<th>$\langle x^4 \rangle_q$</th>
<th>$\langle x^5 \rangle_q$</th>
<th>$\langle x^6 \rangle_q$</th>
<th>$\langle x^7 \rangle_q$</th>
<th>$\langle x^8 \rangle_q$</th>
<th>$\langle x^9 \rangle_q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_k(x; q)$</td>
<td>0.222</td>
<td>0.085</td>
<td>0.041</td>
<td>0.022</td>
<td>0.013</td>
<td>0.0085</td>
<td>0.0056</td>
<td>0.0039</td>
<td>0.0028</td>
</tr>
<tr>
<td>$u_\pi(x; q)$</td>
<td>0.227</td>
<td>0.09</td>
<td>0.045</td>
<td>0.026</td>
<td>0.016</td>
<td>0.011</td>
<td>0.0073</td>
<td>0.0052</td>
<td>0.0039</td>
</tr>
<tr>
<td>$\Delta(%)$</td>
<td>2</td>
<td>6</td>
<td>9</td>
<td>13</td>
<td>17</td>
<td>20</td>
<td>23</td>
<td>26</td>
<td>28</td>
</tr>
</tbody>
</table>

The calculations of the valence $s$ quark in the kaon at the resolving scale, $x s K (x; q_0)$, and at the evolved scale $q = 4.05$ GeV are shown in Fig. 3.21. It also shows the usual trend for the non-singlet evolution. In the case of the pion the valence $u$ quark is the same as the valence $\bar{d}$ quark to a very high approximation (deviation around 0.1%). Thus we did not show the $\bar{d}$ valence distribution for the pion. For the kaon, the $s$ quark is heavier than the $u$ quark, so the equivalence present in the pion is lost here. We thus expect the valence $s$ quark distribution to be different from the valence $u$ quark distribution, and by implication, the valence $u$-quark distribution in the pion. We now test this expectation.

In Fig.3.22 we display the valence $u$- and $s$-quark distributions, and for a baseline comparison, the pion’s valence distribution, all at the resolving scale $q_0 = 0.57$ GeV. It is apparent from the figure that the valence $s$-quark distribution is different from the valence $u$-quark distributions in both the kaon and the pion. The peak is both larger and the distribution more skewed towards higher $x$.

Fig.3.23 shows the three distributions at the evolved scale $q = 4.05$ GeV. It is identical to the valence $u$-quark distributions for both the kaon and pion for $x < 0.2$, after which it becomes systematically higher than both.
Fig. 3.21: Kaon’s valence s-quark distribution, $x_s K(x; q_0)$. The solid line denotes $x_s K(x; q_0)$ calculated using the DSE elements and the full kaon’s Bethe-Salpeter amplitude. The spectator valence u-quark mass is $\bar{M} = 0.40 \text{ GeV}$ and the resolving scale is $q_0 = 0.57 \text{ GeV} = 1/(0.35 \text{ fm})$. The dashed line is the evolved distribution, $x_s K(x; q = 4.05 \text{ GeV})$. 
Fig. 3.22: Kaon and pion valence quark distributions at the resolving scale $q_0 = 0.57$ GeV. Solid line denotes the kaon’s valence $u$-quark distribution, $xu_K(x; q_0)$, dashed line the kaon’s valence $s$-quark distribution, $xs_K(x; q_0)$, and the dash-dotted line the pion’s valence $u$-quark distribution, $xu_\pi(x; q_0)$, respectively. Note that the valence $d$-quark distribution in the pion, to a very good approximation, equals the valence $u$-quark distribution and is thus not displayed.
Fig. 3.23: Kaon and pion valence quark distributions at the evolved scale $q = 4.05\,\text{GeV}$. Solid line is the kaon’s valence u-quark distribution, $xu_K(x; q = 4.05\,\text{GeV})$, dashed line the kaon’s valence s-quark distribution, $xs_K(x; q = 4.05\,\text{GeV})$, while the dashed-dotted line denotes the pion’s valence u-quark distribution, $xu_\pi(x; q = 4.05\,\text{GeV})$, respectively.
Chapter 4

Conclusion

The physics of the hadrons (mesons and baryons) is amazingly complex and rich, both in the depth of its structure and the wide scope of physical phenomena under its ramifications. These phenomena range from low energy manifestations such as bound states to ultrarelativistic high-energy collisions involving hadrons and astrophysics/cosmology. Hadrons, being bound states of strongly interacting quarks and gluons, are subject to the strong interactions, and should thus be describable by Quantum Chromodynamics (QCD), the gauge field theory of the strong interactions. But QCD is a strongly coupled, non-linear, asymptotically free quantum field theory, and is thus notoriously difficult to solve especially in the low energy regime governing bound state structures. Even in the high energy sector, where, due to asymptotic freedom, perturbative techniques are applicable, calculations of physical processes involving hadrons are long and laborious, and in many cases are known to only the next-to-leading order (NLO) in a perturbative expansion. The situation is worse in the low energy region, where perturbative expansions are clearly inappropriate due the magnitude of the strong coupling constant. Therefore a lot of models have been developed to address different aspects of the physics of low energy hadron structure and interactions. These range from nonrelativistic quantum mechanical models (static quark models), models incorporating some symmetries or features of QCD (“QCD-inspired” models like chiral quark models, topological soliton models, etc), phenomenological models incorporating both hadronic and quark/gluon features (QCD Sum Rules), models of the QCD vacuum (Instanton model and its variants), to fully covariant continuum field theoretical approaches like the Dyson-Schwinger Equations (DSEs).

The most fundamental and rigorous approach to strong QCD is lattice QCD, where the Green’s functions of the theory, which contain all relevant information, are evaluated numerically on a lattice. In principle lattice QCD is rigorous enough to address completely problems in strong QCD, but in practice, lattice QCD is beset with technical issues which limit the current scope of applicability.
Since the approach is numerical in nature, computational costs scale dramatically with the size of the lattice. Even more problematic is the issue of taking the continuum limit of the numerical computations, since QCD is a continuum field theory and lattices are always finite. Thus fully covariant continuum field theoretical approaches complement efforts in lattice QCD, and are useful in investigations where the numerical complexity of a lattice calculation is prohibitive (for instance, lattice QCD can calculate moments of structure functions, and not the structure functions themselves).

We have thus employed the Dyson-Schwinger Equations (DSEs) in the present study. The DSEs are an infinite tower of coupled equations for the Green’s functions of QCD, and thus tractable only when suitably truncated. A popular, symmetry-conserving truncation scheme is the rainbow-ladder truncation of the scattering kernel. Using this truncation, we have carried out investigations in two different aspects of non-perturbative QCD in meson physics. Mesons are structurally the simplest hadrons, and are thus an excellent theoretical "laboratory" to investigate and understand strong QCD. We considered the physics of the transition from perturbative to non-perturbative QCD using current-current correlation functions (correlators) and the valence quark distributions in light pseudoscalar mesons (pions and kaons). Both considerations involve non-perturbative inputs furnished by the rainbow-ladder truncated DSEs.

4.1 Correlation Functions and Onset of Non-perturbative Dynamics

In the first part of this study we addressed the question: at what distance or momentum scale is the onset of non-perturbative dynamics? A characteristic signature of non-perturbative dynamics is dynamical chiral symmetry breaking, DCSB. In the chiral (massless) limit QCD exhibits chiral symmetry, which is broken by mass terms in the QCD Lagrangian. To all orders in perturbation theory the symmetry breaking is proportional to the current masses, and thus vanishes in the chiral limit. Therefore the dynamical breaking of the chiral symmetry and its attendant generation of mass in the chiral limit is a genuinely non-perturbative effect. A measure of the dynamically generated mass scale is given by the vacuum expectation value of the two-quark correlation function, the so-called quark condensates. In general current-current correlation functions, which are expectation
values of a product of currents, are excellent tools to investigate various aspects of non-perturbative QCD.

One such aspect is the scale of the onset of dynamical chiral symmetry breaking, which as we know, is non-perturbative. To accomplish this we considered the vector and axial vector current-current correlators. These correlators can be represented by quark loop diagrams, involving quark propagators and proper vertices. The propagators and vertices are the two-point and three-point functions obtainable from the truncated DSEs, and thus are non-perturbative. It is well known that in the chiral limit, vector and axial vector current-current correlators are identical to any finite order of perturbation theory in QCD. Thus their difference vanishes in perturbation theory. Using the non-perturbative inputs from the DSEs, the difference of these correlators, the so-called V-A correlator, is sensitive to non-perturbative effects. The sum of these correlators behaves as a free correlator for distances of up to 1 fm. Therefore the ratio of the difference and the sum, in coordinate or momentum space, probes the onset of DCSB (and by implication non-perturbative dynamics) in coordinate or momentum space respectively.

Our calculations reveal that the onset of DCSB occurs at a distance scale of \( \sim 0.6 \text{ fm} \) or at a momentum scale of \( \sim 3 \text{ GeV}^{-1} \). These values are comparable to those obtained from the ALEPH tau-lepton decay data and the phenomenology of Instanton models. Our calculation shows a steeper rise of the ratio with distance than the fit to the ALEPH data. This signifies room for improvement in some of the model non-perturbative ingredients required for the evaluation of the ratio.

The V-A correlator is also useful in its own right. Various sum rules can be evaluated from this correlator and associated with physical quantities like the pion decay constant and the electromagnetic component of the pion mass difference. Using the non-perturbative input elements from DSEs in this correlator, our results are in good agreement with available experimental results and also competitive with other models. An important derivative from the V-A correlator is the four-quark condensate, which is a useful input in QCD sum rules analysis of various hadronic processes. Our calculations show that the four-quark condensate is not just the square of the two-quark condensate, as naively assumed by the vacuum saturation (vacuum dominance) hypothesis. The calculated ratio of the four-quark condensate to the square of the two-quark condensate at the employed renormal-
The correlators, in particular the V-A correlator, are quite sensitive to the non-perturbative input elements (dressed quark propagators and the Abelian Ball-Chiu vertex Ansatz for both vector and axial vector vertices). We envisage that an even better description can be obtained by going beyond the rainbow-ladder truncation of the DSEs, and also explicitly using the BSEs solution for the required vertices, instead of using the Ball-Chiu Ansatz. Another possible extension is the study of three-point correlators, essential in describing hadronic form factors. One can also, in principle, employ the same techniques to study baryonic properties using the relevant baryonic correlators.

4.2 Quark Distributions in Pions and Kaons

The second focus of the study, contained in Chapter 3, deals with a subject of current vital interest and application: the determination of the quark (parton) distributions in hadrons. These distributions give the probability density for a quark to carry a fraction $x$ of the momentum of a fast moving hadron. Thus even though the hadron under consideration is highly energetic, these distributions are intrinsically non-perturbative in nature since they involve the bound state structure of the hadron. These distributions, along with fragmentation functions (probability distributions describing the creation of a hadron from a fast moving parton), are the essential ingredients needed in perturbative QCD description of relativistic collisions involving hadrons. Due to their non-perturbative nature, these distributions are generally determined from global fits to experimental data from processes such as the deep inelastic scattering (DIS) and Drell-Yan lepton-pair production. Thus it is currently highly desirable to evaluate the quark distributions from “first principles” using non-perturbative techniques.

The pion is the hadron with the simplest valence quark content, and as the Goldstone boson of the dynamical breaking of chiral symmetry, it is intimately related to the vacuum structure of QCD. It is thus interesting to study how non-perturbative and perturbative QCD dynamics are intertwined in deep inelastic scattering (DIS) of the pion. In chapter 3 we have studied the pion valence-quark distribution using the complex-conjugate mass poles parametrization of the solution of the quark
Dyson-Schwinger equation (DSE) and the numerical solution of the pion Bethe-Salpeter equations. The valence-quark mass is $\hat{M} = 0.4$ GeV at the resolving scale $q_0 = 0.57$ GeV $= 1/(0.35)$ fm. The resulting distributions at $q_0$ are evolved to the experimental scale of the Drell-Yan data from Fermilab E-615. The calculated low moments agree with the values obtained from lattice simulations and from a phenomenological fit. We also compare our results to the result of an earlier calculation carried out by Hecht et al. in which the authors used an algebraic parametrisation of the input DSE elements. At the momentum scale $q = 5.2$ GeV, a fit to the available experimental data has been carried out by Wijesooriya et al.. Evolving the Hecht’s result to that scale, we have shown that our model is in better agreement with the fit by Wijesooriya et al. than the result by Hecht et al.. To study the high $x$ behavior we fitted the high $x$ region of the valence-quark distributions at $q = 4.05$ GeV using the functional form $(1 - x)^{\alpha(x)}$. Our result yields $(1 - x)^{2.41}$ which is as much as expected from the DSE model of the pion and pQCD.

We have also studied the kaon structure function. The valence $u$ and $\bar{s}$-quark distribution in the kaon are obtained in a similar manner using the three pairs of complex conjugate mass poles for the dressed-quark propagators and the numerical solution of kaon BSE, at the resolving scale $q_0 = 0.57$ GeV. We evolved the distributions of $u$ and $\bar{s}$ valence quark to $q = 4.05$ GeV. The heavy $s$-quark carries a larger fraction of kaon momentum than the light $u$-quark, as expected. It is interesting to compare the valence $u$-quark distribution of the kaon to that of the pion. Due to the influence of the larger constituent mass of the spectator $s$-quark in the kaon than $d$-quark in the pion, the kaon $u$ valence-quark distribution decreases faster than that of the pion for $x > 0.5$. The pion $u$-quark carries more momentum fraction than the kaon $u$-quark. The $x$ dependence of the ratio of kaon to pion $u$ valence-quark distributions is found to be in reasonable agreement with the Drell-Yan experiment data.

Our calculations can be improved by using a dressed-quark form for the spectator propagator instead of using a constituent quark mass approximation. Further improvements can also be achieved by going beyond the ladder-rainbow truncation employed in this study. We can also extend this type of investigations to the case of the proton where there is a vast amount of experimental data on the quark distribution. Lastly, it could be interesting to consider applying the same techniques to the
fragmentation functions, since they have a similar structure to the quark distribution functions. A study in this direction is presented in [85].
APPENDIX A

KINEMATICS

We collect the basic relevant formulae in this Appendix, starting with the usual Euclidean space four-vectors.

A.1 Euclidean four-vectors and scalar product

Metric tensor: $g_{\mu\nu} = \text{diag}(1,1,1,1)$

Contravariant 4-vector: $a^\mu = (a_0, \vec{a})$

Covariant 4-vector: $a_\mu = g_{\mu\nu}a^\nu = (a_0, -\vec{a})$

Scalar product: $a^2 = a_\mu a^\mu = a_0^2 + |\vec{a}|^2, \quad a \cdot b = a_\mu b^\mu = a_0 b_0 + \vec{a} \cdot \vec{b}$

Euclidean space 4-dimensional volume element

The momentum 4-vector, $q$, is given by:

$$q = |q|(\cos\phi\sin\theta\sin\beta, \sin\phi\sin\theta\sin\beta, \cos\theta\sin\beta, \cos\beta) \quad (A.1)$$

The integration over momentum 4-space is defined as:

$$\int d^4q = \int_0^\infty q^3dq \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \int_0^\pi \sin^2\beta d\beta. \quad (A.2)$$

If we take the direction of $P_\mu$ to be along the 4th axis, then the integral is independent of $\phi$ and $\theta$.

Thus Eq. A.2 reduces to

$$\int d^4q = 4\pi \int_0^\infty q^3dq \int_0^\pi \sin^2\beta d\beta, \quad (A.3)$$

and with the substitution $z = \cos\beta$ we have

$$\int d^4q = 4\pi \int_0^\infty q^3dq \int_0^1 \sqrt{1 - z^2}dz \quad (A.4)$$
A.2 Gamma Matrices in Minkowski space

Anticommutation relations: \( \{ \gamma^\mu, \gamma^\nu \} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \), \( \{ \gamma^5, \gamma^\mu \} = 0 \)

Definition of \( \gamma^5 \): \( \gamma^5 \equiv i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = -i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \)

Hermitian conjugates: \( \gamma^0 \dagger = \gamma^0 \), \( (\gamma^k)\dagger = -\gamma^k \), \( (\gamma^5)\dagger = \gamma^5 \), \( (\gamma^\mu)\dagger = \gamma^0 \gamma^\mu \gamma^0 \)

Squares: \( (\gamma^0)^2 = -(\gamma^k)^2 = (\gamma^5)^2 = I \), \( k = 1, 2, 3 \) \( I \equiv 2 \times 2 \) identity matrix

Trace of Gamma matrices:

\[
Tr(\gamma_\mu \gamma_\nu) = 4 \delta_{\mu\nu}, \tag{A.5}
\]

\[
Tr(\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta) = 4[\delta_{\mu\nu} \delta_{\alpha\beta} - \delta_{\mu\alpha} \delta_{\nu\beta} + \delta_{\mu\beta} \delta_{\nu\alpha}]. \tag{A.6}
\]

A.3 Light-cone coordinates

Light-cone coordinates are defined by a change of variables from the usual \((t, x, y, z)\) or \((0, 1, 2, 3)\) coordinates. Given a vector \(V^\mu\), its light-cone components are defined by

\[
V^+ = V^0 + V^3, \quad V^- = V^0 - V^3, \quad \vec{V}^T = (V^1, V^2), \tag{A.7}
\]

and \(V^\mu = (V^+, V^-, \vec{V}^T)\). Thus for example,

\[
\gamma^+ = \gamma^0 + \gamma^3. \tag{A.8}
\]

Some authors prefer to include the \(1/\sqrt{2}\) factor in Eq. (A.7). It can easily be verified that Lorentz invariant scalar products have the form

\[
V \cdot W = \frac{1}{2}(V^+ W^- + V^- W^+) - \vec{V}^T \cdot \vec{W}^T, \\
V \cdot V = V^+ V^- - (V^T)^2. \tag{A.9}
\]

What are the motivations for defining such coordinates, which evidently depend on a particular choice of the \(z\) axis? One is that these coordinates transform very simply under boosts along the \(z\)-axis. Another is that when a vector is highly boosted along the \(z\) axis, light-cone coordinates nicely
show what are the large and small components of momentum. Typically one uses light-cone coordinates in a situation like high-energy hadron scattering. In that situation, there is a natural choice of an axis, the collision axis, and one frequently needs to transform between different frames related by boosts along the axis. Commonly used frames include the rest frame of one of the incoming particles, the overall center-of-mass frame, and the center-of-mass of a partonic subprocess.

A.4 Euclidean Space Formulation

Lattice gauge theory and the numerical solution of DSEs are typically formulated in Euclidean space.

Our Euclidean space conventions are as follows: 1) we use a non-negative metric for Euclidean four-vectors

\[ a \cdot b = \delta_{\mu\nu} a_\mu b_\nu = \sum_{i=1}^{4} a_i b_i \]  

(A.10)

where \( \delta_{\mu\nu} \) is the Kronecker delta; 2) our Dirac matrices are hermitian and satisfy the algebra

\[ \{ \gamma_\mu, \gamma_\nu \} = 2 \delta_{\mu\nu} ; \]  

(A.11)

and we have

\[ \gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 \]  

(A.12)

so that

\[ \text{tr}[\gamma_5 \gamma_\lambda \gamma_\mu \gamma_\nu \gamma_\rho] = 4 \epsilon_{\lambda\mu\nu\rho} \]  

(A.13)

where \( \epsilon_{\lambda\mu\nu\rho} \) is the completely antisymmetric Levi-Civita tensor in \( d = 4 \) dimensions. One realisation of this algebra is

\[ \gamma_4^E = \gamma^0 \quad \text{and} \quad \gamma_j^E = -i\gamma^j, \quad j = 1, 2, 3, \]  

(A.14)

where \( \gamma^0 \) and \( \gamma^j \) can be any one of the commonly used Minkowski space representations of the usual Dirac algebra. We note that with these conventions a spacelike four-vector, \( p_\mu \), has \( p^2 > 0 \).
A straightforward transcription procedure can be employed to determine the action for the Euclidean field theory that corresponds to one formulated in Minkowski space:

\begin{align}
\int^M d^4x^M &\to -i \int^E d^4x^E, \\
\phi &\to i\gamma^E \cdot \partial^E, \\
A &\to -i\gamma^E \cdot A^E, \\
A_{\mu} B^\mu &\to -A^E \cdot B^E,
\end{align}
(A.15) (A.16) (A.17) (A.18)

where, as usual, \( A \) represents \( g_{\mu\nu} \gamma^\mu_M A^\nu_M \). These transcription rules can be used as a blind implementation of an analytic continuation in the time variable, \( x^0 \): \( x^0 \to -ix^4 \) with \( \vec{x}^M \to \vec{x}^E \), etc.

In studying DSEs it has been commonplace to obtain DSEs from the Minkowski space generating functional and then use transcription rules in momentum space [i.e., \( k^0 \to -ik_4 \) and \( \vec{k}^M \to \vec{k}^E \)]

\begin{align}
\int^M d^4k^M &\to i \int^E d^4k^E, \\
\vec{k} &\to -i\gamma^E \cdot k^E, \\
k_\mu q^\mu &\to -k^E \cdot q^E, \\
k_\mu x^\mu &\to -k^E \cdot x^E,
\end{align}
(A.19) (A.20) (A.21) (A.22)

[the last of which appears in four-dimensional Fourier transforms and follows since \( x^0 \to -ix^4 \) and \( \vec{x}^M \to \vec{x}^E \)] which are analogues of Eqs. (A.15)-(A.18), to obtain equations in Euclidean metric which are assumed to be the Euclidean space counterparts of the original equations. This approach is often argued to be connected with the “Wick Rotation” (Wick, 1954) and is discussed in most text books in association with dimensional regularisation. [the inverse of the free-fermion propagator is \( (i\gamma \cdot k + m) \).] As remarked in Itzykson and Zuber (1980, pg. 485), this procedure is easy to justify in perturbation theory when one neglects the possibility of dynamically generated singularities in the first and third quadrants of the complex \( p^0 \) plane, however, in connection with nonperturbative studies this simple assumption is highly nontrivial and in model studies is often incorrect.
A.5 Short-distance behavior of correlator in x space

The Fourier transform of the free boson propagator is [19]

$$\Delta(x) = \int \frac{d^4P}{(2\pi)^4} e^{iP.x} \frac{1}{P^2 + m^2} = \frac{1}{4\pi^2 x} \int_0^\infty dl J_1(lx) \frac{l^2}{l^2 + m^2} = \frac{m}{4\pi^2 x} K_1(mx), \quad (A.23)$$

where $K_1$ is the modified Bessel function of the second kind. The Fourier transform of the free fermion propagator is

$$S(x) = \int \frac{d^4P}{(2\pi)^4} e^{iP.x} \frac{m - i\gamma.P}{P^2 + m^2} = (m - \gamma.\partial)\Delta(x) = \frac{m^2}{4\pi^2 x} [K_1(mx) + \frac{\gamma.x}{x} K_2(mx)] \quad (A.24)$$

$$K_n(x)|_{x \to 0} = \frac{1}{2} \Gamma(n)(\frac{x}{2})^{-n} \quad (A.25)$$

From Eq. A.24 and Eq. A.25 the free quark propagator at $x \to 0$ is

$$S_0(x)|_{x \to 0} = \frac{m}{4\pi^2 x^2} + \frac{\gamma.x}{2\pi^2 x^4} \quad (A.26)$$

In chiral limit

$$S_0(x)|_{x \to 0} = \frac{\gamma.x}{2\pi^2 x^4} \quad (A.27)$$

At short distance, asymptotic freedom implies that the correlators are determined by free quark propagation.

$$\Pi^0_{\mu\nu}(x) = tr[\gamma_\mu S_0(x) \gamma_\nu S_0(-x)] = \frac{x^2 \delta_{\mu\nu} - 2x_\mu x_\nu}{\pi^4 x^8}, \quad (A.28)$$

$$\Pi^0_{\mu\nu}(x) = (\delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}) \frac{1}{\pi^4 x^6}. \quad (A.29)$$

This shows that at short distance the correlation functions behave as

$$\Pi(x) \sim \frac{1}{x^6} \quad (A.30)$$

Fourier transform to momentum space is written as

$$\int d^4x e^{iP.x} \Pi_{\mu\nu}(x) = \Pi(P^2)(P^2 \delta_{\mu\nu} - P_\mu P_\nu), \quad (A.31)$$
\[\Pi_{\mu\nu}(P) = \int d^4x e^{iP \cdot x} (x^2 \delta_{\mu\nu} - 2x_\mu x_{\nu}) \frac{1}{\pi^2 x^8}\]

\[= \int d^4x e^{iP \cdot x} \frac{1}{12\pi^4} \left\{ \frac{\partial}{\partial x} \frac{\partial}{\partial x} \delta_{\mu\nu} - \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_{\nu}} \right\} \frac{1}{x^4},\] (A.32)

therefore

\[\Pi_{\mu\nu}(P) = (P^2 \delta_{\mu\nu} - P_\mu P_{\nu}) \int d^4x e^{iP \cdot x} \left[ \frac{-1}{12\pi^4 x^4} \right].\] (A.33)
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