A STUDY OF PRE-SERVICE ELEMENTARY TEACHERS’ CONCEPTUAL UNDERSTANDING OF INTEGERS

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by

Carol J. Steiner

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A dissertation written by

Carol J. Steiner

B. A., University of Akron, 1971
M. Ed., Kent State University, 1974
Ph.D., Kent State University, 2009

Approved by

___________________________, Co-director, Doctoral Dissertation Committee
Genevieve A. Davis

___________________________, Co-director, Doctoral Dissertation Committee
Michael Mikusa

___________________________, Member, Doctoral Dissertation Committee
Laura Smithies

Accepted by

___________________________, Interim Director, School of Teaching,
Alexa L. Sandmann Learning, and Curriculum Studies

___________________________, Dean, Graduate School of Education,
Daniel F. Mahoney Health, and Human Services
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Co-Directors of Dissertation: Genevieve A. Davis, Ph.D.
              Michael Mikusa, Ph.D.

The purpose of this qualitative study was to examine how pre-service elementary teachers’ conceptual understanding of integer addition and subtraction understanding is impacted by the use of a novel teaching model. Two models currently exist for teaching integers: the number line model, which emphasizes ordinality; and the neutralization model, which emphasizes cardinality. The novel model incorporated both ordinality and cardinality.

Seventy-nine pre-service teachers took the original survey during the fall semester of 2007 and six of these students made up the sample that was used for this study. All of these students were chosen from a mathematics content course that is required of elementary education majors.

The research design was a blend of a phenomenological study and a teaching experiment. Data was collected using a survey, videotapes of four interview sessions for each pair of participants, and written material provided by the participants. NVivo 7, a qualitative software program, was used to help organize the data.

This study showed that the novel model helped participants to better understand which numbers make up the set of integers and the novel model aided the participants’ understanding of the algorithms for addition and subtraction of integers.
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# TABLE OF CONTENTS

ACKNOWLEDGEMENTS.............................................................................................................. iv

LIST OF TABLES.......................................................................................................................... ix

CHAPTER I: INTRODUCTION ...................................................................................................... 1
  Statement of the Problem ........................................................................................................... 2
  Why Students have Difficulty with Integers ........................................................................... 6
  Models Used for Addition and Subtraction of Integers ............................................................ 7
  Types of Models ......................................................................................................................... 8
  Relating Models to Integers and Integer Operations ............................................................... 10
  Key Misconceptions of Models ................................................................................................. 11
  Overview of this Study .............................................................................................................. 12
  Significance of the Study .......................................................................................................... 13
  Research Questions ................................................................................................................... 15

CHAPTER II: REVIEW OF THE LITERATURE .......................................................................... 16
  Introduction ............................................................................................................................... 16
  History of Zero and the Set of Integers ..................................................................................... 16
  Constructivism in Mathematics Education .............................................................................. 19
  The Role of Representation ...................................................................................................... 24
  Models Used to Understand Integers ...................................................................................... 28
  Studies Involving Integer Addition and Subtraction of Integers ............................................ 34
  Disadvantages of the Two Models Used for Integers ............................................................. 40
  Other Obstacles to Understanding Integers ............................................................................ 44
  Common Mistakes Made When Adding and Subtracting Integers ........................................ 48
  Summary ................................................................................................................................... 50

CHAPTER III: METHODOLOGY ............................................................................................... 52
  Introduction ............................................................................................................................... 52
  Behaviorism and Constructivism .............................................................................................. 52
  Research Methods in Mathematics Education .......................................................................... 53
  Teaching Experiment Methodology ......................................................................................... 55
  Phenomenology as a Research Methodology ........................................................................... 60
  Design of the Study ................................................................................................................... 65
  Pilot Studies .............................................................................................................................. 66
  The Present Study .................................................................................................................... 67
    Selection of the Sample .......................................................................................................... 69
Protocol for Each Session .................................................................................. 69
Tools and Data Gathering .................................................................................. 70
The Survey Instrument ....................................................................................... 76
Validity .................................................................................................................. 79
Reliability ............................................................................................................. 81
Timeline for the Study ........................................................................................ 81
Limitations of the Study ....................................................................................... 84

CHAPTER IV: RESULTS ....................................................................................... 88
Introduction .......................................................................................................... 88
The Initial Survey .................................................................................................. 95
The Novel Model for Integers ............................................................................. 96
Data to Support Research Question Number One ............................................ 101
  Data from the Initial Survey ........................................................................... 101
  Data from Interviews ....................................................................................... 105
  Summary of Responses that Informed Research Question Number One ....... 121
Data to Support Research Question Number Two ........................................... 127
  Introduction ...................................................................................................... 127
  Addition of Integers ......................................................................................... 127
  Subtraction of Integers .................................................................................... 141
Data to Support Research Question Number Three ...................................... 164
  Introduction ...................................................................................................... 164
  Addition of Integers ......................................................................................... 164
  Subtraction of Integers .................................................................................... 188
Analysis of Individuals ....................................................................................... 196
Misconceptions and Use of Rote Procedures .................................................. 196
Participants' Perceptions of the Novel Model .................................................. 203
Summary of Major Findings .............................................................................. 205

CHAPTER V: ANALYSIS ..................................................................................... 207
Introduction .......................................................................................................... 207
The Role of Representation in the Reform Movement in Mathematics Education.. 207
Overview of the Results ...................................................................................... 210
Results of this Study ............................................................................................ 211
  Students Cannot State What Elements Make up the Set of Integers ............. 212
  Student Examples of Where Integers are Used .............................................. 215
  Models Used to Represent Integers ................................................................. 216
  Initial Examples for Addition and Subtraction of Integers ......................... 230
  Student Difficulty with Addition and Subtraction of Integers ...................... 232
  Student Misconceptions about Addition and Subtraction of Integers ......... 233
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Description of Each of the Four, Thirty-Minute Interview Sessions</td>
<td>83</td>
</tr>
<tr>
<td>2. A Description of the Sixteen Codes Used in the Study</td>
<td>91</td>
</tr>
<tr>
<td>3. Students’ Perception of Where Integers are Used</td>
<td>102</td>
</tr>
<tr>
<td>4. Student Reasons Why $-2$ is Greater than $-5$</td>
<td>104</td>
</tr>
<tr>
<td>5. Incorrect Student Responses to “How Much Greater Than $-4$ is $9$?”</td>
<td>104</td>
</tr>
<tr>
<td>6. Participant Indication of Understanding of Integers</td>
<td>126</td>
</tr>
<tr>
<td>7. Student Responses to Integer Addition Problems on Initial Survey</td>
<td>129</td>
</tr>
<tr>
<td>8. Results from Initial Survey Concerning Subtraction of Integers</td>
<td>142</td>
</tr>
</tbody>
</table>
CHAPTER I: INTRODUCTION

Research has shown that students do not understand addition and subtraction of integers at a conceptual and logical level (Shore, 2005; Bolyard, 2005; Wilkins, 1996; Ferguson, 1993; Lytle, 1992). Instead, they have learned procedures that will guide them to correct solutions to problems. Peled, Mukhopadhyay, and Resnick (1988) found, however, that children construct internal representations of negative numbers before they have formal school instruction about them. According to Carraher (1990), children and adults can add two negative integers without previous instruction. They tend to reason just as they would with counting numbers and then insert the negative sign in their response. These self-taught algorithms and problem-solving strategies are based on life experiences (Carpenter, Hiebert, & Moser, 1982). Once students learn procedures in school, these are seen as different from the mathematics that they formulated for themselves. They do not see the connection between their problem solving and “school math”. Therefore, the two become separated and understanding is lost. To overcome this disconnect, it may help for the teacher to have various children show their method for finding the solution to a given problem. This would emphasize the fact that problems can be solved in a variety of ways and the best way is the one that makes sense to the individual problem solver. Students need to understand that the emphasis is on the process, not the final result. There are many ways of thinking about the solution to a
problem and as long as what the child says is not incorrect mathematically, he should be permitted to use his method until he adopts a more efficient method. Connections should be made between the math that is learned in school and the math that is used in solving daily problems. In order to explore student understanding of integers and the operations of addition and subtraction with integers, this researcher decided to use a qualitative study. The goal of the study is to use students’ intuitive sense for integers to develop a model that will bridge the gap between what is learned in school and these intuitive notions about integers.

The purpose of this study is to investigate how the use of a novel model can be used to help students conceptually understand addition and subtraction of integers. This model will exhibit features of the number line model and the neutralization model. In this manner, the concept of cardinality is addressed as well as the concept of ordinality. This study also will examine how this model influences and supports a student’s ability to understand and interpret the procedures involved in adding and subtracting integers.

Statement of the Problem

It has been well documented in the research that students have difficulty with integer operations. According to the Second Mathematics Assessment of the National Assessment of Educational Progress (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981), 75% of 13-year olds correctly added two negative integers but only about 43% correctly added a negative integer and a positive integer. Only 27% of the 13-year olds could correctly subtract with positive and negative integers. Looking at 17-year olds, it
was found that 80% could correctly add two negative integers and about 78% could correctly add a positive integer and a negative integer. Approximately 50% could correctly subtract with positive and negative integers (Carpenter et al. 1981). Students are usually able to verify their answers for addition problems involving integers because they only need to determine whether they have more positives or more negatives. For subtraction problems dealing with integers, most students just memorize the rule about changing signs. It is suggested that students who are unsuccessful are those who have forgotten the rule and they have no physical model to help them (Lytle & Avraam, 1990).

Students need to be proficient with operations on integers in order to be successful in advanced mathematics courses since “this proficiency forms the basis for much of advanced mathematical thinking, as well as the understanding and interpretation of daily events” (National Research Council, 2001, p. 247). Integers are needed, not only in algebra and advanced mathematics, but also in such daily activities as balancing a checkbook, understanding temperatures, and keeping score when playing some games.

Students of all ages find that operations with integers are difficult. In many cases, students try to use previous strategies that they used for whole number addition and subtraction when solving problems with integers. For example, students may overlook the sign that indicates direction (“-”) in a problem such as -3 + 7 and get 10 as the answer because they add the magnitudes of the integers, without regard to direction. Davis, McKnight, Parker, and Elrick (1979) looked at these as frames of processing and
identified five such frames. These frames of processing allowed the researchers to predict 68% of the errors on a test involving integer addition and subtraction.

Hativa and Cohen (1995) pinpoint three reasons for the difficulty that students have with integers and integer operations. First, there is a conflict between what children first learn about magnitude of numbers and the idea of negative numbers. Since negative objects do not exist, children cannot identify the magnitude of negative integers as they can with counting numbers. Second, the same sign, “−” is used to denote two different ideas when integers are discussed. It is used to denote direction, as in (−3), and also it is used to show the operation of subtraction. Third, there does not exist a realistic, familiar model that can be used to model operations involving integers.

According to Hativa and Cohen (1995), there are five types of problems and misconceptions about integers. In the first type, students have trouble subtracting a positive number from zero. They tend to think that you cannot take something from nothing so zero will be the answer. In reality we cannot see a negative number of things. However, we can think of debt as a negative amount of money and assets as a positive amount of money. These concepts do exist in daily life and should be related to integers. The second type of problem deals with subtracting a positive number from a lesser positive number. For example, students tend to solve 3 – 8 in the same way that they would solve 8 – 3. The third type of difficulty occurs when students add two negative integers. They tend to think that −4 + (−5) should be −1 because they assume that if there is a “−” they need to subtract. The fourth type of misconception can be found in
problems where one adds and subtracts the same amount. As an example, a student employing this type of misconception would add \(-5 + 5\) to get 10 because when they see an addition sign, they want to add magnitudes. The fifth type of misconception could be illustrated by a student adding a positive number to a negative number, where the negative number has a greater absolute value. For example, a student might add \((-8)\) to 3 to get 5 instead of \((-5)\). These misconceptions emphasize the fact that to be successful with integer operations, students must have some sense of negative numbers (Bell, 1983).

Many students experience difficulty with beginning algebra due to their lack of understanding of integers. “The extension of the numerical domain from natural numbers to integers during the process by which twelve to thirteen year old students acquire algebraic language constitutes an essential element for achieving algebraic competence in the resolution of problems and equations” (Gallardo, 2002, p. 171). In algebra, students deal with negative numbers as coefficients, constants, and as solutions to equations. Without a good algebra background, other areas of advanced mathematics cannot be conceptually understood.

The National Council of Teachers of Mathematics (NCTM), in their publication, *Principles and Standards for School Mathematics* (NCTM, 2000) recommends that three activities be included in the fifth through eighth grade curriculum to provide students with a better understanding of negative integers. The first activity requires the inclusion of situations in which students understand that negative numbers are logical extensions of the positive numbers. In this sense, students should solve problems of the form \(x - y\),
where \( x \) is less than \( y \) and both \( x \) and \( y \) are positive integers. The second type of activity requires that students compare two negative integers to determine how they are similar to and how they are different from positive integers. This should help students understand order within the negative integers. In this way, students could determine the significance of magnitude of integers. The third type of activity allows students to determine that operations with negative integers are just extensions of the properties and operations with counting numbers.

Why Students have Difficulty with Integers

Students learn about numbers as they count objects. Working with negative numbers in the form of symbols forces students to free themselves of the concrete meaning of words that are contained in real life problems (Gallardo, 2002). For example, a student can imagine twenty feet below sea level or twenty steps backward, but \((-20)\) has little or no meaning; it is an isolated bit of information. Positive integers and zero do not pose these problems because students see positive integers as counting numbers and they try to think of zero in the same way that they did with whole numbers. To them zero means “having nothing” (Levenson, Tsamir, & Tirosh, 2007; Lytle, 1992). To be successful with integer operations, students need to understand that zero can be represented many different ways. Since zero is the identity element for addition, zero may be added to any number without changing the number’s value. Zero is represented as the same number of positives as negatives and as long as the number of positives added is the same as the number of negatives added to a number, the value of the number does not
change. Seeing zero in many ways, such as two neutral pairs, is a different meaning than having nothing and this can be confusing to students. Until a student encounters negative integers, zero has always meant the same thing as “nothing”.

Models Used for Addition and Subtraction of Integers

“The ways in which mathematical ideas are represented are fundamental to how people can understand and use ideas” (Goldin, 2003). Representation is both a process and a product. Representations can be internal or external. Those that are internal exist in the mind of the individual and can be represented externally though the use of signs and symbols (Goldin & Shteingold, 2001). “The fundamental goals of mathematics education include representational goals: the development of efficient (internal) systems of representation in students that correspond coherently to, and interact well with, the (external) conventionally established systems of mathematics” (Goldin & Shteingold, 2001, p. 3). English (1997) states that reasoning in mathematics begins with concrete experiences and observations which the learner then draws upon to “transform into models for abstract thought” (p. 4).

Physical models help to make abstract ideas more concrete, allow new knowledge to relate to previous concepts, enhance insight into new ideas, increase motivation, and increase student achievement (Fennema, 1972; Parham, 1983; Sowell, 1989; Suydam & Higgins, 1977). Models have been used to help students gain a better understanding of integers and integer operations. Models should be chosen that are motivating to students and these models should connect mathematics to the world in which the learner is
immersed. In this way, abstract ideas are put into a more concrete form to which students can relate. Models also help teachers to reach students with different learning styles and allow teachers better tools for assessment and meeting individual needs of students (Ross & Kurtz, 1993).

Types of Models

There are two basic types of models that are used to build an understanding of integers and the operations performed on them. One type of model is the number line in which zero serves as the separator between the positive integers and the negative integers. The positive integers are written either above (on a vertical number line) or to the right (on a horizontal number line) of zero and the negative integers are written either below or to the left of zero.

When modeling addition of integers using a number line, adding a positive amount is shown as a movement to the right (or above) and addition of a negative amount is shown as a movement to the left (or below). Subtraction can be modeled by finding the missing addend in going from the given addend to the sum.

The other type of model for integers is the neutralization model. For this type of model, two-color counters are used most frequently. One side of each counter is yellow and the other side is red. Yellow represents positive integers and red represents negative integers. The most important idea inherent in this model is the fact that a number of one color of counters neutralizes the same number of the other color of counters to produce zero. The student must understand that when pairs of opposite colors of counters are
added to a quantity, the value of the original counters remains unchanged (Van de Walle, 2004). For example, if a set contains ten positive (yellow) counters and twelve negative (red) counters, ten pairs of zero are formed. Each pair consists of a positive and a negative. The final result is the same as having two negative counters. In the same manner, if two positives and two negatives join a set, the cardinality of the set is not changed.

When working with two-color counters, sometimes students get confused with the idea of neutralization. Since they see some number of counters, it is difficult for them to understand how this could represent the same amount as having nothing. The idea that zero can be represented in multiple ways is fundamental to understanding this model.

Van de Walle (2004) claims that students need experience with both the number line model and the neutralization model in order to make sense of integers and integer operations. Cardinal conception of number means that the number represents a given number of objects, while ordinal conception of number means that a given number represents a position relative to other numbers. The number line model emphasizes ordinal understanding of number while the neutralization model emphasizes cardinal understanding of number. Since children usually learn to count numbers of objects before understanding their order or position, cardinal understanding of number comes before ordinal understanding of number (Wilkins, 1996). For this reason students should be introduced to the neutralization model before the number line model (Birenbaum &
Tatsuoka, 1981; Liebeck, 1990). However, in order to completely understand integers, one must understand both ordinal and cardinal meanings of number (Davidson, 1987).

Relating Models to Integers and Integer Operations

Concrete representations contribute to the development of mathematical ideas that are well grounded and interrelated (Stein & Bovalino, 2001). In order for models to have their greatest potential for contributing to learning, students must communicate their thought processes as they work with the models (Robitaille & Travers, 1992) and they should reflect on and justify solutions to problems (Clements & McMillen, 1996).

After the introduction of concepts with a model, students perform operations on numbers instead of concrete objects. There is disagreement as to student reliance on models. Some teachers believe that models should be used to introduce the idea of integers and then students should learn the abstract rules; others believe that the model should be referred to for all problems (Janvier, 1983).

Care must be taken to relate models to concepts (Kaplan, Yamamoto, & Ginsburg, 1989). Students should be encouraged to go beyond the concrete stage of understanding but enough time should be spent with the models to make sure that the student understands the concepts that the model is trying to portray. In too many cases the student is not given enough exposure to the model before he is expected to conceptualize and abstract rules. When this happens, the student is left only with rules that do not make sense (Shore, 2005). The purpose of the model is to allow the student to build a mental representation that resembles the physical model (Wilkins, 1996). Whatever model is
chosen, operations involving integers should be consistent with operations on whole numbers so that students’ previous understanding of number and operations are supported and extended.

It is important to note that models in and of themselves will not teach the concepts of integers and operations on integers. Factors that determine the effectiveness of a model include appropriateness of the model for the concept to be learned; the teacher’s understanding of how the model works and how this relates to the concept; and the amount of emphasis that is placed on understanding as opposed to just being able to perform a procedure (Clements & McMillen, 1996).

In summary, models should be chosen that allow a student to represent a situation in a way that he understands. Representation is fundamental to the study of mathematics (NCTM, 2000). Finding the best representation for a student to understand topics in mathematics is the key to success.

**Key Misconceptions of Models**

When models are used correctly, they lead to increased understanding of concepts and better academic achievement in mathematics. However, as Ernest (1985) points out, if models are used incorrectly they can force students to engage in rote learning of procedures without connections to mathematical ideas (Clements & McMillen, 1996). For students who have misconceptions about topics in mathematics, it has been shown that it is difficult to change students’ misconceptions (Mestre, Gerace, Hardiman, & Lockhead, 1987).
One of the difficulties of working with the number line is the fact that students must understand that it is the unit between the hash marks that is to be considered, not the hash marks themselves (Carr & Katterns, 1984). By counting the hash marks a student will end up with one more (or one less) than anticipated. It is assumed that students know how to use a number line because it was used when they learned operations with the whole numbers. However, the “number line model does not have any compelling inner logic. Instead it assumes familiarity with underlying representational conventions, which are to some extent arbitrary” (Ernest, 1985, p. 418).

Overview of this Study

Research has been done with elementary and middle school children using integers. Most of the studies were quantitative studies that used pre-tests and post-tests to determine the effectiveness of models or teaching methods. Results from the quantitative studies indicate that children in the experimental group did better on the post-test than those children in the control group. Some of these studies examined the effectiveness of different types of models. The qualitative studies that were done indicate that children are familiar with negative integers in the context of temperature or games but were unfamiliar with how to add and subtract with integers. Since most of the research about integers has been done with young children, more qualitative research needs to be done with adult learners, especially pre-service elementary teachers, to shed more light on how students develop concepts for these ideas.
For students to develop meaning for integers there needs to be some intuitive feeling for negative integers (Bell, 1983). Students need to experience situations with a relative zero such as above and below sea level, “going in the hole” in card games, having a golf round that is “below par”, relating assets and debt, and dealing with negative temperatures in the winter time.

This dissertation reports of the use of a novel model that combines the important features of the number line model with those of the neutralization model. This novel model also relates to familiar “everyday” situations that are experienced by college students such as credit and debt. This model will be used with pre-service elementary teachers to determine its impact on their understanding of integers and integer operations. Since these pre-service teachers will impact many future students, it is especially important that they have a deep conceptual understanding of integers and are able to add and subtract with integers in significant, logical ways.

Significance of the Study

Students need a firm understanding of the mathematics that they learn. Whatever degree of meaning we want students to have, it cannot be learned all at once (Brownell, 1947/2004). In the past, students have been forced to memorize rules for operations with integers and we find that they can only perform the mathematics learned in situations that are relatively similar to those in which they first learned them. They often get confused about which rule to follow and resort to their instincts to solve problems dealing with integers (Bolyard, 2005; Ferguson, 1993).
At a young age students are taught to count and relate each number with an object. They develop a one-to-one correspondence between word name and object counted. There are many models for operations with counting numbers and fractions. These models should make sense to students so they can remember the rules that are generalized for carrying out the operations (Hiebert & Carpenter, 1992). When students encounter negative numbers, the student can no longer “see” the numbers. Models are used that can only be used in abstract ways to relate negative integers to what the student has previously learned. These models include the number line and neutralization models which are represented by positive and negative charges, two-color counters, helium balloons, and computer environments (Battista, 1983; Liebeck, 1990; Reeves & Webb, 2004). The type of model that is needed is one that students can understand and that can be related to everyday activities.

Students are often introduced to integers and the operations of addition and subtraction before they have had any experience with what an integer really is. Teachers do not always relate integers to what students have already experienced such as temperature or being above or below sea level. Even if students can relate integers to these situations, they cannot see the “positiveness” or “negativeness” of the situation (Bell, 1983; Hackbarth, 2000; Werner, 1973). They also get confused because the same signs (“+” and “−”) are used to indicate direction and the operations of addition and subtraction (Shawyer, 1985).
Although the application of rules for addition and subtraction of integers allows students to get correct answers, students often do not comprehend why these rules work. Research has shown that students sometimes have difficulty remembering the rules, especially for the operation of subtraction (Bolyard, 2005; Wilkins, 1996). If students have a model to show why a problem yields a given solution, they are more likely to relate the model to the procedure.

This study attempts to allow students to see a concrete model that somewhat mimics the operations of addition and subtraction of whole numbers and relates them to the operations of addition and subtraction with integers. The model relates positive and negative integers to assets and debt, topics that are familiar to most college students.

Research Questions

The following research questions guided this research:

1. How do pre-service elementary teachers interpret and make sense of integers?
2. Does the use of a novel model impact student understanding of addition and subtraction of integers? If so, how?
3. Do pre-service elementary teachers relate the use of a novel model for addition and subtraction of integers to the rule-based procedures that they use to add and subtract integers? If so, how do they develop meaning for these relationships?
CHAPTER II: REVIEW OF THE LITERATURE

Introduction

This chapter will review the literature about integers and the models used to help students understand integers. First, the chapter will review the historical development of integers. The next section will describe the constructivist movement in mathematics education. The third section will discuss the importance of representation as it relates to constructivism and the final section will examine the models that are used for understanding integers and integer operations.

History of Zero and the Set of Integers

The set of integers is made up of the set of natural numbers, zero, and the opposites of the natural numbers. When one considers the set of integers, it is worthwhile to look at the history of zero and the negative integers. The development and acceptance of zero and the negative integers took a long period of time.

Although they did not have a symbol for zero, the Babylonians, around 1700 B.C., used a space between symbols when a placeholder was necessary. Later, about 300 B.C., they used a double wedge (YY) to act as a placeholder. The Babylonians never used their symbol that represented zero to mean an absence of something. It was only used as a placeholder. The Hindus, in the seventh century, were the first to use the symbol “0” for zero. Because of trading, use of the symbol “0” spread to Arabian
countries. Eventually, the use of the “0” spread to western Europe in the twelfth century (Gundlach, 1989). Zero was not needed in most practical situations, so only mathematicians who were interested in operations and solving equations used it.

Integers were first used by the Chinese about 200 BC. They used red rods to represent positive integers and black rods to represent negative integers. Later, a slash was put through the last non-zero digit of the numeral to indicate that it was negative (Musser, Burger, & Peterson, 2008). Chinese rules for adding and subtracting were simple. When adding two numbers with the same sign, add the absolute values together and apply the common sign. When subtracting, a positive subtracted from nothing gives a negative; a negative subtracted from nothing yields a positive (Yan & Shiran, 1987). The first book that mentions negative numbers was *Nine Chapters on the Mathematical Art* that was written during the Han period, which was from 200 BC until 200 AD (M. Gardner, 1989). In India about 620 AD, the negative integers were used to show debts and the work of Brahmagupta was the first to provide written evidence of negative integers. Brahmagupta used negative integers to produce the quadratic formula that we use today (Yan & Shiran, 1987). The Hindus used a dot or small circle above or next to the numeral to indicate negative integers (Musser, Burger, & Peterson, 2008). For example, they would write $\bullet 4$ to represent $-4$. The Hindus were the first to use positive and negative numbers to indicate credits and debts. They were also the first to state that every positive number had a negative square root, as well as a positive square root (M. Gardner, 1989).
Chuquet wrote the first European book that mentioned negative numbers in the fifteenth century (Gallardo, 2002). The symbols “+” and “-” were used in this book but they referred to surpluses and deficits in business problems, rather than the operations of addition and subtraction or as they relate to positive and negative integers. In Italy in the sixteenth century, Cardano recognized negative roots for equations and used m to denote them (Musser, Burger, & Peterson, 2008). For example, m:4 was used to represent −4.

Some mathematicians throughout time thought of integers as invalid numbers since they can’t be represented by concrete objects. For example, Diophantus, an Alexandrian mathematician in the fourth century, said that an equation that needed to have a negative value for a variable was “absurd”. Leonardo de Pisa, who was born in 1175 AD and is more commonly known today as Fibonacci, recognized negative solutions in financial problems, as these were attributed to debt. However, formal use of positive and negative integers did not appear in Europe until the sixteenth or seventeenth century (Musser, Burger, & Peterson, 2008). “The resistance to negative numbers can be seen as late as 1796, when William Frend, in his text, Principles of Algebra, argued against their use” (Bennett & Nelson, 2007, p. 259).

In summary, the acceptance of zero and negative integers took a long time. Their development spread from the east to the western world and although integers were used as early as 620 AD in India, they were not totally accepted by mathematicians until the eighteenth century (Crowley & Dunn, 1985). Even then, some mathematicians viewed
them as not really existing because they could not be seen. By the 1900s mathematicians focused on the ideas of algebra and integers were required for solving equations.

**Constructivism in Mathematics Education**

Arithmetic in the early 1900s was difficult and not related to the mathematics needed in everyday life. It was believed that it didn’t matter what the subject matter was, the brain had to be trained to memorize things in order for it to work. The emphasis was on the product that was produced, rather than on the process of producing that product.

In the 1920s the emphasis in education began to shift to the process of learning (Kliebard, 1986). At this time understanding mathematics was more important than merely producing answers. For students to make sense of any mathematical idea, they must be able to relate to it in some meaningful way. Practice was still important but it was realized that the practice was more beneficial when students understood what they were doing rather than just performing rote procedures (Brownell, 1935). It was believed that mathematics was more meaningful when models were used and when the teacher took the time to help students conceptually develop the mathematics that was to be learned. If teachers were unsure of how students were thinking, they could interview the student. In this manner, teachers and researchers had a better understanding of what students were thinking and students’ incorrect understanding could be studied. By looking at what the student was thinking as well as what mathematics was being produced, a more descriptive idea could be attained as to what the student was thinking as he did mathematics.
Teachers understood that there was more to a student than his brain. They also needed to deal with the emotions, attitudes, and abilities of students. Interests and needs of students came to be seen as important in their understanding of mathematics. Students “must also have experiences in using the arithmetic they learn in ways that are significant to them at the time of learning” (Brownell, 1954). The stage was set for constructivism.

Constructivism has its roots in cognitive psychology. Notable theorists of this branch of psychology are Jean Piaget and Jerome Bruner. Constructivists state that learners use past experiences to create new learning (Ertner & Newhy, 1993). There are two types of constructivism in mathematics education, radical constructivism and social constructivism. Von Glasersfeld is generally associated with radical constructivism and Vygotsky is associated with social constructivism (Jaworski, 1994). Radical constructivism is termed “radical” because it breaks away from the traditional perspective of knowledge in that radical constructivists do not believe that there is an objective reality waiting to be learned. Each individual constructs his own reality, but this does not mean that his reality is the only reality that exists. The world is made up of multiple realities (Goldin, 1990). Social constructivists view knowledge construction as a social system that is influenced by one’s culture. Vygotsky stressed the role of language in developing one’s own construction of knowledge (Phillips, 2000). Knowledge connects social interaction and the individual’s perception of ideas.

This psychology is in contrast to behaviorists, who believed that learners’ minds are like blank slates upon which one can transfer any learning that is deemed essential.
Mathematical knowledge and skills were broken into small pieces that could be “digested” by students “in the belief that mastery of each piece in the learning hierarchy would eventually lead to complex thinking” (Battista, 1999b). When mathematics is learned rote, it is learned without connections to previous knowledge (Goldin, 1990). For example, rote learning consists of memorizing the basic facts of addition without understanding the operation of addition or the properties of addition. Students blindly follow a given procedure for finding a solution to a given problem. Without the understanding of place value and addition, a student could be taught to memorize the basic facts $2 + 4$, $3 + 6$, and $5 + 1$ to add $532 + 164$ but the student may not understand what his answer means. He may not even realize whether his answer is reasonable or not.

Students come to school with much informal knowledge about integers that is learned through their life experiences with temperature, games, golf, above and below sea level, etc. (Carpenter, 1985; Carraher, Carraher, & Schliemann, 1985; Mack, 1990; Peled et al., 1988). Children are excited about learning and these contexts allow them to want to know about new ideas so that they can communicate mathematically with others (Falkner, Levi, & Carpenter, 1999). Once students learn mathematical procedures by rote they lose access to their informal knowledge and thereby lose the ability to build upon understanding (Hiebert & Wearne, 1988; Mack, 1990; Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989).

Constructivism describes what knowing is and it advances the idea that knowledge is constantly evolving based on the experiences of the learner (Cobb, Wood,
& Yackel, 1990). This notion is in agreement with the *Principles and Standards for School Mathematics* that was written by the National Council of Teachers of Mathematics (2000). In this book it states, “Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge” (p. 20). In a constructivist environment the major emphasis is on student understanding and the fact that mathematics must be learned in a way that builds on previous understanding. In this type of environment, the teacher helps each student build more complex, abstract, and powerful mathematical structures than they currently have in place (Battista, 1999b). Open-ended, challenging, and meaningful tasks that engage the learner are required in constructivist environments (Fox, 2001).

In a constructivist environment, the notion has been advanced that learners should go from the concrete, using models, to the semi-concrete (using pictures), to the abstract (using symbols) as they learn topics in mathematics (Crowley, 1987; Lunkenbein, 1985). This theoretical model has been proposed by Jerome Bruner. Other theorists have taken these ideas. It is the job of the teacher to match instruction with the thinking of students in the class (Brownell, 1935; National Council of Teachers of Mathematics, 2000; Dewey, 1963; Piaget, 1970). Schools should not only transmit knowledge and skills, but should also cultivate a sense of what is made possible by acquiring knowledge (Bruner, 1984). The teacher must assess students to make sure that the model connects to the abstract concept; otherwise students are simply applying rote procedures using physical objects (Chappell & Strutchens, 2001; Leitze & Kitt, 2000; Clements, 1999). Individual
problems allow students to make generalizations about why the solution makes sense, thus creating a web of mathematical meaning (Russell, 1999). Teachers “attempt to construct a web of meaning to make what happened in their mathematics classroom intelligible” (Cobb, Yackel, Wood, 1992, p.2).

The greatest amount of learning takes place when the student experiences a perturbation, in which schema must be altered in order to pursue a solution. The student draws upon previous knowledge to devise a solution to the problem through accommodation, which occurs when a schema does not give the expected result, and assimilation, which occurs when new information is gleaned from previous knowledge (Baroody & Ginsburg, 1990). For example, if a student knows that $6 \div 6$ is 1 and then he is asked what $0 \div 0$ is, he may think it is 1 because he has learned that a number divided by itself is 1. He checks his answer using a related multiplication fact and confirms that $1 \times 0$ is 0. If asked if $0 \div 0$ could possibly be equal to be some other amount, the student thinks about previous knowledge that he has about division. He discovers that any number will work as a solution for $0 \div 0$. Because his current way of interpreting things does not work, a breakdown in the assimilation process occurs and a perturbation results (Battista, 1999b). “All accommodations are triggered by perturbations” (von Glasersfeld, 1991). In this way the student in the previous example used accommodation to build upon previous understanding. Suppose for a moment that the student had said there is no solution for $0 \div 0$. When he thinks about ways to show that there is no solution, he is met
with a perturbation because he cannot show that there is no solution. In attempting to show that there is no solution to the problem, the student discovers that there are many answers that would make this statement true. In this way the student has reacted to a perturbation and has become more knowledgeable in the process.

In summary, “all knowledge is constructed and all learning involves constructive processes” (Goldin, 1990, p. 38). Constructivists believe that the power to learn lies with the individual learner and “the teacher’s main function is to establish a mathematical environment” (Noddings, 1990, p. 13). “We need to get thinking out into the open, to encourage students to conceive their own mathematical purposes and execute their own plans, and to provide situations and objects that may trigger conflict (disequilibrium) and reflective abstraction (Noddings, 1990, p. 16). Appropriate models and representations should be used to assist with this understanding so that students conceptually understand the mathematics to be learned. In this sense it is clear that behaviorist approaches to teaching do not yield the kind of understandings that are necessary and appropriate in mathematics education. Suitable models and representations should be developed to assist learners in their conceptual understanding of mathematics.

The Role of Representation

“An important part of learning mathematics is learning to use the language, conventions, and representations of mathematics” (NCTM, 2000, p. 362). A representation can be a symbol or way of showing a particular idea or it can be the idea itself. Students use representations as tools for thinking about and showing their
reasoning as they communicate their thoughts. In the past, many teachers placed such emphasis on representations and students learned these without truly understanding the mathematics that was being represented. Representation was taught as an end in itself. With this in mind, the National Council of Teachers of Mathematics revised their 1989 document with a more detailed version of the Standards in 2000. In this later version the standards are more clearly enumerated and five process standards, one of which is representation, are fully explained. The earlier version of the standards document, which was released in 1989, allowed teachers to enact the standards “without sufficient attention to students’ understanding of the mathematics content” (NCTM, 2000, p. 6). It was realized that students must have both mental and physical representations for the mathematics that they need to learn.

There are three stages in developing one’s internal systems of representation (Piaget, 1970). The first stage is the inventive-semiotic stage in which an individual perceives a new symbol. This symbol, then, is associated with some idea and forms the core of meaning for this idea. In the second stage the earlier system is used as a pattern for the structure of the new system. In this stage students chunk together related ideas, and rules are developed in concert with the earlier system along with meanings that have been added. The third stage occurs when the new system becomes independent and can be separated from the patterns that created it. In this way, new meanings are added. These new meanings may be different from those that were first given to it (Goldin & Shteingold, 2001). The ability to move freely between various representations of the
same concept is seen as the development of conceptual understanding and should be a goal of instruction (Lesh, Behr, & Post, 1987).

The interaction between internal and external representation is fundamental to effective teaching and learning (Goldin, 2003). Whatever meanings and interpretations the teacher may bring to an external representation, it is the nature of the student’s developing internal representation that must remain of primary interest. The student uses internal representations to make sense of external representations and he transforms external representations into meaningful internal representations that can exist with present mental images.

Limitations in some children’s understandings are a result of internal systems of representation that are only partially developed (Goldin & Shteingold, 2001). These leave the student with a lack of skill and an attitude that is less than desirable. Representational tools must be found to overcome these obstacles. The tools can be acquired when we focus explicitly on them in the learning process. Therefore the fundamental goals of mathematics education include representational goals: the development of efficient internal systems of representation in students that correspond logically to, and interact well with, the external conventionally established systems of mathematics (Goldin & Shteingold, 2001). “We teach mathematics most effectively when we understand the effects on students’ learning of external representations and structured mathematical activities. To do this, we need to be able to discuss how students are representing concepts internally – their assignments of meaning, the structural relationships that they
develop, and how they connect different representations with one another” (Goldin & Shteingold, 2001, p. 19).

Students should be encouraged to use representations that make sense to them even if they are not the conventional representations that are used by mathematicians (Davis & Maher, 1993). Teachers can learn a great deal about students’ ways of understanding and thinking about mathematics as they consider students’ representations. Much emphasis should be placed on student understanding of their representations as well as the ideas involved in conventional representations. Models should be chosen that allow the student to build on previous learning and they should use materials with which they are familiar. When are used, students need to conceptually understand the mathematics that the model represents. Otherwise, the model will have as little meaning as abstract mathematical rules that can be memorized without understanding (Gregg & Gregg, 2007).

Hiebert and Wearne (1988) found that there are four major processes in going from a model to a symbol. These are connecting symbols with what they represent (in other words, building a model), learning procedures to manipulate symbols that relate the model to a mental representation, elaborating and routinizing the rules for symbols through practice, and using symbols and rules to refer to other more abstract processes. For example, when using the neutralization model, \( -4 \) can be represented as four red counters and \( +5 \) can be represented as five black counters. The student can then combine four red counters with five black counters to find that after combining pairs of red and
black counters, the left over amount is one black counter. He later can relate the model to the addition problem $-4 + (+5)$, thinking about the model as the solution is determined. After solving several similar addition problems the student can find a procedure for addition and practices, thereby leading to the algorithm for addition of integers. At the fourth level, the student can relate addition and subtraction of integers to algebraic manipulation of symbols. At this level the student can relate the problem $-4 + 5$ to the problem $-4x + 5x = 9$. The first two processes described by Hiebert and Wearne (1988) are semantic processes. “Tasks are solved by reflecting the symbol expressions of the problem into the referent world and selecting strategies based on the meanings associated with the expressions” (Hiebert and Wearne, 1988, p. 373). Most of the time, students who have routinized syntactic rules without making connections between symbols and what the symbols refer to, are more likely to decline opportunities to engage in the semantic processes than students who are encountering symbols for the first time Wearne and Hiebert (1988).

Models Used to Understand Integers

There are several studies involving two types of models used to build understanding of integers and the operations performed on them. These are the number line and neutralization models. In this section the researcher will discuss studies that used these two types of models, along with variations of these two types.

Janvier (1983) referred to the two types of models as equilibrium models and number line models. Some applications of the number line model are subway stations.
dealing with direction and weather problems dealing with temperature. These models allow one to apply more concrete meaning to the numbers on the number line since students are familiar with the idea of opposites conveyed by “east” and “west” and “above” and “below”. Some other applications of the neutralization model discussed in the literature are voting, dancing, and the game of scores and forfeits.

Peled (1991) studied two different types of number line conceptions. One type involved a continuous number line where numbers were ordered from lesser to greater. The other type of number line involved a divided number line that was disjointed at zero. Actions were taken either toward or away from zero. The student would decide how much was needed to get to zero and then continue from there.

Although the number line is the model that most children use when they are first introduced to integers, it seems that the neutralization model is a better fit for understanding integers and integer operations (Davidson, 1987). Cardinality, which deals with amount or number of things, is understood by children before ordinality, which is concerned with position (Davidson, 1987). It would seem, then, that more researchers should investigate the neutralization model. In recent years there have been many more studies that utilize a neutralization model. Some applications of the neutralization method are voting, dancing, attitude, and hot air balloons. Students are familiar with these topics and it gives them a chance to apply previous understanding to a new topic.

Liebeck (1990) introduced third and fourth grade students to integers using a game called “scores and forfeits”. In this game each player could use up to five black
counters, which were referred to as “scores” and five red counters, which were referred to as “forfeits”. For each student’s turn a card was drawn and the student had to follow the directions on the card using their red and black counters. For example, one card read, “You won a treasure hunt. 3 scores.” In this case the student would put out three black counters. If the card indicated that the student had some forfeits, they would put out the indicated number of red counters. At the end of each turn students indicated what their progress was at that point. Students were encouraged to write number sentences that indicated their move. The game was over when one player had a total of five red or black counters. Since students only had five of each type of counter, they quickly realized that putting out two red counters had the same effect as taking away two black counters. Since students were able to relate common situations with positive and negative integers, they were able to learn and retain how to add and subtract integers. When the model relates to experiences that students are familiar with, it is easier for them to relate situations in the game with situations using positive and negative integers.

Peled (1991) classified knowledge about integers into four levels. The first level is simply an extension of the number line to include negative integers. If one looks at the neutralization model, this first level introduces the student to negative integers as amounts, which are generally unfavorable amounts. The second level involves subtracting two counting numbers. A negative integer results when the subtrahend is greater than the minuend. At this level, one uses the same procedure that was used with counting numbers but now one can go on the other side of zero and obtain a negative
result. If one looks at the neutralization method, this level allows students to subtract a greater whole number from a smaller one by taking away what they can and determining the missing amount to give a negative value. The third level involves using the fact that adding and subtracting with integers is the same for some types of problems as adding and subtracting with counting numbers. For example, \(-4 + (-7)\) is just like adding 4 and 7 and appending a negative sign. In a similar example, one can think of \(-5 - (-3)\) as similar to 5 – 3 and appending a negative sign. If one looks at the neutralization model, the type of problem again only deals with those situations in which one has enough of one type of integer to do the subtraction. This process is just an extension of subtraction as “take away”. Level four involves the other types of problems which do not yield the same sort of result as counting numbers such as \(4 + (-7)\) or \(4 - (-7)\). If one is looking at the neutralization model, he must determine, for the operation of addition, whether there are more positives or more negatives. When looking at subtraction, the student must determine how he can obtain enough of the type of integer represented by the sum (minuend) to take away as many as the addend (subtrahend) describes.

Sheffield and Cruikshank (2005) suggested that middle school students be introduced to integers using the stock market, population growth or decline, and speed when driving. They suggest a card game that can be played by students to give practice with addition of integers. The black cards from a regular deck of cards (without the face cards) represent positive integers and the red cards represent negative integers. Five cards are dealt face down to each player. Each player picks up his cards. Each player adds the
amounts on the cards in his hand. On each turn, the player draws one card from the player to his left. The goal is to have a score that is the greatest in absolute value. The game is over when a player has a score of 50 or more. The authors conclude that to be proficient with integer addition and subtraction, students need to study problems that make sense to them and they need to discuss solutions to problems with their peers.

In the Algebra Project (Moses, Kamii, Swap, & Howard, 1989) students learned about integers through the simulation of a trip. Four units were used to build upon this trip. In the first unit students were introduced to the idea of direction, displacement, and equivalence. In the second unit, students created a new meaning for subtraction in which they compared the end points of pairs of displacements. In the third unit students developed a meaning of relative coordinate systems and related this to integers as displacements that have magnitude and direction. In the final unit, students related combining displacements to the concept of integer addition.

Dienes (2000) discussed six stages of integers. In one of the first stages he considers a dance situation. To enter the dance floor, one must enter through a waiting area. A couple consists of a male and a female, and only couples are permitted to dance. As soon as a couple was in the waiting area, they had to enter the dance floor. In this manner, there was only one type of person (male or female) in the waiting area. These ideas could then be displayed using positive and negative integers. Put in this context, addition and subtraction are made more concrete and students can visualize how they operate. They do not need to be given rules and procedures for the operations.
Reeves and Webb (2004) described the story of Crazy Larry. He wanted to “fly” so he attached forty-two helium balloons to his lawn chair. By doing so, he was raised into the air over Los Angeles International airport. Sandbags could be dropped from the lawn chair if Larry wanted to take away some weight in order to go higher. Fifth grade students related the helium balloons to positive integers and the sandbags corresponded to negative integers. Students drew pictures of objects and then had to figure out how many helium balloons would be required to make the object float. They also had to determine what would happen if a given number of sandbags and helium balloons were attached to the object.

Shore (2005) used the voting context to discuss the neutralization model. Voters could vote “yes” or “no” for an issue. Only those who were present in the voting area were to be counted and as soon as a person voted, they left the area. It could easily be seen that if a “yes” voter left the area before voting, this would have the same effect as a “no” voter entering the area. Students could also see that one only needed to see whether there were more “yes” voters or “no” voters to determine which way the issue would be decided. The number of “no” voters determined the amount of negative and the number of “yes” voters determined the amount of positive. In this way students could take a familiar situation and apply it to a conceptual understanding of integers.

In summary, two types of models are used to represent integers. These are the number line and the neutralization models. Each type, however, has several variations. The number line model emphasizes ordinality at the expense of cardinality, while the
neutralization model emphasizes cardinality at the expense of ordinality. No studies available for review combine the two types of models.

Studies Involving Integer Addition and Subtraction of Integers

Several studies have been done advocating the use of the number line or the neutralization model. In each case there was enthusiasm about a particular model and its ability to facilitate conceptual understanding about integers. Again, no one model included both conceptualizations of integers, ordinality and cardinality.

Freemont (1966) used bent pipe cleaners to represent positive and negative integers. A positive one was represented by a pipe cleaner that curved to the right, since the P in positive has a curve to the right. A negative one was represented by a pipe cleaner that curved to the left, since a negative is the opposite of a positive. When these pipe cleaners were put together, they formed a zero. When modeling addition of integers on the number line, students could see the cancellation of opposites.

Sawyer (1973) conducted a study using four different methods of subtraction with integers. When students use the complement method, zero is added to the expression in the form of a number and its additive inverse. For example, if a student wanted to subtract \(-5\) from \(+7\), he would add \(-7\) to the \(+7\) and add \(-7\) to \(-5\) to obtain \(-12\). The form of zero to be added is always the opposite of the addend and itself. To use the ordered pair method to find \(-5 - (+7)\) would involve the student renaming \(-5\) as \((0,5)\) and \(+7\) as \((7,0)\). But the student then must rename \((0,5)\) as an equivalent ordered pair such as \((7,12)\). Then he performs the subtraction to get \((0,12)\) which is equivalent to \(-12\). Using
the systems method, the student recognized that \( x - y \) is equivalent to \( x + (\neg y) \). There were eight types of subtraction problems on the test given to the subjects of the study. It was found that the group using the complement method was better able to remember how to subtract and those using the systems method were able to remember how to subtract even better than the complement method group. The authors suggest that a combination of methods would probably work best.

Gibbs (1977) used the idea of holes (H) and plugs (P) to form a blank. Addition is seen as combining holes and plugs, where a hole and a plug form a blank. Subtraction is seen as removing holes or plugs. This model is in agreement with the “take-away” model for subtraction. Sentences are written using Hs and Ps and it is assumed that the student will relate these to positives and negatives.

Jencks and Peck (1977) used the idea of a witch’s brew into which hot cubes or cold cubes could be placed. Each cube represented a change of one degree. In this model, neutralization is not mentioned. It was obvious to students that when hot cubes were added to the mixture, the temperature increased and when cold cubes were added, the temperature decreased. In the same manner, when hot cubes were taken out of the mixture, the temperature decreased and when cold cubes were taken out, the temperature increased. Some students were confused with problems such as \(-4 - (+6)\) because they did not understand how one could take away hot cubes from a cold temperature and still get a temperature. The units did not seem to make sense. Students had to convince their classmates of their result, rather than relying on rules applied to the cubes or the numbers.
Harvey & Cunningham (1980) assessed 163 eighth grade students using 46 questions that had various types of addition and subtraction problems involving one- and two-place integers. He found that students used a physical model to help them with addition problems but those who were successful with subtraction added the additive inverse of the addend (subtrahend). Students were not able to neither relate this method to a physical model nor explain why it worked.

Tobias, Anderson, Cahill, Carter, Duffin, and Goodwin (1982), along with Galbraith (1974), used the idea of ordered pairs and equivalence classes to discuss integers. In this model equivalence classes are stressed. All ordered pairs of the form \((m + 5, m)\) are considered to be equivalent to \((5,0)\) and can be represented as the integer, \(5^\dagger\). Addition is done using ordered pairs and ordered pairs become the mental model. The neutralization ordered pair \((0,0)\) is written as the addition of two ordered pairs, such as \((0,6) + (6,0) = (6,6) = (0,0)\). In each case, when addition is performed, one element in each ordered pair is 0. As an example, \(4 + (-7)\) could be shown as \((4,0) + (0,7) = (4 + 0, 0 + 7)\), which is equal to \((4,7)\) or \(-3\). Subtraction is done by finding the missing addend of the ordered pair. For example, \(-5 - (2) = (0,5) - (2,0)\). The ordered pair \((0,5)\) is equal to \((2,7)\) so that one can replace \((0,5)\) with \((2,7)\) to obtain \((2,7) - (2,0) = (0,7)\) or \(-7\). This model is another type of neutralization model. The trouble with this model is that each integer has an infinite number of representations when written as an ordered pair.

Bell (1983) studied 25 students who were fifteen years of age. Twenty of these students were successful on all of the addition problems that dealt with integers. These
students relied on some physical model such as the number line to find solutions. Only ten of the students were successful with integer problems that dealt with subtraction, and these students relied on rules rather than understanding. Students had a tendency to ignore signs of integers and they combined the magnitudes by referring to the operation sign. The intent of the study was to determine students’ understanding of integers and then “design teaching which would provoke cognitive conflict with the misconceptions shown in the interviews” (Bell, 1983, p. 67).

Some researchers (Battista, 1983; Cotter, 1969; Kohn, 1978; Sherzer, 1969; and Whimbey & Lochhead, 1981) suggest that students use positive charges and negative charges to represent integers after they have used the two-color counters. For example, \(-4 + (6)\) would be represented with four red chips and 6 yellow chips. Then the student would write \(- - - -\) and \(+ + + + + +\). He could then match a “-” and a “+” four times to create four sets of zero, making the final answer positive two. In this manner, the student would move from the concrete to the semi-abstract level.

According to a study done by Hativa and Cohen (1995), some types of problems showed a greater proportion of intuitive knowledge on the part of the fourth grade participants. For example, prior to instruction, more than fifty percent of the students were successful in subtracting a positive number from zero. These authors also stated that using negative numbers in operations was especially difficult for lower-achieving students. The problems that involved the most pre-instructional knowledge involved subtracting a positive number from zero and subtracting a positive number from a smaller
positive number. More than fifty percent of the non-treatment group and more than three-fourths of the experimental students did these problems correctly when asked in interviews prior to the experiment. This study reinforces the idea that integers can be understood by young children because they have had prior experiences with integers.

In another study, two-color counters were used to solve problems correctly 28.6% of the time (Wilkins, 1996). Students used a number line to correctly solve problems 23.1% of the time. When presented with symbolic problems dealing with integers, students tend to use a number line model. When students are presented with contextualized situations dealing with integers, the students were better able to find and understand solutions. When students were presented with problems dealing with integers that had little or no meaning, they modified the rules that were used for addition and subtraction of whole numbers. When students were allowed to choose which method to use, mental models were used 50.3% of the time, the number line was used 22.6% of the time, and two-color counters were used 26.7% of the time. When students used mental strategies they were successful 64.5% of the time. When they used the number line, students were successful 69.6% of the time, and when students used two-color counters they were successful 75.3% of the time. For this study, the students who used the number line and two-color counters were more successful than those who used mental models.

In this study, Wilkins (1996) found that students tended to use the number line for temperature and waterline problems and they tended to use the two-color counters for
postal worker and weather problems. Stated differently, the students used a continuous model for continuous situations and they used a discrete model for discrete situations.

McCorkle (2001) conducted a study of seventh grade students who were taught addition and subtraction of integers using an experimental group that used a relational approach in which students manipulated a thermometer scale using hot cubes and cold cubes. The control group was taught in the traditional manner using procedural algorithms for addition and subtraction of integers. Two weeks of instruction was provided for each group. Students were given a posttest at the end of the two weeks and the same test was given three weeks later to determine the amount that students remembered. Results suggest that students who learned conceptually had higher scores on the posttest and were able to remember the material better than those who were taught in the traditional manner.

Hackbarth (2000) did an eight-day study that involved sixty-eight seventh graders. The study employed two experimental groups and one control group. One experimental group used plus and minus pieces while the other experimental group used two-color chips. The students in the control group were taught rules without the use of manipulatives. A posttest was given two months after the beginning of the study and no statistical differences were found between the three groups.

Gallardo (2002) claimed that students aged twelve and thirteen need to be proficient with operations involving integers in order to develop skills used in algebra.
For example, to solve for $x$ in the equation $x + 7 = 4$, one must be able to make sense of adding $-7$ to both sides of the equation.

Disadvantages of the Two Models Used for Integers

Various researchers have discussed the advantages and disadvantages of each of the two models for integers. The number line requires students to represent integers with a point and a vector. This representation is confusing for some students because they must be concerned with magnitude and direction, whereas with whole numbers they only had to look at magnitude. For example, $+4$ is represented on the number line with a point that is four units to the right of zero and a vector to the right from zero that is four units in length. With the number line, there is also confusion between the sign associated with an integer (to indicate whether it is positive or negative) and the operations of addition and subtraction (Baily, Barton, Blakey, Brighton, Bromby, & Davey, 1974). As an example, some students confused the integer $-3$ with the operation of subtraction. Also, the number line has its primary focus on movement, rather than on reasoning (Lytle, 1992). One must remember which way to move to perform the operations and there is no connection between the movement and the sign of the operation. Since the number line is a continuous model, greater integers are located to the right and integers of lesser value are located to the left. One of the advantages of the number line is that students can easily verify that $+4$ is greater than $-2$, since $+4$ is to the right of $-2$ (Hart, 1981). However, when using a number line, some students count hash marks corresponding to numerals, rather than counting the units between these numerals (Carr & Katterns, 1984). For
example when adding \(-2 + (+5)\), a student might get a sum of \(+3\) because he counted the numerals beginning with \(-2\) when going five units in the positive direction from \(-2\), instead of counting units between the numerals to get a sum of \(+3\).

In one study, students were generally successful in using the number line model with addition of integers but those who were successful with subtraction of integers used the rule of adding the opposite. Those who could not remember the rule had no physical model to help them make sense of the situation (Lytle & Avraam, 1990). Some authors believe that the number line should not be used when providing instruction about integers (Carr & Katterns, 1984; Ernest, 1985).

In another study, students’ understanding of subtraction was limited and the number line may have contributed to this lack of understanding. Subtraction of a negative amount requires shifts that are more complicated. The researcher recommends that a discrete model should be used when introducing integers (Hart, 1981).

Operations with integers should be consistent with operations with whole numbers. With the neutralization model, addition is still modeled as a combining of elements and subtraction is still modeled as “take away”, unlike the number line model in which new meanings are assigned to addition and subtraction (Fremont, 1966; Bennett & Musser, 1976; Gibbs, 1977). The neutralization model is easy to explain and it is uncomplicated for students (Dirks, 1984). It is also easy to replicate and is very visual (Cotter, 1969; Haner, 1947). The neutralization model has been successful when teaching elementary teachers (Gibbs, 1977). This, in turn, equips the teachers with a better
understanding to teach these concepts to their future students. With the neutralization model, students make sense of integer situations rather than relying on rules (Fremont, 1966). Although only the operations of addition and subtraction are discussed in this research, the neutralization model could also be used to understand multiplication and division of integers (Battista, 1983). Also, with the neutralization model, the primary focus is on reasoning and it is more intuitive than the number line model.

Fischbein (1987) found that the neutralization model cannot easily be extended to operations on rational numbers or real numbers. The neutralization model allows for multiple representations of an integer, which may be confusing to some students. For example, \(-3\) could be represented using three negatives or as five negatives and two positives. Also, the neutralization model does not emphasize order related to direction. There is no difference in the number of objects that represent \(+4\) and \(-4\). The only difference is the color of the objects.

In summary, there are two types of models used to introduce integers, the neutralization model and the number line model. There are four reported advantages of the neutralization model. First, it is in agreement with other fields of study. For example, in science positive and negative charges are attracted to each other and balance each other. This is similar to the two-color counters in which a black counter neutralizes a red counter. The second advantage lies in the fact that the representation of integers as ordered pairs is mathematically sound. Using two-color counters mimics the representation by ordered pairs in that one can see from either representation whether
there are more positives or negatives. Third, the neutralization appeals to the student’s intuitive understanding of integers. Fourth, the neutralization model places emphasis on cardinality, which tends to develop before ordinality (Wilkins, 1996). By using two-color counters, students can concentrate on the actions taking place rather than worrying about the order of the integers on the number line.

The major disadvantages of the neutralization model include its inability to model all types of integer multiplication and division problems. It is also impossible to extend the use of the neutralization model to operations involving rational numbers.

The number line model has some advantages and disadvantages. It allows one to observe order within the set of integers because numbers that are to the right are greater than those that are to the left. The operations of addition, subtraction, multiplication, and division can all be modeled using the number line. Also, operations on the number line can be extended to use with rational numbers. The disadvantages of the number line model include the disagreement as to the way that subtraction. For modeling subtraction, some studies explored finding a missing addend while others suggested going backward on the number line to indicate subtraction. This can cause confusion for students because prior to experiences with integers, subtraction was associated with “take away” and they cannot take away things when using a number line. Some students also have a difficult time determining what it is that they are counting on the number line. Some students count hash marks naming the integers rather than the spaces that represent the units (Carr & Katterns, 1984).
Other Obstacles to Understanding Integers

An integer can be thought of in two ways: as a directed distance or as an additive inverse (Creswell & Forsythe, 1979). When thinking of integers as directed distances, it is understood that movement to the right or upward is positive and movement to the left or downward is negative. In order to understand integers as additive inverses, one must understand the idea of absolute value (Creswell & Forsythe, 1979). The absolute value of a number is the magnitude of the number without regard to sign. In this way, one knows that the additive inverse of a number is a number that is the same distance away from zero as the given number but on the opposite side of zero. For example, two is the additive inverse of negative two. A complete understanding of absolute value is lacking if absolute value is only considered as an integer without a sign (Creswell & Forsythe, 1979).

According to Davis et al. (1979), the signs of integers have little or no meaning for students when they are first encountered. This points to the fact that research should be done with a different model than has been used in the past to learn why the signs of integers have little meaning to students. In the study by Davis et al. (1979), it was found that “result-unknown” problems had a slightly lower percentage of correct responses than “start-unknown” or “change-unknown” problems. If students had a model with which they could model all of these types of problems in a meaningful way, they may be more proficient with operations involving integers.
According to Lytle (1992) there are five fundamental things that must be understood about integers. First, students must understand that zero is a separator between the positive integers and the negative integers, and not an absolute zero, as it is in the whole numbers. Zero is considered neutral and it is also the name for “nothing”, since it has no magnitude. Second, there is a distinction between the sign that indicates an operation and the sign that indicates an integer. The sign that indicates an integer should be above and to the left of the number. For example, for the integer $-3$, the “-” shows that it is negative but with $0 - 3$, the “-” represents the operation of subtraction. Third, students must understand what it means to be an opposite, or additive inverse. They need to understand that when one adds a number to its opposite, the result is zero. Fourth, integers are ordered by their direction. This requires students to understand the idea of absolute value. Fifth, addition and subtraction are inverse operations that “undo” each other. Even young children are familiar with the inverse operations of buttoning and unbuttoning, zipping and unzipping, etc. It is important to note that the answer to an addition or subtraction problem involving integers may be greater than, less than, or equal to the original amount.

Many children have dealt with the idea of negative numbers long before instruction of integers begins (Davidson, 1987). Young children have played card games where they “go in the hole” and they may have heard of temperatures that were “below zero”. When negative numbers are mentioned in contextualized situations, children are
able to make more sense of the numbers than when problems were presented using mathematical notation and equations (Peled et al., 1988).

Adler (1972) found that one of the major obstacles to learning integers occurs if the idea of negative integers is not well developed before the student is taught how to perform operations with them. If students do not understand what negative integers are, they tend to memorize rules and apply them. When they do so, the student does not have a way to determine if his solution is feasible for a problem. Without understanding what integers are, students tend to apply rules that worked for the operations in the set of whole numbers. Hence, the student has no way of knowing in advance if their answer to a given problem is reasonable.

Hall (1974) states that students are often confused about integers by teachers’ incorrect use of vocabulary. The word “minus” should only be used when referring to the operation of subtraction and “opposite” should only be used to refer to a number that is added to another to obtain zero. Also, “negative” should only be used when referring to a number that is less than zero. Too many times teachers are careless and refer to a negative number, such as –3, as “minus three”. This adds to the confusion experienced by students due to the use of the same symbol to represent two ideas.

In another study (Nunes, 1993), it was noted that the “–” sign has three meanings. It can indicate the operation of subtraction, and this is typically the meaning that students initially apply to the “–” sign. It can also be used to designate the magnitude of a number to show debt. With this meaning students understand that the
greater the negative number, the greater the debt. The third meaning associated with the “–” is inversion, where one finds an unknown starting point given intervening steps. To find the starting point one inverts operations, and works backward, to get to the starting point.

Hativa and Cohen (1995) found that students had knowledge of negative integers prior to entering school. Out of twenty-eight students in the study, forty-two percent stated that they knew negative numbers because of explanations by family members in the context of temperatures from watching weather reports and educational programs on television. Fourteen percent of the students had heard about negative numbers through games in which it was possible to lose more points than one had available to them. Seven percent of the students claim to have heard about negative numbers from enrichment activities provided outside of school. In this study, students who were given instruction about integers improved significantly on three tasks involving negative integers. These tasks included the ability to locate where negative integers are located on the number line, the ability to mentally compare negative integers, and the ability to estimate the size of integers and the interval between two given integers. Of these tasks students found comparing two integers the easiest to understand.

Lave (1977) and Reed and Lave (1981) found that tailors who learned arithmetic in school could work arithmetic problems with large numbers by using algorithms what they learned in school but when they made mistakes, they did not realize that their answers did not make sense. Tailors who learned arithmetic through solving problems in
their work environment were more accurate but could not work problems with large numbers. In another study (Nunes, Schliemann, & Carraher, 1993) it was also found that rules interfered with problem solving when third grade students who lacked understanding tried to follow rules they had learned in school. Attempts to provide problems for students to connect their informal understanding of integers to more formal procedures needs to be the focus of integer instruction.

**Common Mistakes Made When Adding and Subtracting Integers**

There are several types of error that students make when adding and subtracting integers. One type is referred to as symmetric subtraction, in which the student subtracts the number with the lesser magnitude from the one with the greater magnitude (Davis, et al., 1979). Sometimes when children are taught how to subtract, the teacher may incorrectly tell them that one should always subtract the smaller number from the bigger number or that the problem can’t be done. In the study done by Davis, et al., it was found that symmetric subtraction explained about 50% of all the answers that were given by the 28 fifth graders involved in the study.

Another type of error is the type that is obtained by the use of primary-grade frames (Davis et al., 1979). Children have been told that addition is a binary operation so they need two numbers to add. If they only have one number in the left-hand most place with no regrouping, they tend to go to the second place over so that they have two numbers to add. For example, if the child is given 217 + 42 + 26 to add in column form, they know they need to add two numbers together. Seeing the 2 in hundred’s place, they
add the 2 to what they got for a total in the ten’s place. For this example, the student would get 105, which is smaller than one of the addends. This can be even more complicated when dealing with integers.

A third type of error that students make is the sequential interpretation of symbol strings and it is related to the way students try to read. They go from the left to the right carrying out the arithmetic as they go. A child making this type of error would get − 7 for the example − 5 + 2 because they would add the 5 + 2 and append a negative sign since this was given to them at the left hand side of the problem.

Ferguson (1993) suggests that subtraction of integers is a challenge for students and there are five implications it has for instruction. First, she suggests that instruction address the informal knowledge of integers that students already possess. Students should understand that negative integers do not always represent bad things. For example, in golf below par is good, but “-” in a checking account is not good. Second, operations with integers should take advantage of student understanding of operations with whole numbers. The meaning of the operations in the set of whole numbers should be developed and related to integers. Third, students should generalize the workings of integer operations rather than be given procedures for operating with integers. Fourth, students need time to make generalizations about operations and these generalizations can be brought to fruition through practice. Finally, students who have misconceptions about integers need to have these misconceptions addressed in a manner that allows them to rebuild and strengthen concepts using prior knowledge and experiences.
Rathmell (1980), studied responses to NAEP (National Assessment of National Progress) examples where students had to match a number line with positive and negative numbers that modeled addition or subtraction to a symbolic statement of the problem. It was found that 45% of the 9-year olds and 39% of the 13-year olds students added the numbers that were found at the ends of the arrow for addition problems. For subtraction, only 14% of the 9-year olds and 33% of the 13-year olds could match the drawing with the correct subtraction problem. This finding points to the fact that many children do not understand the use of a number line as a model for addition or subtraction (Rathmell, 1980).

In summary, students make several types of mistakes when adding and subtracting integers. Some students use the same rules for addition and subtraction of integers that they applied to addition and subtraction of whole numbers. Others work the problem from left to right and append the sign that is to the left of the first integer. Students are less likely to make mistakes when they are first introduced to a topic because they work more slowly to apply their new learning. Students also make fewer errors when they are at the expert stage because they have had much practice with the procedure by the time that they reach this point. Students make most of their mistakes when they are in the intermediate stages (Anderson, 1983).

Summary

This chapter provided a brief historical development of zero and the set of integers. It then discussed the constructivist movement in mathematics education and the
role of representation in mathematics teaching and learning. The researcher then reviewed the research literature related to the models commonly used with integer addition and subtraction. Next, studies involving addition and subtraction of integers were reviewed. Finally, the researcher described the research related to obstacles to understanding integers and common mistakes made when adding and subtracting integers.

This review of the literature provides a background for the reader to understand how the researcher approached this study. Chapter three will examine the methodology employed by the researcher in this study.
CHAPTER III: METHODOLOGY

Introduction

This chapter contains a brief description of behaviorism and constructivism, a pertinent introduction to qualitative research in mathematics education, the methodology used for this research study, and a description of the study. The purpose of this study was to research how students understand integers and the operations of integer addition and subtraction. This purpose was accomplished as the researcher listened to and observed six pre-service elementary teachers as they explored integers and the operations of addition and subtraction throughout four interview sessions using a novel model that was created by the researcher for this study.

Behaviorism and Constructivism

The long-standing debate between behaviorists and proponents of constructivist theory has been well documented over the past several decades (Bauersfeld, 1988; Noddings, 1990; Ball, 1993; Schoenfeld, 2002; Klein, 2007; von Glasersfeld, 1990). This background provides the reader with a better understanding about how research in mathematics education has evolved over the past three decades. According to behaviorists, drill and practice lead to mastery of arithmetic. Consistent with this perspective, knowledge of facts is the key (Gardner, 1987).
Those who believe in constructivist theory believe that students need more than just knowledge of arithmetic. Students need to draw upon past learning and experiences to piece together important bits of information so that they can most efficiently solve problems. Learning is constructed by students so that it makes sense to them (Schoenfeld, 2002).

According to Piaget, assimilation permits the learner to add new information to previously learned material and accommodation refers to adaptations that must be made to previous knowledge and experiences to adjust to the new idea (Crowther, 1997). “The process of abstraction is one that involves reaching the autonomous stage in the functioning of a representational system” (Goldin, 2002, p. 216). Abstraction involves getting away from a particular situation so that the idea or concept can be seen apart from the original problem (Battista, 1999a). Reflective abstraction involves thinking about the actions involved in a process.

Research Methods in Mathematics Education

Research methods were created in the last three decades which allow a researcher to investigate student thinking as the student solves problems. Some of these methods are “observational and experimental studies, teaching experiments, clinical interviews, the analysis of ‘out-loud’ protocols, computer modeling, and more” (Schoenfeld, 2002, p. 442). By designing carefully thought out procedures, these methods permit the researcher to look at in-depth problem solving techniques in tandem with student knowledge of mathematics.
Qualitative research is used when the researcher desires to understand how participants think and feel about topics (Bogdan & Biklen, 1998). The present researcher determined that no single type of qualitative research methodology matched the intent of this study. Therefore the present study embodied some qualities of a teaching experiment and some qualities of a phenomenological study.

In a general sense, a phenomenological study does not involve teaching. Instead it attempts to extract the essence of the phenomenon as it is experienced by each participant in the study. By looking at each student’s perception of the phenomenon, a generalization is made concerning the phenomenon. In contrast, in a teaching experiment the researcher determines how each student understands a topic and then leads each student toward a more mature mathematical conception of the topic. The researcher attempts to change the way students think about mathematical topics by asking questions that create perturbations for the students so that incorrect thinking is challenged. A teaching experiment is not used to generalize the thinking of students.

The emphasis in a teaching experiment is attempting to understand the thought processes of students and funneling these processes to create a more mature conceptualization of the topic, whereas the emphasis in a phenomenological study is describing the common essence of the phenomenon. The next section will include a discussion of each of these types of research in more detail and it will explain how elements of these methodologies were implemented in the present study.
Teaching Experiment Methodology

Teaching experiments were introduced in the United States about 1970 because a gap existed between the research being conducted in mathematics education and the concerns of classroom teachers. Prior to teaching experiments, experimental designs were created outside of the classroom to be implemented using a manipulation of variables in a classroom setting. The goal of this approach was to determine the effects of the variables on education and learning. The way that students made sense of mathematics was not generally considered in the research (Steffe & Thompson, 2000). Since the 1970s student mathematical learning and reasoning has been seen as important when teaching students. If the teacher can determine how a student is thinking and pose problems that expose incorrect processes, student learning can be enhanced.

A teaching experiment consists of teaching sessions in which the researcher explores how students are thinking and learning and the study evolves as it proceeds. A teaching experiment may last a few weeks or a whole year. The purpose of a teaching experiment is to change the way that participants think about a topic so that it makes more sense to them and is correct mathematically (von Glasersfeld, 1987). It is also concerned with moving a student along toward adult understanding of a topic in mathematics (Battista, 1999b). The primary goal of a teaching experiment is a focus on the nature of developing ideas (Lesh & Kelly, 2000). “Because the goal of most teaching experiments is to go beyond descriptions of successive states of knowledge to hypothesize the processes and mechanisms that promote development from one state to
another, it is important to create rich research environments that induce changes in the subjects whose knowledge or abilities are being investigated while minimizing uninteresting influences that are imposed by authority figures” (Lesh & Kelly, 2000, p.200). The following paragraphs detail how this research study utilizes various components of the teaching experiment.

There are three steps in conducting a teaching experiment. These steps are as follows: listing learning goals, preparing activities that will meet these goals, and reflecting before the activities about how students will think about a situation (Simon, 1995). Each of these steps is discussed in the following paragraphs.

The first step in conducting a teaching experiment is listing learning goals for the students. In the present research, the researcher focused on how participants developed a conceptual understanding of integers and the operations of addition and subtraction. The goal of the present research was to determine how participants conceptually understand integers and the operations of addition and subtraction so activities could be constructed using a novel model to increase participants’ conceptual understanding of integers. The goal for each day of this study is listed in the next paragraph.

The researcher’s goal for the first session was participants’ recognition of the two different colors of bills as they relate to “debt” and “money to spend”. Once participants recognized the significance of the two colors of money, they could then model various amounts of money and debt using the novel model. The goal for the second session was for participants to model addition of integers so they could relate the algorithm for
addition of integers to their representations using the novel model. The goal for the third session was for participants to understand subtraction as “take away” as they modeled various subtraction problems using the novel model. The goal for the fourth interview session was for participants to relate their representations with the novel model to an understanding of the algorithm subtraction of integers.

The second step in conducting a teaching experiment is planning learning or instructional activities that will meet the desired objectives. In the present study, the researcher planned activities using a novel model that consisted of a number line that was red and white and play money that was also red and white. The participants were first introduced to the novel model using money, which is a familiar context to college students. Problems were presented that required participants to add and subtract amounts of “debt” and “money to spend”.

In the interviews, day one was spent identifying integers as they relate to debt and money that can be spent. Participants were shown the novel model and they used the novel model to show amounts of “money to spend” and “debt”. Participants later related “money to spend” to positive integers and they related “debt” to negative integers.

In the first session, participants were also asked to model amounts of “money to spend” and “debt” in more than one way using the novel model. Participants added the same number of each color of bill to the given amount and reported that the value did not change. For example, to name \( -3 \) in another way, the participants could put three red bills on the number line portion of the novel model. To name this integer another way,
participants could add one more red bill and one white bill to the given amount to have another way of naming - 3. This activity proved to be helpful later when modeling addition and subtraction using the novel model.

In the second interview session, participants described the elements that make up the set of integers and they used the novel model to show addition of integers. Participants recognized the two colors of money and associated the white money with “money to spend” and the red money with debt. The participants modeled various amounts of money to spend and debt on the number line portion of the novel model and then added the two given amounts. Participants first counted the number of red or white bills indicated by the first addend and placed these on the number line portion of the novel model. Then, they counted the number of red or white bills indicated by the second addend and placed these on the number line portion of the novel model. The final step required participants to remove equal amounts of red bills and white bills so that only one color of money remained on the number line portion of the novel model. This process of removing equal amounts of red and white bills was referred to as removing “zeroes”.

After participants modeled several examples using the novel model, the researcher helped participants explore the algorithm for addition of integers. In other words, the participants conjectured that when integers with the same sign are added the sum is the sum of the absolute values of the addends and the common sign is appended to the sum. When the two integers have different signs, the absolute values of the addends are subtracted and the sum is given the sign of the addend that has the greater absolute value.
In the third interview session participants modeled subtraction of integers using the novel model. Subtraction was modeled on the novel model as “take away” so that participants could transfer their understanding of subtraction with whole numbers to subtraction with integers. The participants first modeled the sum and, if possible, took away the required amount. If it was not possible to take away the required amount, the participants added zeroes (an equal number of red and white bills) until the required amount could be taken away. The fourth interview session related participants’ use of the novel model to the algorithm for subtraction of integers. Through the researcher’s questions the participants were led to discover that when subtracting integers the subtraction sign is changed to addition and the given addend is changed to its opposite. The rules for addition of integers, which was the focus of interview session three, are then followed. The fourth interview session also provided addition and subtraction problems that involved integers that could not be modeled on the number line portion of the novel model because the absolute value of each of the integers was greater than fifteen. For these problems the participants were given red and white bills of greater denominations to model addition and subtraction problems.

The third component of a teaching experiment requires that the teacher-researcher anticipate how participants will think about a situation. The researcher “must not only have a model of the student’s present conceptual structures but also an analytical model of the adult conceptualizations towards which his guidance is to lead” (von Glasersfeld, 1987, p. 13). In this study, prior to each interview session and throughout the session, the
researcher reflected on how participants made sense of the model and the problems provided. In the present study the researcher first asked participants questions to determine their current understanding of integers. Problems were then presented to participants so that perturbations were created in participants’ existing mental schemes. The researcher attempted to “understand the originality of reasoning, to describe its coherence, and to probe its robustness or fragility in a variety of contexts” (Ackermann, 1995, p. 346).

Although a teaching experiment is well thought out with pre-designed tasks, it evolves as the study progresses. Student feedback to tasks determines what future tasks will be included and how present tasks are altered to push students in the correct direction. “The emerging interpretive framework that guides the researchers’ sense-making activities is of central importance and influences profoundly what can be learned in the course of a teaching experiment” (Cobb, 2000, p. 320). In the present study the researcher altered her original questions in response to student thinking and inserted new questions to allow participants to reconstruct their conceptual understanding of integers.

**Phenomenology as a Research Methodology**

Phenomenological research involves describing the meaning of a phenomenon for several individuals (Marshall & Rossman, 2006). The researcher collects data through interviews and surveys and then analyzes the data to determine common themes in participant understandings (Creswell, 2007). The goal of phenomenological research is to describe the phenomenon that participants experience, how they experience the
phenomenon, and then combine these to describe the essence of the phenomenon (Creswell, 2007). In the present study the researcher captured the essence of how participants make sense of integers as they relate to the algorithms for addition and subtraction. The researcher was interested in how the six participants made logical connections between the use of a novel teaching model and the derivation of the algorithms for addition and subtraction of integers. She did this by asking questions that allowed her to see how the participants were thinking and understanding integers. Answers to the questions also helped to support participants’ developing conceptualizations of integers and the operations of addition and subtraction.

When a researcher wishes to explore how participants view experiences differently, a phenomenological study can be used (Creswell, 2007). It is important that the researcher views the experiences and determines the essence of the common phenomenon. Edmund Husserl, a mathematician, is the founder of phenomenology. Phenomenology is viewed as a philosophy consisting of observing experiences that encompass one’s life. It does not look at analysis of situations but instead views the situation through the consciousness of the individual. An assumption made by this perspective is that “the reality of an object is only perceived within the meaning of the experience of an individual” (Creswell, 2007, p. 59). The paramount purpose of phenomenology is to study the phenomenon of human experiences in various acts of consciousness, especially cognitive acts. Phenomenology is interested in how people
experience and interpret facts and events that take place. It attempts to determine how individuals make sense of different situations in daily life.

When using a phenomenological approach, the researcher attempts to find the meaning of the phenomenon as participants in selected activities construct it. For example, in the present research the researcher asked participants questions in an attempt to extract the phenomenon of integers as it had meaning for the participants. In this manner, the researcher is devoted to “the subjective aspects of people’s behavior” (Bogdan & Biklen, 1998, p. 23). Van Manen (1990, p. 11) explains:

Phenomenology is a human science (rather than a natural science) since the subject matter of phenomenological research is always the structures of meaning of the lived human world (in contrast, natural objects do not have experiences which are consciously and meaningfully lived through by these objects). When using the phenomenological method, “subject and object are integrated – what I see is interwoven with how I see it, with whom I see it, and with whom I am. My perception, the thing I perceive, and the experience or act interrelate to make the objective subjective and the subjective objective” (Moustakas, 1994, p. 59).

When using a phenomenological approach to study participants, one goes from the specific to the general (Creswell, 2007). It is important to study each individual’s understanding of the phenomenon and then work to find a common way of thinking about the phenomenon. For example, in the study by Mukhopadhyay, Resnick, & Schauble
(1990), the researchers were interested in whether students would use calculations or just repeat parts of the story to indicate Sam’s financial condition. They were able to ask the students questions to gain increased insight into their thought processes. Thus they were able to determine that students had difficulty when the result involved crossing the zero boundary. In the present study, the researcher reflected on each individual’s understanding of integer addition and subtraction to find the common ways that participants thought about these operations. Such an approach to analysis is consistent with phenomenological research.

Researchers in phenomenological studies use questioning and interviews to obtain information (Creswell, 2007). These are used to establish the meaning that a conscious person attaches to an event (Willis, 2007). It is believed that when one looks at these experiences, certain essences of the phenomenon will be shared by the participants and by carefully studying these, the phenomenon can be described in greater detail. In the present study the researcher studied participants’ responses to tasks so she could better determine how participants made sense of integers and the operations of addition and subtraction. In many instances, the researcher asked questions so that she could better understand how participants were thinking and a common essence of this phenomenon could be extracted.

According to Seidman (1998), the researcher should examine her own experiences with the topic in order to bracket off her experiences from those of the participants. In order to do this she needs to consider past experiences with the
phenomenon, then present experiences, and finally weave the past and present experiences together to form her essential experience with the phenomenon. This phase is called the epoche. In order to help determine the essence of the phenomenon, the researcher then conducts interviews of participants using this same three-step process. This is called the phenomenological reduction stage. The structural synthesis, which is the final stage, allows the researcher to consider all possible connotations and differing perspectives in order to state the essence of the phenomenon. In this particular study, the phenomenon under examination was how participants made sense of integers and the operations of addition and subtraction.

In the present study, phenomenology was used in the portions of the study that required the researcher to bracket off her experiences from those of the participants. In doing so the researcher could determine the essence of sense-making involving integers from the perspective of the participants. The researcher tried to “get inside the heads” of the participants to see integers as the participants did so that she could understand how they were making sense of integers and the operations of addition and subtraction. This allowed the researcher to see how participants made mistakes and how they followed procedures in correct and incorrect ways.

Sometimes it was difficult to determine exactly how the participants were thinking about integers. In an attempt to make sense of the participants’ thinking, the researcher tried to go back and forth between the theoretical framework delineating how conceptual understanding of integers develops in students and the actual conversations to
try to match the two. At times participants’ thinking lead the participants down a wrong path. In these cases the researcher allowed the participants to carry on the discussion without interference from the researcher to see if they could get back on a viable path. At other times the researcher asked the participants to start problems over again so they could rethink the situation.

Design of the Study

A research design is used to convey the researcher’s procedure for a study. Bogdan and Taylor (1975) interpret design to include conceptualization of the problem as well as the topics of method selection, data collection, and analysis. Upon reviewing the various types of qualitative research, this researcher determined that she would study participants’ initial conceptual understanding of integers using a survey designed by the researcher. She then designed a teaching experiment with six participants utilizing a novel model. The teaching experiment was designed so that the researcher could move beyond uncovering something about the participants’ mathematical consciousness regarding integers and into a more pedagogical domain. As part of the study, participants were asked questions that focused on integers and the operations of addition and subtraction of integers. Many components of the design were discussed in the previous sections related to teaching experiments and phenomenology. The study helped the researcher understand how the participants constructed meaning for integers and the operations of addition and subtraction of integers.
The mathematics topics related to integers that were used during the four sessions were determined by analyzing a research-based mathematics survey that was designed by the researcher. This survey will be discussed in more detail in a later section of this chapter. The survey was completed by seventy-nine students from the researcher’s mathematics content courses in the fall semester of 2007. From the analysis of this survey, questions for four interview sessions relating to concepts of integers and the operations of addition and subtraction were developed. After the initial survey was completed by seventy-nine students, six participants were chosen for the study. Each of three pairs of participants was interviewed four different times, with each session lasting approximately thirty minutes.

Pilot Studies

The researcher conducted two pilot studies during the fall and spring semesters of the 2006 – 2007 school year. In each of these semesters, approximately one hundred students completed the initial survey about integers and the operations of addition and subtraction. Questions for four interview sessions were developed by analyzing the survey results. Each semester a group of six students volunteered to participate in a series of meetings where they were given a variety of tasks involving integers and the novel model. These participants were placed in pairs based on their availability and four interviews were conducted with each pair of participants. The novel integer model was introduced and students used it to model problems and share their thinking about the various integer problems with the researcher.
Through this process the researcher was able to further refine the survey and interview questions to grasp students’ understanding and conceptualization of integer operations. For example, in the pilot studies, the researcher concluded that participants did not see the relationship between the standard algorithm for subtraction with integers and what they had done on the novel model. Participants were able to model subtraction on the novel model but they did not connect it to the standard algorithm for subtraction of integers. This researcher, therefore, asked the participants in the present study to list columns on their paper to show the various stages involved in the subtraction process in an attempt to help them make appropriate connections. The columns were added as the participants tried to record their actions. These columns helped the participants to formulate a framework that helped them organize their thinking.

In the pilot studies, participants did not write comments about the novel model and its impact on their understanding of integer addition and subtraction. The researcher determined that this would also be a beneficial addition to the present study because this would help to uncover participant thinking about the novel model and it would also contribute to the literature about student conceptual understanding of integers using the novel model in an attempt to help them make appropriate connections.

The Present Study

It is well documented that students have experienced difficulty with integers and the operations of addition and subtraction. These claims are well documented in the researcher’s teaching experiences as well as cited in existing research (Carpenter et al.,
The purpose of the present study, then, is to describe and analyze the perspectives of pre-service elementary teachers regarding integers and the operations of addition and subtraction. The components of the present study are the survey, the novel model, the journal entries, and the discussions of the participants.

The mathematics topics relating to integers and the operations of addition and subtraction were addressed in the present study. These topics were determined by analyzing a literature-based survey that was completed by seventy-nine students from the researcher’s mathematics content courses in the fall semester of 2007. This course is the first of two, four-credit hour mathematics content classes that are required of elementary education majors at a public university in northeast Ohio. Total time allowed to complete the survey was approximately twenty minutes and students were not informed as to how they performed on the survey.

From the analysis of this survey, questions relating to integers and the operations of addition and subtraction were developed to be asked of participants in four interview sessions. These sessions were conducted with six of the seventy-nine students. Data collected for this study included the initial survey of problems involving integers, videotapes of the interview sessions, written work done in the sessions, and journal entries collected at the end of two of the interview sessions.
Selection of the Sample

This study involved a convenience sample selected from the researcher’s mathematics content courses. This course is required of all elementary education majors at a university in northeast Ohio. During fall semester of 2007 all students in the researcher’s three sections were asked to answer a survey consisting of twenty-four problems dealing with integers and integer addition and subtraction. A copy of the survey is included in Appendix B. After looking at responses to these surveys, six participants were selected from this larger group for the interview portion of the study. The students selected had to have the time to devote to the study and were willing to be participants. Participants were motivated to be in this study because they recognized that these sessions could increase their understanding about integers. Pairs of participants were interviewed four times, with each session lasting approximately thirty minutes.

Protocol for Each Session

A list of questions that was asked of the participants for each of the sessions can be found by looking at the transcripts which are listed in Appendix D. According to Lincoln and Guba (1985, p. 225) “the design of a naturalistic study . . . cannot be given in advance; it must emerge, develop, unfold . . .” Therefore, the questions were appropriately modified during the sessions to gain more information as the researcher saw fit to add to the research base. All participants completed human subject forms which are housed in the researcher’s locked desk in the mathematics building at the university.
Students were informed that they could withdraw from the study at any time and for any reason.

All of the sessions were videotaped. Participants were informed that the sessions would be videotaped and the researcher allowed the participants to view the videotape and read the transcriptions to make sure that the participants’ thoughts were adequately portrayed. This approach to validating the information is referred to as member checking (Lincoln & Guba, 1985; Miles & Huberman, 1994).

The researcher encouraged the participants to be truthful about any questions that they were asked. In this study codes were used instead of real names of the participants to keep the identity of the participants anonymous.

**Tools and Data Gathering**

To this point in time only two types of models have been used to introduce integers. These are the number line model, which emphasizes ordinality at the expense of cardinality, and the neutralization model, which emphasizes cardinality at the expense of ordinality. The novel model that was used in this study emphasizes both ordinality and cardinality. It also draws upon the participants’ previous understanding of money, a practical application of integers for most college-age students.

The following research questions are addressed in this study:

1. How do pre-service elementary teachers interpret and make sense of integers?
2. Does the use of a novel model impact student understanding of addition and subtraction of integers? If so, how?

3. Do pre-service elementary teachers relate the use of a novel model for addition and subtraction of integers to the rule-based procedures that they use to add and subtract integers? If so, how do they develop meaning for these relationships?

These questions were investigated through a qualitative design that initially consisted of surveys and later involved interview sessions and written work with six participants. At the beginning of each session each participant was given a sheet of paper so they could write the problems that they were given and show how they would solve the problems using the manipulative. The paper was provided so that participants could record their thinking but the researcher did not give any specific directions as to how the paper was to be used. The researcher asked the participants questions related to the topic identified for the session. By working through questions that were provided by the researcher, participants discussed their thought processes by talking out loud so the researcher could hear how they were thinking. Much can be learned by listening to others’ thinking (Ball, 1988). Each pair of participants then came to a consensus for their solution to the problem.

Participants worked in pairs so that they could discuss their thinking during the interview sessions. When a new problem was given, participants were expected to model the problem using the novel model. The novel model consisted of white dollar bills and
red dollar bills that were referred to as “money to spend” and “money owed” to represent positive and negative integers, respectively. The model also included a number line that was white to the right of zero and red to the left of zero. If the participants had difficulty modeling the problem using the novel model, the researcher asked questions which would allow her to determine the source of the participant’s difficulty. She instructed participants to use the novel model so that they could conceptually model integers and demonstrate their understanding of the operations of addition and subtraction. Since very little empirical research exists concerning adult learning of integers, a qualitative approach was deemed to be the most reasonable approach for answering the research questions put forth.

Data Used in this Research

Data consisted of surveys that were completed by seventy-nine students in the researcher’s mathematics content courses. The researcher also collected written work from each participant for every session. Written comments related to the novel model were also included in the data. These comments were recorded by the participants during the second and fourth interviews. These data were evaluated to determine participants’ conceptual understanding of integers and the operations of addition and subtraction. By collecting data from more than one source, triangulation was employed to validate information.

The purpose of this study was to reveal how the understanding of integers develops so that assessments can be designed and materials can be developed to support
that understanding. Because integers are important to the understanding of future mathematics, this study may be of interest to others teaching math content courses. This researcher is not concerned with generalizing results to other populations. This is common in qualitative studies in which the researcher is exploring some phenomenon, rather than showing how variables are related (Creswell, 2002). Human behavior is too complex to gather all the facts about it. Instead, “the qualitative researcher tries to grasp the processes by which people construct meaning and describe what those meanings are” (Bogdan & Biklen, 1998, p. 38).

Analysis of data.

Upon completion of the videotaping, the researcher transcribed the tapes and coded the transcriptions in an attempt to uncover emerging themes. Coding was done to break the data into smaller, more manageable parts. This researcher attempted to shed light on emerging themes or explanations in light of participant actions by viewing the transcriptions and she remained open to renaming categories as the coding required. After the data was coded, it was then reassembled to look for common themes. NVivo7 was the software that was used to help with the analysis.

NVivo7 is a software program that was developed in the United Kingdom specifically to help qualitative researchers make sense of their data by coding data at nodes related to codes that the researcher has created. Although the software can be used to auto code data, this researcher used the software only to store codes and print out excerpts that were related to the same code. In this manner the researcher looked for
interesting relationships among the data. The codes that were developed by the researcher were chosen because they provide a path to understanding how the participants perceive integers and the operations of addition and subtraction.

The researcher used NVivo7 to find common phrases, relationships, and themes in the transcripts and journal entries. She then answered her three research questions by drawing conclusions and from analyzing these common phrases and themes. This process was accomplished by looking back at the data and revisiting the videotapes of the participants’ dialogue. She then examined the survey instrument and the transcripts to explore and question the findings that were found in the literature. Differences were expected based on the fact that this study was conducted with pre-service teachers, while most of the previous research on integers was done with elementary and middle school children.

The researcher determined that a qualitative research design was the best methodology for this type of research because she wanted to study how pre-service students developed conceptual understanding of integer addition and subtraction. Details about the survey, selection of the participants, and data obtained from the interviews will be discussed later in this chapter.

The researcher in the present study is concerned with learning how students think through problems related to integers and how this thinking leads to understanding algorithms for addition and subtraction of integers. The research environment was a classroom, where students are familiar with doing mathematical tasks. Naturalistic
inquiry was employed so that participants would be in a natural, rather than a contrived, artificial environment. When participants experienced difficulty in the interview sessions, they were confident that the researcher would help them make sense of the model and the operations of addition and subtraction.

When students find solutions to problems, the answers themselves can be considered correct or incorrect. The present study is concerned with how students conceptualize integers and whether a novel model will increase their understanding of integers. The researcher is concerned with the underlying thinking that is involved in finding solutions to problems involving integer addition and subtraction. Therefore, a qualitative study was undertaken to find the deeper meaning behind students’ calculations and thought processes. This researcher is interested in exploring how pre-service elementary teachers view integers and the operations of addition and subtraction. She is interested in how students make sense of these operations when a novel model is used. It is possible that this qualitative study may build a foundation for understanding thinking and learning about integer addition and subtraction so that informative quantitative studies on these topics can be designed in the future. It is possible that this qualitative study may build a foundation for understanding thinking and learning about integer addition and subtraction so that informative quantitative studies on these topics can be designed in the future.
In the research on integers, several problems were given to students of various ages to determine difficulties that students have with integers. According to Semadeni (1984) and Wilkins (1996), students had the greatest difficulties when a negative integer was subtracted from another negative integer. Other research (Kuchemann, 1981) indicated that addition problems involving integers were correctly solved by over eighty percent of fourteen-year-old students, but subtraction problems caused difficulty. The researcher felt that this difficulty arose because “most children tried to solve the items by making use of, or inventing, rules” (Kuchemann, 1981, p. 82). Liebeck (1990) gave students ten problems on a post-test dealing with addition and subtraction of integers. All ten of these problems were analyzed by Ernest (1985). She found that students had more difficulty with subtraction than they did with addition of integers. The problem that caused difficulty for most students was one that involved more than one subtraction, for example $-2 - (-2) - (-1)$. Five of nine in the group that was taught using scores and forfeits and seven of nine in the group that was taught using a number line missed this problem. From this research it was concluded that students use meaningless rules in order to find answers to integer problems. The present study attempted to allow participants to connect the rules for operations with integers to conceptual understanding.

Because prior research involving integers investigated problems dealing with only one operation at a time and because this study involved college students rather than elementary school students, this researcher determined that there should be some
two-step problems and some problems that had more than one place. Therefore, problems 4(l) and 4(m) involving addition and subtraction of more than one two-place numbers were included on the survey that was given to students. By doing so, she contributes to the research literature. The other questions on the survey included examples of most types of integer addition and subtraction problems that are noted in the research (Davis et al., 1979; Hativa & Cohen, 1995; Mukhopadhyay et al., 1990; Wilkins, 1996). As a check on content validity, the problems chosen for the survey were based on a careful analysis and reflection of the literature about integers and the errors and misconceptions that students make. Because Wilkins (1996) found that students did most poorly on subtraction problems where a negative integer was subtracted from a positive integer, this researcher included example 4(g) on the survey. Problem 4(k) was the same type of problem and it was included on the survey to determine whether students had more difficulty when the problem involves integers with greater place values.

Wilkins (1996) found that students did significantly better on problems where a negative integer was added to a positive integer in which the sum was positive, where a negative integer was added to a positive to yield a negative integer, and where a positive integer was subtracted from a positive integer to yield a negative integer. For this reason, problems 4(a), 4(c), and 4(f), respectively, were included on the survey for this study. Problems 4(h) and 4(i) were included on the survey to determine if students had more difficulty when the integers had greater place values. Davis et al. (1979) and Hativa & Cohen (1995) found that children have difficulty when adding two negative integers so
this researcher included problem 4(b) on the survey. The researcher also included problem 4n on the survey to determine whether students have difficulty when the problem included numbers with greater place values. Davis et al. (1979) found that fifth graders could not correctly subtract a negative integer from a negative integer when the difference was a negative integer so this researcher included problem 4(e) on the survey. Davis et al. (1979) found that students had difficulty when a positive integer was subtracted from a negative integer to yield a negative integer. Problems 4(d), 4(i), and 4(o) were included on the survey to determine if college students had this same difficulty, where 4(i) and 4(o) were problems that had greater place values. Davis et al. (1979) and Hativa & Cohen (1995) found that students had difficulty with subtraction problems involving the difference between two positive integers that yielded a negative solution. Problems 4(f), 4(h), and 4(j) were placed on the survey to determine whether college students could solve this type of problem, where problems 4(h) and 4(j) involved integers with greater place values. Because Mukhopadhyay et al. (1990) found that students have misconceptions about addition and subtraction of integers when they need to “cross over zero”, the researcher included problems 4(a), 4(c), 4(f), 4(h), 4(j), 4(l), and 4(m) on the survey.

The survey also included problems 5 through 10 in which students were asked to make up word problems that would be solved using integer addition and subtraction. The inclusion of such problems by this researcher was an attempt to add to the research base about student understanding about integers and the operations of addition and subtraction.
In the first attempt at this survey the researcher was trying to find out how students might make up problems in a pilot study.

Validity

In quantitative studies, problems are carefully chosen so that a claim can be made about how much someone knows. This type of approach is meant to verify content validity. Content validity means that the study measures what it claims to measure. In such a study questions are well chosen and validated through research. In qualitative research, however, the word “validity” translates to mean trustworthiness (Creswell, 2007). Trustworthiness concerns the believability of the results of the study. Trustworthiness for this study was addressed through triangulation and member checking.

One of drawbacks of a teaching experiment is researcher bias. Researcher bias can be a threat to the internal validity of a study. Researcher bias exists when the researcher affects the behavior of the participants because of the researcher’s previous knowledge of the participants (Gay & Airasian, 2003). To counter this tendency in the present study, the researcher made use of reflexivity. Reflexivity means that the researcher is aware of her own biases, values, and experiences (Creswell, 2007). This researcher consistently reflected on what participants said and sought to clarify any explanations that were not clear. As a further check on researcher bias, as soon as a session was transcribed, the transcript was shown to the participants to ensure that the events were recorded correctly. When discrepancies occurred, the videotape was
reviewed and corrections were made to the transcription. This is a form of member checking common to qualitative research (Lincoln & Guba, 1985; Miles & Huberman, 1994).

Triangulation is used to reinforce credibility of a study (Opie, 2004). It involves the use of multiple sources of data. In the present study, data was collected through observation of video recordings of the participants and their written work. Findings from the interview sessions were compared with what was found in the research about integers and the operations of addition and subtraction.

Reactivity bias, which exists “when participants in a study respond differently than they normally would in the face of experimental conditions” (Marczyk, DeMatteo, & Festinger, 2005, p. 186), can be a threat to external validity. The present researcher attempted to negate this bias by instructing participants that their participation would not impact their grade in the course. Participants were informed throughout the study that the researcher wanted truthful answers to the questions provided so that she could better understand how the participants were thinking about problems. Participants were also informed that they could choose to withdraw from the study at any time. The six participants attended each of the four sessions. They were motivated to participate in the study because they believed it would help them to better understand integers.

The present researcher is aware of biases that can appear in qualitative research. Through reflexivity, member checking, and triangulation, she countered potential biases of the study.
Reliability

Reliability refers to the credibility of a study. In the present study, a variety of sources was used to support the researcher’s claims. Member checking, as described in the previous section, was employed in this study. Also, a colleague who has a PhD in mathematics education coded three sections of the transcript to verify the researcher’s coding process. The researcher reviewed the codes with this person and gave instruction about the codes. Where there were differences in coding, the researcher further reviewed the coding that was used. Another sample was then given to the examiner to determine if the codes were used correctly. This is standard protocol for determining reliability in a qualitative research study (Silverman, 2006).

Reliability means that the study generates “descriptions, explanations, and models that are meaningful, shareable, and useful to others in situations beyond those in which they were developed” (Lesh, Lovitts, & Kelly, 2000, p. 30). Because the generalizability of the study is important, the present researcher has included the transcripts and details about the novel model so that a future researcher may conduct a similar study.

Timeline for the Study

Each pair of participants met with the researcher for four sessions, with each session lasting approximately thirty minutes. In the first session, the participants discussed what an integer is and modeled it on the modified number line. Participants were asked how they determine where an integer is located on the number line and the researcher recorded their explanation. In the second session, the participants determined
how the model could be used to model addition of integers. The researcher looked for indications that participants saw a connection between using the novel model and the standard algorithm. In the third session, participants determined how the model could be used to model subtraction of integers. During this session, the researcher looked for evidence that the participants were able to relate this to the algorithm for subtraction. In the fourth session, participants were given some addition examples and some subtraction examples to determine that they understood the connection between the model and the algorithms for addition and subtraction. One participant was asked to model addition and subtraction examples on the novel model while the other participant(s) attempted to determine what addition or subtraction example was modeled. Participants then reversed roles. Also in this session the researcher gave participants some integer problems that required bigger bills. Some of these five, ten, twenty, fifty, and one hundred dollar bills were white and some were red to denote positive and negative amounts. When working these problems the participants did not have access to an appropriate number line. These problems were given so that the researcher could determine if participants needed to have both a number line representation and a two-color model to understand problems involving integer addition and subtraction. Table 1 details the particulars of each interview session.
Table 1

Description of Each of the Four, Thirty-Minute Interview Sessions

<table>
<thead>
<tr>
<th>Session Number</th>
<th>Objective</th>
<th>Activity</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>Participants explain what they know about integers.</td>
<td>Participants locate various integers using the novel model.</td>
<td>How would you model four dollars of debt?</td>
</tr>
<tr>
<td>Two</td>
<td>Participants explain how they understand integer addition.</td>
<td>Participants model addition of two integers.</td>
<td>How would you model four dollars of debt added to seven dollars of money to spend?</td>
</tr>
<tr>
<td>Three</td>
<td>Participants explain how they understand integer subtraction.</td>
<td>Participants model subtraction of two integers.</td>
<td>How would you model seven dollars of money to spend subtract nine dollars of money to spend?</td>
</tr>
<tr>
<td>Four</td>
<td>Participants explain how they understand problems involving integer addition and subtraction with big bills.</td>
<td>Participants model addition and subtraction of integers using big bills.</td>
<td>How would you model four dollars of debt subtract six dollars of debt followed by an addition of 40 dollars of money to spend?</td>
</tr>
</tbody>
</table>

For this study, a qualitative study was undertaken to determine how participants conceptually understand the operations of addition and subtraction with integers. The researcher determined that this was the best method to use to obtain answers to her three research questions because a teaching experiment takes advantage of the experiences of the researcher, as well as the personal experiences of the participants (Marshall &
Rossman, 2006). The researcher attempted to extract the essence of the phenomenon as she watched and listened to the participants across the four interview sessions.

**Limitations of the Study**

This qualitative study, reflecting characteristics of phenomenological research and a teaching experiment, was conducted with a small number of volunteer participants. The generalizability of the study, therefore, is threatened and is a limitation of this study. However, this researcher is concerned with how the students in this study understand integers and the operations of addition and subtraction with integers. She is not concerned with generalizing results to other populations nor is she concerned with how variables are related. Researchers who implement qualitative studies commonly explore some phenomenon, rather than showing how variables are related (Creswell, 2002). “Human behavior is too complex to gather all the facts about it so the qualitative researcher tries to grasp the processes by which people construct meaning and describe what those meanings are” (Bogdan & Biklin, 1998, p. 38).

There are several limitations that stem from the methodology used in this study. One limitation involves researcher bias, where the researcher may have interpreted participants’ understanding differently from what was actually taking place (Steffe & Thompson, 2000). Even though the researcher used multiple sources of data such as videotapes of the sessions, journal entries, and participant verification of transcripts to support her conjectures about student thinking, it is possible that the interpretation is not correct. In all research of this type, there is that possibility.
Another limitation is the small sample size for this study and how it was chosen. The researcher used a convenience sample, rather than a random sample. Since this study was concerned with pre-service elementary teachers, it would not make sense to do a randomized study. Due to other commitments, not all students were available to participate in the interview sessions. Also, the researcher wanted participants who were eager to participate and willing to share their thought processes.

The original survey was given to all seventy-nine students in the researcher’s classes during the fall semester of 2007. When students took the survey they were asked if they would like to participate in the interview portion of the study. Of the ten students who volunteered to be interviewed, a set of six participants was chosen based on their availability to attend the interview sessions. Because this study only examined six participants, the generalizability is limited. Also, since the researcher was the instructor for the math content course from which the participants were selected, researcher bias is also considered a limitation affecting the validity of the study. The researcher was aware that instructor bias was possible. These limitations were countered through multiple research techniques and will be discussed in depth in the portion of this chapter that deals with validity.

Since this study was done with adult learners, it was impossible to know in advance what previous instruction and experiences the participants had about integers or what they conceptually understood. This limitation was countered by first giving the
initial survey to determine if students could order integers and perform the operations of addition and subtraction with integers.

This researcher is concerned with how the students in this study understand integers and the operations of addition and subtraction. One goal of this study was to use a unique model that would allow students to understand integers and to see how the algorithms for integer addition and subtraction can be shown to make sense. Participants were first asked to model integers on the novel model, which allowed students to connect the ideas of ordinality and cardinality for integers. Once participants successfully modeled integers on the number line, they were given addition and subtraction problems involving integers so they could visually make sense of the operations using the novel model. After spending two sessions modeling addition and subtraction of integers using the novel model, participants were asked questions which allowed them to make a connection between the standard algorithms for addition and subtraction of integers and their work on the novel model.

In summary, in describing the methodology used in this research design, this chapter detailed the difference between quantitative and qualitative research. It included the different types of questions that are addressed in each type of research. It also discussed teaching experiments and phenomenological studies, the two types of qualitative research that were pursued by the researcher.

The difference in theoretical perspectives of behaviorism and constructivism was detailed so the reader can see the conflicting views that exist about how students learn
and develop meaning about mathematics. Pilot studies that were done by the researcher were then discussed and the researcher showed how these studies impacted the present study. Next, the present study was described, including a discussion of the survey instrument and the four interview sessions that made up the study. Validity and reliability of the study were discussed and limitations of the study were also mentioned. Finally, a timeline for the study was given.

Chapter four will elaborate on the results of this study. The researcher will examine each of her research questions in light of data from the present study. Excerpts from the participants will substantiate claims made by the researcher.
CHAPTER IV: RESULTS

Introduction

This chapter contains a description of the data collected from the initial survey of seventy-nine students along with data that was collected from interviews with six of these students. The six students were interviewed in pairs four different times, with each session lasting approximately thirty minutes. The data for this chapter is organized according to the researcher’s three research questions. The three research questions are:

1. How do pre-service elementary teachers interpret and make sense of integers?
2. Does the use of a novel model impact student understanding of addition and subtraction of integers? If so, how?
3. Do pre-service elementary teachers relate the use of a novel model for addition and subtraction of integers to the rule-based procedures that they use to add and subtract integers? If so, how do they develop meaning for these relationships?

Data for this chapter will include all three sources: the initial survey, interviews, and written work collected from the participants. For the first research question, the data will be presented from the initial survey and excerpts from interview sessions one and two. For the second research question, data from the initial survey and excerpts from interviews two and three will be included. Excerpts from interviews three and four will be used to answer the third research question. The initial survey, titled “Pre-service Teachers’ Understanding of Integers”, can be found in Appendix B.
The initial survey was given to seventy-nine students in the researcher’s math content course in the fall of 2007. Students were given approximately twenty minutes at the end of one class period to complete the survey. They were told that the survey would not count as a graded assignment for the course and if the student was interested in participating in the interview portion of the study, they could indicate a day and time that they were available to meet for each of the next four weeks.

Ten of the seventy-nine students who completed the initial survey showed an interest in participating in the interviews. Two of these students, however, could not commit to the amount of time required by the study. One of the others failed to come to the first interview and so was dropped from the study. Another person participated in the first interview but failed to come to the second interview so he was also eliminated from the study. The remaining six students participated in all four of the interviews.

Participants were interviewed in pairs and the sessions were videotaped. The sessions for Group 1 were held in a small, windowless classroom in the education building of the university. The sessions for the other two groups were held in a small room in the university library. In each of these rooms, a table was placed so that the participants were seated next to each other on the same side of the table, while the researcher sat on the opposite side of the table facing the participants. The novel model was placed so that it was viewed right-side up by the participants and upside down by the researcher. The video camera was placed on a tripod to the right of the participants to capture facial expressions of the participants and to show the participants’ use of the novel model.
At the beginning of each interview session, participants were given a sheet of lined paper. Participants used the piece of paper to answer questions that were posed by the researcher and to record comments about the interview session. Each piece of paper was collected at the end of the session.

Each pair of participants was interviewed four different times, with each session lasting approximately thirty minutes. After each interview session, the researcher transcribed and coded the video tapes according to qualitative research protocol (Creswell, 2007). Coding is a process in which labels, or names, are given to chunks of information. From a thorough review of the data, sixteen codes emerged to classify the major topics discussed by the participants. The sixteen codes that were used were as follows: two colors, meaning of integers, ordering integers, modeling integers with the novel model, meaning of addition, absolute value, participants’ initial examples of addition, modeling addition on the number line, meaning of subtraction, participants’ initial examples of subtraction, modeling subtraction on the number line, relating the novel model to the algorithms for integer addition and subtraction, big bills, writing prompts, use of procedures, and misconceptions. After determining the codes, the researcher read through the transcripts to assign these codes to the data and used the NVivo 7 software to help her quickly access excerpts related to particular categories. NVivo 7 allowed the researcher to store the data and keep track of codes while still maintaining the original documents intact. The researcher then searched the codes and text to look for commonalities and differences related to participants’ conceptual understanding of integers and the operations of addition and subtraction involving
integers. In this manner the researcher captured the essence of participant conceptualization of integers and the operations of addition and subtraction. Table 2 gives examples of how participants’ comments were coded using the sixteen codes.

Table 2

*A Description of the Sixteen Codes Used in the Study*

<table>
<thead>
<tr>
<th>Name of the Code</th>
<th>Frequency of Use in the Four Interview Sessions</th>
<th>Example of the Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two Colors</td>
<td>8</td>
<td>BH: You can use the white bills for the positives and the red bills for the negatives.</td>
</tr>
<tr>
<td>Meaning of Integers</td>
<td>8</td>
<td>I: What are integers?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>LC: Numbers.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>I: (to RV) Do you agree?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>RV: Yeah.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>I: Just numbers? Any number?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>LC: They’re whole numbers. (Not very confident of answer she’s given)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>I: (to RV) Do you agree?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>RV: Whole numbers from the number line.</td>
</tr>
<tr>
<td>Ordering Integers</td>
<td>5</td>
<td>I: Which is greater: four dollars of money to spend or six dollars of debt?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>LB &amp; BH: Four dollars of money to spend.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>I: How do you know?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>LB: Because negative is bad and positive is good. If you have positive, you have more and if you have negative, then you’re short.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>BH: I would say it’s farther to the right of the zero.</td>
</tr>
<tr>
<td>Modeling Integers with the Novel Model</td>
<td>33</td>
<td>I: RV, let’s see three dollars of debt.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>RV: Three dollars of debt? (She put three red bills on the number line.)</td>
</tr>
<tr>
<td>Meaning of Addition</td>
<td>5</td>
<td>I: What does addition mean to you?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>LC: Add something to another thing.</td>
</tr>
<tr>
<td>Absolute Value</td>
<td>7</td>
<td>I: What’s meant by absolute value?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>BH: That is always a positive number. Like if you have the absolute value of negative four, that’s positive four.</td>
</tr>
</tbody>
</table>
Table 2 (continued)

**A Description of the Sixteen Codes Used in the Study**

<table>
<thead>
<tr>
<th>Name of the Code</th>
<th>Frequency of Use in the Four Interview Sessions</th>
<th>Example of the Code</th>
</tr>
</thead>
</table>
| Participants’ Initial Examples of Addition | 3 | I: Give me an example of an addition problem that deals with integers.  
LB: Two plus two. |
| Modeling Addition on the Number Line | 29 | I: Let’s look at six dollars of money to spend (RV placed six white bills on the number line as LC wrote the problem on her paper.) plus five dollars of debt (LC then placed five red bills on the number line as RV wrote the problem on her paper.) So what’s your solution? (LC removed five white bills and five red bills from the number line.)  
LC: One.  
I: And how many zeroes?  
RV: Five. |
| Meaning of Subtraction | 4 | I: What does subtraction mean?  
NB: To take away. |
| Participants’ Initial Examples of Subtraction | 4 | I: Let’s have an example for subtraction of integers. (Each participant wrote an example on their paper.) LC, tell me about your example.  
LC: It’s four minus two equals two. You just have four and subtract, take away, two and you’re left with two. |
| Modeling Subtraction on the Number Line | 11 | I: And let’s try one dollar of money to spend (RV put one white bill on the number line.) subtract one dollar of debt. (LC put a red bill on the number line.) Is that taking away one dollar of debt? (RV had her hands on the stack of white bills, thinking about what to do.)  
LC: No. I am so confused. |
| Use of Procedures | 10 | RV: In the problem it’s four minus a negative one. Um, and that’s really, in the end, adding.  
LC: Right.  
RV: Cause two negatives are a positive. |
### A Description of the Sixteen Codes Used in the Study

<table>
<thead>
<tr>
<th>Name of the Code</th>
<th>Frequency of Use in the Four Interview Sessions</th>
<th>Example of the Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>Misconceptions</td>
<td>10</td>
<td>I: And let’s try one dollar of money to spend (RV put one white bill on the number line.) subtract one dollar of debt. (LC put a red bill on the number line.) Is that taking away one dollar of debt? (RV put her hands on the pile of white bills, thinking about what to do.) LC: No. I am so confused.</td>
</tr>
<tr>
<td>Big Bills</td>
<td>11</td>
<td>I: If you look through those, you’ll see all kinds of bills but I want you to show what thirty dollars of money to spend would look like. What’s thirty dollars of money to spend? (VF put down a $20 white bill and NB put down a $10 white bill.) Okay, that’s thirty dollars of money to spend. From that, I want you to subtract fifty dollars of money to spend. (NB put down a $50 red bill. She started to put it on the number line.) NB: Oh, there’s no number line. I: No number line. NB: Subtract fifty from thirty? I: Subtract fifty from thirty. NB: You’re gonna come up with negative twenty. I: How? NB: Because you’re subtracting fifty from twenty. Wait. Oh my God. Never mind. (VF put down two red ten dollars bills.) She’s right. VF: I forget how, though.</td>
</tr>
</tbody>
</table>
Table 2 (continued)

A Description of the Sixteen Codes Used in the Study

<table>
<thead>
<tr>
<th>Name of the Code</th>
<th>Frequency of Use in the Four Interview Sessions</th>
<th>Example of the Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relating the Novel Model to the Algorithms for Integer Addition and Subtraction</td>
<td>15</td>
<td>LB: And what was your question again? How was the number of zeroes . . .</td>
</tr>
<tr>
<td></td>
<td></td>
<td>I: Related to the addition algorithm. Remember when you did the chart on the back?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>LB: The number of . . .</td>
</tr>
<tr>
<td></td>
<td></td>
<td>BH: Oh yeah.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>LB: It indicates your positive (pause), yeah, your positive addend.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>BH: That would be the . . .</td>
</tr>
<tr>
<td></td>
<td></td>
<td>I: Do you agree BH that the number of zeroes is the positive addend?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>BH: No. It would be equal to the one whose absolute value is</td>
</tr>
<tr>
<td></td>
<td></td>
<td>LB: That’s right. Absolute value.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>BH: The absolute value of, wait, the negative number is larger than the absolute value of the positive number it’s equal to the positive number.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>I: And is that always going to be the case?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>BH: No. If the absolute value of the positive number is greater than the absolute value of the negative number, um, then the number of zeroes is the difference between the two absolute values.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>I: (to LB) Do you agree?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>LB: Yes, I just said it in my head as you were doing it.</td>
</tr>
</tbody>
</table>

| Writing Prompts                                      | 3                                              | BH: It (the novel model) made the idea of why you add when you're actually subtracting a negative number more clear. |

The independent reviewer, whose description was given in Chapter 3, coded data from the first interview for each of the three groups of participants. This sample represents twenty-five percent of all interview data that was collected. The coded data were then compared to the coding done by the researcher. The rate of agreement between the reviewer and the researcher was 92.6%. The items that were coded differently were discussed and agreement was reached for these items prior to the coding of the remaining
interview items. This form of triangulation, known as investigator triangulation, is used to determine inter-rater reliability (Silverman, 2006). The present study also used data triangulation by considering more than one source of data in determining codes and results. These sources of data included a survey, videotapes of the interview sessions, and written work produced by the participants. In qualitative research, triangulation is used to determine consistency of analysis of data by finding multiple sources of confirmation so the researcher can draw a conclusion (Willis, 2007).

The next section includes a discussion of the initial survey that was used in this study. Following that discussion is an introduction to the novel model and its use in this study. The next part of the chapter describes how the initial survey and the interview data were used to support answers to each of the three research questions posed by the researcher. The final section of the chapter explains how the participants viewed their experiences with the novel model.

The Initial Survey

The initial survey was designed by the researcher and its content was based on the literature (Davis et. al, 1979; Hativa & Cohen, 1995; Kuchemann, 1981; Mukhopadhyay, 1990; Semadeni, 1984; Wilkins, 1996). It consisted of ten questions, some of which had multiple parts involving integers and the operations of addition and subtraction with integers. The first three questions of the initial survey provided background information about students’ conceptual understanding of integers. The purpose of the survey was to determine students’ interpretations of the practicality of integers, what students know about integers, and where they use integers. The survey was also used to determine
students’ ability to add and subtract with integers. For more detailed information about the survey instrument the reader should refer to the section in chapter three titled “The Survey Instrument”. A list of the problems on the survey is presented in Appendix B.

In the first question on the initial survey, students were asked where they use integers in their everyday lives. The purpose of this question was to determine students’ sense of practicality of integers. The next two questions required students to order integers. The fourth question was a series of fifteen problems involving symbolic addition and subtraction of integers. The remaining six questions of the survey asked students to give examples of word problems that would be solved using addition or subtraction of integers. For example, one of the questions asked students to make up a word problem whose solution would be $17 - (-2)$. The results from the survey were used to partially answer research questions one and two.

The Novel Model for Integers

The novel model for integer addition and subtraction is a combination of the two existing models currently used to teach integers. These two models are the number line model and the two-color chip model. By using this blended innovative model, the participants were able to see ordinality of integers as usually shown on a number line, as well as cardinality of integers, which is typically demonstrated using the two-color chip model. This novel model was designed to maximize the contextualization and conceptual understanding of integers.

The novel model for integers includes a number line that is white to the right of zero and red to the left of zero. This researcher made a deliberate choice to make zero a
different color than the positive or negative integers. She chose green to show that zero is neither positive nor negative. Zero, instead, separates the positive integers from the negative integers. The model also includes play money in two colors, white to represent positive amounts and red to represent negative amounts. In this study, positive amounts were referred to as “amounts of money to spend” and negative amounts were called “debt”. Each bill has Velcro on the back in the middle of the bill. So that the bills will stick, the number line also has Velcro in the middle of each interval between two consecutive integers. Figure 1 shows the novel model.

![Image of the number line portion of the novel model for integers]

*Figure 1. The Number Line Portion of the Novel Model for Integers*

The participants were interviewed in pairs, except for session 1 with group 1, which had three members. For the first interview session, Group #1 consisted of LC, RV, and CA. CA dropped out of the study after the first interview because he could not find time to participate. Group 2 consisted of BH and LB; and Group 3 consisted of NB and VF. All of the students were females except for CA. Approximately fourteen percent of the initial survey respondents were male and this ratio is reflected in the composition of
the class from which this sample was drawn. Since most elementary education majors at this university are female, it is not unusual to have a class that has no male members.

The dialogues use name codes to represent participants and “I” to represent the researcher. Name codes were used to maintain anonymity of participants. This is consistent with recommendations for qualitative research (Bogdan & Biklen, 1998).

The questions for each of the four semi-structured interviews were determined by reviewing and reflecting on student responses to items on the initial survey and research regarding how students develop these ideas. Specific questions and tasks for the interview sessions were designed prior to the interview sessions to explore how participants make sense of integers. During the course of the interview sessions questions were added as needed to facilitate participants’ conceptual understanding of integers. The sequences of questions reflected the developmental levels of learning about integers and uncovering the reasons for the algorithms for addition and subtraction with integers.

In the interviews, day one was spent identifying integers as they relate to debt and money that can be spent. Participants were shown the novel model that was designed by the researcher for use in this study. It consisted of two colors of money and a color-coded number line that was the same two colors. Participants used the novel model to show amounts of money to spend and debt as they relate to positive and negative integers. The researcher’s goal for the first session was participants’ recognition of the two different colors of bills as they relate to “money to spend” and “debt”. Once participants recognized the significance of the two colors of money, they could then model various amounts of money and debt using the novel model.
In the first session, the participants were also asked to model amounts of “money to spend” and debt in more than one way using the novel model. Participants added the same number of each color of bill to the given amount and reported that the value did not change. For example, to name $-3$ in another way, the participants could put three red bills on the number line portion of the novel model. To name this integer another way, participants could add one more red bill and one white bill to the given amount to have another way of naming $-3$. This activity proved to be helpful later when modeling addition and subtraction using the novel model.

In the second interview session, participants described the elements that make up the set of integers and they used the novel model to show addition of integers. Participants recognized the two colors of money and associated the white money with “money to spend” and the red money with debt. The participants modeled various amounts of money to spend and debt on the number line portion of the novel model and then added the two given amounts. Participants first counted the number of red or white bills indicated by the first addend and placed these on the number line portion of the novel model. Then, they counted the number of red or white bills indicated by the second addend and placed these on the number line portion of the novel model. The final step required participants to remove equal amounts of red bills and white bills so that only one color of money remained on the number line portion of the novel model. This process of removing equal amounts of red and white bills was referred to as removing “zeroes”. After participants modeled several examples using the novel model, the researcher helped participants explore the algorithm for addition of integers. In other words, the participants
conjectured that when integers with the same sign are added the sum is the sum of the absolute values of the addends and the common sign is appended to the sum. When the two integers are different signs, the absolute values of the addends are subtracted and the sum is given the sign of the addend that has the greater absolute value. The goal for this second session was for participants to model addition of integers so they could relate the algorithm for addition of integers to their representations using the novel model.

In the third interview session participants modeled subtraction of integers using the novel model. Subtraction was modeled on the novel model as “take away” so that participants could transfer their understanding of subtraction with whole numbers to subtraction with integers. The participants first modeled the sum and, if possible, took away the required amount. If it was not possible to take away the required amount, the participants added zeroes (an equal number of red and white bills) until the required amount could be taken away. The goal for this session was for participants to understand subtraction as take away and to model various subtraction problems using the novel model.

The fourth interview session related participants’ use of the novel model to the algorithm for subtraction of integers. Through the researcher’s questions the participants were led to discover that when subtracting integers the subtraction sign is changed to addition and the given addend is changed to its opposite. The rules for addition of integers, which was the focus of interview session three, are then followed. The fourth interview session also provided addition and subtraction problems that involved integers that could not be modeled on the number line portion of the novel model because the
absolute value of each of the integers was greater than fifteen. For these problems the participants were given red and white bills of greater denominations to model addition and subtraction problems.

For the first example, the participants were asked to model thirty subtract fifty. Because the number line portion of the novel model only had integers between negative fifteen and positive fifteen, participants could only rely on the two-color money portion of the novel model for this part of the interview. In this fourth interview session the researcher sought to see if the participants could apply their thinking from earlier sessions when they used the novel model with representations for lesser amounts. The researcher wanted to determine if participants could visualize these problems in the same manner as when they had both aspects of the novel model. In other words, the researcher wanted to determine if both parts of the novel model were necessary for the participants to make sense of the problems.

Data to Support Research Question Number One

The data collected to answer the first research question came from questions one through three of the initial survey along with coding from the first two interviews for each pair of participants. Each of these sources will be discussed separately to determine their significance in answering the researcher’s first question.

Data from the Initial Survey

The first research question is “How do traditional pre-service elementary teachers interpret and make sense of integers?” The researcher wanted to determine if students had a conceptual understanding of integers or if they only use them in symbolic form. The
first three questions of the initial survey addressed this question. Question #1 asked participants to state where they use integers. Responses to this question were categorized and tabulated.

In a preliminary analysis of responses to question #1, the following categories emerged: school-related uses of integers and practical applications. There were also such disparate responses that it was decided to include a category of “other”. Some of the most common responses considered in this category were “everywhere” and “every day”. Thus there were three categories considered when classifying students’ responses to this question: school related, practical applications, and other. Eleven students had no response and one student wrote “I don’t know”. Thirty-eight of the students who listed activities where integers are used had a response that involved only one category, twenty-four listed activities from two of the categories, and five listed activities from all three of the categories. Table 3 summarizes the responses to question #1.

Table 3

Students’ Perceptions of Where Integers are Used

<table>
<thead>
<tr>
<th>School-related</th>
<th>Practical Applications</th>
<th>Other</th>
<th>No response or “I don’t know”</th>
</tr>
</thead>
<tbody>
<tr>
<td>37</td>
<td>40</td>
<td>24</td>
<td>12</td>
</tr>
</tbody>
</table>

Number of students with this response, N = 79

Note. Some students gave more than one response.
Only one student reported that integers are used when recording temperatures. Of the responses given, this response was the only practical application of integers that could require negative integers but not other types of rational numbers such as fractions or decimals. Of those who mentioned school-related activities, twenty-seven students mentioned that integers are used in math class. Thirteen students responded that integers are used “everywhere” and were included in the “other” category.

Several students listed activities that only involved non-negative integers. For example, seven students listed “counting” and two mentioned “phone numbers”. Other students listed practical applications that could use integers but that usually involve rational numbers. For example twenty-seven of the seventy-nine students mentioned banking or money, but did not elaborate on what this had to do with integers. Five others listed carpentry but did not tell how this involved integers. The researcher determined that during the interview portion of the study, presentation of the novel model would allow participants to explore zero and negative integers, as well as the positive integers with which they were already familiar.

Question #2 of the initial survey asked students to determine which is greater, \(-2\) or \(-5\), and to state the reason for their response. The researcher wanted to know if students understood ordering in the set of integers. The researcher used the interview sessions to determine if ordering of integers is something that students have memorized or if there is some meaningful understanding behind it. Depending on how participants responded to this question, in the interview portion of the study the researcher probed with other questions or problems to get participants to think differently about their ideas.
Of the seventy-nine student responses, two had an incorrect response and another student did not give an answer but stated, “yes, because it is”. One student gave an incorrect response but provided no reason for her answer. Another person who had an incorrect response stated that $-5$ is greater than $-2$ because $5$ is greater than $2$. The seventy-six correct responses to this question are summarized in Table 4.

Table 4

Student Reasons Why $-2$ is Greater Than $-5$

<table>
<thead>
<tr>
<th>Reason</th>
<th>Number of Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$ is closer to 0,</td>
<td>51</td>
</tr>
<tr>
<td>or closer to the whole numbers,</td>
<td></td>
</tr>
<tr>
<td>or closer to the counting numbers</td>
<td></td>
</tr>
<tr>
<td>$-2$ is to the right of $-5$,</td>
<td>3</td>
</tr>
<tr>
<td>or $-5$ is to the left of $-2$</td>
<td>9</td>
</tr>
<tr>
<td>$-5$ is taking more away</td>
<td>2</td>
</tr>
<tr>
<td>$0 &gt; -2 &gt; -5$</td>
<td>1</td>
</tr>
<tr>
<td>No reason for the response</td>
<td>10</td>
</tr>
</tbody>
</table>

Number of responses, $N = 76$

Question #3 of the initial survey asked students to state how much greater $9$ is than $-4$. Sixty-one of the seventy-nine students had the correct response of $13$. The incorrect responses given by seventeen students are summarized in Table 5.

Table 5

Incorrect Student Responses to “How Much Greater Than $-4$ is $9$?”

<table>
<thead>
<tr>
<th>Response</th>
<th>Number of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>1</td>
</tr>
<tr>
<td>14</td>
<td>6</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
</tr>
</tbody>
</table>

Number of students with this response, $N = 17$
The researcher asked this question to determine if students would use a mental number line to count hash marks of the integers or if they counted the units between the given integers. Almost twenty-two percent of the students had incorrect responses to this question. Therefore, during the interview portion of the study, the researcher asked participants questions about absolute value so they could better determine distances between two integers.

Data from Interviews

Data from the first and second interviews of each group of participants was used to answer the first research question. For clarity of analysis, the discussion is broken into eight sections. These sections are “participants’ initial reaction to two colors of money”, “participants’ reaction to the color-coded number line”, “participants give examples of integers”, “participants’ initial introduction to the novel model”, “participants model zero”, “participants model zero in more than one way”, “participants name integers in more than one way”, and “participants’ understanding of absolute value”. Each section includes a brief explanation of the researcher’s rationale, each group’s response to the prompt, and a summary.

Participants’ Initial Reaction to Two Colors of Money

Participants were asked why they thought two colors of money were used. The researcher was interested in whether participants would make the connection to debt and money to spend without seeing the color-coded number line. Their responses follow.

Group 1’s response:

I: (Placing a white bill on the table) Has either one of you ever seen one of those?
CA: A real one?
I: The real one. Have you seen that?
LC: Yes.
I: What is it?
CA: A dollar.
I: (Placing a red bill on the table) What’s that?
LC: One dollar.
I: Are they both the same?
CA: They’re different colors.
I: They’re different colors. Why do you suppose they’d be different colors?
CA: No idea.
LC: I don’t know.

Group 2’s response:

I: I have some stuff in here (opening the container that contains red bills and white bills). I want you to tell me if you’ve ever seen any of those (place a white bill on the table).
LB: No.
BH: No.
I: You don’t know what that is?
LB: Well, it’s a fake dollar bill.
I: So you’ve seen some money before? You’ve had some experience with money?
LB & BH: Oh yeah.
LB: Just a little. (Putting a red bill in front of the participants) I: I also have that. What do you notice about those things that I showed you?
BH: They’re all the same except they’re different colors?
I: How many different colors?
BH: Two.
I: Why do you suppose that I would have two different colors of money?
LB: So you could tell the difference in what you were doing.
I: In what way?
LB: Whatever you’re using it for, you could separate your uses?
I: What sorts of uses would you have?
BH: If you’re teaching someone how to subtract.

Group 3’s response:

I: (Placing a white bill on the table) Have you ever seen one of these?
NB: Uh huh.
I: What is it?
VF: A dollar bill.
I: So you’ve seen one of those at some time? Maybe even a couple of them?
VF: Uh huh.
I: (Placing a red bill on the table) What about that?
NB: The same thing.
I: The same thing except . . .
NB: It’s red.
I: Why do you suppose that I would have two different colors of money?
NB: I don’t know. You’ll come up with something.
I: You don’t have any idea why I would have two different colors of money?
NB: The only thing I can think of is like some kind of problem you’re gonna like (putting hands together again and again) I don’t know.

In summary, all students were familiar with money and recognized that the money was two different colors. However, without seeing the color-coded number line, none of the participants could initially indicate the reason for two colors of money. The researcher assumed that the participants would recognize that there were two colors of money but she did not expect them to know why there were two colors. At this point she did not create problems dealing with the white and red money because the participants had not yet seen the number line portion of the novel model that was also red and white. The researcher instead listened for participants’ explanations for the two colors. By listening to participants’ ideas about the reason for two colors the researcher intended to gain a better understanding of how participants were thinking about the two colors of money and to determine if they were thinking about “debt” and “money to spend” as the two types of money. Prior research indicates that in many cases school learning is distilled from reality and that is the reason students find it difficult to remember how to operate with integers (Bruner, 1966). The researcher intended to relate participants’
understanding of “money to spend” to positive integers and their understanding of “debt” to negative integers. She determined that this might increase their conceptual understanding of integers by applying previous practical applications with money. Although other researchers (Bolyard, 2005; Crowley & Dunn, 1985; Mukhopadhyay et al., 1990) used money to discuss integers, no other study incorporated the concept of two colors of money together with ordinality of integers, as depicted on the novel model that is featured in the present study.

**Participants’ Reactions to the Color-Coded Number Line**

By presenting the color-coded number line, the researcher was interested in whether participants had seen a number line before and if they would relate the two colors of money with the two colors found on the number line. Furthermore, the researcher wanted to determine if the participants would associate the two colors with “money to spend” and “debt”. Tasks were designed by the researcher to help participants transition from the words “money to spend” and “debt” to “positive integers” and “negative integers”. The following responses were given when participants were shown the color-coded number line.

**Group 1’s response:**

I: Is this like the number line you’ve seen before?
LC: Yes.
I: Exactly?
LC: No. It’s two different colors.
I: Why do you suppose it’s two different colors?
CA: Negative and positive.
I: What do you notice about those two different colors? What are the two colors that are there?
LC: Red and white.
I: Does that sort of relate to the money?
CA: Yeah, the red’s negative and the white’s positive.
I: So, if I said to you, “How is this different?” You said it’s two different colors and you can see that negative’s going to be red and white’s going to be positive. So how would you think about the money in those terms? What would the red money signify?
LC: I would think money you don’t have.
I: Money you don’t have?
CA: Yeah.
I: What would the money you don’t have be called?
LC: Debt.
I: And what would the white money be?
LC: The money you have.

Group 2’s response:

I: What do you notice about that number line that is different from other number lines that you’ve seen before?
LB: It’s color coded.
I: How is it color coded?
LB: Negatives are red; positives are white.
I: And do you think that has anything to do with those two different colors?
BH: You can use the white bills for the positives and the red bills for the negatives.

Group 3’s response:

I: So the colors are different. What do you notice about the colors? How many colors are there?
NB & VF: Two.
I: And what colors are they?
NB: Red and white.
I: Does that sound like something you’ve seen in the not so distant past?
VF: The money.
I: So what do you suppose that red money means?
NB: Negative.
I: And what would negative money be?
NB: You don’t have any.
I: Okay, and worse than that . . .
VF: Debt.
I: Debt. And white money, which is more like the money that you normally see, would be . . .
NB: Positive.

In summary, all of the participants recognized that the money and the number line were the same two colors, red and white. When the participants in Group 1 replied that the number line was like other number lines, the researcher questioned their comment to make sure they understood the significance of the two colors. All participants also related the red money with the negative integers on the red side of the number line and the white money with the positive integers on the white side of the number line. All participants in groups 1 and 3 also recognized that red money represented “debt”.

Participants Give Examples of Integers

During the first interview session, participants modeled amounts of debt and money to spend using the novel model but the researcher did not mention the word “integer” because she wanted the participants to concentrate on the two types of money so they could relate this to integers in the second interview session. The important aspect of the first interview session was understanding zero as a divider between “debt” and “money to spend” since this was reported as being important in prior research (Lytle, 1992).

At the beginning of the second interview session, the researcher asked the participants in all three groups what numbers make up the set of integers. The researcher expected the participants to state their responses orally and she did not expect them to give a precise answer to this question because the initial survey indicated that students do
not understand what elements make up the set of integers. A search of prior research did not report any studies where students were asked to identify members of the set of integers.

For participants to completely understand the operations of addition and subtraction with integers, it is important to know what numbers can be considered for these operations. To demonstrate that participants understood what numbers make up the set of integers, the researcher asked them about integers. Their responses follow.

Group 1’s response:

I: What are integers?
LC: Numbers.
I: (to RV) Do you agree?
RV: Yeah.
I: Just numbers? Any number?
LC: They’re whole numbers. (Not very confident of answer she’s given)
I: (to RV) Do you agree?
RV: Whole numbers from the number line.

Group 2’s response:

I: What are integers?
BH: I think they’re whole numbers.
I: Whole numbers?
LB: We just went over this too. They're numbers that can be negative, positive or I always want to call them fractions too but that's not it.
BH: Are fractions integers?
LB: No, those are irra…rational numbers.
BH: I keep wanting to say like the set that includes…
LB: Exactly. No, they're counting numbers. Those are whole numbers. You can't have zero, negative, or positive.
BH: Negative one’s still an integer, though.
LB: Yeah, but if the whole numbers include zero, the counting numbers do not. That’s what I said.(Both wrote what they think integers are.)
I: So what are integers?
BH: I put the set that includes the whole numbers, negative or positive.
LB: I put whole numbers on the number line that can be positive or negative.

Group 3’s response:

I: So what are integers?
(Long pause) VF: Numbers.
I: Numbers?
NB: That’s what I kept thinking.
I: Just any numbers?
NB: Well, it’s like the thing you were explaining to us like whole numbers, that whole . . . that thing we did on the board (sets of numbers and which are subsets of other sets) and you had like every, like . . .
I: Oh, the sets of numbers?
NB: Yeah. For some reason when I think of integers that’s what I think of. Which I know is completely wrong but . . .
I: OK, can you give me some examples of integers?
VF: Positive or negative numbers.
I: OK, what about something like one-half? Is one-half an integer?
VF: Could be.
NB: I don’t know. I kinda think it is though.
VF: I think so.

In summary, none of the participants was sure what numbers make up the set of integers. Participants in Groups 1 and 3 initially mentioned that integers were numbers. Upon further questioning, Group 1 claimed that integers are whole numbers and Group 3 was not sure whether the integers included fractions but knew that integers could be positive or negative numbers. Group 2 indicated that the terms “integers” and “whole numbers” were synonymous. In the next portion of the interview, the researcher introduced the participants to the novel model so they could better understand which numbers make up the set of integers.
**Participants’ Initial Introduction to Novel Model**

During the first interview session, participants were asked to model four dollars of money to spend, zero dollars, and four dollars of debt using the new model. Figure 2 shows how three dollars of debt could be modeled using the novel model.

![Three dollars of debt](image)

*Figure 2: One way to model three dollars of debt (negative three) on the novel model*

The researcher wanted to make sure that the participants understood that debt should be modeled using red bills on the red side of the number line and money to spend should be modeled using white bills on the white side of the number line, again emphasizing that zero is the divider between positive and negative amounts. Excerpts from the interviews follow.

**Group 1’s response:**

I: How would you show four dollars of debt on the number line? (CA put four red bills on the number line.)

**Group 2’s response:**

How would you model two dollars of money to spend? (LB placed two white bills on the number line.)

I: (to BH) Do you agree?

BH: Yes.
Group 3’s response:

I: So if I ask you to look at this (number line) and model three dollars of money to spend using your number line, how would you model three dollars of money to spend? (VF placed two white bills on the number line and she directed NB to place another white bill on the number line.)

In summary, when participants were shown the color-coded number line, they quickly related the white money with money to spend and the red money with debt. All three groups were able to model money to spend and debt on the color-coded number line using the white and red money, respectively. All participants consistently modeled debt as red money to designate negative integers and they modeled money to spend as white money to show positive integers. Each group understood that zero was the divider between “money to spend” and “debt”. In the next task, the researcher asked participants to model zero using the novel model.

*Participants Model Zero on the Novel Model*

Participants were asked how zero could be modeled using the novel model. The researcher anticipated that participants would initially model zero as the number line with no bills because this relays a student’s earliest conception of zero (Hativa & Cohen, 1995; Levenson, Tsamir, & Tirosh, 2007; Lytle, 1992). The following excerpts from the interviews present participants’ understanding of zero.

Group 1’s response:

I: How would you model zero dollars on the number line?
CA: It would be just that. (The number line without any bills on it.)

Group 2’s response:
I: How would you model zero dollars?
BH: Just don’t put anything on it (the number line).

Group 3’s response:

I: How could you model zero dollars on the number line?
VF: You can. Just don’t put anything on the number line.
I: Do you agree?
NB: Yes.

Initially, all participants in the three groups thought of zero as representing nothing. This finding is consistent with prior research (Hativa & Cohen, 1995; Levenson, Tsamir, & Tirosh, 2007; Lytle, 1992).

Because of research by Lytle (1992), the researcher determined that participants should be able to model zero in more than one way. Zero is an important concept when modeling addition and subtraction of integers with the novel model and will be discussed at greater length in later sections. Figure 3 shows one way that zero can be modeled is by placing the same number of red bills as white bills on the number line.

Figure 3: Zero modeled on the novel model
Participants Model Zero in More Than One Way

Since participants initially modeled zero as the absence of objects, the researcher wanted to know if this was the only meaning that they attached to zero. Participants were then asked to model zero in other ways. Their responses follow.

Group 1’s response:

I: Is there another way you could model it (zero)? (Without any hesitation LC put a red bill and a white bill on the number line.)
I: Do you agree CA?
CA: Yes.
RV: You could leave it like it is or you could just do that (She put a red bill and a white bill on the number line.)

Group 2’s response:

I: Is there another way you could model zero dollars?
(LB placed a white bill on the number line as BH placed a red bill on the number line.)
I: Is there another way you could model zero dollars?
(LB placed another white bill on the number line and BH placed another red bill on the number line.)
I: So what do you notice? How are you modeling zero?
BH: You have to have the same amount of negative as positive.

Group 3’s response:

I: Could you show zero another way? (NB took the bills off the number line.) And another way?
NB: There’s another way?
I: I don’t know. Is there another way?
VF: Two here (pointing to the positive side of the number line) and two there (pointing to the negative side of the number line).
I: So what’s the important thing? Showing zero, it’s important that you are putting on what?
VF: Equal amounts.
In summary, all participants in Groups 1 and 2 seemed to already have a notion of zero and could model it as “nothing” or as matching sets of white and red bills. There was no hesitation from any of the participants except for NB in Group 3. NB did not initially know of another way to model zero but VF was able to model zero in other ways and she was able to state that one would need to add the same number of positives and negatives to zero to maintain a value of zero.

*Participants Name Integers in More than One Way*

Recall that participants were previously required to respond to questions such as, “How would you show three dollars of money to spend on the number line?” In each group participants were asked if there were other ways to model a given integer. This prompt was designed to delve deeper into the students’ understandings of integers.

Although prior research did not stress the ability to name integers in multiple ways, this researcher determined that the ability to name integers in many ways is fundamental to understanding addition and subtraction of integers when using the novel model.

When discussing addition and subtraction of integers, this researcher focuses on two important ideas related to zeroes. First, when integers are added using the novel model, it is important to remove zeroes, thus naming the sum in a different but equivalent way. Second, when integers are subtracted using the novel model, it is important for students to recognize that zeroes can be added to integers without changing their value so that the required amount can be taken away. In the next task, the researcher asked the participants to name integers in more than one way using the novel model. The following
excerpts show participants’ responses when they were asked to name integers in more than one way.

Group 1’s response:

I: Can you show three dollars of money to spend another way?  
(CA put three red bills on the number line and LC put three more white bills on the number line.)  
I: Is that three dollars of money to spend?  
CA: Uh huh.  
I: Could you show three dollars of money to spend another way?  
(LC put another white bill on the number line as CA put another red bill on the number line.)  
I: What are you really doing?  
CA: A negative and a positive.

Group 2’s response:

I: Could you show two dollars of money to spend another way?  
BH: You want to add two and I’ll add two. (BH placed two red bills on the number line as LB placed two white bills on the number line.)  
I: So, in effect, what are you putting on there to keep the same value?  
BH: The same amount on both sides of zero.  
I: Which is really pairs of positive and negative which is really . . . What is a positive and a negative?  
LB: Zero. I just remembered that.

Group 3’s response:

I: Is there another way you could model three dollars of money to spend?  
NB: You could just pull the money off and put the three over here?  
I: Put the three over where?  
NB: Just put the one on at three and not have the other two on there.  
I: You could but using those three, is there something else that you could do to make it three dollars of money to spend?  
VF: Could you put one (a white bill) here (on the positive side of the number line) and a negative over there (on the red side of the number line)?  
I: Do you agree NB?  
NB: Yeah that works.
In summary, all of the participants in all three groups modeled various integer amounts using the novel model and they named these integers in more than one way. The participants in Group 1 and Group 2 initially thought that the number of positives and negatives that needed to be added had to be the same as the initial amount given. For example, they put three white bills on the number line to show three dollars of money to spend. They then added three more white bills and three red bills to show another amount that was equal to three dollars of money to spend.

Participants knew to add the same number of red bills and white bills to the previous amount in order to keep the same integer value. In Group 3, NB did not initially grasp the idea but agreed that there was a second way to model positive three when it was modeled by her partner, VF. The researcher determined that naming integers in various ways would allow participants to make sense of the addition algorithm. For example, when the signs of the addends in an addition problem are different, the participants could better understand why the magnitude of the sum is the difference between the absolute values of the addends.

The addition algorithm also requires that students understand what is meant by absolute value. Creswell and Forsythe (1979) state that understanding absolute value is crucial to understanding integers as additive inverses. For example, zero is obtained when an integer is added to its additive inverse. This understanding is also crucial when modeling addition and subtraction with the novel model. To find the solution to an addition problem involving integers of opposite signs one must determine the absolute value of each addend and then subtract these absolute values. The sign of the sum is
determined by the addend that has the greater absolute value. To find the solution to a subtraction problem involving integers, the subtraction sign is changed to addition and the addend is changed to its additive inverse. Rules for addition of integers are then followed. In later sections the researcher will discuss how the participants used the novel model to conceptually understand these algorithms.

Participants’ Understanding of Absolute Value

In the first interview session, after participants modeled various amounts of money to spend and amounts of debt, they were asked the meaning of absolute value. The following quotes show how participants responded to the question “What is meant by absolute value?”

Group 1’s response:

I: What’s meant by absolute value?
LC: A number without, like even if it’s negative, you just take the negative off. It’s like negative one, the absolute value is one.
I: (to CA) Do you agree?
CA: Yeah.
I: What’s meant by absolute value, RV? What’s the absolute value of negative four?
RV: Four.
I: How did you know?
RV: Because even though it’s on the negative side, it’s still showing that you have four dollars in some way.

Group 2’s response:

I: What’s meant by absolute value?
BH: That is always a positive number. Like if you have the absolute value of negative four, that’s positive four.

Group 3’s response:
I: Do you know what absolute value is? If I said, “What’s the absolute value of negative four?”
NB & VF: Four.
I: How did you know that?
NB: It’s the opposite of whatever that is. I mean if it’s a negative. You turn the negative into a positive.

In summary, five of the six participants defined absolute value as the magnitude of the integer. One student, NB, gave directions one should follow to find the absolute value of a number. All participants in all groups could find the absolute value of an integer but did not define absolute value in any terms except for looking at the magnitude of the integer. The researcher noted that participants defined absolute value as the distance from zero on the number line. This definition is typical of the definition for absolute value that is given in textbooks. Therefore for the next set of tasks the researcher asked the participants for their definition of addition. She wanted to determine if participants used their definition of absolute value to understand addition of integers when one of the addends is a positive integer and the other addend is a negative integer.

Summary of Responses that Informed Research Question Number One

In summary, the survey revealed that students use integers most of the time for school activities and practical applications such as banking. Although most students were able to give at least one example where integers are used, almost eighteen percent indicated that they didn’t know where integers are used. Also, the majority of practical applications for integers that were listed by the students on the survey involved only positive integers. In the interview portion of the study, participants were introduced to the novel model using “money to spend” and “debt”. Through questioning by the researcher
and use of the novel model, these words evolved into “positive integers” and “negative integers”, suggesting that participants’ introduction to the novel model expanded their notion of integers to include both positive and negative integers. The researcher consistently used the words “money to spend” and “debt” when using the novel model but the participants consistently wrote the amounts using “+” and “–”. This may have been done because these symbols were more convenient and quicker for them to write.

On the initial survey, seventy-six of the seventy-nine students stated that – 2 is greater than – 5 and the most frequent reason given involved proximity to zero. This caused the researcher to reflect on the importance of ordinality of integers. Consistent with research on integers (Creswell & Forsythe, 1979; Lytle, 1992), this researcher deemed that questions about absolute value would be more relevant to conceptual understanding of integer addition and subtraction than merely knowing which of two negative integers was greater. Therefore participants were asked about the absolute value of several integers and then were asked which of two integers had the greater absolute value.

In question #3 of the initial survey the researcher attempted to find out if students could determine how much greater 9 is than – 4. She asked this to see if students counted numbers between the two amounts given by including both of the integers, or just one of the integers, or neither of the integers. She found that six of the seventy-nine students recorded an answer of fourteen, presumably by counting forward from – 4 to 9 to include both integers in their count. This finding is consistent with research involving construction of mental number lines, since the number line stresses the concept of
ordinality (Peled, et al., 1988). The researcher reflected on this finding during the interview sessions and she asked participants to model various integers using the novel model. Integers were written on the number line portion of the novel model to indicate their ordinality. Because the novel model incorporated ordinality with cardinality, participants were able to count out the required number of red bills or white bills (indicating cardinality of the integer) and place them on the number line without concerning whether they should count hash marks or spaces. The researcher noted that all participants were able to correctly model the given integers. During the interview session the researcher also asked participants which of two given integers is greater. She then asked them to explain their answer and tell which of the two integers had the greater absolute value.

Five fundamental things that one must understand in order to have a meaningful interpretation of integers were advanced by Lytle’s research in 1992. First, students must understand that zero is neither positive nor negative but instead separates the positive and negative integers. Second, students must be able to order integers. Third, students should know that “+” and “-” can show direction for integers as well as indicate the operations of addition and subtraction. Fourth, students must know what it means to be an additive inverse and understand that each integer has an additive inverse. Her fifth requirement requires students to understand that addition and subtraction are inverse operations.

When the present researcher tried to understand the meaning of her findings, comparisons were made to Lytle’s conjectures. In this research study, however, the last two requirements were meshed with ideas of absolute value as it relates to the algorithm
for subtraction. Because of the research that has been established on understanding of integers, this researcher pursued this line of inquiry in order to shed more light on how integers are conceptually organized and used. These ideas were considered in research questions two and three of the present study.

From the initial survey the researcher determined that students were not able to name relevant applications of integers. Almost forty-seven percent of the students reported that integers are used in school activities and most of the practical applications that they recorded involved only positive integers. This issue was addressed in the interview sessions using a novel model that used money as a practical application of integers.

This researcher used the interview sessions to uncover several fundamental concepts about students’ conceptual understanding of integers. She wanted to see if the ability to name integers in more than one way aided this conceptual understanding. Participants were able to name integers in many ways during the interview sessions. By naming integers in more than one way, subtraction as “take away” could be modeled for all cases using the novel model. In this manner participants could visualize zero as some number of positives and an equal amount of negatives, rather than as the absolute zero they may have thought of in the past.

Using Lytle’s framework, participants’ responses are categorized in Table 6 below. The data from the survey and the interviews were sorted according to participant responses relating to each of the five concepts mentioned by Lytle. Support of these concepts from the survey is indicated by an S and the item number from the survey.
Support from the interviews is indicated by line numbers from the transcripts of the videotaped interviews. Responses that were incorrect are shown in parentheses. A section that is not filled in indicates that no evidence supports that particular concept for that particular participant. The complete transcripts can be found in Appendix C. All of the categories except for the fourth column, “Integers can be represented in many ways” were cited in the research by Lytle as pre-requisite skills needed to understand integers and the operations of addition and subtraction. The fourth column was added by this researcher to enhance the analysis of the data.
Table 6

Participant Indication of Understanding of Integers

<table>
<thead>
<tr>
<th>Participant</th>
<th>Lytle’s categories</th>
<th>Integers are the counting numbers, their opposites, and zero</th>
<th>Zero separates positive and negative integers</th>
<th>Ordering Integers</th>
<th>Integers can be represented in many ways</th>
<th>Absolute value</th>
</tr>
</thead>
<tbody>
<tr>
<td>LC</td>
<td>(308-313)</td>
<td>S2, S3</td>
<td>(43-49), 76-86</td>
<td>(122-123)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>RV</td>
<td>(315)</td>
<td>(S2), S3</td>
<td>(177-185), 233-237</td>
<td>(259-265)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>LB</td>
<td>(1718), (1726)</td>
<td>1579-1580</td>
<td>S2, S3, 1576-1577</td>
<td>(1602-1610)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>BH</td>
<td>(1711), (1725)</td>
<td>S2, S3, 1576-1577</td>
<td>1653-1666, 1653-1666, 1816-1821</td>
<td>(1619-1621)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NB</td>
<td>(3649-3666)</td>
<td>S2, (S3), 3552-3555, 3562-3564, 4217-4218</td>
<td>(3624-3641), 3851-3855, 3601-3603, 4201-4202</td>
<td>(3565-3572)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>VF</td>
<td>(3649-3666)</td>
<td>(S2), S3, 3559-3560</td>
<td>3619-3623</td>
<td>(3591-3593)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>


The data collected in this study support Lytle’s categories. The table above shows how some change occurred between the time that the survey was given and the end of the interview sessions. For example, the parentheses in the category labeled “ordering integers” indicate that NB had a very narrow view of how to order integers before the interview sessions but by the end of the study, she was able to tell which of two integers
was greater. The implications of this table data, as they relate to the research questions, will be elaborated on in chapter five.

Data to Support Research Question Number Two

Introduction

The second research question is “How does the use of a novel model impact student understanding of addition and subtraction of integers?” In order to answer this question, this researcher wanted to know if students could symbolically add and subtract integers. She also wanted to know which definition of addition and subtraction the students employed. Question four of the initial survey, along with coding from the second and third interviews with each pair of participants, addressed this question. In order to see if students understood the operations of addition and subtraction, students were asked to write word problems that would be solved using addition and subtraction of integers. This type of question was asked in problems six through ten of the initial survey. The operation of addition will be discussed first and then the operation of subtraction will be discussed.

Addition of Integers

Addition may be thought of as a joining of sets. The number of sets to be combined is represented by the number of addends (Musser, Burger, & Peterson, 2008). Addition can also be thought of symbolically as an operation in which an algorithm is used (Wilkins, 1996). This researcher was interested in determining the distinction between these two ways of thinking about addition and the students’ evidence of each. During the interview portion of the study, the researcher was also interested in whether
the novel model helped participants to conceptually understand and make sense of the algorithms for addition and subtraction with integers. “Aspiring elementary teachers must begin to acquire a deep conceptual knowledge of the mathematics that they will one day need to teach, moving well beyond mere procedural understanding” (Greenberg & Walsh, p. 23, June 2008).

**Data from the Initial Survey**

The researcher used the initial survey to determine students’ ability to add integers. After reflecting on the results of the survey, the researcher developed tasks to be performed in the interview sessions to help determine how participants make sense of the operation of addition of integers.

Problems in question four of the initial survey were included to determine if students could correctly compute answers to addition examples when they were given in symbolic form. Question four, parts (a) through (c), of the initial survey asked students to find the sum of two, single digit integers while part (n) of the question asked students to find the sum of a three-digit integer and a two digit-integer. Problems involving more than one digit were inserted to verify if integers with more places caused increased difficulty for students and to help the researcher define tasks that would be used during the interview sessions.

Almost all of the students were able to correctly solve addition problems when they were written in symbolic form. The researcher concluded that the students correctly used the algorithms for addition of integers to obtain these solutions. Results are given in Table 7.
Table 7

Student Responses to Integer Addition Problems on Initial Survey

<table>
<thead>
<tr>
<th>Problem number from the initial survey</th>
<th>Problem</th>
<th>Number of students who had a correct answer (out of 79)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4(a)</td>
<td>6 + (−3)</td>
<td>79</td>
</tr>
<tr>
<td>4(c)</td>
<td>−8 + 6</td>
<td>76</td>
</tr>
<tr>
<td>6</td>
<td>Make up a word problem whose solution would be</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>−5 + 3</td>
<td></td>
</tr>
<tr>
<td>4(b)</td>
<td>−4 + (−5)</td>
<td>77</td>
</tr>
<tr>
<td>8</td>
<td>Make up a word problem whose solution would be</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>−8 + (−5)</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Make up a word problem whose solution would be</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>8 + (−4)</td>
<td></td>
</tr>
<tr>
<td>4(n)</td>
<td>−2 6 3 + (−79)</td>
<td>74</td>
</tr>
</tbody>
</table>

Ninety-six percent of the students were able to symbolically add a negative integer and a positive integer as asked in Problem 4(c). Problem 4(a), which involved the addition of a positive integer and a negative integer, was answered correctly by all of the students. Problem 4(b) asked students to find the sum of two negative integers and was answered correctly by ninety-seven percent of the students. The researcher concluded that most students were able to symbolically solve addition problems involving integers and it didn’t matter whether the addends were positive integers, negative integers, or one of each type.
Questions six, eight, and nine on the initial survey asked students to make up a word problem that should be solved using addition of two, single digit integers. These questions were included to determine if students could give practical examples of addition problems that involved addition of integers. For the word problems to be considered correct, the student had to show that addition of a negative amount is different from subtracting a positive amount. An operation means that something is being done. Positive and negative integers, on the other hand, are used to show direction. The students also had to understand that addition is an operation in which things are combined.

Problems six and nine on the initial survey asked students to write a word problem that would be solved by adding a negative integer and a positive integer. In problem six the positive integer appeared as the second addend, and in problem nine the positive integer appeared as the first addend. The researcher wanted to know if the position of the negative addend made a difference in students’ ability to write appropriate word problems.

Problem six asked students to write a word problem for $-5 + 3$. Twenty of the seventy-nine students wrote an appropriate word problem. An example of what this researcher considered a correct word problem for problem six is, “The temperature was $-5$ degrees today but then it went up three degrees. What is the temperature?” An example of what this researcher considered incorrect for this problem is the following: “John had 5 cats. Mary took those cats away from John. He was left with $-5$ cats. Mary then gave him 3 cats. How many cats does John have?” Twenty-three of the seventy-nine
students did not write any word problem for this question. Four of these twenty-three students wrote the correct solution of $-2$, indicating that they used an algorithm to obtain an answer but they did not know where they would use this information. Data from previous symbolic addition problems on the survey indicated that students provided correct solutions to addition of integers ninety-six percent of the time.

The researcher wanted to know if students had more difficulty making up problems in which the negative addend appeared as the second addend instead of the first addend. Problem nine on the initial survey asked students to make up a word problem for the expression $8 + (-4)$. Only one student was able to write an appropriate word problem. That student wrote, “Sam and Joe are playing a game and are on the same team. Sam has eight points and Joe has negative four. What is their total?” The most common example that was considered incorrect for problem nine was the following type: “Zach has eight puppies. He sells four of them. How many does he have left?” This type of example was given by thirty-seven of the seventy-nine students. Although the correct solution of “four” would be obtained by solving this word problem, the problem requires the operation of subtraction rather than addition of a negative amount. Thirty-four students of the seventy-nine students left the problem blank. This finding indicated that knowledge of integers and operations on them does not help participants to connect with familiar contexts to provide satisfactory examples of operations with integers.

The researcher wanted to determine if writing a word problem for addition involving two negative integers was more difficult for students than writing a word problem involving addition of a positive integer and a negative integer. Problem eight of
the initial survey asked students to give a word problem whose solution would be $8 + (-5)$. Twenty-one of the seventy-nine students wrote an appropriate word problem. One student whose solution was considered to be correct used the following example: “I owe you eight dollars, but borrow five more.” An example of a student whose solution was considered incorrect is “Bob has thirteen pens. I took eight on Monday and five on Tuesday. I owe him thirteen pens.”

*Summary of data from the initial survey.*

In general, students were able to correctly solve problems involving symbolic addition of integers but were unable to write a word problem for the same type of problem. Although all of the students could correctly perform the rule-based symbolic addition of a positive integer added to a negative integer, only one of the seventy-nine students could make up a word problem using a real world scenario. Most of the students who made up an incorrect word problem wrote a problem that required subtraction instead of addition. Five of the students used numbers other than those given in the problem to obtain the correct solution but the researcher considered these to be correct interpretations since the problem did not state that only the given integers could be used when making up the word problem.

Ninety-six percent of the seventy-nine students were able to correctly perform the symbolic addition of a negative integer added to a positive integer but only twenty-five percent were able to write a similar word problem. Two of the students whose answers were considered correct used numbers other than those given in the problem, but that yielded the correct solution. The researcher concluded that although students can use the
algorithm to correctly add integers, they do not relate these processes to practical applications of integers.

Although ninety-seven percent of the students could correctly perform the symbolic addition of two negative integers, only twenty-six percent could make up a word problem involving addition of two negative integers. Of the twenty-one students who had correct answers for this word problem, twelve wrote problems related to banking or money and six wrote problems related to football. Forty of the seventy-nine students left this problem blank. This finding implies that students do not understand practical applications of negative integers even though they can use the algorithm to correctly add negative integers.

Data from the Interviews

Interviews two and three provide the substance for this section. The section will be broken into two parts: “participants’ definition of addition”, and “participants model addition of integers using the novel model”. In these sessions the researcher asked participants for their definition of addition so she could determine if participants used ideas related to addition of whole numbers or if they thought about additive inverses when adding integers. To think about additive inverses when using the novel model, would imply that participants remove pairs of red and white bills until only one type of bill remained on the number line portion of the novel model. If participants thought about addition of whole numbers they would simply place the correct number of bills corresponding to each of the addends on the number line portion of the novel model.
Participants’ definitions of addition.

This researcher was interested in participants’ definition of addition because conceptual understanding of an operation requires one to know what the operation means. To determine how participants viewed addition, the participants were asked what addition is. Responses from this question would help determine tasks to increase conceptual understanding of addition. The following responses convey participants’ definitions of addition.

Group 1’s response:

I: What does addition mean to you?
LC: Add something to another thing.

Group 2’s response:

I: What does addition mean?
BH: Adding one thing to another.

Group 3’s response:

I: What does addition mean to you?
NB: Adding, putting two and two together.
I: (to VF) Do you agree?
VF: Yeah.

In summary, all participants of all groups agreed that addition meant a union, or joining, of exactly two groups of objects and they all mentioned “add” in their definition. Unilaterally, there was no hesitation on their part. They were certain of the definition of addition.

The researcher was concerned that all participants defined the operation of addition using the word “add”. Because this definition did not reveal exactly what
addition meant to the participants, the researcher asked the participants to give an example of integer addition. The researcher listened to find out if participants used both positive and negative integers in their initial example for addition. If they only used positive integers, this might indicate that participants don’t understand what elements make up the set of integers. Participants’ responses follow:

Group 1’s response:
I: Give me an example and tell me what you’d do for adding integers. (Long pause) Write one down, maybe. (Each wrote an addition problem on their paper. LC wrote $2 + 4 = 6$; RV wrote $17 + 3 = 20$.)

Group 2’s response:
I: Give me an example of an addition problem that deals with integers. LB: Two plus two.

Group 3’s response:
I: Give me an example of an addition problem with integers. VF: Five plus negative two.

In summary, all participants in two of the three groups gave addition examples that involved only natural numbers (i.e., positive integers). Participants in these two groups may have thought that the set of integers was the same as the set of counting numbers. The third group gave an example of a positive integer added to a negative integer. Participants in this group understood that the set of integers includes negative integers. In the next set of tasks, the researcher asked participants to model addition of two amounts of debt. The purpose of this type of question was to create an intentional
broadening of participants’ interpretations of the set of integers. This approach is consistent with the design of a teaching experiment.

Participants model addition of integers using the novel model.

Participants were asked to model three dollars of debt plus four dollars of debt using the novel model. The researcher started with an example where both of the addends were negative because she wanted to determine if participants would generalize from their experience with adding two positive integers. In other words, the sign of the sum would be the same as the sign of each of the addends and the magnitude of the sum would be the sum of the magnitudes of the addends. In this example participants used amounts of “debt” to represent negative integers.

Group 1’s response:

I: And let’s show three dollars of debt (LC put three red bills on the number line) plus four dollars of debt. (LC then put four more red bills on the number line.) Write it down. (Both wrote \[-3 + (-4),\] and the solution (Both wrote \[-7\].

Group 2’s response:

I: What if we have three dollars of debt, plus four dollars of debt? And let's see it on the number line. (LB placed three red bills and then four more red bills on the number line.) Does that make sense? And you had zero number of zeroes. (Each had already written this on their paper.)

Group 3’s response:

I: And let’s show three dollars of debt (LC put three red bills on the number line) plus four dollars of debt. (LC then put four more red bills on the number line.) Write it down. (Both wrote \[-3 + (-4),\] the solution (Both wrote \[-7\].)
All of the participants were able to model addition of two negative integers using the novel model. For the example given, the participants first put three red bills on the number line portion of the novel model and then they placed four more red bills on the number line. They could verify that their answer should be negative seven because there were seven red bills on the number line. No zeroes needed to be removed to determine the solution.

Since participants did not have any difficulty with addition involving integers of the same sign, in the next set of tasks participants were asked to model addition examples where one addend was positive and the other addend was negative. The researcher wanted to find out if participants were able to model both addends using the novel model and then remove zeroes so that only one color of bill remained on the number line portion of the novel model. Participants modeled the first addend by placing the appropriate number of red or white bills on the number line portion of the novel model. Then they modeled the second addend by placing the appropriate number of red or white bills on the novel model.

Because all participants evidenced an understanding about various names for positive and negative amounts, the researcher used this information to help the participants structure their thinking. If there were two colors of bills on the number line, participants removed the pairs of red and white bills until only one color of bill remained on the number line. This process of removing pairs of positive and negative was referred to as removing pairs of zero. The researcher conjectured that if participants could name
integers in many ways by adding zeroes, it should not be any more difficult for them to remove zeroes in order to name integers in other ways.

Participants were also told to write their solutions (and the number of zeroes removed from the number line) on a piece of paper that was collected by the researcher. The number of zeroes was later used to help participants determine why the algorithm for addition of integers makes sense.

The following dialogues show how participants modeled addition on the number line using the two colors of money.

Group 1’s response:

I: Let’s look at five dollars of money to spend (RV placed five white bills on the number line.) plus seven dollars of debt. (LC wrote the problem, the solution and the number of zeroes and then placed seven red bills on the number line. As she was placing the seven red bills on the number line, RV wrote values in the chart.)
I: And, is that your answer (referring to the number line)? (LC counted to herself pointing to red bills with her pencil.)
LC: Yes.
I: What is your answer?
LC: Negative two.
I: How could you show negative two? I know that’s one way you could show negative two on the number line but is there some way that everybody could see that it’s two dollars of debt? (LC removed a red bill and a white bill five times to leave two red bills on the number line.)
I: How many zeroes?
RV: Five.

Group 2’s response:

I: Let’s see six dollars of money to spend plus five dollars of debt. (BH put six white bills on the number line and LB put five red bills on the number line.)
BH: And take away five zeroes (as she and LB removed five red bills and five white bills from the number line).
I: So your answer was one. Let LB do this one all by herself. Take that one off. Four dollars of money to spend plus six dollars of debt. (LB put four white bills and six red bills on the number line. She then removed four red bills and four white bills from the number line.) And how did you know how many to take off?
LB: Well you have positive four and negative six so you have to subtract them because you cannot add them because negative six is larger so you have to subtract them and you end up with negative two because six is negative and it’s bigger.
I: Negative six is bigger?
LB: Negative six is bigger than positive four. Yes. Well, sorry, in the negative . . .the number that has the negative sign that’s larger ends up being the sign of the answer.

Group 3’s response:

I: And let’s do five dollars of money to spend (VF placed five white bills on the number line.) plus seven dollars of debt. (VF placed seven red bills on the number line.) So your problem would be. . .
VF: Five plus negative seven. (She wrote $-2$ for her solution.)
I: And how many zeroes?
VF: Five zeroes.
I: And how did you know that?
VF: Because you need equal amounts of negative and positive numbers. Those five positive numbers (referring to the white bills) so you need equal amounts so you have to have five negative numbers.
I: OK, so could you show how you got the five zeroes? Could you show me how you’re getting the negative two for your final answer?
VF: So you have one pair of zeroes, two pair of zeroes, three pair of zeroes, four zeroes, five zeroes. (She pointed to one red and one white bill when she said each of these.) So you have two left and they’re negatives.
I: So show me, by taking off . . .
VF: The zeroes?
I: Yeah. (VF removed a red bill and a white bill, then another red bill and another white bill, and continued this until she was left with only two red bills on the number line. She removed those closest to zero so the answer
was not visible by looking only at the number line.)

VF: You have two negative numbers left.

In summary, the researcher always asked participants to model problems involving dollars of debt and dollars of money to spend, instead of using negative and positive integers. For example, she said: “Let’s show five dollars of money to spend plus seven dollars of debt.” On their paper, the participants always wrote dollars of debt as negative integers and dollars of money to spend as positive integers. The participants then modeled the problem on the novel number line and wrote their solution on the paper. For the example given above, the participants wrote 2 as the solution. In other words, even though the researcher used the language associated with the novel model, the participants translated this language to the more traditional language using the words “positive” and ”negative” integers. This abstraction helped participants to relate the novel model to their previous knowledge of integers.

Participants in two of the three groups had some difficulty with addition examples involving a positive integer and a negative integer. Only one student in one of the groups modeled this type of addition so that only one color of bills remained on the number line at the completion of the problem. In the other two groups, the participants were able to give the correct answer to the addition problem even though both red and white bills remained on the number line. Modeling addition in this way was consistent with modeling addition of whole numbers.

When participants had both red bills and white bills on the number line, the researcher questioned the participants about the bills that remained on the number line. All participants then stated that zeroes could be removed. Using this technique, only one
color of money remained on the number line and the solution could be determined quickly by glancing at the number line. Through the researcher’s questioning, the participants were able to use the model to show how they were thinking.

Subtraction of Integers

A common definition of subtraction involves the idea of “take away” where a number of objects is taken away from a given number of objects (Musser, Burger, & Peterson, 2008). This conceptualization involves cardinality and is typically modeled using two-color counters. A second conceptualization of subtraction involves making a comparison between the cardinality of two sets to determine how many more elements one set has than another. This second conceptualization, in which the missing addend is the unknown amount, relies on ordinality and is usually modeled using a number line. This researcher was interested in which of these conceptualizations participants used to solve subtraction examples involving integers. In the interview sessions, if participants have previously used the number line model for subtraction, the researcher asked more questions related to the two-color counter model. If participants have previously used the two-color model for subtraction, the researcher asked more questions related to the number line model. In this manner, participants used the novel model to its fullest potential, with emphasis on both cardinality and ordinality of integers.

Data from the Initial Survey

On the initial survey, problems four, five, seven, and ten required that students solve several subtraction problems involving integers. These problems were of varying types and difficulty. Problem four asked students to solve problems symbolically and
problems five, seven, and ten required students to make up word problems that require integer subtraction. The researcher used these problems to determine whether students knew how to symbolically subtract integers and if they knew practical applications for subtraction of integers. For the word problems to be considered correct, the student had to show that subtraction of a negative amount is different from adding a positive amount. In other words, a student could not just follow the procedure for subtraction of integers. Some understanding had to be in place for students to respond to this prompt. The results are presented in Table 8.

Table 8

*Results from Initial Survey Concerning Subtraction of Integers*

<table>
<thead>
<tr>
<th>Problem number from the initial survey</th>
<th>Problem</th>
<th>Number of students who had a correct answer (out of 79)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4(d)</td>
<td>$-8 - 2$</td>
<td>71</td>
</tr>
<tr>
<td>10</td>
<td>Make up a word problem whose solution would be $-3 - 8$</td>
<td>8</td>
</tr>
<tr>
<td>4(i)</td>
<td>$-104 - 56$</td>
<td>67</td>
</tr>
<tr>
<td>4(o)</td>
<td>$-147 - 100$</td>
<td>73</td>
</tr>
<tr>
<td>4(g)</td>
<td>$9 - (-4)$</td>
<td>70</td>
</tr>
<tr>
<td>4(k)</td>
<td>$209 - (-75)$</td>
<td>60</td>
</tr>
<tr>
<td>5</td>
<td>Make up a word problem whose solution would be $17 - (-2)$</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 8 (continued)

Results from Initial Survey Concerning Subtraction of Integers

<table>
<thead>
<tr>
<th>Problem number from the initial survey</th>
<th>Problem</th>
<th>Number of students who had a correct answer (out of 79)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4(e)</td>
<td>$-8 - (-3)$</td>
<td>70</td>
</tr>
<tr>
<td>8</td>
<td>Make up a word problem whose solution would be $-10 - (-3)$</td>
<td>0</td>
</tr>
<tr>
<td>4(l)</td>
<td>$9 - 16$</td>
<td>70</td>
</tr>
<tr>
<td>4(h)</td>
<td>$9 - 82$</td>
<td>68</td>
</tr>
<tr>
<td>4(j)</td>
<td>$483 - 592$</td>
<td>70</td>
</tr>
</tbody>
</table>

Although seventy of the seventy-nine students (almost eighty-nine percent) correctly solved the subtraction problems symbolically, no students could make up a word problem in which a negative integer was subtracted from another integer. This was the case when the initial integer (the sum) was positive as well as when the initial integer was negative. Knowledge of integers and operations with integers does not help participants to connect with familiar contexts so they can provide satisfactory examples of operations with integers. The researcher used the preceding information to design the next set of tasks.
Data from the Interviews

To further answer research question 2, excerpts from interviews three and four were extracted. In these interviews, the six participants were asked questions concerning the operation of subtraction so the researcher could better understand how participants think about subtraction. This section is broken into the following categories: “participants’ definition of subtraction”, “participants’ initial examples of subtraction”, and “participants model subtraction using the novel model”.

Participants’ definitions of subtraction.

This researcher was interested in participants’ definitions of subtraction because she wanted to know if participants thought about subtraction as take away, or as finding a missing addend, or if they had no conceptualization of subtraction. If participants understood subtraction as “take away”, the novel model would allow them to extend this idea to integers. To do so would require that participants name integers in more than one way by adding zeroes so the required amount could be taken away. Participants had shown in prior tasks that they were able to name integers in many ways.

If participants thought about subtraction as finding a missing addend, this idea could also be extended to include integers using the novel model. The participants would count out the number of bills indicated by each of the addends and mentally determine how far and in which direction one would need to go in order to get from the addend to the sum. If participants had no conceptualization of subtraction, the researcher would provide examples to help participants develop a conceptual approach to subtraction.
Therefore, in the interviews, participants were asked what subtraction is and their responses follow.

Group 1’s response:

I: OK. What does subtraction mean?
LC: To take something away.

Group 2’s response:

I: So what is subtraction? (Long pause) You told me what addition is. What is subtraction?
BH: The difference between two numbers.
I: How do you subtract integers?
(long pause) LB: You have a whole number and you take away a certain amount from that whole number.

Group 3’s response:

I: What does subtraction mean?
NB: To take away.

In summary, each participant in each group perceived subtraction as “take away”.

This is the simplest type of subtraction for students to understand (Grouws, 1992) and this type of subtraction can be easily modeled using the novel model. The researcher used this information to help participants expand their conceptualization of subtraction to include integers.

Participants’ initial examples of subtraction.

After determining participants’ definitions of subtraction, the researcher asked participants to give an example of subtraction with integers. She listened carefully to answer three important questions. The first question was “Did the participants use word problems or problems listing only numbers?” Participants’ use of word problems that
correctly involve subtraction would be an indication that a good working definition of subtraction is in place and could be drawn upon when using the novel model. If they used word problems that did not involve subtraction, a better foundation would need to be built before subtraction could be discussed using the novel model.

The second question was “Did the participants use negative integers in their initial examples?” Since most of the participants included initial addition examples that involved only counting numbers, the researcher wanted to know if participants would have this same type of initial example for subtraction. Participant use of only positive integers could expose a limited definition of integers. The researcher was prepared to ask participants questions to help them better understand that subtraction could involve negative integers.

The third question that the researcher wanted to answer in this portion of the interview was “What conception of subtraction did participants use when stating their initial examples?” Use of subtraction as take away could indicate that the participant followed through with their initial definition of subtraction (from the last set of excerpts). Participant responses follow.

Group 1’s response:

I: OK. How do you subtract integers? Let’s have an example for subtraction of integers. (Each participant wrote an example on their paper.) LC, tell me about your example.
LC: It’s four minus two equals two. You just have four and subtract, take away, two and you’re left with two.
I: And RV, what did you have?
RV: I have five minus three equals two.

Group 2’s response:
I: How do you subtract integers? (long pause)
LB: You have a whole number and you take away a certain amount from that whole number.
I: Give me an example.
LB: You’ve got 10 apples,
BH: Take away five.
LB: And you take away five. You’re subtracting. You’re making the five a negative number. Ten minus five equals five apples are left.

Group 3’s response:

I: Both of you, write down a subtraction problem dealing with integers. And put the solution. And then tell me how you got it. NB, what did you have for yours?
NB: I don’t think I’m even getting it right. I just have twelve minus six.
I: Is “twelve minus six” a subtraction problem with integers?
VF: Yes.
NB: It is?
I: Yes, so you’re cool. And what did you have VF?
VF: I put five minus negative three.
I: And five minus negative three is . . .
VF: Eight.
I: How did you get that?
VF: Um, well I was taught to add, like if it’s two negatives, you add both of them. You add five plus three.

In summary, five of the six participants gave a subtraction example that involved only positive integers and whose solution was also a positive integer. Group 2 used the words “whole numbers” instead of “integers”, indicating that they thought these two sets contained the same elements. One participant, VF, gave an example that involved a positive integer subtract a negative integer. Her solution was correct and she probably applied the algorithm to get the solution.
Participants model subtraction using the novel model.

In the next set of tasks, the participants were asked to model subtraction examples using the novel model. In some situations it is difficult for students to model subtraction with integers as “take away” unless one recognizes that integers can be named in more than one way. For this reason, participants were initially given examples in which the required amount could not be taken away from the sum that was given. In these cases, the participants had to add “zeroes” in the form of an equal number of white bills and red bills so that the required amount could be taken away. An example would be \(-2 - 4\).

Using the novel model for this example, a participant would first place two red bills on the number line portion of the novel model. Since four white bills cannot be taken away, the participant should name negative two in another way by adding zeroes. If he adds four more red bills and four white bills to the number line, the value of the integer is not changed but four white bills are available to be taken away. When these are taken away six red bills remain. The answer, then, is negative six. The following dialogues portray participants dealing with this type of example.

Group 1’s response:

I: Let’s model two dollars of debt. (LC put two red bills on the number line.) And I want you to subtract now, you told me that subtraction was take away, so I want you to subtract four dollars of money to spend. (LC picked up some white bills but seemed confused. She started to put them on the number line.) Is that subtracting? Is that taking away? (She removed the white bills she had put on the number line.)

LC: I don’t know.
I: Well, if you’re putting them on there, are you taking them away?
LC: No.
I: I mean, you told me that subtraction was take away so, . . .
RV: So . . . (long pause) (LC put two red bills on the number line.)
I: Is that take away?
LC: Yes.
I: What are you taking away?
LC: I don’t know. Empty space, right here. (She slapped her hand down on the positive side of the number line. She then removed the two red bills that she had put on the number line.)
LC: Hum. You said minus four dollars that you could spend?

At this point LC knew that the answer should be negative two but she could not find a way to take away four white bills to leave two red bills. Therefore, she placed two red bills and some white bills on the number line portion of the novel model so that she could take away the required amount. The researcher questioned LC, reminding her that she had previously defined subtraction as “take away”. She needed to have four white bills to take away and the researcher asked her a question to trigger her ability to name integers in more than one way.

I: Yes. So, to help you out, can you think of another way to name two dollars of debt?
LC: Negative two.
I: That would help you out. Besides negative two.
LC: Um.
RV: The absolute value of two.

LC was grasping at straws. She knew the answer had to be negative two but probably thought back to a previous interview where absolute value was mentioned. In that interview LC said that absolute value was “a number without, like if it’s negative, you just take the negative off”. The researcher continued the interview, thinking that this might trigger something for LC.

I: Okay, the absolute value of two. I see two things there.
LC: Well, if you made it negative six and four, you could use the zeroes
and you would get. . .Oh, no, that doesn’t work. Never mind. (short pause) I don’t know.

LC was on the right track when thinking about negative two as negative six and four (adding four zeroes to the original amount) but then did not see how this would help her to solve the problem. The researcher conjectured that LC thought she should immediately see two red bills as the only amounts on the number line portion of the novel model. The researcher continued, trying to get LC to rename two dollars of debt in another way. When the researcher determined that LC could not make sense of the situation, she reminded LC that integers can be named in more than one way, but emphasized that only zeroes could be added so that the value of the integer was not changed. With this prompt LC was able to solve the problem.

I: Think about what you just said before you said “I don’t know”. LC: If you make it negative six?
I: Okay. And so you’re really putting on what?
LC: Negative.
I: Four negatives, and if you put on four negatives and you put on . . .
LC: Four positives.
I: Four positives. Would that be the same thing? (LC put five red bills and RV put four white bills on the number line.)
LC: Yes.
I: Except that you put on how many?
RV: (interjecting for LC) You put on five.
LC: (She took off one of the red bills.) Just kidding.
I: Just checking. Now can you take away your four dollars of money to spend?
RV & LC: Yeah.
I: So take away your four dollars of money to spend. (RV removed the four white bills from the number line.) So what was the problem that you had? I said two dollars of debt subtract four dollars of money to spend. So what did you write in the column that says “Problem”?
LC: Negative two minus four.
I: Okay. What’s your solution?
RV & LC: Negative six.

Group 2’ response:

I: You want to take four dollars of money to spend from your two dollars of debt.
BH: You’ve got to reverse that (referring to changing the subtraction sign to addition on LB’s paper). It’s negative two minus four.
LB: That’s what I had the first time. And then I tried negative two plus four?

LB was not sure about how to subtract with integers but she knew that the algorithm states that something must be changed. She also was not sure what the answer should be. In the following dialogue BH demonstrated that she did not understand what needed to be changed when subtracting integers. The researcher continued to ask BH questions so that she had a better understanding of subtraction using the novel model.

BH: No. Negative two minus plus four so that negative and the plus makes that a negative, I think.
I: Does that make sense to you?
BH: I just know the rule about a negative and a positive is negative.
I: If you have two dollars of debt, (pointing to the number line with the two red bills) there’s two dollars of debt.
LB & BH: Uh huh.
I: Can you take away four dollars of money to spend as it is right now? (Pause) Do you see four dollars of money to spend that you can just take away?
BH: Well, you mean without it being here (She pointed to the positive side of the number line.)
I: Yeah.
LB: But why would you write it negative two minus positive four? Why wouldn’t you just take four minus two?
BH: Cause that’s different.
LB: But if you have four positive amounts of money to spend, you have four dollars of extra money, and you have two dollars of debt so you take
your four dollars of money to spend. You pay off your debt. How much are you left with?

BH: Negative two.

BH knew that the answer should be negative six because she knew how to apply the algorithm but she did not know why the algorithm works. LB was not sure what the answer should be and she confused the operations of addition and subtraction. Since participants had earlier stated that subtraction meant to take away, the researcher asked more questions that forced the participants to think about the meaning of subtraction.

LB: No, but if you have four of these (She picked up some white bills.) You know what I mean? (She placed four white bills on the number line.)
I: Where are these four coming from? (referring to the four white bills)
LB: Cause you said we have four dollars.
I: I want you to take away four dollars of money to spend.
BH: But if you take away even more money to spend you’re still making yourself more in debt.
LB: So if you’re taking away money to spend, (She placed four more red bills on the number line.)
I: And what did you just do?
LB: I added four more dollars of debt.
I: Can you do that?
LB: (after a short pause) Yes.

At this point LB confused debt with subtracting a positive amount. By adding four red bills, she increased the debt by four dollars but she did not take away four dollars of money to spend, as indicated in the problem. Even though the answer will be the same for both of these transactions, the researcher tried to convey the meaning of subtraction by asking more questions.

I: What allows you to do that?
BH: Um, credit.
LB: Or an IOU. Um, because if you don’t have any positive money, if you don’t have any, yeah, positive money, then you can’t take from your positive amount and pay off your debt so therefore if you can’t do that then you must have more debt. (Long pause) So that’s wrong? I’m just trying to think logically about it.

LB and BH were trying to make sense of the situation in terms of their practical experiences with money. They confused addition of a negative amount with subtraction of a positive amount. The researcher determined that it was best to start the problem over again from the beginning.

I: Okay. Let’s take off those four red bills (the last ones that LB had put on the number line). Let’s take everything off. Let’s wipe the slate clean. Two dollars of debt. (LB put two red bills on the number line.)
I: I want to see two dollars of debt. (LB placed two red bills on the number line.) We all three agree that’s two dollars of debt.
BH & LB: Yes.
I: So you want to take away four dollars of money to spend. Do you see any money to spend that you can take away?
BH: No.
I: So think of . . .
BH: (very emphatically) Oh! I’m sorry. Because you’re not going to be . . . since negative (is) less than zero you can’t take away zeroes so it automatically has to go over here (pointing to the negative side of the number line).
LB: So you’re adding on the right track?

BH knew that adding debt was similar to taking away money to spend. She tried to make sense of what she knew and how to model it using the novel model. The researcher helped her develop a conceptual understanding of subtraction by asking more questions related to naming integers in more than one way.

I: Remember the exercises we did where I said “What are different ways of showing these amounts?”
BH: Uh huh.
I: Are there different ways that you could show two dollars of debt?
BH: Yeah. Uh huh.
I: That would help you out?
BH: Okay, I have an idea. One, two (She counted the red bills on the number line.) You want to be two dollars in debt, right?
LB: Right. (BH placed five white bills on the number line, counting “one, two, three, four, five”) Add one more to that (pointing to the negative side of the number line). (LB placed another red bill on the number line.)
I: Now, what did you add on there?
LB: Why did you only add five?

BH modeled two dollars of money to spend rather than two dollars of debt. When BH put white bills on the number line portion of the novel model, LB was confused and she questioned BH about her actions. BH then thought about her previous actions and determined that she was wrong. Through questioning by LB, BH was able to see her error and stated that two dollars of debt means two more red bills than white bills on the number line portion of the novel model.

BH: Oh no, wait. We’re three dollars in debt. Actually that’s not right. You have to add a couple more (referring to the red bills) because you have to have two more than I have.
LB: But why do you only have five? 
BH: Because I have to take away four so I just figured it would be easier.
LB: Oh, Okay.
BH: So if you go to seven on that (meaning if you have seven red bills) that’s two dollars in debt.
LB: Is that correct? Oh, the light bulb actually gets plugged in!
I: And now what are you going to do?
BH: Take away four dollars of money to spend. (She removed four white bills from the number line.)
LB: Yes!
BH: Okay.
I: And is that your answer?
BH & LB: Yes.
I: What is your answer?
BH: (as LB was counting the red bills and then the white bill) Negative six.
LB: Yes.

Both LB and BH were able to determine the answer even though both red bills and while bills remained on the number line. The researcher wanted to make sure that they understood “removing zeroes”.

I: Is that the way that you would normally show six dollars of debt on the number line? Usually when we show the answer we have only one color left over, right? (BH took one red bill and one white bill from the number line.)

BH: Just cancel out a zero.

Group 3’s response:

The first thing I want to see is two dollars of debt. (NB put two red bills on the number line.) That’s two dollars of debt, right? (NB nodded in agreement.) I want you to subtract four dollars of money to spend. NB: It’s negative, right? (long pause) I don’t like the subtraction. I haven’t done this in so long. Aren’t you supposed to add the values together?

NB knew that subtraction of a positive amount could be thought of as addition of a negative amount but she was not sure how to model this using the novel model. The researcher asked questions to get the participants to make sense of the operation of subtraction. Her goal was to get the participants to understand why the algorithm for subtraction works.

I: That’s what the algorithm says to do but I’m trying to get you to see where that algorithm comes from, first of all, so that it will make sense to you. Okay, so you see your two dollars of debt, right? NB: Yeah.

I: Can you take four dollars of money to spend away from that as it is now?

NB & VF: No.

I: So what can you do? Can you get some money to spend? VF: Can you write it a different way?
I: Okay, let’s see it. (VF put three white bills on the number line.)
VF: Two dollars of debt. (She put another white bill on the number line and two more red bills on the number line. Then she put two more white bills on the number line. This was done very slowly and VF was concentrating very hard.) So then you take away four dollars.

VF knew the algorithm for subtraction and tried to find a way to place four white bills on the number line so they could be taken away. The amount that she modeled was positive two instead of negative two, but she knew that positive four had to be taken away. The researcher questioned VF about the display on the number line portion of the novel model.

I: Is that two dollars of debt right there? (She counted the white bills and then counted the red bills.) On your number line, do you have two dollars of debt? (VF then put two more red bills on the number line.) Now you can take away four dollars of money to spend. And what are you left with?
VF: Six dollars of debt. (She only counted the red bills.) (Picking up the two white bills) These would be gone too. So you have six dollars of debt left. (She didn’t seem convinced.) Could be just the four dollars of . . .

The researcher was not sure that the participants understood their actions with the novel model so she asked them to show the problem again.

I: Let’s see it again. Let’s start all over again. Two dollars of debt. (VF put four white bills and six red bills on the number line.) So, is that what two dollars of debt looks like? Do you agree, NB? Is that another name for two dollars of debt?
NB: Yeah.
I: Now can you take away your four dollars of money to spend?
VF & NB: Yes.
I: And you’ll be left with . . . (VF took off four white bills.)
NB: Six.
I: Six dollars of . . .
NB & VF: Debt.
In summary, after several examples participants were able to see the need for adding equal amounts of positive and negative in order to perform the given subtraction. This was called “adding zeroes”. This researcher wanted students to determine the number of zeroes needed so they could relate this to the algorithm used for subtraction. This will be discussed more when looking at the third research question.

After several examples similar to the one above were given to the participants, examples such as “negative four subtract negative two” were given. Up to this point, the researcher had given examples where the number of zeroes that needed to be added was the same as the given addend. The researcher wanted to determine if participants used the same method to solve all subtraction problems involving integers. Participant responses are indicated below.

Group 1’s response:

I: Let’s see the four dollars of debt again. (LC removed the three white bills from the number line.) Now I want you to subtract three dollars of debt.
LC: Okay. (She put three white bills on the number line.)
RV: If you’re subtracting debt, then you don’t have to add anything to the negative side.
I: Oh, we don’t.
LC: Do you have to add to the positive then?
I: But we could add three dollars of money to spend as long as we add in what? (LC & RV looked confused.) If we want to keep it all balanced, . . .
RV: We could add three negatives on there (pointing to the negative side of the number line). Right?

The researcher wanted to make sure that the participants understood that adding an equal number of red bills and white bills was the same as adding zero, which does not change the value of an integer. She continued with the interview session.
I: So add the three dollars of debt (LC put three more red bills on the number line.) Because what you did was you added what? The only thing that you can add to a number . . .
RV: More debt.
I: And not change its value is . . .
LC: Zero.
I: So you added how many zeroes?
RV & LC: Three.
I: Three zeroes. Now, can you take away your three dollars of debt?
LC: Yes. (She removed three red bills from the number line. Both participants looked confused to find money of both colors on the number line.)
I: When you took away three dollars of debt, that was basically the same as adding what? (Long pause) If you have a debt and I take away three dollars of your debt, that’s basically the same thing as . . .
LC: Adding a positive.
I: Adding three dollars of money to spend, isn’t it? So now what are you going to do? Your answer on the number line, you’re going to, (LC removed three red bills and three white bills from the number line.) One more.

Group 2’s response:

I: Let’s try negative four minus negative two.
BH: On the number line?
I: Sure. (LB put four red bills on the number line. She then put two more red bills on the number line.)

At this point, LB confused subtracting negative two with adding negative two but the researcher did not interrupt because she wanted to see if LB would discover her mistake.

BH: Minus negative two?
I: Uh huh.
LB: And then you put up two and you take away . . .
BH: Wait a minute. (She put two white bills on the number line.) I think we need to take away two of these (pointing to the red bills).
BH knew that two white bills had to be placed on the number line because LB previously placed two red bills on the number line. She understood that only zero can be added to a number without changing its value. She also knew that two red bills had to be taken away because the problem required two to be subtracted.

LB: Uh huh.
BH: And then if I do plus two (She put two white bills on the number line.), and then we bring the zeroes (she removed two white bills from the number line and LB removed two red bills from the number line.) We take away two and we’re left with negative two.
LB: Yes.
I: Okay.

Even though there were enough red bills to take away two of them, the participants still added two zeroes, as in the previous examples. LB tried to make sense of the previous transactions.

LB: So we actually took up six cause you took up your two (white bills) and I took up my four (red bills) and four and two is six.
BH: No. This answer’s two (pointing to the last column).
LB: You have six minus two
BH: You should’ve started with four and then added two when I add. Wait. It worked on the number line.

BH was unsure of what she had done but was convinced that she must have done it correctly because she got the correct answer.

LB: Uh huh. You took up your two (pointing to the positive side of the number line) and I took up (long pause).
BH: We had minus four take away two dollars of money to spend so I had my two dollars of money to spend. . .
LB: So four minus two is two. So that’s right.
BH: And then we got rid of those two dollars.
BH corrected LB when she said “four” instead of “negative four” but BH mistakenly said “take away two dollars of money to spend” instead of “two dollars of debt”. Several more examples were given so that BH and LB could clarify what subtraction means and so they could see what it means to add zeroes.

Group 3’s response:

I: Okay. Let’s see that three dollars of debt again. (VF put two more red bills on the number line so that there was a total of three red bills on the number line.) And let’s see if we can do it the same way that we did those ones in the very beginning (of this session). So I want you to subtract two dollars of debt. What did you do in the very beginning? You added some zeroes, didn’t you?
VF: Uh huh.

The researcher wanted to make sure that VF knew how many zeroes to add and that she understood why zeros had to be added.

I: So how many zeroes would you add in?
VF: Three zeroes.
I: How many? If you want to subtract two dollars of debt.
VF: Oh, two zeroes.
I: Two zeroes. (VF put two red bills on the number line.) Are those zeroes? (VF removed two white bills from the number line.) Those are money to spend so what else do you have to do?
VF confused adding negative two with subtracting positive two. The researcher asked more questions so that VF would find her mistake.

VF: It means to take away two of them (pointing to the red bills and white bills; meaning to make two zeroes).
I: Well, now wait. You put on the two white bills, right? (VF nodded in agreement.) Two dollars of money to spend. Can you just throw money on there?
VF: Oh add two more. (She put two more red bills on the number line.)
VF was able to correct her mistake. The researcher asked more questions to clarify the meaning of the algorithm for subtraction.

I: So you’ll add two red ones and now take away the two dollars of debt (VF removed two red bills from the number line.) And what problem do you see in front of you now?
VF: Negative three plus two.
I: Negative three plus two. Right? (NB nodded in agreement.) And then you go through the same things that you did for addition. You make those zeroes and make it so you have just one color.
VF: Okay. (VF removed a red bill and a white bill, then another red bill and white bill from the number line.)
I: Is it making any more sense? (VF and NB nodded in agreement.) Okay.

Sometimes the participants didn’t know what to do to find a solution using the novel model. For example, given the problem “four dollars of money to spend subtract seven dollars of money to spend”, the following dialog took place between LB and the researcher:

I: Okay, so now can you subtract your seven dollars of money to spend? (LB stared at the number line.) Do you have seven dollars of money to spend that you can subtract?
LB: Oh yes. (She removed seven white bills from the number line.)
I: Is that your answer?
LB: No.
I: So now what do you have to do?
LB: I have absolutely no idea.
I: You want one color left on the number line.
LB: So that means I need to continue to take these off. (She removed four white bills from the number line.)
I: Can you just take four off like that?
LB: No, not without taking off four from here. (She removed four red bills from the number line.)
I: And so your answer is. . .
LB: Negative three.
In this case LB was confused as to how to proceed. The researcher asked questions that allowed this participant to scaffold her learning using the novel model.

In summary, participants were able to determine how many zeroes were required to perform subtraction examples. After adding the number of zeroes that corresponded to the addend that was given in the problem, the participants were then able to take away the amount indicated by the addend. After the subtraction was done, an equal number of positives and negatives were removed to determine the solution. Following this process, only one color of money remained on the number line.

Summary of Responses that Informed Research Question 2

This researcher reflected on participants’ definitions of addition and subtraction of integers. Addition was defined by all participants as combining sets of objects or quantities while subtraction, in most cases, was thought of as “taking away”. One participant thought of subtraction as a difference, but none of the participants viewed subtraction as a comparison. When the researcher stated the addition and subtraction problems, she usually used the terms “debt” and “money to spend” rather than “negative integers” and “positive integers”. The participants, however, always wrote their answers using positive and negative integers. This may have been because it is easier and quicker to write integers than to write amounts of “debt” and “money to spend”.

This researcher was also interested in whether students could perform symbolic addition and subtraction, without a context. For this part of the research, question four of the initial survey was considered. The researcher then wanted to know if students could supply a context for a given addition or subtraction problem that involved integers.
Questions six through ten addressed this part of the question. Data from the initial survey and coding from interviews two and three was used to determine whether students conceptually understood the operations of addition and subtraction or if they were merely carrying out learned procedures.

All of the participants readily modeled addition on the novel model when the signs of the two addends were the same. However, when adding addends of different signs, two of the three groups gave the correct answer but had both colors of bills on the number line. About half of the time these participants felt no need to remove zeroes but instead looked to see which side of the number line had more bills. They determined the solution in their mind and they were accurate using this method. When the participants of these groups were questioned about having both colors of money on the number line, they were able to remove a positive and a negative as many times as needed so that only one color of money remained on the number line.

All of the participants knew the rule-based algorithm for subtraction of integers, where one changes the subtraction sign to addition and the sign of the addend to its opposite. When participants did not have enough of one color of bills to take away the required amount, they showed some hesitancy and didn’t know where to begin. They tried to figure a way to put bills on the novel model so that it would show the correct solution. When this occurred, the researcher asked participants questions about naming integers in more than one way by adding “zeroes”.

Initially, the participants relied on their use of the algorithm for subtraction. However, after several examples, all participants renamed numbers using zeroes, and
they then used the “take away” definition of subtraction. The researcher designed the problems so that the initial problems required the participants to add as many zeroes as the given addend. This was done so that the participants could relate the algorithm for subtraction to what they were doing with the novel model.

Data to Support Research Question Number Three

*Introduction*

Question number three has two parts: “How do pre-service elementary teachers relate the use of a novel model for addition and subtraction of integers to the rule-based procedures that they use to add and subtract integers? How do they develop meaning for these relationships?” This question was answered using the data from interviews three and four. Addition and subtraction will be treated separately.

*Addition of Integers*

When participants were given addition examples to display on the novel model, they were also asked to create three columns on their paper. These columns were labeled “Problem”, “Solution”, and “Number of Zeroes”. By providing these columns for the participants to fill in, the researcher anticipated that participants would relate these columns to the algorithm for addition of integers. Participants placed bills on the number line portion of the novel model to correspond to the amount of each addend. They then removed an equivalent number of red and white bills from the number line so that only red or white money remained on the number line. Addition examples from prior tasks provided a good background for understanding how the addition algorithm works. If both of the addends were of the same sign, no zeroes needed to be removed and the sum would
be the sum of the absolute values of the addends. The sum would have the same sign as each of the addends. If the signs of the addends were different, zeroes had to be removed and the number of zeroes was the same as the addend that had the lesser absolute value. The sum would have the sign of the addend that had the greater absolute value.

Discussions from the three groups as they worked through some addition examples are provided below. The first example for each group is an example where the addends are both positive and do not require the removal of zeroes. The second example is one in which one addend is positive and the other addend is negative. In this case, participants had to remove some zeroes.

Group 1’s response:

I: And on your paper, let’s turn the paper over and I want you to make three columns on the back, three, evenly spaced columns if you can. At the top of the first one I want you to write “Problem”. At the top of the second column I want you to write “Solution” and the third one I want you to write “Number of Zeroes”, which may or may not make sense at this point in time. And as we do these problems I want you to model them on the number line and fill in the chart that you’ve just created. So I want you to show four dollars of money to spend plus two dollars of money to spend. (Both participants wrote 4 + 2 under “Problem” and 6 under “Solution”. LC and RV each put two white bills on the number line. Then each put another white bill on the number line.) And your solution is. . .

LC & RV: Six.
I: And did you have to make any zeroes?

LC: No.
I: So you’ll put a zero in that last column.

Since LC gave the answer without removing zeroes from the number line, the researcher wanted to make sure that she could remove zeroes so that only one color of
bills remained on the number line. Therefore, the researcher asked more questions about removing zeroes.

I: How could you show negative two? I know that’s one way you could show negative two on the number line but is there some way that everybody could see that it’s two dollars of debt? (LC removed a red bill and a white bill five times to leave two red bills on the number line.)

I: How many zeroes?
RV: Five.
I: LC, do you see how she got five zeroes?
LC: Yeah, I just didn’t write it down. (She wrote “5” in the proper column in her chart.)

Group 2’s response:

I: Turn your paper over and I want you to create three columns evenly spaced and the heading for the first column I wanted to write “problem”. Second column, I want you to write “solution”. And the third column “number of zeros”.
BH: Number of zeroes?
I: Yes. It makes absolutely no sense at this point, but that’s okay. Hopefully it will. So we’re going to look at four dollars of money to spend plus two dollars of money to spend. What would that problem be?
BH: Four plus two.
LB: Yeah. That’s what you want us to write?
I: Yes. Then I want you to show it on the number line. (BH put four white bills on the number line and then two more white bills on the number line.) And your answer would be?
BH & LB: Six.
I: And were there any zeroes, any cancellations? (BH & LB shook their heads.) So that last column will be zero.

Group 3’s response (NB and VF were interviewed separately):

I: Now, on the back of your paper you’re going to have three columns. The first one is going to say “Problem”; second one’s going to say “Solution”, and the third one will say “Number of zeroes”.
NB: Number of zeroes?
I: Uh huh. And as we do these problems I’m going to have you fill in the
chart and show it on the number line. And let’s show four dollars of money to spend plus two dollars of money to spend.

NB: Am I just solving it or am I using it on here (pointing to the number line)?

I: You’re going to solve it, whichever order you want to do it. You can either solve it and then show it on the number line or you can show it on the number line and then solve it. The last column doesn’t make any sense at this point.

NB: One, two, three, four. (She put four white bills on the number line.) And then you add two more. (She put two more white bills on the number line.) It’s going to give you six.

I: So it would be six. You didn’t have any zeroes there that you had to worry about, did you?

NB: No.

I: On the back of your paper I want three columns. The first column I want you to label “Problem”, the second one “Solution”, and the third one “Number of zeroes”. That probably doesn’t make any sense now but it’s okay. The first thing we’re going to do is show four dollars of money to spend. (VF placed four white bills on the number line.) And then we’re going to add to that two dollars of money to spend. (VF placed two more white bills on the number line.) So what problem was that?

VF: Four plus two.

I: OK so in the “Problem” section you’ll write 4 + 2 and then in the “Solution” you’ll write

VF: Six.

I: And then in the “Number of zeroes” did you have to cancel anything in order to get your solution?

VF: No.

I: So you’ll put 0.

In summary, all three groups were able to model the addition examples on the novel model and they were able to remove the appropriate number of zeroes. All groups were able to write the solution to the problem and complete the columns on their paper.

The next example for each group involved addition of a positive integer and a negative integer. This type of example involved removing zeroes from the number line.
portion of the novel model. Participants recorded the number of zeroes that needed to be removed.

Group 1’s response:

I: Let’s look at five dollars of money to spend (RV placed five white bills on the number line.) plus seven dollars of debt. (LC wrote the problem, the solution and the number of zeroes and then placed seven red bills on the number line. As she was placing the seven red bills on the number line, RV wrote values in the chart.)
I: And, is that your answer (referring to the number line)? (LC counted to herself pointing to red bills with her pencil.)
LC: Yes.

LC was able to give find her answer even though both red bills and white bills remained on the number line portion of the novel model. The researcher asked LC to verbally state her result to verify that she understood the problem.

I: What is your answer?
LC: Negative two.

Group 2’s response:

I: Let’s see five dollars of money to spend plus seven dollars of debt. So one of you do the five dollars of money to spend and the other one do the seven dollars of debt. (BH put five white bills on the number line and LB put seven red bills on the number line.) Is that your answer?

The researcher asked this question to verify that participants understood the configuration of bills on the number line portion of the novel model.

LB: No, because then what you’ll do is pick up one of each (meaning a red bill and a white bill) until . . . because you have more debt than you have real money.
I: And so how many of those things. . . How many zeroes did you pick up?
BH: Five.

Group 3’s response:
I: And let’s do five dollars of money to spend (VF placed five white bills on the number line.) plus seven dollars of debt. (VF placed seven red bills on the number line.) So your problem would be. . .
VF: Five plus negative seven. (She wrote −2 for her solution.)

VF applied the algorithm to find the solution. The researcher asked VF questions to clarify how she was thinking about subtraction.

I: And how many zeroes?
VF: Five zeroes.
I: And how did you know that?
VF: Because you need equal amounts of negative and positive numbers. Those five positive numbers (referring to the white bills) so you need equal amounts so you have to have five negative numbers.
I: OK, so could you show how you got the five zeroes? Could you show me how you’re getting the negative two for your final answer?
VF: So you have one pair of zeroes, two pair of zeroes, three pair of zeroes, four zeroes, five zeroes. (She pointed to one red and one white bill when she said each of these.) So you have two left and they’re negatives.
I: So show me, by taking off . . .
VF: The zeroes?
I: Yeah. (VF removed a red bill and a white bill, then another red bill and another white bill, and continued this until she was left with only two red bills on the number line. She removed those closest to zero so the answer was not visible by looking only at the number line.)
VF: You have two negative numbers left.
I: And normally we’d take them off from the ends so you could see the negative two.
VF: Oh yeah.

After several examples were given, the participants were asked to determine how the number of zeroes was related to the problem. In this manner, the researcher tried to get the participants to relate the algorithm for addition of integers to what was done when modeling the example on the number line portion of the novel model. Dialogue from the each of the groups is given below.
Group 1’s response:

I: So now what I want you to do is look at that and see if you can come up with some statement relating the way that you add with integers related to the number line that you just used. (Both participants seem confused.) So look at the number of zeroes and see if that has any relationship to what’s going on or maybe if the solution has a relationship to something you know, or what’s going on? (Very long pause.)

The researcher realized that the participants did not understand what she wanted them to do. Therefore, she reframed the question.

I: Do you see any relationship between the number of zeroes and anything else that you had? (Long pause)
LC: Um. Yeah.
I: What is it?
LC: Well, when both problem, or when both
I: Addends?
LC: addends are the same, like both are positive or both are negative, there’s no zeroes. And when they’re not positive, when they’re positive and negative, the smaller, like the absolute, you take the absolute value of both, the smaller number is the one that shows up in the zeroes.
I: Do you agree RV?
RV: Yeah.
I: Do you see what she’s saying?
RV: Yeah.
I: And how do you find the solution to your problem? You said when they’re the same sign there’s no zeroes. How do you know what the solution is in that case?
LC: It’s, (pause), well it’s greater than the addends and if both are positive then the solution’s positive.
I: And what’s the number part of the solution.
LC: What do you mean, the number part?
I: The number part for your solution. The solution has two parts, the number part and the direction, right?
LC: Right.
I: So your solution for the first one was. . .
LC: Six.
I: How did you get the number part for that solution?
LC: Because the addends are both positive.
I: So what did you do to get the six?
LC: Added.
I: OK. And what about when they were both negative? You said there were no zeroes.
LC: You added.
I: You added those to get the number part. How did you figure the sign in each of those cases?
LC: Uh, well if they were both negative, if both addends were negative, then the solution is negative.
I: And if they’re both positive?
LC: Then the solution is positive.
I: And RV, what do you notice if they aren’t both positive or both negative?
RV: If they’re not both positive or both negative um, there’s always going to be some number of zeroes.
I: OK.
RV: And, um, the one that is the higher integer, um, the lower integer is going to be the number of zeroes for the number.

The researcher asked RV to clarify what she meant by “lower” integer so she could verify that she considered “absolute value” of the addends.

I: What do you mean by the “lower” integer?
RV: Like, we had seven of, we had negative seven and five, and the five was the number of zeroes that we had.
I: OK, so you’re saying five is the lower integer?
RV: Yes. Like, in a sense.
I: In what sense?
RV: Cause if you take the absolute value of both.
I: How about “when the signs are the same…”
RV: When the signs are the same, the solution’s gonna be, um, the same sign as the problem.
I: And how do you get the number part in that case?
RV: You just add them together.
I: OK, and what if the signs are different?
RV: If they’re different, um, (long pause) it’s, like you (very long pause)

The researcher was not sure that the participants understood the algorithm so she asked LC to clarify what RV was trying to say.

I: Help her out LC.
LC: Uh, then you take the integer with the greatest absolute value and whatever sign it had, that’s what the problem, the solution is gonna have. So if, uh, (looking at her paper for problems discussed) you took six plus negative five, six has the greater absolute value and your answer is going to be positive.
I: OK, and how did you get the number part when the signs are different for the solution? How did you figure out what that number part was? You told us how you’re going to figure out if the sign is going to be positive or negative.
LC: Um, well, I just took six and subtracted five and got one.

The researcher asked LC to clarify what was subtracted. This then was related to the number of zeroes that were removed from the number line portion of the novel model.

I: OK, so you really subtracted what?
LC: Five.
I: You subtracted five from six but what is that?
LC: I don’t know.
I: RV help her.
RV: It’s like canceling out five dollars of debt.
I: OK.
LC: In addition.
I: Is it addition?
RV: Subtraction.
I: You’re subtracting those. . .
RV: Negative?

RV was not sure what she was subtracting, so the researcher gave her a hint.

I: Two words. A, V.
RV: Absolute values.
I: So does that make any sense?
LC: Yes.

Group 2’s response:

I: So now I want you to look at that last column and see if you can relate that to the rule that you used for adding integers. See if that relates at all to what you were doing. What is the rule for adding integers. When you added those... when you had something like four plus negative six, both of you knew immediately that your answer was negative two. How did you know that and can you find some way that the last column would help you?
(There was a long pause as BH and LB looked at their papers.)
I: When do you have zero zeroes?
LB: When either the answer is positive or you have an equal number of... 

The researcher asked LB and BH more questions and this created a perturbation for the participants. By showing a counterexample, the researcher was able to get LB and BH to make a correct generalization.

BH: When both your addends are negative or positive... the same sign?
I: Do you agree LB? See, if you look at your next to the last problem, you have a positive answer.
LB: Uh huh.
I: But you didn’t have zero zeroes did you?
LB: No.
I: So do you agree with BH’s idea?
LB: (emphatically) Yes.
I: OK. When do you have some number of zeroes then? That you have to neutralize a positive and a negative?
LB: When one of your... when only one of your addends is negative.
I: And the other one is?
LB: Positive.
I: How many zeroes did you have and can that be related to the problem in any way?
BH: Oh. It made sense for a minute but never mind.
The researcher allowed LB and BH to discuss among themselves what they thought was going on. She wanted to see if they could state the algorithm in light of the working with the novel model.

LB: Well, if you have either both addends are positive or both addends are negative, you will have the number of zeroes be zero. But if you have one addend that is different from the other you will always have. Sorry, always is a bad word. You will most likely have a zero. At least one.

LB did not want to say “always”, even though it was correct. The researcher again directed the participants to relate the numbers on their papers to the algorithm for addition of integers.

I: And how many zeroes? Can you relate that number of zeroes to the numbers that you are adding?
BH: If your negative number is larger … the absolute value of your negative number is larger then you have the same number of zeroes as your positive number.
LB: What about in the second to last one though? That’s the only difference.
BH: Because then the absolute value of the larger number was a positive number is bigger than the negative number. Then you would just subtract them.

Instead of listing two separate cases for when the positive integer has the greater absolute value and then when the negative integer has the greater absolute value, the researcher thought that it would be easier for the participants if they thought of just one situation in which the difference of the absolute values of the addends is found. She asked the participants to state their conjecture again.

I: So let’s state your theory here again.
BH: If your absolute value of your negative number is larger than your positive number your number of zeroes is going to be equal to your
positive number. Or am I over thinking this a lot.
LB: Say it again. No, say it one more time. It takes me sometimes a while.
BH: Ok. If your absolute value of your negative number is larger than your positive number,
LB: (She repeated and looked at her paper trying to make sense of it)
Absolute value of your negative number is larger than your positive number
BH: This one and this one. (BH was pointing to numbers on LB’s paper to explain what she meant) Well this here the number of zeroes is one. (LB had an incorrect number of zeroes for one of the problems.)
LB: Well that’s maybe why I screwed it up. (She corrected her mistake.)
BH: Then your number of zeroes is equal to your positive number.
LB: Except for the second to last one we get
BH: That’s because six is larger than the absolute value of five.
LB: OK. I just want to make sure that I was totally understanding what you were saying.
I: What happens if the absolute value of the positive number is greater than the absolute value of the negative addend?
BH: Then it’s just the difference of the two.
LB: I agree with that. Yeah, because the difference between six and five is one, no matter whether they’re negative or positive.
I: And does that relate to the way that you add integers? When you have something like negative seven plus positive five
BH: Oh.
I: You know that you would have how many zeroes?
BH: Five.
I: And how do you find your answer? What is your answer?
BH: Negative two.
I: Because you . .
BH: Got rid of five zeroes.
I: So looking at the negative seven and the five, how did you get, first of all the two?
BH: I guess seven minus five?
I: Is two. And how did you know it was negative?
LB: Because the larger addend is negative.
I: Is negative seven greater than five?
LB: No. But the addend itself is larger.
I: The addend’s negative seven.
LB: OK, the absolute value, dang it, If the absolute value of your negative number is larger than your positive number, then that equals the same amount of zeroes that you pick up. Is that right?
BH: Close. The amount of zeroes is the same as what number?
LB: The amount of zeroes equals your positive number. Or the absolute value of your positive number.

LB was incorrect when she stated that the amount of zeroes is the same as the positive addend. The researcher asked her some questions to create a perturbation.

I: So you’re saying if the absolute value of the negative one is greater than the positive one then you’re going to take the difference of those absolute values and that’s going to be the number part. How do you determine the sign
LB: By the non-absolute value. By whichever integer is . . . I can’t say that. Whichever integer is larger. That’s what I want to say but it’s not true. Whichever absolute value is larger.
I: So if we’re adding and the signs are the same, our sum is going to be what?
BH: Just combine the two numbers.
I: So you just add the two numbers and give it what sign?
BH: Whatever sign they both have.
I: So if they’re different signs, how do you determine the number part for your answer?
BH: Take the difference of the absolute values.
I: And what sign do you give that sum?
LB: The absolute value of the larger one.

Group 3’s response (the two participants were interviewed separately):

I: So now what I want you to do is look at that column where you have “Number of zeroes” and see if you can relate that along with the problem and the solution to the way that you work through the problem. Because normally you don’t have those things. You know what the rule is so see if you can relate what you did to the algorithm that you use for adding integers. (Long pause as she studied her results. I don’t think she understood the question.) So when do you have zero zeroes?
VF: Um. When you have either two positives together or two negatives together.
I: OK. And how do you determine what the sign is of your sum when they’re both positive or both negative?
VF: Oh, the sign of the solution?
I: Uh huh.
VF: If they’re both negative then you get a negative number.
I: OK. And if they’re both positive?
VF: You get a positive.
I: And how do you determine the number part for that solution when they’re both the same sign?
VF: The number of zeroes.

The researcher repeated the question for VF because her answer indicated that she did not understand the question.

I: The number part for the solution.
VF: You just add them up regularly and then if it’s a negative just add the negative sign into it.
I: OK.
VF: Cause they’re both negative.
I: Good. And how do you determine your solution if you don’t have them both the same sign?
VF: If you don’t have the same sign, you’re just subtracting from . . . No (She looked again at her paper.)
I: Well, see if you can relate that number of zeroes to something about the problem.
VF: The number of zeroes is one of the addends.

The researcher asked more questions so that VF could be more precise in her conclusion and thereby understand the algorithm for addition of integers.

I: OK, which addend?
VF: The positive addend if the bigger number is negative then the number of zeroes is the positive addend.
I: If the bigger number is negative?
VF: Yes.
I: So . . .
VF: Then the number of zeroes will be the positive addend, which is the smaller number.
The researcher asked more questions to get participants to better understand the algorithm. She needed VF to understand that “absolute value” was a better choice of words than “bigger number”.

I: OK. Is the positive number really smaller?
VF: Oh, it should be the bigger number. (Long pause as VF studied the chart again.) It will be the bigger number because it’s positive and positives are more than negatives. (Long pause.)
I: When you had the one that said “Two dollars of debt plus one dollar of money to spend”, you have negative two plus one, right?
VF: Yeah.
I: Which of those is greater, negative two or one?
VF: One.
I: OK. So the number of zeroes you said though was going to be one. Is it always the greater number that determines that number of zeroes?
VF: Yes.
I: Is it? What about when you had three dollars of . . .
VF: Oh, like six plus negative five.
I: Yeah. Six plus negative five.
VF: But that one is the smaller number.
I: See if you can determine some way of looking at them, looking at the number of zeroes, how can you relate that to one of those numbers that you’ve got? The number of zeroes is always going to be one of those numbers, isn’t it?
VF: Yeah.
VF: Um, I’m trying to think because sometimes it’s the positive number that is the bigger number and sometimes the number of zeroes is the smaller one and I’m trying to relate how that (she waved her hands). . . I’m still not sure.
I: Sometimes it’s the positive one and sometimes it’s the negative one. What if you just looked at the number part? What do we call that when we look just at the number part of the number? Do you know? (Long pause) Two words. First one starts with A, second one starts with V. (VF was still confused.)
VF: I’m not sure.
I: Absolute . . .
VF: Oh, absolute value.
I: OK, so does that help you?
VF: Yeah.
I: So now state your profound theory. What is that number of zeroes? It’s always the . . .
VF: Absolute value of the (long pause as she continued to look at her chart) always the smaller number.
I: And then how do you determine your solution?
VF: Subtract the absolute value of the smaller number
I: From the . . .
VF: Bigger number.
I: From the absolute value
VF: Yeah. Of the other one.
I: The one that has the greater absolute value. And what sign do you give that solution?
VF: The negative sign?
I: Always negative?
VF: Negative if the greater number was negative?
I: If the greater . . .
VF: Absolute value is negative.
I: And it will be positive when
VF: The greater absolute value is positive.
I: And then how do you get the number part? (Long pause) So now you know how to get the sign, how do you determine the number part?
VF: By subtracting the smaller absolute value from the greater absolute value.

NB was interviewed separately for this session.

I: I want you to look at that “Number of Zeroes” column and see if that relates to anything in the problem.
NB: (Pause) Yes, it’s the positive number for each one.

Since NB stated that the number of zeroes is always the same as the positive addend, the researcher asked questions that created a perturbation for NB.

I: Is it always the positive number?
NB: (She looked down at her paper and studied it.) No. I have one that’s a negative number.
I: Okay, but what is that number of zeroes?
The researcher asked NB to relate the number of zeroes to the original problem.

I: What is the number of zeroes? Is that related to the original problem in any way?
NB: Yeah that. . . (She put her index fingers parallel to each other.) What do you call it? Where you have the negative five, then it’s always a positive five, what is that called? I don’t remember what it’s called.
I: Absolute value?
NB: Yes!
I: So it’s the absolute value of . .
NB: Negative five.
I: In every case?
NB: Five.

By asking more questions, the researcher helped NB relate the number of zeroes to the algorithm for addition of integers.

I: Well, what is that number of zeroes? It’s always the absolute value of something. The absolute value of what?
(NB was confused.) If you look at the first one, it was five for the number of zeroes.
NB: Yeah.
I: That was the same as what in the problem?
NB: The positive.
I: Okay. And in the next one, the one was the same as . .
NB: The positive. And the third one’s the negative.

NB did not see a connection between the number of zeroes and the addends. The researcher asked questions about the number of zeroes for each of the examples that were done.

I: Okay. So what is that number of zeroes? Can you sort of tie together how many zeroes you had in each case related to the problem? Why was that one, the last one, why was it four? Why wasn’t it six? And the next to the last one, why was it five, not six? The third one from the bottom, why was it one and not two? (long pause) And just to get you thinking even
more, the ones where you had zero number of zeroes, when did that happen? When did you have zero zeroes? (long pause) In that first one, did you have any zeroes?
NB: Because they’re both negative.
I: Okay, or they’re both . . .
NB: Positive.
I: So, if they’re both positive or both negative . . .
NB: You won’t have any zeroes.
I: So now we have to look at how do you know how many zeroes you’ve got? Is that related to the problem at all? You saw that the four in the last one was the same as one of your addends. Right? (NB nodded in agreement.) Which addend was that number of zeroes? In every case, it was . . .
NB: I’m lost.

NB admitted that she didn’t understand the connection between the number of zeroes and the algorithm so the researcher asked more questions that would lead NB to make sense of the examples that she had done.

I: Okay. In that last one, we had negative six plus four.
NB: Yeah.
I: And you said there’s four zeroes.
NB: Wait. The only one that’s not five is the only one with a positive number.
I: Do you think that will always work?
NB: Seems like.
I: The second one down. After the five there’s a one.
NB: Yeah. There’s a negative one. The answer comes out to be a negative every answer, like the one has five zeroes comes out to be negative two, one . . .

The researcher asked NB to do an example using the novel model to help NB see how the novel model clarified the algorithm for addition of integers. By looking at a single example, NB could focus more on how the number of zeroes was related to the algorithm.

I: Okay. Let’s try negative three plus five. (NB filled out the chart before
modeling on the number line.)
NB: This actually helps. (She put three red bills on the number line.)
I: What part of it helps? (She pointed to the number line.) The number line and the color?
NB: Now that I’m getting used to it. (She put five white bills on the number line. She was somewhat confused. She took off three red bills.) I have three zeroes.
I: Because you’re really trying to get it so it’s just one color of money, right?
NB: Yeah.
I: And so if you take away your three white bills along with (She removed three white bills.) the three red bills that’s three pairs of zero isn’t it? So look at those numbers. Look at the number of zeroes and see if you can relate that to the problem itself. Where did you see that number before? On that one (the last problem done) you had three zeroes and your two addends were negative three and five. Right? In the previous one, it was negative six plus four. And you said you had four zeroes.
NB: Yeah.
I: So what is that number of zeroes? If you’re looking at those two addends, how do you know how many zeroes you’re going to have if you’re adding and the signs are different? (Very long pause) Is it always one of the addends?
NB: I’m sorry. Is it . . .
I: Is the number of zeroes always the same as one of the addends?
NB: Yes.
I: Okay. Which addend?
NB: In this last problem, the negative three.
I: And in the previous one, it was the same as the positive four.
NB: Yes.
I: Looking at those numbers, you said something about absolute value?
NB: Yeah.
I: Can you say something about absolute value related to that number of zeroes?
NB: Well, yeah, because whenever there’s a negative three in absolute value it’s going to be the opposite of that which is going to be positive three.
I: Okay. So now I want you to figure out how you get your solution. If you didn’t have this number line, how could you figure out that negative three
plus five is two?
NB: Well, because when you think in your, well, when I think in my head, you know, like with you, like how you’re giving examples of how to spend money. I have five dollars to spend. I spent three dollars. How much do I have left over? Two. That’s how I think of it.
I: So you’re taking the difference between those things?
NB: Yeah.
I: And how do you know what sign to give the answer?
NB: Because the larger number is what.
I: The larger number?
NB: Yeah, well. The larger number of the two between the negative and the positive, that’s going to be the sign. So if it’s like positive six plus negative three. Six is greater than negative three so therefore, it’s going to be three.
I: Which is greater, negative two or positive one?
NB: One.
I: So why was it a negative for your answer?

The researcher asked questions to help NB understand that it is the absolute value of the addends that one must subtract, rather than subtracting the “smaller number” from the “greatest number”.

NB: Because there’s negative two and I added plus one, which leaves me okay, then explain it. I can’t explain it. I won’t say the greatest number. But the larger number of the two. The higher the number, whether it’s negative or positive, but the higher the number.
I: Just the number.
NB: It’s going to be what that particular sign.
I: Okay. Good. So you’re really looking at absolute value?
NB: Yes.
I: Of those two numbers, aren’t you?
NB: Yeah.
I: And you’re taking, if the signs are different, then you’re taking the difference between those absolute values and if the signs are the same, then you’re just.
NB: Solving the problem. As it is.
I: And what sign do you give the answer in that case if they’re both, say
negative?
NB: It’ll be negative.

One week later, the researcher wanted to see if the participants remembered what they had done the previous week. To assess this ability to remember, she either asked each group to give an example involving integer addition or she gave them an example involving addition of integers. When one participant gave an example, the other participant in the group was then asked to model the given problem on the novel model and explain the number of zeroes as it related to the algorithm. Their responses follow.

Group 1’s response:

I: So RV, make up a problem. An addition problem. (RV wrote a problem on her paper.) You want to share it with us?
RV: Two plus negative one.
I: And LC, if you have two plus negative one, how many zeroes would there be?
LC: One.
I: And do you see that some place in the problem?
LC: Yes.
I: So how is that number of zeroes related to something in the problem?
RV: It’s the absolute value of the second addend.
I: Is it always the second addend?

The researcher asked more questions to get the participants to understand that the number of zeroes was the same as the lesser of the absolute values of the addends.

LC: It’s the smallest. Well, . . .
RV: It would be the smallest, um, the smallest absolute value of the problem.
I: Okay, the smaller absolute value. The one that has the smaller absolute value. And then, if those signs are different, how do you determine your answer?
LC: Whichever number has the greatest absolute value, that’s the sign you use.
I: Okay, and how do you get the number part?
LC: Um. You take, you subtract them.

Group 2’s response:

I: How would you model negative five plus four on the number line or five dollars of debt plus four dollars of money to spend? (BH put five white bills on the number line and LB put four red bills on the number line.) And would that be your final answer?
BH: Five dollars of debt and four dollars of money to spend you say?
I: Uh huh.
BH: Yeah.
I: And so, what would be your final answer?
BH: Our final answer would be (She removed four white bills as LB removed four red bills from the number line.)
I: And so, your answer would be . . .
BH & LB: Negative one.
I: Okay. Or one dollar of debt. How is the number of zeroes . . .
Remember when we did the chart with the number of zeroes? You had the problem, the solution, and the number of zeroes.
LB & BH: Uh huh.

The researcher asked questions to allow participants to relate what they had just done to the work that they did the week before. To aid in the statement of their conjecture, the researcher gave them their papers from the previous week’s session.

I: How was that number of zeroes related to the addition algorithm? Do you remember?
BH: You mean like that theory?
I: Yeah. Let me get out your papers here (giving back the papers that were done in the previous session).
BH: How’s it related to zero?
LB: It indicates your positive (pause), yeah, your positive addend.
BH: That would be that the . . .
I: Do you agree BH that the number of zeroes is the positive addend?
BH: No. It would be equal to the one whose absolute value is . . .
LB: That’s right. Absolute value.
BH: The absolute value of, wait, the negative number is larger than the
absolute value of the positive number it’s equal to the positive number.
I: And is that always going to be the case?
BH: No. If the absolute value of the positive number is greater than the
absolute value of the negative number, um, then the number of zeroes is
the difference between the two absolute values.
I: (to LB) Do you agree?
LB: Yes. I just said it in my head as you were doing it.

Group 3’s response:

I: Okay, so how would you model five dollars of debt plus four dollars of
money to spend, on the number line? (NB put one white bill on the
number line and VF put red bills on the number line.)
NB: What was the problem again? (She then put three more white bills on
the number line.)
I: Is that your final answer?
NB: Yes.
I: Remember that your final answer you want only one color.
VF: It should be one red. (VF took off four red bills and NB took off four
white bills from the number line.)
I: So you’re going to take off four of those zeroes, right? And you’ll be
left with .
NB: One.
I: One dollar of debt. Okay. How is the number of zeroes related to the
addition algorithm? Remember, you just did that, NB, so you have an
advantage. (NB had just completed session 2 before this session.)
Remember you found how many zeroes you had. Like in that case (the
example just completed), you said that you had four zeroes. How is that
related to the problem, negative five plus four?
NB: It’s the absolute value.

The researcher asked questions to allow the participants to be more specific in
their response.

I: Okay. What about the absolute value? It’s the absolute value of . . .
NB: Negative four.
I: Of negative four because negative four . . .
NB: You have four zeroes and with negative four (inaudible), which is
going to be the absolute value of four anyway so it’s going to be four
zeroes.
I: How did you know it wasn’t going to be the absolute value of the negative five?
NB: I’m starting to see it as whatever is more on the number line. Like this one had four (She pointed to the positive side of the number line.) and this one had five. (She pointed to the negative side of the number line.) So you were taking away all four to make one dollar of debt.
I: Okay. That makes sense, right?
NB: Yeah, I think.
I: So you had the four zeroes because that was the one that had the lesser absolute value, right?
NB: Yeah.

In summary, participants related the addition algorithm to the work they had done with the novel model. Prior research did not relate the use of a model to the understanding of the algorithm as they use it to add integers. Participants in the present study were able to state the algorithm in the following manner: when adding two addends whose signs are the same, the sum is the sum of the addends’ absolute values and the common sign is given to the sum. Participants had some difficulty in the beginning understanding some of the researcher’s questions but when they were asked the following week to model addition examples, they were able to relate what was done on the novel model to the familiar algorithm for addition of integers. In the third interview, one of the participants, BH, mistakenly said that the number of zeroes was the difference between the absolute values of the addends when the signs are different. However, she was always able to remove the correct number of zeroes on the novel model. Participants showed that when adding two integers whose signs are different, the sum is the difference of the absolute values of the addends and the sign given to the sum is the sign of the addend that has the greater absolute value.
Subtraction of Integers

During the third interview when participants were given subtraction examples to display on the novel model, they were also asked to create three columns on their paper. These columns were labeled “Problem”, “Solution”, and “Number of Zeroes”. Participants recorded the problem on their paper. Then they placed the appropriate number and color of bills on the number line portion of the novel model, as dictated by the sum in the problem. Next, the amount indicated by the addend was taken away or, if it couldn’t be taken away, some “zeroes” were added to make the subtraction possible. The number of zeroes indicated the number of pairs of positives and negatives that were needed in order to perform the subtraction on the number line. The solution was the result showing on the number line after any pairs of positives and negatives were removed.

Discussions from the three groups are provided below.

Group 1’ response:

I: Now, I want you to look at the number of zeroes that you added in each of those cases and see if you can relate that to anything in the problem.
(LC removed the bills from the number line as RV studied the problems.)
The researcher asked questions to allow participants to see that the number of zeroes to be added is the same as the addend that is given.

RV: It’s everything that’s being subtracted from the original amount.
I: Okay, it’s what was subtracted from the original amount. Do you agree, LC?
LC: (not really confident of what has transpired) Uh huh.
I: Are you sure?
LC: Uh huh.
I: Positive?
LC: Yeah.
I: And when you looked at that, do you see any relation between what you
were doing there (on the number line) and your algorithm for subtraction? (RV and LC were confused by the question.)

The researcher asked the following question to make sure that the participants had the correct algorithm in place.

I: What’s your algorithm for subtraction? When you normally have four subtract negative one, what do you usually do?
LC: Four plus negative one.

LC incorrectly stated the related addition problem but was corrected by RV.

I: Okay, so you do four plus . . .
RV: Four plus one.
I: Four plus one. Right?
LC: Wait. What was the problem?
I: Four subtract negative one.
LC: Oh yeah, it’s four plus one.

Group 2’s response:

I: I want you to look at those problems and figure out how many zeroes had to be added for each of those problems.
BH: (very quickly) It’s the same as the absolute value of the second, uh, what you’re subtracting from the first.
LB: Addend.
BH: That’s what it’s called.
I: So it’s the absolute value of the second amount?
LB: Yes.
I: And how did you determine the solution then? (long pause)
BH: You mean like if we had two negatives, you add?
LB: If you have two negatives, you have a positive.
I: What if you had, okay, let’s try this for your theory. Let’s take those off. (LB and BH removed the bills from the number line.) And let’s write down this problem. Negative four subtract negative two.

BH and LB looked at their papers to decide if there was some pattern.

BH: That’s a negative two. Wait a minute. I’m seeing a pattern here. Four, one, one, two, four, three, five, three. (She was reading the numbers from
the last column on her paper – the number of zeroes that needed to be added.)
LB: Four, five, six minus two is four. (She was reading down the number of zeroes column.)
BH: The number of zeroes that you add with the first amount that you’re using equals your solution.
LB: Say it again.
BH: Okay. Wait a minute. Cause I was looking at two and four as six, one and one is two, one and four is five, two and three is five. (long pause) Do you see what I mean?

The participants stated their conjecture. They applied the rules for addition of integers after changing the subtraction to addition and changing the sign of the second amount (the addend).

LB: Here it is. Yeah. Six minus four is your first addend. Negative six plus four is negative two. Two minus one is one. Five minus one is four. Negative five plus two is negative three. Negative five plus four is negative one. Negative three plus three is zero. Seven minus five is two. Negative seven plus three is negative four.
I: Sounds good.

Group 3’s response:

And let’s do two dollars of debt subtract four dollars of debt. (NB put two red bills on the number line.)
NB: Two dollars of debt.
I: Okay. And subtract four dollars of debt. (NB put four more red bills on the number line and four white bills on the number line.) And now you want to subtract your four dollars of debt. (NB removed four red bills from the number line.) See, in order to have that four dollars of debt to take off, you added in four dollars of money to spend along with it, didn’t you? And that’s why that problem’s going to become negative two plus four. Does that make sense?
NB: Yeah.
I: So it’s not just slopping together a rule, an algorithm, to figure out what’s going on. What is that number of zeroes?
The researcher asked for a general answer to this question so that participants could explore why the algorithm for subtraction of integers works. Because the participants were not able to see a connection between the algorithm and what was done with the novel model, the researcher asked the participants to look at their papers to locate where the number of zeroes was found.

NB: In this one?
I: In any of them. (VF and NB looked intently at their papers, but seemed confused by the question.) Where do you see a two in your original problem? (Pointing to the problem on NB’s paper) If this was your original problem, do you see a two?
NB: Yeah.
I: (pointing to another problem) Do you see a three for this one?
NB: Yeah.
I: (pointing to a previous problem) Do you see a five for this one?
NB: Yeah.
I: (pointing to a previous problem) Do you see a three?
NB: Yeah.
I: (pointing to a previous problem) A four?
NB: Yeah.
I: What is that number every single time?

After the researcher asked the participants about the integer whose sign was changed, they stated that the second number (the addend) determined the number of zeroes that had to be added.

NB: The right hand number.
I: Okay. It’s the thing that you’re subtracting, isn’t it? The absolute value of the thing you’re subtracting.
NB: Yeah.

Near the beginning of interview four, participants were given a subtraction problem involving integers and they were asked to model it on the novel model. They
were asked to put columns on their paper to allow them to see how the subtraction algorithm was related to what they did on the model. The columns that participants made on their paper the week before were changed so that they made more sense to the participants. As the researcher was working with the participants in interview three, she noticed that it was a giant leap for Groups 1 and 3 to go from finding the number of zeroes that needed to be added to finding the solution and relating the number of zeroes to the algorithm. Therefore, she had participants in Groups 1 and 3 and one of the participants in Group 2 adjust the columns on their papers to say “Problem”, “What needs to be done?”, “What’s left on the number line?”, and “What’s your solution?” A sample of one participant’s paper showing these columns and the participant’s responses is shown in Appendix C.

Group 1’s response:

I: Let’s have four subtract seven. (LC put five white bills on the number line but then took one off when she realized she had too many.
RV: Oh. Do you want to subtract . . .
I: Subtract seven. So, four subtract seven. Thank you LC. (LC wrote the problem on her paper.)
RV: Alright then, seven zeroes. (She put seven red bills on the number line and LC put seven more white bills on the number line.)
I: Okay. Now can you take away your seven?
RV: (pointing to the positive side of the number line) Over there.
I: Okay.
LC: Why? I don’t understand.

The researcher asked RV to explain her actions to find out if she understood what she was doing and also to see if she could explain her actions to LC.

I: Okay. Explain it RV. Why are you taking away seven?
RV: Cause we’re taking away seven there (pointing to the positive side of
the number line). Um and we’re taking away the seven zeroes that we had. But wait.

RV incorrectly said she was removing seven zeroes when she took away seven white bills. The researcher asked her questions about what she was doing.

I: Taking away seven zeroes?
LC: Oh, I get it. I take seven off of here. (She removed seven white bills from the number line.)
I: Right. Because it says “subtract seven”.
LC: Right. And before I didn’t have seven so I had to . . .
I: So, you really had to put seven zeroes on there didn’t you?
LC: Uh huh.
I: So now she’s taking off her seven positives. Right, RV?
RV: Uh huh.
I: So what’s on the number line?

The researcher asked the participants this question so they could conceptually understand the subtraction algorithm for integers.

RV: Negative seven plus four.
I: Okay, or four plus negative seven. And then what are you going to do?

This question prompted the participants to remove zeroes as they had done in previous addition examples.

RV: Take off the zeroes.
I: Okay. (RV removed four red bills from the number line as LC removed four white bills.) And your final solution would be . . .
RV: Negative three.
I: Can you actually see it on the number line when we’re doing it that way?
RV: Yeah. (LC nodded in agreement.)
I: Which is why I forced you to do it RV’s way. You could’ve done it your way. .
LC: Uh huh.
I: But then we wouldn’t get the algorithm.
Group 2’s response (BH and LB were interviewed separately):

The researcher asked BH to model four subtract negative two. BH understood that in order to have negative two to subtract, two zeroes had to be added.

BH: So positive four (She put four white bills on the number line.) and you’re going to take away negative two so you need to add negative two on but you also need to add two positives. (She put two red bills and two more white bills on the number line.) to make that zero and then take away the negative two. (She removed two red bills from the number line.) So I have six.
I: Okay.
BH: So I have two zeroes there.
I: Okay.
BH: And the solution is six.

Group 3’s response:

I: And for the next one let’s do four dollar of money to spend subtract seven dollars of money to spend. (VF put four white bills on the number line. She then put seven more white bills on the number line. NB then put seven red bills on the number line.) Okay, now can you subtract your seven dollars of money to spend?
NB: Yes.
I: Okay. Take it off. (NB and VF both looked at their papers trying to figure out what to write.) Don’t write anything down there yet. (VF removed seven of the white bills from the number line.) Just take off the seven dollars of money to spend because that’s taking it away, or subtracting, true?
NB: Yeah.
I: And what do you have left on the number line?
NB: Three. Negative three.

Even though NB had more bills on the number line than three red bills, she was able to state the correct answer of negative three. The researcher wanted to make sure that
she understood that four zeroes could be removed from the number line portion of the novel model.

I: Okay, but write down how many reds and how many whites. And then your final solution would be. . . What do you have to do to get your final solution?
NB: Take away three zeroes.
I: Three?
NB: Four zeroes.
I: Four zeroes. Okay. So you get negative three, true?
NB: Yes.

In summary, all but one of the participants was able to see the relationship between the algorithm for subtraction of integers and the way it was modeled on the novel model. Some participants were confused with the initial problem because they could not “take away” the required amount. However, after participants recalled that integers could be named in more than one way, all but one of them were able to perform the subtraction using the novel model. By adding the column titled “What’s on the number line?”, participants were more clearly able to see the connection between the algorithm for subtraction and the way that subtraction was modeled on the novel model. All but one of the participants could see why the subtraction sign was changed to addition and the addend was changed to its opposite. For this one participant a link was not made to the subtraction algorithm.

Analysis of Individuals

Although the basic level of analysis of the data presented so far has been collected by groups, individual participant responses were considered in the analysis of the data. Because there was no research relating the use of a model for addition and subtraction of
integers to the algorithms for addition and subtraction of integers, it was the intention of
this researcher to have participants determine this relationship between the modeling and
the algorithms. Throughout this process, participants had some misconceptions about the
use of rote procedures for addition and subtraction of integers.

Misconceptions and Use of Rote Procedures

In interview one, two of the participants explained what they knew about addition
of integers. Pertinent aspects of the interview follow:

LC: Negative six plus two equals negative four.
I: How did you know that?
LC: Because. . . I don’t know. I took negative six and sub, well, added
two. I don’t know how to explain it. Like if you have a negative number
and add two positives, you get, you’re left with negative four.
RV: Um. Negative eleven plus four so I’m just gonna get negative seven.
I: And how did you know that?
RV: Um. (Long pause)
I: Did you have something scratched out there?
RV: Yeah. I wasn’t watching. I don’t know what I was doing for a minute.
I: What did you have at first?
RV: At first I thought I had just put eleven but it was supposed to be
negative eleven.
I: Oh. So how did you know it was negative seven?
RV: (Long pause) It’s just like basic math, pretty much.

Both participants were able to determine the solution to the problem. In the first
case, LC knew that she should subtract to find the solution to the addition problem and in
the second case RV said “it’s just like basic math”. In other words, they had learned
procedures but could not fit their previous rules with their work using the novel model.

During interview two, LC and RV were asked to show subtraction using the novel
model. Their responses follow.
LC: Negative two minus four.
I: Okay. What’s your solution?
RV & LC: Negative six.
I: What are you taking away?
LC: I don’t know. Empty space, right here. (She slapped her hand down on the positive side of the number line. She then removed the two red bills that she had put on the number line.)
I: And let’s try one dollar of money to spend (RV put one white bill on the number line.) subtract one dollar of debt. (LC put a red bill on the number line.) Is that taking away one dollar of debt? (RV had her hands on the white bills, thinking about what to do.)
LC: No. I am so confused.
RV: Wait. You’re subtracting, you’re really just adding another dollar. (She put another white bill on the number line and then removed a red bill.)
I: So what did you really add on there?
RV: You added one more dollar to spend.
I: And you added . . .
LC: One you can’t spend.
I: One dollar of debt.
LC: Yeah.
I: So you were really adding how much to your dollar of money to spend that you had?
LC: One. I don’t know.

RV: In the problem it’s four minus a negative one. Um, and that’s really, in the end, adding.
LC: Right.
RV: Cause two negatives are a positive.

In each case the participants did not relate their earlier work in the sessions to subtraction. They instead knew rule-based procedures that worked. In interview two, most said that subtraction means “to take something away” but did not use this when first using the novel model.
During interview four, LB was interviewed separately from BH. She was given several addition problems to model using the novel model. Her responses follow.

I: Four dollars of money to spend plus six dollars of debt. (LB put six red bills and then four white bills on the number line. LB wrote four minus six on her paper. She corrected it to four plus negative six. She then removed four white bills and four red bills from the number line.)

LB: Negative two.

I: Three dollars of money to spend plus one dollar of debt. (LB wrote the problem on her paper as 3 – 1).

In each case when LB was given an addition problem involving integers of different signs, she wrote the problem as a subtraction problem. She was using the algorithm for addition instead of writing the problem as given.

Another example of participants applying algorithms without understanding is portrayed in the following discourse where LB and BH were asked to solve 3 – (– 2).

I: Three subtract negative two.

LB: So then you would add them.

I: So why did you say you would add them, LB?

In the following discussion LB confused the rule for multiplication of two negatives with the subtraction algorithm for integers.

LB: Because you change the . . .since you have two negatives; you can’t have two negatives, it’s a double negative kind of like in English. You have to add them together and then make the negative a positive because you can’t have two negatives. So that means three plus positive two equals positive five.

I: And that’s what you were taught?

LB: Yes.

I: And that made sense to you?

LB has relied on rules and procedures to succeed in mathematics. She has learned that mathematics is rule-based and does not necessarily make sense.
LB: After a long time. Not right away no. It was a rule you had to follow. It was accepted because it was written by somebody who knew more than me.

I: So when you were adding and subtracting with integers, what would you say was your reason for having an answer? You figured out the problem, you had an answer. Did it make sense all the time?

LB & BH: No.

I: So you just, I mean, how did you know it was right then?

(Long pause) LB: Well, we learned our basic facts of addition and subtraction in grade school. So you take your basic facts and you tweak them in to fit whatever operation is required.

BH: I’d like to know why you add when you have two negatives so I can tell my daughter.

LB: I can’t tell you. It’s one of those things that I refuse to ask why anymore because it confuses me. I just accept it. I learned that the hard way.

During interview four, LB was also given some subtraction problems involving integers. Her responses follow.

I: Seven dollars of debt subtract two dollars of money to spend. (LB put seven red bills on the number line. She then put two white bills on the number line. She wrote on her paper and then removed two white bills and two red bills from the number line.) So that’s seven dollars of money of debt subtract two dollars of money to spend?

LB: Yeah. No, because I have to have two more (She put two red bills on the number line.) Subtract two dollars of money to spend. I still have seven dollars of debt.

I: So if you have seven dollars of debt subtract two dollars of money to
spend, where did you take away the two dollars of money to spend?

The researcher asked LB some questions so that a perturbation was created.

LB: (as she put two white bills on the number line) I took them from here.
I: Where did they come from?
LB: Where did the positive, or the money to spend, come from?
I: Uh huh. (LB looked blankly into space.) You had seven dollars of debt and I said subtract from that two dollars of money to spend.
LB: Oh, I gotcha. (She removed two white bills from the number line. She then removed two red bills from the number line.) I shouldn’t have had any ones on there in the first place (She meant white bills.)
I: What did you just take away?

LB’s actions indicated that red bills represented money to spend so the researcher asked her questions about what she was doing.

LB: Two dollars of money to spend.
I: Is that what red bills indicate?
LB: No. (She put the two red bills back on the number line.) So these go back. Seven dollars of debt minus two dollars of money to spend. (As she put two white bills on the number line) I would’ve had two dollars on here. I: Where did they come from?

Again the researcher asked questions so that LB might make sense of what she was doing. LB should have added two zeroes so that she had two white bills to take away.

LB: (She paused a few seconds.) They didn’t come from anywhere. (as she removed the two white bills) They shouldn’t even be here because if you only have seven dollars of debt, you have to take two dollars of money to spend and if you don’t have two dollars of money to spend, then you can’t take away two dollars of money to spend. You can only add two dollars of debt.
I: So, does that mean that problem can’t be done?
LB resorted to the algorithm for subtraction of integers without meaning because she could not make sense of what she was doing.

LB: Oh, it can be done. Negative seven minus positive two. Should be plus two (She changed numbers on her paper.) Negative seven plus two should be negative five. Or two from seven, you can’t do that.
I: Why not?
LB: Because you can’t take seven. . . you can’t take seven from two.
You’ll end up with negative five.

The researcher asked LB some questions to allow her to recall what subtraction meant and why there was a need to add some zeroes. Asking LB to rename seven dollars of debt might trigger the idea of adding zeroes.

I: Is there another way that you could show seven dollars of debt?
LB: Uh huh. (LB put two more red bills and two white bills on the number line.)
I: And does that help you? (LB put her fingers on the red bills and she counted them.) Yeah, ‘cause then you can take away these two dollars of money to spend. (She removed the two white bills from the number line as she said this.) And you will be left with nine dollars of debt. No that can’t be right. Seven minus two. Seven dollars of debt. . . So it would be nine. . . two minus nine. Two mi. . .no wait. Nine. . . eleven. It would be negative seven so you have seven dollars of debt and from that you’re supposed to. . .

When LB heard the word “subtract”, she applied the algorithm to find her answer.

She then attempted to make the novel model comply with the determined answer.

In the first interview, LC confused subtraction with addition of a negative amount.

This is shown in the following discussion.

I: How do you know that that is the same thing as three dollars of money to spend?
CA: Because there’s three of those (pointing to the red bills) and four of those. . . I don’t know.
LC: It’s the same as six (pointing to the white bills) minus three (pointing to the red bills).
I: It’s the same as six minus three?
LC: Or six plus negative three.

In interview four, the researcher was interested in whether the participants could transfer what they had learned using the two colors of money and the two colors on the number line to using bigger bills that were two colors but without the number line. The bills that were used were ones, fives, tens, twenties, and fifties. There were white bills and red bills of each denomination.

When LC and RV were working with the big bills, they applied the algorithm for subtraction of a negative instead of relying on what they had done with the novel model in previous sessions. They didn’t have the number line portion of the model to help find the solution. This can be seen in the following portions of the interview.

I: How about fourteen dollars of money to spend (LC put down a white $10 bill and four white $1 bills.) subtract two dollars of debt?
RV: So wait. Don’t you have to add . . . You have to add two. (She put two red bills with the other bills. LC picked up two of the white $1 bills.)
LC: No.
RV: Oh no. Wait.
LC: And you just take away (She removed two white $1 bills.) They cancel each other out.
RV: Yeah, so twelve.
I: So, fourteen subtract negative two is twelve?

I: How about thirteen dollars of debt (RV put a red $10 bill and three red $1 bills on the table.) subtract twenty-six dollars of debt?
RV: Subtract twenty-six so (LC put a two red $10 bills and a red $1 bill on the table.) Oh, wait, you need a five.
LC: Uh huh. (RV found a red $5 and LC put it on the table with the other bills.)
RV: Now you need twenty-six. (LC found two white $10 bills, a white $5 bill, and a white $1 bill to put on the table.)
I: So now you want to take away twenty-six dollars of debt, LC. (long pause)
LC: Uh. So (she removed the two red $10 bills, the red $5 bill, and the red $1 bill from the table.) Take away. Wait. It’s adding. I’m so confused.
Participants’ Perceptions of the Novel Model

After the second interview session participants were asked to write about whether the novel model helped or hindered their understanding of integers. Their responses follow:

LC: The number line and the colored money is very helpful in solving problems quickly and accurately. I could probably solve problems with just one color of money just fine because I use green money all the time. When I figured out that I could represent a problem more than one way I was excited.

RV: The number line and colored money don’t help as much. It is more of a visual aid but can help to describe what we’re trying to say. I was excited to find more than one way to show a solution to a problem.

BH: This model makes more sense to me because of the cancelling out of zeroes. I have never thought of it this way before because I’m so used to just memorizing everything to learn. It is more clear to have some theory presented.

LB: This model makes sense because you can visualize and have your learning hands-on. Not only are you using 2 different manipulatives to satisfy the individuals learning needs, but you are also keeping it simple.

VF: The model of the money line and money was very helpful. I understand that the number of zeroes was obtained by having equal amounts of money on both the positive and negative sides. With only the above chart (listing the addition problem, number of zeroes to be removed, and solution) I had trouble figuring out the relationship of the problem, solution, and the number of zeroes. Now I understand a lot better.

NB: The number line is a helpful way of learning integers because it helps you visually see what’s happening through each problem. I do think it gets a bit confusing at times but overall I think it’s a great way of learning integers. Memorizing just the rules can be confusing.

Participants were also asked to write about their experiences with the novel model after session four. Their comments follow.
LC: The color helped and so did the number line. Both of those things helped teach integers in a new way. I understand why you add and subtract integers now. Before I just did it. I understand now how the algorithm came about.

RV: The number line helped focus. Also the colored money was very helpful in distinguishing debt from money to spend. I enjoyed getting a better understanding of the integer system.

BH: It made the idea of why you actually add numbers when you are subtracting a negative number. The combination of the number line and the color of money helped with this idea. These both helped me understand the algorithm. They both also made the idea of addition more clear.

LB: I first took a while to not do it (the math problem) in my head, but because of the number line, I was able to adjust my way of thinking to be able to solve the problems given to me.

NB: When we were finished with the number line me moved onto larger bills. The number line helped a lot with the addition & subtraction problems. However, when we moved on to the larger bills, we don’t have the number line. I became a bit confused. I started focusing on the color of the bills instead of imagining the line. I was still a bit confused when focusing on the colors. I do now have a better understanding of integers. So overall despite some of the confusion, I feel I understand it better than I have ever before.

VF: Going to the big bills I thought about the rules since we couldn’t use the number line. The number line was very helpful. I think that this whole study was very useful and will be useful in my math class. I understand better the actual process of adding and subtracting integers. I hope this knowledge will stay with me so I can pass it along to others. Thanks for the opportunity.

In summary, some of the participants may have been reluctant to use the novel model to show integers at the beginning of the study, but by the end of the study most participants felt that both the color of the money and the number line helped them to better understand integers and the operations of addition and subtraction. All participants expressed their feeling of increased knowledge about integers.
Summary of Major Findings

Each section of this chapter dealt with data that supported each of the researcher’s three questions. Major findings from the data indicate that students do not know what numbers belong to the set of integers and they believe that school activities and practical applications such as banking are instances where integers are used most of the time. In designing tasks for the first interview session, the researcher considered five prerequisite skills that Lytle (1992) suggests must be understood before a complete conceptualization of integers can be achieved.

To model addition and subtraction of integers in the interview portion of this study, participants used a novel model consisting of a red and white number line and play money that consisted of red bills and white bills. Red bills and the red side of the number line represented “debt” and white bills and the white side of the number line represented “money to spend”. These terms became synonymous with negative integers and positive integers, respectively. Although some of the participants were reluctant to use the novel model at the beginning of the study, by the end of the study all participants were able to use the novel model to show addition and subtraction of integers. All participants claimed that the novel model gave them a more conceptual understanding of integers and the operations of addition and subtraction.

Participants named integers in many ways using the novel model. This served as a prerequisite skill for adding and subtracting integers. Subtraction of integers was viewed as “take away” so that participants could apply the same procedure to subtraction of integers as they had previously done with subtraction of whole numbers. Participants
were able to apply their work with the novel model to an understanding the algorithms for addition and subtraction of integers.

In the last interview session the researcher lead participants to discover how the work with the novel model was related to the algorithms for addition and subtraction of integers. The last session also required participants to use “big bills”, which were larger denominations than could be modeled on the number line portion of the novel model.
CHAPTER V: ANALYSIS

Introduction

This chapter presents a summary of the findings of this research and its significance is explained in the context of existing research. The chapter is organized as five sections. Because representation was a major focus of this study, the first section of this chapter describes the importance of the role of representation in the current reform movement in mathematics education. The second section provides an overview of the results determined by answering each of the three research questions that guided this study and analyzes the findings from this research as it relates to existing research. The third and fourth sections of this chapter discuss the limitations and implications of this study. The final section gives recommendations for further research.

The Role of Representation in the Reform Movement in Mathematics Education

NCTM, in their document, *Principles and Standards for School Mathematics* (NCTM, 2000), along with the National Mathematics Advisory Panel (United States Department of Education, 2008) and the National Council on Teacher Quality (Greenberg & Walsh, 2008) have led a charge to change the way that mathematics is taught in our schools. All three of the groups mentioned above suggest that instead of teaching mathematics as a series of algorithms that students must memorize, the mathematics curriculum must allow students to make sense of the mathematics that is to be learned. These organizations advocate a constructivist view of learning in which students connect
the mathematics that they have learned in the past to the mathematics that they need to
learn in the future.

The NCTM *Principles and Standards for School Mathematics* (*NCTM, 2000*) provides a philosophical framework for the five process standards, which are problem
solving, reasoning and proof, communication, connections, and representation. These
principles suggest important aspects of learning and doing mathematics. Although all of
these process standards are important, the focus of the present research was
representation.

A representation describes the signs that stand in for and take the place of
something else (Mitchell, 1995). Representations are used as “tools for thinking and
instruments for communicating” (*NCTM, 2000*, p. 206). A representation can be a
symbol or an internal creation. Students constantly construct representations in
mathematics. Some representations are found in textbooks and others are created by the
students themselves. Students use representations as tools for thinking about mathematics
and also as they communicate their ideas to others. Representations “help to portray,
clarify, or extend a mathematical idea by focusing on essential features” (*NCTM, 2000,*
p. 206). The ways that ideas are represented are crucial to student understanding. When
students are given opportunities to explore multiple representations for a concept, student
ability to think mathematically is increased (*NCTM, 2000*).

Internal representations are developed by students as they seek meaning for a
concept and these internal representations are then transformed into external
representations so that communication about mathematical ideas can take place. In the
same manner, external representations are transformed into internal representations so that the student can make sense of the concept. Limitations in some students’ understanding are a result of internal systems of representation that are not completely developed. For example, students may think that integers and whole numbers contain exactly the same elements. They continue to apply previously learned rules for whole numbers for addition and subtraction of integers. These rules may have included the idea that when a number is subtracted from another number, the result must be less than the initial amount and when a number is added to another, the result must be greater than either of the original amounts. The novel model allows students to see that the result for a subtraction problem or an addition problem can be less than, equal to, or greater than the original amount given.

A goal of mathematics education should include the development of efficient internal systems of representation in students that are consistent with conventional systems of mathematics (Goldin & Shteingold, 2001). Modifying representations to better suit the content, the learners, learning, or the context is an important part of education (Ball, 1988). New models may be needed that allow students to relate past experiences to the mathematics to be learned. “The goal is to put students in situations where they express their current ways of thinking in forms that will be tested and revised in the direction of increasing power” (Lesh & Doerr, 2003). The novel model created for this study is based on students’ past experiences with money and bridges their understanding of money to include integer addition and subtraction. It is believed that the novel model will help students build necessary mental representations to construct meaning for
integers, even for situations that are not intuitive such as getting smaller sums when adding and getting greater differences when subtracting.

The present study was an attempt to determine if participants could use a representation of integers to conceptually understand integer addition and subtraction. This representation was a novel model that combined ordinality and cardinality of integers. The model consisted of a number line and money that were color-coded. Red represented “debt” or negative integers, and white represented “money to spend” or positive integers. Research from the past three decades does not show that this type of model exists (Bolyard, 2005; Davidson, 1987).

Overview of the Results

This researcher chose to use a qualitative design for this study so she could better understand how students think about integers. This study introduced participants to a model which combined the features of the number line and two-color counters, hence representing cardinality and ordinality of integers in a single model. This model was used to introduce participants to integers and the operations of addition and subtraction. Three research questions guided this study. They are:

1. How do pre-service elementary teachers interpret and make sense of integers?
2. Does the use of a novel model impact student understanding of addition and subtraction of integers? If so, how?
3. Do pre-service elementary teachers relate the use of a novel model for addition and subtraction of integers to the rule-based procedures that they use to
add and subtract integers? If so, how do they develop meaning for these relationships?

When discussing the impact of the study, the researcher uses the word “student” to refer to the seventy-nine students who took the initial survey and she uses the word “participant” to refer to the six from the larger group of students who participated in the interview phase of the study. Only the six participants used the novel model that is discussed in this study. The larger group of students only completed the initial survey.

Results of this Study

Previous research explored students’ knowledge of integers prior to instruction about integers. Most of the studies were done with elementary school students, whereas the present study engaged college-age students. Previous research also discussed the effectiveness of the number line model and the two-color model when introducing integers. The present study involved a novel model that combined the qualities of the number line model and the two-color model. No other study has used a model that combines the qualities of both the number line and neutralization models.

The present study sought to determine how pre-service elementary teachers conceptualize integers and the operations of addition and subtraction with integers. The researcher studied what conceptual understanding of integers that students had, and in the case of the six students who participated in the interview portion of the study, drew on participants’ previous understanding of integers to allow them to build a foundation for future mathematics involving integers. The researcher attempted to understand the interview sessions through the eyes of the participants so she could better understand
their misconceptions and difficulties. Throughout the study, she asked the participants questions to help remedy their conceptual understanding. Rorty (1982) claimed that the goal of constructing knowledge is not to achieve closer and closer matches with reality but to cope with or adapt to reality. Therefore, when teachers search for ways to maximize student learning in the mathematics classroom, “they are attempting to construct a web of meaning and thus develop an ontology to make what happened in their mathematics classroom intelligible” (Cobb, Yackel, & Wood, 1992, p. 2). The present researcher attempted to make sense of what happened in the interview sessions as it related to the “web of meaning” offered by the participants.

The researcher studied the survey, videotapes and written work of the students to analyze their conceptual understanding of integers and integer addition and subtraction. Specific comments made by the participants were reflected upon by the researcher to determine how the present study relates to existing research. The following sections outline the conjectures this researcher drew from the data and the justification for these conjectures.

*Students Cannot State What Elements Make up the Set of Integers*

Throughout this study, participants revealed several misconceptions about integers. Also, as evidenced from the survey data, none of the participants in this study initially understood what numbers were included in the set of integers. At the beginning of this study, when asked what integers are, three of the six participants said that they are whole numbers and two of the six said that fractions are included in the set of integers. One of the six participants defined integers as “the set that includes the whole numbers,
negative or positive”. Although this last definition is not totally correct, it was the closest to identifying all members of the set of integers.

Evidence from the survey data also revealed that none of the participants in the current study mentioned that negative integers belong to the set of integers. Two of the participants thought that all fractions were also integers. Also, when asked to give examples of where integers are used, two of the six participants indicated that they used integers to balance their checkbook. The researcher concluded that this does not usually involve negative integers, and sometimes involves decimals, which are not always integers. Another misconception about negative integers is presented in the following statement by one of the participants.

I: So what do you suppose that red money means?
NB: Negative.
I: And what would negative money be?
NB: You don’t have any.

This participant (NB) evidently thought that negative integers were just other names for zero. Her definition of negative integers was incorrect. If one does not know what negative integers are, it is unlikely that he or she can conceptually understand how to perform operations using them with meaning. Students have learned how to use procedural rules for integers without understanding where the rules come from and, more importantly, without meaning. These results concur with existing research (Bolyard, 2005; Lytle, 1992; Wilkins, 1996). Adler (1972) found that one of the major obstacles to learning integers occurs if the student does not understand negative integers. He found that if students do not understand what negative integers are, they tend to memorize rules
for operating with integers; therefore, he suggests that negative numbers be introduced as early as the first grade. Likewise, Smith (2002) asserts that if students do not understand integers, operations with integers become procedure oriented and meaningless.

In the beginning of the study, participants were unsure about what numbers make up the set of integers but by the end of the study all of the participants recognized that the set of integers included the counting numbers, the opposites of the counting numbers, and zero. This knowledge may have been facilitated by the markings on the number line portion of the novel model that was used in the study. The participants were able to relate their previous use of positive and negative integers to the novel model which served as their physical representation for integers. After seeing the physical representation for integers with the novel model, they were able to internalize this representation and provide external representations through their written expressions and discussion involving integers and the operations of addition and subtraction. For example, in interview session two LC and RV modeled several addition problems. After modeling these problems on the novel model they were able to state that when both addends are the same, there are no zeroes and when one addend is positive and the other addend is negative, the number of zeroes to be removed is the same as the magnitude of the addend that has the lesser absolute value.

Understanding that zero separated the positive and negative integers may also have been facilitated by the number line portion of the novel model. This idea is fundamental if one is to understand the concept of absolute value. In the beginning of the study participants could find the absolute value of an integer but they were not able to
define absolute value in a meaningful way. As the study progressed, participants were able to name integers in many ways by adding “zeroes”. Because the participants added the same amount of positives and negatives to a number without changing the number’s value, participant conception of absolute value became more enriched.

Student Examples of Where Integers are Used

Students in the present study were asked where they use integers. Forty-eight of the seventy-nine students said that integers were used in school related activities, with the majority of those stating that they were used in math class. They did not make connections between what they learned in school and what they needed to know in life. If one does not see applications of material learned in school, it is easily forgotten and is considered meaningless by students (Bolyard, 2005; Hackbarth, 2000; Lytle, 1992). When this happens and students later need to build on what they have learned, the foundation is not strong.

From the initial survey, those who wrote “school related” activities did not indicate that they possessed a conceptual understanding of integers. They listed such activities as computing grade point average and counting. These activities do not require negative integers and this researcher concluded that the students in this study had a limited definition and understanding of integers. This omission of activities which require negative integers could seriously impede their understanding of what numbers make up the set of integers. Previous research did not ask students where they used integers. In the present study the researcher specifically asked students about their use of integers to try to verify some of the things that other researchers had noted or found in different
circumstances. The present researcher found that students do not conceptually understand or rote know what elements belong to the set of integers nor do they understand where they are used.

*Models Used to Represent Integers*

Bolyard (2005) reported on a review of research conducted by Suydam & Higgins (1977) which investigated the use of manipulative materials in kindergarten through eighth grade mathematics. They found that in almost half of the studies reviewed (eleven of twenty-three studies) students using manipulative materials scored significantly higher on achievement tests than students who did not. Ten studies reported no significant results. They conclude, “In a simple manipulative vs. non-manipulative comparison, non-manipulative lessons cannot be expected to produce superior achievement. Lessons using manipulative materials have a higher probability of producing greater mathematics achievement than do non-manipulative lessons” (Suydam & Higgins, 1977, p. 57.) One must make certain that students understand the manipulative and how it relates to the mathematics. One must guard against using the manipulative in a manner which is as devoid of meaning as memorizing abstract mathematical procedures (Gregg & Gregg, 2007). Students should be encouraged to discuss their answers to the problems with their peers (Smith, 2006; NCTM, 2000). For this reason participants in the present study were asked to discuss and confirm their answers with their partner. During all of the interview sessions, the researcher asked questions of the participants and encouraged the participants explain their solutions to the problems. When a participant expressed frustration or confusion about a problem, the researcher asked the other participant to
explain how she was thinking about the problem. The researcher only interfered if there was a serious mathematical error in what the participant was explaining. This process allowed both of the participants to view the problem in another way and to make sense of the solution that they found.

The present study used a novel model that consisted of a number line that was red on the left side and white on the right side along with red money and white money. The red side of the number line and the red money represented negative amounts or “debt”; and the white side of the number line and the white money represented positive amounts or “money to spend”. Each integer was clearly marked on the number line, with zero as the separator between the positive integers and negative integers. According to the survey data, none of the six participants knew what numbers belonged to the set of integers. Through using the novel model, participants learned that the set of integers includes the counting numbers, zero, and the negative integers. Because of the participants’ use of the novel model, they were able to establish a better definition for the set of integers. Also because of the novel model, participants were able to understand that zero is neither negative nor positive but instead serves to separate the negative integers and the positive integers.

The participants used the novel model to represent addition and subtraction with integers. Throughout chapter four excerpts depicted participants’ increased conceptual understanding of addition and subtraction of integers. Use of the novel model increased participants understanding of these operations.
Participants remarked in their comments after the final session that the novel model helped them make sense of the operations of addition and subtraction with integers. One participant, BH, wrote “It (the novel model) made the idea of why you actually add numbers when you are subtracting a negative number. The combination of the number line and the color of money helped with this idea. These both helped me understand the algorithm.” Another participant, VF, wrote “The number line was very helpful. I think that this whole study was very useful and will be useful in my math class. I understand better the actual process of adding and subtracting integers.” These comments, along with excerpts from the interview sessions that were provided in chapter four provide evidence that the novel model was effective in affording the participants a conceptual understanding of integers and the operations of addition and subtraction with integers.

In interview session four, NB and VF were given the problem “six subtract eight”. After they decided that eight white bills could not be taken away from six white bills, they added eight “zeroes” so that the eight white bills could be taken away. After taking away the eight white bills the number line portion of the novel model displayed six white bills and eight red bills. The participants then realized that this was the same as the problem “$6 + (-8)$”, which is the same as that yielded by the algorithm for subtraction of integers. One of the participants, LC, wanted to simply take away the six white bills from the eight white bills. This action would not lead to understanding the algorithm and explains why the researcher did not have this type of example as one of the first examples given. Instead, the first examples that were given to the participants required them to add
as many “zeroes” as the given addend. After doing several examples of this type, most of
the participants continued to add as many “zeroes” as the addend and this led to their
making sense of the algorithm for subtraction of integers.

In the past, two types of representation have been used to facilitate understanding
of integers. These are the number line and the neutralization model. In most studies the
neutralization method employed counters that were two colors, one color on one side and
another color on the other side. The number line emphasizes ordinality at the expense of
cardinality and two-color counters emphasize cardinality at the expense of ordinality. A
cardinal conception of number is a representation of the amount of objects in a collection.
An ordinal conception of number is one that represents a number as a position or order
relative to other numbers.

When students have been introduced to integers using the number line,
misconceptions can arise because numbers are used to represent positions as well as
transformations (Janvier, 1983; Hativa & Cohen, 1995). Operations with negative
numbers are done by first turning toward the negative direction and then moving forward
to add a negative amount or backward to subtract a negative amount (Bolyard, 2005). The
greatest difficulty for students is modeling subtraction of negative integers (Liebeck,
1990) because they use the idea of jumping forward and backward and spinning around
(Carson & Day, 1995). When using the number line model, there is also the possibility
that the student will count the number of points separating two integers rather than the
number of spaces between two integers (Bolyard, 2005). None of the participants in the
present study had difficulty modeling addition or subtraction on the novel model.
The first subtraction problems in this study required participants to take away an amount that could not initially be seen on the novel model such as “four dollars of debt subtract one dollar of money to spend”. Participants in the present study expressed confusion when subtraction examples were given in which they could not take away the required amount. For example BH and LB were asked to model two dollars of debt subtract four dollars of money to spend. Initially LB and BH knew that there had to be four white bills on the number line so that the subtraction could be done.

Both participants thought that the answer should be negative two, probably because they were thinking of addition rather than subtraction. They were confused about how to model it. The participants knew that subtraction could be modeled as “take away” but were unsure how this could be modeled using the novel model. The participants did not see a way to take away four dollars of money to spend since there were no white bills on the number line. LB then confused addition of a negative amount with subtraction of a positive amount so the researcher continued in her questioning. Through the researcher’s questioning, LB was able to make sense of what she was modeling through the use of the novel model.

The ability to model addition and subtraction using the novel model may have been due to the fact that the researcher first began by using just the two colors of money. Later, the participants related these two types of money to “debt” and “money to spend”. In the beginning problems that the researcher gave, she used the terms “debt” and “money to spend” to emphasize the two types of money. Later, the participants related “debt” to negative integers and “money to spend” to positive integers. In this way they
were able to change their vocabulary to be in line with standard terminology, and at the same time, their conceptual understanding was enhanced. This was true even in the last portion of interview four when the participants were given bills depicting greater amounts of debt and money to spend and they were not able to use the number line portion of the novel model since the number line only displayed integers from negative fifteen to positive fifteen. Because participants were able to model all previous problems using both the number line portion of the novel model and the two colors of money, they were confused when given the first example involving the big bills. After their initial confusion the researcher continued to prompt their exploration so that meaning could be made. Because of this interaction with the researcher, all participants were able to model these addition and subtraction examples using only the two-color portion of the novel model. For example, when given the example “twenty-four dollars of debt subtract eighty-six dollars of money to spend”, BH counted out a white twenty dollar bill and four white one dollar bills. She knew that she could not subtract eighty-six dollars of money to spend from this so she counted out eighty-six dollars of money to spend and eighty-six dollars of debt, thereby adding eighty-six “zeroes”. She could then take away eighty-six dollars of money to spend and she was left with the problem “twenty-six dollars of debt plus eighty-six dollars of debt”. Combining these amounts, BH achieved her final answer of one hundred ten dollars of debt. In this manner, BH was able to relate her previous understanding to the new context of bigger bills.

The ability to model addition and subtraction using the novel model may also have been because the participants in this study were older than the students in previous
research or it may have been due to questioning technique employed by the researcher. Also, using the novel model did not require the abstract manipulation of spinning around to indicate subtraction because the model allowed participants to model integers in many ways and the novel model allows participants to visualize subtraction as “take away”. It may also have been because the novel model combined the characteristics of the two-color counters and the number line model.

The researcher constantly thought of questions based on participants’ responses so they could establish a better conceptual understanding of integers. For example, in interview session three, when LC was asked to show two dollars of debt subtract four dollars of money to spend, she put two red bills on the number line portion of the novel model and then picked up some white bills. To try to correct LC’s line of reasoning, the researcher asked LC if picking up white bills was taking away money to spend. When LC’s confusion persisted, the researcher asked if two dollars of debt could be named another way. This served as a trigger for LC to add some number of white bills and the same number of red bills. Through the researcher’s questioning, in response to LC’s understanding, LC was able to change her initial thoughts about subtraction to incorporate what she had previously modeled using the novel model. This is consistent with teaching experiment methodology.

LB and BH have had experience with money and debt. They were trying to apply their practical experience to the novel model. LB and BH got the correct answer but the researcher was not satisfied that they had a conceptual understanding of what they were modeling so she asked them to start over again. After thinking about the problem, BH
realized that to model this she needed to have two more red bills than white bills showing on the number line.

Eventually, through the researcher’s questioning techniques, the participants were able to model the example correctly. Initially they wanted to put the amount that needed to be taken away on the novel model without thinking about naming the integer in another way. They knew what the answer should be and they tried to put bills on the number line so they could get the correct answer.

In the past, when students have been introduced to integers using two-color counters, there has been no distinction between the counters except for the two colors. Unless they understand that a counter of one color negates a counter of the other color, students may get confused and consider the total number of counters without regard to opposite pairs. Janvier (1983) claims that even though the two-color counter model is logical, students have difficulty using it to model addition and subtraction of integers.

Lytle (1992) noted that although students can perform addition and subtraction using the two-color counters, they were not able to transfer this manipulation to symbolic computations. In the present study, however, only one of the participants was unable to transfer what was done using the novel model to her use of the standard algorithm for subtraction of integers. All six of the participants were able to relate addition of integers to what they did on with the novel model. Participants connected the algorithms for integer addition and subtraction to their work with the novel model. This connection was evidenced when the participants put amounts in the “Number of zeroes” column on their
paper. Later, by looking at the columns on their paper, they were able to relate the number of zeroes in the problem to the algorithms for addition and subtraction.

Two studies where items were all symbolic have shown two-color counters to be more effective than the number line model (Birenbaum & Tatsuoka, 1981; Liebeck, 1990). Those students who have seen a number line before seem to have lower scores for subtraction problems than students who said they had not seen the number line before. Also, students with some prior experience with negative numbers and two-color counters seem to have slightly higher scores than students who have not seen them before (Wilkins, 1996).

Davidson (1987) suggests that a complete conceptualization of integers must have both ordinal and cardinal meanings. Based on Davidson’s findings, a new model is needed that combines the two types of models previously used so that more students can create better representations for integers and thus, conceptually understand what integers mean. Therefore, this researcher created a new model that combines the qualities of the number line and those of the two-color counters. With this new novel model, both cardinality and ordinality of integers are emphasized. This novel model helped the participants understand why the subtraction sign is changed to addition and the addend is changed to its opposite. In the third interview RV explained to LC how to model 

\[ -3 - (-2) \]

I: Let’s do three dollars of debt (LC put three red bills on the number line.) subtract two dollars of debt (LC put two more red bills on the number line and RV put two white bills on the number line.)

RV: You just take these off (referring to the two white bills).

I: Two dollars of debt? Is that what you just took off?
LC: No. Positive.
RV: You put on two.
I: You put on two zeroes so you had two dollars of money to spend and two
dollars of debt that you added on. Right?
RV: Yeah. So how many zeroes was that?
RV: Two. (to LC) This is what the problem would look like. (She showed her the
problem on RV’s paper.) You have negative three minus a negative two. So it’s
really adding two.
LC: I can do them when I look at them
I: I’m trying to get you to understand why you do what you do.

From this example, LC understood what happens and why the algorithm for
subtraction works as long as she is using the novel model. At this point, she was not able
to relate her work with the novel model to the algorithm for integer subtraction. After
working through two more problems she was able to relate her work on the novel model
to the algorithm.

In the fourth interview participants were asked to solve addition and subtraction
problems that involved amounts with greater absolute values such as $20 - $50. They
were given red bills in denominations of $1, $5, $10, $20, $50, and $100, as well as white
bills in the same denominations. They were not given a number line, so they were asked
to use only the two-color part of the novel model. Only one of the six participants, BH,
was able to relate what she had done before without the use of the number line to find the
solution. This approach is shown in the following excerpt.

I: How about fifteen dollars subtract sixty-seven dollars?
BH: Fifteen dollars (She counted out fifteen dollars of white bills.).
I: Is that fifteen? Over there. Oh, okay.
BH: Yeah. Fifteen dollars subtract sixty-seven, you said?
I: Yes.
BH: Sixty-five, sixty-six, sixty-seven. (She counted out amounts with the red bills.) Um. And I take away fifteen zeroes from both sides. (She removed fifteen dollars of red bills and fifteen dollars of white bills.) Negative fifty-two?

When working out this problem, BH knew to first count out fifteen white bills but then she counted out a total of sixty-seven dollars in red bills to represent the subtraction of $67. This action showed that she was confusing the operation of subtraction with the idea of adding a negative amount. Instead of subtracting sixty-seven, which would have been shown by taking away sixty-seven white bills, BH added negative sixty-seven. She should have added sixty-seven red bills and sixty-seven white bills to show that she was adding zero to the original amount. Then she could take away sixty-seven white bills.

This researcher concluded that BH had this difficulty because she did not have the number line portion of the model because when she used the number line portion of the novel model, she did not exhibit this confusion between the operation of subtraction and addition of a negative amount. However, it is possible that BH did understand the problem and used the model to show a different problem, which is one that she has changed to adding the opposite. She solved the problem $15 + (−67)$ instead of $15 − 67$.

The researcher searched the transcripts to make sure BH understood subtraction using the novel model. The following excerpt shows that BH had no difficulty modeling subtraction using the novel model.

I: Okay. What if you have positive four subtract negative two?
BH: Do you want me to (motioning toward the number line to remove the bills)?
I: Yeah. Take those off. (BH removed the bills from the number line.)
BH: So positive four (She put four white bills on the number line.) and you’re going to take away negative two so you need to add negative two on but you also need to add two positives. (She put two red bills and two more white bills on the
number line.) to make that zero and then take away the negative two. (She removed two red bills from the number line.) So I have six.

This evidence indicates that BH may have internalized the fact that subtraction of a positive is just like adding a negative. This demonstrates that BH made the connections between what was done on the novel model and her use of the subtraction algorithm for integers.

The other five participants had difficulty without both types of models present. One example of this difficulty is evidenced by the writing of one of the participants in which she said: “The number line helped a lot with the addition and subtraction problems. However, when we moved on to the larger bills, we didn’t use the number line. I became a bit confused. I started focusing on the color of the bills instead of imagining the number line. I was still a bit confused when focusing on the colors.”

This researcher wanted the participants in this study to develop a conceptual understanding of what integers are and where they are used in everyday situations. Using integers to add and subtract was the primary focus of the study. Many contexts could have been used to study integers but the one that is embedded in the novel manipulative that was created for this study was the idea of debits and credits in the form of red money and white money. College students are familiar with debt and money to spend, and this terminology was used to introduce the participants to positive and negative integers. The white money was called “money to spend” and was always located on the white side of the number line. The red money was called “debt” and was always located on the red side of the number line. The participants themselves related “money to spend” to positive
integers and “debt” to negative integers. The researcher did not initiate the use of the words “positive” and “negative” and at the beginning of the study always referred to the white money as “money to spend” and the red money as “debt”. The novel model allowed the participants to experience both the cardinal representation of integers, as depicted with two-color counters, and the ordinal representation of integers, as depicted with the number line model. With red money representing debt and white money representing money to spend, participants could count the number of bills as well as see their order on the number line portion of the novel model. They understood that the only additional amount that could be placed on the number line was zero, which is represented by a given amount of white bills and the same amount of red bills. By modeling problems involving integer addition and subtraction on the novel model, participants were able to conceptually understand integer addition and subtraction. However, when the number line portion of the novel model was taken away and participants only had red and white bills to indicate positive and negative amounts, they were not able to relate what they had done on the number line. The mental constructions that they had made when using the novel model did not exist when the number line portion of the novel model was not present. This researcher concludes that if more time had been spent using smaller dollar amounts without the number line portion of the novel model prior to using big bills, the mental constructions may have been more efficient. Spending more time without the number line portion of the novel model may help students create more efficient mental representations for subtraction of integers.
By relating integers to previous understanding, the researcher anticipated that students would make connections to previously learned algorithms for addition and subtraction of integers. When participants had difficulty with the initial subtraction example involving big bills, the researcher asked participants if they could rename the given amount in a different way. This process is shown in the following excerpt.

VF: You didn’t have, um, fifty to give so you only had thirty so you had to have twenty negative.
I: Okay, but if you put in twenty negative, you also have to put in . . .
VF: Twenty positive.
I: Twenty positive and that’s where you took out the twenty positive along with that thirty to get your fifty to take away. True?
VF: Yes.

After a few examples with this questioning technique, all except one of the participants were able to use the big bills without the use of the number line. These participants seem to have difficulty solving problems without the number line. After prompting them with questions about naming the integer another way, participants were then able to model the problems with only the big bills. In this manner the present study makes contributions to the literature about representation as well as the literature of integers. Future research could be conducted where at least one session could have students work with smaller amounts using the bills and not the number line portion of the novel model. The next step would involve withdrawing even this portion of the model so that students perform integer addition and subtraction using only conventional symbolic notation.
Initial Examples for Addition and Subtraction of Integers

When participants were asked to give an initial example of integer addition, participants in two of the three groups gave only examples containing positive integers. This was also the case when participants were asked for an initial example for subtraction. This finding could be due to the fact that participants thought that integers and counting numbers were synonymous. If this was the case, participants’ understanding of integers is flawed. It could also have indicated that the participants were more familiar with addition and subtraction of counting numbers. This type of definition for integers could lead to an answer of 3 for the problem $2 - 5$. Since 5 can’t be taken from 2, elementary students take 2 from 5 (Gallardo, 2002).

Participants in the present study may have been reticent to give an example involving negative integers for fear that the researcher would ask them to explain their solution. Later, after exploring the novel teaching model, when participants were asked to give an addition example and a subtraction example using negative integers, they were able to give appropriate examples. The following excerpt illustrates one of the participants giving an appropriate problem.

I: What if they weren’t both positive? Could you make up one where they weren’t both positive?
LC: Negative six plus two equals negative four.
I: How did you know that?
LC: Because . . . I don’t know. I took negative six and sub, well, added two. I don’t know how to explain it. Like if you have a negative number and add two positives, you get, you’re left with negative four.

In this example, LC did not use the novel model, nor did she have a mental representation for subtraction. She only knew the rule. There were no other cases in the
literature where students were asked to give an initial example of integer addition and subtraction. Because of this lack of prior data, this researcher did not know if the responses of the participants in this study were typical of responses that could have been elicited in other studies.

In this study, participants were exposed to new vocabulary for integers. Negative integers were represented by red bills and they were referred to as “debt”. Positive integers were represented by white bills and they were referred to as “dollars of money to spend”. Participants understood that one way to model dollars of debt was to place that many red bills to the left of zero. They also understood that one way to model a given amount of dollars of money to spend was to place that many white bills to the right of zero. Throughout the study participants used the words “negative” and “positive” even though the researcher consistently used “debt” when referring to negative amounts and “money to spend” when referring to positive amounts. Thus, in this study, participants demonstrated that negative amounts were located to the left of zero and positive amounts were located to the right of zero. Saying “negative four” is different from seeing $-4$. In the present study participants could see the $-4$ on the number line to confirm their correct placement of four red bills. This phenomenon may have been due to the structure of the novel teaching model designed for this research because research conducted by Davis et al. (1979) indicates that the signs of integers have little or no meaning for students when they are first encountered. It is important to remember that the participants in this study were college students and not elementary school students, with whom most of the previous research was concerned. The participants in this study had been introduced to
integers many times before and had performed integer addition and subtraction many times, usually only in a procedural manner.

**Student Difficulty with Addition and Subtraction of Integers**

On the initial survey, this researcher found that students had more incorrect answers for subtraction than for addition. This finding adds to the literature that students have more difficulty with the operation of subtraction than with the operation of addition (Ferguson, 1993; Kuchemann, 1981; Nunes, 1993). This is true for all sets of numbers, not just integers.

Question four of the initial survey asked the larger group of students who were surveyed to symbolically add and subtract with integers. Overall results from the initial survey administered to seventy-nine students in the fall semester of 2007 indicated that more than ninety-three percent of the students (or seventy-four students), were able to correctly compute an answer to a problem involving addition of integers but only seventy-six percent, or sixty students, were able to correctly compute the answer to a problem that involved subtraction of integers. The results of this study continue to reflect results similar to past research on addition and subtraction of integers with elementary students (Bolyard, 2005; Hackbarth, 2000; Hativa & Cohen, 1995; Liebeck, 1990; Wilkins, 1996). Even though the present study did not involve elementary school students, the participants seem to have the same difficulties as found in prior research.

Research suggests that subtraction is more difficult than addition because of the multiple uses of the “–” sign (Hall, 1974; Nunes, 1993; Shawyer, 1985). This “–” sign can indicate the operation of subtraction or it can be used to designate a negative integer.
This confusion is further complicated by teachers’ carelessness in naming integers. For example, some teachers use “minus three” as a name for negative three (Hall, 1974). This researcher attempted to alleviate this problem by stating a subtraction problem using the word “subtract”. She did not use the term “minus” to refer to a negative integer or the operation of subtraction. As reported consistently in chapter four, the researcher in this study initially asked questions such as: “What is four dollars of money to spend subtract three dollars of debt?” or “What is two dollars of debt subtract four dollars of money to spend?”

Because the participants in the present study had previous experience with money, they found it natural to relate the concept of taking away money as the same thing as increasing debt. This idea is shown in the following dialogue.

I: I want you to take away four dollars of money to spend.
BH: But if you take away even more money to spend you’re still making yourself more in debt.

This idea helped them understand that subtracting a positive amount would be the same as adding a negative amount. When participants used the novel model they were able to take a symbolic problem and relate it back to the operations of addition and subtraction.

*Student Misconceptions about Addition and Subtraction of Integers*

Hativa and Cohen (1995) found that students have five types of misconceptions when adding and subtracting integers. Of these five types, this researcher determined, based on her twenty years of teaching mathematics to pre-service elementary teachers, that the students in the present study would not have the first or fourth types of
misconception which include not having trouble with problems that involve subtracting a positive amount from zero or problems where a number is added to its opposite. Therefore, for the purposes of this study, the assumption was made that students would not have these sorts of misconceptions and there were no problems of these types on the initial survey. The researcher will discuss those misconceptions that were presented on the initial survey and she will briefly discuss the first and fourth types of misconceptions.

The first type of misconception portrayed in Hativa and Cohen’s study (1995) involved subtraction of a positive number from zero. For example, when given the problem 0 – 6, students might get 6 (the number that was subtracted), or 0 (since they have rotely learned that you can’t take something from nothing), or 4 (because they were thinking 10 – 6 because they learned that with whole numbers you need to regroup if you don’t have enough to subtract). In a study involving fourth graders, during pre-treatment interviews, it was found that more than one-half of no-treatment students and more than three-fourths of experimental students correctly answered symbolic integer problems where a positive integer was subtracted from zero. Problems were presented to the fourth grade students using a computer (Hativa & Cohen, 1995). As mentioned before, students in the present study were not given any examples on the initial survey where a positive integer was subtracted from zero. This researcher reflected on past experiences teaching pre-service elementary teachers and recalled that whenever students were asked to subtract a positive integer from zero they knew the result had to be negative. They probably knew this because of their previous experience with playing games where they could “go in the hole” or other practical situations. During the interviews, only one of the
participants had difficulty modeling this type of problem using the novel model. However, after this participant was asked if zero could be renamed in another way, she was able to add zeroes and model the correct solution.

The second type of mistake delineated by Hativa and Cohen (1995) involved subtraction of a positive integer from a lesser positive integer. For example, when given the problem $9 - 16$, students might get 7 (because they just subtracted the lesser amount from the greater amount), or 25 (because they added the numbers together), or $-25$ (because they added the numbers together and appended the negative sign), or 9 (because they couldn’t take a greater amount from a lesser amount), or 16 (because that was one of the numbers given). Problem 4f of the initial survey asked students to write the answer for $9 - 16$. In the present study, none of the students had any of the incorrect answers resulting from this type of misconception for problem 4f. However, two of the seventy-nine students did have this misconception for problem 4h. Problem 4h asked students to find the solution to $9 - 82$. On problem 4h, these two students recorded 73 as the answer for $9 - 82$.

By the end of this study, all of the participants were able to correctly model and conceptually understand problems of this type using the novel model. The researcher did not give participants problems in symbolic form without the novel model present because they had demonstrated in the original survey that they could do problems in symbolic form by applying rule-based procedures. The intent of the present study was to get participants to demonstrate their conceptual understanding of integer addition and subtraction by connecting rule-based procedures to their use of the novel model. In
Hativa and Cohen’s 1995 study involving fourth grade students, it was found that one-half of the no-treatment group and more than three-fourths of students in the experimental group were able to find a solution to this type of problem during pre-treatment interviews (Hativa & Cohen, 1995). This finding indicates that most fourth graders have pre-instructional knowledge for doing this type of subtraction.

The third type of error that Hativa and Cohen (1995) list involves adding two negative numbers. For example, when given $-4 + (-5)$, students might get $-1$ (because they subtracted 4 from 5 and since both were negative, the answer must be negative), or 1 (because they subtracted 4 from 5), or 9 (because they added the number parts together), or 4 (because that was one of the number parts). On the initial survey, two of the seventy-nine students recorded 9 as the answer for $-4 + (-5)$. However, all six participants were able to correctly solve this type of addition problem using the novel model.

Hativa and Cohen’s (1995) fourth type of error resulted when a number was added to its additive inverse. An example of this type would be $4 + (-4)$. As mentioned before, there were no problems of this type of misconception on the initial survey in the present study. Pre-service teachers have had much experience with money and know that two dollars of money to spend is needed to balance a two dollar debt. For this reason, this researcher chose to not include this type of problem on the initial survey. However, examples of this type were given to the six participants in the study. When participants in the present study were given problems containing this type of problem, all of them correctly solved the problems using the novel model. The following excerpts show that
participants understand that zero means some amount of positive bills is balanced by the same number of negative bills.

Group 1:

LC: I like canceled out (pointing with her pencil) four dollars and then added four but it looked like I added eight. But I didn’t. The negative ones cancel out the positive ones.

Group 2:

I: So, in effect, what are you putting on there to keep the same value? BH: The same amount on both sides of zero. I: Which is really pairs of positive and negative which is really... What is a positive and a negative? LB: Zero. I just remembered that.

Group 3:

VF: Because you need equal amounts of negative and positive numbers. Those five positive numbers (referring to the white bills) so you need equal amounts so you have to have five negative numbers. Participants knew that to rename an integer they needed to add an equal amount of red bills and white bills. Thus, participants used the idea of additive inverses when removing zeroes so that just one color of bills remained on the number line. Participants also used this information when they renamed integers in order to subtract. Thus, the novel model helped participants to see addition and subtraction of integers in a different way from what they had in the past. This ability to see meaning for addition and subtraction of integers was noted in all of the participants’ writings at the end of the study. For example, LC wrote: “The color helped and so did the number line. Both of those things helped teach integers in a new way. I understand why you add and subtract integers now. Before, I just did it.”
LC and RV showed in the following excerpt how the algorithm for addition of integers is related to their work with the novel model.

I: So how is that number of zeroes related to something in the problem?
RV: It’s the absolute value of the second addend. (She meant the addend that was given.)
I: Is it always the second addend?
LC: It’s the smallest. Well, . . .
RV: It would be the smallest, um, the smallest absolute value of the problem.
I: Okay, the smaller absolute value. The one that has the smaller absolute value. And then, if those signs are different, how do you determine your answer?
LC: Whichever number has the greatest absolute value, that’s the sign you use.
I: Okay, and how do you get the number part?
LC: Um. You take, you subtract them.
I: Okay, you subtract their . . .
LC: Absolute values.

Prior to this study, the participants had relied on rule-based procedures that gave them answers without conceptual understanding. The researcher gave the participants problems to model using the novel model and she asked them questions about what they were doing with the novel model.

The fifth type of misconception cited by Hativa and Cohen (1995) involved adding a positive integer to a negative integer when either has a greater absolute value than the other. Applying this misconception for \(-8 + 6\) students might think the answer is 2 (because they subtracted 6 from 8), or 14 (because they added 8 and 6), or \(-14\) (because they added 8 and 6 and appended a negative sign) or \(-8\) (because that’s the value of one of the numbers given), or 6 (because that’s one of the numbers given). Problem 4c of the initial survey asked students to find the solution for \(-8 + 6\). Four of the seventy-nine students thought the answer was 2 and one student thought the answer was \(-14\),
demonstrating that five of the seventy-nine students had this type of misconception mentioned by Hativa and Cohen (1995).

Writing Word Problems Involving Integers

Liping Ma (1999) did research on elementary teachers in the United States and China and found that most of the teachers in the United States could not give a real life scenario for problems involving division with fractions. She claimed that this inability demonstrates a lack of deep understanding of these concepts. This researcher found that students in the present study found it more difficult to correctly write word problems for integer addition and subtraction than to perform the operations symbolically. Less than twenty-seven percent of the seventy-nine students were able to create a real life scenario involving an addition problem that contained integers. Subtraction of integers was even more problematic. While close to seventy-six percent of the students were able to compute an answer to a problem involving symbolic subtraction of integers, only ten percent could make up a word problem that would be correctly solved using the same integers. Students especially found it difficult to make up a word problem that involved subtracting a negative integer. It didn’t matter whether the negative integer was subtracted from a positive or a negative integer. None of the seventy-nine students could make up an appropriate word problem for this type of problem on the initial survey.

Although the present study involved integer addition and subtraction and Ma’s research concerned subtraction with regrouping, multi-digit multiplication, division by fractions, and relating perimeter and area, both studies were done with pre-service elementary teachers and both show a reliance on rule-based procedures without
understanding. Just as the participants in Ma’s study knew to invert and multiply when dividing fractions but could not write appropriate word problems that required division of fractions, the participants in the present study knew the rules for finding solutions to addition and subtraction problems involving integers but they could not provide meaningful word problems that required addition and subtraction of integers. Previous research did not ask students to give meaningful examples of addition and subtraction involving integers but this researcher was concerned about whether students could relate their past experiences to integers and the operations of addition and subtraction.

Participants in the present study knew that to add integers of different signs, they simply subtracted the magnitudes of the integers and appended the sign of the one with the greater absolute value. They also knew that to subtract integers, they only need to change the subtraction to addition and the second number to its opposite. They then followed the familiar rules for addition of integers. In this study, because participants had already been introduced to integers many times using rule-driven procedures, their ability to understand the model in its entirety many have been jeopardized. Although five of the six participants were able to model problems using the novel model, they also knew how to find the correct solution using the rules. They sometimes thought about what the solution would be and then proceeded to model it on the novel model so they would get the correct solution. For example, in session three, BH and LB were asked to model three dollars of debt subtract two dollars of money to spend. The following excerpt shows how she modeled the solution using the novel model.

BH: It’s positive two so just negative three minus two which is . . . Sorry. We have to add two zeroes. (She removed a white bill as LB removed a red bill from
the number line.) Wait a minute. (LB put the red bill back on the number line.)
Negative three subtract two dollars of money to spend . . .
LB: I have five (referring to the number of red bills on the number line).
BH: I have negative five.

Previous research confirms the fact that students memorize rules for addition and
subtraction of integers rather than conceptually understanding the operations (Hackbarth,
2000; Lytle, 1992; Smith, 2002; Bolyard, 2005). Memorizing rules for integer addition
and subtraction without conceptually understanding the process damages one’s ability to
learn mathematics with meaning (Battista, 1999a; Kamii, 1994; Kamii & Dominick,
1998; O’Brien, 1999; Sfard, 2003). It is difficult for students to overcome their previous
behaviorist training. Even though some of the participants were reluctant to use the novel
model because they already knew how to add and subtract integers, the present study
shows that students were more able to conceptually understand why the procedures for
integer addition and subtraction make sense.

In order to get elementary students to conceptually understand integers, it is
important that pre-service teachers be given opportunities to develop logical
understandings of mathematics so they can teach their students in this manner. Teachers
need to have a deep conceptual understanding of how and why the procedures for integer
addition and subtraction work. Using a model like the one used in the present study
allows students to scaffold their learning while they make sense of the operations of
addition and subtraction. Teaching only procedures for the operations, on the other hand,
will result in students’ knowledge being a mile wide and an inch deep (Ma, 1999).
Subtraction Instead of Addition

When students were asked to give real-life scenarios for problems requiring addition of a negative integer, many of the students in this study wrote problems that instead required the operation of subtraction. For example, students were asked to make up a word problem for $8 + (-4)$. Only one student was able to write an appropriate word problem and thirty-seven of the seventy-nine students wrote word problems similar to the following: “Zach has eight puppies. He sells four of them. How many does he have left?” This further illustrates that students confuse negative integers with the operation of subtraction. The fact that students confuse negative integers with the operation of subtraction was also noted in the research (Gallardo, 1995; Lytle, 1992; Wilkins, 1996).

Difficulty with Problems Containing More Places

This researcher also found that participants tended to give incorrect solutions more frequently when the numbers in the problems had two and three digit numbers. For example, adding $-4 + (-5)$ was answered correctly by seventy-seven out of seventy-nine of the students but the problem $-263 + (-79)$ was answered correctly by seventy-four of the students in the study. Four students left this question blank and one subtracted $-79$ from $-263$. Five of the students who missed this problem had 342 for their answer, instead of $-342$. This finding supports the research of Davis, et al. (1979) who found that students do not attach significance to the sign of integers.

For the problem $9 - (-4)$, seventy of the seventy-nine students correctly solved it, but only sixty of the students were able to correctly solve $209 - (-75)$. Four of the students who missed this problem had an answer of 286, rather than 284. This error may
have been due to the fact that they subtracted 9 from 15, instead of adding the 9 and 5. These students seemed to combine rules for addition and subtraction in their quest to find the solution. Four other students had 134 as the solution for $209 - (-75)$. This second group of students did not understand how to subtract negative integers. Instead of changing the second amount to its opposite and following rules for addition, these students subtracted 75 from 209. This finding concurs with the research of Gallardo (1995), who found that students have difficulty with subtraction when a negative integer is subtracted from a positive integer.

_Difficulty with Addition of Integers_

In the present study, problems involving addition of a positive and a negative integer that yielded a negative result were missed by more students than examples where the answer was positive. For example, on the initial survey students were asked to find $6 + (-3)$ in question 4(a). Only one of the seventy-nine students had an incorrect answer for this problem. That student reported that the answer is $-3$. However, in question 4(c) where students were asked to find $-8 + 6$, four of the seventy-nine students had an incorrect answer. This finding could be due to the fact that students think of $6 + (-3)$ as $6 - 3$. They were then confused by $-8 + 6$ because the negative sign was assigned to the first addend instead of the second addend. Three of the four students who had an incorrect solution for this problem subtracted 6 from 8 to get an answer of 2. One other student had an incorrect answer of $-14$, presumably by subtracting 6 from $-8$. The “−” triggered each of these students to think of subtraction rather than allowing the sign to
designate a negative integer. This last finding was consistent with previous research (Davis et al., 1979; Gallardo, 1995; Lytle, 1992; Wilkins, 1996).

**Participants Use the Novel Model**

The researcher used money that was red and white along with a number line that was also red and white. The number line to the left of zero was red to indicate negative integers. Red bills were used to denote debt or negative integers. To the right of zero the number line was white and white bills denoted positive integers. The researcher did not expect the participants to immediately attach significance to the color of the bills. She did, however, expect the participants to relate the two colors with positive and negative when the number line was revealed to them.

When participants were first shown the red money and the white money on the novel model, they could see the resemblance to real money and they indicated that there were two different colors of money. However, not surprisingly, without seeing the color-coded number line, none of the participants could initially indicate the reason for two colors of money. When they were shown the number line that was also red on one side and white on the other, all participants immediately associated the red money with debt and the white money with money that could be spent.

All participants were able to represent a given amount of “dollars to spend” or debt on the novel model with white money and red money, respectively. The participants were able to represent their ideas about debt and money to spend based on their past experience with money. The researcher anticipated that the participants’ past experience with money would allow them to build a conceptual foundation for integers, and later for
integer addition and subtraction. In this manner, the participants were able to conceptually understand positive and negative integers so that the groundwork for integer addition and subtraction could be laid.

The model helped participants relate past experiences of debt and having money to spend with positive and negative integers so that a more concrete way of thinking about integers was expended. “Working in real-world contexts may help students make sense of underlying mathematical concepts and may foster an appreciation of those concepts” (NCTM, 2000, p. 297).

Understanding integers is important for laying the groundwork of more advanced mathematical topics such as algebra (Bolyard, 2005; Ponce, 2007). For example, when solving an equation for a variable, one must know how to add and subtract with integers. It is important for all students to have access to appropriate representations so that a solid foundation is built and future learning is enhanced (NCTM, 2000).

Participants’ responses across this study indicated that they initially had a narrow understanding about integers and how they are used. This researcher determined that it is important for students to rename integers in many ways in order to conceptually understand addition and subtraction of integers. Naming integers in many ways is important. When adding integers of different signs, it is important to recognize that a given number of positives will “cancel” the same number of negatives. Thus, it is helpful to know that $4 + (−7)$ can be thought of as $4 + (−4 + −3)$ and then, using the associative property for addition, one can add $4 + (−4)$ to get 0, giving a final answer of $−3$ for the problem. In this example it is important to understand that another name for 0 is $4 +$
In fact, by modeling integers in many ways, students realized that the representation used for various integers depends on the given problem. Likewise, in subtraction, sometimes zeroes need to be added so that the required amount can be taken away. For example, given the problem $2 - (-5)$, one could rename $2$ as $2 + (-5) + 5$ so that $-5$ could be taken away. The solution is the same as $2 + 5$, or $7$. The participant could see the seven white bills and also that these bills took up all the places from $0$ to $7$ on the number line. In this manner, both the cardinality and ordinality could be seen using the novel model. Both of these qualities were used by the participants to help them make sense of the problems given. When the participants were introduced to the big bills, the number line was not helpful because it only included integers from negative fifteen to positive fifteen. In effect, participants could only use the attribute of cardinality when using the big bills. All of the participants had some difficulty adjusting to using only the part of the model that dealt with cardinality. At this point, the questioning technique of the researcher became very important in guiding the participants to an increased conceptual understanding of integers.

When first introduced to the novel model, all participants displayed the required amount of money correctly on the model. For example, when participants were asked to model two dollars of money to spend on the number line, all participants were successful. Participants were then asked if the amount could be shown another way. Two of the three groups initially thought that in order to name the same integer, the amount of red bills and white bills that needed to be added had to be the same as the absolute value of the integer to be named. In other words, they would rename two dollars of money to spend
by placing two more white bills and two red bills on the number line. However, when asked for another way of naming the amount, all participants were able to add just one more of each color of bill to the amount already displayed on the novel model. When asked about this, one participant said that after the first “other way”, you could just add one more of each color of bill but for the first time you had to add on equal amounts of the same amount that was displayed on the number line. To get this participant to understand that this was not true, the researcher asked if adding one more of each type of bill changed the value of the amount on the number line. For example, the researcher asked this participant what amount is displayed on the number line if are two red bills and then the researcher placed one more red bill and a white bill on the number line. In this way the participant determined that just one more of each color bill could be placed on the novel model to have an equivalent representation for the integer. By the researcher’s question, the participant was able to reflect upon previous misconceptions so that conceptual understanding was further developed. When representing integers on the number line, all participants realized that even the first time, any number of additional red bills and an equivalent number of white bills could be on the novel model to represent any given integer. This concept development was a very important step needed to understand integer addition and subtraction.

**Student Disinterest in the Novel Model**

Two of the participants, NB and LC, indicated disinterest in the novel model at the beginning of the study by saying things such as “I just know the rule” and “I was seeing it in my head” rather than modeling problems on the novel model. They did not
see the need for the novel model because they had already learned the rules and were able to get correct solutions to the problems without using a model. They felt that using a model instead of using a standard algorithm was a waste of time. These two participants did not realize that elementary school teachers must have a conceptual understanding of the mathematics that they teach and they must understand how to use manipulatives appropriately to support student conceptual understanding in the classroom. This belief is not unusual among college students when working with manipulatives (Smith, 2002). Wearne and Hiebert (1988) found that students who have routinized syntactic rules without making the connections between the symbols and referents are less likely to engage in the semantic processes of connecting and developing than those who have not been exposed to the mathematics topic before.

After NB and LC realized that there could be understanding attached to previously memorized algorithms, they were more interested in how the novel model helped them to conceptualize the meaning behind the algorithms. Their attitude toward the novel model and learning about the operations of addition and subtraction improved as the study progressed. This understanding was evidenced in their writing at the end of sessions two and four, as reported in chapter four. It is also evidenced in the transcripts as the study progressed.

Using models to get students to understand mathematics can sometimes be discouraging when students feel that they already know how to do the mathematics. They think that the answer is the most important thing. This happens at all levels of learning from middle school through the college level. Although students may be able to get a
correct answer for the problem, they are left without a check or sense of whether the problem is right or wrong. They only know the procedure-based rule. Later, when the students are required to use previous understandings to build a mathematical foundation, they find that the foundation is weak. Because of this lack of conceptual understanding, we find that many college students are not ready to take college-level mathematics courses.

Also important to this study was the questioning technique employed by the researcher. Teachers play a key role in guiding students’ learning. “This guidance is often in the form of carefully posed questions by the teacher accompanied by the selection of responses that further the movement along the path that the teacher has chosen towards a convergence of mathematical meaning” (Lesh & Doerr, 2003). The researcher posed questions throughout the interviews that required participants to think about how a solution to a problem was ascertained. A solution, in and of itself, even if it was correct, was not good enough for this study. The researcher wanted students to demonstrate their conceptual understanding of integer addition and subtraction so that a solid foundation could be built. She wanted to make sure that these future teachers would not just teach rule-based procedures to their students. Therefore, thought processes were deemed at least as important as answers in this study.

The researcher asked the participants questions that allowed them to think about the novel model as it related to integer addition and subtraction. For example, if a participant wanted to find the solution for $4 - (-2)$, they might put four white bills on the novel model. If they appeared to be puzzled because they couldn’t take away two red
bills, the researcher would ask if four could be represented in another way. This questioning usually triggered the participant’s previous experience with renaming integers in many ways and they would add some zeroes to the novel model.

This process forced all participants to reevaluate their thinking about the problems and LC and NB decided to use the manipulative to show their thought processes. Through this action, they evidenced the importance of their work with the novel model and its relationship to the operations of addition and subtraction of integers.

**Participants Scaffold their Thinking**

The novel model allowed the participants in this study to make more sense of integers and how integers are added and subtracted. Participants were able to scaffold their own thinking by naming integers in several ways. Also, because integers were represented first as two colors of money, later as debt and money to spend, and finally as negative and positive integers, participants were able to continue to develop their conceptual understanding of integers in gradual steps. Because of their past experience with money, participants realized that each dollar of money to spend canceled a dollar of debt. The novel model allowed them to clearly visualize this concept.

In the beginning of the study participants were surprised that integers could be named in more than one way. This may have been due to their shallow understanding of integers. As participants modeled integers on the novel model, they quickly realized that adding an equal amount of red bills and white bills to a given amount on the model did not change its value. This evidenced sense-making on the part of the participants and was visually portrayed using the novel model. Participants used this basic conceptual
information to help them understand the algorithms for addition and subtraction of integers. For example, when participants needed to subtract an amount that required more bills than what they had on the number line, they needed to rename the integer in another way by adding “zeroes”.

The researcher’s line of questioning also aided participants’ conceptual understanding of integers. When participants experienced difficulty or frustration with their ability to model a given problem, the researcher asked questions that allowed them to see how a broad conceptual understanding of integers helped them to make sense of the processes and procedures with integers.

In order to help participants scaffold their thinking about addition of integers using the novel model, the researcher asked the participants to make up a chart with three columns. These three columns were labeled “Problem”, “Number of Zeroes”, and “Solution”. Participants modeled each of the addends on the novel model and, when possible, they removed equivalent amounts of positive and negative. By completing the chart as they worked the problems on the novel model, participants were able to connect what they were doing on the model with what they do when adding integers using the algorithm. In other words, if both addends are the same sign, one only needs to add the magnitudes and append the common sign. When one addend is positive and the other is negative, one needs to subtract the magnitudes of the addends and append the sign of the addend that has the greater absolute value. This generalization is essential when working with integers.
The participants also transitioned from one representation to another to achieve a better conceptual understanding of integers. They were initially shown the novel model and by using it to model integer addition and subtraction, the participants were able to develop an internal representation for integers that was consistent with symbolic notation expressed in their external representations of integers. They were also able to translate internal representations of integers into commonly applied procedures. For example, the participants were able to relate their actions on the novel model to previously learned procedures for integer addition and subtraction.

In the same way, when participants worked subtraction problems, they created a chart that had four columns. These columns were labeled “Problem”, “Number of Zeroes that Need to be Added”, “What is on the Number Line?”, and “Solution”. The researcher chose initial problems that required some number of zeroes to be added. An example would be “seven dollars of debt subtract two dollars of money to spend”. Participants first put seven red bills on the number line to show the seven dollars of debt. They then added as many zeroes as was indicated by the addend. Since there were no white bills to take away, they put two white bills and two more red bills on the number line. When participants took away the two white bills that indicated money to spend, they were left with nine red bills on the number line. When participants completed the chart for the problems, they could see that they always added the number of zeroes that was the same as the magnitude of the amount to be taken away. When they took away the amount indicated in the problem, the remaining amount on the number line was the same as adding the opposite of the amount that was to be taken away. This is exactly what the
algorithm for subtraction indicates but by using the novel model, the participants could see why the sign is changed and addition is used. For example, in session three the following dialog took place between NB and the researcher.

I: And let’s do two dollars of debt subtract four dollars of debt. (NB put two red bills on the number line.)
NB: Two dollars of debt.
I: Okay. And subtract four dollars of debt. (NB put four more red bills on the number line and four white bills on the number line.) And now you want to subtract your four dollars of debt. (NB removed four red bills from the number line.) See, in order to have that four dollars of debt to take off, you added in four dollars of money to spend along with it, didn’t you? And that’s why that problem’s going to become negative two plus four. Does that make sense?
NB: Yeah.
I: So it’s not just slopping together a rule, an algorithm, to figure out what’s going on. (in a subtraction problem) What is that number of zeroes?
NB: In this one?
I: In any of them. (VF and NB looked intently at their papers, but seemed confused by the question.) Where do you see a two in your original problem? (Pointing to the problem on NB’s paper) If this was your original problem, do you see a two?
NB: Yeah.
I: (pointing to another problem) Do you see a three for this one?
NB: Yeah.
I: (pointing to a previous problem) Do you see a five for this one?
NB: Yeah.
I: (pointing to a previous problem) Do you see a three?
NB: Yeah.
I: (pointing to a previous problem) A four?
NB: Yeah.
I: What is that number every single time?
NB: The right hand number.

With this model, participants were able to conceptually understand the algorithms for integer addition and subtraction. This understanding is essential when developing
ideas in mathematics so that students have a strong foundation on which to build. This is one of the goals of the NCTM (NCTM, 2000).

Participants’ language about integers allowed them to scaffold their learning. Initially the researcher used everyday language such as “money to spend” and “debt” to represent positive and negative integers, respectively. From the beginning of the study to the end, participants instead referred to “money to spend” as positive integers and “debt” as negative integers. In this manner they took charge of their own learning using the novel model. They made the transition from the language used with the novel model to a more mathematical language. In doing so, the participants made progress in cognitive thinking about integers.

Participants in the study reported that both parts of the novel model, the color-coded number line and the color-coded dollar bills, helped them to understand integers and the operations of addition and subtraction. After session two, one of the participants claimed that she could use just one color of money to understand integers. However, at the end of the study she admitted that having the two-colors of money along with the two colors on the number line aided her understanding of integer addition and subtraction.

In the beginning of the study, one of the participants, RV, preferred to use the rules that she had learned to add and subtract integers, rather than trying to conceptually understand these operations. At the end of interview two, RV wrote the following: “The number line and colored money don’t help as much. It is more of a visual aid but can help to describe what we’re trying to say.” By the end of the study her attitude about the manipulative changed and she understood why the algorithms for integer addition and
subtraction worked. At the end of interview four, she wrote: “The number line helped focus. Also the colored money was very helpful in distinguishing debt from money to spend. I enjoyed getting a better understanding of the integer system.”

If these participants had been introduced to the novel model when they first encountered integers, this study suggests that it might have been easier for them to model addition and subtraction of integers without thinking about the procedures that they had memorized for these operations. This researcher asked questions throughout the study to get the participants to relate the model to their previous understanding. By the end of the study all participants could relate the novel model to addition and subtraction of integers and they expressed a feeling of greater confidence in their conceptual understanding of integers.

**Limitations of this Study**

This study involved seventy-nine students from a mathematics content course for pre-service teachers from a four-year university in the Midwest. The six participants for the interview phase of the study were volunteers from the larger group of students. The researcher used a qualitative study so that she could learn more about how students think about integers and the operations of addition and subtraction. Researchers who implement qualitative studies commonly explore some phenomenon, rather than showing how variables are related (Creswell, 2002). “Human behavior is too complex to gather all the facts about it so the qualitative researcher tries to grasp the processes by which people construct meaning and describe what those meanings are” (Bogdan & Biklin, 1998, p. 38).
By conducting a qualitative study, this researcher was able to ask questions to delve deeper into how participants made sense of integers. When participants gave responses that did not make sense, the researcher asked more questions to more accurately assess participants’ inaccurate thought processes. Then she was able to ask more questions to get the participants to expose a perturbation and build a better mathematical foundation. The researcher was not concerned about generalizing results of the study to larger populations and she acknowledges that these results will not generalize to other populations.

Another limitation is the limited amount of time involved in the study. There were only four interview sessions with each group and each session lasted approximately thirty minutes. More revealing data about individual participants may have surfaced if more time could have been spent with the participants.

Implications of this Study

This study clearly showed that the novel model helped pre-service teachers build conceptual understanding make sense of integers and the operations of addition and subtraction. Prior to using the novel model, pre-service elementary teachers in this study revealed that they did not conceptually understand what integers are and where they are used. The majority of the students in this study reported that integers were only used in school and school-related activities such as computing grade point average. The novel model helped participants relate what they were doing when using the model to the algorithms for addition and subtraction. Participants in this study were able to see the logic involved in the algorithms for addition and subtraction of integers. They were able
to make sense of what they modeled using the novel model and then relate this to the algorithms.

For example, in interview session three (in lines 2690 through 2697), BH was given the problem “negative two subtract negative three”.

BH: Negative two (She put two red bills on the number line.) and you want to take away negative three so (She put three more red bills on the number line.) And add three there too. (She put three white bills on the number line.) So the amount of zeroes would be three. Take away negative three. (She removed three red bills from the number line.) And I can still take away two more zeroes. (She removed two white bills and then two red bills from the number line.)
I: Okay, because it’s an addition problem now.
BH: Yeah. And I have positive one.

In this way, conceptual understanding guides the use of algorithms. This conceptual understanding will be helpful when these pre-service teachers teach their future students about integers.

When integers are introduced in the curriculum, the results of this study reveal that emphasis should be placed on their practical applications. Common activities should be used as stepping stones for advanced topics. Since money is a topic that is understood by many students, it can be used as a starting point when discussing integers. Students should be expected to create and correctly solve problems using integers so they can see the practicality of integers. The novel model that was used in this research has proven to be an ideal tool for relating integers to the real world situations of money to spend and debt. It also allowed participants to visualize the ordinal and cardinal representations of integers at the same time.
Students in this study indicated that they had learned algorithms for addition and subtraction of integers without conceptually understanding them. They could get a solution to the problem but they had no sense about the correctness of their solution. Modifications must be made to the curriculum of the college-level mathematics content courses that pre-service teachers take as part of their preparation for licensure to teach. For pre-service teachers to best help children understand the algorithms, they need to understand the workings of the algorithms rather than just teaching the procedures. It is much easier to remember how to do something when it makes sense. A model which incorporates both the ordinal and cardinal representations of integers can help students to conceptualize integers. This conceptualization would allow students to make their own meaning for integers and the operations of addition and subtraction. To this end, the novel model could be introduced to the students to help develop the algorithms for addition and subtraction with integers.

Textbooks and other curriculum materials could include the novel model as another way to introduce integers to elementary education majors. By combining both the cardinal and ordinal representation for integers in one model, students could develop a better conceptual understanding of integers using the novel model. It appears clear from this research that combining the two elements of previous models has a greater impact on student conceptual understanding of integers. The novel model maximizes opportunities for more students to learn integers in a meaningful way.

This study also has ramifications for elementary school mathematics programs. Use of the novel model allowed the researcher to get at students’ thinking and how they
related to operations involving integers. She continually tried to place herself in the place of the participants to see how they were making sense of the material. Her questions were designed to be responsive to the participants’ levels of understanding and to elicit their level of understanding and move them forward in their conceptual development of integers. If teachers would take the time and effort to be responsive to students, it would help them to teach students in a manner more consistent with the way that students think. Questioning students in a way that leads to perturbations allows students to discover their misconceptions without their feeling inadequate in mathematics.

More emphasis should be placed on what integers are and where they are used in practical situations. Also, more time needs to be spent with modeling integers and the operations of addition and subtraction using models that incorporate cardinality as well as ordinality. Time should be spent in the upper elementary years developing conceptual understanding of the algorithms rather than memorizing rules (NCTM, 2000). To this end, the use of the novel model illustrated in this study would be a good starting point.

It is important for elementary teachers to think about the mathematics that they teach. If teachers reflect on how they understand integers when using a model that incorporates both cardinality and ordinality, this may help them to think through the types of problems that would help their students make connections between the procedures for operations and why they work. For example, initial problems for subtraction should be of the type where the required amount cannot be taken away such as negative three subtract four. More emphasis should be placed on understanding rather than on procedures to find answers.
Teachers should understand that a meaningful knowledge of integers involves more than just the use of procedures for addition and subtraction. With the right problems and the right model, teachers will be able to ask the right questions to help students conceptually understand the operations of addition and subtraction of integers. After students acquire a deep conceptual understanding of integers, they will be able to abstract these ideas to more advanced mathematics such as solving algebraic equations and inequalities.

Recommendations for Further Research

This researcher would recommend more widespread use of the novel model to see if other students react in similar ways. Studies could involve middle school students who have not been introduced to integers and could involve the entire class rather than just a few participants.

This researcher would recommend that more studies be conducted with pre-service elementary teachers using the novel model for integers. These pre-service teachers could be asked to model some problems with the novel model and explain how the model helped or hindered their conceptual understanding of integers. If the novel model helps these students to better understand integers conceptually, they will be better prepared to teach integers to children in their future classrooms.

In future studies the researcher would recommend that more time be spent with just the two colors of bills after the students understand the operations of addition and subtraction. Initial problems without the number line should include one-place numbers similar to the examples that were modeled using the novel model. For example, take
away the number line portion of the novel model and ask students to model five dollars of money to spend subtract two dollars of debt. This approach would ease the transition to larger bills without a number line. Students could then concentrate on the concept of additive inverses to understand zeroes. For example, a student could be asked to model three dollars of debt. Then he could be asked what amount of money to spend would be required to cancel this debt of three dollars. By placing the three white bills on the novel model, students could see that there is a one-to-correspondence between the number of dollars of debt and the number of dollars of money to spend. After several problems of this type, students could be asked to generalize their understanding of zero. Of course students must verbalize this as zero and understand that zero doesn’t always mean “nothing”. However, by removing one dollar of debt for every dollar of money to spend allows students to interpret zero in the traditional manner.

Studies could be done with elementary students using the novel model. Although elementary students will not have had as much experience with money as adults, they could begin to see connections between everyday mathematics and the mathematics that they are learning in school. These students could also use the novel model to conceptually understand the algorithms for addition and subtraction of integers. The novel model would help to scaffold students’ learning to allow better access to this conceptual understanding.

The novel model could also be used with a larger group of students to determine its usefulness in a regular classroom setting. A quantitative study could be done with a
control group and an experimental group where pre-test and post-test results are used to determine the effectiveness of the novel model.

In conclusion, there is much more that students could be learning about integers than just procedures to find solutions. The model used in this study served as a motivation for the participants to connect previously learned procedures with conceptual understanding of integers. If the model had been used before students learned the procedures for addition and subtraction of integers, this researcher believes that students would find the procedures to be more meaningful and exciting. We need to help students overcome their fear of mathematics and their tendency to rely on meaningless procedures to find solutions.
APPENDIXES
APPENDIX A: CONSENT FORM
Understanding of Integers by Pre-Service Elementary Teachers

I want to do research on operations with integers. I want to do this because I would like to understand how students think about integers and operations and how they reason with them. I would like you to take part in this project. If you decide to do this, you will be asked to give input in surveys and possibly answer some interview questions concerning integers and operations. The interview process would require no more than five sessions of approximately thirty minutes each. These would be conducted at your convenience.

Confidentiality will be maintained to the limits of the law.

If you take part in this project, you would learn more about how you think about elements and how you perform operations with them. Taking part in this project is entirely up to you, and no one will hold it against you if you decide not to do it. If you do take part, you may stop at any time.

If you want to know more about this research project, please call me at (330) 672-2663. The project has been approved by Kent State University. If you have questions about Kent State University's rules for research, please call Dr. John L. West, Vice President and Dean, Division of Research and Graduate Studies (Telephone: (330) 672-2704).

You may keep this copy of the consent form.

Sincerely,

Carol Steuer, instructor and coordinator of MATH 14001

___________________________________________
Please sign and return the bottom portion of this letter.

I agree to take part in this project. I know what I will have to do and that I can stop at any time.

_________________________  __________________
Signature  Date

-----------------------------------------------

Department of Mathematical Sciences

F.O. Box 0190 • Kent, Ohio 44242-0190
330-672-2600 • Fax: 330-672-2609 • http://www.math.kent.edu
APPENDIX B: SURVEY INSTRUMENT
Pre-service teachers' understanding of integers

This information about integers is for a research project. None of this information will influence your grade and your input is greatly appreciated.

1. Where do you, personally, use integers?

2. Which is greater $-2$ or $-5$? Why?

3. How much greater than $-4$ is $9$?

4. Write the answer below each of the following:
   (a) $8 + 3$
   (b) $-4 - 3$
   (c) $8 + 6$
   (d) $8 - 2$
   (e) $-8 - 3$
   (f) $9 - 16$ (g) $9 - (-4)$
   (h) $9 - 82$
   (i) $104 - 56$
   (j) $483 - 592$
   (k) $209 - (-73)$
   (l) $34 + (-8 - 8)$
   (m) $18 + (42 - 8) + (-58)$
   (n) $263 + (-79)$
   (o) $-147 - 100$

5. Make up a word problem whose solution would be $17 - (-2)$.

6. Make up a word problem whose solution would be $5 + 3$.

7. Make up a word problem whose solution would be $16 - (-3)$.

8. Make up a word problem whose solution would be $8 + (-3)$.

   Make up a word problem whose solution would be $8 + (-4)$.

10. Make up a word problem whose solution would be $-3 - 8$. 


APPENDIX C: ONE STUDENT’S COLUMNS FOR SUBTRACTION
<table>
<thead>
<tr>
<th>Problem</th>
<th>What needs to be done</th>
<th>What's on the number line</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>-7 + -2</td>
<td>nothing</td>
<td>7 red bills + 2 white bills</td>
<td>-9</td>
</tr>
<tr>
<td>4 + 3</td>
<td>nothing</td>
<td>4 white bills + 3 white bills</td>
<td>7</td>
</tr>
<tr>
<td>5 + 2</td>
<td>BRING 7 to 2</td>
<td>5 red bills</td>
<td>-3</td>
</tr>
<tr>
<td>4 + (-6)</td>
<td>Take off 4 zeros</td>
<td>4 white bills</td>
<td>-2</td>
</tr>
<tr>
<td>-3 + 6</td>
<td>Take off 3 zeros</td>
<td>6 red bills</td>
<td>3</td>
</tr>
<tr>
<td>3 + (-4)</td>
<td>Take off 4 zeros</td>
<td>3 white bills</td>
<td>2</td>
</tr>
<tr>
<td>-7 - 2</td>
<td>Add 2 zeros</td>
<td>9 red bills</td>
<td>-9</td>
</tr>
<tr>
<td>4 - (-1)</td>
<td>Add 1 zero</td>
<td>1 red bill, 5 white bills</td>
<td>5</td>
</tr>
<tr>
<td>-1 - 3</td>
<td>Add 3 zeros</td>
<td>4 red bills</td>
<td>-4</td>
</tr>
<tr>
<td>8 - (-5)</td>
<td>Add 5 zeros</td>
<td>7 white bills</td>
<td>7</td>
</tr>
<tr>
<td>-5 - (-2)</td>
<td>Add 3 zeros</td>
<td>7 red bills</td>
<td>-3</td>
</tr>
<tr>
<td>4 - 7</td>
<td>Take off 4 zeros</td>
<td>7 red bills, 4 white bills</td>
<td>-3</td>
</tr>
<tr>
<td>6 - 8</td>
<td>Take off 6 zeros</td>
<td>8 white bills</td>
<td>-2</td>
</tr>
<tr>
<td>8 - 6</td>
<td>Take off 6 zeros</td>
<td>8 red bills</td>
<td>2</td>
</tr>
<tr>
<td>-1 - (-6)</td>
<td>Add 6 zeros</td>
<td>6 white bills</td>
<td>5</td>
</tr>
</tbody>
</table>
APPENDIX D: TRANSCRIPTS OF THE INTERVIEW SESSION
Phase 3  Session 1  CA, LC, & RV

1 I: (Placing a white bill on the table) Has either one of you ever seen one of those?
2 CA: A real one?
3 I: The real one. Have you seen that?
4 LC: Yes.
5 I: What is it?
6 CA: A dollar.
7 I: (Placing a red bill on the table) What’s that?
8 LC: One dollar.
9 I: Are they both the same?
10 CA: They’re different colors.
11 I: They’re different colors. Why do you suppose they’d be different colors?
12 CA: No idea.
13 LC: I don’t know.
14 I: (Putting the number line on the table) Suppose that I have this. Have you ever seen one
15 of these?
16 CA: Yep.
17 I: What is it?
18 LC: A number line?
19 I: Is this like the number line you’ve seen before?
20 LC: Yes.
21 I: Exactly?
22 LC: No. It’s two different colors.
23 I: Why do you suppose it’s two different colors?
24 CA: Negative and positive.
25 I: What do you notice about those two different colors? What are the two colors that are
26 there?
27 LC: Red and white.
28 I: Does that sort of relate to the money?
29 CA: Yeah, the red’s negative and the white’s positive.
30 I: So, if I said to you, “How is this different?” You said it’s two different colors and you
31 can see that negative’s going to be red and white’s going to be positive. So how would
32 you think about the money in those terms? What would the red money signify?
33 LC: I would think money you don’t have.
34 I: Money you don’t have?
35 CA: Yeah.
36 I: What would the money you don’t have be called?
LC: Debt.
I: And what would the white money be?
LC: The money you have.
I: Money you have or money that you can spend, right?
LC: Right.
I: So, if I asked you to model four dollars of money to spend on the number line, how would you model four dollars of money to spend?
(LC put four white bills on the number line.)
I: Notice that they stick because they have Velcro on the back. Do you agree CA?
(CA nodded in agreement.) So that’s four dollars of money to spend. Could you show four dollars of money to spend another way?
CA: (After a long pause) No.
I: Okay, let’s take those off then. (Participants removed the bills from the number line.)
I: How would you show four dollars of debt on the number line?
(CA put four red bills on the number line.)
I: As you look at this number line, there’s something I want you to say. I want you to tell me if you’re looking at the color of money as the most important thing or the number line as the most important thing. When you put that four dollars of debt on there, which caught your attention most, the number line or the color of the money?
CA: The number line.
I: Do you agree with that LC?
LC: Yes.
I: Take those off. (Participants removed the four red bills from the number line.) How would you model zero dollars on the number line?
CA: It would just be that. (number line without any bills on it)
I: Is there another way you could model it?
(Without any hesitation LC put a red bill and a white bill on the number line.)
I: Do you agree CA?
CA: Yes.
I: Could you show zero another way?
(CA put another red bill and another white bill on the number line.)
I: How many different ways are there of showing zero?
CA: Two hundred. Or a hundred?
I: Just a hundred?
CA: Or would it. . . I don’t know.
I: What if I had a number line that went on forever and ever?
CA: Then it would be forever.
I: Let’s take those off. (Participants took the bills off the number line.) Show three dollars of money to spend.

(LC put three white bills on the number line.)

I: Can you show three dollars of money to spend another way?

(CA put three red bills on the number line and LC put three more white bills on the number line.)

I: Is that three dollars of money to spend?

CA: Uh huh.

I: Could you show three dollars of money to spend another way?

(LC put another white bill on the number line as CA put another red bill on the number line.)

I: What are you really doing?

CA: A negative and a positive.

I: And what is a negative and a positive?

CA: When you add them?

I: Yes.

CA: I don’t know.

I: How do you know that that is the same thing as three dollars of money to spend?

CA: Because there’s three of those (pointing to the red bills) and four of those. . .I don’t know.

LC: It’s the same as six (pointing to the white bills) minus three (pointing to the red bills).

I: It’s the same as six minus three?

LC: Or six plus negative three.

I: When I say, “Let’s see another way of showing three dollars of money to spend. What are you doing?

CA: You’re grabbing a negative and a positive.

I: And a negative and a positive together make . . .

CA: Zero.

LC: Oh.

I: And you can add how many zeroes to a number and not change its value?

CA: Right.

I: As many as you want, right? Let’s take those off. (CA and LC took the bills off the number line.)

I: When you did that, were you looking at color of money or number line?

LC & CA: Number line.

I: Which is greater four dollars of money to spend or six dollars of debt?

CA: Four dollars.
I: How do you know?
CA: ‘Cause you have four dollars and you don’t have negative six.
I: So you think four dollars of money to spend is greater than six dollars of debt? Do you agree LC?
LC: (hesitating) Yes.
I: You’re not real sure though?
LC: No. I know I know this. I just can’t remember what they are.
I: What’s meant by absolute value?
LC: A number without, like even if it’s negative, you just take the negative off. It’s like negative one, the absolute value is one.
I: (to CA) Do you agree?
CA: Yeah.
I: So what’s the absolute value of negative four?
CA & LC: Four.
I: How do you know?
CA: Isn’t the distance between zero and the number?
I: So you say the distance between zero and the number?
CA: Not between zero and the number. It could be . . . couldn’t it be like between any number?
I: Give me an example.
CA: Like if it was negative ten and negative five, that would be five? That would be the absolute value.
(LC nodded in agreement.)
I: What are you looking at the absolute value of?
CA: Negative ten and negative five?
I: What’s the absolute value of negative ten?
CA: Ten.
I: What’s the absolute value of negative five? . . . of negative five?
CA: Five. Oh, so it would be fifteen?
I: It depends on what you’re doing with those things.
CA: Yeah.
I: So the absolute value of negative four you said was four. What’s the absolute value of positive four?
CA & LC: Four.
I: So let’s put on negative four (meaning to model it on the number line). Let’s put on four dollars of debt.
(CA put four red bills on the number line.) And you said the absolute value of that is four.
CA: Right.
Think about that for a minute. Now, let’s put on the four dollars of money to spend. (LC put four white bills on the number line.) What’s the absolute value of four dollars of money to spend?

LC: Four.

I: Four. What do you notice? (Participants looked confused.) You said both of those had an absolute value of four.

CA: Uh huh.

I: So what is absolute value? (There was a long pause.)

CA: What I don’t get. I mean . . .

I: You can do it but you don’t know what it is, right?

CA: Right.

I: RV come on in. (RV joined the group.) We’re looking at this money so LC how about if you explain to RV what we’re doing here.

LC: (As she picked up a red bill) We’re talking about negative and positive numbers and the red are the negative and the white are the positive money. I don’t know. She’s asking us questions.

I: CA, do you have anything to add to that?

CA: Nope.

I: So you guys wrapped it up in about twenty seconds what I took fifteen minutes to say.

CA: No.

I: So let’s let RV see if she understands what’s going on here. (I took all the bills off the number line.) So, RV, if we ask you to model on the number line, we’re going to ask you to concentrate on whether you are looking at the number line or if you’re looking at the colors of the money. Model four dollars of money to spend on the number line.

RV: Money to spend.

I: Yes, four dollars of money to spend.

(RV put four white bills on the number line.)

I: (to LC and CA) Do you guys agree? (CA & LC nodded in agreement.) How could you show four dollars of money to spend another way RV, or is there another way? (long pause) Or is that the only way?

RV: That would be the only way because you haven’t spent the money yet.

I: Okay. LC, what do you think?

LC: I think there’s another way.

I: How would you show it another way? (LC put four red bills and four more white bills on the number line.)

Do you agree RV?

RV: Yeah you could do that.
I: What did she really add on there RV? (RV looked puzzled.) What did she put on the number line?
RV: She just, um, it looks like she made it look like she spent some and then she just added another four dollars to it.
I: Okay, so what’s that like adding to the number itself? (RV didn’t understand the question.)
RV: It’s just adding another, um, an extra four dollars to it.
I: Is it adding four dollars? (Long pause.) If she were just adding four dollars to it, where would that four dollars have been? LC, what did you really do?
LC: Um, well I don’t know how I was supposed to say this but I like canceled out (pointing with her pencil) four dollars and then added four but it looked like I added eight. But I didn’t. The negative ones cancel out the positive ones so it’s like I took away four and then I added four. I don’t know. I really spent four dollars. I don’t know how to explain.
I: RV, are you confused?
RV: No, I’m starting to just look at things. I think I understood what she did.
I: Okay, what did she do?
RV: Like we had the four dollars and then she made it look like it was spent so there was another four dollars added onto the number line but then she added another four so she still had the four dollars.
I: Okay, so by spending four dollars and adding on four dollars she was really adding how much to her total amount?
RV: Eight.
I: Eight? Eight bills but was that eight dollars of money to spend that she adding on or eight dollars of debt?
RV: It was still four dollars of money to spend.
I: So she still had four dollars of money to spend. What’s the only thing that you can add to a number and not change its value? (long pause) In real life. (RV was still confused.)
I: What’s the only amount that you can add to something and still have the same amount that you started with?
RV: Nothing.
I: Okay. And nothing is what mathematically?
RV: Mathematically, it’s um, (long pause)
I: Four letter word.
RV: Zero.
I: Zero. So do you see how you added four pairs of a positive and a negative, which is four zeroes, and so you still have the same value as what you started with?
RV: Yeah.
I: Take those all off. (The participants took the bills off the number line.) RV let’s see three dollars of debt.

RV: Three dollars of debt? (She put three red bills on the number line.)

I: Could three dollars of debt be shown another way? And were you looking at color or the number line or a combination of both?

RV: At the numbers.

I: Do you mean the numbers down below (beneath the number line)?

RV: Yes. (RV placed three white bills on the number line and directed CA to put three more red bills on the number line.)

I: Did you have to add three pairs of zeros to that to get the same thing? (CA shook his head in disagreement.)

CA says no.

CA: Did you have to?

I: Did she have to?

CA: She could have put four.

I: So if you would’ve put four down, what would that have looked like? Four pairs of zeroes with the three. What would that look like CA?

CA: It would look like you had just one more.

I: Can you show it? (CA put one more red bill and one more white bill on the number line.) RV, do you agree?

(RV nodded her head in agreement.) LC, do you agree?

LC: Yes.

I: So how many ways are there of showing three dollars of debt?

RV: Lots of ways.

I: So let’s take those off. (Participants took all bills off the number line.) RV, how would you show zero dollars on the number line?

RV: You could leave it like it is or you could just do that (She put a red bill and a white bill on the number line.)

I: And you could probably get more representations of zero than that, true? (RV nodded in agreement.) What’s meant by absolute value, RV? (RV looked confused.) If I ask you to find the absolute value of negative four, let’s see negative four. (RV placed four red bills on the number line.) What’s the absolute value of negative four?

RV: Four.

I: How did you know?

RV: Because even though it’s on the negative side, it’s still showing that you have four dollars in some way.
I: Okay. And when you did that, were you looking at the number line or were you looking at the colors of the bills to figure out how many of those, or what were you using? Or a combination?

RV: I used the number line.

I: Okay. What about the absolute value of positive four or four dollars of money to spend?

RV: It would still be four.

I: Why?

RV: Um, because again it’s showing how much you have in some kind of representation but you still have four dollars that you can spend.

I: And if you’re looking at absolute value, if you’re trying to find what the absolute value is, what is absolute value? This is where we were a little bit confused before.

RV: Absolute value? (long pause) It would be, um, I don’t know how to define it.

I: What is it CA?

CA: I have no idea. I was just waving at the camera. I mean, I know what it is but. . . Isn’t it like just, I don’t know. . .

I: LC, what do you think?

LC: I don’t know.

RV: It’s like the positive value of any number, um, greater than or equal to zero.

I: Okay. Which has the greater absolute value: three dollars of money to spend or six dollars of debt?

CA: Six dollars of debt.

I: Which would have the greater absolute value: four dollars of debt or six dollars of money to spend?

RV: Six to spend.

I: So you understand absolute value? (All three participants nodded in agreement.) Let’s look at two dollars of debt. (LC put two red bills on the number line.) And can you name that another way rather than having just two red bills on the number line? (CA put two more red bills on the number line and LC put two white bills on the number line.) Can you show it another way? (CA put one more red bill on the number line and LC put one more white bill on the number line.) Do you agree?

RV: Yes.

I: So we can add zero to a number and not change the value and zero is going to be what?

What is zero in terms of this model?

CA: Negative two and two?

I: Okay. Negative two and two or negative one and . . .

CA: One.
Phase 3  
Session 2  
RV & LC

303  I: What are integers?
304  LC: Numbers.
305  I: (to RV) Do you agree?
306  RV: Yeah.
307  I: Just numbers? Any number?
308  LC: They’re whole numbers. (Not very confident of answer she’s given)
309  I: (to RV) Do you agree?
310  RV: Whole numbers from the number line.
311  I: Where are the integers used?
312  (Long pause) LC: Everywhere?
313  I: Like . .
314  LC: In money.
315  RV: And everyday math.
316  I: Give me an example.
317  RV: Like going to the store. You have to spend money.
318  I: How were you first introduced to integers? (Long pause.) Or were you introduced to
319  integers?
320  LC: Yes, but I don’t know when. I was little and like when my parents showed me how to
321  pay for things with money, that’s when I was first introduced to integers. (RV nodded in
322  agreement.)
323  I: (to RV) Same with you?
324  RV: Yeah.
325  I: What does addition mean to you?
326  LC: Add something to another thing.
327  I: When you were introduced to integers, when were you introduced to negative integers?
328  LC: Sixth grade. (RV nodded in agreement.)
329  I: And how?
330  LC: In math class. They just taught us about them.
331  I: How did they do it? Do you remember?
332  LC: Um, I think they showed us on a number line.
333  I: RV, what about you?
RV: In sixth grade they showed us on a number line the difference between positive and negative integers.

I: And did that make sense to you?

RV: At first, no. Um. But then after a while it started to make sense.

I: What was confusing about it?

RV: Um. I think it was just the way the teacher was teaching it that just confused everyone.

I: OK. What does subtraction mean?

LC: To take something away.

I: (to RV) Do you agree? (RV nodded in agreement.) How do you subtract integers? Or better yet, how do you add integers? (Long pause) Give me an example and tell me what you’d do for adding integers. (Long pause) Write one down, maybe. (Each wrote an addition problem on their paper. LC wrote 2 + 4 = 6; RV wrote 17 + 3 = 20.) You both have a problem written down. Did you put an answer for yours? (Both nodded.) And how did you know that?

LC: Because I just added two plus four more. You could count that out.

I: (to RV) And what did you have?

RV: I had seventeen plus three.

I: And so you both chose positive numbers to add. What if they weren’t both positive?

LC: Negative six plus two equals negative four.

I: How did you know that?

LC: Because... I don’t know. I took negative six and sub, well, added two. I don’t know how to explain it. Like if you have a negative number and add two positives, you get, you’re left with negative four.

I: And what did you have RV?

RV: Um. Negative eleven plus four so I’m just gonna get negative seven.

I: And how did you know that?

RV: Um. (Long pause)

I: Did you have something scratched out there?

RV: Yeah. I wasn’t watching. I don’t know what I was doing for a minute.

I: What did you have at first?

RV: At first I thought I had just put eleven but it was supposed to be negative eleven.

I: Oh. So how did you know it was negative seven?

RV: (Long pause) It’s just like basic math, pretty much.

I: OK. How do you subtract integers? Let’s have an example for subtraction of integers.

(Each participant wrote an example on their paper.) LC, tell me about your example.
LC: It’s four minus two equals two. You just have four and subtract, take away, two and you’re left with two.

I: And RV, what did you have?

RV: I have five minus three equals two.

I: So both of you again were dealing just with things before you got to integers. Can you give an example of one where you would have some negative integers? (LC & RV each wrote a problem on their paper.) LC, what do you have?

LC: Negative seven minus two equals negative nine.

I: And how did you know that?

LC: Um. You, I just, well I don’t know. I just took like negative seven minus two more is negative nine. I don’t know.

I: RV what did you have?

RV: Negative seven minus seven is negative fourteen.

I: OK and how did you know that?

RV: Um, well I know that seven and seven is fourteen but it’s just on the negative side.

I: OK. I’m going to put out this number line that we worked with the other day and we’ll do some things on here, reviewing what we did last week. So how would you model three dollars of debt on the number line?

(LC put three red bills on the number line.) Do you agree? (Both participants shook their heads in agreement.) Could you do it another way? (LC put another red bill on the number line as RV put a white bill on the number line.) And what did you really do to that three dollars of debt to make it the same amount on there?

RV: Um, we zeroed out one of the negative dollars and then added another negative dollar to the other side.

I: So you zeroed. . . What do you mean by zeroed out?

RV: We added one dollar but then it, um, but because we already had one dollar on there, um, it, um, I don’t know how to explain it.

I: LC, can you help her?

LC: We took away a dollar of debt so then we had to add another dollar (pointing to the red bill that she placed on the number line) of debt there.

I: Where did you take away the dollar of debt?

LC: (Pointing to the white bill) Right here, by adding a positive dollar that canceled out the negative dollar of debt.

I: Let’s take those off. (LC and RV removed the bills from the number line.) How would you model zero dollars on the number line? (LC put a red bill on the number line as RV put a white bill on the number line.) So what did you do?
LC: We had negative debt or a negative dollar and a positive dollar and they cancel each other.

I: Let’s take those off. (LC and RV took the bills off the number line.) And on your paper, let’s turn the paper over and I want you to make three columns on the back, three, evenly spaced columns if you can. At the top of the first one I want you to write “Problem”. At the top of the second column I want you to write “Solution” and the third one I want you to write “Number of Zeroes”, which may or may not make sense at this point in time. And as we do these problems I want you to model them on the number line and fill in the chart that you’ve just created. So I want you to show four dollars of money to spend plus two dollars of money to spend. (Both participants wrote 4 + 2 under “Problem” and 6 under “Solution”. LC and RV each put two white bills on the number line. Then each put another white bill on the number line.) And your solution is...

LC & RV: Six.

I: And did you have to make any zeroes?

LC: No.

I: So you’ll put a zero in that last column. Take those off. (LC & RV took the bills off the number line.) And let’s show three dollars of debt (LC put three red bills on the number line) plus four dollars of debt. (LC then put four more red bills on the number line.) Write it down. (Both wrote \(-3 + (-4)\), the solution (Both wrote \(-7\).) How many zeroes?

LC: None. (Both wrote 0 in the third column of their chart.)

I: Take those off. Let’s look at five dollars of money to spend (RV placed five white bills on the number line.) plus seven dollars of debt. (LC wrote the problem, the solution and the number of zeroes and then placed seven red bills on the number line. As she was placing the seven red bills on the number line, RV wrote values in the chart.)

I: And, is that your answer (referring to the number line)? (LC counts to herself pointing to red bills with her pencil.)

LC: Yes.

I: What is your answer?

LC: Negative two.

I: How could you show negative two? I know that’s one way you could show negative two on the number line but is there some way that everybody could see that it’s two dollars of debt? (LC removed a red bill and a white bill five times to leave two red bills on the number line.)

I: How many zeroes?

RV: Five.

I: LC, do you see how she got five zeroes?

LC: Yeah, I just didn’t write it down. (She wrote “5” in the proper column in her chart.)
I: Take those off. And let's have two dollars of debt (LC put two red bills on the number line.) plus one dollar of money to spend. (RV put one white bill on the number line.) So write it as a problem, your solution, and whenever we show the solution on the number line we typically only want one color of money left over. (LC removed one red bill and one white bill from the number line, so that only one red bill remained.) And how many zeroes?

LC: One.

I: Let's take that off. (RV removed the red bill.) Let's look at six dollars of money to spend (RV placed six white bills on the number line as LC wrote the problem on her paper.) plus five dollars of debt (LC then placed five red bills on the number line as RV wrote the problem on her paper.) So what's your solution? (LC removed five white bills and five red bills from the number line.)

LC: One.

I: And how many zeroes?

RV: Five.

I: And the last one. Four dollars of money to spend (RV placed four white bills on the number line.) plus six dollars of debt. (Both wrote the problem and solution on the paper. Then LC put six red bills on the number line. Then RV removed four white bills as LC removed four red bills from the number line.) You each had four zeroes on your paper, right? So your answer was negative two. So now what I want you to do is look at that and see if you can come up with some statement relating the way that you add with integers related to the number line that you just used. (Both participants seemed confused.) So look at the number of zeroes and see if that has any relationship to what's going on or maybe if the solution has a relationship to something you know, or what's going on? (Very long pause.) Do you see any relationship between the number of zeroes and anything else that you had? (Long pause)

LC: Um. Yeah.

I: What is it?

LC: Well, when both problem, or when both

I: Addends?

LC: addends are the same, like both are positive or both are negative, there's no zeroes. And when they're not positive, when they're positive and negative, the smaller, like the absolute, you take the absolute value of both, the smaller number is the one that shows up in the zeroes.

I: Do you agree RV?

RV: Yeah.

I: Do you see what she's saying?

RV: Yeah.
I: And how do you find the solution to your problem? You said when they’re the same sign there’s no zeroes. How do you know what the solution is in that case?

LC: It’s, (pause), well it’s greater than the addends and if both are positive then the solution’s positive.

I: And what’s the number part (LC looked confused as to what was asking.) of the solution.

LC: What do you mean, the number part?

I: The number part for your solution. The solution has two parts, the number part and the direction, right?

LC: Right.

I: So your solution for the first one was...

LC: Six.

I: How did you get the number part for that solution?

LC: Because the addends are both positive.

I: So what did you do to get the six?

LC: Added.

I: OK. And what about when they were both negative? You said there were no zeroes.

LC: You added.

I: You added those to get the number part. How did you figure the sign in each of those cases?

LC: Uh, well if they were both negative, if both addends were negative, then the solution is negative.

I: And if they’re both positive?

LC: Then the solution is positive.

I: And RV, what do you notice if they aren’t both positive or both negative?

RV: If they’re not both positive or both negative um, there’s always going to be some number of zeroes.

I: OK.

RV: And, um, the one that is the higher integer, um, the lower integer is going to be the number of zeroes for the number.

I: What do you mean by the “lower” integer?

RV: Like, we had seven of, we had negative seven and five, and the five was the number of zeroes that we had.

I: OK, so you’re saying five is the lower integer?

RV: Yes. Like, in a sense.

I: In what sense?

RV: Cause if you take the absolute value of both.
I: OK, there you go. I think that’s what you really meant to say. (RV nodded in agreement.) So, what I want you to write now is whether this number line and the red and white money helped or confused you? So write a little bit and be truthful. (LC and RV wrote their comments on their paper.) As you’re writing, write down things that were great “Aha” moments using the number line and things that made it more confusing for you using this model. (LC and RV continued to write.) And then we’ll get each of you to state what you just figured out about addition of integers. RV, what did you discover about addition with integers? (pause) What’s the algorithm for adding integers?

RV: I don’t know how to describe it.

I: OK. Well, try.

RV: Um, (very long pause)

I: How about “when the signs are the same. . .”

RV: When the signs are the same, the solution’s gonna be, um, the same sign as the problem.

I: And how do you get the number part in that case?

RV: You just add them together.

I: OK, and what if the signs are different?

RV: If they’re different, um, (long pause) it’s, like you (very long pause)

I: Help her out LC.

LC: Uh, then you take the integer with the greatest absolute value and whatever sign it had, that’s what the problem, the solution is gonna have. So if, uh, (looking at her paper for problems that were discussed) you took six plus negative five, six has the greater absolute value and your answer is going to be positive.

I: OK, and how did you get the number part when the signs are different for the solution?

How did you figure out what that number part was? You told us how you’re going to figure out if the sign is going to be positive or negative.

LC: Um, well, I just took six and subtracted five and got one.

I: OK, so you really subtracted what?

LC: Five.

I: You subtracted five from six but what is that?

LC: I don’t know.

I: RV help her.

RV: It’s like canceling out five dollars of debt.

I: OK.

LC: In addition.

I: Is it addition?

RV: Subtraction.

I: You’re subtracting those. . .
RV: Negative?
I: Two words. A, V.
RV: Absolute values.
I: So does that make any sense?
LC: Yes.

Phase 3  Session 3  RV & LC

I: I’d like you to model two dollars of debt. (LC put two red bills on the number line.)
Can you model it a different way? (LC put another red bill on the number line as RV put
a white bill on the number line.) And a different way? (LC put another red bill and RV
put another white bill on the number line.) Let’s take those off. (RV and LC removed the
bills from the number line.) And, let’s model zero. (LC put a red bill on the number line
as RV put a white bill on the number line.) And another way? (LC put another red bill
and RV put another white bill on the number line.) And you both agree that that’s zero?
(Both nodded in agreement.) So there are many ways to model a number. (Both nodded
in agreement.) Let’s take those off. (RV and LC removed the bills from the number line.)
Tell me, what is addition? (long pause) You have added before, right?
RV: Yeah.
LC: You add one integer to another integer.
I: Okay. Could you give me an example of an addition problem?
RV: (after a short pause) One plus two.
I: One plus two. So how would you model five dollars of debt plus four dollars of money
to spend, on the number line? (LC put a red bill at negative five on the number line and
then four more red bills while RV put four white bills beginning at positive one. So,
what’s your answer?
LC: Negative one.
I: How do you know?
LC: Because negative five plus four is negative one?
I: Could you see that on the number line immediately?
LC: Yes.
I: Where?
LC: Um. I don’t know. Right here. (She pointed to the left of the last red bill on the
number line.)
I: But I see more than just one red bill.
LC: There’s five of them.
I: Yes, there are. But you said the answer would be one red bill. Right? Negative one.
LC: Uh huh. Well, the white bills cancel out the red bills.
I: Okay, so show the canceling out. (LC removed four red bills as RV removed the four white bills.) So, how many zeroes did that make?
RV: Four.
I: Four zeroes. So, remember from last week, we looked at the number of zeroes that were made and how could that be related to the way that you add integers? For example, if you write down on your paper “Negative five plus four”, you said there were four zeroes. And your answer was negative one. How is that number of zeroes related to the addition problem, negative five plus four? (Long pause)
LC: I don’t understand the question.
I: Okay. You said you had four zeroes.
LC: Uh huh.
I: Do you see a four any place in that problem?
RV & LC: Yes.
I: Do you suppose that the number of zeroes is related to that number somehow?
LC: Yes.
I: Let’s look at another problem. Let’s look at four plus negative three. (Both wrote the problem on their paper.) How many zeroes would you have for that one?
LC: Three.
I: And is that related to the problem at all?
LC: Yes.
I: So give me another addition problem and you figure out how many zeroes there are and if it’s related to the problem. So RV, make up a problem. An addition problem. (RV wrote a problem on her paper.) You want to share it with us?
RV: Two plus negative one.
I: And LC, if you have two plus negative one, how many zeroes would there be?
LC: One.
I: And do you see that some place in the problem?
LC: Yes.
I: So how is that number of zeroes related to something in the problem?
RV: It’s the absolute value of the second addend. (She meant the addend that was given.)
I: Is it always the second addend?
LC: It’s the smallest. Well, . . .
RV: It would be the smallest, um, the smallest absolute value of the problem.
I: Okay, the smaller absolute value. The one that has the smaller absolute value. And then, if those signs are different, how do you determine your answer?
LC: Whichever number has the greatest absolute value, that’s the sign you use.
I: Okay, and how do you get the number part?
LC: Um. You take, you subtract them.
I: Okay, you subtract their . . .

LC: Absolute values.

I: Good. Now tell me, what is subtraction?

LC: (short pause) Taking away something?

I: Okay. Taking away. That’s the easiest type of subtraction that we can have. So, I want you, on your paper, to create three columns. The first column I want you to label “Problem”. The second column, “Solution”, and the third column, “Number of zeroes”.

RV: Same thing as last week.

I: Yes, it is. And on the number line, I’d like you, take off the one bill that’s there (LC removed the red bill from the previous problem), and let’s model two dollars of debt. (LC put two red bills on the number line.) And I want you to subtract now, you told me that subtraction was take away, so I want you to subtract four dollars of money to spend. (LC picked up some white bills but seemed confused. She started to put them on the number line.) Is that subtracting? Is that taking away? (She removed the white bills she had put on the number line.)

LC: I don’t know.

I: Well, if you’re putting them on there, are you taking them away?

LC: No.

I: I mean, you told me that subtraction was take away so, . . .

RV: So . . . (long pause) (LC put two red bills on the number line.)

I: Is that take away?

LC: Yes.

I: What are you taking away?

LC: I don’t know. Empty space, right here. (She slapped her hand down on the positive side of the number line. She then removed the two red bills that she had put on the number line.)

LC: Hum. You said minus four dollars that you could spend?

I: Yes. So, to help you out, can you think of another way to name two dollars of debt?

LC: Negative two.

I: That would help you out. Besides negative two.

LC: Um.

RV: The absolute value of two.

I: Okay, the absolute value of two. I see two things there.

LC: Well, if you made it negative six and four, you could use the zeroes and you would get . . . Oh, no, that doesn’t work. Never mind. (short pause) I don’t know.

I: Think about what you just said before you said “I don’t know”.

LC: If you make it negative six?

I: Okay. And so you’re really putting on what?
LC: Negative.

I: Four negatives, and if you put on four negatives and you put on . . .

LC: Four positives.

I: Four positives. Would that be the same thing? (LC put five red bills and RV put four white bills on the number line.)

LC: Yes.

I: Except that you put on how many?

RV: You put on five.

LC: (She took off one of the red bills.) Just kidding.

I: Just checking. Now can you take away your four dollars of money to spend?

RV & LC: Yeah.

I: So take away your four dollars of money to spend. (RV removed the four white bills from the number line.) So what was the problem that you had? I said two dollars of debt subtract four dollars of money to spend. So what did you write in the column that says “Problem”?

LC: Negative two minus four.

I: Okay. What’s your solution?

RV & LC: Negative six.

I: And what was the number of zeroes that you added? (Long pause) What did you put on the number line besides that two dollars of debt?

LC: Four dollars of debt and four dollars to spend.

I: And how many zeroes was that?

LC: Four.

I: Let’s take those off. (LC and RV removed the bills from the number line.) And let’s try one dollar of money to spend (RV put one white bill on the number line.) subtract one dollar of debt. (LC put a red bill on the number line.) Is that taking away one dollar of debt? (RV had her hands on the white bills, thinking about what to do.)

LC: No. I am so confused.

RV: Wait. You’re subtracting, you’re really just adding another dollar. (She put another white bill on the number line and then removed a red bill.)

I: So what did you really add on there?

RV: You added one more dollar to spend.

I: And you added . . .

LC: One you can’t spend.

I: One dollar of debt.

LC: Yeah.

I: So you were really adding how much to your dollar of money to spend that you had?

LC: One. I don’t know.
I: If you added a dollar of debt,
LC: Uh huh.
I: And a dollar of money to spend, you really added, . .
RV: Nothing.
I: Nothing, or zero. True? How many zeroes did you add for that problem?
RV: One.
I: One. Does that make sense, LC?
LC: Uh huh. Wait. What was the original problem?
I: The original problem was one dollar of money to spend subtract one dollar of debt.
LC: I had the problem wrong.
I: Oh, okay. What did you write?
LC: One minus one.
I: Oh, that explains the confusion then.
LC: Yeah.
I: Does this make sense now?
LC: Yep.
I: Take those off. (RV removed the bills from the number line.)
LC: So how many zeroes were there?
RV: Just one zero.
I: Let’s try this one. Four dollars of money to spend. So let’s see the four dollars of money to spend. (RV put four white bills on the number line.) Subtract one dollar of debt.
(LC put a red bill on the number line and RV put another white bill on the number line.)
RV: That would be a zero then.
I: Does that make sense, LC, what you just did?
LC: Well, I understand what I did. Oh yeah, now I get it.
I: You got it?
LC: Wait. No.
I: RV explain to her what’s going on.
RV: In the problem it’s four minus a negative one. Um, and that’s really, in the end, adding.
LC: Right.
RV: Cause two negatives are a positive.
LC: But don’t these (pointing to a red bill and a white bill) just cancel each other out?
RV: (Pointing to the red bill and the farthest white bill from zero) These two do.
LC: Then how could it be five? I don’t get it.
RV: Because in the end you’re going to be adding.
LC: I don’t get it.
RV: The problem’s pretty much saying four plus one.
LC: Right. So, that’s five. Why do we have a negative?
RV: Well, that’s going to cancel out.
LC: But why are we canceling it? I don’t get it. Seriously.
I: Okay. Let’s take off that one red bill and one white bill. (LC took off a red bill and RV took off a white bill.) Do you agree that that’s four dollars of money to spend?
LC: Yes.
I: Since we want to take away a dollar of debt, do we have any dollars of debt that we can take away?
LC: No.
I: So, we have to have some dollars of debt to take away. How can we get dollars of debt for us to take away? The only thing that we can add to a number and not change its value is . . .
LC: I don’t know.
I: What’s the only number that you can add to . . .
LC: Zero.
I: Zero. So what does zero look like? (LC put a red bill on the number line and RV put another white bill on the number line.) A positive and a negative, true?
LC: Uh huh.
I: Now can you take away the one dollar of debt?
LC: Uh huh.
I: So, when you take away that dollar of debt (LC removed the red bill from the number line.) you’re left with . .
LC: Five.
I: Five dollars of money to spend. Does that make any more sense?
LC: (very emphatically) Yes.
I: Let’s take those off. (RV removed the bills from the number line.) This time LC we’re going to let you do this one all by yourself. Three dollars of debt (LC put three red bills on the number line.)
LC: Uh huh.
I: Subtract two dollars of money to spend. (LC put two more red bills and two white bills on the number line.) So basically you added what?
LC: Two zeroes.
I: Two zeroes. Now can you take away your two dollars of money to spend?
LC: Yes. (She removed two white bills from the number line.)
I: And you’ll be left with . . .
LC: Five, negative five or five dollars of debt.
I: Okay, good. So how many zeroes was that?
LC: Two.
I: Good. Okay. Take those off. (LC removed the bills from the number line.) And let’s have one dollar of debt (LC put one red bill on the number line.) subtract four dollars of money to spend. (LC put four white bills and four more red bills on the number line.) Now can you take away the four dollars of money to spend? (RV removed the four white bills from the number line.) So you added how many zeroes?

LC: Four.

I: Good. Let’s take those off. (LC removed the bills from the number line.) And we want zero dollars subtract three dollars of money to spend. (LC put three red bills on the number line and RV put three white bills on the number line.) RV: Take away the three (She removed the three white bills from the number line.)

I: And you’re left with . . .

LC: Negative three.

I: And how many zeroes was that?

RV & LC: Three.

I: Good. Let’s take those off. (LC removed the bills from the number line.) And let’s have two dollars of debt (LC put two red bills on the number line.) subtract three dollars of money to spend. (LC put three red bills on the number line as RV put three white bills on the number line.) RV: And then take away the three. (RV removed the three white bills from the number line.)

I: And how many zeroes?

RV: Three.

I: (LC looked confused.) LC, what’s wrong?

LC: Oh, I didn’t have the problem down (She hadn’t written it on her paper) and I couldn’t remember what it was.

I: Oh, okay. So it was two dollars of debt subtract three dollars of money to spend. Makes sense?

LC: (very confidently) Uh huh.

I: Let’s take those off. (LC removed the bills from the number line.) And let’s have four dollars of debt (LC put four red bills on the number line. Long pause as I waited until LC wrote the problem on her paper.) subtract three dollars of money to spend. (RV put three white bills on the number line and LC put three more red bills on the number line.) RV: Then (She removed three white bills from the number line.)

I: So how many zeroes was that?

RV: Three.

I: Three. So when you added those three zeroes, do you see that, in order to take away that three dollars of money to spend, you were really adding on what to the other side?
After you took away that three dollars of money to spend, you really had what more on the other side?

RV: Three more.

I: Three more of . . .

RV: The negative.

I: Okay. Of the debt, which was the opposite of that three dollars that you needed to spend. Does that make any sense?

RV: Yeah.

I: Now, I want you to look at the number of zeroes that you added in each of those cases and see if you can relate that to anything in the problem. (LC removed the bills from the number line as RV studied the problems.)

RV: It’s everything that’s being subtracted from the original amount.

I: Okay, it’s what was subtracted from the original amount. Do you agree, LC?

LC: (not really confident of what has transpired) Uh huh.

I: Are you sure?

LC: Uh huh.

I: Positive?

LC: Yeah.

I: And when you looked at that, do you see any relation between what you were doing there (on the number line) and your algorithm for subtraction? (RV and LC were confused by the question.) What’s your algorithm for subtraction? When you normally have four subtract negative one, what do you usually do?

LC: Four plus negative one.

I: Okay, so you do four plus . . .

RV: Four plus one.

I: Four plus one. Right?

LC: Wait. What was the problem?

I: Four subtract negative one.

LC: Oh yeah, it’s four plus one.

I: So do you see how the algorithm can be . . . you can use this to look at what the algorithm is doing? (LC nodded in agreement.) Using that same sort of idea, I want you to look at five dollars of money to spend (RV put five white bills on the number line.) subtract three dollars of money to spend. (LC picked up some red bills and paused.

RV: (RV put three more white bills on the number line.) Put three on. (LC put three red bills on the number line.) Wait. Subtract or just . . .

LC: I don’t get it.

RV: We have five to spend.
LC: Now I just . .. Maybe . . . (RV started to take off three white bills.) Yeah, take those off. Then they cancel each other out.

RV: Then we have three that cancel each other out. (RV removed three white bills and LC removed three red bills from the number line.)

I: So how many zeroes did you add in there?

RV & LC: Three.

I: So write that down.

LC: What was the problem?

I: The problem was five dollars of money to spend subtract three dollars of money to spend.

LC: Okay.

I: Let’s take those off. (RV removed the bills from the number line.) Let’s do three dollars of debt (LC put three red bills on the number line,) subtract two dollars of debt (LC put two more red bills on the number line and RV put two white bills on the number line.)

RV: You just take these off (referring to the two white bills).

I: Two dollars of debt? Is that what you just took off?

LC: No. Positive.

RV: You put on two.

I: You put on two zeroes so you had two dollars of money to spend and two dollars of debt that you added on. Right?

RV: Yeah.

I: Let’s see that again. (She put the two white bills back on the number line.) So that’s what you had before. Now I want you to subtract two dollars of debt. (LC removed two red bills and two white bills from the number line.) Now look at what you have in your hand, LC. Is that two dollars of debt?

LC: No, cause they cancel each other out. This is zero.

RV: No, it would end up being negative four.

LC: I’m so confused.

RV: These stay on. (RV took two white bills from LC.) But then these two (pointing to the two white bills) are gonna end up canceling out two of these (pointing to the red bills). So it would be negative one.

LC: Okay.

I: Guess who’s going to do the next one?

LC: Me.

I: Take those off. (LC removed the bills from the number line.) So how many zeroes was that?
RV: Two. (to LC) This is what the problem would look like. (She showed her the
problem on RV’s paper.) You have negative three minus a negative two. So it’s really
adding two.
LC: I can do them when I look at them
I: I’m trying to get you to understand why you do what you do. Okay, four dollars of
debt, LC. (LC put four red bills on the number line.) Subtract three dollars of debt. (LC
put three more red bills on the number line. She picked up some white bills but seemed to
be confused.)
RV: It’s like the last problem we just did.
LC: I don’t get it.
I: Help her RV.
RV: All right. (RV removed three of the red bills from the number line.) Last time we
just did . . . It was subtracting a negative so it’s really adding. So you would add on (She
put three white bills on the number line.)
LC: It’s the word “debt” that keeps throwing me off.
RV: Yeah, okay.
I: But now (LC started to take off the three white bills.) Wait, wait, wait. Can you just put
on three dollars of money to spend like that?
LC: (after a long pause) Yes?
I: So if you put on three dollars of money to spend that’s the same as what you had
before? (Both LC and RV seemed confused.) Let’s see the four dollars of debt again. (LC
removed the three white bills from the number line.) Now I want you to subtract three
dollars of debt.
LC: Okay. (She put three white bills on the number line.)
RV: If you’re subtracting debt, then you don’t have to add anything to the negative side.
I: Oh, we don’t.
LC: Do you have to add to the positive then?
I: But we could add three dollars of money to spend as long as we add in what? (LC &
RV looked confused.) If we want to keep it all balanced, . . .
RV: We could add three negatives on there (pointing to the negative side of the number
line). Right?
I: So add the three dollars of debt (LC put three more red bills on the number line.)
Because what you did was you added what? The only thing that you can add to a number
. . .
RV: More debt.
I: And not change its value is . .
LC: Zero.
I: So you added how many zeroes?
RV & LC: Three.
I: Three zeroes. Now, can you take away your three dollars of debt?
LC: Yes. (She removed three red bills from the number line. Both students looked confused to find money of both colors on the number line.)
I: When you took away three dollars of debt, that was basically the same as adding what?
(Long pause) If you have a debt and I take away three dollars of your debt, that’s basically the same thing as . . .
LC: Adding a positive.
I: Adding three dollars of money to spend, isn’t it? To find your answer on the number line, you’re going to, (LC removed three red bills and three white bills from the number line.) One more. How about two dollars of debt (LC put two red bills on the number line as RV wrote $2$ on her paper.) subtract four dollars of debt. (LC put four white bills and four more red bills on the number line.). So again, it was two dollars of debt subtract four dollars of debt.
LC: It doesn’t make sense to me.
I: Why not?
LC: Because if you do the zeroes, you still have negative two. You still have two dollars of debt. I don’t know. I’m so confused.
I: With what you have in front of you now, can you take away four dollars of debt?
LC: Yes. (She removed four red bills from the number line.)
I: And what you see in front of you is the same problem as what? Two dollars of debt plus . . .
LC: Four dollars.
I: Four dollars of money to spend. Looking at your subtraction problem, is that how you would solve that problem normally? If you have negative two subtract negative four, . . .
LC: It would be negative six.
I: Negative two subtract negative four?
RV: It would be negative two plus four. (LC put her hands to her face as she recognized her mistake.)
LC: Right.
I: So, in your problem, if you were doing negative two subtract negative four, that’s the thing that you would write down is what? Negative two . . .
LC: Plus four.
I: Negative two plus four. Do you see that on the number line in front of you?
LC: Right.
I: But then to figure out what that is, you have to go through and do the same thing that you did for addition, getting rid of those things so that you have just one color remaining. So you’re getting rid of some zeroes.
LC: Uh huh. (LC removed two red bills and RV removed two white bills from the number line.)

I: So you’re simplifying it and your answer would be. . .

LC: Two.

I: Positive two or two dollars of money to spend. Does that make any more sense?

LC: Uh huh.

I: Let’s take those off. (RV removed the bills from the number line.) Let’s look at five dollars of money to spend (LC did not write down the problem. She put five white bills on the number line.) subtract two dollars of debt. (LC put two more white bills and two red bills on the number line.) Does that look like the problem that you would usually solve to answer that problem? Five subtract negative two. Well, let’s take off, you want to take away the two dollars of debt, don’t you? (LC removed the two red bills.)

LC: Uh huh.

I: And what you’re left with is. . .

LC: Seven.

I: Does that make any sense at all?

LC & RV: Uh huh.

I: So can you relate what you’re doing to the algorithm that you use for subtraction? Does it make any more sense than it did half an hour ago?

LC: Yes.

I: I want you to ponder that for next week because next week we’re going to do some addition problems. We’re going to do some subtraction problems on the number line. You have to remember what you’re doing. Just remember that subtraction we’re treating as take away.

Phase 3  Session 4  RV & LC

I: So what is addition?

RV: It’s two addends added together.

I: Okay. And what is subtraction?

LC: An addend subtract an addend. Take, well, you could take, oh wait. I don’t know.

RV: Um.

LC: It’s take away.

I: Okay. It’s take away. (to RV) You agree? (RV nodded in agreement.) Take away. It’s the easiest type of subtraction, isn’t it? Okay, so on your paper I want you to first of all put your name and the date.

LC: It’s the thirtieth, right?

RV & I: Yes.
I: And then you’re going to put four columns. The first column you’re going to call “Problem”, second column “What needs to be done?”, third column “What’s on the number line?”, and the fourth column is “Solution”. So some of these will be using the columns. You know, when we get to a real problem we’ll be using some of those things.

The first thing I want you to do, though, is show four dollars of debt with the money on the number line. So you don’t have to write anything down. (RV put four red bills on the number line.) (to LC) Do you agree? (LC nodded in agreement.) Could you show it another way? (RV put another red bill on the number line as LC put a white bill on the number line.) Okay, now I want you to remember this that that’s another way of writing four dollars of debt. Think you can remember that for at least thirty minutes? (RV shook her head.) Could you even model it (four dollars of debt) another way? (RV put another red bill on the number line and LC put another white bill on the number line.) So you could keep on doing that. What are you really adding when you each put on one of those bills?

LC: Zeroes.

I: Zeroes. Okay. So let’s take those off. (RV and LC removed the bills from the number line.) And how would you model seven dollars of debt plus two dollars of debt? (RV put seven red bills on the number line.)

RV: There’s seven. (She then put two more red bills on the number line.) And another two.

I: Okay, so on your paper, you could write down what the problem is. So it was seven dollars of debt plus two dollars of debt. How would you write that with numbers?

LC: Negative seven plus negative two.

I: Good. And then what are you going to write in the next column. The next column says “What needs to be done?” Do you need to do anything to figure out the solution? Do you have to add zeroes, or cancel anything, or do anything to find out what your solution is?

LC: No.

I: So you’ll put “Nothing”. And what’s on the number line put down how many reds, how many whites. And then your solution. Okay, so let’s take those off the number line. (RV removed the bills from the number line.) And this time let’s do four dollars of money to spend (LC put four white bills on the number line.) plus three dollars of money to spend. So you’ll write that one down on your paper. Four dollars of money to spend plus three dollars of money to spend. (LC put three more white bills on the number line.) Can you fill in the other columns. What needs to be done? What’s on the number line? (LC counted the white bills.), and your solution.

And, how about negative five plus two. (LC removed the bills from the number line.) Yeah, take those off. (RV put five red bills on the number line and LC put two white bills
So, what’s the problem? Negative five plus two. What needs to be done? You don’t really need to introduce any zeroes or anything, do you?

RV: No.

I: However, you could do what?

RV: (Pointing to two red bills and two white bills that are on the number line) Take these off.

I: Okay, so you’ll take off two zeroes so you could take off two reds and two whites.

RV: Now?

I: Yeah. (RV removed two red bills and two white bills from the number line.) So you’re really taking off two zeroes, right?

RV: Yeah.

I: And, what’s your solution?

LC: Negative three.

I: Good. Let’s take those off. (RV removed the bills from the number line.) And let’s do four plus negative six. (LC put four white bills on the number line and RV put six red bills on the number line.) And so, what needs to be done? (RV removed four red bills from the number line as LC removed four white bills.)

LC: How would you write that? Like take away zeroes?

I: Take off as many zeroes as you had to do. Or take off this many reds, that many whites. And then your solution.

LC: Negative two.

I: Good. Let’s take those off. (RV removed the bills from the number line.) And let’s do negative three plus six. (RV put three red bills on the number line and LC put six white bills on the number line.) (RV then removed three red bills from the number line.)

LC: How do you spell “zeroes”?

I: I think it’s “zeroes”.

LC: That’s what I thought.

I: But, it doesn’t really matter. So what did you take off RV? (LC filled in the columns on her paper this whole time.)

RV: I took off the three. They’re still there. (meaning that she expected LC to remove three white bills when she removed three red bills. She pointed to the positive side of the number line and then removed three of the white bills.)

LC: I’m sorry.

I: Okay, do you see what she did?

LC: Yeah.

I: Okay. So you’re left with . . .

LC: Three.
I: Positive three. Okay, how about (LC started to remove the bills from the number line.)
three plus negative one. (When LC heard “three” she left the three white bills on the
number line. Then she put a red bill on the number line. LC then removed one of the
white bills and one of the red bills from the number line.) So you’ve got addition pretty
well down, right?
LC & RV: Uh huh.
I: Okay, let’s take those off. (RV removed the bills from the number line.) Remember
we’re looking at subtraction as take away so the first thing I want to see is seven dollars
of debt. (RV put seven red bills on the number line.) And from that I want you to take
away two dollars of money to spend. So what’s the problem?
LC: Seven, negative seven minus two.
I: Good. (LC put two white bills on the number line and then started to
take them away as
RV took away two red bills.) Wait. (Both LC and RV put the bills back on the number
line.) Is that legal? What did you put on there, LC?
LC: Two dollars, oh I . . .
I: Can you just slap on two dollars like that?
LC: No.
RV: You have to add these two (holding two red bills).
I: Okay. (RV put the two red bills on the number line.)
LC: I thought she already did. That’s why I . . .
I: Okay. Now what are you going to do?
RV: Take away two dollars to spend.
I: Okay. So leave those two dollars on there just so we can write down what’s on the
number line. So what did you have to do? You had to . . .
LC: Add zeroes.
I: Add how many zeroes?
RV: Two zeroes.
I: So add two zeroes. That’s what you’ll put in the “What needs to be done?” And then,
what do you have on the number line? You know what, let’s take the two dollars off and
then we’ll look at what’s on the number line.
LC: (pointing to the two white bills on the number line) These?
I: Yeah. Because you’re taking your two dollars away. Now write down what you’ve got
on the number line. So for “What’s on the number line?”, you’ll write down . . .
LC: Nine red.
I: Nine reds. And then your solution would be . . .
LC: Negative nine.
I: Negative nine. Good. Let’s take those off. (RV removed the bills from the number line.) You’re going to have a bunch of these. LC was smart to make the columns all the way down. So let’s do four subtract negative one. (LC put four white bills on the number line. RV put one red bill on the number line and LC put another white bill on the number line.) What needed to be done? LC: Add one zero. RV: A negative. I: Negative one. What’s your answer then? “What’s left on the number line?” And then “What’s your solution?” LC: Five. I: Okay. Positive five. Not a problem. Okay, let’s take those off. (LC removed the bills from the number line.) And let’s do negative one subtract three. RV: Negative one. (She put a red bill on the number line.) Then we put three zeroes on. (She put three more red bills on the number line as LC put three white bills on the number line.) I: Good. (LC removed three white bills from the number line.) So you knew what needed to be done. How many zeroes did you have to add in? LC & RV: Three. I: And you’re left with . . . RV: Negative four. I: Good. Let’s take those off. (RV removed the bills from the number line.) And let’s do two subtract negative five. (LC put two white bills on the number line.) So, LC, what are you going to have to do? LC: Um. Add negative five? I: Can you just add negative five? LC: Well, you have to put (pointing to the positive side of the number line) there too. I: So, you’re really putting on how many zeroes? LC: Five. I: Five zeroes. (LC put five more white bills on the number line.) RV: Then these (She picked up the first and last red bills on the number line.) are going to come off. I: Okay, yeah. That’s what you want to take off. And so, what’s left on the number line? (LC counted the white bills by pointing at each of them.) RV: Seven. I: Could you have just looked down here (pointing to the numeral “7” on the number line)? LC: Yeah, but I get confused.
I: Yeah, that would make it way too easy. Okay, let’s take those off. (LC removed the bills from the number line.) And let’s do negative five subtract negative two. (RV put five red bills on the number line.

RV: And then, (looking to LC) put on two zeroes. (RV put two more red bills on the number line and LC put two white bills on the number line.)

I: So you need to put on two zeroes.

LC: Why?

I: Why, RV?

RV: Because we’re subtracting the negative two and we still need the zeroes to get the negative two.

LC: (not convinced) Okay.

I: What did you want to do LC?

LC: (pointing to the two white bills) Not put these on.

I: Okay, you just wanted to take off the two red bills?

LC: Uh huh.

I: Okay. And that would have been perfectly okay except that I was leading you to it this way because I’m trying to get you to see something.

LC: Okay.

I: So, what is, as you look at the number line right now, that’s going to be negative five subtract, well, you haven’t subtracted your negative two. Subtract your negative two.

RV: Sub. . .

I: It was negative five subtract negative two, right?

RV: Right. (RV looked slightly confused because there will be two colors of bills left on the number line, not one, as in the other examples.)

I: So subtract negative two.

RV: So take away these? (pointing to the red bills)

I: Uh huh. (RV removed two of the red bills from the number line.) And then I want you to write down what’s on the number line right now. You could write it as red bills, white bills or you could write it as a problem. And what problem did you write, or what did you write down for that, LC?

LC: Five reds and two whites.

I: Okay, what did you write down, RV?

RV: Five reds, two whites.

I: Okay. Let’s see if we could write it another way, rather than saying reds, let’s say what? (LC and RV appeared confused.) What is five reds?

RV: Negative five.

I: Okay. Negative five plus two, right?

RV: Yeah.
I: Okay, so let’s put negative five plus two where it says, “What’s showing on the number line?” or “How is it on the number line?” or whatever it is about the number line. (referring to a column on their paper) And sort of keep that in the back of your mind. You guys both got a solution? Now you’re going to do what? You want your answer to be just one color, don’t you?

RV & LC: Right.

I: So what do you have to do now?

RV: Take off the zeroes.

I: Okay, take off your zeroes. (RV removed two red bills and two white bills from the number line.) And you’re left with . . .

RV: Negative three.

I: Negative three. (LC looked confused.) LC, is that okay?

LC: Uh huh. (She appeared to be still upset that two zeroes were added in the first place.)

I: Just bear with us for a brief moment. I know what you want to do. Okay, take those off. (RV removed the bills from the number line.) Let’s have four subtract seven. (LC put five white bills on the number line but then took one off when she realized she had too many.

RV: Oh. Do you want to subtracting . . .

I: Subtract seven. So, four subtract seven. Thank you LC. (LC wrote the problem on her paper.)

RV: Alright then, seven zeroes. (She put seven red bills on the number line and LC put seven more white bills on the number line.)

I: Okay. Now can you take away your seven?

RV: (pointing to the positive side of the number line) Over there.

I: Okay.

LC: Why? I don’t understand.

I: Okay. Explain it RV. Why are you taking away seven?

RV: Cause we’re taking away seven there (pointing to the positive side of the number line). Um and we’re taking away the seven zeroes that we had. But wait.

I: Taking away seven zeroes?

LC: Oh, I get it. I take seven off of here. (She removed seven white bills from the number line.)

I: Right. Because it says “subtract seven”.

LC: Right. And before I didn’t have seven so I had to . . .

I: So, you really had to put seven zeroes on there didn’t you?

LC: Uh huh.

I: So now she’s taking off her seven positives. Right, RV?

RV: Uh huh.

I: So what’s on the number line?
RV: Negative seven plus four.
I: Okay, or four plus negative seven. And then what are you going to do?
RV: Take off the zeroes.
I: Okay. (RV removed four red bills from the number line as LC removed four white
bills.) And your final solution would be . . .
RV: Negative three.
I: Good. Okay, let’s take those off. (RV removed the bills from the number line.) Let’s do
six minus eight. (LC put six white bills on the number line. She then put eight more white
bills on the number line as RV put eight red bills on the number line.) Now what are you
going to do?
RV: Now let’s subtract your eight. (LC removed eight white bills from the number line.)
I: So what did you do before you subtracted the eight? You had to . . . or you put on . . .
LC: Eight zeroes.
I: Eight zeroes. And then, once you subtracted your eight, what’s on the number line?
RV: Negative eight plus six.
I: Okay, or six plus negative eight.
RV: Yeah. And then (She removed seven red bills from the number line as LC removed
six white bills from the number line.) Wait. I just pulled off more than I needed. (She put
one of the red bills back on the number line.) You take off your six (meaning for LC to
take six white bills from the number line.) And we’re left with negative two.
I: Okay. How about . . . Take those off. (RV removed the bills from the number line.) And
we’ll have one minus six. (LC put one white bill on the number line. LC then put six
more white bills on the number line as RV put six red bills on the number line.) So you
put on . . . And now what are you going to do?
RV: (LC removed six white bills from the number line.) So you’re taking off yours. So
it’s one plus negative six.
I: Yes.
LC: Hang on.
RV: (pointing to the white bill) One plus (pointing to the red bills) negative six. (LC
removed the white bill from the number line as RV removed a red bill from the number
line.) And we get negative five.
LC: Okay.
I: Yeah, so what’s showing on the number line is one plus negative six.
LC: Okay.
I: Now, as you look at the original problem, it was one minus six, do you see the
algorithm, why it’s working. The algorithm says change that subtraction to addition,
change that second thing to its opposite. Can you actually see it on the number line when
we’re doing it that way?
RV: Yeah. (LC nodded in agreement.)

I: Which is why I forced you to do it RV’s way. You could’ve done it your way. . .

LC: Uh huh.

I: But then we wouldn’t get the algorithm.

LC: Right.

I: So I sort of led you that way but you were just too smart for me. You said, “No. . .”

And your way would work. Okay, let’s take those off. (RV removed the bills from the number line.) Negative one subtract six. (RV put one red bill on the number line. (LC put six white bills on the number line.)

RV: So then I’m putting six on also?

LC: Yeah.

RV: Okay. (RV put six more red bills on the number line.)

I: What did you have to do?

RV: We added six zeroes.

I: Okay. And now you’re going to subtract your six. (LC took six white bills from the number line.) And, what’s left on the number line?

RV: Negative seven.

I: And does that one also work with the algorithm?

LC: Uh huh.

I: It’s a miracle. Okay, how about if we have. . . Take those off. (RV removed the bills from the number line.) And we’ll have negative one (RV put one red bill on the number line.) subtract negative six.

LC: Negative one. (LC wrote the problem on her paper.)

I: Subtract negative six.

RV: We’re going to have to add six zeroes. (RV put six more red bills on the number line as LC put six white bills on the number line.)

I: Is that what you were about to say LC?

LC: Nope.

RV: Um. So then I’m taking off the six here (on the negative side of the number line).

Yeah. (She removed six red bills from the number line.)

I: Because you’re subtracting negative six.

RV: And then we have a zero there. So it’s five.

I: Okay, LC, show us how you would have done that one. Let’s take those off. (LC removed the bills from the number line.) Let’s see the LC way of doing this.

LC: Um. Well, I don’t know how I’d show it on the number line. I would just. . .

I: Well, let’s try it.

LC: Okay.

I: Negative one subtract. . .
LC: Oh, okay, I get it.
I: Negative one subtract negative six.
LC: (She put one red bill on the number line.) I would take negative one and I would make it into an addition problem. Is that bad?
I: Well, then it’s not showing “take away”. I want to see the take away.
LC: Oh, well, I don’t know how to show that.
I: Oh, okay, then we won’t worry about what I thought you were going to do. Okay, well, I happen to have some other bills.
LC: No.
I: Oh yeah. Other bills. These don’t have Velcro and I don’t have a number line to go with them. (placing the bills on the table) But what do you notice about those bills?
LC: They’re like real money.
I: (looking at the bills that they have been using up to this point) So are these.
RV: Oh, these are. . . These have. . . It’s ones, tens, and fifties.
I: Big bills, right?
RV: Yeah. Twenties and fives.
LC: I like playing with the real money.
I: I do too.
RV: Green ones.
I: So how would I model thirty subtract fifty with that money?
RV: Here’s a fifty. (She took a red $50 from the stack of money.) Maybe I could . . .thirty (She put down a $20 white bill and a $10 white bill.)
I: Is that thirty subtract fifty?
RV: No.
I: Just checking. How would I show thirty subtract fifty?
RV: (putting her hands on the $20 white bill and the $10 white bill) There’s thirty. So now (long pause) (LC grabs the stack of bills from RV.) Go ahead. (LC took two $10 red bills along with two $10 white bills.)
LC: Something like that.
RV: Wait. What?
LC: I don’t know. I added zeroes. Oh wait. Thirty subtract fifty.
RV: Yeah.
LC: So now we can subtract fifty. (She removed the red $50 bill.)
I: Is that subtracting fifty?
LC: No. I’m confused. (She pushed the bills to RV.)
RV: Alright. So then . . . (She removed the red $50 bill.) I don’t think we needed that one.
I: We’ve got a ten, a twenty, a ten, a ten (white bills), red ten, red ten.
RV: There’s thirty (pointing to the white $20 and $10 bills). We added two (meaning two tens worth of zeroes). Subtract fifty (She removed the $50 of white bills.) And we’re left with negative twenty.

LC: (She saw that I am looking at her to get her reaction to RV.) What she said.

I: I’m going to let you do the next one. Okay. So you don’t have to write those down. You can write them on the back if you want. How about twenty dollars of debt plus thirty dollars of money to spend? (LC put three white $10 bills in front of herself. She also put a red $20 bill and a red $10 bill out. She canceled the red $20 bill with two of the white $10 bills.) Okay, and you’re left with. . .

RV: Ten dollars.

I: Ten dollars. Okay. How about ten dollars of debt (LC put a red $10 bill out.) plus five dollars of debt? (She put out five red $1 bill.) And that would give you. . .

RV: Negative fifteen.

I: Okay, or fifteen dollars of debt. True? (RV and LC nodded in agreement.) Okay. How about fourteen dollars of money to spend (LC put down a white $10 bill and four white $1 bills.) subtract two dollars of debt?

RV: So wait. Don’t you have to add . . . You have to add two. (She put two red bills with the other bills. LC picked up two of the white $1 bills.)

LC: No.

RV: Oh no. Wait.

LC: And you just take away (She removed two white $1 bills.) They cancel each other out.

RV: Yeah, so twelve.

I: So, fourteen subtract negative two is twelve?

LC: No, you’re right.

RV: Now wait. (She put two red $1 bills on the table. No, you have to add two more (meaning that LC should have put two more white $1 bills on the table. LC then looked through the stack of bills to find two more white $1 bills. She placed them on the table.) So you add those two and I take away these (She removed two red $1 bills.) and we’re left with sixteen.

I: Does that make sense LC? (LC looked confused.) Guess who gets to do the next one?

Okay, take those away. One hundred twelve dollars of money to spend (LC put a white $100 bill, a white $10 bill and two, white $1 bills on the table.) Okay, there’s your one hundred twelve dollars. Subtract twenty-three dollars of debt. (LC put a red $20 bill and three red $1 bills on the table.)

RV: No, you have to. . .

I: Can you just add in negative twenty-three?

LC: No. Oh, I get it. Wait. No, I don’t get it. What was the. ..
I: What’s the only legal thing...  
LC: Let me write it down.  
I: Okay. One hundred twelve dollars of money to spend subtract twenty-three dollars of debt.  
RV: Subtract negative twenty-three. (LC had written positive twenty-three on her paper.) (LC put a white $20 bill on the table.) You have some more here. (LC was looking for more white $1 bills in the stack of money. She then put three of the white $1 bills on the table.)  
I: Do you understand what you’re doing, LC?  
LC: No.  
I: Okay. She... You needed to take away twenty-three dollars of debt.  
LC: Uh huh.  
I: In order to do that you needed twenty-three dollars of debt to take away.  
LC: Okay.  
I: So, the only legal thing that you can add to a number and not change its value is what? What can you add to a number and not change its value?  
LC: Zero.  
I: Zero. So, by putting that negative twenty-three in there, you also had to add in positive twenty-three.  
LC: Okay.  
I: Now you can take away your twenty-three dollars of debt (LC removed the red $20 bill and four red $1 bills).  
LC: Okay.  
I: Does that make sense?  
LC: Uh huh.  
I: Not to worry. Let’s try this one. (LC removed the bills from the table.) How about thirteen dollars of debt (RV put a red $10 bill and three red $1 bills on the table.) subtract twenty-six dollars of debt?  
RV: Subtract twenty-six so (LC put a two red $10 bills and a red $1 bill on the table.) Oh, wait, you need a five.  
LC: Uh huh. (RV found a red $5 and LC put it on the table with the other bills.)  
RV: Now you need twenty-six. (LC found two white $10 bills, a white $5 bills, and a white $1 bill to put on the table.)  
I: So now you want to take away twenty-six dollars of debt, (long pause) LC.  
LC: Uh. So (she removed the two red $10 bills, the red $5 bill, and the red $1 bill from the table.) Take away. Wait. It’s adding. I’m so confused.
I: Okay. You had what? Thirteen dollars of debt, right? And you want to take away twenty-six dollars of debt. (RV put her hands on the two red $10 bills, the red $5 bill, and the red $1 bill.)
LC: Uh huh.
I: Do you have twenty-six dollars of debt that you can take away?
LC: No.
I: You didn’t before, so you added in twenty-six dollars of debt and twenty-six dollars of money to spend.
LC: Right.
I: Because that was adding zero.
LC: Uh huh.
I: Now, can you take away your twenty-six dollars of debt?
LC: Yes. (RV removed the two red $10 bills, the red $5 bill, and the red $1 bill.)
RV: And then what you need to do is you have to take away thirteen from that side (pointing to the white bills). So pretty much . . . (She replaced white $5 bill for five white $1 bills.) Take away thirteen from there. (LC removed a white $10 bill and three white $1 bills.)
I: You see why you’re taking away thirteen, LC?
LC: Uh huh.
I: You sure?
LC: Yeah.
I: You just don’t want me to give you another problem.
LC: No, I don’t care.
I: Okay, one more just for “funsies”. So you can figure out how much money you had left, right?
LC: Yeah. (RV removed the bills from the table.)
I: Okay, let’s do this one. Let’s take two dollars of money to spend. (LC put two white $1 bills on the table.) Subtract thirteen dollars of debt. (RV located a red $10 bill and three red $1 bills.) What are you going to do LC?
LC: Add thirteen dollars over here. (She and RV have separated the table into a “number line like” model. LC put a white $10 bill and three white $1 bills on the table.)
I: Because I am. No, because you have . . . because you can’t take away thirteen that’s not there.
I: Okay, so she put in thirteen of the red bills, which is debt. You put in thirteen of the money to spend, which balanced it out
LC: Uh huh.
I: So it became zero. And now you’re going to . . . Did you already subtract your thirteen dollars of debt?
RV: There’s your thirteen (as RV removed the red $10 and three red $1 bills.)

I: There’s your thirteen dollars of debt. She’s subtracting those.

LC: Oh. I get it now.

I: And you’re going to be left with . . .

LC: Okay, uh, fifteen dollars.

I: Okay. Does that make sense?

LC: Yeah, I got it.

I: Okay. On the back of your paper, I want you to write down, since you’re all done with

this experiment, . . .

I want you to write down did this help make things more clear or did it confuse you?

What, if anything, was valuable about this? Um, you know, did the color help or hurt?

Did the number line help or hurt? Was it a combination of those two things or what do

you think? Cause I really appreciate you guys doing this.

LB & BH

LB & BH

I: I have some stuff in here (opening a container that contains red bills and white bills). I
want you to tell me if you’ve ever seen any of those ( A white bill was placed on the

 table).

LB: No.

BH: No.

I: You don’t know what that is?

LB: Well, it’s a fake dollar bill.

I: So you’ve seen some money before? You’ve had some experience with money?

LB & BH: Oh yeah.

LB: Just a little.

(Putting a red bill in front of the participants) I: I also have that. What do you notice

about those things that I showed you?

BH: They’re all the same except they’re different colors?

I: How many different colors?

BH: Two.

I: Why do you suppose that I would have two different colors of money?

LB: So you could tell the difference in what you were doing.

I: In what way?

LB: Whatever you’re using it for, you could separate your uses?

I: What sorts of uses would you have?

BH: If you’re teaching someone how to subtract.
I: Well, suppose that we have this thing (rolling out the number line). Have you ever seen one of these?

LB: Number line.

I: So you’ve seen that before.

LB & BH: Yeah.

I: In what sort of stuff did you see a number line?

BH: I’ve seen it in my math classes as well as in my kids’ math classes.

LB: I second that.

I: What do you notice about that number line that is different from other number lines that you’ve seen before?

LB: It’s color coded.

I: How is it color coded?

LB: Negatives are red; positives are white.

I: And do you think that has anything to do with those two different colors?

BH: You can use the white bills for the positives and the red bills for the negatives.

I: As you work with this model I want you to tell me how this might be different from other models that you’ve used. How would model four of the red dollars on the number line?

(BH placed a red bill at –4, LB placed a red bill at –1, BH placed a red bill at –3, and LB placed a red bill at –2.)

I: In real life how would you think of that? Would that be a good thing or a bad thing?

BH & LB: It would be a bad thing.

I: What would it be like?

LB: In your checkbook. Negative. That’s what I equate it to.

BH: Owing somebody four dollars.

I: Ok, owing somebody four dollars or four dollars of debt. Take those off. How would you model four dollars of money to spend?

(LB placed a white bill at 4; BH placed a white bill at 1; BH placed a white bill at 2; and LB placed a white bill at 3.)

I: Take those off. How would you model zero dollars?

BH: Just don’t put anything on it (meaning the number line).

I: Is there another way you could model zero dollars?

(LB placed a white bill on the number line as BH placed a red bill on the number line.)

I: Is there another way you could model zero dollars?

(LB placed another white bill on the number line and BH placed another red bill on the number line.)

I: So what do you notice. How are you modeling zero?

BH: You have to have the same amount of negative as positive.
I: Take those off. When you did that were you concentrating on the color of the money or on the number line?
LB & BH: Colors of the money.
I: Which is greater: four dollars of money to spend or six dollars of debt?
LB & BH: Four dollars of money to spend.
I: How do you know?
LB: Because negative is bad and positive is good. If you have positive, you have more and if you have negative, then you’re short.
BH: I would say it’s farther to the right of the zero.
I: So in that case you’re using the number line more than the color of the money?
BH & LB: Yes.
I: You’re using the colors but you’re also using the number line, right?
LB & BH: Yes.
I: If you’re focusing on one or the other (number line or color) at any time, I want you to let me know.
What’s meant by absolute value?
BH: That is always a positive number. Like if you have the absolute value of negative four, that’s positive four.
I: So the absolute value of negative four is four. What is that in terms of this number line?
BH: Would it be the number that makes it zero?
LB: Come again.
I: If you’re looking at the absolute value of negative four, you said that’s four.
LB: Correct.
I: Is there another number that would have an absolute value of four?
BH: Just four.
I: And what do you notice about four and negative four?
LB: They’re equidistant from each other (as she placed her hands out from four to negative four).
I: So how would you define absolute value? (There was a long pause.) Since you said that the absolute value of four and negative four are the same, how could you say what absolute value is?
BH: Would it be the number that brings you to zero? Like opposites of each other? I don’t want to use the word opposite.
LB: I think it would be whatever the number, it’s the true number of whatever you’re trying to find. It’s the base number of what you’re trying to find.
I: Or just the number part?
LB: The number part. Yes. No matter what the sign is.
I: Which has a greater absolute value: three dollars of money to spend or two dollars of debt?

BH: Three dollars.

I: How did you know that?

LB: Because three is greater than two.

I: So you didn’t have to put anything on the number line, you didn’t have to look at colors of bills, you just knew that?

LB: Yeah.

I: Which has a greater absolute value: four dollars of money to spend or seven dollars of debt?

BH: Seven dollars of debt.

I: Let’s see if you can show three dollars of debt on the number line.

(BH placed a red bill at –3; LB placed a red bill at –1; and BH placed a red bill at –2.)

I: Can you show three dollars of debt another way?

(LB put three red bills at 1, 2, and 3. BH put three red bills at –4, -5, and –6. LB then placed another red bill at 4.)

BH: You have to have three there (meaning on the positive side of the number line) and six here (meaning on the negative side of the number line).

LB: That’s right. I’m sorry. (She took off the red bill that was at 4.)

BH: (pointing to the red bills on the positive side of the number line) You need these to be white?

LB: If we’re doing how many dollars of debt?

BH: Three (she pointed to the three red bills at –1, -2, and –3.) You need three dollars of money to spend.

LB: Yep. (She removed the three red bills at 1, 2, and 3.)

(BH put a white bill at 1, LB put a white bill at 3 and another at 2.)

I: Which did you use more, the color of the money or the number line?

BH: I was using the number line. (LB nodded in agreement.)

I: If you look at that, what do you notice, how do you know that’s the same as three dollars of debt?

BH: (pointing at a red bill and a white bill) One of these would cancel each other out.

LB: Uh huh.

I: Because those three are really three what?

LB: Three values.

I: Three values of . . .


BH: Ones?

LB: Ones. I’m grasping at straws.
BH: (pointing to a red bill and a white bill on the number line) You mean like these are the absolute value?

I’m not sure what you’re asking.

I: Show me again the three dollars of debt just with red bills.

(LB and BH took off three red bills and three white bills from the number line.)

I: And then what did you do to show another way of showing three dollars of debt?

BH: (As they placed three red bills and three white bills on the number line) We added three red bills and three white bills.

LB: And the three white bills cancel the red bills out. Therefore, you have three dollars left of red bills which is on the left part of the number line.

I: So you could actually show that as showing a positive and a negative off to make . . .

What is it when you take a positive and a negative off?

BH: You mean zero?

I: Can you show three dollars of debt another way?

BH: You mean besides the first way?

I: Yes.

BH: So you’d add three (meaning three white bills) and I’d add three (meaning three red bills). (LB placed three white bills on the number line and BH placed three red bills on the number line. LB then placed three more white bills on the number line as BH placed three more red bills on the number line.)

LB: One, two, three. One, two, three.

I: Does it always have to be threes that you put on?

LB: Yes, because if you’re going to cancel three out with three others, then it takes three to cancel three out.

I: Could you put just one more red bill on the number line and have the same value?

BH: You’d have to put a white bill with it.

LB: It’s a one-to-one correlation.

I: Before you were putting three on at a time. Did you have to put three on at a time?

Three positives and three negatives.

LB: To start out.

I: When you had the three dollars of debt, and then I said “Show it another way”, did you have to put on three more dollars of debt and three more dollars of money to spend?

BH: Actually, no. You could have just added one. Right? (looking at LB) If you have four here (referring to the red bills) and you have one there (referring to a white bill), we’d still be three dollars in debt.

LB: Because four minus one is three.
I: So you could just add as many zeroes as you need to have the same value for the number. Take those things off. (LB and BH removed the bills from the number line.)
How would you model two dollars of money to spend?
(LB placed two white bills on the number line.)
I: (to BH) Do you agree?
BH: Yes.
I: Could you show it another way? So show two dollars of money to spend another way.
(BH placed a red bill on the number line as LB placed a white bill on the number line.)
Remember to tell if you’re looking at the number line or if you’re looking at colors.
BH: I’m looking at the number line.
LB: Uh huh. Number line.
I: Could you show two dollars of money to spend another way?
BH: You want to add two and I’ll add two. (BH placed two red bills on the number line as LB placed two white bills on the number line.)
I: So, in effect, what are you putting on there to keep the same value?
BH: The same amount on both sides of zero.
I: Which is really pairs of positive and negative which is really. . .What is a positive and a negative?
LB: Zero. I just remembered that.

Phase 3 Session 2 LB & BH

I: What are integers?
LB: Do we answer that or do we write?
I: You can write. You can write and then answer. Whatever.
BH: I think they’re whole numbers.
I: Whole numbers?
LB: We just went over this too. They're numbers that can be negative, positive or . . .I always want to call them fractions too but that's not it.
BH: Are fractions integers?
LB: No, those are irra…rational numbers.
BH: I keep wanting to say like the set that includes. . .
LB: Exactly. No, they're counting numbers. Those are whole numbers. You can't have zero, negative, or positive.
BH: Negative one's still an integer, though.
LB: Yeah, but if the whole numbers include zero, the counting numbers do not. That’s what I said.
(Both wrote what they think integers are.)
I: So what are integers?
BH: I put the set that includes the whole numbers, negative or positive.
LB: I put whole numbers on the number line that can be positive or negative.
I: Okay. Where are integers used?
LB: Where?
BH: Everywhere.
I: Can you give me some practical examples of where you'd use integers?
LB: Balancing your checkbook.
I: On the thermostat.
BH: When you're driving, your speed.
I: When were you first introduced to integers? When did you first study them?
BH: You mean actually using them or dealing with? Learning to count.
I: OK.
LB: Yeah, but did you even know what a negative number was when you were learning
to count?
BH: No. I just used counting numbers. So probably in first grade when we were learning
how to subtract.
LB: That would have to be it. See that's the thing. We didn't even know that they were
called integers.
I: What were they called then?
BH: Just numbers.
I: So how were you first introduced to negative integers?
BH: I guess it would be just like if you had 10 apples and take away five.
LB: Do you mean when we knew that an integer was an integer?
I: When were you first introduced to negative integers? (Participants wrote on notebook
paper.) What does addition mean?
BH: Adding one thing to another.
I: (To LB) Do you agree? (LB nodded in agreement.)
BH: Joining things.
I: How do you add integers?
LB: You add everything that's in the ones place, and then you add everything the tens
place and then you add everything that's in the hundreds place.
I: (To BH) Do you agree?
BH: Yeah. Just combine the two numbers together.
I: Give me an example of an addition problem that deals with integers.
LB: Two plus two.
I: And you would just add the two and the two? (Both participants nodded in agreement.)
What if you had two negative integers that you were adding?

I: OK. Give me an example.

LB: Then it would be a negative number in the end.

I: Negative two plus negative two.

LB: Negative four.

I: And you would get . . .

LB: Negative four.

I: What does subtraction mean?

LB: Take away.

BH: That’s what I think of.

I: How do you subtract integers?

LB: You have a whole number and you take away a certain amount from that whole number.

I: Give me an example.

LB: You’ve got 10 apples,

BH: Take away five.

LB: And you take away five. You’re subtracting. You’re making the five a negative number. Ten minus 5 equals five apples are left.

I: What if you had something like positive three subtract negative two?

BH: You’d add them.

I: Three subtract negative two.

LB: So you’ll have one left over. Because three is a positive number, a whole positive number, and then you’re subtracting two away from that whole three. So when you take two away from that three, you’ll have one left.

BH: It was three minus negative two. (LB had written $3 - (-2) = 1$)

LB: So then you would add them.

I: So why did you say you would add them, LB?

LB: Because you change the . . . since you have two negatives; you can’t have two negatives, it’s a double negative kind of like in English. You have to add them together and then make the negative a positive because you can’t have two negatives. So that means three plus positive two equals positive five.

I: And that’s what you were taught?

LB: Yes.

I: And that made sense to you?

LB: After a long time. Not right away no. It was a rule you had to follow. It was
accepted because it was written by somebody who knew more than me.

I: So when you were adding and subtracting with integers, what would you say was your
reason for having an answer? You figured out the problem, you had an answer. Did it
make sense all the time?

LB & BH: No.

I: So you just, I mean, how did you know it was right then?

(Long pause) LB: Well, we learned our basic facts of addition and subtraction in grade
school. So you take your basic facts and you tweak them in to fit whatever operation is
required.

BH: I'd like to know why you add when you have two negatives so I can tell my
daughter.

LB: I can't tell you. It's one of those things that I refuse to ask why anymore because it
confuses me. I just accept it. I learned that the hard way.

(Placing the number line on the table in front of the participants)I: So we are going to
review a little bit of what you did before. I want you to model three dollars of debt on the
number line. (The participants put three red bills on the number line.) Is that the only way
that three dollars of debt can be modeled on the number line? (LB & BH shook their
heads.) Let's see another way. (LB placed another red bill on the number line as BH
placed a white bill on the number line.) And we could probably even do it more ways
than that?

LB: Yes.

I: What is it that you added on the number line without changing the value of your three
dollars of debt?

BH: As many as you put on the positive side you have to add to the negative side.

I: And what is that that you're adding in each case? A positive one and a negative one
makes

BH: Zero.

I: And you can add zero to any amount and not change the value.

LB: Correct.

I: Let's take those off. How would you model zero dollars on the number line?

BH: You could do (placing a red bill and a white bill on the number line).

I: Is that the only way you could do it?

LB & BH: No.

I: How else could you model zero on the number line?

BH: As long as you have a balanced amount on both sides. (BH and LB placed another
red bill and another white bill on the number line.) You could do it all day.

I: Cool. Let's take those off. Turn your paper over and I want you to create three columns
evenly spaced and the heading for the first column I wanted to write “problem”. Second
column, I want you to write “solution”. And the third column “number of zeros”.

BH: Number of zeroes?

I: Yes. It makes absolutely no sense at this point, but that's okay. Hopefully it will. So we’re going to look at four dollars of money to spend plus two dollars of money to spend. What would that problem be?

BH: Four plus two.

LB: Yeah. That's what you want us to write?

I: Yes. Then I want you to show it on the number line. (BH put four white bills on the number line and then two more white bills on the number line.) And your answer would be?

BH & LB: Six.

I: And where there any zeroes, any cancellations? (BH & LB shook their heads.) So that last column will be zero. Take those off. What if we have three dollars of debt, plus four dollars of debt? And let's see it on the number line. (LB placed three red bills and then four more red bills on the number line.) Does that make sense? And you had zero number of zeroes. (Each had already written this on their paper.) Let's take those off. Let’s see five dollars of money to spend plus seven dollars of debt. So one of you do the five dollars of money to spend and the other one do the seven dollars of debt. (BH put five white bills on the number line and LB put seven red bills on the number line.) Is that your answer?

LB: No, because then what you'll do is pick up one of each (meaning a red bill and a white bill) until... because you have more debt than you have real money.

I: And so how many of those things... How many zeroes did you pick up?

BH: Five.

I: Five. And you understand that? (BH & LB nodded in agreement.) So your answer was negative two.

LB: Yes. (She erased what she had on her paper.) I had two (meaning number of zeroes). I wasn’t thinking of the problem; I was thinking of the answer.

I: Let's take those off. How about two dollars of debt plus one dollar of money to spend. (LB placed two red bills on the number line as BH put one white bill on the number line.)

BH: So we take away these (She took off a red bill and a white bill from the number line.)

I: And you’re really taking those off because they are... 

BH: Zeroes.

I: So that you're left with just one color of money for your answer. And your answer’s going to be

LB & BH: Negative one.

I: OK, let’s take that off. I. Let's see six dollars of money to spend plus five dollars of
debt. (BH put six white bills on the number line and LB put five red bills on the number line.)

BH: And take away five zeroes (as she and LB removed five red bills and five white bills from the number line).

I: So your answer was one. Let LB do this one all by herself. Take that one off. Four dollars of money to spend plus six dollars of debt. (LB put four white bills and six red bills on the number line. She then removed four red bills and four white bills from the number line.) And how did you know how many to take off?

LB: Well you have positive four and negative six so you have to subtract them because you cannot add them because negative six is larger so you have to subtract them and you end up with negative two because six is negative and it’s bigger.

I: Negative six is bigger?

LB: Negative six is bigger than positive four. Yes. Well, sorry, in the negative . . . the number that has the negative sign that’s larger ends up being the sign of the answer.

I: So you’re really looking at the . . .

LB: Integer?

BH: Is that like the absolute value?

I: Absolute value.

LB: All right, there we go.

I: What I want you to do now is look at that column where it says “Number of zeroes”, so for that last one how many zeroes did you have, LB?

LB: Eight. One, two, three, four, five, six, seven, eight. (She was counting the four red bills and the four white bills.) No, sorry. Yeah, eight.

I: Eight.

LB: Yeah, cause I had four over here (pointing to the positive side) and six here. Four and four is eight. What am I doing wrong?

BH: The groups, you want to group them to make a zero and then (pointing with one hand on the positive side of the number line and the other on the negative side) One group, two groups, three groups, four groups.

LB: Oh, I see what you’re saying. So I’m not doing it individually; I’m doing it as a group.

I: How many zeroes did you have for the problem before that one?

LB: Oh that was. I didn’t have anything before that. Six minus five is one so that means I took off one, two, three, .. I took off one group.

I: (to BH) Is that what you had?

BH: Uh huh.

I: So now I want you to look at that last column and see if you can relate that to the rule that you used for adding integers. See if that relates at all to what you were doing. What
is the rule for adding integers. When you added those... when you had something like
four plus negative six, both of you knew immediately that your answer was negative two.
How did you know that and can you find some way that the last column would help you?
(There was a long pause as BH and LB looked at their papers.)
I: When do you have zero zeroes?
LB: When either the answer is positive or you have an equal number of...
BH: When both you addends are negative or positive... the same sign?
I: Do you agree LB? See, if you look at your next to the last problem, you have a positive
answer.
LB: Uh huh.
I: But you didn’t have zero zeroes did you?
LB: No.
I: So do you agree with BH’s idea?
LB: (emphatically) Yes.
I: OK. When do you have some number of zeroes then? That you have to neutralize a
positive and a negative?
LB: When one of your...when only one of your addends is negative.
I: And the other one is?
LB: Positive.
I: How many zeroes did you have and can that be related to the problem in any way?
BH: Oh. It made sense for a minute but never mind.
LB: Well, if you have either both addends are positive or both addends are negative, you
will have the number of zeroes be zero. But if you have one addend that is different from
the other you will always have . Sorry, always is a bad word. You will most likely have a
zero. At least one.
I: And how many zeroes? Can you relate that number of zeroes to the numbers that you
are adding?
BH: If your negative number is larger ... the absolute value of your negative number is
larger then you have the same number of zeroes as your positive number.
LB: What about in the second to last one though? That’s the only difference.
BH: Because then the absolute value of the larger number was a positive number is
bigger than the negative number. Then you would just subtract them.
I: So let’s state your theory here again.
BH: If your absolute value of your negative number is larger than your positive number
your number of zeroes is going to be equal to your positive number. Or am I over
thinking this a lot.
LB: Say it again. No, say it one more time. It takes me sometimes a while.
BH: Ok. If your absolute value of your negative number is larger than your positive
number,
LB: (She repeated and looked at her paper trying to make sense of it) Absolute value of
your negative number is larger than your positive number
BH: This one and this one. (BH was pointing to numbers on LB’s paper to explain what
she meant)
Well this here the number of zeroes is one. (LB had an incorrect number of zeroes for one
of the problems.)
LB: Well that’s maybe why I screwed it up. (She corrected her mistake.)
BH: Then your number of zeroes is equal to your positive number.
LB: Except for the second to last one we get
BH: That’s because six is larger than the absolute value of five.
LB: OK. I just want to make sure that I was totally understanding what you were saying.
I: What happens if the absolute value of the positive number is greater than the absolute
value of the negative addend?
BH: Then it’s just the difference of the two.
LB: I agree with that. Yeah, because the difference between six and five is one, no matter
whether they’re negative or positive.
I: And does that relate to the way that you add integers? When you have something like
negative seven plus positive five
BH: Oh.
I: You know that you would have how many zeroes?
BH: Five.
I: And how do you find your answer? What is your answer?
BH: Negative two.
I: Because you. . .
BH: Got rid of five zeroes.
I: So looking at the negative seven and the five, how did you get, first of all the two?
BH: I guess seven minus five?
I: Is two. And how did you know it was negative?
LB: Because the larger addend is negative.
I: Is negative seven greater than five?
LB: No. But the addend itself is larger.
I: The addend’s negative seven.
LB: OK, the absolute value, dang it, If the absolute value of your negative number is
larger than your positive number, then that equals the same amount of zeroes that you
pick up. Is that right?
BH: Close. The amount of zeroes is the same as what number?
LB: The amount of zeroes equals your positive number. Or the absolute value of your
I: So you’re saying if the absolute value of the negative one is greater than the positive one then you’re going to take the difference of those absolute values and that’s going to be the number part. How do you determine the sign?

LB: By the non-absolute value. By whichever integer is . . . I can’t say that. Whichever integer is larger. That’s what I want to say but it’s not true. Whichever absolute value is larger.

I: So if we’re adding and the signs are the same, our sum is going to be what?

BH: Just combine the two numbers.

I: So you just add the two numbers and give it what sign?

BH: Whatever sign they both have.

I: So if they’re different signs, how do you determine the number part for your answer?

BH: Take the difference of the absolute values.

I: And what sign do you give that sum?

LB: The absolute value of the larger one.

I: Now, I’d like you to write about this model. Does it make it more clear about what you’re doing or does this model mess you up?

Phase 3 Session 3 LB & BH

I: I’d like you to model two dollars of debt. (LB put two red bills on the number line.)

Can you model it a different way? (LB started to remove a red bill as BH started to put on four white bills.)

LB: Oh well, I’ll just leave that one on (referring to the red bill she was about to pick up).

I: So, is that two dollars of debt?

BH: Oh, no. (She removed two white bills from the number line.) I’m doing this backwards. I’m sorry, I’m still sleeping. (She then put more red bills on the number line.)

LB: Yeah I don’t know what . . . (she started to laugh)

I: Can you do it a different way?

BH: Do you want to add one more white one? (BH put another white bill on the number line and LB put another red bill on the number line.)

I: And take those off. And let’s model zero. (BH put a white bill on the number line and LB put a red bill on the number line.) And another way. (LB placed another red bill on the number line and BH placed another white bill on the number line.) And that makes sense?

LB & BH: Yes.

I: Let’s take those off. What is addition?

LB: One addend plus one addend equals a sum.
I: Okay.
BH: I was thinking the same thing.
I: Exactly. And how would you model addition, not necessarily on the number line but can you give an example of an addition problem?
BH: Two plus four equals six.
I: Okay. And you would know how to model that.
BH: Uh huh.
I: How would you model negative five plus four on the number line or five dollars of debt plus four dollars of money to spend? (BH put five white bills on the number line and LB put four red bills on the number line.) And would that be your final answer?
BH: Five dollars of debt and four dollars of money to spend you say?
I: Okay. And you would know how to model that.
BH: Uh huh.
I: And so, what would be your final answer?
BH: Our final answer would be (She removed four white bills as LB removed four red bills from the number line.)
I: And so, your answer would be . . .
BH & LB: Negative one.
I: Okay. Or one dollar of debt. How is the number of zeroes . . . Remember when we did the chart with the number of zeroes? You had the problem, the solution, and the number of zeroes.
LB & BH: Uh huh.
I: How was that number of zeroes related to the addition algorithm? Do you remember?
BH: You mean like that theory?
I: Yeah. Let me get out your papers here (giving back the papers that were done in the previous session).
BH: How’s it related to zero?
LB: Oh, I was thinking in class. I’m going “What? What’s she talking about?” And then I just remembered. And what was your question again? How was the number of zeroes . . .
I: Related to the addition algorithm. Remember when you did the chart on the back?
LB: The number of . . .
BH: Oh yeah.
LB: It indicates your positive (pause), yeah, your positive addend.
BH: That would be that the . . .
I: Do you agree BH that the number of zeroes is the positive addend?
BH: No. It would be equal to the one whose absolute value is . . .
LB: That’s right. Absolute value.
BH: The absolute value of, wait, the negative number is larger than the absolute value of the positive number it’s equal to the positive number.

I: And is that always going to be the case?

BH: No. If the absolute value of the positive number is greater than the absolute value of the negative number, um, then the number of zeroes is the difference between the two absolute values.

I: (to LB) Do you agree?

LB: Yes. I just said it in my head as you were doing it.

I: So what is subtraction? (Long pause) You told me what addition is, what is subtraction?

BH: The difference between two numbers.

I: Okay. And let’s put those papers on here (referring to last week’s papers that the participants did). And on your paper I’d like you to create three columns. One column that says “Problem”, another one that says “Solution”, and “Number of Zeroes”. So the first one that I’d like you to do is two dollars of debt subtract four dollars of money to spend.

LB: And four dollars of money to spend, did you say?

I: Yes. And I’d like to see it on the number line.

BH: Four dollars of debt (LB put four red bills on the number line.) (BH started to put four white bills on the number line.)

I: What are you doing, BH?

BH: Um, I don’t know. (She removed the four white bills from the number line.)

I: What did you tell me that subtraction was? How do you model subtraction?

BH: The difference. So that would be negative two.

I: How? How could it be negative two?

BH: Oh. (She was very confused at this point.)

I: How do we usually model subtraction? What’s the easiest way to look at subtraction?

If you have a problem like six, not that this relates to that problem, but if you have six minus four, what do you think of?

BH: Two.

I: Okay, and how do you know that?

LB: Because if you take four away from six, . . .

I: Think about what you just said.

BH: Okay. Take away four from negative two you get negative six.

I: Okay. So there’s your negative two (on the number line) or two dollars of debt. I want you to take away four dollars of money to spend.

LB: You can’t.

BH: Yeah, I still think it will be negative six.
LB: Is that right?
I: But why?
BH: Because when you have the negative sign it makes that negative.
LB: So you have four dollars of money to spend and two dollars of debt. Correct?
I: You want to take four dollars of money to spend from your two dollars of debt.
BH: You’ve got to reverse that (referring to changing the subtraction sign to addition on LB’s paper). It’s negative two minus
LB: That’s what I had the first time. And then I tried negative two plus four?
BH: No. Negative two minus plus four so that negative and the plus makes that a negative, I think.
I: Does that make sense to you?
BH: I just know the rule about a negative and a positive is negative.
I: Okay. Did it make sense to you when you first learned it?
BH: No.
I: So does it make sense to you now?
BH: It’s starting to.
I: If you have two dollars of debt, (pointing to the number line with the two red bills) there’s two dollars of debt.
LB & BH: Uh huh.
I: Can you take away four dollars of money to spend as it is right now? (Pause) Do you see four dollars of money to spend that you can just take away?
BH: Well, you mean without it being here (She pointed to the positive side of the number line.)
I: Yeah.
LB: But why would you write it negative two minus positive four? Why wouldn’t you just take four minus two?
BH: Cause that’s different.
LB: But if you have four positive amounts of money to spend, you have four dollars of extra money, and you have two dollars of debt so you take your four dollars of money to spend. You pay off your debt. How much are you left with?
BH: Negative two.
LB: No, but if you have four of these (She picked up some white bills.) You know what I mean? (She placed four white bills on the number line.)
I: Where are these four coming from? (referring to the four white bills)
LB: Cause you said we have four dollars.
I: I want you to take away four dollars of money to spend.
BH: But if you take away even more money to spend you’re still making yourself more in debt.
LB: So if you’re taking away money to spend, (She placed four more red bills on the number line.)
I: And what did you just do?
LB: I added four more dollars of debt.
I: Can you do that?
LB: (after a short pause) Yes.
I: What allows you to do that?
LB: The smart answer way or the real way? Like I’ve got a whole pile of them right here (picking up some red bills).
BH: Um, credit.
LB: Or an IOU. Um, because if you don’t have any positive money, if you don’t have any, yeah, positive money, then you can’t take from your positive amount and pay off your debt so therefore if you can’t do that then you must have more debt. (Long pause) So that’s wrong?
LB: I’m just trying to think logically about it.
I: Okay. Let’s take off those four red bills (the last ones that LB had put on the number line). Let’s take everything off. Let’s wipe the slate clean. Two dollars of debt. (LB put two red bills on the number line.) That’s what you have.
BH: Uh huh.
I: I want you to take away four dollars of money to spend.
BH: I think I’m getting hung up on the colors.
LB: Take away four dollars of money to spend.
I: So where do you want to take it away from? Where would you have money to spend? On what side would you have money to spend? (LB & BH both pointed to the positive side of the number line.) Do you have any there?
LB & BH: No.
I: How could you get some there without changing the value of what you have?
BH: Oh, you mean the value of the answer?
LB: Put down four whites. . .
BH: No, if I put . . .
LB: Or two whites I mean. No, four whites.
BH: Well, like if I had six dollars to spend and then I took away four and put them on that side (referring to the negative side of the number line)
LB: Yeah, put six. There you go. Put six there (BH put six white bills on the number line.)
I: So what are you really adding to that side?
BH: I’m adding six and then I’m going to take them away to . . . (LB put six more red bills on the number line.) Put them.
BH: Then I take away four. Wait. One, two, three, four, five, six. (She then took four white bills off the number line, so she was left with two white bills and four red bills on the number line.)

I: Take those off. I want to see two dollars of debt. (LB placed two red bills on the number line.) We all three agree that’s two dollars of debt.

BH & LB: Yes.

I: So you want to take away four dollars of money to spend. Do you see any money to spend that you can take away?

BH: No.

I: So think of . . .

BH: (very emphatically) Oh! I’m sorry. Because you’re not going to be . . . since negative then less than zero you can’t take away zeroes so it automatically has to go over here (pointing to the negative side of the number line).

LB: So you’re adding on the right track?

I: Remember the exercises we did where I said “What are different ways of showing these amounts?”

BH: Uh huh.

I: Are there different ways that you could show two dollars of debt?

BH: Yeah. Uh huh.

I: That would help you out?

BH: Okay, I have an idea. One, two (She counted the red bills on the number line.) You want to be two dollars in debt, right?

LB: Right. (BH placed five white bills on the number line, counting “one, two, three, four, five”) Add one more to that (pointing to the negative side of the number line). (LB placed another red bill on the number line.)

I: Now, what did you add on there?

LB: Why did you only add five?

BH: Oh no, wait. We’re three dollars in debt. Actually that’s not right. You have to add a couple more (referring to the red bills) because you have to have two more than I have.

LB: But why do you only have five?

BH: Because I have to take away four so I just figured it would be easier.

LB: Oh, Okay.

BH: So if you go to seven on that (meaning if you have seven red bills) that’s two dollars in debt.

LB: Is that correct? Oh, the light bulb actually gets plugged in!

I: And now what are you going to do?

BH: Take away four dollars of money to spend. (She removed four white bills from the number line.)
LB: Yes!

BH: Okay.

I: And is that your answer?

BH & LB: Yes.

I: What is your answer?

BH: (as LB was counting the red bills and then the white bill) Negative six.

LB: Yes.

I: Is that the way that you would normally show six dollars of debt on the number line?

Usually when we show the answer we have only one color left over, right? (BH took one red bill and one white bill from the number line.)

BH: Just cancel out a zero.

I: Okay. So, how many zeroes did you really have to add there?

LB: (counting the red bills) One, two, three, four, five. Cause we had two.

I: Did you really have to . . . Okay, you had two dollars of debt and I added these (referring to the additional red bills that were put on the number line) and we took away those four dollars to spend and then we also took away another zero so that’s why we have one less.

I: So how many zeroes did you really have to add?

LB: One.

BH: A minimum of, uh, two.

I: Let’s see it again. Take those off. (LB removed the bills from the number line.)

LB: We’re not doing very well.

BH: I’m very tired.

I: Two dollars of debt. (LB placed two red bills on the number line.)

LB: Two dollars of debt.

I: And I want to take away four dollars of money to spend.

LB: Hey, let’s put it this way. Instead of doing two dollars of debt like this (pointing to the two red bills) . . .

BH: Let’s show it another way? (She put two white bills on the number line.) And add two more (meaning two more red bills). (LB put two more red bills on the number line.)

LB: I was thinking four there (pointing to the positive side of the number line) and six here (pointing to the negative side of the number line).

I: Do you have four dollars of money to spend?

BH: No, I need to add two more. (She put two more white bills on the number line.) So you add two more (meaning that LB should put two more red bills on the number line).

LB put two more red bills on the number line.

LB: Yeah. Yeah. There you go. Now we have two dollars of debt.

I: How many zeroes did you really have to add?
BH & LB: Four.

I: Four. That’s what I want you to put down in that column. And your final answer was?

BH: Negative six.

I: Does that make sense?

LB & BH: Yes.

I: Let’s take all those off. (LB & BH removed the bills from the number line.)

BH: That was hard.

I: Hopefully it’ll get easier. One dollar of money to spend subtract one dollar of debt.

LB: So that would actually be . . .

BH: (placing one white bill on the number line) One dollar of money to spend. . .

LB: (placing one red bill on the number line) And one dollar of debt.

BH: To subtract this (pointing to the red bill). . .

LB: You add them both together. Because a negative plus a negative equals a positive. (Long pause) It’s one dollar of debt and you have one dollar. . .

BH: Take away one dollar of debt?

I: Take away one dollar of money to spend.

BH: Uh huh. That’s zero.

LB: (looking at her paper) Tell me the problem again, please.

I: One dollar of money to spend subtract one dollar of debt.

LB: So never mind. That’s right. No.

BH: I can see where you’re going.

LB: Cause you can’t have two negatives next to each other. They cancel each other out. (long pause) Because if you were to say “One, or one dollar to spend, subtract one dollar of debt. Well that’s already a negative one. So, you’re subtracting one minus and then you’re subtracting a negative dollar. So then you put it in parentheses and you subtract a negative dollar. So then you add those together.

BH: See I was looking at it as the word of taking it away the dollar just as a minus one.

LB: Normally I would have done that but she said “a dollar positive and then a dollar negative”.

BH: Okay. I can see that.

LB: But you’re subtracting. So that’s why I say that . . .

BH: It would be two dollars.

LB: Yes, that would be your solution. So that means that (She dropped her pencil in frustration.)

I: Is that what the number line is showing right now?

LB & BH: No.
I: Let’s take those off again. (LB & BH removed the bills from the number line.) Let’s try it again. One dollar of money to spend.

BH: (She placed a white bill and then another white bill on the number line.) (to LB) Why don’t you put a negative one there (meaning for LB to put a red bill on the negative side of the number line).

I: Subtract one dollar of debt. (LB removed a red bill from the number line.)

BH: Oh, you’re right LB.

LB: Subtract one dollar of debt.

BH: Cause then you have two dollars left.

LB: Uh huh. And you don’t have any zeroes (meaning that no zeroes need to be canceled after taking away the dollar of debt).

BH: Oh, wait.

LB: No, you have one zero. Oh, sorry. Sorry. You have one zero because we took away the one dollar. . .

BH: But we didn’t take one away from this side (pointing to the positive side of the number line).

LB: Oh, that’s true, so I was right the first time. (She changed her number of zeroes on her paper.) Shucky dogs.

I: But are we taking away zeroes or are we adding zeroes?

BH: We added one zero.

LB: Yes.

I: Okay. So that’s what I want to know.

BH: Okay.

I: The number of zeroes that you added.

BH: Okay, that would be one. I’m going to write “Zeroes Added” here (referring to column labeled “Number of Zeroes”).

LB: Yeah.

I: Okay, are we ready for the next one?

BH: This one took a lot less time that the first one so we’re making progress.

I: That’s right. Okay. Four dollars of money to spend. . . (BH placed four white bills on the number line.)

BH: Uh huh. (LB put a red bill on the number line.)

I: Wait. What are you doing?

LB: Well, I’m just showing it another way. (She removed the red bill from the number line.) Anyway, go on.

I: Subtract one dollar of debt.

BH: Okay, so wait a minute. (LB put a red bill on the number line and BH put another white bill on the number line.) Let’s start here (LB removed the red bill from the number line).
line.) We’re going to add . . . No, go ahead and put that on (referring to the red bill that
LB had removed). We’re going to add one zero to make up for that (pointing to the red
bill) and then we’re going to subtract that and we have five dollars. With one zero. Do
you agree?
LB: Doink. Doink. Five. One. (She was writing the amounts in the “Solution” column
and the “Zeroes Added” column.)
I: You both agree?
LB & BH: Yes.
I: Okay. Let’s take those off. (BH removed the bills from the number line.) Three dollars
of debt subtract two dollars of money to spend.
BH: Okay. Say, you have three dollars of debt (She put two white bills on the number
line as LB put three red bills on the number line.)
I: Where’s your three dollars of debt? (BH removed the two white bills from the number
line and looked at her paper.)
BH: I don’t know if I need to put any positive money . . .
LB: No, you don’t. (LB put two more red bills on the number line.)
BH: No, I don’t. Then subtract two dollars of debt. Wait. Why do you have five on there?
(Referring to the red bills)
LB: Three dollars of debt subtract two dollars of money to spend.
BH: No, no, no. (to I) Can you read that again?
I: Three dollars of debt subtract two dollars of money to spend.
BH: Okay, then I do need them. (She placed three white bills on the number line.)
LB: Then you’re going to subtract . . .
BH: It’s a positive two so just negative three minus two which is . . . Sorry. So we add
two zeroes. (She removed a white bill as LB removed a red bill from the number line.)
Wait a minute. (LB put the red bill back on the number line.) Negative three subtract two
dollars of money to spend . . .
LB: I have five (referring to the number of red bills on the number line).
BH: I have negative five.
LB: It will be negative five.
BH: You have five on there right now?
LB: Yes.
BH: If I subtract two dollars of money to spend (pointing to the two white bills). . . We
added two zeroes then, right?
LB: We will add two zeroes. Yes.
BH: Negative five and two is . . .
I: You both agree?
BH & LB: Yes.
I: Let’s take those off. (BH and LB removed the bills from the number line.) Let’s look at one dollar of debt subtract four dollars of money to spend. (BH put four white bills on the number line.) Where’s your one dollar of debt?

LB: I haven’t gotten there yet. (She put one red bill on the number line.)

BH: Sorry. You have to add four more zeroes.

LB: Yes. (She put four more red bills on the number line.) One, two, three, four.

BH: Wait a minute. Let me figure something out. (She wrote the number of zeroes and solution on her paper.)

LB: Okay. One dollar of debt subtract . . .

BH: No, that’s right.

LB: Four dollars of money to spend.

BH: You added four zeroes.

LB: (pointing to the red bills on the number line) One, two, three, four, five.

BH: (She removed the four white bills from the number line.) And I’m going to take away my four dollars to spend so that leaves us with negative five. (long pause)

I: LB’s confused. Let’s take those off and let LB do the whole thing. (BH and LB removed the bills from the number line.)

LB: No, LB’s not confused. LB just likes to just . . . I’m double checking.

I: Let’s let LB do it just to make sure that we know what’s going on. Let’s show one dollar of debt LB. (LB put one red bill on the number line.) And now I want you to subtract four dollars of money to spend. (LB put more red bills on the number line.) Oh well I didn’t put any on there (referring to the positive side of the number line) in the first place so . . . but if I put them down (She put four white bills on the number line.) . . . then I’m just going to have to take them all right back up. (She removed the four white bills from the number line.)

I: So that makes sense? You see how she got the four zeroes?

LB: Yes.

I: Let’s take those off. (LB removed the bills from the number line.)

LB: Sometimes it just takes me a little longer.

I: That’s fine. As long as you get it. Let’s show zero dollars subtract three dollars of money to spend. (LB put three red bills on the number line.)

LB: We didn’t add any zeroes.

I: Is that right?

BH: Well, it seems right but if I was looking at the, like the rest of them, I thought I had a thing figured out about how many zeroes there were and then it wouldn’t make sense. (She had written “3” for the number of zeroes added.)
I: So where did your three dollars of debt that you subtracted come from, LB? (BH put a white bill on the number line and then removed it.) Where did those three red ones come from?

LB: That’s your three dollars of debt. You don’t have anything to subtract it from and since you’re taking zero dollars of money to spend . . .

I: Where are you taking the three dollars of money to spend?

LB: There is no three dollars of money to spend. It’s three dollars of debt minus zero dollars. Isn’t that right?

I: It’s zero dollars minus three dollars of money to spend.

LB: Oh, sorry! I heard it wrong.

BH: Oh. (She put two white bills on the number line and then removed them.)

LB: Zero minus three is negative three, technically.

BH: You have zero dollars and take away three dollars of money to spend?

I: Yes.

BH: So it would (She put three white bills on the number line.)

LB: It would be negative three.

BH: Yeah, it would be negative three. I wonder if we have to do this. (She put three white bills and LB put three more red bills on the number line.) No, because you had zero dollars.

LB: So it’s still negative three and you’re not taking away any. I mean you’re not adding any zeroes.

BH: So I can’t have anything in the positive here (pointing to the positive side of the number line) because to have money to spend you have to have it over here first (pointing to the positive side of the number line). (long pause) So take away three dollars of money to spend.

LB: So put three ones there (pointing to the positive side of the number line). (BH put three white bills on the number line.) And then you’re still, but see, then I have to add three (meaning three red bills). (She put three more red bills on the number line.) There. Cause here’s what makes zero. (She put her hands to cover the three white bills and three of the red bills.) And then there’s your three dollars of money to spend. No, zero dollars of money minus three dollars of money to spend. Right? So you have three dollars, you have zero dollars of money to spend, which is right here (She covered three white bills and three red bills on the number line.)

BH: But you already have that in debt. Oh.

LB: So you subtract three dollars of money to spend, right? Is that what the question was?

I: Zero dollars and subtract three dollars of money to spend.
BH: So we’ll both take away three (She removed the three white bills and LB removed three red bills from the number line.) to make that zero to cancel out my three dollars of money to spend.

LB: No, then I would just take up three and you would leave your three here (pointing to the positive side of the number line). No, that’s not right. You are right.

BH: I don’t know. Starting with zero is kind of confusing to me.

LB: Yeah, that’s where I’m at, too. Zero dollars of money to spend.

BH: The answer is definitely negative three.

LB: Yeah. Well if you have your zero dollars and then you take away three dollars of money to spend, which would be taking out three zeroes then, and I would have six here (She put three more red bills on the number line so that it now had six red bills.) and I would have to take up three.

BH: To cancel out my three zeroes.

LB: Correct. So that means that we actually added six zeroes.

BH: No, three.

LB: But if you took up your three (meaning three white bills) and I took up my three (meaning three red bills) (long pause)

BH: No because we had positive three and negative three equals zero We added three more.

LB: So that’s why we take off three more. That’s right.

BH: Is that right? She won’t tell us.

I: Let’s see it one more time.

LB: We could be totally off base.

I: Let’s see zero dollars. (LB put three red bills and BH put three white bills on the number line.)

BH: Okay, this is zero dollars.

I: Now take off three dollars of money to spend.

BH: You have to have three more there. Wait.

LB & BH: No.

BH: All’s I have to do is take . . .

LB: Take up those three (referring to the three white bills). Take away three dollars of money to spend.

BH: But then we’re not left with any zeroes and we never added.

LB: Yeah we did.

BH: We added three because we had to make that original problem zero.

I: So, it’s three zeroes?

LB & BH: Yes.
I: Take those off. (LB & BH removed the bills from the number line.) Two dollars of money to spend subtract five dollars of debt. I want to see the two dollars of money to spend. (BH put two white bills on the number line.) And now I want you to subtract five dollars of debt. (LB put five red bills on the number line.)

BH: Yeah, but you added two more... Now wait.

LB: You need to add, cause the answer’s seven.

BH: But we need to do the amount of zeroes, which is five. I have five zeroes to counteract that... (she meant five positives). (She put five more white bills on the number line.)

LB: Yes.

BH: Take away five dollars

I: Of debt.

BH: Yeah, take away five dollars of debt. (LB removed five red bills from the number line.) So you have seven dollars.

LB: Five zeroes.

I: Do you agree?

BH: Yes. (LB and BH removed the bills from the number line.)

I: And one more, just to make sure. Let’s have four dollars of debt subtract three dollars of money to spend. (LB put four red bills on the number line.)

BH: Okay, you have four dollars of debt and you need to add three zeroes. (BH put three bills and LB put three more red bills on the number line.) And take away three dollars of money to spend. (She removed three white bills from the number line.) (LB reached out to take away some red bills). No, you don’t take away anything.

LB: Four dollars of debt (She pointed to the red bills on the number line.) subtract positive three.

BH: Yeah, I took away the positive three.

LB: That’s right.

BH: So you’re left with negative seven.

I: Okay, and how many zeroes?

BH: Three.

I: I want you to look at those problems and figure out how many zeroes had to be added for each of those problems.

BH: (very quickly) It’s the same as the absolute value of the second, uh, what you’re subtracting from the first.

LB: Addend.

BH: That’s what it’s called.

I: So it’s the absolute value of the second amount?

LB: Yes.
I: And how did you determine the solution then? (long pause)
BH: You mean like if we had two negatives, you add?
LB: If you have two negatives, you have a positive.
I: What if you had, okay, let’s try this for your theory. Let’s take those off. (LB and BH removed the bills from the number line.) And let’s write down this problem. Negative four subtract negative two.
BH: That’s a negative two. Wait a minute. I’m seeing a pattern here. Four, one, one, two, four, three, five, three. (She was reading the numbers from the last column on her paper – the number of zeroes that needed to be added.)
LB: Four, five, six minus two is four. (She was reading down the number of zeroes column.)
BH: The number of zeroes that you add with the first amount that you’re using equals your solution.
LB: Say it again.
BH: Okay. Wait a minute. Cause I was looking at two and four as six, one and one is two, one and four is five, two and three is five. (long pause) Do you see what I mean?
LB: Here it is. Yeah. Six minus four is your first addend. Negative six plus four is negative two. Two minus one is one. Five minus one is four. Negative five plus two is negative three. Negative five plus four is negative one. Negative three plus three is zero. Seven minus five is two. Negative seven plus three is negative four.
I: Sounds good. Let’s try negative four minus negative two.
BH: On the number line?
I: Sure. (LB put four red bills on the number line. She then put two more red bills on the number line.)
BH: Minus negative two?
I: Uh huh.
LB: And then you put up two and you take away . . .
BH: Wait a minute. (She put two white bills on the number line.) I think we need to take away two of these (pointing to the red bills).
LB: Uh huh.
BH: And then if I do plus two (She put two white bills on the number line.), and then we bring the zeroes (she removed two white bills from the number line and LB removed two red bills from the number line.) We take away two and we’re left with negative two.
LB: Yes.
I: Okay.
LB: So we actually took up six cause you took up your two (white bills) and I took up my four (red bills) and four and two is six.
BH: No. This answer’s two (pointing to the last column).
LB: You have six minus two
BH: You should’ve started with four and then added two when I add. Wait. It worked on
the number line.
LB: Uh huh. You took up your two (pointing to the positive side of the number line) and
I took up (long pause).
BH: We had minus four take away two dollars of money to spend so I had my two dollars
of money to spend. . .
LB: So four minus two is two. So that’s right.
BH: And then we got rid of those two dollars.
I: Take those off. (LB and BH removed the bills from the number line. And let’s try
negative three subtract negative two. (LB put three red bills on the number line. She then
put two more red bills and BH put two white bills on the number line.)
BH: Wait a minute. How many zeroes? We had two zeroes. Now take those two off
(meaning to remove two red bills from the number line).
LB: Yeah. Yeah.
BH: We put two zeroes on.
LB: Yes.
BH: And then we take away the zeroes. Take off two (meaning for LB to take off two red
bills. (She removed two white bills from the number line and LB removed two red bills.)
I: How many zeroes did you add in?
BH & LB: Two.
I: And your answer was. . .
BH: Negative one.
I: Let’s take that one off. (LB removed the red bill from the number line.) And let’s see
four dollars of debt (LB put four red bills on the number line.) subtract three dollars of
debt.
BH: I’ll add on three zeroes on this side (She placed three white bills on the number line.)
Now take away the three zeroes. (She removed three white bills and LB removed three
red bills from the number line.
I: Wait. What did you add to your four dollars of debt, and you put the three dollars over
here (referring to the positive side of the number line), was that three zeroes? (Pause)
BH: If I put three and then she cancels out three?
LB: Then it’s technically no zeroes. But it should be three zeroes.
I: There’s your four dollars of debt (referring to the four red bills on the number line).
LB: Uh huh.
I: I want you to subtract three dollars of debt. (LB started to put some red bills on the
number line.)
BH: No. No. No. You subtract debt you actually add. (Long pause) Cause if I owe you four dollars and you’re going to subtract three dollars of what I owe you, I’d only owe you a dollar.

LB: Right. So you need to put up your three there (meaning that BH needed to put three white bills on the number line).(BH put three white bills on the number line.)

I: Can you just put on three dollars of money to spend? (long pause)

LB: You could.

I: Legally?

BH: No! If you’re taking away, can’t you just take away three of these (referring to the red bills)? But then you’re not adding any zeroes.

LB: Right.

BH: Oh shoot, that blew my whole theory then.

I: Could you do it using your theory? What would you do using your theory, BH?

BH: I would have to give myself money to spend.

I: Okay. And if you give yourself money to spend, the only way that that’s legal is if . . .

LB: I counter it (meaning that she needs to put down the same number of red bills).

I: Okay. Let’s do it. (BH put three white bills and LB put three red bills on the number line.)

LB: Uh huh.

I: And now you can take away . . .

LB: Three dollars of debt.

BH: And that might give us the right answer then.

I: Take away your three dollars of debt. (LB removed three red bills from the number line.) That’s your answer. (four red bills and three white bills) Now you want to simplify it . . .

BH: Yeah. (She and LB removed three white bills and three red bills from the number line.)

I: So that you’re going to just make it one color of money.

LB & BH: Uh huh.

I: Let’s think about what you did. What did you have down? You had your four dollars of debt, so you had your negative four . . .

LB: Right.

I: Then you had your plus three.

BH: Oh. Right. I understand, I think.

LB: And to counteract that plus three I added my negative three.

I: And then you took away your three dollars of debt . . .

LB: And she took away her three positive monies to spend.

I: Did she?
BH: No. I actually added it because I got rid of it.


BH: (continuing her thought) Three dollars of debt, that means I had three dollars to spend to get rid of it.

LB: Correct.

I: If you had four dollars of debt, (LB put four red bills on the number line.) and then you added three dollars of debt and three dollars of money to spend . . .

BH: But you don’t want to add three dollars of debt cause that would give us a different answer.

I: Well now if you added three dollars of debt . . .(LB put three more red bills on the number line.)

BH: That’s right.

I: Because you’re balancing that three dollars of money to spend, right?

LB: Exactly.

I: Now you’re taking away your three dollars of debt. So what you really did. . .

BH: Oh! Okay.

LB: See, now we have four (She pointed at the red bills.) minus three (She pointed at the white bills.)

BH & LB: Now you balance out the zeroes.

I: Now I want you to relate that to the algorithm that you use for subtraction of integers.

So, in other words, see if your theory still works.

BH: Yeah.

I: What was your theory?

BH: That the number of zeroes you add is equal to the absolute value of the second amount.

I: Of the second one each time?

BH: Uh huh.

I: Okay. And then, how do you get your solution? You figured out the number of zeroes that you have to add, . . .

BH: (to LB) Now do you see that?

LB: I’m sorry I was thinking of your thing at the same time. (BH leaned over and looked at LB’s paper.)

BH: What theory do you have about how this (number of zeroes added) relates to this (the problem).

LB: If you take your solution, whatever your solution is, you don’t take the absolute value of it, you take whatever it is. (Looking at the first problem given) You take your negative six, you add four and you get negative two, which is your first addend. You take your solution again on the second one, two minus one is one.
I: Try that with the very last one we did.
LB: See now it’s not going to work on the very last one because we had three, we added
three zeroes. . .
BH: Oh.
LB: So negative one, well actually, plus three would still equal negative two, or I mean
positive two.
BH: If you subtract three, though, it would give you negative four.
I: What about the next to last one?
BH: (Looking at the sheet of problems) Negative one minus negative two is negative
three. Minus negative two, minus two is negative four
LB: Seven, six, five, four
BH: But then if you do negative seven minus a negative three, that would be . . .
LB: But it isn’t a negative three. It’s a positive three. So then you add, I’m sorry, you
subtract. Negative one minus three.

Phase 3      Session 4      BH Only

I: I’d like you to show three dollars of debt. (BH put three red bills on the number line.)
BH: Just one way?
I: And then another way. (BH put a white bill and another red bill on the number line.) So
you could keep on doing that for a long time.
BH: Yeah.
I: Okay, let’s take those off. (BH removed the bills from the number line.) And what is
addition? What does it mean to add?
BH: Adding two numbers together to get a sum.
I: Okay. So, what if you had positive five plus negative six?
BH: Go ahead and do that?
I: Uh huh. (BH put five white bills on the number line.)
BH: Plus negative six. (She put six red bills on the number line.) So this is one way. We
can take off zeroes.
I: And how many zeroes will you be taking off? (She removed a red bill and a white bill
five times so that a red bill was left on the number line.)
BH: Five.
I: Good. And your answer would be. . .
BH: Negative one.
I: Good. Okay. What’s subtraction? (BH removed the red bill from the number line.)
Yeah, you can take that one off.
BH: Okay. I’m finding the difference between two numbers.
I: Okay. So, if I have, uh, negative two subtract three. (BH put two red bills on the number line. She then put three more red bills on the number line.)

BH: Actually, you’re adding negative three.

I: Can you just add negative three to a number and have the same thing?

BH: Subtract three.

I: Is subtracting three the same as adding negative three? (BH put three white bills on the number line.)

BH: Add three zeroes.

I: Okay.

BH: And then you’re taking away three (She removed three white bills from the number line.) so you’re left with five negative.

I: Okay. So, on your piece of paper, let’s put your columns for “Problem”, then “Number of zeroes”...

BH: Do you want me to write it this way? (vertically, instead of horizontally)

I: It doesn’t matter. That’s fine the other way.

BH: Problem.

I: Then “Number of zeroes”, and then “Solution”. So for that first one, we had negative two minus three. (BH had written it incorrectly.) Negative two minus three.

BH: Alright, sorry. “Zeroes that you’re taking away”.

I: Well, the number of zeroes in this case that you had to add, because you had to add some zeroes so that you had some to take away, right?

BH: Which would be three.

I: Okay. And your solution then was...

BH: Negative five.

I: Okay. What if you have positive four subtract negative two?

BH: Do you want me to (motioning toward the number line to remove the bills)?

I: Yeah. Take those off. (BH removed the bills from the number line.)

BH: So positive four (She put four white bills on the number line.) and you’re going to take away negative two so you need to add negative two on but you also need to add two positives.(She put two red bills and two more white bills on the number line.) to make that zero and then take away the negative two. (She removed two red bills from the number line.) So I have six.

I: Okay.

BH: So I have two zeroes there.

I: Okay.

BH: And the solution is six.

I: Good. Take those off. (BH removed the bills from the number line.) And, let’s have negative two subtract negative three.
BH: Negative two (She put two red bills on the number line.) and you want to take away negative three so (She put three more red bills on the number line.) And add three there too. (She put three white bills on the number line.) So the amount of zeroes would be three. Take away negative three. (She removed three red bills from the number line.) And I can still take away two more zeroes. (She removed two white bills and then two red bills from the number line.)

I: Okay, because it’s an addition problem now.

BH: Yeah. And I have positive one.

I: Good. So that all makes sense to you.

BH: Uh huh.

I: I’m going to skip some of the problems that I had everybody else do because you’ve sort of got that. I don’t know if it was from class today . . .

BH: Part of it. Yeah.

I: Yeah. Okay, so suppose that I have some other bills. I have big bills. So we don’t have the number line any more.

BH: Okay.

I: Now we have big bills. Can you imagine what the red bills are and what the white bills are?

BH: Red would be negative; white would be positive.

I: Okay. So, how would you handle negative, or twenty-four dollars of debt subtract eighty-six dollars?

BH: (counting out red bills) Twenty-four dollars in debt . . .

I: Uh huh.

BH: And then subtract eighty-six dollars?

I: Uh huh.

BH: (counting out white bills) Eighty. I’ve got eighty-six positive and eighty-six negative (counting out red bills).

I: Okay.

BH: So now I have eighty-six plus twenty-four, which is a hundred and ten. I’m sorry. What was the question?

I: Twenty-four dollars of debt subtract eighty-six dollars. (short pause)

BH: Oh, I have to take that away because that’s the eighty-six positive.

I: Okay.

BH: So now I have one hundred and ten dollars of debt.

I: Okay. Good. Okay. How about fifteen dollars subtract sixty-seven dollars?

BH: Fifteen dollars (She counted out fifteen dollars of white bills.).

I: Is that fifteen? Over there. Oh, okay.

BH: Yeah. Fifteen dollars subtract sixty-seven, you said?
I: Yes.
BH: Sixty-five, sixty-six, sixty-seven. (She counted out amounts with the red bills.) Um.
And I take away fifteen zeroes from both sides. (She removed fifteen dollars of red bills
and fifteen dollars of white bills.) Negative fifty-two?
I: So, if you have fifteen dollars and then you subtract sixty-seven. . . Yes, it would be
what kind of fifty-two?
BH: Negative.
I: Negative fifty-two. Okay. How about fourteen dollars plus a debt of thirty-six dollars?
(BH counted out fourteen dollars in white bills.)
BH: Okay, fourteen dollars. And a debt of?
I: Thirty-six. (She counted out thirty-six dollars in red bills.)
BH: Then I would take away fourteen here . . . It might be easier if I do this. Fourteen zero
BH: Okay. What if you had fifteen dollars subtract fourteen dollars of debt?
BH: (BH counted out fifteen dollars in white bills.) Fifty dollars. And subtract fourteen
dollars in debt?
I: Right. (BH removed fourteen dollars in red bills and fourteen dollars in white bills.) So
I’d have one dollar.
I: So, if you have fifteen dollars. . .
BH: Oh, wait a minute. Okay, I did that wrong. I’m sorry. Okay. Fifteen dollars and if
I’m subtracting fourteen dollars of debt here (pointing to the fourteen dollars of red bills
she had counted out) I also have to add fourteen dollars on that side. (She put fourteen
white bills with the original fifteen dollars.) And I have twenty-five, twenty-nine dollars.
I: Okay. So you understand the difference between addition and subtraction.
BH: Yeah.
I: Okay, what I’d like you to write on your paper, . . . You can either do it on that side or
on the other side. What has this manipulative done? Has it made things more clear? If it
hasn’t made things more clear, what could be done to improve it? If it did help, what
helped about it?
BH: Okay. Um. Trying to think how I’m going to write this. It made the idea of why you
add when you’re actually subtracting a negative number more clear.
I: Okay. And was that because of the color of the money or was that the number line, or
was it a combination of the two, or . . .
BH: Um. I’d say both of them.
I: Okay. (BH wrote on her paper for several seconds.) Did the number line and two colors
help you to understand the algorithm that, you know, change the subtraction to addition
and change the second number to its opposite? Also, in addition, do you see why you’re
taking the difference when the signs are different?
BH: Uh huh. They both helped me to understand the algorithm as a combination.

Phase 3      Session 4      LB Only

I: What is addition?
LB: You’re . . . I have a hard time using the same word to describe it. You are taking two numbers and making them equal another number.
I: Okay. What’s subtraction?
LB: Taking a certain amount of. . . or taking a number . . . taking a certain amount of something away from a larger number to get a sum. Or difference. I’m sorry, a difference.
I: Okay. I want you to have four columns on your paper. The first one I want you to put “Problem”, second one “What needs to be done?”, third one “How is it written”, in other words “What’s on the number line?”, and then the last one is “Solution”. Okay, how can you model four dollars of debt? (LB put four red bills on the number line.) Can you model it another way? (LB put two more red bills and two white bills on the number line.) Let’s take those off. (LB removed the bills from the number line.) And how would you model seven dollars of debt plus two dollars of debt? (LB put seven red bills on the number line and then she placed two more red bills on the number line.) And don’t forget to write it down on your paper.
LB: Oh. What was the first one?
I: Seven dollars of debt plus two dollars of debt. (LB wrote on her paper.) Take those off.
(LB removed the bills from the number line.) Four dollars of money to spend plus three dollars of money to spend. (LB put four white bills on the number line and then she placed three more white bills on the number line. She wrote on her paper.) Okay, take those off. (LB removed the bills from the number line.) Five dollars of debt plus two dollars of money to spend. (LB put five red bills and then two white bills on the number line. She then wrote on her paper.) And would that be your final answer?
LB: No. (She removed two white bills and two red bills from the number line.) That would be my final answer. (There were three red bills on the number line.) Negative three.
I: Take those off. Four dollars of money to spend plus six dollars of debt. (LB put six red bills and then four white bills on the number line. LB wrote four minus six on her paper. She corrected it to four plus negative six. She then removed four white bills and four red bills from the number line.)
LB: Negative two.
I: How about taking those off. Three dollars of debt plus six dollars of money to spend. (LB put six white bills on the number line and then she put three red bills on the number line. Then she removed three bills of each color to leave three white bills on the number line.)
344

line. She then wrote the correct answer on her paper after filling in the other columns. LB started to remove the bills from the number line but stopped when she heard “three dollars of money to spend.” Three dollars of money to spend plus one dollar of debt. (LB wrote the problem on her paper as 3 -1. She then put a red bill on the number line. LB removed a red bill and a white bill from the number line to leave two white bills on the number line.)

I: Take those off. Seven dollars of debt subtract two dollars of money to spend. (LB put seven red bills on the number line. She then put two white bills on the number line. She wrote on her paper and then removed two white bills and two red bills from the number line.) So that’s seven dollars of money of debt subtract two dollars of money to spend? LB: Yeah. No, because I have to have two more (She put two red bills on the number line.) Subtract two dollars of money to spend. I still have seven dollars of debt.

I: So if you have seven dollars of debt subtract two dollars of money to spend, where did you take away the two dollars of money to spend?

LB: (as she put two white bills on the number line) I took them from here. I: Where did they come from?

LB: Where did the positive, or the money to spend, come from?

I: Uh huh. (LB looked blankly into space.) You had seven dollars of debt and I said subtract from that two dollars of money to spend.

LB: Oh, I gotcha. (She removed two white bills from the number line. She then removed two red bills from the number line.) I shouldn’t have had any ones on there in the first place (She meant white bills.)

I: What did you just take away?

LB: Two dollars of money to spend.

I: Is that what red bills indicate?

LB: No. (She put the two red bills back on the number line.) So these go back. Seven dollars of debt minus two dollars of money to spend. (As she put two white bills on the number line) I would’ve had two dollars on here.

I: Where did they come from?

LB: (She paused a few seconds.) They didn’t come from anywhere. (as she removed the two white bills) They shouldn’t even be here because if you only have seven dollars of debt, you have to take two dollars of money to spend and if you don’t have two dollars of money to spend, then you can’t take away two dollars of money to spend. You can only add two dollars of debt.

I: So, does that mean that problem can’t be done?

LB: Oh, it can be done. Negative seven minus positive two. Should be plus two (She changed numbers on her paper.) Negative seven plus two should be negative five. Or two from seven, you can’t do that.
I: Why not?

LB: Because you can’t take seven... you can’t take seven from two. You’ll end up with negative five.

I: Is there another way that you could show seven dollars of debt?

LB: Uh huh. (LB put two more red bills and two white bills on the number line.)

I: And does that help you? (LB put her fingers on the red bills and she counted them.)

Yeah, ‘cause then you can take away these two dollars of money to spend. (She removed the two white bills from the number line as she said this.) And you will be left with nine dollars of debt. No that can’t be right. Seven minus two. Seven dollars of debt... So it would be nine... two minus nine. Two mi... no wait. Nine... eleven. It would be negative seven so you have seven dollars of debt and from that you’re supposed to subtract two dollars of money to spend. So you have to have nine dollars of debt and two dollars of money to spend. Then you take away the two dollars of money to spend and you’re left with nine dollars of debt. (LB stared at her paper wondering what to write in the third column.)

I: So what did... Show me the two dollars of money to spend on the number line before you took it off. (LB put two white bills on the number line.) Because you added what?

LB: Two positives.

I: And...

LB: Nine negatives.

I: Just now? You had seven... 

LB: (She placed her hands on the seven red bills.) I had seven but in order to show that we had seven dollars of debt and you could take away two dollars of money to spend, then you had to have nine dollars of debt first and then you can take away your two dollars of money to spend. (She put her hand on the two white bills on the number line.) And you’re left with nine dollars of debt.

I: So what I want to see in that column that you’re trying to figure out what to put, is what’s on the number line now before you take away the two dollars of money to spend.

LB: Two minus nine.

I: Is it two minus nine?

LB: Two minus seven.

I: Is it two minus seven? What do those red bills indicate? (LB stared at her paper trying to make sense of what she was doing.) How many red bills are there?

LB: Two plus negative nine.

I: Okay.

LB: There we go.

I: And now you’re going to...

LB: Take away two dollars of debt... I mean two dollars of money to spend.
I: And what’s left?
LB: Negative nine.
I: (Let’s take those off. LB removed the bills from the number line.) How about four
dollars of money to spend subtract one dollar of debt? (She wrote 4 – 1 on her paper. LB
put four white bills on the number line. She then put a red bill on the number line and for
a few seconds she thought about what to do.) Four dollars of money to spend subtract one
dollar of debt. (As she said this she put a white bill on the number line.) I’m all kinds of
second guessing myself. What needs to be done? You subtract . . .
I: Do you have the problem written down right? You have four minus one.
LB: Four dollars of money to spend and subtract. . . So it’s one minus four not four minus
one.
I: What would the one indicate? If you had one minus four, the one would indicate . . .
LB: Plus four. Because you want to take away one dollar of debt from four dollars of
money to spend.
I: So what is the original problem? What did you start with?
LB: Four minus one.
I: Is it four minus one?
LB: (She stared at her paper for a few seconds.) Four plus negative one. (LB is talking as
she filled out the chart on her paper.) Subtract negative one from four. How is it written
on the number line? Five minus or five plus. . .
I: Just put five white bills and one red bill.
LB: Okay. So then the solution if I want to take one dollar of debt away (She removed
the red bill from the number line.) I would take my one dollar of debt away. Four plus
negative one is negative three. I mean is positive three. I’m sorry. (She reached to take
another white bill from the number line.) So this shouldn’t have been on there. Five. . .
I’m confused. (She put the white bill back on the number line.) So now I’m not exactly
sure.
I: Okay. Let’s go back to the beginning then. Four dollars of money to spend. (LB
removed one of the white bills from the number line so that four remained.)
LB: Four dollars of money to spend.
I: Subtract one dollar of debt. (LB put a red bill on the number line.) Is that subtracting
one dollar of debt?
LB: Technically, no. But it does cancel out one.
I: But what’s the only thing that you can add to that four dollars of money to spend
without changing the value at all? (LB stared at the number line.) In other words, what’s
another way to show four dollars of money to spend? (LB put another white bill on the
number line.)
LB: You cancel the red one, the negative bill or the debt, with the positive bill.
I: So you’re really adding . . . zero to that.

LB: Yeah.

I: It’s the same value as four dollars of money to spend, true?

LB: True.

I: Now, can you subtract one dollar of debt? (LB removed the red bill from the number line.) And you’re left with . . .

LB: Five positives.

I: Okay. Let’s take those off. (LB removed the bills from the number line.) And let’s have one dollar of debt (LB put a red bill on the number line.) subtract three dollars of money to spend.

LB: (LB picked up some white bills.) Your one dollar of debt. And subtract from that three dollars of money to spend. (LB put three white bills on the number line and picked up some red bills. She then wrote on her paper.) So negative one plus three. One dollar of debt and subtract from that three dollars of money to spend so plus three (She wrote this on her paper in the first column.) So negative one minus three. (She put another white bill on the number line and pushed the red bills against her hand.) Yes. (LB wrote on her paper as she talked.) What needs to be done is you need to subtract one dollar of debt from the three dollars of money to spend. Five . . . oops, four whites plus one red. (LB looked at the number line, then back at her paper. She started to take off a white bill.

I: Wait. Before you take those off. (LB left the white bill on the number line.) Is that showing . . . What was the original problem?

LB: One dollar of debt (She pointed to the red bill on the number line.) And subtract from that three dollars of money to spend. So I need four red bills and one white bill. (She looked at her paper.) Yeah, because it’s just the opposite of the other one.

I: Let’s take those off before we get you all confused. (LB removed the bills from the number line.) We want to show one dollar of debt (LB put a red bill on the number line.) subtract three dollars of money to spend.

LB: (She picked up the white bills and put them down.) Subtract three dollars of money to spend. (LB started to pick up some white bills.)

I: So, in other words, what’s another way of showing one dollar of debt so that you’ll have three dollars of money to spend to take away? (As this was being said, LB put three more white bills on the number line and then three red bills.) What did you just put on there? You put on . . .

LB: Four white bills and three red bills. And that way I still have one dollar of debt. But I need three dollars, or I need two dollars of money to spend (She removed a white bill from the number line.)

I: Let’s take them all off again. (LB removed the bills from the number line.) One dollar of debt.
LB: One dollar of debt. (LB put a red bill on the number line.)

I: Subtract three dollars of money to spend. (LB hesitated and fingered the white bills.)

LB: You can’t take three dollars of money to spend away from that. (She put three white bills on the number line.)

I: So if you want to take those away, what’s the only way that you can do that?

LB: You have to counteract that. (LB then put three red bills on the number line.)

I: Okay. With. . .

LB: Your negative bills.

I: And you agree that that’s another way of naming one dollar of debt?

LB: Yes.

I: Now can you take away your three dollars of money to spend?

LB: Yes. (She removed three white bills from the number line.) There’s our three dollars of money to spend.

I: And you’re left with. . .

LB: Four red bills or negative four. (LB wrote on her paper.)

I: So on the next line write what you really did.

LB: So we did three white bills plus four red. We’re left with four red bills.

I: So in the “What needs to be done” column, what did you need to do in order to solve the problem?

LB: Subtract. You needed four dollars of debt from three dollars of money to spend, which is what I had up here to begin with.

I: You didn’t take four dollars of debt from three dollars of money to spend, you had the four dollars of debt and the three dollars of money to spend. Right?

LB: Correct.

I: And then you just canceled your zeroes. . .

LB: Yes.

I: No, you took off your three dollars of money to spend.

LB: Money to spend. Yes.

I: So take those off. (LB removed the bills from the number line.) How about two dollars of money to spend (LB put two white bills on the number line.) subtract five dollars of debt. (LB wrote 2 – 5 on her paper. She then picked up the red bills and placed five red bills on the number line. She hesitated for a few seconds.)

LB: If I have two dollars of money to spend and I have five dollars of debt.

I: But I said two dollars of money to spend and I want you to subtract five dollars of debt.

(LB then picked up the stack of white bills and put five more on the number line, pausing to think after putting two of them on the number line. When she had a total of five white bills on the number line, she paused and then put one more on the number line. She
counted the white bills. She put two more white bills on the number line so she had a total of eight white bills on the number line.)

LB: So that would be seven. . . That would be two dollars of money to spend and then I have to take away from that five dollars of debt.

I: So what did you put on the number line?

LB: I put seven dollars of money to spend and five dollars of debt.

I: How many dollars of money to spend? (LB counted the white bills.)

LB: Oops, I meant seven dollars of money to spend. (She removed one of the white bills from the number line.) I put eight.

I: Okay, so you really put on what? Five dollars of money to spend. . .

LB: Yes.

I: And . . .

LB: Five dollars. No, and two dollars of money in debt.

I: Where’s the two dollars of money in debt?

LB: It’s not there yet. (She picked up several red bills.) Five, six, seven.

I: Let’s start all over again. (LB removed the bills from the number line.) Remember, as you’re doing these, that you’re renaming that original amount some other way to accommodate your subtracting what you need to subtract.

LB: Okay.

I: So we have two dollars of money to spend (LB put two white bills on the number line.) And we want to subtract five dollars of debt. (LB first picked up a stack of white bills, then a stack of red bills as she thought about what to do. She placed five red bills on the number line. Then she quickly picked up the stack of white bills but stared at the number line.) So that’s what you want to take away, right?

LB: Yes.

I: But what allows you to have that to take it away? (LB stared at her paper.) Can you just throw those five dollars of debt on the number line?

LB: No, I have to have two dollars (She put her right hand on the two white bills that were on the number line) . . . I have to have the two zeroes or two dollars to be able to counteract my other two dollars here (She placed both her hands on the last two red bills that she put on the number line.) to have my three dollars of debt that’s left over.

I: Let’s take all that off. (LB removed the bills from the number line.) Let’s show two dollars of money to spend. (LB put two white bills on the number line.) What’s another way to show two dollars of money to spend? (LB put two red bills and two more white bills on the number line.) What’s another way to show two dollars of money to spend? (LB put two more white bills and two more red bills on the number line.) What’s another way to show two dollars of money to spend? (LB put two more white bills and two more
red bills on the number line.) Do I always have to add two dollars of negative and two  
dollars of positive to have zero?  
LB: Yes.  
I: Could I just add one to each side. One dollar of money to spend and one dollar of debt  
and have zero?  
LB: Yes.  
I: Let’s take everything off again. (LB removed the bills from the number line.) And let’s  
show what you just said while you’re doing this. Before we do that though, can you tell  
me why you thought you had to always put two dollars of debt with two dollars of money  
to spend on the number line?  
LB: Because you use the number two. It could have been... I could have put one. I could  
have put four. I could have put three. It didn’t matter as long as I did the exact same  
amount on both sides, it does not matter.  
I: Okay, so let’s see your two dollars of money to spend. (LB put two white bills on the  
number line.) And show me another way to show two dollars of money to spend. (LB put  
seven more red bills and seven more white bills on the number line.) Okay, let’s take all  
those off again. (LB removed the bills from the number line.) And again, I want the two  
dollars of money to spend (LB put two white bills on the number line.) subtract five  
dollars of debt. (Without hesitation, LB put five red bills five more white bills on the  
number line.) So write down what you’ve got on the number line.  
LB: Take five dollars of debt and add seven dollars of money to spend.  
I: Did you really add seven dollars of money to spend?  
LB: No, I did not add it. I subtracted seven dollars of money to spend.  
I: Is that what the problem said to do?  
LB: No, it said to take two dollars of debt, or two dollars of money to spend, and subtract  
five dollars of debt. So I need to put two dollars of money to spend. (She wrote in the  
second column on her paper.) Take two dollars of money to spend and subtract five  
dollars of debt. (Then she wrote in the third column of her paper.) I should put seven  
white plus five red. And then you’re left with plus two. (She wrote 2 in the last column.  
She stared at her paper for several seconds.  
I: Where did you show taking away the five dollars of debt? Have you taken away the  
five dollars of debt?  
LB: No, I have not. (She then removed five red bills from the number line.)  
I: Then you’re left with...  
LB: So then I have plus seven.  
I: Okay. Let’s take those off. (LB removed the bills from the number line.) Let’s have  
four dollars of money to spend (LB put four white bills on the number line.) subtract
seven dollars of money to spend. (LB put four red bills on the number line and then
stopped. She paused for several seconds; then she put another red bill on the number line)
LB: Okay, so hold on. Four dollars of money to spend and I want to subtract seven
dollars of money to spend. (She stopped and stared at the number line for several
seconds.) Seven dollars of money to spend. Okay, if I have four dollars of money to
spend, and I want to take away seven dollars of money to spend? (She stared at the
number line for several seconds.) I’m going to take away seven dollars of money to
spend. (She put two more red bills on the number line.) Seven. So that means that I have
(Shes stared at the number line for several seconds.) Three dollars of debt and that can’t be
right because I’m supposed to have seven. . . . Take away seven dollars of money to spend.
So, no, actually that’s right.
I: Let’s take those seven red bills off. (LB removed the seven red bills from the number
line.)
LB: If I add . . .
I: Just forget everything you know.
LB: See, that’s my problem. It’s messing me up completely because this goes against
everything that I was taught.
I: There’s your four dollars of money to spend.
LB: Uh huh.
I: Show me another way to show four dollars of money to spend that would help me
when I want to take seven dollars of money to spend away.
LB: So, I have four dollars of money to spend. If I continue to do this (She put a red bill
and a white bill on the number line.) all the way to seven, that means that I will still have
four dollars of money to spend. But if I’m taking seven dollars of money from that, that
means that I need to add negatives. One, two, three, four, five, six, seven (She counted
the white bills on the number line by pointing at each.) One, two, three (She counted the
red bills on the number line. She then picked up several red bills and placed three more of
them on the number line.) four, five, six. Unless you start easier than this (she removed
three white bills from the number line. She added another red bill to the number line and
recounted them.) One, two, three, four, five, six, seven. That’s what I did the first time,
isn’t it? Yeah. Okay. (She put two more white bills on the number line.) So now I have
zero dollars. If I want seven dollars . . . (From the number line she removed all the bills
except for four white bills.) Four dollars of money to spend and I have to take away seven
dollars. (She put another white bill and then a red bill on the number line.) One. (She then
put another bill of each color on the number line.) Two. (She put another bill of each
color on the number line.) Three. (She stared at the number line for several seconds.)
That’s eight. (She then removed one white bill from the number line. She stared at the
number line for several seconds. Then she put the white bill that she had removed back
on the number line. She put three more white bills and three more red bills on the number
line without hesitation. She smiled.)

I: So what do you have on the number line?

LB: A lot. I have eleven dollars of money to spend and seven dollars of debt. (In the third
column on her paper LB wrote seven reds and eleven whites.)

I: In that second column, what needed to be done?

LB: Well, if I have four dollars of money to spend and then I add seven dollars of money
to spend, that gives me eleven but I’m supposed to subtract seven dollars of money to
spend so then I need to counteract those extra seven that I put on there with negatives.

I: Okay, so now can you subtract your seven dollars of money to spend? (LB stared at the
number line.) Do you have seven dollars of money to spend that you can subtract?

LB: Oh yes. (She removed seven white bills from the number line.)

I: Is that your answer?

LB: No.

I: So now what do you have to do?

LB: I have absolutely no idea.

I: You want one color left on the number line.

LB: So that means I need to continue to take these off. (She removed four white bills
from the number line.)

I: Can you just take four off like that?

LB: No, not without taking off four from here. (She removed four red bills from the
number line.)

I: And so your answer is...

LB: Negative three.

I: Let’s take those off. Let’s see eight dollars of money to spend (LB sighed. She put
eight white bills on the number line.) subtract six dollars of money to spend. (LB grabbed
the stack of white bills and stared at the number line for a few seconds.)

LB: So I have eight dollars of money to spend and I want to take away from that six
dollars of money to spend?

I: Yes. (LB put six red bills on the number line. She looked at her paper for a few
seconds, then at the number line. She put six more white bills on the number line. She
looked at her paper for several seconds.)

LB: One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen,
fourteen. (She counted the white bills on the number line and then looked at her paper for
several seconds.) One, two, three, four, five, six. (She counted the red bills on the number
line. She looked at her paper for several seconds.) So if I were to take away six dollars of
money to spend. One, two, three, four, five, six. (She counted the bills as she removed six
white bills from the number line.) Six dollars of money to spend.
I: So what’s left on the number line?
LB: I have (She counted them.) eight dollars of money to spend and (She counted them.) six dollars of debt. So that means I actually have two dollars of money to spend. (LB stared at her paper for a few seconds and then looked at me.)
I: Okay, so how do I show that on the number line?
LB: That you have two dollars of money to spend?
I: Right. (LB removed two white bills and then two red bills. Then she removed a red bill and a white bill four more times until only two white bills remained on the number line.)
So you’re left with two dollars of money to spend.
LB: Yes. (She filled in the columns on her paper.) Oh. (She made this comment after putting the solution and looking at the columns on her paper.)
I: So looking at what you had on the number line before you took that off (meaning the six pairs of positives and negatives) you had. . .
LB: I had six reds and I had fourteen whites (She wrote this in the third column on her paper and stared at her paper for several seconds.)
I: So if you were writing that out, it was really fourteen. . .
LB: Minus six. . .
I: I think you had eight. (She scratched out what she had in the third column.)
LB: I think I did too. Yeah, because those numbers just weren’t working out for me. Yes, so you would have eight minus six.
I: Or. . .Eight minus six? I think you had fourteen. . .
LB: Right.
I: And then. . .
LB: Eight reds. (LB stared at her paper.)
I: Eight reds?
LB: No, six reds. (She was looking at her paper in the third column.) I had six reds. (She counted the pieces of Velcro on the number line where the six red bills would have been.) One, two, three, four, five, six. Yeah I had six reds and I had fourteen whites.
I: So it was really fourteen plus . . .
LB: Negative six. (She wrote in the third column of her paper as she talked.) Fourteen plus negative six. It had to have been eight.
I: You ended up with two, right?
LB: (She stared at her paper for a few seconds.) Yes.
I: So what did we have on there (meaning the number line)? We had fourteen whites and eight. . .
LB: It must have been eight. Right. (LB wrote in the third column on her paper.)
I: So it’s fourteen plus negative eight.
LB: There we go.
I: And then you simplified that . . .
LB: Yes.
I: To get . . .
LB: Two dollars of money to spend.
I: Let’s take those off. (LB removed the bills from the number line.) How about six dollars of money to spend (LB put six white bills on the number line.) subtract eight dollars of money to spend. (LB put three more white bills on the number line and then immediately put three red bills on the number line. She then put three more red bills on the number line.)
LB: So I had six dollars of money to spend subtract eight dollars of money to spend, correct?
I: Correct. (LB then counted six white bills on the number line and placed her hand on the sixth one away from zero. She then counted the six red bills that were on the number line and took off the other three white bills that were to the right of her hand. I’ve got to look here at my whites. She then put two more red bills on the number line.)
LB: So I had my six dollars of money to spend and then I had to take away from that eight dollars of money to spend.
I: Right.
LB: Yes. So I have my eight dollars of money to spend, or my six dollars of money to spend and I took away eight (She pointed to the eight red bills on the number line.) dollars of money to spend. If I do that, (She removed six pairs of zero, removing a red bill and a white bill six times.) I have two dollars of debt. Or two dollars of . . .
I: Okay. (LB wrote in the columns on her paper.) I know the problem is that you know too much.
LB: (laughing) That’s never been the problem.
I: Before you write anything else, let’s try this again. Take those two red bills off. (LB removed the bills from the number line.) Six dollars of money to spend. (LB put six white bills on the number line.)
LB: Six dollars of money to spend.
I: And I want you to take away eight dollars of money to spend. (LB stared at the number line for several seconds.) What can you do. . . (Even before I said this LB started to put two more white bills and six red bills on the number line.) How many red bills are you putting on there?
LB: Six.
I: And how many white ones did you put on there to balance those six red bills?
LB: Eight. (She counted the white bills on the number line by putting her finger on each as she counted them.) Yes, I have eight on there now.
I: But you had six on there before. You had six white ones on there to start out, right?
LB: That’s true. So I need to add, how many do I have? (She then put six more white bills on the number line.) There we go.

I: How many red ones did you put on there?

LB: I put six red ones.

I: And how many white ones?

LB: (She counted all the white bills that were on the number line by pointing her pencil at them.) Fourteen.

I: Beyond the six dollars of money to spend?

LB: Yes, beyond the six dollars of money to spend.

I: You put fourteen more?

LB: No, I have a total of fourteen.

I: You have a total of fourteen. But if you put on six red ones and you put on how many additional white ones?

LB: (She looked at her paper for a few seconds.) I had six to begin with so I put on eight additional white ones.

I: So will eight white ones balance the six red ones that you put on?

LB: No, I need to add, wait, two more red ones. (She then put two more red bills on the number line.)

I: So that’s still showing six dollars of money to spend, true?

LB: (With no hesitation) True.

I: So now you want to take off your eight dollars of money to spend. (LB removed eight white bills from the number line counting them as she did so.)

LB: There’s my eight dollars of money to spend that I just took off. So I have one, two, three, four, five, six, seven, eight - eight dollars of debt and one, two, three, four, five, six – six dollars of money to spend.

I: And so your final answer would be. . .

LB: Negative two (She did not removed pairs of zeroes.)

I: And how could you show that so you have only one color of money on the number line? (LB removed a white bill with her right hand and a red bill with her left hand until she only had two red bills left on the number line.) So on the number line you really saw what?

LB: I really saw fourteen dollars of money to spend and eight dollars of debt.

I: So it was really fourteen plus negative eight. (LB wrote that in the third column of her paper.)

LB: Yes. So I have negative two (She wrote this in the fourth column of her paper. Then she wrote in the second column.) Subtract fourteen dollars of money to spend from eight dollars of debt.

I: Would that give you the same thing as six plus negative eight?
LB: Yes.
I: Isn’t that really what you had on the number line because you had . . .
LB: Yes. Six plus negative eight (She wrote this in the third column of her paper.)
I: Let’s try one more. (LB removed the two red bills from the number line.) How about
one dollar of money to spend subtract (LB put a white bill on the number line.) six dollars
of money to spend. (LB wrote the problem as one plus negative six in the first column on
her paper. She then picked up the stack of red bills and put six of them on the number
line. She then picked up the stack of white bills and six more white bills on the number
line. She stared at her paper for several seconds.)
LB: I have one dollar of money to spend and you take away six dollars of money to
spend. (She stared at her paper for a few seconds.)
I: So, have you taken away your six dollars of money to spend?
LB: No. (She started to remove some red bills from the number line.)
I: What are you taking away?
LB: I’m not exactly sure why I started down there (referring to the negative side of the
number line) (LB then removed six white bills from the number line.) One, two, three,
four, five, six. Take away my six dollars of money to spend.
I: So, now on the number line, you see one plus negative six.
LB: Yes.
I: That was not your original problem, was it?
LB: No.
I: It was one minus six.
LB: Yes.
I: So, maybe in your problem column (referring to the first column on her paper) you
could write one minus six.
LB: I can do that. (LB wrote in the first column of her paper.)
I: But on the number line now, you have one plus negative six (She wrote that in the third
column on her paper.) and you’re going to end up with. . .
LB: Negative five or five dollars of debt.
I: Because you will. . .
LB: (LB removed a red bill with her left hand and, at the same time, she removed a white
bill with her right hand.) And take these away and I will have five dollars of no money to
spend.
I: Suppose that we took away your number line and we gave you new money. (Placing
other money that was red and white in front of LB.) Big bills. (LB removed the five red
bills that were on the number line.)
You don’t have a number line any more.
LB: Okay.
I: I want you to do thirty dollars of money to spend subtract fifty dollars of money to spend.

LB: (She wrote on her paper as she talked.) So I have thirty dollars of money to spend . . .

I: Minus fifty dollars of money to spend. (LB place a white twenty dollar bill and a white ten dollar bill to her right. She then put a red fifty dollar bill next to the other two bills.)

LB: (She stared at the bills for a few seconds.) So I have thirty dollars of money to spend and I want to take away fifty dollars of money to spend.

I: Right.

LB: (LB stared at the bills for several seconds.) (She picked up the white bills.) So I have my thirty dollars and then I have my fifty dollars (pointing to the red fifty dollar bill) (She stared at the red and white bills that remained in a pile.) And (She picked up a white fifty dollar bill from the pile.) I have my fifty dollars.

I: And now can you take away your fifty dollars?

LB: Of money to spend. Yes.

I: And what are you left with?

LB: I am left with fifty minus . . . negative fifty plus thirty.

I: Which is?

LB: Negative twenty.

I: Did that make more sense to you than the number line with the other monies? (LB looked at the money and the number line for several seconds.)

I: Or is it because of what you did before . . .

LB: Yes, it's because of what I did on the number line. On the number line, that got me to the point that I could do this. (She pointed to the big bills.)

I: Okay, let's try another one. How about twenty dollars of debt plus thirty dollars of money to spend. (She wrote the problem on her paper.)

LB: So I have (She picked up a red twenty dollar bill.) twenty dollars of debt and thirty dollars of money to spend. (She picked up a white ten dollar bill and a white twenty dollar bill.)

I: And you're adding those?

LB: Yes.

I: So, what's your answer?

LB: (She removed the white twenty dollar bill and the red twenty dollar bill.) Ten dollars.

I: Good. Okay, let's try another one. How about fourteen dollars subtract two dollars of debt?

(LB took four white bills and a white ten dollar bill from the pile of bills.)

I: Two dollars of debt. (LB wrote the problem on her paper as fourteen minus two.) When you wrote fourteen minus two, . . .
LB: I don’t remember what you said the original problem was.
I: Fourteen dollars of money to spend subtract two dollars of debt. How is that written?
LB: Fourteen plus negative two.
I: Fourteen dollars of money to spend subtract two dollars of debt.
LB: Oh. So (She stared at her paper for a few seconds as she wrote.) fourteen minus two.
I: Fourteen dollars of money to spend subtract two dollars of debt.
LB: So two minus fourteen.
I: Fourteen dollars of money to spend subtract two dollars of debt.
LB: So two minus fourteen.
I: Fourteen dollars of money to spend subtract two dollars of debt.
LB: Fourteen dollars of money to spend.
I: Subtract two dollars of debt. (LB wrote subtract negative two. She then picked up two red one dollar bills and two more white dollar bills.) Now, subtract your two dollars of debt. And you’re left with . . .
LB: I should have sixteen. (She counted the six white bills.)

VF & NB

Phase 3 Session 1 NB & VF

I: (Placing a white bill on the table) Have you ever seen one of those?
NB: Uh huh.
I: What is it?
VF: A dollar bill.
I: So you’ve seen one of those at some time? Maybe even a couple of them?
VF: Uh huh.
I: (Placing a red bill on the table) What about that?
NB: The same thing.
I: The same thing except . . .
NB: It’s red.
I: Why do you suppose that I would have two different colors of money though?
NB: I don’t know. You’ll come up with something.
I: You don’t have any idea why I would have two different colors of money?
NB: The only thing I can think of is like some kind of problem you’re gonna like (putting hands together again and again) I don’t know.
I: What if I showed you this thing and ask you if you’ve ever seen one of these things before. (Unrolling the number line) Have you ever seen one of those?
NB & VF: Yes. A number line.
I: So as you look at this number line, is it like the number lines that you’ve seen in the past?
VF: Yes.
I: Exactly?
NB: I’ve seen them.
I: The way you’ve seen them, they look just like that?
VF: Except for the colors.
I: So the colors are different. What do you notice about the colors? How many colors are there?
NB & VF: Two.
I: And what colors are they?
NB: Red and white.
I: Does that sound like something you’ve seen in the not so distant past?
VF: The money.
I: So what do you suppose that red money means?
NB: Negative.
I: And what would negative money be?
NB: You don’t have any.
I: Okay, and worse than that . . .
VF: Debt.
I: Debt. And white money, which is more like the money that you normally see, would be . . .
NB: Positive.
I: The positive money or the money that you have to spend. Right? So if I ask you to look at this and model three dollars of money to spend using your number line, how would you model three dollars of money to spend?
(VF placed two white bills on the number line and she directed NB to place another white bill on the number line.)
I: So that’s three dollars of money to spend. And as you look at this it is important that you tell me if you are using the number line or if you’re using the colors or if you’re using both those things, what is sort of hitting you at any given moment. So when you put the three dollars of money to spend on there, what were you thinking? Were you thinking more number line or counting the dollar bills and knowing that there were three or were both of those ideas hitting you at the same time or what was happening?
VF: For me both were hitting me at the same time cause you have to look at the number line to see where you want to put the money but you have three dollars of money to spend so it would be positive so it would be white money.
I: What about you NB?
NB: The number line.
I: So you agree that that’s how you could model three dollars of money to spend?
NB: Yes.
I: Is there another way you could model three dollars of money to spend?

NB: You could just pull the money off and put the three over here?

I: Put the three over where?

NB: Just put the one on at three and not have the other two on there.

I: You could but using those three, is there something else that you could do to make it three dollars of money to spend?

VF: Could you put another one (a white bill) here (on the positive side of the number line) and a negative over there (on the red side of the number line)?

I: Do you agree NB?

NB: Yeah that works.

I: Is there another way that you could show three dollars of money to spend?

(VF: Just add another one (meaning a positive and a negative).

NB: (Placing another white bill and a red bill on the number line) Just add another one and put a negative one.

I: And so you could do that how many different times?

NB: Forever.

I: And so those are all different ways of showing three dollars of money to spend. That is going to be extremely important. Don’t forget that. Let’s take those off. (NB and VF took the bills off the number line.) And let’s see how you would show four dollars of debt. (NB and VF placed four red bills on the number line.) So that’s four dollars of debt. Can you show four dollars of debt in another way? (VF put another red bill and a white bill on the number line.) Now, as you’re doing that are you thinking number line or colors or what are you thinking?

NB: I’m subtracting.

I: You were subtracting?

NB: And the number line.

I: And could you show four dollars of debt another way?

NB: Just add another one to each side.

I: So you could put another one on each side. And you could do that all day too, couldn’t you? Let’s take those things off the number line. How could you model zero dollars on the number line?

VF: You can. Just don’t put anything on the number line.

I: Do you agree?

NB: Yeah. Unless you do (grabbing a white bill and a red bill and placing them on the number line).

I: Oh. Do you agree?

VF: Uh huh.
I: Could you show zero another way? (NB took the bills off the number line.) And another way?

NB: There’s another way?

I: I don’t know. Is there another way?

VF: Two here (pointing to the positive side of the number line) and two there (pointing to the negative side of the number line).

I: So what’s the important thing? If you’re showing zero, it’s important that you are putting on what?

VF: Equal amounts.

I: Equal amounts of positives and negatives. And did you use the same idea when you were showing four dollars of debt in a different way? When you put on the red one and the white one, what were you really adding to that four dollars of debt?

VF: A positive.

I: And . . .

VF: A negative.

I: A positive and a negative, which is really what?

VF: Zero.

I: So you were really just adding zeroes before, weren’t you? So that’s important. Which is greater, four dollars of money to spend or six dollars of debt?

NB: Four dollars.

I: Okay, why?

NB: Because negative six is less than four.

I: And you just figured that out in your head? You didn’t use the number line or anything to figure that out, true?

NB: Yeah.

I: Which is greater, four dollars of debt or two dollars of money to spend?

VF: Two dollars of money to spend.

NB: I didn’t get the question.

I: Okay. Which is greater, what did I say, four dollars of debt or two dollars of money to spend?

NB: Two dollars of money to spend.

I: Do you know what absolute value is? If I said, “What’s the absolute value of negative four?”

NB & VF: Four.

I: How did you know that?

NB: It’s the opposite of whatever that is. I mean if it’s a negative. You turn the negative into a positive.

I: What’s the absolute value of positive four?
NB & VF: Four.

I: So if you looked at four dollars of debt on the number line. Let’s look at four dollars of
debt on the number line. (VF put four red bills on the number line.) So if you look at that,
can you sort of see why that would have an absolute value of four?
NB: I just know the rule.
I: And what would the rule tell you? That it’s going to be just . . .
NB: Four.
I: Can you see four by looking at that negative four?
NB: Oh yeah I see four.
I: So you can sort of see a relationship between the absolute value and that negative four.
NB: Yeah.
I: Let’s take those off. (NB & VF took the bills off the number line.) And let’s look at
four dollars of money to spend on the number line. (NB put two white bills on the
number line and VF put two more white bills on the number line.) And what would be the
absolute value of four?
VF & NB: Four.
I: Do you see that that’s the same as the absolute value of negative four?
NB & VF: I do.
I: How are they the same? In other words, why do they have the same absolute value?
VF: They each have four.
I: Okay, they each have four . . .
NB: I’m going with her (pointing to VF).
I: So each has four. In that case you were looking at what – the number line or the colors
or . . .
VF: Colors with the objects.
NB: I was seeing it in my head.
I: Seeing it in your head so you’re saying “Forget this”. That’s fine. You’re allowed to do
that. Which has a greater absolute value, three dollars of money to spend or two dollars of
debt?
NB & VF: Three dollars of money to spend.
I: Why?
NB: Because the absolute value of negative two is two so three is bigger than two.
I: Which has a greater absolute value, four dollars of money to spend or seven dollars of
debt?
VF: Seven dollars of debt.
I: Why?
VF: Because of absolute value. Because income is a positive seven.
I: And seven is greater than . . .
VF: Four.
I: And why?
VF: Because it’s obvious. It’s a bigger number.
I: In terms of the number line what could you say about seven and four?
VF: Seven is farther along the number line.
I: So seven is farther to . . .
VF: The right than four.
I: Let’s look at two dollars of money to spend. (VF took off two white bills from the number line, leaving two white bills.) Is there another way to represent two dollars of money to spend? (VF put two more white bills on the number line and NB put two red bills on the number line.) You added how many zeroes to that?
VF: Two.
I: I want to see two dollars of money to spend with four zeroes added to it.
NB: (After a very long pause, pointing to the negative side of the number line) And those two (referring to the two white bills) that would be four. That would be two zeroes.
I: What does one zero look like? What pair of things make a zero?
VF: A positive one and a negative one. (She placed a red bill on the number line and another white bill on the number line.) (NB looked confused.) What’s wrong NB?
I: Okay. So that would be one zero, right? (VF then took off one white bill and placed another red bill on the number line. Then she placed another white bill on the number line.) You saw the two dollars of money to spend, right? So now I’m saying I want you to add four zeroes. A zero is a positive and a negative. So you’re just putting on four positives and four negatives in addition to the two dollars of money to spend. (As I was talking, VF continued putting combinations of a red bill and a white bill until she had four of each. She then put two more white bills on the number line.)
NB: I was thinking two dollars in my head. Two point zero zero and I was getting the two zeroes from that so I was thinking four zeroes and 20 000 and that’s not right.
I: Oh. When I say a zero, I mean a positive and a negative.
NB: Okay. I feel so aweful.
I: So that’s another way of representing two dollars of money to spend. Right? By putting four more zeroes. Does that make sense?
NB: Yeah.
I: That’s going to be a real crucial thing next week or the week after that but if you can get that idea you’re going to be cool.

Phase 3  Session 2  VF & NB

I: So what are integers?
(Long pause) VF: Numbers.
I: Numbers?
NB: That’s what I kept thinking.
I: Just any numbers?
NB: Well, it’s like the thing you were explaining to us like whole numbers, that whole . . .
. that thing we did on the board (sets of numbers and which are subsets of other sets) and
you had like every, like . . .
I: Oh, the sets of numbers?
NB: Yeah. For some reason when I think of integers that’s what I think of. Which I know
is completely wrong but . . .
I: OK, can you give me some examples of integers?
VF: Positive or negative numbers.
I: OK, what about something like one-half? Is one-half an integer?
VF: Could be.
NB: I don’t know. I kinda think it is though.
VF: I think so.
NB: I have a feeling it is.
I: Where are integers used?
NB: Where?
I: Where do you use integers?
NB: If integers are numbers, everywhere.
VF: Yeah.
I: Give me an example.
NB: My checkbook.
I: Oh, your checkbook?
VF: I’m gonna say a bank account.
I: OK. When were you first introduced to negative integers?
NB: That was a long time ago.
VF: In elementary school.
I: How were you introduced to them? Do you remember?
NB: I don’t remember. (VF nodded in agreement.) I think it was like taking things away.
What I think it was like . . .
VF: Yeah.
I: So taking things away?
NB: Yeah I think they were using like different . . . I can’t remember.
I: What does addition mean to you?
NB: Adding, putting two and two together.
I: (to VF) Do you agree?
VF: Yeah.

I: OK. How do you add integers? (Silence) Give me an example of addition problem with integers.

NB: I don’t like that word.

I: You don’t like that word?

VF: Five plus negative two.

I: Five plus negative two?

NB: Yeah.

I: And what would be the answer for five plus negative two?

NB & VF: Three.

I: And how do you know?

NB: Don’t you subtract two from five?

I: Oh, you subtract two from five?

VF: Yeah.

I: What if you had two positive numbers to add together?

VF: You get a positive number.

I: What if you had two negative numbers and you were adding? How would you find your answer and what would it be? Give me an example and then tell me what the answer would be and how you figured it out.

VF: Negative one plus negative one.

I: And how much is that?

VF: Negative two.

I: And how did you figure that out?

VF: They’re both negative so you keep adding the negatives and keep the negative the same (meaning to keep the negative sign for the sum).

(to NB) I: Do you agree?

NB: Uh huh.

I: What does subtraction mean?

NB: To take away.

VF: Yeah.

I: Take away? So that’s the type of subtraction that you’ve learned. And how do you subtract integers?

(Long pause) NB: What she (meaning VF) said. Five minus two, negative two.

VF: I think of a timeline.

I: So five plus negative two is the same thing as subtraction?

NB: No. (NB seemed confused.) You can take it away because I’m (inaudible)

I: Both of you, write down a subtraction problem dealing with integers. And put the solution. And then tell me how you got it. NB, what did you have for yours?
NB: I don’t think I’m even getting it right. I just have twelve minus six.
I: Is “twelve minus six” a subtraction problem with integers?
VF: Yes.
NB: It is?
I: Yes, so you’re cool. And what did you have VF?
VF: I put five minus negative three.
I: And five minus negative three is . . .
VF: Eight.
I: How did you get that?
VF: Um, well I was taught to add, like if it’s two negatives, you add both of them. You
add five plus three.
I: Oh. So what if you had five minus seven?
VF: Then you would put the bigger number in cause the negative would go to the bigger
number so it would be negative two.
I: Did you find those rules confusing? How did you know those rules?
NB: I remember it was confusing to me.
I: Was it? So how did you get through school?
NB: Just remember the rules.
VF: Right.
I: So just memorizing the rules?
VF: Yeah.
I: And you really didn’t care whether you understood them or not? As long as you could
do it and got the right answer? (NB nodded in agreement.)
VF: It kinda made sense but like with the negative, putting this with the bigger number.
I: Why would that make sense?
VF: It just seemed like it would. Just negative and a bigger number so it’ll be more
negatives. I don’t know.
NB: I get what she’s trying to say.
I: OK, say it again.
NB: Well, I can’t say it either. It’s hard to translate it like to words. She’s trying to say,
what’s your problem?
VF: Five minus negative three.
NB: Well minus five is in the negative hole already. It’s already. . . (She made a
movement to the left with her hands.)
I: She doesn’t have negative five. She just has positive five.
NB: Oh, you do that addition thing. You do the two negative signs (She put both index
fingers parallel to each other.) or is that multiplication?
VF: You make them both positive?
NB: Yeah, that’s multiplication, isn’t it?
VF: Yeah, that’s multiplication.
I: Why do you do that?
NB: I don’t know. I get confused. I don’t even know which set you’re supposed to use for multiplication of what. I just know you’re supposed to do that so . . .
I: So you just did it because it was the thing to do?
NB & VF: Yeah.
I: To get the right answer.
NB & VF: Yes.
I: And that was the important thing? What if we said, “How would you model three dollars of debt?”
I forgot the number line and money so the second part had to be rescheduled. VF came at one time and NB came at a different time.
VF Only
I: We were looking at the numbers and we were talking about how we first learned about integers and you said that . . . when did you first learn about negative integers? Do you remember?
VF: Probably somewhere in elementary school.
I: Do you remember in what context you learned about them?
VF: No.
I: We were playing with the money on the number line so let’s see how you would show three dollars of debt? (VF put three red bills on the number line.) Is there another way that you could show three dollars of debt?
VF: Yeah. (She put four white bills on the number line, then paused with another white bill in her hand. Then she placed the additional white bill on the number line. Then she removed the white bill from the number line.)
I: Tell me what you’re thinking as you do it. (She then took two white bills off the number line.)
VF: Um. I’m trying to get more negatives than the positives. (She then placed three more red bills on the number line. There was a total of three white bills and six red bills on the number line.)
I: Can you show it another way? (She placed three more red bills and three more white bills on the number line – one red, then a white, and so on.) I noticed that you always put on three red ones for three white ones.
VF: Yeah.
I: Did you have to do that?
VF: Um (pause) Yes.
I: So you couldn’t just put on one positive and one negative?
VF: No because . . . from just this (pointing to a red bill and a white bill on the number line)?
I: If you have what you have in front of you now, could you put one red one and one white one and still have the same amount?
VF: Yes.
I: OK, so you really didn’t need three more of each, did you?
VF: No.
I: Let’s take those things off. (VF took all the bills off the number line.) How could you show zero dollars of money to spend? (VF placed one red bill and one white bill on the number line.)
VF: I put equal amounts on the number line.)
I: OK. So you could keep on . . .
VF: Going.
I: OK. Let’s take those off. (VF removed the bills from the number line.) On the back of your paper I want three columns. The first column I want you to label “Problem”, the second one “Solution”, and the third one “Number of zeroes”. That probably doesn’t make any sense now but it’s okay. The first thing we’re going to do is show four dollars of money to spend. (VF placed four white bills on the number line.) And then we’re going to add to that two dollars of money to spend. (VF placed two more white bills on the number line.) So what problem was that?
VF: Four plus two.
I: OK so in the “Problem” section you’ll write $4 + 2$ and then in the “Solution” you’ll write
VF: Six.
I: And then in the “Number of zeroes” did you have to cancel anything in order to get your solution?
VF: No.
I: So you’ll put 0. Let’s take those off (referring to the bills on the number line). And let’s model three dollars of debt (VF placed three red bills on the number line.) plus four dollars of debt. (VF placed four more red bills on the number line.) What will you write down for your problem?
VF: Negative three plus negative four.
I: And your solution?
VF: Negative seven.
I: OK, and your number of zeroes?
VF: Zero.
I: Let’s take those off. And let’s do five dollars of money to spend (VF placed five white bills on the number line.) plus seven dollars of debt. (VF placed seven red bills on the number line.) So your problem would be...

VF: Five plus negative seven. (She wrote -2 for her solution.)

I: And how many zeroes?

VF: Five zeroes.

I: And how did you know that?

VF: Because you need equal amounts of negative and positive numbers. Those five positive numbers (referring to the white bills) so you need equal amounts so you have to have five negative numbers.

I: OK, so could you show how you got the five zeroes? Could you show me how you’re getting the negative two for your final answer?

VF: So you have one pair of zeroes, two pair of zeroes, three pair of zeroes, four zeroes, five zeroes. (She pointed to one red and one white bill when she said each of these.) So you have two left and they’re negatives.

I: So show me, by taking off...

VF: The zeroes?

I: Yeah. (VF removed a red bill and a white bill, then another red bill and another white bill, and continued this until she was left with only two red bills on the number line. She removed those closest to zero so the answer was not visible by looking only at the number line.)

VF: You have two negative numbers left.

I: And normally we’d take them off from the ends so you could see the negative two.

VF: Oh yeah.

I: But you see that? (She nodded in agreement.) Let’s take those off. What about if we have two dollars of debt (She placed two red bills on the number line.) plus one dollar of money to spend? (VF placed a white bill on the number line.) So what’s the problem?

VF: Negative two plus one. (She wrote -1 in the solution column.) And one pair of zeroes.

I: And show that. (She removed one red bill and one white bill from the number line.) Because when you want your answer, you want just one color left, right? Good. Let’s take that off. (VF removed the red bill from the number line.) And let’s do six dollars of money to spend (VF placed six white bills on the number line.) plus five dollars of debt. (VF placed five red bills on the number line.) What’s the problem?

VF: Six plus negative five.

I: And your solution?

VF: One. And five zeroes.

I: And can you show them? (VF removed one white bill from the number line.)
VF: And you just have those zeroes left (pointing to the pairs of red and white bills).
I: OK so you just took off the one you wanted left on there, right?
VF: Yeah. (She laughed and put the white bill back on the number line.)
I: Show how you’d get the five zeroes. (VF removed a red bill, then a white bill, etc. until only one white bill was left on the number line.) So that makes sense to you, right.
VF: Uh huh.
I: Four dollars of money to spend (VF placed four white bills on the number line.) plus six dollars of debt. (VF placed six red bills on the number line.)
VF: That would be four plus negative six is negative two and you’d get that (She removed a red bill and a white bill at a time until only two red bills were left.)
I: And how many zeroes?
VF: Two zeroes.
I: How many zeroes?
VF: Wait. Four zeroes.
I: And your solution was negative two.
VF: Yes.
I: So now what I want you to do is look at that column where you have “Number of zeroes” and see if you can relate that along with the problem and the solution to the way that you work through the problem. Because normally you don’t have those things. You know what the rule is so see if you can relate what you did to the algorithm that you use for adding integers. (Long pause as she studied her results. I don’t think she understands the question.) So when do you have zero zeroes?
VF: Um. When you have either two positives together or two negatives together.
I: OK. And how do you determine what the sign is of your sum when they’re both positive or both negative?
VF: Oh, the sign of the solution?
I: Uh huh.
VF: If they’re both negative then you get a negative number.
I: OK. And if they’re both positive?
VF: You get a positive.
I: And how do you determine the number part for that solution when they’re both the same sign?
VF: The number of zeroes.
I: The number part for the solution.
VF: You just add them up regularly and then if it’s a negative just add the negative sign into it.
I: OK.
VF: Cause they’re both negative.
I: Good. And how do you determine your solution if you don’t have them both the same sign?

VF: If you don’t have the same sign, just you’re subtracting from . . . No (She looked again at her paper.)

I: Well, see you if can relate that number of zeroes to something about the problem.

VF: The number of zeroes is one of the addends.

I: OK, which addend?

VF: The positive addend if the bigger number is negative then the number of zeroes is the positive addend.

I: If the bigger number is negative?

VF: Yes.

I: So . . .

VF: Then the number of zeroes will be the positive addend, which is the smaller number.

I: OK. Is the positive number really smaller?

VF: Oh, it should be the bigger number. (Long pause as VF studied the chart again.) It will be the bigger number because it’s positive and positives are more than negatives. (Long pause.)

I: When you had the one that said “Two dollars of debt plus one dollar of money to spend”, you have negative two plus one, right?

VF: Yeah.

I: Which of those is greater, negative two or one?

VF: One.

I: OK. So the number of zeroes you said though was going to be one. Is it always the greater number that determines that number of zeroes?

VF: Yes.

I: Is it? What about when you had three dollars of . . .

VF: Oh, like six plus negative five.

I: Yeah. Six plus negative five.

VF: But that one is the smaller number.

I: See if you can determine some way of looking at them, looking at the number of zeroes, how can you relate that to one of those numbers that you’ve got? The number of zeroes is always going to be one of those numbers, isn’t it?

VF: Yeah.

I: Which number is it? (VF spent much time studying the chart.) If you could do some out loud thinking, that would be really good.

VF: Um, I’m trying to think because sometimes it’s the positive number that is the bigger number and sometimes the number of zeroes is the smaller one and I’m trying to relate how that (she waved her hands). . .
I’m still not sure.

I: Sometimes it’s the positive one and sometimes it’s the negative one. What if you just looked at the number part? What do we call that when we look just at the number part of the number? Do you know? (Long pause) Two words. First one starts with A, second one starts with V. (VF was still confused.)

VF: I’m not sure.

I: Absolute . . .

VF: Oh, absolute value.

I: OK, so does that help you?

VF: Yeah.

I: So now state your profound theory. What is that number of zeroes? It’s always the . . .

VF: Absolute value of the (long pause as she continued to look at her chart) always the smaller number.

I: OK, it’s always the one who has the smaller

VF: Absolute value.

I: The number of zeroes is always the same as the one that has the lesser absolute value.

VF: Yeah.

I: And then how do you determine your solution?

VF: Subtract the absolute value of the smaller number

I: From the . . .

VF: Bigger number.

I: From the absolute value

VF: Yeah. Of the other one.

I: The one that has the greater absolute value. And what sign do you give that solution?

VF: The negative sign?

I: Always negative?

VF: Negative if the greater number was negative?

I: If the greater . . .

VF: Absolute value is negative.

I: And it will be positive when

VF: The greater absolute value is positive.

I: And then how do you get the number part? (Long pause) So now you know how to get the sign, how do you determine the number part?

VF: By subtracting the smaller absolute value from the greater absolute value.

NB only

I: What are integers?

NB: Uh, numbers.
I: What kind of numbers? Just any kind?
NB: It could be negative numbers, positive numbers.
I: Would a fraction be an integer?
NB: I think so, yeah.
I: Where are integers used?
NB: Everyday life.
I: Everyday life. And when were you first introduced to negative integers?
NB: In elementary schools?
I: How were you introduced to negative integers? Do you remember?
NB: No.
I: You don’t remember. Okay. What does addition mean?
NB: To add, to put two sums together.
I: And how do you add integers? Can you give me an example of an addition problem with integers and then how you would solve it?
NB: With addition?
I: Uh huh.
NB: Um. Simple problems could be like two plus two.
I: Okay. How about if you had some negative integers in there? Could you make one up with negative integers?
NB: Uh, I mean if I add negative three plus negative two it’s going to be negative five.
I: How do you know?
NB: Well, because you have two negative numbers. You’re adding those numbers so there’s only going to become a larger negative number.
I: And what does subtraction mean?
NB: To take away.
I: Okay. Remember that.
NB: Okay.
I: Subtraction means. . . The easiest way to show subtraction is to take away. How do you subtract integers?
NB: (pause) The same thing you would with addition. You’re just like. . . Say you’re adding four plus, or I’m sorry four minus one. You’re going to take one away, which is going to give you three.
I: Okay. So, let’s model three dollars of debt on the number line. (NB put three red bills on the number line.) And can it be modeled another way?
NB: Yes. (NB put one white bill and one more red bill on the number line.)
I: So, how many ways are there to model three dollars of debt?
NB: I could go on forever.
I: Okay. Remember that. Let’s take those off and let’s see how you would model zero dollars.

NB: Well, one way is just to keep it at zero. Or, that way. (She put one red bill and one white bill on the number line.)

I: And you could show it many different ways?

NB: I could just keep going. (She motioned to the left and to the right with her hands.)

I: What’s the important thing? If you’re modeling zero, what’s the important thing to watch?

NB: You always have zero. You always want to have the equal sides of positive as you do negative (meaning equal amounts of positive and negative).

I: Okay. Now, on the back of your paper you’re going to have three columns. The first one is going to say “Problem”; second one’s going to say “Solution”, and the third one will say “Number of zeroes”.

NB: Number of zeroes?

I: Uh huh. And as we do these problems I’m going to have you fill in the chart and show it on the number line. So let’s take those bills off the number line. (NB removed the red bill and the white bill from the number line.)

And let’s show four dollars of money to spend plus two dollars of money to spend.

NB: Am I just solving it or am I using it on here (pointing to the number line)?

I: You’re going to solve it, whichever order you want to do it. You can either solve it and then show it on the number line or you can show it on the number line and then solve it.

The last column doesn’t make any sense at this point.

NB: One, two, three, four. (She put four white bills on the number line.) And then you add two more. (She put two more white bills on the number line.) It’s going to give you six.

I: So it would be six. You didn’t have any zeroes there that you had to worry about, did you?

NB: No.

I: Let’s take those off. (NB removed the bills from the number line.) And let’s see three dollars of debt plus four dollars of debt. (NB put three red bills on the number line.)

NB: I have negative four.

I: Okay.

NB: And I give myself three more dollars of debt. (She put three more red bills on the number line.) Which is going to give me seven dollars of debt.

I: Okay. And take those off. (NB removed the bills from the number line.) So far, so good. How about five dollars of money to spend plus seven dollars of debt?
NB: Okay. (She put five white bills on the number line.) I have five dollars of money to spend. *And I spent seven dollars. (She put seven red bills on the number line.) Which gives me a negative two.

I: Do I see negative two right there?

NB: Well, actually I could just (She removed five red bills and then five white bills from the number line.) Take all these off (referring to the bills she had just removed).

I: So what are you taking off?

NB: I’m taking off . . . Well, I spent five dollars and I spent seven dollars, which is two dollars more than I had so therefore it’s going to be negative two.

I: When you took off the five red bills and then the five white bills, you were taking off what? (Pause) What is it when you have a positive and a negative? What is that? How much?

NB: Negative two?

I: If you have a positive and a negative, how much is that? One positive and one negative.

NB: Zero.

I: Zero. So what did you take off? You took five pairs of positives and negatives off, didn’t you?

NB: Yeah.

I: So you took off how many zeroes?

NB: One?

I: How many did you take off? How many pairs of positive one and negative one did you take off?

NB: I took five zeroes off.

I: Five of them. So you’re going to put five in that “Number of Zeroes” column. Does that make sense?

NB: Yeah. I never (inaudible)

I: This is different from anything you’ve ever done before. Okay. How about two dollars of debt plus one dollar of money to spend? (NB put one white bill on the number line.)

NB: One dollar of money to spend. (She put two red bills on the number line.) And I’m going to spend two dollars. So therefore (She took one red bill and one white bill off the number line.) I’m taking away one zero.

I: Okay, so what are you going to write in that last column?

NB: One.

I: Okay. Take that one off. (NB removed the red bill from the number line.) And how about six dollars of money to spend plus five dollars of debt? (NB put six white bills on the number line.)

NB: I have six dollars to spend. (She then put five red bills on the number line.) I spent five, which gives me (She removed five white bills and then five red bills from the
one, two, three, four, five zeroes (She pointed to the red bills as she counted. She then removed the five red bills.)
I: And how about four dollars of money to spend plus six dollars of debt? (She put a total of four white bills and six red bills on the number line.)
NB: Okay. I have four dollars to spend. Instead I spent six. So, therefore, I see what you’re getting at. (NB seemed confused. She removed four red bills from the number line.) Oh, okay, now my four zeroes. (She removed five white bills from the number line.) I was thinking like, I wasn’t thinking at first.
I: So let’s write that down and then you know I’m going to make you think. I want you to look at that “Number of Zeroes” column and see if that relates to anything in the problem.
NB: (Pause) Yes, it’s the positive number for each one.
I: Is it always the positive number?
NB: (She looked down at her paper and studied it.) No. I have one that’s a negative number.
I: Okay, but what is that number of zeroes?
NB: Huh?
I: What is the number of zeroes? Is that related to the original problem in any way?
NB: Yeah that. . . (She put her index fingers parallel to each other.) What do you call it?
Where you have the negative five, then it’s always a positive five, what is that called? I don’t remember what it’s called.
I: Absolute value?
NB: Yes!
I: So it’s the absolute value of. . .
NB: Negative five.
I: In every case?
NB: Five.
I: Well, what is that number of zeroes? It’s always the absolute value of something. The absolute value of what?
(NB was confused.) If you look at the first one, it was five for the number of zeroes.
NB: Yeah.
I: That was the same as what in the problem?
NB: The positive.
I: Okay. And in the next one, the one was the same as . . .
NB: The positive. And the third one’s the negative.
I: Okay. So what is that number of zeroes? Can you sort of tie together how many zeroes you had in each case related to the problem? Why was that one, the last one, why was it four? Why wasn’t it six? And the next to the last one, why was it five, not six? The third
one from the bottom, why was it one and not two? (long pause) And just to get you thinking even more, the ones where you had zero number of zeroes, when did that happen? When did you have zero zeroes? (long pause) In that first one, did you have any zeroes?

NB: Because they’re both negative.

I: Okay, or they’re both . . .

NB: Positive.

I: So, if they’re both positive or both negative . . .

NB: You won’t have any zeroes.

I: So now we have to look at how do you know how many zeroes you’ve got? Is that related to the problem at all? You saw that the four in the last one was the same as one of your addends. Right? (NB nodded in agreement.) Which addend was that number of zeroes? In every case, it was . . .

NB: I’m lost.

I: Okay. In that last one, we had negative six plus four.

NB: Yeah.

I: And you said there’s four zeroes.

NB: Wait. The only one that’s not five is the only one with a positive number.

I: Do you think that will always work?

NB: Seems like.

I: The second one down. After the five there’s a one.

NB: Yeah. There’s a negative one. The answer comes out to be a negative every answer, like the one has five zeroes comes out to be negative two, one . . .

I: Okay. Let’s try negative three plus five. (NB filled out the chart before modeling on the number line.)

NB: This actually helps. (She put three red bills on the number line.)

I: What part of it helps? (She pointed to the number line.) The number line and the color?

NB: Now that I’m getting used to it. (She put five white bills on the number line. She was somewhat confused. She took off three red bills.) I have three zeroes.

I: Because you’re really trying to get it so it’s just one color of money, right?

NB: Yeah.

I: And so if you take away your three white bills along with (She removed three white bills.) the three red bills that’s three pairs of zero isn’t it? So look at those numbers. Look at the number of zeroes and see if you can relate that to the problem itself. Where did you see that number before? On that one (the last problem done) you had three zeroes and your two addends were negative three and five. Right? In the previous one, it was negative six plus four. And you said you had four zeroes.

NB: Yeah.
I: So what is that number of zeroes? If you’re looking at those two addends, how do you know how many zeroes you’re going to have if you’re adding and the signs are different? (Very long pause) Is it always one of the addends?
NB: I’m sorry. Is it . . .
I: Is the number of zeroes always the same as one of the addends?
NB: Yes.
I: Okay. Which addend?
NB: In this last problem, the negative three.
I: And in the previous one, it was the same as the positive four.
NB: Yes.
I: Looking at those numbers, you said something about absolute value?
NB: Yeah.
I: Can you say something about absolute value related to that number of zeroes?
NB: Well, yeah, because whenever there’s a negative three in absolute value it’s going to be the opposite of that which is going to be positive three.
I: Okay. So now I want you to figure out how you get your solution. If you didn’t have this number line, how could you figure out that negative three plus five is two?
NB: Well, because when you think in your, well, when I think in my head, you know, like with you, like how you’re giving examples of how to spend money. I have five dollars to spend. I spent three dollars. How much do I have left over? Two. That’s how I think of it.
I: So you’re taking the difference between those things?
NB: Yeah.
I: And how do you know what sign to give the answer?
NB: Because the larger number is what. . .
I: The larger number?
NB: Yeah, well. . . The larger number of the two between the negative and the positive, that’s going to be the sign. So if it’s like positive six plus negative three. . .Six is greater than negative three so therefore, it’s going to be three.
I: Which is greater, negative two or positive one?
NB: One.
I: So why was it a negative for your answer?
NB: Because there’s negative two and I added plus one, which leaves me . . . Okay, then explain it. I can’t explain it. I won’t say the greatest number. But the larger number of the two. The higher the number, whether it’s negative or positive, but the higher the number.
I: Just the number.
NB: It’s going to be what that particular sign.
I: Okay. Good. So you’re really looking at absolute value?
I: Of those two numbers, aren’t you?
NB: Yeah.
I: And you’re taking, if the signs are different, then you’re taking the difference between those absolute values and if the signs are the same, then you’re just . . .
NB: Solving the problem. As it is.
I: And what sign do you give the answer in that case if they’re both, say negative?
NB: It’ll be negative.
I: So now you see what you’re doing when you add integers?
NB: Right.

Phase 3  Session 3  NB & VF

I: I want you to model two dollars of debt.
VF: On the . . .
I: On the number line. Two dollars of debt.
VF: Debt is red. (VF & NB each put one red bill on the number line.)
I: Okay. Can you model two dollars of debt a different way? And sort of remember that this is really important. (NB put two white bills on the number line.)
NB: Oh, two dollars of debt?
I: Yeah, I want you to show two dollars of debt another way. (NB put two more red bills on the number line.) (to VF) Do you agree?
VF: (after studying the number line for a few seconds) Yes.
I: Okay. Could you show it another way? (VF put two more red bills and two more white bills on the number line.) And did you have to put on two red bills and two white bills each time when you’re showing two dollars of debt a different way?
NB: You could add just one.
I: You could just add one, couldn’t you? But do you understand that’s just another name for two dollars of debt?
VF: Uh huh.
I: You got that.
NB: Yes.
I: That’s good. And you can see the two dollars of debt because you could do those zeroes (meaning to cancel out to make zeroes), couldn’t you? (NB & VF nodded in agreement.) Let’s take those off. (VF and NB removed the bills from the number line.) And, let’s model zero. (VF put a red bill on the number line and NB put a white bill on the number line.) And you could do that many different ways, couldn’t you?
VF: Yes. (NB nodded in agreement.)
I: And so, just keep that in the back of your mind. That’s just another way to show zero.
Take those off. (VF and NB removed the bills from the number line.) What is addition? If I say I want you to add something, what does that mean?
NB: To add two and two together to equal. . .
I: To add two things together, probably. Right? Okay, so how would you model five dollars of debt plus four dollars of money to spend, on the number line? (NB put one white bill on the number line and VF put red bills on the number line.)
NB: What was the problem again? (She then put three more white bills on the number line.)
I: Is that your final answer?
NB: Yes.
I: Remember that your final answer you want only one color.
VF: It should be one red. (VF took off four red bills and NB took off four white bills from the number line.)
I: So you’re going to take off four of those zeroes, right? And you’ll be left with . . .
NB: One.
I: One dollar of debt. Okay. How is the number of zeroes related to the addition algorithm? Remember, you just did that, NB, so you have an advantage. (NB had just completed session 2 before this session.) Remember you found how many zeroes you had. Like in that case, you said that you had four zeroes. How is that related to the problem, negative five plus four?
NB: It’s the absolute value.
I: Okay. What about the absolute value? It’s the absolute value of . . .
NB: Negative four.
I: Of negative four because negative four . . .
NB: You have four zeroes and with negative four (inaudible), which is going to be the absolute value of four anyway so it’s going to be four zeroes.
I: How did you know it wasn’t going to be the absolute value of the negative five?
NB: I’m starting to see it as whatever is more on the number line. Like this one had four (She pointed to the positive side of the number line.) and this one had five. (She pointed to the negative side of the number line.) So you were taking away all four to make one dollar of debt.
I: Okay. That makes sense, right?
NB: Yeah, I think.
I: So you had the four zeroes because that was the one that had the lesser absolute value, right?
NB: Yeah.
I: Okay. What’s subtraction?
NB: To take away. (VF nodded in agreement.)
I: Taking away is the easiest form of subtraction. Remember that. So don’t try to do anything else. When you’re subtracting, I want to see taking away. So, I want you to create three columns on your paper. The first one is going to be “Problem”, the second one is going to be “Solution”, and the third one is going to be “Number of Zeroes”. So, the first problem I want you to model on the number line and write down and figure out the solution and then write down the number of zeroes, which doesn’t make any sense right now is two dollars of debt subtract four dollars of money to spend. The first thing I want to see is two dollars of debt. (NB put two red bills on the number line.) That’s two dollars of debt, right? (NB nodded in agreement.) I want you to subtract four dollars of money to spend.
NB: It’s negative, right? (long pause) I don’t like the subtraction. I haven’t done this in so long. Aren’t you supposed to add the values together?
I: That’s what the algorithm says to do but I’m trying to get you to see where that algorithm comes from, first of all, so that it will make sense to you. Okay, so you see your two dollars of debt, right?
NB: Yeah.
I: Can you take four dollars of money to spend away from that as it is now?
NB & VF: No.
I: Can you take four dollars of money to spend away from that as it is now?
I: So what can you do? Can you get some money to spend?
VF: Can you write it a different way?
I: Okay, let’s see it. (VF put three white bills on the number line.)
VF: Two dollars of debt. (She put another white bill on the number line and two more red bills on the number line. Then she put two more white bills on the number line. This was done very slowly and VF was concentrating very hard.) So then you take away four dollars.
I: Is that two dollars of debt right there? (She counted the white bills and then counted the red bills.) On your number line, do you have two dollars of debt? (She then put two more red bills on the number line.) Now you can take away four dollars of money to spend.
I: And what are you left with?
VF: Six dollars of debt. (She only counted the red bills.) (Picking up the two white bills) These would be gone too. So you have six dollars of debt left. (She didn’t seem convinced.) Could be just the four dollars of... I: Let’s see it again. Let’s start all over again. Two dollars of debt. (VF put four white bills and six red bills on the number line.) So, is that what two dollars of debt looks like?
I: Do you agree, NB? Is that another name for two dollars of debt?
NB: Yeah.
I: Can you take away your four dollars of money to spend?
VF & NB: Yes.
I: And you’ll be left with . . . (VF took off four white bills.)
NB: Six.
I: Six dollars of . . .
NB & VF: Debt.
I: Does that make sense? So, in your “Solution” (column) you’ll write “- 2 – 4 = - 6. And how many zeroes did you have to add on?
VF: Four zeroes.
I: Four zeroes. Did you see that NB? Did you see how she got the four zeroes? Because we’re going to make you do the next one.
NB: Yeah. I actually do. Yeah. So let’s take those off. (VF took the bills off the number line.) And NB’s going to show us if we have one dollar of money to spend and we want to subtract one dollar of debt. So I want to see the one dollar of money to spend. (NB put one red bill and one white on the number line.) Don’t . . . Just the one dollar of money to spend right now. (She then took the red bill off.) There we go. Now, I want you to take away one dollar of debt. (NB had a red bill in her hand but was confused and didn’t know what to do so she put it on the number line.) Can you just put the dollar of debt on there? (NB shook her head to say no.) What do you have to do?
NB: (Putting her hands on the bills that were on the number line) I have to put both of them on there. (She took the white bill off the number line.) I can’t just do this because I need another one.
I: Right. (NB was confused.) Okay, let’s start over. (NB took the bills off the number line.) You have one dollar of money to spend. (NB put one white bill on the number line.) Now I want you to take away one dollar of debt. (NB started to take the white bill off the number line but she constantly watched me to see if she was correct.) One dollar of debt. (She let go of the white bill and left it on the number line. She had a red bill in her hand and she made a circling motion with her hand that held the red bill. She looked confused.) Do you have anything there? (meaning red bills)
NB: No.
I: How can you get something there?
NB: By putting it down?
I: Can you rename a dollar of money to spend another way?
NB: One dollar of money to spend another way? (She then put one more white bill and one red bill on the number line.)
I: Isn’t that another way of naming it?
NB: Yeah.
I: Now can you take away one dollar of debt? (NB took one red bill from the number line.) And you’re left with?
NB: Two.

I: Two dollars of money to spend, aren’t you?

NB: Yeah.

I: So how many zeroes did you have to add in?

NB: Two. (She looked at me to see if she was right.) One.

I: Just one, right? Because you put on one white bill and one red bill, right? (NB took a long time to look at her paper.) You aren’t believing it yet. Okay, let’s see it again. One dollar of money to spend. Let’s just see that. (NB took one white bill from the number line so only one white bill was left on the number line.) And you want to take away one dollar of debt. So what did you do? (NB appeared confused.) You gave (NB put one red bill and another white bill on the number line.) one dollar of money to spend another name like that, true?

NB: Yeah.

I: And then you took away the one dollar of debt (NB removed the red bill from the number line.) and you were left with

NB: Two.

I: Two dollars of money to spend. So how many zeroes did you actually add in there?

(NB held up one finger.) Just one. Yes. Does that make sense? (NB still looked confused.) I know it’s different than anything you’ve ever done before.

(After a long pause) NB: I’m coming up with a totally different answer.

I: Okay, what did you come up with?

NB: Minus one. One minus negative one.

I: Which is?

NB: I keep . . . minus one plus one is. . . I keep like . . . At first I was coming up with zero and now I’m coming up with negative two, not positive two.

I: Okay, and what’s the real answer?

NB: (very confidently, after looking at the number line) Positive two.

I: Uh huh.

NB: But I keep coming up with negative two.

I: Why?

NB: I don’t know.

I: Okay. Well, let’s try another one. Let’s try four dollars of money to spend. (NB put two more white bills on the number line so there were four altogether on the number line.) Subtract one dollar of debt. (NB put one red bill on the number line.)

NB: (She started to take a white bill off the number line.) So you take. . .

I: If you subtract, look at your paper though, you didn’t write “four subtract a dollar of debt”. Would that be the same as “four subtract one”? (NB scribbled out what she had written. She was confused.) Where did those four white bills come from? (There were
four white bills and one red bill on the number line.) Can they just... Does money just
float in from the sky?
NB: (She looked up after studying her paper for a long time) It’s going to be five.
I: Gotta be five?
NB: Yeah. Cause negative one minus... oh I hate subtraction.
I: Let’s see the one dollar of debt (NB removed all but one red bill from the number line.)
We can see that real easy.
NB: I do see that.
I: Good, then. Now, let’s subtract four dollars of money to spend. (NB put four more red
bills on the number line.) Can you just put four dollars of debt on there? (NB then put
four white bills on the number line.) Now, what did you do? You put on four dollars of
money to spend so that you could take it off but you also had to put on four dollars of
debt to balance that, didn’t you?
NB: Yes.
I: So that was how many zeroes?
NB: Four.
I: Four zeroes. So, in that “Zeroes” column, put “4”. And now, what are you doing?
You’re looking at that and you’re making it so that you have only one color left.
NB: I have to take off (She put her hand over the white bills.)
I: Yes, so you’re going to take off the four white bills, four dollars of money to spend, ...
NB: It’s going to be negative five.
I: It’s going to be five dollars of debt, isn’t it?
NB: Yeah.
I: Does that make sense?
NB: Yes.
I: So, let’s look now at zero dollars. So let’s show zero dollars. Take those things off. (VF
removed the bills from the number line.) And I want to subtract three dollars of money to
spend. So I want zero subtract three is the problem, right?
NB: Zero subtract three?
I: Uh huh. (NB wrote – 3 – 0.) Not negative three subtract zero. Zero subtract three. (NB
scribbled out what she had written before and wrote 0 – ( - 3)). So you’ve got zero dollars.
Can you subtract three dollars of money to spend as it is right now?
NB: No.
I: So what are you going to do? (VF reached for the bills to place on the number line.)
NB: Put three on each side.
I: Three on each side. That makes sense, true?
NB: Yeah.
I: So you’re adding on three zeroes. (NB nodded in agreement.) (VF put three white bills and three red bills on the number line.) And then you’re going to subtract your three dollars of money to spend (VF removed the three white bills from the number line.), taking it away. (NB nodded in agreement.) And you’ll be left with . . .

NB & VF: Negative three.

I: Negative three, or three dollars of debt. Okay. Let’s try this one. Take those off. (NB removed the bills from the number line.) Let’s have two dollars of debt subtract three dollars of money to spend. (NB put three white bills on the number line.) Is that two dollars of debt? Let’s see the two dollars of debt first. (NB removed the white bills and put two red bills on the number line.) And then subtract three dollars of money to spend. (NB put three more red bills and three white bills on the number line.) And now subtract, take away, the three dollars of money to spend (NB removed the three white bills from the number line. She glanced at her paper and looked confused.) And you’re left with. . .

NB: Five.

I: What kind of five?

NB: Negative five.

I: Negative five and there were three zeroes weren’t there?

NB: Yes.

I: Is this making any sense?

NB: It is, actually. (VF removed the bills from the number line.)

I: Okay. Let’s have two dollars of money to spend subtract five dollars of debt. So the first thing I want to see is two dollars of money to spend. (VF put two white bills on the number line.) And then I want to take away five dollars of debt. (VF put five more white bills and five red bills on the number line.) Okay, and then you can take away five dollars of debt (VF removed five red bills from the number line.) so that you’re always left with just one color of money, and so your final answer would be . . .

VF: Seven dollars.

I: Seven dollars of money to spend. And you had five zeroes. True?

VF & NB: Uh huh.

I: Okay. Let’s try five dollars of money to spend. Let’s see that first. (NB left five of the previous white bills on the number line.) And subtract three dollars of money to spend. (NB put three more white bills on the number line.)

VF: You didn’t have to put that on there. You just take the three off of here (pointing to the white bills that were on the number line before.) Cause you’re just subtracting three.

I: But it’s okay the way she did it, right? (VF seemed confused.) If we wanted to see. . .

NB: I was going to put them on (motioning toward the negative side of the number line).

I: And you’re going to put three red ones on, right?

NB: Yeah.
VF: Oh
I: You’re perfectly right, VF, but what I want you to see is where this algorithm comes from for the subtraction. (NB put three red bills on the number line.) And now you’re going to subtract your three dollars of money to spend. (NB started to take off the three red bills.) Three dollars of money to spend. (NB put the three red bills back on the number line and took three white bills off the number line.) And you’re going to be left with . . .
NB: Five. (She was just counting the white bills.)
I: Is it five?
NB: Eight?
I: Is it eight?
NB: Two.
I: Are you just pulling numbers out of a hat?
NB: No, it’s two.
I: It’s two. Because you need to do what? You need to make it so that you have just one color of money left, which means you’re going to make some zeroes now and take them off, aren’t you with what’s left on there (the number line). You took off your three dollars of money to spend and now you’re going to take off. (NB took off two white bills and VF took off three red bills and one white bill.) But don’t write down the three zeroes. You were just making it so that you had just one color of money left, right?
NB: Yeah.
I: Okay. Let’s try this one. Let’s take those off.
VF: So how many zeroes did you have?
I: There were three zeroes for that one.
VF: Okay, I see.
I: According to your method, you didn’t have any but I sort of led you this way because I wanted you to do things like that. Let’s take those two dollars off. (NB removed the bills from the number line.) And let’s look at three dollars of debt subtract two dollars of debt. (VF put three red bills on the number line.) There’s three dollars of debt. And now, take off two dollars of debt.
NB: Take off?
I: Take away two dollars of debt, right, because I said “three dollars of debt subtract two dollars of debt”.
NB took two red bills off the number line.) So that’s one way to do it. Let’s see it a different way.
NB: That’s right then. (She filled in the columns on her paper.)
I: Yeah.
NB: Okay.
I: Okay. Let’s see that three dollars of debt again. (VF put two more red bills on the number line so that there was a total of three red bills on the number line.) And let’s see if we can do it the same way that we did those ones in the very beginning (of this session). So I want you to subtract two dollars of debt. What did you do in the very beginning?
VF: Uh huh.
I: So how many zeroes would you add in?
VF: Three zeroes.
I: How many? If you want to subtract two dollars of debt.
VF: Oh, two zeroes.
I: Two zeroes. (VF put red bills on the number line.) Are those zeroes? (VF removed two white bills from the number line.) Those are money to spend so what else do you have to do?
VF: It means to take away two of them (pointing to the red bills and white bills; meaning to make two zeroes).
I: Well, now wait. You put on the two white bills, right? (VF nodded in agreement.) Two dollars of money to spend. Can you just throw money on there?
VF: Oh add two more. (She put two more red bills on the number line.)
I: So you’ll add two red ones and now take away the two dollars of debt (VF removed two red bills from the number line.) And what problem do you see in front of you now?
VF: Negative three plus two.
I: Negative three plus two. Right? (NB nodded in agreement.) And then you go through the same things that you did for addition. You make those zeroes and make it so you have just one color.
VF: Okay. (VF removed a red bill and a white bill, then another red bill and white bill from the number line.)
I: Is it making any more sense? (VF and NB nodded in agreement.) Okay. Let’s take that one off. (VF removed the red bill from the number line.) And let’s look at five dollars of money to spend subtract two dollars of debt. (VF put five white bills on the number line.) And you want to subtract two dollars of debt. (VF put two red bills and two more white bills on the number line. She then took off one red bill and one white bill. She seems confused.) Is that subtracting two dollars of debt? (VF put the red bill and white bill back on the number line.)
VF: Oh. (She removed the two red bills from the number line.)
I: And you would be left with. . .
NB & VF: Seven.
I: Seven dollars of money to spend. True?
VF: Uh huh. (NB seemed confused.)
I: NB, we’re going to make you do the next one.

NB: I’m confused on the last problem we did then. Five minus three dollars of debt, which we came up with the answer of two.

I: I think it was, um, which one was that?

NB: Eight.

I: We had five subtract three. It was five dollars of money to spend subtract three dollars of money to spend.

VF: Oh, (looking at NB’s paper) that’s a double negative.

NB: Oh, I see.

I: Because if it was the other way, we would’ve had what you said. Okay, let’s try this one. NB, let’s take those off. (NB removed the bills from the number line.)

NB: How many zeroes on the other one?

I: How many zeroes were there on that one?

VF: Two zeroes.

NB: Yeah.

I: Okay. Let’s try this one, NB: Four dollars of debt. So let’s see the four dollars of debt first before I even tell you what you have to take away. (NB put four red bills on the number line.) And now you’re going to take away three dollars of debt. (NB took three red bills from the number line.) And you’re going to be left with . . .

NB: One. Negative one.

I: Negative one or a dollar of debt. So if you had negative four subtract negative three, that’s equal to . . .

NB: Negative one.

I: Negative one. Now, unfortunately, the way that you did it is correct, but could I also have looked at it as four dollars of debt (NB put enough red bills to make a total of four red bills on the number line.) and in order to take away the three dollars of debt (NB put three white bills on the number line and started to put another white bill on the number line as she studied her paper. She then tossed the other white bill aside.)

NB: That’s right.

I: This was three dollars of money to spend (pointing to the three white bills). Can you just put on money to spend? (NB shook her head to say no.) What do you always have to do? (NB then put three more red bills on the number line.) Okay. So not as I look at that, can I take away my three dollars of debt? (NB removed three red bills from the number line.) What problem do you see in front of you as it stands right now?

NB: Negative four minus three.

I: Negative four minus three?
NB: Plus three.

I: Negative four plus three. Look at the subtraction problem that I gave you, “negative four subtract negative three”. Do you see now why you change that subtraction to addition and the second number to its opposite? Does it make any more sense? (VF nodded to say yes. NB seemed confused.)

VF: Yeah.

I: NB, we’re going to make you do another one. Let’s take those off. (NB removed the bills from the number line.) And let’s do two dollars of debt subtract four dollars of debt. (NB put two red bills on the number line.)

NB: Two dollars of debt.

I: Okay. And subtract four dollars of debt. (NB put four more red bills on the number line and four white bills on the number line.) And now you want to subtract your four dollars of debt. (NB removed four red bills from the number line.) See, in order to have that four dollars of debt to take off, you added in four dollars of money to spend along with it, didn’t you? And that’s why that problem’s going to become negative two plus four. Does that make sense?

NB: Yeah.

I: So it’s not just slopping together a rule, an algorithm, to figure out what’s going on. What is that number of zeroes?

NB: In this one?

I: In any of them. (VF and NB looked intently at their papers, but seemed confused by the question.) Where do you see a two in your original problem? (Pointing to the problem on NB’s paper) If this was your original problem, do you see a two?

NB: Yeah.

I: (pointing to another problem) Do you see a three for this one?

NB: Yeah.

I: (pointing to a previous problem) Do you see a five for this one?

NB: Yeah.

I: (pointing to a previous problem) Do you see a three?

NB: Yeah.

I: (pointing to a previous problem) A four?

NB: Yeah.

I: What is that number every single time?

NB: The right hand number.

I: Okay. It’s the thing that you’re subtracting, isn’t it? The absolute value of the thing you’re subtracting.

NB: Yeah.
I: So what is addition?
VF: Adding two things or adding together.
I: Okay, adding two things together. What is subtraction?
NB: Taking away.
I: Taking away. So, on your piece of paper I want four columns. First column I want you to write “Problem”, the next one “What needs to be done”, third one is “What’s on the number line?”,
NB: What’s on the number line?
I: Yeah, What’s on the number line, and the last one’s going to be “Solution”. Is that four?
NB: Uh huh.
I: Okay. That’s good. So, how would you model four dollars of debt on the number line?
(NB put four red bills on the number line.) Can you do it another way? (VF put a white bill on the number line and studied it for a long time, then picked up another white bill, thought about it, and then picked up a red bill and put the red bill on the number line.) So you could keep that up forever, right?
NB: (nodded her head) I guess.
I: Let’s take those off. (NB and VF removed the bills from the number line.) And, let’s show seven dollars of debt plus two dollars of debt. (NB put seven red bills on the number line.)
NB: Are we doing the zeroes thing again?
I: Maybe. That’s seven dollars of debt. Now I want plus two dollars of debt. (VF handed NB two red bills and she put these red bills on the number line.) And your final answer would be . . .
VF: Nine dollars of debt.
I: Negative nine, right? Or nine dollars of debt. So what I want you to do is . . . “How that appears on the number line” is the next to the last column. The second column where it says “What needs to be done?”, did you have to do anything?
NB: No.
I: No. So nothing had to be done. And in that third column, “How does it appear on the number line?” probably write seven red bills and then two red bills would be nine red bills. And, let’s take those off. (NB and VF removed the bills from the number line.) And then, how about four dollars of money to spend plus three dollars of money to spend. (VF put seven white bills on the number line.) Do you agree?
NB: Uh huh.
I: Okay. Let’s take those off and fill in the columns that you need there (on their papers).
You probably did that huh? You’re ahead of the game. You’re good. (VF and NB removed the bills from the number line.) And then I want to see five dollars of debt plus two dollars of money to spend. (NB put five red bills on the number line. Then she put two white bills on the number line.)

NB: There’s the problem. It’s not the solution.
I: Okay. So, that’s the problem. That’s what appears on the number line, right?
NB: Yes.
I: So in the “What appears on the number line?”, you’ll write five red, two white. And, what do you have to do?
NB: You have to take. (She removed two red bills and two white bills from the number line.)
I: Okay, so those are really two zeroes, aren’t they, that you took off? (NB nodded in agreement.) So in your column where it says “What needs to be done?” you’ll say “cancel two zeroes” or “take off two zeroes”, or something about two zeroes. And then put your solution in the “Solution” column. Okay, take those off. (VF and NB took the bills off the number line.) And let’s have four dollars of money to spend plus six dollars of debt. (VF put four white bills on the number line and NB put six red bills on the number line.) And what do you have to do?
NB: Take four off (she motioned to both sides of the number line) so you have four zeroes.
I: Okay. So, you got that part of it down pat, right?
NB: Yeah, I think so.
I: Okay, and it makes sense what you’re doing?
NB: Yeah.
I: Okay, so your answer would be . . .
VF: Negative two.
I: Right. So let’s take those off. (VF and NB removed the bills from the number line.)
How about three dollars of debt plus six dollars of money to spend? (NB put three red bills on the number line and VF put six white bills on the number line.) Okay, so then what do you have to do?
NB: Take three zeroes away.
I: Okay. And your final answer would be . . .
VF & NB: Three.
I: Three. How about . . . Take those off. (VF and NB removed the bills from the number line.) We’ll have three dollars of money to spend plus one dollar of debt. (NB put one red bill on the number line and VF put three white bills on the number line.) And do you see what needs to be done?
VF: Take off one zero.
I: Okay. And you’ll be left with a solution of . . .
VF: Two.
I: Two. So you see how, when you’re adding, you can tell how many zeroes you’re going
to have.
NB: Yeah.
I: How do you know? How can you tell by looking at the problem, when you’re adding,
how many zeroes you’re going to have to pull off?
VF: The positive number.
I: Is it always the positive number? (VF studied her paper.)
VF: It’s . . 
NB: The larger number? (She whispered “the smaller one”. She made some motions with
her pencil and it was obvious that she was trying to think about it.)
VF: The smaller number.
I: Smaller number? Which is smaller, negative five or two, from that very first one?
NB: You did this to me last week. (VF continued to study her paper.)
I: And what did you say last week? What did you finally come up with?
NB: Um, I think I said (long pause) not the greatest number but the. . .
VF: Absolute value?
I: Absolute value?
NB: Yeah.
I: Yeah. So it’s going to be the one that has the lesser absolute value, right?
NB: Yes.
I: Okay, let’s take those off. (NB removed the bills from the number line.) And remember
that addition is putting things together. And you told me that subtraction was taking
away.
NB: Yeah.
I: So when I say “seven dollars of debt subtract two dollars of money to spend”. . .
NB: Seven dollar of debt subtract two . . .
I: Subtract two dollars of money to spend. I want to see the seven dollars of debt.
VF: What was it again?
I: Seven dollars of debt subtract two dollars of money to spend.
VF: Okay. (NB put eight red bills on the number line.)
I: Is that seven?
NB: I don’t know. (She counted the bills.) No. (She took one of the red bills off the
number line.)
I: Close. Okay, so that’s your seven dollars of debt. Now, from that, I want you to take
away two dollars of money to spend. (VF put two white bills on the number line.)
VF: Add two over there. (meaning to put two more red bills on the number line) Add two
to the negative seven. (NB put two more red bills on the number line.)
I: Because what are you really putting on there, NB?
NB: Nine.
I: Well, you’ve got nine negatives there but what are you really...She put on two
positives. You put on two negatives. So you were really adding in...  
NB: Seven plus two?  
I: When you put the two positives and the two negatives, just that part of it, what were
you actually adding to that seven dollars of debt?
NB: Two zeroes.
I: Two zeroes. So you weren’t changing the value at all. That’s still seven dollars of debt
showing on there, right? (NB nodded in agreement.) So now can you take away the two
dollars of money to spend?
NB: Yeah.
I: (NB took two white bills from the number line.) Okay, so you’re going to take that
away. And you’ll be left with...
NB: Seven. (She started to take two red bills from the
number line – to balance the two
white bills)
VF: Keep that on there.
NB: Huh? All right.
I: Yeah, because it’s going to be nine dollars of debt, isn’t it? So you just took away the
two dollars of money to spend. I didn’t say to take away two dollars of money to spend
and two dollars of debt. Okay?
VF: For the part “What’s on the number line?” do we put what’s left on the number line?
I: What you had, uh, before. It really doesn’t matter. You can just put the nine red bills.
That’s going to come into play a little bit later.
NB: So, technically, we canceled out two zeroes.
I: No, you put on two zeroes so that you could take away the two...  
NB: So I don’t need to do anything for “What needs to be done?”
I: Right. I mean you did have to add in two zeroes. So what needs to be done, you had to
add in two zeroes, right? Because otherwise you didn’t have money to take away.
NB: Okay.
I: Okay, let’s take those off. (NB and VF took the bills from the number line.) And, let’s
do four dollars of money to spend subtract one dollar of debt. (VF put four white bills on
the number line. NB put one red bill on the number line and VF put another white bill on
the number line.) Do you see what she’s doing, NB? You put the one dollar of debt...
NB: She’s adding one so she can take the zero away.
I: Right. Be. . She’s not going to take the zero away. You put the one dollar of debt on there so that it could be taken away but in order to put the one dollar of debt on there, you also had to put one dollar of money to spend, which is what she did. Okay, so what do you see on the number line? You had to add in one zero, right? And what do you see on the number line? Just say how many reds and how many whites.

VF: One red and five whites.

I: Okay. And then you can take away your one red, yeah, your one red (NB took the red bill off the number line.) and you’ll be left with . . .

VF: Five white bills.

I: Or five dollars of money to spend. Is that making any sense? (NB and VF nodded to indicate that it did make sense.) Okay, let’s take those off. (VF removed the bills from the number line.) And, let’s do one dollar of debt subtract three dollars of money to spend.

(VF put one red bill on the number line. NB picked up some white bills and thought for a few seconds.)

NB: I don’t like subtraction.

VF: (pointing to the positive side of the number line) Yeah, you put the three on these. (pointing to the red bills) And add three of those over here too (indicating that NB should put three more red bills on the number line. VF put three red bills on the number line.) cause you have those zeroes. (NB looked confused.)

I: Because you need to take away the three dollars of money to spend. . .

NB: Yeah.

I: So you had to have it to take it away. And the only way you could have it was to also bring in three dollars of debt.

NB: Oh.

I: Okay, so you had to add some zeroes. How many zeroes did you have to add in?

NB: Three.

I: Three. And then write down what appears on the number line. And then subtract your three dollars of money to spend. (VF removed three white bills from the number line.) And you’re left with . . .

NB: Negative four.

I: Negative four or four dollars of debt. Okay, let’s take those off. (VF removed the bills from the number line.) And we’ll let NB do this one all by herself. Two dollars of money to spend subtract five dollars of debt. (NB picked up two red bills.)

I: Where’s your two dollars of money to spend? (NB put two white bills on the number line, while still holding onto the red bills.) Because, basically, that’s all you have to start with, isn’t it?

NB: Yeah.
I: Okay, now you want to take away five dollars of money to spend. (NB put five red bills on the number line. She then put two more red bills on the number line.)

NB: And then take away the two dollars of money to spend. What am I doing? And I’m left with negative seven.

I: Is that the answer to two subtract negative five?

NB: That’s what I have. Apparently, it isn’t right.

I: Is it? Think about it. If you have two dollars of money to spend and you’re taking away some debt. . Okay, so let’s see the two dollars of money to spend again. Let’s take all those other things off. Two dollars of money to spend. (NB put two white bills on the number line.) Now, you want to sub. . ., you want to take away five dollars of debt. (NB put five red bills on the number line.)

I: Can you just throw in five dollars of debt? (NB stared at her paper.) What’s the only thing you can add to something and not change its value? (long pause) What’s the only number that you can add to something and have the same thing as what you started with? (NB tossed her pen down and rubbed her head.)

NB: Oh, my brain is not functioning today.

I: Okay, well suppose I told you that you have three dollars.

NB: Okay.

I: And I say, “I’m going to give you some amount and you’re still going to have three dollars when you’re done”. How much must I have given you?

NB: Six.

I: Three plus six.

NB: You gave me three.

I: No, I didn’t give you anything. You had three dollars.

NB: Oh, okay.

I: And I want to know when I’m done giving you this money, you’re still going to have three dollars. How much did I give you? (long pause – NB was confused.) You have three dollars.

NB: Okay.

I: And I’m going to say, “You’re going to end up with three dollars. How much did I give you?”

NB: Nothing.

I: Nothing. Zero. So, by putting those five dollars of debt on the board, that’s illegal. You can’t just add five dollars of debt. What do you also have to bring in with your five dollars of debt?

NB: (pointing to the positive side of the number line) Five dollars.

I: Five of those. Yes. So put five over there because you’re adding in five zeroes, aren’t you?
NB: Yeah.

I: Okay. (NB put five more white bills on the number line.) Now you can take away your five dollars of debt. (NB removed five red bills from the number line.) And you’ll be left with . . .

NB: Seven.

I: Seven dollars of money to spend.

NB: I’m getting confused between which side it should be is where I’m getting . . .

I: Addition and subtraction? You’re getting confused with what?

NB: Like, I . . . (NB was confused and didn’t know how to say what she meant. There is a long pause.)

I: (pointing to the positive side of the number line) This is going to be the positive side or the money to spend side, right?

NB: Yeah.

I: (pointing to the negative side of the number line) And that’s the debt side.

NB: I understand but the last problem was the same style but it was going to be on the negative side as the last one was.

I: Yeah, we had one dollar of debt and then we had to take away three dollars of money to spend. So let’s see that one again. Let’s take all these guys off. So you ended up with positive seven or seven dollars of money to spend for that one. Let’s take those off. (NB & VF removed the bills from the number line.) Let’s redo that last one. We’ll let NB just do that last one. You have one dollar of debt. (NB put one red bill on the number line.)

Now, from that one dollar of debt we want to take away three dollars of money to spend.

(NB put three white bills on the number line.)

NB: Then I have to add three to this. (She picked up three red bills and placed them on the number line.)

I: That’s right. Now you can take away your three dollars of money to spend. (NB removed the three white bills from the number line.) And that’s why we had all negatives over there. Does that make more sense?

NB: Yeah.

I: Okay. (NB removed the bills from the number line in preparation for the next problem.) Here’s the next one: five dollars of debt subtract two dollars of debt.

NB: Five dollars of debt. (She put five red bills on the number line.)

I: Uh huh.

NB: Minus two dollars of debt. (She picked up two more red bills but had to think a few seconds before she put them on the number line.) I have to put two on that side (pointing to the positive side of the number line).

I: Uh huh. (VF put two white bills on the number line.) Okay, so what’s on the number line right now? You have the . . .
VF: Seven red bills.
I: Seven red bills.
VF: Two white bills.
I: Two white bills. And now, can you take away your two dollars of debt?
NB: Yes.
I: Okay, so take away two dollars of debt. (NB removed two red bills from the number line.) Now what’s on the number line? Let’s rewrite, cross out what you wrote before and let’s rewrite this. So you have five dollars of debt and you have two dollars of money to spend, right?
NB: Yeah.
I: Okay and what’s your solution then?
NB: Three.
VF: Negative three.
I: Is it going to be positive three or negative three?
NB: Negative three.
I: Okay, negative three because you need to. . .
NB: Take off the two zeroes.
I: That’s right. Okay. So you see that.
NB: Yeah.
I: Okay, let’s take those off. (VF and NB removed the bills from the number line.) And for the next one let’s do four dollar of money to spend subtract seven dollars of money to spend. (VF put four white bills on the number line. She then put seven more white bills on the number line. NB then put seven red bills on the number line.) Okay, now can you subtract your seven dollars of money to spend?
NB: Yes.
I: Okay. Take it off. (NB and VF both looked at their papers trying to figure out what to write.) Don’t write anything down there yet. (VF removed seven of the white bills from the number line.) Just take off the seven dollars of money to spend because that’s taking it away, or subtracting, true?
NB: Yeah.
I: And what do you have left on the number line?
NB: Three. Negative three.
I: Okay, but write down how many reds and how many whites. And then your final solution would be. . . What do you have to do to get your final solution?
NB: Take away three zeroes.
I: Three?
NB: Four zeroes.
I: Four zeroes. Okay. So you get negative three, true?
NB: Yes.

I: Okay, let’s take those off. (NB and VF removed the bills from the number line.) And let’s do six dollars of money to spend subtract eight dollars of money to spend. (VF put six white bills on the number line. NB put eight red bills on the number line as VF put eight more white bills on the number line.) Okay, so now can you take away your eight dollars of money to spend? (NB nodded to say that this can be done. VF removed eight of the white bills from the number line.) (referring to the noise made by the Velcro as bills are removed from the number line) My husband says it sounds like bags of potato chips. Okay, and now write down what’s on the number line. (VF and NB wrote on their papers.) So if you combined those things that would be your solution, right?

NB: Yes.

I: So now what do you have to do to figure out what the answer is?

NB: You take the six away (referring to the white bills) and you take six away from here (referring to the negative side of the number line).

I: Because you’re taking off the six zeroes. Right. Does that make sense?

NB: Yes.

I: Cool. And let’s take those off. (VF and NB removed the bills from the number line.) And, let’s have eight dollars of money to spend.

NB: Eight dollars?

I: Eight dollars of money to spend subtract six dollars of money to spend. (VF put eight white bills on the number line. NB put six red bills on the number line. VF put six more white bills on the number line.) Now you can subtract your six dollars of money to spend. (VF removed six white bills from the number line.) And then you’ll write down what you have on the number line. As you look at what’s on the number line, you really have eight plus negative six, true?

NB: Yes.

I: If you look at the problem “eight subtract six”, do you see why you need... Well, what is the algorithm for subtraction? How do you subtract with integers? What did you learn before you came to this class? (NB and VF did not understand the question.) If you would... Let’s look at that six minus eight because that’s even more interesting. Six minus eight. What did you do to figure out the solution to that before you had money and stuff (referring to the red and white money and number line) to do this?

NB: Well, you can’t take eight from six, so.

I: So what did you do?

NB: So I just turned it into a negative.

I: So you did eight minus six and put a negative with it?

NB & VF: Yeah.

I: Really what you did with it was you had six plus negative eight, didn’t you?
VF: Uh huh.

I: And on the number line, if you looked at that particular problem, you had six reds and then eight... I'm sorry. Six whites and eight reds, right? For six minus eight..

NB & VF: Yes.

I: Isn't that the same as six plus negative eight, which is the algorithm that you've always used.

NB: Yeah.

I: Does this make sense of the algorithm?

NB: Yes.

I: One more and then we're going to get you some other bills. So let's do... Well, we didn't even take off the whatever it was, the six dollars and then we cancelled those zeroes and all that. Let's take those off. (VF and NB removed the bills from the number line.) Let's do one dollar of debt subtract six dollars of debt.

NB: I don't like these problems.

I: Okay, in that case we'll let NB do it.

NB: I had to say that didn't I? Okay. (NB put one red bill on the number line.)

I: There's your one dollar of debt. And you want to take away six dollars of debt. (NB put six white bills on the number line, two at a time. (She paused for a few seconds. She then picked up some red bills.)

NB: I have to put six down here too, though (pointing to the negative side of the number line).

I: Uh huh. Cause you're just adding zeroes, aren't you? Now you can take away your six dollars of debt. (NB removed six red bills from the number line.) And so, on the number line you have... (NB wrote on her paper.) And then, you do what?

NB: Take one zero away.

I: Okay. And then what's your final answer? (NB removed one red bill and one white bill from the number line.)

NB: Five.

I: Five. Positive five. Well, I happen to have some big bills. No number line for these but I have big bills. (VF removed the bills from the number line and the bills with Velcro on the back were removed.) If you look through those, you'll see all kinds of bills but I want you to show what thirty dollars of money to spend would look like. What's thirty dollars of money to spend? (VF put down a $20 white bill and NB put down a $10 white bill.)

Okay, that's thirty dollars of money to spend. From that, I want you to subtract fifty dollars of money to spend. (NB put down a $50 red bill. She started to put it on the number line.)

NB: Oh, there's no number line.

I: No number line.
NB: Subtract fifty from thirty.
I: Subtract fifty from thirty.
NB: You’re gonna come up with negative twenty.
I: How?
NB: Because you’re subtracting fifty from twenty. Wait. Oh my God. Never mind. (VF put down two red ten dollars bills.) She’s right.
VF: I forget how, though.
I: Okay, well how can you get negative twenty from what you had? Can you relate what you’re doing there to what you were doing on the number line?
NB: Yeah.
I: How?
VF: You didn’t have, um, fifty to give so you only had thirty so you had to have twenty negative.
I: Okay, but if you put in twenty negative, you also have to put in . . .
VF: Twenty positive.
I: Twenty positive and that’s where you took out the twenty positive along with that thirty to get your fifty to take away. True?
NB: Yes.
I: Okay, let’s try this one. How about if we have twenty dollars of debt. (NB put down a red twenty dollar bill.) Added to thirty dollars of money to spend. (VF put down a white twenty dollar bill and a white ten dollar bill.) How much is that?
NB: Ten.
I: Why?
NB: Because thirty’s larger than twenty. So, therefore, simple problem, thirty minus twenty is ten.
I: But it wasn’t thirty minus twenty. It was twenty dollars of debt plus thirty dollars of money to spend.
NB: Yeah, ten dollars.
I: Why? Relate that to what you were doing on the number line.
VF: Um, you add ten to this (pointing to the negative side of the number line. She then studied her paper for a few seconds.) So it’s twenty minus thirty.
I: It was twenty dollars of debt plus thirty dollars of money to spend. So, negative twenty plus thirty.
VF: So you subtract that twenty (meaning the red twenty dollar bill) from this (the white $30).
I: Not subtract but they would cancel out just like they did on the number line, wouldn’t they? So, you’d have your positive twenty and your negative twenty, making, like,
twenty zeroes and then you’d be left with a positive ten. True? (VF nodded in agreement.) Now, what I want you to write... This is the last time that you ever have to come so I want you to write on the back, if going to the big bills if the number line helped you at all or if you were concentrating on just the color or just the number line or, you know, just your thoughts over all. Was it a good plan, bad plan, or should I even graduate?
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