QUARK DYNAMICS AND CONSTITUENT MASSES IN HEAVY QUARK SYSTEMS

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by

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Chapter 1

Introduction

QCD is that part of the standard model of particle physics that deals with the strong interaction. The standard model has passed every test, among the many thousands that it has been subjected to, compared to experiment. Most of these have been in the short distance regime where perturbation theory is relevant. In the large distance regime, only lattice simulations of QCD have produce results and the agreement with experiments is within statistical error. Direct solutions on a continuum of space-time at distance scales relevant to hadrons does not exist due to the strength of the running coupling at such scales. One must rely on models in this regime. This work addresses a particular model approach.

1.1 QCD as a theory of strong interactions

Quantum mechanics is the theory for the description of physical processes in atomic or smaller scales. At atomic scales, a non-relativistic approach proves to be adequate for most of the system’s physics. There are phenomena though, like the Lamb shift, that can not be accounted for by using the quantum mechanics tools. For that purpose, the unification of quantum mechanics with relativity is needed. Dirac was the person who first quantized the relativistic Hamiltonian of a particle and there were a few surprises from this process. It was found that the equations of the system allow negative energy solutions as well and that was interpreted as being the energy of a sea of antiparticles. That means the number of the particles of the
theory is infinite and not fixed. Additionally there is another problem lurking in the background and that is the violation of causality. The way to solve the problems arising from this picture is through the introduction of a mathematical tool called field. The quantum field essentially plays the same role for the relativistic theory as the classical field/functions we get in classical mechanics when from a discrete system with finite degrees of freedom we move to a continuous one with infinite degrees of freedom. Now, although the introduction of the relativistic quantum field solves many problems of the relativistic quantum mechanics, there are some new ones. A common technique for extracting information about physical quantities from a field theory is perturbation theory. So the necessary renormalization of the theory is implemented in a perturbative way. We still don’t have any information though for the nonperturbative behavior of the renormalization constants. If we want to consider the theory as a mathematically well defined physical theory we must make sure the renormalization constants have no abnormal nonperturbative behavior.

Quarks and leptons are considered so far as the fundamental units of matter. They both have three families of particles and antiparticles and they are all fermions. Leptons can have $\pm e$ or zero electric charge while the quarks can have $\pm (1/3)e$ or $\pm (2/3)e$. Quarks have an additional internal degree of freedom called color and there are three types: red, green, blue. All charged particles interact electromagnetically through the exchange of massless photons and the theory that describes that kind of interaction is Quantum Electrodynamics (QED). QED is an Abelian local gauge theory with a coupling constant that is small enough to allow a perturbation treatment to calculate physical observables with very high accuracy. On the other hand the color interaction between quarks overshadows the electromagnetic forces at the subatomic
scale of hadrons. The strong interaction is different from the Electro-Magnetic (EM) and the reason is that the mediators (the massless gluons) carry color charge too, therefore they can also interact with each other. So the EM field term in the QED Lagrangian density has no self interaction component while color field has, signifying a crucial difference in the nature of the interaction. With that extra term the theory becomes nonlinear resulting in a richer, more interesting and more complicated physical world. QCD is a theory for the strong interactions and uses, as QED, the gauge principle to build its Lagrangian. The algebraic structure of the two theories is of course very different and essentially QCD can be considered as a Yang-Mills theory with an SU(3) local non-Abelian color gauge symmetry compared to the very simple U(1) gauge symmetry of QED or the SU(2)×U(1) of the electroweak theory. That has a profound impact in the behavior of the running coupling of the two theories. While the QED running coupling is getting smaller with the distance and therefore we can easily and safely apply perturbative methods to study different physical phenomena, the QCD coupling has exactly the opposite behavior resulting in the confinement of elementary QCD excitations: quarks and gluons. That means perturbative analysis can be applied only for small scales (distances less than 0.1 fm or momenta larger than 2 GeV). Unfortunately the most important QCD phenomena happen in the infrared region (0-2 GeV momenta) and perturbation theory can not help us understand them. As a last comment it is worth noticing here that an SU(2) theory will provide a running coupling with weaker dependence than SU(3) in small momenta while an SU(4) with stronger dependence, so it appears that SU(3) results in a running coupling with the right momentum dependence.
1.2 Nonperturbative QCD studies

For the purpose of studying QCD observables in a non-perturbative way there are several, mostly phenomenological, approaches. Lattice-QCD is the most complete non-phenomenological and direct method.

Very early (1974) [14] the MIT phenomenological bag model was developed in order to explain the hadron spectrum and different properties of the particles. The phenomenological assumptions of the model are based on the fact that the quark and gluons are confined in a very small space where they can be considered as free particles. The quarks are massless inside the bag but infinitely heavy outside. This is translated into a very rigid boundary condition that is not Lorentz invariant and it also creates some spurious motions e.g. oscillations of all quarks with respect to the bag. Confinement results from the balance of the pressure on the bag walls from the outside and the pressure resulting from the kinetic energy of the quarks inside the bag. The fundamental problem of course with that crude approximation is how close is that picture with the real physical system. One can easily solve the equations to get the wave functions and calculate hadrons spectrum and other properties. After fitting the parameters of the model there is a difference up to 30 % between calculated and experimental data like masses, magnetic moments etc. (for some applications see [15], [16], [17], [18], [19], [20], [21] and references therein).

The QCD sum rule approach is another phenomenological way to study non-perturbative phenomena and appears to work well for the calculation of the masses of the lowest hadronic states and effective coupling constants. For these calculations essentially one tries to separate the contributions into perturbative ones that can
be evaluated from first principles and non-perturbative ones that are handled pheno-
menologically. By performing an operator product expansion a general form for
the required matrix elements is implemented. (see [22], [23], [24], [25], [26], [27], [28],
[29] for a sample of a variety of applications of the approach).

For the heavy quark mesons, effective field theories can usually simplify their
study. Since the effects of a very heavy particle are not so important for certain
low energy phenomena one can remove the corresponding degrees of freedom and
construct an effective low energy field theory for the light degrees of freedom. For
that purpose one has to identify the heavy quark fields and ”integrate them out” in
the generating functional. Unfortunately the action of the theory is not local any more
since in the full theory there is a possibility to have involvement of the heavy quark in
virtual processes where it can propagate for a short distance inversely proportional to
the effective mass (\(\Delta x \sim 1/M\)). Then an application of the procedure followed in sum
rules has to be used and the non-local action is expressed as an infinite series of local
terms in an operator product expansion (this is more or less an expansion in terms
of \(1/M\)). At this point we essentially have separation of long- from short- distance
physics and the first one corresponds to low energy interactions that are the same in
both the full and the effective theory. On the other hand the short distance effects,
because of quantum corrections that involve large virtual momenta of the order \(M\),
and since the heavy particle has been integrated out, are not described correctly. So
in the last step of the procedure these effects have to be added in a perturbative
way by applying renormalization-group techniques. \(c\) and \(b\) quark meson masses and
decay constants have been calculated in this way ([30], [31], [32],[147]).
Another very popular category of phenomenological models involves the introduction of effective potentials in a Hamiltonian formalism. The method uses some constituent masses for the quarks that are fitted to experimental data. The quark is treated in a non-relativistic way and the effects of the gluonic field in the properties of the hadrons are represented by an effective instantaneous potential. The simplest potential model employed to study heavy quarks states has a Coulomb-like and a linear term. The Coulomb-like term is for the short distance interactions and the linear term for the strong coupling limit. More sophisticated potential models take into account spin-spin and spin-orbit type of interactions. All these models provide an easy and fast way to calculate hadron observables, but they are not Poincare invariant, they don’t possess other QCD symmetries and the large number of parameters to be fitted limits their ability to provide deep understanding of QCD phenomena (a small sample of the vast number of papers in this area are [33], [34], [35], [36], [37], [38], [39], [40]).

From the generating functional of a field theory and by taking the appropriate functional derivatives over the sources one can get a tower of infinite, coupled, non-linear integral equations for the n-point functions of the physical system. These are called Dyson-Schwinger equations and their solution will have all the information for the physical properties of the bound states of quarks. Unfortunately solving that system of equations is an impossible task. The way to extract some information is by truncation in such a way that selected QCD symmetries are preserved and the corresponding Ward-Takahashi identities are still satisfied. The effect of the missing equations is usually represented by phenomenological models with few parameters fitted to experimental data. One of the simplest models is a delta function approximation for
the gluon propagator and although it misrepresents the ultraviolet behavior of QCD it has been used extensively to qualitatively study the effects of different truncations on hadronic properties. With this model the integral equations involved are reduced to an algebraic system of equations that is more easily solved numerically. The natural generalization of the delta interaction is a Gaussian distribution of infrared interaction strength. Such a distribution, combined with the calculated UV behavior from perturbative QCD, is the Maris-Tandy model that will be used in this work. A wide range of properties of light quark mesons have been studied in that way and their relation to certain very important QCD phenomena, like chiral symmetry breaking (CSB) and confinement, has been understood. This is a rather direct approach to qualitatively or quantitatively study the influence of different QCD mechanisms on the value of physical observables, and it is the approach we are going to use in our studies ([53],[54],[55],[56],[57],[58],[59],[60],[61],[62],[63],[64],[65],[66],[67],[68],[69],[70] and references therein).

Over the last 20 years or so lattice-QCD has built a reputation as the most reliable and direct way to study QCD from first principles. The application of lattice calculations is based on the fact that QCD has a path-integral or functional integral representation. For the lattice calculations one has to discretize space and time. In that way we get a grid where the nodes or the "sites" of the grid have a spacing distance $a$ from each other and the lattice has total length $L$ (so we have a 4-D lattice hypercube with sides of length $L$). The quarks then exist at these points and the gluons on the lattice links. Essentially the "links" are replacing the color field, and they transform in a very simple way under SU(3) gauge transformation. There are many
technical issues involved in the lattice QCD calculations and some of these can introduce an unknown amount of systematic error. The calculations for the light quarks or in the chiral limit require a very small spacing $a$, therefore the number of points in the lattice is very large making the numerics prohibitively time- and money-consuming. The calculation of different quantities in this case is done by extrapolation and that too will introduce an unknown amount of optimistic error. For the studies of the light quark properties, and the chiral limit phenomena, modeling the quark interaction in the Dyson-Schwinger equation formalism has been proved a very effective, fruitful and very fast and inexpensive approach. There is now a huge number of papers on lattice-QCD and hundreds are produced each year from research groups from all over the world. Next is a small number of them for studies in a diversity of topics in QCD ([41], [42], [43], [44], [45], [46], [47], [48]).
Chapter 2

QCD essentials

We introduce some basic and fundamental elements of QCD and discuss two key phenomena of the theory, namely: *dynamical chiral symmetry breaking* and *confinement*. Dynamical chiral symmetry breaking (DCSB) is of crucial importance in QCD and it is a non-perturbative effect that is rather difficult to be studied by using lattice-QCD.

2.1 QCD Lagrangian density and generating functional.

The QCD Lagrangian density contains the free fermion Lagrangian density

\[ L_0(x) = \bar{q}(x)(i\gamma_\mu \partial_\mu x - m_0)q(x), \]

where \( m_0 \) is actually a diagonal matrix whose entries are the bare quark masses. A requirement of invariance of the theory under SU(3) local non-Abelian gauge transformation on the quark fields,

\[ q(x) \longrightarrow q'(x) = e^{-ig\hat{\theta}(x)}q(x), \]

forces the introduction of a gauge field that should transform in the following way:

\[ \hat{A}_\mu(x) \longrightarrow \hat{A}'_\mu(x) = e^{ig\hat{\theta}(x)}(\hat{A}_\mu(x) + \frac{i}{g} \partial_\mu) e^{-ig\hat{\theta}(x)}, \]

where,

\[ \hat{A}_\mu(x) = \frac{\lambda_a}{2} A_\mu^a(x), \quad \hat{\theta}(x) = \frac{\lambda_a}{2} \theta^a(x). \]
In terms of field components, Eq. (2.3) then takes the form:

\begin{equation}
A_\mu^a(x) \rightarrow A_\mu^a(x) = A_\mu^a(x) - \partial_\mu \theta^a(x) + g f^{abc} \theta^b(x) A_\mu^c(x),
\end{equation}

where \(a,b,c\) are color indexes taking the values 1,2,...,8. The \(3 \times 3\) matrices \(\hat{\lambda}_a\) are a representation of the Hermitian generators of the SU(3) Lie algebra with the properties

\begin{equation}
Tr[\hat{\lambda}_a \hat{\lambda}_b] = 2 \delta_{ab}, \quad [\hat{\lambda}_a, \hat{\lambda}_b] = 2i f^{abc} \hat{\lambda}_c,
\end{equation}

where \(f^{abc}\) are the structure constants of SU(3) algebra.

Gauge symmetries are probably the most important and deepest symmetries of nature having a geometrical interpretation in the very abstract realm of fiber bundles, connections and differential forms. Fundamental mathematical aspects of gauge transformations and related issues can be found in refs. [13],[49],[50],[51].

Gauge invariance leads to the QCD Lagrangian density having the form:

\begin{equation}
\mathcal{L}_{QCD} = \mathcal{L}_0' - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = \mathcal{L}_0 + g \bar{q} \gamma^\mu \frac{i}{2} \lambda_a A^a_\mu - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu},
\end{equation}

where the \(\mathcal{L}_0'\) is the same as in Eq. (2.1) but with the partial derivative \(\partial^\mu_x\) replaced by a covariant one \(D^\mu_x\) given by :

\begin{equation}
D^\mu_x = \partial^\mu_x - i g \frac{\lambda_a}{2} A^a_\mu(x).
\end{equation}

One can see the field tensor \(F^a_{\mu\nu}\) is

\begin{equation}
[D_\mu, D_\nu] = -i g \frac{\lambda_a}{2} F^a_{\mu\nu},
\end{equation}

which give us the following expression for \(F^a_{\mu\nu}\):

\begin{equation}
F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu.
\end{equation}
The last term in the above expression is the self-interaction (non-linear) term responsible for the totally different behavior of the gluonic field from that of the EM field ([2], [5], [7], [8]).

Since now we have a theory with a gauge symmetry, in order to avoid infinities in the functional integral of the action due to integration over equivalent gauge configurations, we need to choose and specify the gauge. For that reason we need to introduce some gauge fixing fields $\omega, \bar{\omega}$ that are called Fadeyev-Popov ghost fields. Finally to define the generating functional, in order to produce the different Green functions, we have to introduce source terms for the fields of the theory. So at the end the generating functional of QCD looks like:

$$Z[\eta, \bar{\eta}, J^{\mu a}, \phi^a, \bar{\phi}^a] = \int DqD\bar{q}DA^{a\mu}_\mu D\omega D\bar{\omega} \exp\{iS_k[q, \bar{q}, A^{a\mu}_\mu, \omega, \bar{\omega}] + i \int d^4x[\eta\bar{q} + \bar{\eta}q + A^{a\mu}_\mu J^{\mu a} + \phi^a \omega^a + \bar{\phi}^a \bar{\omega}^a]\}. \tag{2.11}$$

$S_k$ is the gauge fixed action:

$$S_k[q, \bar{q}, A^{a\mu}_\mu, \omega, \bar{\omega}] = \int d^4x[L_{QCD} + \{\partial^\mu \bar{\omega}^a (\delta_{ab} \partial^\mu \omega^b - gf^{abc} \omega^c) - \frac{\lambda^a}{2k} (\partial^\mu A^{a\mu}_\mu)^2\}], \tag{2.12}$$

where the added term to $L_{QCD}$ inside the integral is for fixing the gauge. The parameter $k$ in the last term in the integral is the unrenormalized gauge fixing parameter where, for our studies, $k=1$ (Landau or transverse gauge).

The generating functional can be used to produce the equations of motion or the different Ward-Takahashi identities of QCD. For example, the 2-point function of quarks (quark propagator), which is related to the probability of the quark to
propagate from a point $y$ of space-time to a point $x$ is given by

$$S(x - y) = (-i)^2 \frac{\delta^2 (\ln Z)}{\delta \eta(x) \delta \bar{\eta}(y)} \bigg|_{\eta = \bar{\eta} = 0}.$$  

Before we move on and introduce the equations of our study we should also mention that for practical reasons we are going to work in Euclidean metric and not in Minkowski metric. That means we should analytically continue the time variable $t \rightarrow -i t'$ or in other words perform a $90^\circ$ Wick rotation. For the calculation of physical observables one needs to reach the time-like region and as a consequence extend the numerics into the complex plane. From here on the Euclidean metric is adopted. In this metric

$$\{\gamma_\mu, \gamma_\nu\} = 2 \delta_{\mu\nu}, \quad \gamma_\mu^\dagger = \gamma_\mu, \quad a \cdot b = \sum_{i=1}^{4} a_i b_i.$$  

Expressions in Euclidean metric can be transcribed from those in Minkowski metric by the following replacements in Minkowski quantities:

$$\emptyset \rightarrow -i \emptyset, \quad \emptyset' \rightarrow -i \emptyset', \quad v.u \rightarrow -v.u, \quad d^4x \rightarrow -id^4x,$$

supplemented by a factor of $(i)$ for each loop integral.

2.2 Two Dyson-Schwinger Equations: quark propagator gap equation and Bethe-Salpeter bound states equation.

We introduce the equations that form the basis of our studies. The starting point is the so called gap equation for the quark propagators and it is acquired from the Dyson-Schwinger equation of the generating functional, by applying the functional derivative operator $\frac{\delta}{\delta (\eta(x))}$ and using Eq. (2.13). The expression we get contains also the 2-point function of gluons and the 3-point quark-gluon function. For practical
reasons it is more convenient to work in momentum space so we Fourier transform the integral equation we got to finally arrive at the following nonlinear integral equation (see for example [1]):

\begin{equation}
S(p)^{-1} = Z_2 (i p + m_{bm}) + Z_1 \int_q ^{\Lambda} g^2 D_{\mu \nu}(k) \gamma_\mu \frac{\lambda^i}{2} S(q) \Gamma_\nu(p, q)
\end{equation}

where \( Z_1, Z_2 \) are the quark-gluon vertex and quark field renormalization constants correspondingly, \( m_{bm} \) is the quark bare mass, \( \Lambda \) is the regularization mass scale for the translationally invariant integral, \( D_{\mu \nu}(k) \) is the renormalized dressed gluon propagator, and \( \Gamma_\nu(p, q) \) is the renormalized dressed quark-gluon vertex. The second term in the right of the equation is the self-energy term which is due to the interaction of the quark with the QCD vacuum. At the end of the calculations the regularization is removed by taking \( \Lambda \rightarrow \infty \). The quark propagator has the general form:

\begin{equation}
S(p)^{-1} = i p A(p^2, \mu^2) + B(p^2, \mu^2) = A(p^2, \mu^2)(i p + M(p^2))
\end{equation}

where \( M(p^2) \) is the quark mass function which is independent of the renormalization point \( \mu \). We require \( A(\mu^2, \mu^2) = 1 \) and \( B(\mu^2, \mu^2) = m_r(\mu) \) where \( m_r(\mu) \) is the renormalized current quark mass at scale \( \mu \). One can solve Eq. (2.16) numerically after reducing it into a system of coupled nonlinear integral equations for \( A \) and \( B \) and the solution is uniquely determined under the above renormalization conditions.

The renormalized, homogeneous BSE for a bound state of a quark of flavor \( a \) with an antiquark of flavor \( b \) having total momentum \( P \) is:

\begin{equation}
[\Gamma_M^{ab}(p, P)]_{tu} = \int_\Lambda \frac{d^4 \tilde{q}}{(4\pi)^4} K_{tu}^{rs}(p, \tilde{q}, P) \times [S^a(\tilde{q} + \eta P)\Gamma_M^{ab}(\tilde{q}, P)S^b(\tilde{q} + \bar{\eta} P)]_{sr},
\end{equation}

\footnote{We should stress at this point that a proof of the existence of a unique solution of the gap equation in the real axis or complex plane, in general or in the special case of the Rainbow-Ladder truncation with the Maris-Tandy model that we are going to use for our studies, under the above boundary conditions does not exist, but for all practical purposes we accept this to be true.}
where $\eta$ ($\bar{\eta}$) is the momentum partitioning parameter for the quark (anti-quark) with $\eta - \bar{\eta} = 1$, and $\Gamma_{M}^{ab}(p, P)$ is the BS amplitude. The index M is for the type of the meson: scalar, pseudo-scalar, vector or axial vector while the indexes $r, s, t$ and $u$ are for the combined color and Dirac matrix indexes, $K_{\mu}^{rs}(p, \bar{q}, P)$ is the renormalized amputated irreducible quark-antiquark scattering kernel. The way to solve this equation numerically proceeds first by multiplying the left side of (2.18) with a constant $\lambda(P^2)$ that depends on the meson momentum $P^2$, transforming it into an eigenvalue problem as a function of $P^2$. After taking projections over the covariants and Chebyshev moments of the invariants and calculating the traces of both sides, we get a set of coupled integral equations that can be solved numerically. When the eigenvalue is one we have reached the meson mass shell and the solution is the Bethe-Salpeter amplitude (BSA) of the meson.

For the pseudoscalar mesons the most general form of the BSA has four invariants:

$$\Gamma_{(PS)}^{ab}(q, P) = i\gamma_5 E(q, P) + \gamma_5 \gamma.P F(q, P) + \gamma_5 \gamma.q G(q, P) + \gamma_5 \sigma_{\mu\nu} \gamma_\mu P_\nu H(q, P),$$

(2.19)

while for the vector mesons there are eight:

$$\Gamma_{(V)}^{ab}(q, P) = \sum_{j=1}^{8} C_{j\mu}(q, P) F^{ij}(q^2, P^2, q.P).$$

(2.20)

The eight covariants $C_{j\mu}(q, P)$ appearing in the vector BSA can be written

$$C_{1\mu}(q, P) = T_{\mu\nu} \gamma_\nu, \quad C_{2\mu}(q, P) = T_{\mu\nu} q_\nu q_\mu, \quad C_{3\mu}(q, P) = T_{\mu\nu} q_\nu P.q.P,$$

$$C_{4\mu}(q, P) = T_{\mu\nu} \varepsilon_{\nu\alpha\beta\gamma} \gamma_5 \gamma_\alpha q_\beta P_\gamma, \quad C_{5\mu}(q, P) = T_{\mu\nu} q_\nu, \quad C_{6\mu}(q, P) = T_{\mu\nu} \sigma_{\nu\kappa} q_\kappa q.P,$$

$$C_{7\mu}(q, P) = T_{\mu\nu} \epsilon_{\nu\alpha\beta\gamma} q_\alpha q_\beta P_\gamma, \quad C_{8\mu}(q, P) = T_{\mu\nu} q_\nu q_\kappa \gamma_\kappa q.P,$$

(2.21)

The most general form has twelve terms but since the polarization vectors are transverse to the total momentum $P$ the physical BSA contains only components transverse to $P$. 

\[\text{[2]}\]
where \( T_{\mu\nu} = T_{\mu\nu}(P) = \delta_{\mu\nu} - \frac{P_{\mu}P_{\nu}}{P^2} \) is the transverse projection operator. The invariant amplitudes in both cases are Lorentz scalar functions of \( q^2, P^2, q \cdot P, \) and the momentum partitioning parameter \( \eta. \) For our studies a Chebychev polynomial expansion of the invariants over the angle \( \hat{q} \cdot \hat{P} \) will be employed,

\[
X(q^2, P^2, \hat{q} \cdot \hat{P}, \eta) = \sum_{j=0}^{n} X_j(q^2, P^2, \eta) U_j(u), \tag{2.22}
\]

where \( X=\{E, F, G, H, F^i\}, i=1, 2, ..., 8, \) \( X_j(q^2, P^2, \eta) \) \( j=1, 2, ..., n \) are the Chebychev moments of the invariant \( X \) and \( u=\cos \theta = \hat{q} \cdot \hat{P}. \) The Chebyshev Polynomials of the second kind obey the orthonormalization condition

\[
\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2} \, dx = \begin{cases} 
0 & : n \neq m, \\
\pi/2 & : n = m.
\end{cases} \tag{2.23}
\]

For our calculations we keep only the first four polynomials in the expansion and these are,

\[
U_0(x) = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x. \tag{2.24}
\]

Since we are working with the Euclidean metric the meson mass shell is at some time-like \( P^2, \) i.e. of \( P^2 = -m^2, \) where \( m \) is the mass of the meson. That means the quark momenta of the propagators in Eq. (2.18) are in general complex numbers and we need to solve the gap equation in the region where the momenta vary during the integration in the BSE.
2.3 Normalization and decay constants for pseudoscalar and vector mesons.

The normalization condition to ensure unit probability for a bound state in relativistic quantum field theory is

\[
1 = \frac{\partial}{\partial P^2} \left\{ \int_q^\Lambda \text{Tr}_{CD} \left[ \Gamma^{ab}_M(q, -K) S^a(q_+) \times \Gamma^{ab}_M(q, K) S^b(q_-) \right] \right\} + \int_q^\Lambda \int_k^\Lambda \left[ \chi^{ba}_M(k, K) \right]_{ut} \left[ K_{rs}^{rs}(k, q, P) \times [\chi^{ab}_M(q, K)]_{sr} \right] \bigg|_{K = P = -m^2},
\]

(2.25)

where \( \chi^{ab}_M(q, K) = S^a(q_+) \Gamma^{ab}_M(q, K) S^b(q_-) \) is the meson wave function, with \( q_+ := q + \eta P, \ q_- := q + \bar{\eta} P \) the quark momenta, \( \Gamma^{ab}_M(q, K) = [C^{-1} \Gamma^{ab}_M(-q, K) C]^t \) is the anti-meson BSA, \( C = \gamma_2 \gamma_4 \) is the charge conjugation matrix, and \( A^t \) denotes the transpose of the matrix A. The trace in the first term is over both Dirac and color indexes (the last one will just give us a factor of 3), and we should notice that only the kernel \( K_{rs}^{rs}(k, q, P) \) can depend on P in the second term. If the kernel is independent of the total momentum, e.g. ladder kernel, then the derivative of this term vanishes.

A similar expression is true for the vector mesons but now, because of the three spin polarizations, we need to take the polarization average.

Another quantity we are going to calculate is the electroweak decay constant of mesons. The definition of the electroweak decay constant \( f_H \) of a charged pseudoscalar meson is the following [62]:

\[
< 0 | q^b \gamma^\mu \gamma_5 q^a | H^{ab}(P) > = i f_H^{PS} P_\mu,
\]

(2.26)

where \( | H^{ab}(P) > \) is the meson state with total momentum \( P_\mu \). The meson state is normalized according to relation (2.25) and additionally the phase has been chosen in such a way that the decay constant is real and positive. Working out the details in the definition Eq. (2.26) we end up with the following relation for the pseudoscalars decay
constant expressed in terms of the meson normalized BSA and quark propagators:

\[
(2.27) \quad f^P_H = \frac{Z_2 N_C}{P^2} \left\{ \int_\Lambda d^4 q \left( \frac{2\pi}{4} \right)^4 P_\mu \text{Tr}_D \left[ \Gamma^{ab}_M(q, P) S^b(q_-) \gamma_\mu \gamma_5 S^d(q_+) \right] \right\},
\]

where \(P^2 = -m_H^2\) and \(N_C = 3\) is the number of colors, from the trace over the color indexes. \(Z_2\) is the quark propagator renormalization constant and essentially this is almost the same \((Z_2 = 0.970)\) for all quarks under study since, for the range of quark masses \(0 - 4\) GeV, there is a variation in the fourth decimal point only. The decay constant is related to the total electroweak decay rate of the meson through the relation (see [80]):

\[
(2.28) \quad \Gamma(H \rightarrow \nu_l + \nu_l \gamma) = \frac{G_F^2 |V_{qq'}|^2}{8\pi} f_H^2 m_l^2 m_H (1 - \frac{m_l^2}{m_T^2})^2 [1 + \mathcal{O}(\alpha)],
\]

where \(m_l\) and \(m_H\) are the masses of the lepton and the meson, and \(V_{qq'}\) is the Cabbibo-Kobayashi-Maskawa mixing matrix element. The \(\mathcal{O}(\alpha)\) term, where \(\alpha\) is the fine-structure constant, contains higher order corrections due to radiative phenomena. Using Eq. (2.28) one can calculate the experimental pseudoscalar mesons electroweak decay constant from the experimentally measured electroweak decay rate.

For the vector meson decay constant we have in similar way [62]:

\[
(2.29) \quad \langle 0 | \bar{q}^b \gamma_\mu q^a | H^{ab}(P, \lambda) \rangle = i f^V_H m_H \epsilon_\mu^{(\lambda)}(P),
\]

where now we have to include the polarization vector \(\epsilon_\mu^{(\lambda)}(P)\) of the vector meson for which we have, as we already mentioned earlier, \(P \cdot \epsilon_\mu^{(\lambda)}(P) = 0\), and the normalization condition \(\epsilon_\mu^{(\lambda)*}(P) \cdot \epsilon_\mu^{(\lambda)}(P) = 3\), where \(\lambda = 1, 2, 3\), and summation over both indexes is assumed. We can express the last equation as a loop integral:

\[
(2.30) \quad f^V_H = -\frac{Z_2 N_C}{3P^2} \left\{ \int_\Lambda d^4 q \left( \frac{2\pi}{4} \right)^4 P_\mu \text{Tr}_D \left[ \Gamma^{ab}_M(q, P) S^b(q_-) \gamma_\mu S^d(q_+) \right] \right\},
\]
where the only difference between this relation and (2.27) is the factor $1/3$ because of the polarization of the vector meson, and the appearance of matrix $\gamma_\mu$ instead of having $\gamma_\mu\gamma_5$. The pseudoscalar meson can decay through a $W$-boson channel, so finally we end up with a lepton and its anti-neutrino, while the equal quark vector meson becomes a virtual photon that will give us a lepton-antilepton pair. In the last case we can find a simple relation between the experimentally measurable decay rate and the decay constant of the particle [62]:

$$f_H^V = \left( \frac{3m_V\Gamma_{V\to\ell\bar{\nu}}}{4\pi\alpha^2Q_f^2} \right)^{1/2},$$

where $\alpha$ is the electroweak coupling constant and $Q_f$ is the electric charge of quark of flavor $f$. For the decay constants of the $\rho$ and $K^*$ we can use the partial decay width of the decays $\tau \to \rho\nu_\tau$ and $\tau \to K^*\nu_\tau$ given by the relation [62]

$$\Gamma(\tau \to V\nu_\tau) = \frac{G_F^2|V_{ab}^2|}{8\pi} f_V^2 m_\tau m_V^2 (1 - \frac{m_\tau^2}{m_V^2})(1 + \frac{m_\tau^2}{2m_V^2}),$$

where $m_\tau$ and $m_V$ are the masses of the $\tau$ lepton and the vector meson.

2.4 Dynamical chiral symmetry breaking (DCSB) and confinement.

One of the most important challenges of any theory that tries to describe properties of hadrons is to give a satisfactory and consistent explanation for the lack of free quark and gluons [53]. In the case of QCD, the underlying algebraic structure of the theory is believed to be the fundamental source of the confinement phenomenon. A proof of this does not yet exist. By using the gap equation as a starting point with a simple model for the gluon propagator, we will demonstrate how we can have DCSB (a dynamically generated quark mass) and how this is also connected to the confinement of quarks. Before we do so though, it is essential to briefly discuss chiral symmetry and its connection with the mass.
Starting at the fundamental level of the Lagrangian containing quark fields, chiral symmetry implies invariance under the following global gauge transformation (assuming only two flavors):

\begin{equation}
q(x) \rightarrow q'(x) = e^{-i\gamma_5 \hat{\phi}} q(x)
\end{equation}

where \( \hat{\phi} = \frac{\vec{s}}{2} \cdot \vec{\phi} \) with \( \tau^i \) are the Pauli isospin matrices. There is a quantum number called helicity \( h \) describing the relative orientation of the spin of a particle and its momentum \( \vec{p} \), defined as \( h = \hat{s} \cdot \hat{p} \) (where the hat over the vectors is to show that these are unitary vectors). It is obvious that a massless particle will have a definite helicity, either positive or negative, and it is a Lorentz invariant quantity, but for all other particles, helicity will depend upon the frame of reference. The particle’s velocity can be reversed by a boost to a new frame if it is moving at a speed less than the speed of light and therefore has some rest mass. In the case of fermions it is convenient to introduce the chirality projection operators \( \hat{P}_{L/R} \) (the justification will be given later) to get the two chirality components of the field \( q(x) \):

\begin{align}
\hat{P}_R q(x) &= \frac{1}{2}(I + \gamma_5)q(x) = q_R(x), \\
\hat{P}_L q(x) &= \frac{1}{2}(I - \gamma_5)q(x) = q_L(x)
\end{align}

where \( I \) is the \( 4 \times 4 \) unitary matrix, \( q_R(x) \) is the right-handed component and \( q_L(x) \) the left handed component. For a massless fermion chirality is identical with helicity. For massive fermions, helicity and chirality differ by an amount that decreases with increasing momentum. Since \( [\gamma_5, \gamma_\mu] = 0 \), it is obvious that all terms of the QCD Lagrangian (2.7), except the mass term \( \bar{q}(x)m_0 q(x) \), do not mix right- and left-handed fermion components. The massless limit of the QCD Lagrangian is invariant to independent internal rotations of the type (2.33) applied to the right- and
the left-handed components of \( q(x) \). This symmetry is described by the group an 
\( SU(N_f)_R \times SU(N_f)_L \) in the case of \( N_f \) flavors. When \( m_0 \neq 0 \) we have explicit chiral symmetry breaking and one can consider the mass as a measure of the breaking of the symmetry. The same can been seen through the free fermion propagator

\[
S(p) = \frac{-i \not{p} + m_0}{p^2 + m_0^2},
\]

which under the chiral transformation gives

\[
S(p) \rightarrow S'(p) = e^{i\gamma_5} S(p) e^{i\gamma_5} = \frac{-i \not{p} + m_0 e^{2i\gamma_5}}{p^2 + m_0^2}.
\]

The reason we mention this is because the fermion condensate is intimately related to the behavior of the chiral limit fermion propagator, and especially the chiral limit mass function, since by definition:

\[
<\bar{q}q>^0_{\mu} = -N_c \lim_{\Lambda \rightarrow \infty} [Z_4(\mu^2, \Lambda^2) \int_\Lambda d^4q \frac{1}{(2\pi)^4} Tr_D(S_{\mu=0}(q))] \tag{2.37}
\]

where \( N_c = 3 \) is the number of colors, \( Z_4 \) is the quark mass renormalization constant that depends on the renormalization point \( \mu \) and on the regularization scale \( \Lambda \). The quark condensate depends on the renormalization point \( \mu \), but not on the gauge parameter or the regularization mass scale \( \Lambda \). From the last expression we get,

\[
<\bar{q}q>^0_{\mu} = -4N_c \lim_{\Lambda \rightarrow \infty} [Z_4(\mu^2, \Lambda^2) \int_\Lambda d^4q \frac{1}{(2\pi)^4} \frac{M_0(q^2)}{A_0(q^2) q^2 + M_0^2(q^2)}], \tag{2.38}
\]

with the mass function in the chiral limit \( M_0(q^2) \) appearing also in the numerator of the integrand. If the mass function is not zero then we have DCSB and a non-zero quark condensate. The value of the condensate therefore is a measure of dynamical chiral symmetry breaking. One can also notice that:

\[
\bar{q}q = \bar{q}_L q_R + \bar{q}_R q_L \tag{2.39}
\]
so the condensate involves coupling of right and left handed fermions.

In QCD the lightest quarks $u$ and $d$ are believed to have a few MeV current mass, much smaller than the typical non-perturbative scale of QCD where $\Lambda_{QCD} \sim 300 \text{ MeV}$. Hence to first approximation one can consider $u$- and $d$- quarks to be almost massless. As a result we have an approximate chiral symmetry as well as an approximate isospin symmetry (i.e. two SU(2) symmetries in flavor space). Therefore in the case where we have the up and down flavors of quarks included in the Lagrangian the overall symmetry is an $SU_A(2) \times SU_V(2)$. There is also an equivalent way to express the action of the two symmetry operators. We can describe the combined action of the two operations as the action of the transformation:

$$
(2.40) \quad \hat{T} = U_L U_R = e^{-iP_L \hat{\phi}_L} e^{-iP_R \hat{\phi}_R}
$$

which involves the projection operators $\hat{P}_L/R$ and $\hat{\phi}_L/R = \vec{\tau} \cdot \vec{\phi}_L/R$. We can now understand the reasons for the introduction of the two projection operators in (2.34). We also have another way to interpret chiral symmetry using (2.40). We can say that chiral invariance regards an SU(2) symmetry that can be realized in an independent way (since we can have different values for the two infinitesimal parameters $\vec{\phi}_L/R$) in the spaces defined by the two projection operators $P_L/R$ so at the end we are going to have an $SU(2)_R \times SU(2)_L$ symmetry. For this to be true we should have no mass for the fermions and no interaction terms that mix right and left handed states or, in other words, interactions that do not flip the spin of the particles. That means we should have conservation of helicity. So schematically we have:

Conservation of helicity $\longleftrightarrow$ Chiral symmetry + no spin-flip terms.

If one attempt to understand the mass spectrum of baryons and mesons using a
naive constituent mass approach for the quarks, which implies explicit chiral symmetry breaking, then there are unexplained inconsistencies. For example, if we use the proton mass to extract some constituent mass for the up and down quarks, then that mass will be about 350 MeV or so. But then how it is possible to have the pion mass to be only 140 MeV? On the other hand if we assume that the masses of the two quarks is zero then how it is possible at the end to get particles with finite masses and why the masses between the pion and the $\rho$ meson are so different? To answer these and additional questions arising from other experimental data we essentially have to include in the picture the effects of the QCD vacuum. The answer is dynamical chiral symmetry breaking (DCSB). The QCD Lagrangian may not break the chiral symmetry but the QCD vacuum can. The QCD vacuum is a dynamical state where we have, among other things, the creation of quark-antiquark pairs and the energy required to obtain such a pair is very small if they are massless. Since they strongly attract each other we expect, like in the case of the condensate of electron pairs in superconductivity, to have a condensate of quark-antiquark pairs. This state should be characterized by the same quantum numbers as the QCD vacuum, namely, zero total momentum and total angular momentum. So, going back to relation (2.37) for the quark condensate, we can see that the quark propagator should have a nonzero effective mass $M_0(q^2)$ acquired through the interaction with the vacuum. According to Goldstone’s theorem, since we have $N_f^2 - 1$ spontaneously broken symmetry generators, we should have three, spin zero, massless bosons for $N_f = 2$ and eight Goldstone bosons for $N_f = 3$. One can identify the three pions as the Goldstone bosons in the first case and the octet of pseudoscalar mesons for the last one. The fact that the pions have actually a small non-zero mass is an indication of a very small non-zero
current mass for the up and down quarks. The quarks will finally get a very large effective or dressed mass because of dynamical chiral symmetry breaking. It is worth noticing here that if one starts with a massless Lagrangian then, since perturbation theory gives a self-energy of the fermion that is proportional to the quark current mass, the quark will never acquire mass. DCSB is a non-perturbative phenomenon and as we will see next can be explained through the Gap equation of the quark propagators.

Let us consider the approximation where the gluon propagator is just a delta function [106]:

\[ g^2 D_{\mu\nu}(k) = (2\pi)^4 G_0 T_{\mu\nu}(k) \delta^4(k), \]

(2.41)

where \( G_0 \) is the model’s mass-scale. We also consider the quark-gluon vertex to be bare: \( \Gamma_{\mu}^i(p, q) = \gamma_{\mu} \frac{\lambda^i}{2} \). Then the Gap equation will be reduced to the following:

\[ i \not{p} A(p^2) + B(p^2) = i \not{p} + m_0 + G_0 \gamma_{\mu} \frac{-i \not{p} A(p^2) + B(p^2)}{p^2 A^2(p^2) + B^2(p^2)} \gamma_{\mu} \]

(2.42)

for which no regularization mass scale is needed since the model is ultraviolet finite.

We take all dimensional quantities to be in units of \( G_0 \) and give results for \( G_0 = 1 \).

From (2.42) after making a projection and taking traces we get a system of coupled nonlinear algebraic equations for functions \( A(p^2), B(p^2) \):

\[ A(p^2) = 1 + 2 \frac{A(p^2)}{p^2 A^2(p^2) + B^2(p^2)} \]

(2.43)

\[ B(p^2) = m_0 + 4 \frac{B(p^2)}{p^2 A^2(p^2) + B^2(p^2)}. \]

(2.44)

In the chiral limit \( m_0 = 0 \), and Eq. (2.44) becomes:

\[ B(p^2) = 4 \frac{B(p^2)}{p^2 A^2(p^2) + B^2(p^2)}, \]

(2.45)
which has two solutions. The first one is the trivial solution,

\begin{equation}
B(p^2) = 0,
\end{equation}

and (2.43) yields

\begin{equation}
A(p^2) = \frac{1 + \sqrt{1 + \frac{8}{p^2}}}{2}, \quad p^2 > 0
\end{equation}

(the other solution of A is not consistent with the condition \(A(p^2) \to 1\) as \(p^2 \to \infty\) as well as the requirement that the quark self-energy should be real in the space-like region). So this is a result in agreement with that of perturbative QCD and does not give us something new. The second solution though is more interesting. In the other case we have:

\begin{equation}
p^2 A^2(p^2) + B^2(p^2) = 4,
\end{equation}

for which (2.43) gives

\begin{equation}
A(p^2) = 2,
\end{equation}

and finally using this in (2.48) we get

\begin{equation}
B(p^2) = 2\sqrt{1 - p^2}, \quad p^2 \leq 1.
\end{equation}

With the mass function definition \(M = B/A\), then

\begin{equation}
M(p^2) = \sqrt{1 - p^2} \quad for \quad p^2 \leq 1 \quad and \quad M(p^2) = 0 \quad for \quad p^2 > 1.
\end{equation}

So from (2.51) we can right away see the salient features of a DCSB solution of the system. One can start with a massless fermion, and the low-momentum interaction with the vacuum, produces a non-zero momentum dependent mass. We also see that
the mass dressing increases as we go deeper in the time-like region. At this point it is also worth noticing that the value of the parameter $G_0$ of the model does not affect the existence of a DCSB solution of the gap equation. For a model where the gluon propagator reads:

\begin{equation}
(2.52) \quad g^2 D_{\mu \nu}(p - q) = c_g \delta_{\mu \nu} \theta(\Lambda^2 - q^2)
\end{equation}

where $c_g$ is an interaction strength parameter and $\Lambda$ is a cutoff, it was found that DCSB can only happen if the strength of the interaction exceeds a critical value $c_g^{cr} = \sqrt{3\pi}/\Lambda$ [69]. So it appears that whenever a Munczek-Nemirovsky (MN) [106] type of model is used, the interaction is strong enough to guarantee the existence of a DCSB solution.

Now, to the previous observations for the special features of a DCSB solution we should also add one more that can be linked to confinement. We found a DCSB solution for the case where $A$ and $B$ are such that Eq. (2.48) is satisfied. This expression actually is the denominator of the quark propagator amplitudes and the previous notice implies the lack of free particle-like poles for the propagator. One can interpret this as a realization of quark confinement. One can also be tempted to reverse this observation and say that all models providing quark confinement should at the same time exhibit DCSB but the model of Eq. (2.52) is a counter example.

In the general case where $m_0 \neq 0$ one can solve the system of Eq. (2.43) and (2.44) algebraically and get four different closed form solutions one of which satisfies the physically acceptable ultraviolet limit behavior $B(p^2) \to m_0$ as $p^2 \to \infty$ and that solution exhibits the trait noticed and linked to confinement in the chiral limit case.
Extensive studies have been done using the MN model but with a dressed quark-gluon vertex instead [95]. In this case the vertex has a Dirac structure that corresponds to a specific type or subclass of interaction diagrams and in addition to the equations for the quark propagator functions A and B we have a system of algebraic equations for the invariants of the vertex. One can solve this system numerically and obtain qualitative information about the role of the subclass of diagrams in the realization of different physical observables ([12], [52], [53], [66], [78]).

Chiral symmetry can be used to extract a relation between the axial vector vertex, pseudoscalar vertex and quark propagators that is called axial vector Ward-Takahashi identity (AV WTI), but we will leave this discussion for the next chapter where we will show how this identity can be used to construct the Maris-Tandy model.
Chapter 3

Dynamically dressed quark propagator

At this point we are going to introduce a truncation scheme and an effective model for the solution of the gap equation and the BSE. We will briefly explain the guiding constraints followed for the construction of the model and discuss in detail the behavior of the solution of the gap equation, since it will have a decisive role in the possibility of solving the BSE.

3.1 Chiral symmetry and the axial vector Ward-Takahashi identity (AV WTI).

From the equation for the quark propagator (2.16) we see that its solution requires the knowledge of two other functions: the gluon propagator and the quark gluon vertex. So one has to solve the corresponding integral equations which also require the knowledge of higher n-point functions and ultimately that means we should solve an infinite system of integral equations. It is necessary therefore to truncate that system and reduce it to a tractable problem. We also need to construct a model to represent the effects of the missing equations. The driving force for the truncation scheme and the construction of the model we are going to use is chiral symmetry as expressed through the axial vector Ward-Takahashi identity. Since the generating functional should be invariant under the chiral transformation one can show that this is true if the following identity is satisfied for the flavor non-singlet quark states [66]:

\[ -i P_\mu \Gamma^\ell_{5\mu}(q, P) = S^{-1}(q_+) \gamma_5 \mathcal{F}^\ell + \gamma_5 \mathcal{F}^\ell S^{-1}(q_-) - 2 \mathcal{M}^{\ell m} \Gamma^m_5(q, P) \]
where $\mathcal{M}^{\ell m} = tr_F[\{\mathcal{F}^{\ell}, \mathcal{M}\}\mathcal{F}^{m}]$ with the trace taken over the flavor indices. $\mathcal{M}$ is a diagonal matrix $\mathcal{M} = \text{diag}\{m_u(\mu), m_d(\mu), m_s(\mu), m_c(\mu), m_b(\mu), \ldots\}$ of the current quark masses at renormalization scale $\mu$ and $\{\mathcal{F}^{\ell}|\ell = 0, 1, \ldots, N_f^2 - 1\}$ are the generators of the group $U(N_f)$ in the fundamental representation, obeying the orthonormalization condition $\text{tr}(\mathcal{F}^{\ell}\mathcal{F}^{m}) = \frac{1}{2} \delta^{\ell m}$. $P = q_+ - q_-$ is the total momentum and $q$ the relative momentum of the two quarks, while the dressed propagator $S = \text{diag}(S_u, S_d, S_s, S_c, S_b, \ldots)$. $\Gamma^{\ell}_5(q, P)$ is the renormalized axial vector vertex with $\Gamma^{\ell}_5(q, P) = \mathcal{F}^{\ell} \Gamma_5(q, P)$ where $\Gamma_5(q, P)$ satisfy the inhomogeneous equation [60],[66]:

(3.2) \[ [\Gamma_5(p, P)]_{tu} = Z_2 [\gamma_5 \gamma_{\mu}]_{tu} + \int_q^{\Lambda} K_{rs}^{\mu}(k, q, P)[S(q^+)] \Gamma_5(q, P) \Gamma_5(q, P)[S(q^-)]_{rs} \]

With $\Gamma^{\ell}_5(q, P) = \mathcal{F}^{\ell} \Gamma_5(q, P)$ a similar equation for the pseudoscalar $\Gamma_5(q, P)$ can be obtained from (3.2) by just replacing $\Gamma_5$ with $\Gamma_5$, $Z_2$ with $Z_4$ and $\gamma_5 \gamma_{\mu}$ with $\gamma_5$ in the first term in the right of that equation. In the chiral limit the last term in (3.1) is zero and then one can show that the axial vector vertex has the following structure [60]:

\[
\Gamma_5(p, P) = \gamma_5[\gamma_{\mu}F_r(p, P) + \not{p}_\mu G_r(p, P) - \sigma_{\mu\nu}p_\nu H_r(p, P)] +
\]

(3.3) \[ \bar{\Gamma}_5(p, P) = f_{PS} \frac{P}{P^2} \Gamma_{PS}(p, P), \]

involving the pseudoscalar BSA $\Gamma_{PS}(p, P)$, the electroweak decay constant $f_{PS}$, which is actually the residue of the pseudoscalar pole in the axial vector vertex, and $F_r, G_r, H_r$ as well as $\bar{\Gamma}_5$ that are finite as $P^2 \to 0$. It is obvious from this last expression that in the limit $P^2 \to 0$ the behavior of the axial vector vertex will be dictated by that of the pseudoscalar BSA. By using Eq. (3.3) in the (AV WTI) we can also get some very important relations between the invariant amplitudes of the BSA, the $F_r, G_r, H_r$.
amplitudes and functions $A, B$ of quark propagator in the chiral limit [60],[66]:

\begin{align}
(3.4) & \quad f_{PS} E(p, 0) = B(p^2) \\
(3.5) & \quad F_r(p, 0) + 2f_{PS} F(p, 0) = A(p^2) \\
(3.6) & \quad G_r(p, 0) + 2f_{PS} G(p, 0) = A'(p^2) \\
(3.7) & \quad H_r(p, 0) + 2f_{PS} H(p, 0) = 0
\end{align}

In the case of DCSB we have a nonzero function $B(p^2)$ resulting in a nonzero momentum dependent quark mass and from the above equations we see that there is a $P^2 = 0$ pseudoscalar meson solution for the homogeneous BSA. This Goldstone boson will exist if:

\begin{equation}
(3.8) \quad E_{PS}(p, 0) = i\gamma_5 \frac{B(p^2)}{f_{PS}}
\end{equation}

is non-zero, i.e. if there is a non-zero $B(p^2)$ from DCSB.

The above analysis reveals some other aspects and consequences of the DCSB related to the pseudoscalar meson BSE and axial-vector vertex BSE solutions. In the case where there is a small nonzero mass for the quarks that explicitly breaks chiral symmetry both $\Gamma_{5\mu}^\ell (q, P)$ and $\Gamma_{5}^\ell(q, P)$ have a pole at finite mass and the residues on the left and right of Eq. (3.1) must be equal. In the chiral limit the residue of the bound state pole in the flavor non-singlet pseudoscalar vertex is $(- < \bar{q}q >^0_\mu)/f_{PS}^0$, and then the aforementioned relation will give us the well known Gell-Mann-Oakes-Renner relation [60],[66],

\begin{equation}
(3.9) \quad f_{PS}^2 m_{PS}^2 = -[m_a(\mu) + m_b(\mu)] < \bar{q}q >^0_\mu + O(m_q^2),
\end{equation}
where $m_{PS}, f_{PS}$ are the mass and decay constant of the pseudoscalar meson with constituent quarks of flavor $a$, $b$. $< \bar{q}q >^0_\mu$ is the quark condensate in the chiral limit defined in (2.37).

From the two integral equations for the axial-vector and pseudoscalar vertices for equal quark systems we can get, by addition, a new integral equation that reads:

$$[U(p, P)]^\mu_{tu} = -iZ_2[\gamma_5 \gamma_\mu P_\mu + 2\gamma_5 m(\mu)]_{tu} + \int_q^\Lambda K^{sr}_{tu}(p, q, P)[S(q+)]U(q, P)S(q-)]_{rs},$$

(3.10)

where we have set

$$U(p, P) = -iP_\mu \Gamma_5^\mu + 2m(\mu)\gamma_5.$$

(3.11)

The AV WTI can be used to write $U(p, P)$ in terms of the quark propagators so:

$$U(p, P) = S^{-1}(p_+ )\gamma_5 + \gamma_5 S^{-1}(p_- ).$$

(3.12)

After using the last expression in Eq. (3.10) and using the gap equation for the quark propagators to express $U$ in terms of the quarks self-energies we end up with the following relation:

$$Z_1 \frac{4}{3} \int_q^\Lambda \left\{ g^2 D_{\mu\nu}(p_+ - q)[\gamma_\mu S(q)\Gamma_\nu(p_+, q)\gamma_5]_{tu} + g^2 D_{\mu\nu}(p_- - q)[\gamma_5 \gamma_\mu S(q)\Gamma_\nu(p_-, q)]_{tu} \right\}$$

$$= \int_q^\Lambda K^{sr}_{tu}(p, q, P)[S(q_+ )\gamma_5 + \gamma_5 S(q_- )]_{rs}.$$ 

(3.13)

This is, essentially, the equivalent integral form for the AV WTI (3.1) and makes more obvious the restrictions imposed by the chiral symmetry on the BSE kernel $K$, the quark-gluon vertex $\Gamma_\nu$, and gluon propagator $D_{\mu\nu}$. If we are going to make an approximation or a truncation for $\Gamma_\nu$ and $D_{\mu\nu}$ then, through the last equation, we can determine the corresponding truncated BSE kernel $K$ that preserves chiral symmetry.
The rainbow-ladder truncation scheme that we will introduce next and use for our studies satisfies (3.13).

3.2 Rainbow-Ladder Truncation and the Maris-Tandy model.

In the rainbow truncation for the gap equation we approximate the quark-gluon vertex with the bare vertex $\Gamma^i_\nu(k, p) \rightarrow \gamma_\nu \frac{\lambda^i}{2}$ and we set:

$$Z_1 g^2 D_{\mu\nu}(q) \Gamma^i_\nu(k, p) \longrightarrow 4\pi \alpha(q^2) D^\text{free}_{\mu\nu}(q) \gamma_\nu \frac{\lambda^i}{2},$$

where $D^\text{free}_{\mu\nu}(q)$ is the free gluon propagator and $\alpha(q^2)$ is an effective running coupling. From Eq. (3.13) we can get the corresponding truncation for the BSE kernel, called the ladder truncation for the BSE, and it consists of replacing the kernel $K$ with:

$$[K(p, q, P)]_{rs}^{tu} \longrightarrow -4\pi \alpha(q^2) D^\text{free}_{\mu\nu}(q) \left[\frac{\lambda^i}{2} \gamma_\mu\right]^{ru} \otimes \left[\frac{\lambda^i}{2} \gamma_\nu\right]^{ts}.$$

This is equivalent to considering the interaction between the two quarks taking place through the exchange of a single gluon with a modified coupling strength $\alpha(q^2)$ and it is actually the first term in a systematic expansion of the kernel $K$. The coupling function $\alpha(q^2)$ contains dressing effects for both gluon propagator and quark-gluon vertex and it can be considered as an effective dressing for the gluon propagator. The most important feature of this truncation scheme of course, is that it is “self-consistent” in the sense that we get vector and axial vector vertices that satisfy the corresponding Ward-Takahashi identities. The vector Ward-Takahashi identity is related to conservation of the electromagnetic current for quarks.

We are going to use a one parameter model for the running coupling $\alpha(k^2)$ that has two terms. The so called MT model has the form [62]:

$$\frac{4\pi \alpha(k^2)}{k^2} = \frac{(2\pi)^2 k^2 D}{\omega^6} e^{-\frac{k^2}{\omega^2}} + \frac{2(2\pi)^2 \gamma_m F(k^2)}{\ln[\tau + (1 + \frac{k^2}{\Lambda_{QCD}^2})^2]}$$
where D, ω are parameters of the model, \( \gamma_m = 12/(33 - 2N_f) \) is the one-loop anomalous dimension for the mass, \( N_f = 4 \) the number of quark flavors, and \( F(u) = (1 - e^{-(u/4m_t^2)})/u \). The parameter \( \Lambda_{QCD} \) defines a mass scale for the onset of non-perturbative QCD phenomena and is set \( \Lambda_{QCD} = 0.234 \text{ GeV} \), \( \tau = e^2 - 1 \) is used to prevent any singular behavior on the real axis, consistent with gluon confinement. The second term in the model is important in the ultraviolet region and is set up so that it reproduces the 1-loop perturbative QCD running coupling behavior. The first term is a phenomenological one for the behavior of the model in the infrared region \( 0 < k^2 < 1 \text{ GeV}^2 \) and provides the infrared enhancement necessary for the right value of the quark condensate. The amount of infrared enhancement is determined by the two parameters D, ω of the model and it was found that, when you vary ω within region 0.3-0.5 the quark condensate can be reproduced as \( -\langle \bar{q}q \rangle_{\mu=1}^{\text{GeV}} = (0.241 \text{ GeV})^3 \) so long as \( \omega D = (0.72 \text{ GeV})^3 \). Hence we refer to the MT model as a one parameter model in this sense. For our calculations we choose \( \omega = 0.4 \text{ GeV}, D = 0.93 \text{ GeV}^2 \) and \( m_t = 0.5 \text{ GeV} \) and the u/d- and s-current quark masses at the renormalization scale \( \mu = 19 \text{ GeV} \) fitted to the masses of pion and kaon are \( m_{u/d}(19 \text{ GeV}) = 0.00374 \text{ GeV} \) and \( m_s(19 \text{ GeV}) = 0.083 \text{ GeV} \).

The model provides the necessary quark mass dressing for DCSB and gives us quark confinement. Since its introduction, numerous studies of properties of light mesons and comparison with lattice QCD data have verified and established it as a realistic representation of the effective \( \bar{q}q \) interaction in the space-like region and in the low mass time-like region. For example calculations of the masses of the light pseudoscalar and vector mesons agree with the experimental data within 5% ([55], [60], [62]), while the corresponding electroweak decay constants, being more sensitive
to calculational details, agree within 10 % ([55], [59], [63], [79], [93]). Within the
impulse approximation for the elastic form factors ([61], [64], [82]) of the light-quark
pseudoscalar mesons, and the electroweak transition form factors of the pseudoscalar
and vector mesons, the model provided an efficient way for estimating these quantities
([81], [88], [89], [90]). Until recently the use of the model had been limited in the
case of light quark mesons because the solution of the gap equation for heavy quark
propagators is numerically challenging as we approach the heavy quark meson mass
shell. This is a result of the behavior of the model in the complex plane in that region
and maybe the effect of the proximity of singularities of the quark propagator. We
will discuss these issues in more detail in the next section and we will show how we
can solve some of these problems.

3.3 Gap equation solution.

For the numerical solution of the gap equation one has to take the projection
and traces of the equation and reduce it to a system of coupled nonlinear integral
equation for the functions A, B. The unrenormalized quark self-energy term of the
gap equation in the rainbow truncation is:

\[ \Sigma'(p) = i \not{p} \{ A'(p^2) - 1 \} + B'(p^2) = \frac{4}{3} \int_0^\Lambda \frac{d^4q}{(2\pi)^4} \frac{G(k^2)}{k^2} T_{\mu\nu}(k) \gamma_\mu S(q) \gamma_\nu, \]

where we have set \( G(k^2) = 4\pi \alpha(k^2), \) \( k = p - q \) is the gluon momentum and the factor
\( \frac{4}{3} \) comes from the trace over the color matrices. By taking the Dirac trace of the last
equation we get:

\[ B'(p^2) = 4 \int_0^\Lambda \frac{d^4q}{(2\pi)^4} \frac{G(k^2)}{k^2} \sigma_s(q^2), \]
and if we multiply by $\not{p}$ and then take the Dirac trace, we get the second equation:

$$(3.19) \quad p^2 (A'(p^2) - 1) = \frac{4}{3} \int^\Lambda \frac{d^4q}{(2\pi)^4} \frac{G(k^2)}{k^2} \left( p.q + 2 \frac{(k.p)(k.q)}{k^2} \right) \sigma_v(q^2),$$

where we have introduced the quark propagator amplitudes $\sigma_s(q^2), \sigma_v(q^2)$:

$$(3.20) \quad \sigma_s(q^2) = \frac{1}{A(q^2)} \frac{M(q^2)}{q^2 + M^2(q^2)}, \quad \sigma_v(q^2) = \frac{1}{A(q^2)} \frac{1}{q^2 + M^2(q^2)}.$$

The quark propagator in terms of $A', B'$ is then:

$$S^{-1}(p) = i \not{p} A(p^2) + B(p^2) = Z_2(i \not{p} + m_{bm}) + \Sigma'(p) =$$

$$(3.21) \quad i \not{p}(Z_2 + A'(p^2) - 1) + (m_{bm} + B'(p^2)).$$

From the renormalization condition, $S^{-1}(p)|_{p^2=\mu^2} = i \not{p} + m_r(\mu^2)$, we have that:

$$(3.22) \quad Z_2(\mu^2, \Lambda^2) = 2 - A'(\mu^2, \Lambda^2),$$

and

$$(3.23) \quad m_r(\mu^2) = Z_2(\mu^2, \Lambda^2)m_{bm} + B'(\mu^2, \Lambda^2),$$

where $m_r(\mu^2)$ is the renormalized current quark mass at point $\mu^2$ and it will actually be the parameter we will fit to experimental data. From the last two relations we can re-express the renormalization condition for $A, B$ as:

$$(3.24) \quad A(\mu^2, \Lambda^2) = 1 + A'(p^2, \Lambda^2) - A'(\mu^2, \Lambda^2),$$

$$(3.25) \quad B(\mu^2, \Lambda^2) = m_r(\mu^2) + B'(p^2, \Lambda^2) - B'(\mu^2, \Lambda^2).$$

We should notice here that the renormalization condition is applied on the solution of the gap equation on the real axis and in the case of the chiral limit we should have $Z_2(\mu^2, \Lambda^2)m_{bm} = 0$, so Eq. (3.25) becomes $B(\mu^2, \Lambda^2) = B'(p^2, \Lambda^2)$. 
For computational convenience, and without loss of generality, we choose the external quark four-momentum to be \( p = |p|(1,0,0,0) \), i.e. have a component only along the time axis. In this case the four-vector product of \( p \) and \( q \) momenta is:

\[
(3.26) \quad p.q = |p||q| \cos \omega
\]

where \( q = |q|(\cos \omega, \cos \phi \sin \theta \sin \omega, \sin \phi \sin \theta \sin \omega, \cos \theta \sin \omega) \). Since there is no term in the integrands of the two equations with dependence on the other angles, the four dimensional integral reduces to a two dimensional one:

\[
\int_{\Lambda}^{\Lambda} \frac{d^4q}{(2\pi)^4} = \frac{4\pi}{(2\pi)^4} \int_{0}^{\Lambda} q^3 \, dq \int_{0}^{\pi} \sin^2 \omega \, d\omega = \frac{1}{(2\pi)^3} \int_{0}^{\Lambda} x \, dx \int_{-1}^{1} \sqrt{1-u^2} \, du,
\]

where we have set \( x = q^2 \) and \( u = \cos \omega \).

Solving numerically the system of equations (3.18) and (3.19) on the real axis is then an easy and straightforward process. For the purpose of solving the BSE though, as we already noted in chapter 2, we need to know the quark propagator in a parabolic region in the complex plane. The external momentum in the gap equation, formerly \( y = p^2 \), will be a complex number (it will be \( y = q^2_\pm \) with \( q_+ = q + \eta P \) and \( q_- = q + \bar{\eta} P \) ) and that will create different problems in our numerics. The first problem will come in the form of singular behavior of the second term in the parenthesis inside the integral of Eq. (3.19). The first term (infrared dominant) of the MT model is designed to be zero at zero gluon momentum \( k^2 \) in order to remove this singular behavior. However the ultraviolet tail of the model is \textit{small but finite} at \( k^2 = 0 \), and singular behavior in the solution will arise. As a result the solution will have instabilities depending on the integration grid we are using. When both momenta \( p^2, q^2 \) vary in the space-like region the term \( (k.p)(k.q)/k^2 \) has a well-defined analytic behavior, and even in the
case where \( p = q \) the singularity is canceled by the behavior of the numerator. When we continue \( p^2 \) in the complex plane then a non-integrable singularity will appear when \( |p^2| = q^2 \) and for integration purposes we should integrate over \( q^2 \) along a contour that includes the external \( p^2 \) point. Numerically that is a very difficult task to deal with and the suggestion to overcome this problem was to slightly modify the second part of the MT model [10]. The modification regards function \( F(u) \) and now we set \( F(u) = (1 - e^{-(u/4m_t^2)^2})/u \) in the model. The change in the quark condensate because of the modification is less than 3% and, as one can see from fig. (3.1), there is an almost imperceptible difference, on the real axis, for functions \( \sigma_s \) and \( \sigma_v \) in the chiral limit. The same is true for all current quark masses up to a certain time-like momentum. Although there isn’t any difference in the solution of the gap equation with the modified model on the space-like real axis, there is a very important effect for the solution, essentially for all different current quark masses, on the real axis in the low time-like region. Now functions \( \sigma_s \) and \( \sigma_v \), for every quark, appear to collapse and have erratic behavior at about \( p^2 \sim -1 \) GeV\(^2\) (see fig. 3.2). Explicitly that means we can use the propagators of different quarks for the study of mesons with masses only up to 2 GeV. That may be an improvement for the light quark calculations but unfortunately for the heavy quarks this modified model creates an unsurpassed obstacle.

That can be clearly seen in the case of the c quark propagator functions \( \sigma_s/v \) on the real axis (see fig. 3.3). From fig. (3.3) we can see that we have a smooth behavior of the solution with the MT model up to \( q^2 \sim -2 \) GeV\(^2\) momentum but only up to \( q^2 \sim -1 \) GeV\(^2\) for the mMT. In the complex plane we see that the instabilities are after \( q^2 \sim -1 \) GeV\(^2\) in the first case and the solution from the mMT model collapses
Figure 3.1: $\sigma_s$ and $\sigma_v$ functions in the chiral limit using the MT and modified MT (mMT) model.
Figure 3.2: $\sigma_s$ and $\sigma_v$ functions for the u/d, s, c quark using modified MT model. Unstable behavior for all quarks starts at about $\sim -1 \text{ GeV}^2$, limiting the use of the solutions in studying mesons lighter than 2 GeV.
Figure 3.3: $\sigma_s$ and $\sigma_v$ functions for the $c$ quark on the real axis using the MT and mMT model. Unstable behavior for the $c$ quark starts at about $\sim -1 \text{ GeV}^2$, but with the MT model we have stable solution deeper in the time-like region on the real axis.
around that point making essentially the situation worse. So the trick creates a new obstacle limiting even more the available area on the complex plane for the heavy quark meson studies. It was also found that the instabilities on the real axis in the time-like region are due to the first term in the model. That can be qualitatively understood by noticing that once the external momentum $p^2$ varies in the time-like region then the argument of the exponential is a complex number. In detail we have

$$e^{-k^2/\omega^2} = e^{-(x-|y|)}e^{-i \sqrt{(x|y|)}/v}$$

where $k = p-q$, $x = q^2$, $y = p^2$, $v = \hat{q}\cdot\hat{p}$ and that term is a product of an exponential and an oscillating part. The oscillations are faster as the $\sqrt{(x|y|)}$ factor gets larger and that can cause numerical difficulty for both the angular integral and the radial one. It was found that by keeping just the ultraviolet tail of the model, the solution for the $c$ quark is stable on the time-like real axis beyond the $-2 \text{ GeV}^2$ point where the problems start for the complete model solution. In another case it was found that by using a specific number of mesh points distributed in a certain way in the interval $[0,a]$ where $1 \leq a \leq 3$, the solution on the real axis, for all quark masses, will collapse to the solution we get with the modified model (which has unstable behavior beyond $-1 \text{ GeV}^2$). Additionally it was found that the solution is sensitive to perturbations in the mesh points in the infrared region.

On the other hand, since the two functions $\sigma_s(p^2)$ and $\sigma_v(p^2)$ should go to zero for $|p|^2 \to \infty$ in all directions in the complex plane, and since they are not constants, they can not be analytic everywhere, there must be some singularities. We also expect the dominant IR term of the model to have a decisive role in the existence, location and the type of the singularities of the solution. The first pair of these singularities we anticipate to indicate the maximum mass of mesons that can be studied.

If some of these singularities occur inside the integration domain of the BSE then
special numerical methods are needed [67] which are presently developed only for simple poles and BSE modes that permit a mapping of the present two-dimensional integration domain so that one integration is trivial. In the general case we are interested in the limit the mesons we treat to be light enough so that these singularities do not enter the integration domain. For ground state mesons containing u/d- and s-quarks, in particular the pseudoscalar and vectors, this is the case. For the excited states of these light quark mesons, one is limited to a mass of about 1-2 GeV. As the current quark mass is raised, the singularities move further away from the origin and the ground state pseudoscalar and vector meson quarkonia masses increase in a way that keeps the singularities outside the needed integration domain.

At this point we must notice that the calculations in Eq. (3.18), (3.19) involve integrations over the internal quark momenta q. So we should first calculate the functions on the space-like real axis and then use this solution to calculate them in the complex plane. For the last calculation the gluon momentum argument of the MT model will vary in the complex plane and the numerics will mostly be affected by its behavior there. On the other hand we fitted the parameter of the model to the chiral limit condensate which requires the solution of the gap equation on the space-like real axis only. Hence the behavior of the model in the space-like axis is well known, trusted and fitted to a physical observable while there is no information for the behavior in the complex plane. One can avoid all these complications by a simple change in the integration variable in the gap equation integral, from the quark momentum q to the gluon momentum k. However this complicates the numerical solution of the problem. Now we need to know the quark propagator amplitudes inside the integral at grid points in the complex plane that are different from those
required for the solution of the BSE. For that reason we have to use an interpolation technique and express the values of the functions inside the integrals in terms of the other set. The accuracy of the calculation now depends on the external grid being fine enough to accurately determine their values on the grid inside the integral. The calculations are also more cumbersome. Another problem arises from the fact that by discretizing the variables we have different cutoffs for the moments $q$, $p$ and essentially for the calculations of the integrals we need to know the propagator outside the region it is finally needed. That problem can be easily solved by noticing that for such large momenta, the functions are very small, slowly varying, and they can be approximated by a constant. So for the points that are outside the needed region, we use the value at the last point (largest momentum $p^2$) of the external grid. Because of the small values of the two amplitudes there, and also because of the behavior of the MT model, we know that this approximation will have no effect in our calculations of the BSA and the different physical observables. The last and more serious problem is the singularities of the solution. Now we should worry about the existence of singular points inside the integration domain not at the step where we solve the BSE but at the initial step where we solve the gap equation.

3.4 Quark propagator within meson dynamics.

With this approach all numerical and instability problems are solved and the only restriction in the calculation of the propagators in the complex plane comes from its singularities. The type and the location of these points is not known and that means we have to use a trial and error approach to locate them. In the case of the $c$ quark for example with current mass $m_c(19 \text{ GeV}) = 0.88 \text{ GeV}$ and for the parabolic
region determined by $q_+^2 = (\tilde{q} + \eta P)^2 = q^2 - (\eta M)^2 + 2i\eta \sqrt{q^2 M^2} v$ where $\tilde{q}$ is the BSE integration variable, with $\eta = 0.50$ and $P^2 = -M^2 = -14 \text{ GeV}^2$ (the peak of the region will be then at $(\eta P)^2 = -3.5 \text{ GeV}^2$) we have functions $Re(A)$, $Im(A)$, $Re(M)$, $Im(M)$ and $Re(\sigma_s)$, $Im(\sigma_s)$ plotted in Figures (3.4, 3.5, 3.6, 3.7, 3.8, 3.9) correspondingly. For the studies of c-quark systems we need to solve the gap equation for $P^2$ as small as $P^2 = -M_V^2 \sim -9.6 \text{ GeV}^2$, so the peak is at $(\eta P)^2 \sim -2.4 \text{ GeV}^2$ and the singularities are far enough from the BSE integration domain to not cause any trouble. The $P^2 = -14 \text{ GeV}^2$ for these plots was chosen because, for $P^2$ smaller than that, numerical solution of the gap equation is not possible since we are extremely close to the singularities, and moreover these singularities can be clearly seen in the plots for this case.

The two approaches, solving the gap equation with integration over the quark momentum and in the other case over the gluon momentum, are mathematically equivalent and the numerics should give the same answer as the number of mesh points of the grid goes to infinity. For a practical finite grid though, the first approach can still be applied for the chiral limit and light quark studies, using the modified model or the MT model after choosing a value for the parameter $\eta$ such that the parabolic region does not contain instabilities or singularities. For quarks heavier than about 300 MeV one has necessarily to resort to the second method.

From these plots we can see that only function $\sigma_s$ (and the same is true for $\sigma_v$) has singularities, but not A and M. We can conclude then that the singularity is the point where the denominator of $\sigma_{s/v}$ vanishes, i.e. $q_+^2 + M^2 (q_+^2) = 0$. We will discuss this

\footnote{For practical numerical convenience, for the total momentum $P$ of the meson, we apply the same trick we used in the external momentum $p$ for the solution of the gap equation on the real axis and we assume $P = |P|(1,0,0,0)$ without loss of generality [155].}
Figure 3.4: $Re(A)$ for the $c$ quark in a parabolic region in the complex plane ($P^2 = -14 \text{ GeV}^2$, $\eta=0.50$) with integration over the gluon momentum (kcp). There isn’t any unstable behavior or any singularities. In this plot and in the following ones the straight lines connecting the end-points have no significance and should be ignored.
Figure 3.5: $Im(A)$ for the c quark in a parabolic region in the complex plane ($P^2 = -14 \text{ GeV}^2$, $\eta=0.50$) with integration over the gluon momentum (kcp). $Im(A)$ has a smooth behavior too.
Figure 3.6: $Re(M)$ for the c quark in a parabolic region in the complex plane ($P^2 = -14 \text{ GeV}^2$, $\eta = 0.50$) with integration over the gluon momentum (kcp). Like in function A there isn’t any unstable behavior or any singularities.
Figure 3.7: $\text{Im}(M)$ for the $c$ quark in a parabolic region in the complex plane ($P^2 = -14 \text{ GeV}^2$, $\eta=0.50$) with integration over the gluon momentum (kcp).
Figure 3.8: $Re(\sigma_s)$ for the c quark in a parabolic region in the complex plane ($P^2 = -14 \text{ GeV}^2$, $\eta=0.50$) with integration over the gluon momentum (kcp). The peak of the parabolic region is at $(\eta P)^2 = -3.5 \text{ GeV}^2$. There isn’t any unstable behavior but we see indications of the existence of a pair of complex conjugate singularities near the peak of the region, approximately located at $(x_o, y_o) \sim (-2.7, \pm 3.5) \text{ GeV}^2$. The type of the singularities is unknown.
Figure 3.9: $\text{Im}(\sigma_s)$ for the c quark in a parabolic region in the complex plane ($P^2 = -14 \text{ GeV}^2$, $\eta=0.50$) with integration over the gluon momentum ($k_{\text{cp}}$). We have the same singularities as in the real part and no instabilities.
Figure 3.10: $Re(\sigma_s)$ for the $c$ quark in a parabolic region in the complex plane ($P^2 = -8 \, GeV^2$, $\eta=0.50$) with integration over the gluon momentum ($k_{cp}$) and keeping only the first IR term of the kernel. The peak of the parabolic region is at $(\eta P)^2 = -2.0 \, GeV^2$. The solution is different but we still have the general characteristics of the solution from the complete MT model. The first pair of singularities approximately appear to be $(x_o, y_o)\sim(-1.7,\pm2) \, GeV^2$.

issue in detail, as well as the role and importance of the imaginary parts of functions $A$ and $M$ (these are odd functions of the $Im(q^2)$ as they should be) in the solution of the BSE, in chapter 5.

By keeping only the infrared dominant first term of the model the solution for $Re(\sigma_s)$ for the same quark is plotted in fig. (3.10). We can still see the general characteristics of the complete solution but there is also a shift of the singularities closer to the space-like region, and also closer to the time-like real axis, making smaller
Figure 3.11: Projection of $Re(\sigma_s)$ for the c quark onto the real plane xz ($x=Re(q_s^2)$, $z=Re(\sigma_s)$) with integration over the gluon momentum (kcp), with the complete MT model and keeping only the first term in the model. One can clearly see now that in the second case there is a shift of the singularities towards the space-like region. Since it is not so easy to compare graphs in three dimensions we can get some information from a projection of the graphs onto the z-x plane ($z=Re(\sigma_s)$, $x=Re(q_s^2)$). From fig. (3.11) we see that by keeping only the first term in the model we have a pair of complex conjugate singularities around $-1.7 \, GeV^2$ on the real axis. The vertical lines at different $Re(q_s^2)$ represent the variation of the function along the $y=Im(q_s^2)$ direction. The end-points of the vertical lines, that are not connected, correspond to the values of the function when the imaginary part is minimized (for angular grid point closest to zero) and they are essentially the values of the function on the real axis. These points can not be seen in this plot once we
are between the two singularity peaks. They also have been used to compare with the solution of the gap equation on the real axis providing a test for the validity and accuracy of the gap equation solution in the complex plane. Finally this plot can give us a better view of the variation of the amplitude along the imaginary axis and we actually observe that this variation is about the same, for both solutions, for as deep as \( \text{Re}(q_{\perp}^2) \sim -1 \) in the time-like region.

Since the pair of complex singularities moved closer to the space-like region, and to the time-like real axis, we expect to provide a stronger support for the BSE kernel for smaller \(|P|^2\) than that in the complete model calculations, therefore the meson mass will be smaller. Indeed it was found that in this case the pseudoscalar \( c\bar{c} \) meson mass is\(^2\) \( m = 2.408 \text{ GeV} \) which is lighter by 0.627 GeV or 20.7 \% of the mass 3.035 GeV from the full interaction calculation. So the first infrared term accounts for about 80 \% of the meson mass. The decay constant was found to be \( f=0.276 \text{ GeV} \), smaller by 0.111 GeV or by 28.7 \% of the decay constant \( f=0.387 \text{ GeV} \) from the full MT calculation. By using the propagator from the full model in the calculation of the two integrals for the constants \( N \) (normalization constant determined by Eq. (2.25))and decay constant \( f \) (given by Eq. (2.27)) we get \( f=0.260 \text{ GeV} \), so we still have a value smaller by 32.8 \% from that of the full model calculation. The situation is better if we replace the BSA in these two integrals with the one from the full MT calculation. Then \( f=0.317 \text{ GeV} \) and is smaller by 18.9 \%. Therefore most of the difference is due to the IR calculated BSA, and not because of the IR gap solution used in the \( N, f \) integrals. In the case where we keep only the UV term in the MT

\(^2\)Notice again that the mass shell for this case is at \( P^2 = -5.8 \text{ GeV}^2 \) and from the plot of \( \text{Re}(\sigma_s) \) for which \( P^2 = -8.0 \text{ GeV}^2 \) one can see we are safely far enough from the singularities. This is true for the rest of the calculations and the plots presented in this chapter.
Figure 3.12: $Re(\sigma_s)$ for the c quark in a parabolic region in the complex plane ($P^2 = -8 \text{ GeV}^2$, $\eta=0.50$) with integration over the gluon momentum ($k_{cp}$) and keeping only the UV term in the model. The peak of the parabolic region is at $(\eta P)^2 = -2.0 \text{ GeV}^2$. The first singularity is on the real axis around $(x_o, y_o)\sim(-1.9, 0) \text{ GeV}^2$.

model, the solution has the first singularity on the real axis (see fig. 3.12) and a meson mass shell can not be reached.

For the fictional pseudoscalar $s\bar{s}$ system, performing similar analysis, the IR solution gave a mass 0.565 GeV (see Table 3.2), which is smaller by 18.7 % from that of the full model (0.696 GeV), while from the full model decay constant of 0.182 GeV we end up to 0.151 GeV with only the IR part of the interaction. That is a 17.0 % decrease in the decay constant value. If we use the full model gap solution or the full model BSA in the decay constant calculations we get a 0.166 GeV (smaller by 8.8 % than the full model f) and a 0.163 GeV (smaller by 10.4 % than the full model
f) decay constant, revealing that now the IR BSA and the IR gap propagator have about the same responsibility for the smaller value of the decay constant. No meson mass was possible to be reached with the UV tail term.

The same analysis for the b quarkonium on the other hand, revealed many interesting and important things for the role and the interplay of the IR, UV term and the effects of the heavy quark mass behavior of the propagator. We start our analysis by pointing out some important features of the full model solution. Compared to the c quark Re(M) behavior, the b quark Re(M) appears to be almost flat at the peak of the parabolic region. Re(A) also appears to vary little, just above one, in the complex plane. From the b quark Re(σs) plot we found the singularities have not only moved, as was expected, deeper in the time-like region, along the real axis, but also further apart along the imaginary axis, and there is an almost flat area between the singularities peaks. This is exactly the opposite from a constituent-like behavior of a propagator. The importance of these observations will be discussed in more detail in chapter 5.

In Table 3.1 we collect the data for the approximate location of the singularities for the c and b propagators for the full, IR and UV gap solution. In that table we include a qualitative estimation of the quark mass dressing for each case. We assume that the location of the singularities along the real axis can be used to extract that piece of information. From these data we observe that there is an increase in the quark mass dressing as we go from the c quark to the b quark using the full model. For the c quark there is an almost equal dressing from the IR and UV term of the interaction while in the b quark the UV term provides more than twice the mass dressing of the IR term.
Figure 3.13: \(Re(\sigma_s)\) for the b quark in a parabolic region in the complex plane \((P^2 = -72 \text{ GeV}^2, \eta=0.50)\) with integration over the gluon momentum (kcp) with the IR term of the model. The peak of the parabolic region is at \((\eta P)^2 = -18.0 \text{ GeV}^2\). The first pair of singularities now appear much closer to the space-like region and closer to the time-like real axis. Roughly their location is at \((x_o, y_o)\sim(-17.2, \pm7.5) \text{ GeV}^2\).

<table>
<thead>
<tr>
<th>quark</th>
<th>(x_o, y_o)</th>
<th>(M_{\Sigma})</th>
<th>(x_o, y_o)</th>
<th>(M_{\Sigma})</th>
<th>(x_o, y_o)</th>
<th>(M_{\Sigma})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c)</td>
<td>(-2.7,±3.5)</td>
<td>0.763</td>
<td>(-1.7,±2.0)</td>
<td>0.424</td>
<td>(-1.9,0.0)</td>
<td>0.498</td>
</tr>
<tr>
<td>(b)</td>
<td>(-24.0,±12.0)</td>
<td>1.1</td>
<td>(-17.2,±7.5)</td>
<td>0.347</td>
<td>(-23.0,0.0)</td>
<td>0.996</td>
</tr>
</tbody>
</table>

Table 3.1: \(c\) and \(b\) quark propagator approximate location of the singularities in the complex plane for the full MT, IR and UV model gap solution. All data are in GeV units. For the qualitative approximate estimation of the quark mass effective dressing we assume that \((m_Q(19 \text{ GeV}) + M_{\Sigma}^Q)^2 \sim -x_o\).
$ss, cc$ and $bb$ pseudoscalar meson masses

<table>
<thead>
<tr>
<th>meson</th>
<th>full MT</th>
<th>IR only</th>
<th>$\Delta M/M%$</th>
<th>UV only</th>
<th>$\Delta M/M%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ss$</td>
<td>0.696</td>
<td>0.565</td>
<td>−18.7</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>$cc$</td>
<td>3.035</td>
<td>2.41</td>
<td>−20.6</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>$bb$</td>
<td>9.585</td>
<td>8.106</td>
<td>−15.4</td>
<td>9.530</td>
<td>−0.6</td>
</tr>
</tbody>
</table>

$s\bar{s}, c\bar{c}$ and $b\bar{b}$ pseudoscalar meson decay constants

<table>
<thead>
<tr>
<th>meson</th>
<th>full MT</th>
<th>IR only</th>
<th>$\Delta f/f%$</th>
<th>UV only</th>
<th>$\Delta f/f%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ss$</td>
<td>0.182</td>
<td>0.151</td>
<td>−17.0</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>$cc$</td>
<td>0.387</td>
<td>0.276</td>
<td>−28.7</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>$bb$</td>
<td>0.692</td>
<td>0.172</td>
<td>−75.1</td>
<td>0.606</td>
<td>−12.4</td>
</tr>
</tbody>
</table>

Table 3.2: $ss$(fictional), $cc$ and $bb$ pseudoscalar meson masses and decay constants with their relative percentage differences from calculations where only the first infrared (IR) term or only the ultraviolet(UV) perturbative tail in the MT model is retained.

$s\bar{s}, c\bar{c}$ and $b\bar{b}$ vector meson masses

<table>
<thead>
<tr>
<th>meson</th>
<th>full MT</th>
<th>IR only</th>
<th>$\Delta M/M%$</th>
<th>UV only</th>
<th>$\Delta M/M%$</th>
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</thead>
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<tr>
<td>$ss$</td>
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<td>0.949</td>
<td>−11.5</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>$cc$</td>
<td>3.235</td>
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<td>−</td>
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<tr>
<td>$bb$</td>
<td>9.658</td>
<td>8.130</td>
<td>−15.9</td>
<td>9.586</td>
<td>−0.8</td>
</tr>
</tbody>
</table>

$s\bar{s}, c\bar{c}$ and $b\bar{b}$ vector meson decay constants

<table>
<thead>
<tr>
<th>meson</th>
<th>full MT</th>
<th>IR only</th>
<th>$\Delta f/f%$</th>
<th>UV only</th>
<th>$\Delta f/f%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ss$</td>
<td>0.259</td>
<td>0.274</td>
<td>+5.8</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>$cc$</td>
<td>0.415</td>
<td>0.333</td>
<td>−19.8</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>$bb$</td>
<td>0.682</td>
<td>0.340</td>
<td>−50.1</td>
<td>0.510</td>
<td>−25.2</td>
</tr>
</tbody>
</table>

Table 3.3: $ss$, $cc$ and $bb$ vector meson masses and decay constants with their relative percentage differences from calculations where only the first infrared (IR) term or only the ultraviolet(UV) perturbative tail in the MT model is retained.
3.5 Results for quarkonia mesons.

The results for the masses and the decay constants of the three pseudoscalar and vector quarkonia ($Q\bar{Q}$ $Q=s,c,b$) meson systems are collected in Tables (3.2) and (3.3). For the b quarkonium we observe now that even the very weak, tail term, of the interaction can give us a bound state. A very weak attractive force is adequate for such heavy particles to bind them together. The mass of the system is lighter by just 0.055 GeV or 0.57 % of the mass from the full model, while the decay constant decreases by 12.4 %, from 0.692 to 0.606 GeV, when we keep only the UV tail term. On the other hand, since the IR term is much more attractive than the UV term we get a much smaller mass, 15.4% relative decrease, while the decay constant decrease by more than 75 %. This also signifies a dominance of the UV term over the IR, supported by the heavy quark mass behavior of the propagator. Same observations are also true for the corresponding vector mesons data in table 3.3. The only difference we observe for the vector mesons is the greater difference between the full model calculated decay constant and the one calculated using the UV tail term in the b quarkonium. The percentage difference is about twice as much that of the corresponding pseudoscalar meson and the reason is that the vector mesons are more extended objects than the pseudoscalar hence the IR term contributes more in the evaluation of the physical observables.

The first Chebychev moment of the dominant invariant of the pseudoscalar meson BSA from the different calculations and the three equal quark systems appear in fig. (3.14). Similar behavior is observed for the corresponding vector mesons. As the mass increases we observe that the IR amplitude has an increasingly sharper decrease than the full MT one, and the system is more delocalised, consistent with the expectations
Figure 3.14: First Chebychev moment of the dominant invariant for the $s\bar{s}$, $c\bar{c}$, $b\bar{b}$ pseudoscalar systems, from the full MT model solution and the solution with only the IR term. For the $b\bar{b}$ quark system we also have an UV calculated amplitude since we can reach a mass shell even with only the weaker UV tail term in the interaction.
of a smaller dressing of the quark mass provided by the IR term. The UV amplitude of
the b quarkonium on the other hand decreases a little slower than the MT one
and the system is slightly more localized. This is a sign that for this system the UV
term is now providing most of the quark mass dressing and binding force for the two
quarks.

From the last results become obvious that as the current quark mass increases
the IR term became less important in the calculations of the meson observables.
The question is why and how. Is it because the UV term changed somehow and
contributes more in the quark interaction? The answer is no, the effective interaction
is the same. Essentially this is a result of the effect of the large current quark mass in
the behavior of the propagator amplitudes and consequently the combined dynamical
interplay with the MT model. As we raise the quark mass the amplitudes will have
smaller values, even very deep in the time-like region, and we have to get closer and
closer to their singularities to notice some important increase in their values. That in
turn will suppress the contribution and lessen the significance of the IR term while the
UV term of the model will become more relevant in the evaluation of the observables.
That is also related to the extremely fast decay of the IR term and much slower
decrease of the UV one (see fig. 3.15).

To qualitatively see the effect of the heavy quark mass, we plot the product of
the MT rainbow-ladder kernel with the two propagators amplitudes that multiply it
in the BSE. There are four cases: \( \sigma_s(q_+^2) \cdot \sigma_s(q_-^2) \), \( \sigma_v(q_+^2) \cdot \sigma_v(q_-^2) \) and \( \sigma_s(q_+^2) \cdot \sigma_v(q_-^2) \),
but for convenience we consider only the real axis where we have \( \text{Re}(\sigma_{s/v}(\text{Re}(q_{\pm}^2))) = \sigma_{s/v}(q^2 - (\eta P)^2) \), so there are essentially only three amplitude products in the BSE
integrand: \( \sigma_s^2(q^2 - (\eta P)^2) \), \( \sigma_v^2(q^2 - (\eta P)^2) \) and \( \sigma_s(q^2 - (\eta P)^2) \cdot \sigma_v(q^2 - (\eta P)^2) \). The
Figure 3.15: Plot of the full MT model and the model with only the IR or UV tail term. The IR term will mostly determine the strong behavior of the model for $k^2 < 1 \text{ GeV}^2$ and beyond that point the UV dominates.
plots of these products with the MT kernel for the $s\bar{s}$, $c\bar{c}$, $b\bar{b}$ pseudoscalar mesons are in fig. (3.16, 3.17, 3.18).

From fig. (3.16, 3.17, 3.18) we can clearly see the relative effect of the propagator amplitudes and how that changes as we increase the current quark mass. While $\sigma_s^2$ for the $s$ quark has almost no effect in the IR region, it provides IR suppression for increasing quark mass while on the other hand slowly diminishes the pressure in the UV region. The other amplitude $\sigma_v^2$ while initially for the light quarks provides some IR enhancement, very fast provides a lot of pressure there diminishing the importance of the first IR term of the kernel. It also suppresses the UV tail, more than the $\sigma_s^2$ for quarks heavier than the $c$ quark.

Also we observe some mild *increasing* suppression of the MT behavior in the IR region as the current quark mass *increases* for the product $MT\cdot\sigma_s^2$, and a much stronger *increasing* suppression for the second function product due to the $\sigma_v^2$ amplitude. For the last product the situation is somewhere between the first two since it combines the effects of both propagator amplitudes. As we also noticed earlier for the last two cases of products involving $\sigma_v$, we actually have an enhancement in the IR region for the $s$ quark system. For the UV region in all cases we have suppression of the tail term, but that suppression is *decreasing* as the current quark mass *increases*, somehow faster for the $MT\cdot\sigma_s^2$ than in the $MT\cdot\sigma_s \cdot \sigma_v$ product, and for the term involving the $\sigma_v^2$ there is no difference for all quark systems for $q^2 > 10 \text{ GeV}^2$ (the increasing IR suppression just extends a little beyond $1 \text{ GeV}^2$ as the quark mass increase). The difference in the effects between the first and the second type of terms are due to the fact that the $\sigma_s$ amplitude has also the quark mass function in the numerator moderating the suppressing effects of this function.
Figure 3.16: Plot of the product \( MT \cdot \sigma_s^2 \) for the \( s\bar{s}, c\bar{c}, b\bar{b} \) quark systems compared with the MT model behavior for external momentum \( p^2 = 0 \). In all cases the propagator amplitude along the real axis is for the on-shell solution of the BSE. One can clearly see the increasing suppression imposed by the amplitude in the IR region and the decreasing suppression on the UV tail term, as the current quark mass increase. This IR suppression is mild compared to the other two cases since the amplitude \( \sigma_s \) has also the mass function in the numerator, moderating the decrease of the amplitude. Same things are true for every \( p^2 > 0 \) but there is a significant scale down as \( p^2 \) increase.
Figure 3.17: Plot of the product $MT \cdot \sigma_v^2$ for the $s\bar{s}$, $c\bar{c}$, $b\bar{b}$ quark systems compared with the MT model behavior for external momentum $p^2 = 0$. In all cases the propagator amplitude along the real axis is for the on-shell solution of the BSE. Once again we see the increasing suppression imposed by the amplitude in the IR region and the same, for all cases after certain $q^2$, suppression of the UV tail term, as the current quark mass increase. Since the $\sigma_v^2$ amplitude doesn’t also have the quark mass function in the numerator to moderate the decrease, we notice a much stronger suppression in the IR region as the current quark mass increase. This suppression extents somehow in the UV region, but for $q^2 > 10\, GeV^2$ is the same for all cases. Finally notice that for the $s$ quark system we observe an enhancing effect for the MT model in the IR region. Same things are true for every $p^2 > 0$ but there is a significant scale down as $p^2$ increase.
Figure 3.18: Plot of the product $MT \cdot \sigma_s \cdot \sigma_v$ for the $s\bar{s}$, $c\bar{c}$, $b\bar{b}$ quark systems compared with the MT model behavior for external momentum $p^2 = 0$. In all cases the propagator amplitude along the real axis is for the on-shell solution of the BSE. As one would expect, from the two previous plots, the degree of the effects in the IR and UV region is somewhere between the degree of the effects of the other two terms. Same things are true for every $p^2 > 0$ but there is a significant scale down as $p^2$ increase.
Although from these plots one can qualitatively understand why the IR term becomes less important while the UV weak tail term becomes more relevant for the mesons physical observables as we increase the quark mass, the question is how something like that happens. For that reason we have to focus on the behavior of the two amplitudes in the two regions (IR and UV) and on how that behavior changes as we increase the current quark mass. The arguments will be again mostly qualitative.

From the the two coupled gap equations (3.18, 3.19) and using the plots (3.16, 3.17) we can extract some qualitative information about the relative change in the effects of the IR and UV tail term will bring on the quark propagator amplitudes $M(p^2)$ and $A(p^2)$ as we increase the quark mass. We notice that the integrand of the equation for $B'(p^2)$ is just $M_T \cdot \sigma_s$. That function is almost unchanged in the IR region. There is only a small suppression over there as we increase the quark mass. The downwards UV pressure though is getting less. Therefore the values of the product in the UV region will contribute more in the behavior of function $B'(p^2)$. On the other hand $M_T \cdot \sigma_v$ is the function that appears in the other integral equation. There is almost no change of that function in large momenta (larger than about 9-10 $GeV^2$) but up to this point there are very strong suppressive forces greatly diminishing the contribution for the values of $A'(p^2)$. As a result we expect function $A(p^2)$ to have smaller variation in the complex plane and get values closer to one as the quark mass increase. Function $M$ ($M=B/A$) on the other hand, since it is proportional to $B$, will still be receiving important contributions from the self-interaction term and even for the heavy b quark will vary much and not very close to the b current mass of 3.8 GeV. Dressing IR effects will also slowly diminish and will be replaced by smaller UV dressing effects. So it should be $\Delta(M^{IR}_\Sigma) < 0, \Delta(M^{UV}_\Sigma) > 0, |\Delta(M^{IR}_\Sigma)| > |\Delta(M^{UV}_\Sigma)|$, $M_\Sigma = M^{IR}_\Sigma + M^{UV}_\Sigma$
so overall $\Delta(M) < 0$. Notice that the estimated quark mass dressing using the singularities location of the IR and UV gap solutions for the c and b quarks (Table 3.1) do confirm the first two expected changes in the IR and UV quark mass dressing contributions as the quark mass is raised, but don’t confirm the third case, since from these data appears the UV dressing effects increase faster than the IR decrease and the overall dressing increase. It is not clear though if this is because of a disadvantage of the MT model or because of our approach to estimate these dressings.

Next we will try to qualitatively understand how a raising quark mass can have the effects we observed so far. For the light quarks we assume $m_q \ll \Lambda$ where $\Lambda \sim 1$ GeV is the chiral symmetry breaking scale. $m_q \gg \Lambda$ for the heavy quarks and we assume relative small dressing effects even in the IR region so $m_q + M \sim m_q$.

For the $\sigma_s$ amplitude we have $\sigma_s(q^2 - M_H^2/4) = (1/A(q^2 - M_H^2/4))(M(q^2 - M_H^2/4)/(|q^2 - M_H^2/4 + M^2(q^2 - M_H^2/4)|)) = (1/A)((m_q + M)/(q^2 - M_H^2/4 + m_q^2 + M^2 + 2m_q M_H))$, where $M_H$ is the hadron mass.

Let us now focus on the IR region. We are very close to the peak of the parabolic region and for convenience of our analysis we take $q^2 \sim 0$. For the light quarks we have strong dressing effects in that region so $m_q + M \sim M$, and the propagator amplitude is approximately $\sigma_s(q^2 \sim 0) \sim (1/A)M/(M^2 - M_H^2/4)$. Although A is getting larger as we get near the peak, pushing down the values of $\sigma_s$, the hadron mass $M_H$ is mostly due to the quark IR dressing effects and therefore the difference in the denominator at the same time is getting smaller providing the very small IR enhancement we noticed for the product of $\sigma_s$ with the MT model. Since for the other amplitude the mass function is replaced by one ($M \rightarrow 1$) which is larger than $m_q + M$, and since the denominator is the same, we expect for the same
reasons as before to have stronger than the $\sigma_s$ IR support for the MT model. For the heavy quarks in addition to $m_q + M_\Sigma \sim m_q$ we assume $M_H \sim 2m_q$. Then $\sigma_s(q^2 \sim 0) \sim (1/A)(m_q/(M_\Sigma^2 + 2m_qM_\Sigma))$. $M_\Sigma$ is of the order of 1 GeV now, that is why we keep the $M_\Sigma^2$ in the denominator. $A$ has values close to one and does not affect much the behavior of the amplitude near the peak. Because we now have the sum of two positive terms in the denominator, we expect that will lightly push down the values of $\sigma_s$ near the peak. The large quark mass in the numerator inhibits stronger suppressing IR effects from this amplitude but something similar does not exist in $\sigma_v$ resulting in a stronger IR suppression. These observations help us to qualitatively understand the different IR effects we noticed in the last figures and how and why they change as the quark mass increase. We should notice at this point that the behavior of $\sigma_s$ in the IR region is model dependent and for the heavy quarks may not present a realistic behavior, but for the other amplitude, $\sigma_v$ we know for certain there will be a $1/m_q$ suppression for the heavy quarks as we increase the quark mass.

The UV region analysis is easier since we are in the perturbative region and for both light and heavy quarks we have $m_q + M_\Sigma \sim m_q$ and $A \sim 1$. This is essentially model independent analysis. Notice at this point that for our analysis we are talking about dressing effects in the quark mass in the IR or UV region originating either from IR or UV interaction effects. There is though a shift in the momenta for the propagator amplitudes so in reality, when we refer to mass dressing in the IR region, we actually refer to the dressing in the quark mass near the peak (from $\text{Re}(q^2_\pm) = -(\eta M)^2$ to about $\text{Re}(q^2_\pm) \sim - (\eta M)^2 + 2 \text{GeV}^2$ on the real axis) of the meson mass shell parabolic region in the complex plane due to IR and/or UV interaction effects, and when we talk about mass dressing in the UV region, we
actually mean the dressing \textit{in} the quark mass \textit{far from the peak} (approximately for $Re(q^2_{\pm}) > -(\eta M)^2 + 2 \text{GeV}^2$ on the real axis) of the meson mass shell parabolic region in the complex plane \textit{due again to IR and/or UV interaction effects}. There is an insignificant quark mass dressing \textit{in} the UV region (for $q^2 > 4 - 5 \text{GeV}^2$ which for the propagators momenta in the BSE is for $Re(q^2_{\pm}) > -(\eta M)^2 + 5 \text{GeV}^2$ on the real axis in the corresponding parabolic region) for \textit{all} quark masses and that’s why we can assume the last approximation for the quark mass function. The reason for this is that the internal quark propagator momentum in the gap equations depend on the external momentum $p^2$ and as the last one increase provides through the propagator in the gap equations integrand an increasing suppression on IR and UV dressing effects for the quark mass \textit{in} the UV region. For the same reason the MT model provides less quark binding as $p^2$ increase and we move to the UV region, the gluon momentum in the BSE is $k = p - \tilde{q}$ so we have an $e^{-p^2/\omega^2}$ term strongly suppressing IR binding forces \textit{in} the UV region and through a more complicated dependence of the UV tail on $p^2$ we also have suppression of the UV binding effects again \textit{in} the UV region. We have then $\sigma_s(q^2 > 5 \text{GeV}^2) \sim m_q/(q^2 + m^2_q - M_H^2/4)$. For light quarks the numerator is very small and as $q^2$ increase the amplitude will decrease very fast. For $\sigma_v$ though the numerator is one which is larger than the quark mass and will slow down a little that decrease. For the heavy quarks on the other hand since $M_H \sim 2m_q$ the amplitude is further simplified to $\sigma_s(q^2 > 5 \text{GeV}^2) \sim m_q/q^2$. As the current mass further increase will provide more support for the values of $\sigma_s$ in large momenta. If we replace that mass with one to get the other amplitude, since now the quark masses are larger than 1 GeV, there will be less UV enhancement than with the $\sigma_s$. The simple form of the denominator also makes possible for the product of the two functions with the MT
model to follow closer the changes in the behavior of the model, in fig. (3.16, 3.17)
notice the obvious bending at about 2-3 GeV\(^2\) of the c and b quark cases of products.
Finally notice that we have very strong increase in the suppression effects in the IR region as the quark mass is raised and that affects the propagator amplitude products in a small area near the tip of the parabolic region, while there is a comparatively mild decrease in the suppression effects in the UV region as the quark mass is raised in a comparatively much larger area of the mass shell parabolic region in the complex plane, therefore the integrated strength of the decrease in the IR dressing and binding effects should be slightly more than the integrated strength of the increase in the UV dressing and binding effects so that at the end will have an overall decrease in the quark mass dressing and binding energy in both IR and UV regions.

Summing up, we found that as the current quark mass increase the propagator amplitude \(\sigma_v\) and less \(\sigma_s\) will suppress mass dressing and binding IR contributions while initially for light quarks \(\sigma_v\) will enhance IR contributions. At the same time \(\sigma_s\) will provide the support that will enhance UV dressing and binding effects as the current mass is raised. From the location of the singularities of the c quark propagator, from the full model calculations and when we keep the IR or UV term (see Table 3.1), and from plots (3.16, 3.17) qualitatively we may conclude that this is the current quark mass region where this transition, in dressing and binding IR and UV region contributions, takes place. In terms of physics, the long distance interaction between the quarks (IR term) is inhibited by their large mass and now they interact mostly through the exchange of short-range (large momentum) gluons. As the mass is raised becomes increasingly more difficult for the quarks to move further apart with the large current quark mass gradually replacing in that way the
effect of the strong IR term of the interaction. In other words, the increase in the current mass now replaces the strong quark mass dressing IR effects of the model, making that term almost unnecessary, and with only a small attraction we can have a bound state. In chapter 5 additional studies will further establish these conclusions.

From the two plots in fig. (3.10, 3.13) becomes obvious that the MT kernel, because of the infrared term, does not support a single pole-mass constituent-like behavior and actually, from the second plot, it looks like we are moving further away from that type of behavior as the current quark mass increase. We can assume then that a constituent-like behavior can be supported by the effective kernel, as we increase the quark mass, only if there is an important decrease in the strength of the IR term of the kernel.

For the solution of the gap equation one starts by assuming a free quark boundary condition at some large space-like momentum scale well inside the perturbative region and this solution is modified because of the interaction of the quark with the vacuum, mostly in the nonperturbative region. If we ignore the strong nonperturbative effects, expressed through the first term of the MT kernel, and keep only the perturbative UV tail term, then from perturbative QCD we know that the pole of the propagator, will retain some of its initial features determined by the current quark mass. The fact that we still have a first singularity on the real axis (fig. 3.12 and similar plot for the b quark) is a verification of that. On the other hand as we go to heavier quarks the dressing effects will become comparatively smaller, the location of the singularities will be mostly determined by the current quark mass and they will be very close, along the real axis, to the mass-pole singularity of the free propagator. Therefore the dynamics of the interaction, as expressed through the MT model, will have a
lesser role in the *location* of the singularities as the current quark mass is raised, but will be always responsible for their *type* (confinement excludes poles) which still has a very subtle and important role for the calculation of certain meson observables. For the calculations of physical observables of heavy quark systems we have to go very close to these singularities, overshadowing the IR dynamics of the model, which is suppressed elsewhere from the heavy quark mass behavior of the propagator(s) amplitudes. Therefore we expect that a small variation of the parameters of the model, will have insignificant impact in the location of the heavy quark propagator singularities. As a consequence the values of the physical observables will also show indifference to these variations and will be mostly determined by the quark masses. Reversing the reasoning, if the IR term of the model had an explicit quark mass dependence, we should expect a decreasing dependence of the strength of the model on the mass as we go to heavier quarks.
Chapter 4

A dressed quark propagator parametrization

Many times, for practical purposes, it is useful to employ some type of parametrization for the solution of an equation. That is even more imperative in the type of calculations involved in our studies and especially for the case of heavy quark mesons. The results for the masses and decay constants of equal quark mesons, and also $K, K^*$, for such a parametrization are presented here and compared with the results of the dynamical solution. A thorough analysis of the factors responsible for any differences is made.

4.1 Complex conjugate mass pole parametrization.

As we briefly stated in chapter 3 for the studies of mesons with a heavy quark the propagator can be calculated by integrating over the gluon momentum but the calculations become more cumbersome as we go to heavier mesons since we need to cover larger areas in the complex plane. Also, for some calculations, e.g. the decay constants or form factors using the impulse approximation, we need to know the propagators in three different parabolic areas and these calculations are also time consuming. One way to deal with this will be to solve the gap equation in the largest possible region in the complex plane with a very dense grid and then use interpolation to get the propagator at the points needed. The only risk with that kind of approach will be the error introduced in the calculations from the interpolation. Another equally effective way will be the use of a parametrized representation of the solution.
A convenient parametrization is the one that employs a linear combination of free propagators with complex conjugate masses. We use three complex conjugate pairs of masses and we denote this representation as 3ccp.

There are several reasons we employ that type of parametrization. The first is that we have the simplest type of singularities. A different type of singularity e.g., power or logarithmic branch cuts, would severely complicate our calculations. A second reason is that lattice-QCD simulations of quark propagators can be well represented this way. The most important one though is that with this representation we can simulate confinement. It was found that the meson decay width generated in the solution of the BSE from one pole is exactly canceled by that due to the complex conjugate pole preventing the decay to free quarks ([67],[68]). A direct way to verify confinement of a particle is to examine if there is a violation of positivity by the particles' propagator ([140],[141]). If the Swinger function $\Delta(x^2)$ for a propagator over interval x in a Euclidean field theory is not positive definite for $x^2 > 0$ then positivity is violated and we have confinement. As we will find out in the next chapter another very important advantage for the studies of properties of unequal quark mesons is the knowledge of the exact location of the propagator’s singularities.

The 3ccp parametrization for the propagator is:

$$S(q) = \sum_{k=1}^{3} \left( \frac{z_k}{i q + m_k} + \frac{z_k^*}{i q + m_k^*} \right), \quad (4.1)$$

where $m_k, z_k$ are complex parameters. The total number of independent parameters to be determined is then twelve. We can express the different scalar functions of the quark propagator in terms of these parameters and especially for the amplitudes
\( \sigma_s(q^2) \) and \( \sigma_v(q^2) \) we have:

\[
\sigma_s(q^2) = \sum_{k=1}^{3} \left( \frac{z_km_k}{q^2 + m_k^2} + \frac{z_k^*m_k^*}{q^2 + m_k^{*2}} \right)
\]

\[
\sigma_v(q^2) = \sum_{k=1}^{3} \left( \frac{z_km_k}{q^2 + m_k^2} + \frac{z_k^*m_k^*}{q^2 + m_k^{*2}} \right).
\]

For different current quark masses, we fit the parameters, using the last two functions (4.2, 4.3), to the solution of the gap equation for \( q^2 \in [-1, +\infty) \text{ GeV}^2 \). It is assumed that range of values will be adequate to uniquely determine the parameters of the representation and so far as physical observables of interest is concerned. It is interesting to notice here the for the range of current quark masses \( 0.0 \text{ GeV} \leq m_q(19 \text{ GeV}) \leq 6.0 \text{ GeV} \), we found a smooth trajectory for the real parts of the parameters (see fig. 4.1, 4.2) and a less well-determined trajectory for some of the imaginary parts, indicating their lesser importance in the representation. In detail, parameter \( m_{1r} \) is always positive and increases slowly with the current quark mass and the difference \( (m_{1r} - m_q) \) slowly increases also. The parameter \( m_{1i} \) initially is a very small number and very slowly approaches one. \( z_{1r} \) initially is also a small number and increases quite fast for masses up to 2 GeV reaching an almost constant value of about 0.5 for masses larger than that. The imaginary part \( z_{1i} \) is about 0.5 for all masses. \( m_{2r} \) on the other hand is always negative and increases smoothly in absolute value, faster than \( m_{1r} \), while its imaginary part is very small and almost zero in the middle of the region of the studied quark masses. \( z_{2r} \) decreases slower than \( z_{1r} \) increases for masses up to 2 GeV and approaches zero slowly. The parameter \( m_{3r} \) is always positive, increases very fast with small current quark mass and the overall trajectory indicates that it plays a minor role in the parametrization. The
Figure 4.1: Variation of the real parts of parameters $m_i$ ($i=1,2,3$) in the 3ccp representation, for current masses from the chiral limit up to 5.5 GeV
Figure 4.2: Variation of the real parts of weights $z_i$ ($i=1,2,3$) in the 3ccp representation, for current masses from the chiral limit up to 5.5 GeV
corresponding $z_{3r}$ approaches zero very fast. From the last two observations one can conclude there is a very minor contribution of the third pair of poles term in the propagator’s representation for current quark masses $m_q \geq 2 \text{ GeV}$. The same pattern appears to be true for the second pair of poles term but this happens at a much slower pace and is definitely important even for large quark masses (5 or 6 GeV). This can be considered as an indication of a lack of constituent-like behavior for the gap solution propagator in that region of masses. Overall it appears that the first pair of complex conjugate poles is the dominant one for the representation of the solution of the gap equation on the real axis and actually the knowledge of their exact position is of crucial importance for our calculations in the studies of unequal quark mesons.

The parameters of the 3ccp representation for the chiral limit, $u/d \ (m_{u/d}(19 \text{ GeV}) = 0.00374 \text{ GeV})$, $s \ (m_s(19 \text{ GeV}) = 0.083 \text{ GeV})$, $c \ (m_c(19 \text{ GeV}) = 0.88 \text{ GeV})$ and $b \ (m_b(19 \text{ GeV}) = 3.8 \text{ GeV})$ quarks appear in Table 4.1. Comparison of the amplitudes $\sigma_s, \sigma_v$ from the gap equation solution with the ones from the 3ccp representation are presented in figures (4.3, 4.4, 4.5, 4.6, 4.7). For the light quark propagators, for all meson studies, the parabolic region peak never goes deeper than $q^2 \sim -0.8 \text{ GeV}^2$ in the time-like region, for the $c$ we have to go as far as $q^2 \sim -2.6 \text{ GeV}^2$ and for the $b$ quark systems up to $q^2 \sim -22.6 \text{ GeV}^2$.

From these plots we can see that for all quarks there is excellent agreement between the 3ccp representation and the gap equation solution on the real axis for $q^2 > -1 \text{ GeV}^2$, but for the heavy quarks $c$, $b$ (especially for $c$) for $q^2 < -1 \text{ GeV}^2$ we don’t have so good agreement. That is not surprising at all since, we fitted the parameters to the gap equation solution in the region $(-1, \infty) \text{ GeV}^2$. What is
Figure 4.3: $\sigma_{s/v}$ functions for the chiral limit from the DSE equation compared to the 3ccp fit.
Figure 4.4: $\sigma_{s/v}$ functions for the $u/d$ quark with $m_{u/d}(19 \text{ GeV}) = 0.00374 \text{ GeV}$ from the DSE equation compared to the 3ccp fit.
Figure 4.5: $\sigma_{s/v}$ functions for the $s$ quark with $m_s(19 \text{ GeV}) = 0.083 \text{ GeV}$ from the DSE equation compared to the 3ccp fit.
Figure 4.6: $\sigma_{s/v}$ functions for the $c$ quark with $m_c(19 \text{ GeV}) = 0.88$ GeV from the DSE equation compared to the 3ccp fit.
Figure 4.7: $\sigma_{s/v}$ functions for the $b$ quark with $m_b(19\, GeV) = 3.8\, GeV$ from the DSE equation compared to the 3ccp fit.
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<td>-1.2730</td>
<td>0.5310</td>
<td>0.1442</td>
<td>0.0403</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.4914</td>
<td>0.6842</td>
<td>0.1380</td>
<td>-0.0335</td>
</tr>
<tr>
<td>s</td>
<td>1</td>
<td>0.7801</td>
<td>0.4881</td>
<td>0.1831</td>
<td>0.4631</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-1.4958</td>
<td>0.0063</td>
<td>0.1315</td>
<td>5.2021</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2.6360</td>
<td>-0.1058</td>
<td>0.1946</td>
<td>-1.5621</td>
</tr>
<tr>
<td>c</td>
<td>1</td>
<td>1.8314</td>
<td>0.6114</td>
<td>0.4478</td>
<td>0.2632</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-2.9807</td>
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</tr>
<tr>
<td></td>
<td>3</td>
<td>34.8220</td>
<td>-0.1859</td>
<td>-0.0030</td>
<td>-0.0411</td>
</tr>
<tr>
<td>b</td>
<td>1</td>
<td>5.3472</td>
<td>0.7830</td>
<td>0.4899</td>
<td>0.6097</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-9.6500</td>
<td>0.0189</td>
<td>0.0258</td>
<td>-0.1687</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>52.2378</td>
<td>-0.1504</td>
<td>-0.0061</td>
<td>-0.2264</td>
</tr>
</tbody>
</table>

Table 4.1: 3ccp propagator parameters fitted to the gap equation solution for the Chiral, u/d, s, c, b quark propagator on the real axis $(-1, \infty)$ GeV.

Surprising, especially for the $b$ quark propagator, is that we still have a satisfactory agreement. The explanation for this is that the pair of poles closest to the space-like region dominate over the others and mostly determine the behavior of the two amplitudes in the region $(-1, \infty)$ GeV². Actually these poles will be the only ones needed for the calculations near their location area. In solving the BSE, we need to get closer to these poles as we go to heavier quark mesons, and essentially, and because of the behavior of the MT model, they are the ones mostly responsible for the outcome of our calculations.

Although, direct comparison of the 3ccp quark propagator amplitudes with the gap equation solution on the real axis gives a first, qualitative, visual measure of accuracy, it does not guarantee accuracy throughout the sampled complex plane for
the solution of the BSE nor does it tell us how good the representation is for the
calculation of different physical observables. The propagator amplitudes in the BSE
integrand are in an environment that contains other functions that depend on \( P \),
like the BSA invariants and scalar expressions we get after the projections of the
covariants and the traces of the integrands. So depending on the value of \( P \) the error
in the propagator fits can be enhanced or be suppressed. A small difference between
the 3ccp representation and the original DSE solution can sometimes make a big
difference in the calculation of various physical observables. The ultimate criteria for
the quality of the 3ccp representation will be the calculated observables themselves
and some observables will be more sensitive than others. For the cases where we need
to be very close to the first pair of poles we know that, since the type of singularities
of the real solution is different, the 3ccp will not be a good representation and we
expect to introduce more uncertainty in our calculations.

4.2 Meson masses using the 3ccp propagator representation.

The purpose for investigating the utility of the algebraic complex mass pole fit for
quark propagators is to provide a convenient alternative to the more difficult use of
numerically attained propagators within the bound state BSE calculation. This fit
can not replace the gap solution if high accuracy is desired in all details. However it
is only intended to provide physical observables for a qualitative representation and a
certain level of uncertainty is to be tolerated. Our employed ladder-rainbow truncation
model cannot be perfect. Also the mass-pole fit for propagators can help identify
the approximate location of singularities to help understand the full dynamical case.

QCD is a Poincare covariant theory (the Rainbow-Ladder truncation scheme and
MT model are such that this symmetry is preserved) and one can prove that the
physical observables are independent of the parameter $\eta$. However the detailed ingredients (the propagators and invariant amplitudes of the BSE vertex) do depend on $\eta$ (see [62],[63],[64]). We exploit that freedom and choose a value of $\eta$ such that all singularities are outside the integration domain. The knowledge of the exact location of the lowest pair of poles of the 3ccp representation of quark propagators, and the real pole of the heavy quark propagator in the constituent mass approximation in the next chapter, can guide the choice of $\eta$. For equal quark mesons the obvious choice will be $\eta = 0.5$. Even for this case, and for other reasons that we explain below, the calculations for each observable were done for two other symmetric values of $\eta$ as well. The parabolic domain of the $q^2$ plane for the BSE solution has an infinite area, but for our numerics we have to discretize the integral and introduce a hard cut-off value for $\tilde{q}^2$. As a result the area is finite and its value depends on the product ($\eta P)((\bar{\eta}P))$. So if we keep $P$ constant and vary $\eta(\bar{\eta})$ the parabolic domain will change, and propagators’ singularities will be closer or further away. In previous studies ([62],[64]) it was found that without the complete Dirac structure for the BSA and/or if you don’t have convergence with respect to the number of Chebychev polynomials in the invariants’ angle expansion, the calculated observables will show a dependence on the value of $\eta$. Because of the previous observations we decided to repeat our calculations for different grids, several values of $\eta$ within the allowable range, as well as solve the BSE for a range of $P^2$, for the following reasons:

\footnote{Different values of $\eta$ correspond to different relative momenta $q$ and that is equivalent to a shift in the integration variables. For non-anomalous processes, loop integrals are independent of such a shift due to Poincare invariance.}
• Indirectly check convergence with respect to the number of Chebychev polynomials used in the invariants expansion. The direct way requires more polynomials in the expansion, but something like that is computationally not practical and will make the numerics a lot more cumbersome.

• To check the influence from the proximity of the propagators’ singularities to the BSE integration domain and to see if more mesh points need to be used.

• To check if there is a systematic error from the use of the algebraic quark propagator fit.

Additionally for all calculations we use four Chebychev moments in the BSA invariants’ expansion, two of even and two of odd order, but unless we study unequal quark systems or if we use a value $\eta \neq 0.50$ for the equal quark mesons, only specific parity polynomials will be present in the expansions (usually even). So for equal quark mesons with $\eta = 0.50$ we can also test the accuracy of our numerics by inspecting the odd order moments.

Since, for computational convenience, we plan to use the light quark 3ccp propagators in the studies of light-heavy quark mesons, their use in the present calculations will additionally serve in estimating the uncertainty we will have in our numerics there. This simple statement requires some clarification since for each meson we have a different mass and therefore one might think different areas in the complex plane will be sampled for the solution of the BSE. As we will see in detail in the next chapter, the total momentum maybe is different for each case, and the same is true for the parameter $\eta$, their product $\bar{\eta}P$ though, and consequently the parabolic area for the light quark propagators, is always about the same introducing the same uncertainty.
Table 4.2: Pseudoscalar and vector meson masses: experimental data and calculated masses using the gap or the 3ccp fit for the quark propagators. In the fourth column of the table we have the relative percentage differences between gap and experimental meson masses: $\Delta \frac{m}{m^{exp.}} = \frac{m^{gap} - m^{exp.}}{m^{exp.}}$, while in the last column we list the relative percentage differences between 3ccp and gap masses: $\Delta \frac{m}{m^{exp.}} = \frac{m^{3ccp} - m^{gap}}{m^{gap}}$. All masses are in GeV. Experimental data are from [80].

<table>
<thead>
<tr>
<th>meson</th>
<th>Exp.</th>
<th>Gap</th>
<th>$\Delta \frac{m}{m^{exp.}}$</th>
<th>3ccp</th>
<th>$\Delta \frac{m}{m^{gap}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi(u\bar{u})$</td>
<td>$0.139 \pm 3.5 \times 10^{-4}$</td>
<td>$0.138$</td>
<td>$-0.7$</td>
<td>$0.044$</td>
<td>$-68.1$</td>
</tr>
<tr>
<td>$\rho(u\bar{u})$</td>
<td>$0.769 \pm 9.0 \times 10^{-4}$</td>
<td>$0.742$</td>
<td>$-3.5$</td>
<td>$0.750$</td>
<td>$+1.1$</td>
</tr>
<tr>
<td>$K(us)$</td>
<td>$0.494 \pm 1.3 \times 10^{-5}$</td>
<td>$0.497$</td>
<td>$+0.6$</td>
<td>$0.465$</td>
<td>$-6.4$</td>
</tr>
<tr>
<td>$K^*(us)$</td>
<td>$0.896 \pm 2.5 \times 10^{-4}$</td>
<td>$0.936$</td>
<td>$+4.5$</td>
<td>$1.00$</td>
<td>$+6.8$</td>
</tr>
<tr>
<td>$(s\bar{s})(fict.)$</td>
<td>$-0.00$</td>
<td>$0.696$</td>
<td>$-0.696$</td>
<td>$0.663$</td>
<td>$-4.7$</td>
</tr>
<tr>
<td>$\phi(s\bar{s})$</td>
<td>$1.019 \pm 1.9 \times 10^{-9}$</td>
<td>$1.072$</td>
<td>$+5.2$</td>
<td>$1.078$</td>
<td>$+0.6$</td>
</tr>
<tr>
<td>$\eta_c(c\bar{c})$</td>
<td>$2.980 \pm 1.2 \times 10^{-4}$</td>
<td>$3.035$</td>
<td>$+1.8$</td>
<td>$3.007$</td>
<td>$-0.9$</td>
</tr>
<tr>
<td>$J/\Psi(c\bar{c})$</td>
<td>$3.097 \pm 1.1 \times 10^{-9}$</td>
<td>$3.235$</td>
<td>$+4.5$</td>
<td>$3.180$</td>
<td>$-1.7$</td>
</tr>
<tr>
<td>$\eta_b(bb)$</td>
<td>$9.300 \pm 4.0 \times 10^{-2}$</td>
<td>$9.585$</td>
<td>$+3.1$</td>
<td>$9.347$</td>
<td>$-2.5$</td>
</tr>
<tr>
<td>$\Upsilon(bb)$</td>
<td>$9.460 \pm 2.6 \times 10^{-4}$</td>
<td>$9.685$</td>
<td>$+2.4$</td>
<td>$9.440$</td>
<td>$-2.5$</td>
</tr>
</tbody>
</table>

in our calculations.

The calculated masses of the light mesons $\pi$, $\rho$, $K$, $K^*$, fictional pseudoscalar $s\bar{s}$, $\phi$ and equal heavy quark mesons $\eta_c$, $J/\Psi$, $\eta_b$, $\Upsilon$ are presented in Table (4.2).

The first observation we make is that by using the 3ccp fit for the $u/d$ quarks to calculate the pion mass we get a value that is much smaller than the one we get with the full dynamical solution, about $-68\%$ relative difference. This is not so surprising since there is lot of delicate cancellations accuracy in the pion mass calculation due to its Goldstone boson character and the 3ccp fit does not represent to the required degree of accuracy. The full dynamical (gap) calculation for the pion is straightforward and the 3ccp fit is not aimed of this appreciation. Beyond that point we see that the relative difference for the other mesons is from $0.6 - 6.8\%$ compared with the gap calculations. The deviation that stands out is $-6.4\%$, for the
pseudoscalar Kaon, and a little more for the vector \( K^+ \), +6.8 %. There are several reasons for this and we analyze them in more detail since the circumstances should be relevant to other unequal quark mesons and the relative difficulty of calculations in studies of vector mesons properties.

A difference one can notice right away is in the Dirac structure of the BSA of vector and pseudoscalar mesons. Pseudoscalars have four invariant amplitudes while the vector mesons have eight. Accuracy concerns are greater for the latter. Also the equal quark BSA, must have a specific charge parity and their invariant amplitudes can only have even or odd order polynomials in the angle variable \( \hat{q} \cdot \hat{P} \). For the unequal ones we are going to have both even and odd angle dependence and more terms will be present in the expansion. For the vector mesons, since they are heavier, we have to move closer to the first pair of poles and the numerics will be more sensitive to the grid density and the number of Chebychev polynomials for the angle expansion.

If the mass poles are close, the 3cpp fit of the propagator will be less faithful to the dynamical DSE solution. The increase of the importance of the first pair of poles in shaping the BSE solution, as we go to heavier quark mesons, can be seen in the results of table (4.3) where we compare the masses of \( s\bar{s}, c\bar{c}, b\bar{b} \) pseudoscalar mesons, calculated using the 3cpp fit with all three pair of poles, with the ones we get by keeping only the first pair. The \( \eta_b \) meson mass and BSA are almost exclusively determined by the first pair of poles of the \( b \) quark 3cpp propagator. For the decay constants and the \( s\bar{s} \) system we get a value that is smaller by 21.74 % from the 3cpp one but if we use the 3cpp model in the calculations in the \( N,f \) integrals that difference drops to \(-4.9 \%\), therefore the other two pair of poles are important for the calculations of the two integrals for \( N,f \). For the \( c\bar{c} \) the situation has greatly improved.
Table 4.3: $s\bar{s}$, $c\bar{c}$, $b\bar{b}$, pseudoscalar meson masses using all three (3ccp case) and only the 1st pair (1ccp case) of complex conjugate poles (ccp) in the 3ccp fittings for the $s,c,b$ quark propagators. In all cases $\eta = 0.50$. We calculate the decay constants using the BSA we get with only the first ccp and in the decay constant integrals we either use again the first ccp ($f^1$ case) or the complete 3ccp fit ($f^2$ case). $\Delta M/M = (M^{1\text{ccp}} - M^{3\text{ccp}})/M^{3\text{ccp}}$ is the relative difference in the masses and $\Delta f/f = (f_i - f^{3\text{ccp}})/f^{3\text{ccp}}$, with $i = 1, 2$, is the relative difference for the two decay constants we calculate using the 1ccp BSA.

<table>
<thead>
<tr>
<th>meson</th>
<th>$M_H$(GeV)</th>
<th>$\Delta M/M%$</th>
<th>$f/(f^1, f^2)$(GeV)</th>
<th>$\Delta f/f%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(s\bar{s})(\text{fict.})(3\text{ccp})$</td>
<td>0.663</td>
<td>-</td>
<td>0.184</td>
<td>-</td>
</tr>
<tr>
<td>$(s\bar{s})(\text{fict.})(1\text{ccp})$</td>
<td>0.970</td>
<td>+46.3</td>
<td>(0.144,0.175)</td>
<td>(−21.7, −4.9)</td>
</tr>
<tr>
<td>$\eta_c(c\bar{c})(3\text{ccp})$</td>
<td>3.007</td>
<td>-</td>
<td>0.362</td>
<td>-</td>
</tr>
<tr>
<td>$\eta_c(c\bar{c})(1\text{ccp})$</td>
<td>2.963</td>
<td>−1.5</td>
<td>(0.347,0.346)</td>
<td>(−4.1, −4.4)</td>
</tr>
<tr>
<td>$\eta_b(b\bar{b})(3\text{ccp})$</td>
<td>9.347</td>
<td>0.0</td>
<td>(0.547,0.546)</td>
<td>(0.0, −0.2)</td>
</tr>
<tr>
<td>$\eta_b(b\bar{b})(1\text{ccp})$</td>
<td>9.347</td>
<td>0.0</td>
<td>(0.547,0.546)</td>
<td>(0.0, −0.2)</td>
</tr>
</tbody>
</table>

and in both cases the decay constant is smaller by about 4 %, including the other two poles in the calculations of $N,f$ does not improve the result . Finally for the $b\bar{b}$ essentially there is no difference in the decay constants.

The procedure to check the effect for some of the other factors is, as it was explained, by performing the calculations for different numbers of mesh points in the integration grid, different $\eta$’s, and a range of values of the total momentum and then plot the evolution of the eigenvalue. In graph (4.8) we have plotted, for demonstration of our analysis, the $P^2$ evolution of the eigenvalue for the fictional $s\bar{s}$ pseudoscalar meson for three values of $\eta$. Calculations done with another grid, for reasons of simplicity, are not included, but we have no change in the behavior of the eigenvalue and from the plot we see there is no dependence on $\eta$. We can conclude then that the calculated mass is a reliable number within our accuracy criteria. For the vector Kaon it was found that as we go closer to the mass shell there is a small dependence on $\eta$. 
and we can infer that due to the quark asymmetry we need more Chebychev moments in the invariants’ expansion. That is true for all unequal vector mesons studies but this variation is about $1 - 6\%$ and we can ignore it for the present studies.

For the $c, b$ unequal quark mesons, as we will explain in the next chapter, there are deeper reasons that can be responsible in part for the behavior of the BSE solution. We will show that the same factors have a favorable and beneficial effect for the equal quark mesons solution. So for all these reasons we expect to have larger differences in the calculated vector meson observables than the pseudoscalar ones and more for the unequal hadrons than the equal ones, by using the 3ccp fits.

On the other hand an important observation for our analysis is that the meson masses are rather robust, especially for the heavy quark systems. They are less
sensitive to details and uncertainties than other quantities like for example the decay constants. Part of the reason for this is that heavy meson masses calculations are sensitive to the propagator details only in a very small area near the peak of the mass shell region in the complex plane while decay constants are sensitive probes over a larger domain in the parabolic region. For the present we can conclude that the 3ccp fit can be used in the place of the real solution and give us excellent estimations of the mesons masses except the pion. The quality of the mesons’ BSA though is a totally different story as we will see in the case of the decay constants. In the case of b quark pseudoscalar and vector mesons that can be seen directly from the behavior of their Chebychev moments. In figures (4.9) and (4.10) we compare the first moment (zero order) of the first invariants in the mesons amplitudes from the gap solution and the model.

From both plots we observe an excellent agreement between the gap solution and the 3ccp fit calculations for the light quark mesons, there is a slight difference in the case of the c quark mesons, and there is a significant difference in the case of b mesons. The fact that we are so close to the first pair of singularities of the propagators and that they are of different type, is responsible for the result in the last case. The same pattern appears to be true in general for the third moment of the first invariant and the moments of the other invariants. Another piece of information we can extract from these two plots is the rate at which the amplitude drops as $p^2 \to +\infty$ and the quark mass increases. By checking the momenta points $p^2$ where the Chebychev moment drops to half of the initial value (we have normalized them so that this value is one) or just from the general pattern of the plots we see that the amplitude decreases slower as the quark mass increases. If we Fourier transform the amplitude
Figure 4.9: First Chebychev moments of the dominant invariant of the pseudoscalar mesons amplitude from the 3ccp fit compared to the ones from the gap solution. Dashed line is to show the momentum point where the amplitude is half its initial value, which is one.
Figure 4.10: First Chebychev moments of the dominant invariant of the vector mesons amplitude from the 3ccp fit compared to the ones from the gap solution. Dashed line is to show the momentum point where the amplitude is half its initial value, which is one.
Table 4.4: Pseudoscalar and vector meson electroweak decay constants: experimental data and calculated constants using the gap or the 3ccp fit for the quark propagators. In the fourth column of the table are the relative percentage differences between gap and experimental values: \( \Delta \frac{f}{f^{\text{exp.}}} = \frac{f^{\text{gap}} - f^{\text{exp.}}}{f^{\text{exp.}}} \) and in the last column we have the relative percentage differences between 3ccp and gap decay constants: \( \Delta \frac{f}{f^{\text{exp.}}} = \frac{f^{\text{3ccp}} - f^{\text{gap}}}{f^{\text{gap}}} \). All decay constants are in GeV. Experimental data are from [80].

to the spatial coordinates we are going to have a function that is more and more localized as the quark mass increases and one can consider this as the realization of the classical limit approach where if the mass is very large then from Heisenberg’s uncertainty principle we know there is a very small uncertainty in the position of the particle and therefore has a point-like behavior.

4.3 Meson electroweak decay constants using the 3ccp propagator representation.

The decay constants of the mesons appear in Table (4.4). Since they are very small numbers and because they involve the use of the BSA and quark propagators in each of the three steps, we expect them to be more sensitive to details than meson masses.

We notice a surprising accuracy for \( f_\pi \) calculated from the 3ccp fit. We attribute this to canceling effects taking place in the calculations of the normalization and decay
constant integrals. We next observe that while the relative difference with the gap results for the light pseudoscalar mesons is small, as we go to heavier mesons the 3ccp fit produces an increasing discrepancy which is about 20% for the \( \eta_b \) meson. For the vector mesons, even with light quark, the difference is about 10 – 15%. There are several reasons for the discrepancies in the decay constants introduced through the use of the 3ccp fit. Firstly the resulting meson mass occurs in the decay constant integral. Secondly the meson BSA appears in the normalization integral and the decay constant integral. Finally the quark propagator representation occurs directly in all integrals. Most important the normalization integral involves the derivative of the propagators, enhancing the integral’s sensitivity to propagators details.

In order to isolate the effect of the propagator fit, either directly or indirectly, through the BSA, we will calculate the decay integrals by keeping the 3ccp BSA but using the gap propagator solutions instead. Secondly we will use the full dynamical solution for the BSA but we will keep the 3ccp fit for the propagator in the subsequent steps of calculations in the \( N \) (normalization) and \( f \) (decay constant) integrals. The results are presented in Tables (4.5,4.6)

From the relative differences in the fifth column of Table 4.5 and from the last column in Table 4.6 we can conclude that the quark propagator fit, in general, works well in the calculations of the \( N,f \) integrals even for heavy quark mesons. That can be attributed to the lack of enhancement of the behavior of the propagators near the peak of the parabolic region since now the integrals kernel does not include a MT model-like behavior in the infrared. On the other hand from the last column of Table 4.5 and the fifth column of Table 4.6 we can extract some information about the effect on the decay constants from the differences in the BSA from the two approaches.
Table 4.5: Pseudoscalar and vector meson decay constants from gap calculations, as well as from calculations using the 3ccp BSE solution and 3ccp or Gap propagators in the N,f integrals. The third column has the results from calculations using the 3ccp fit while the fourth one has the decay constants calculated by using the 3ccp BSA but the gap propagators (instead the 3ccp) in the N,f integrals. The next two columns have the percentage difference between the last ones and the 3ccp: $\Delta f / f_{3ccp}^0 = (f_0 - f_{3ccp}) / f_{3ccp}$ or the gap: $\Delta f / f_{gap}^0 = (f_0 - f_{gap}) / f_{gap}$ calculated decays. $f_0$ is in GeV.

<table>
<thead>
<tr>
<th>meson</th>
<th>Gap (GeV)</th>
<th>3ccp (GeV)</th>
<th>Gap in N,f($f_0$)</th>
<th>$\Delta f / f_{3ccp}^0%$</th>
<th>$\Delta f / f_{Gap}^0%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi(u\bar{u})$</td>
<td>0.131</td>
<td>0.131</td>
<td>0.130</td>
<td>+0.8</td>
<td>+0.8</td>
</tr>
<tr>
<td>$\rho(u\bar{u})$</td>
<td>0.207</td>
<td>0.240</td>
<td>0.237</td>
<td>+1.3</td>
<td>-14.5</td>
</tr>
<tr>
<td>$K(us)$</td>
<td>0.155</td>
<td>0.155</td>
<td>0.153</td>
<td>+1.3</td>
<td>+1.3</td>
</tr>
<tr>
<td>$K^*(us)$</td>
<td>0.241</td>
<td>0.218</td>
<td>0.221</td>
<td>-1.4</td>
<td>+8.3</td>
</tr>
<tr>
<td>$(ss)(fict.)$</td>
<td>0.182</td>
<td>0.184</td>
<td>0.182</td>
<td>+1.1</td>
<td>0.0</td>
</tr>
<tr>
<td>$\phi(s\bar{s})$</td>
<td>0.259</td>
<td>0.279</td>
<td>0.272</td>
<td>+2.5</td>
<td>-5.0</td>
</tr>
<tr>
<td>$\eta_c(c\bar{c})$</td>
<td>0.387</td>
<td>0.362</td>
<td>0.378</td>
<td>-4.4</td>
<td>+2.3</td>
</tr>
<tr>
<td>$J/\psi(c\bar{c})$</td>
<td>0.415</td>
<td>0.340</td>
<td>0.382</td>
<td>-12.4</td>
<td>+7.9</td>
</tr>
<tr>
<td>$\eta_b(bb)$</td>
<td>0.692</td>
<td>0.547</td>
<td>0.571</td>
<td>-4.4</td>
<td>+17.5</td>
</tr>
<tr>
<td>$\Upsilon(bb)$</td>
<td>0.682</td>
<td>0.517</td>
<td>0.535</td>
<td>-3.5</td>
<td>+21.6</td>
</tr>
</tbody>
</table>

The $b$ quark mesons decay constants reveal the impact of the differences in the BSA amplitudes noticed in graphs (4.9) and (4.10) and the larger relative difference in the vector mesons from the pseudoscalar ones confirms the conclusions of our analysis in the previous paragraph about the impact of the larger number of invariants in the BSA and the closeness of the singularities in the integration domain.

A final observation is that by fitting the $c$, $b$ quark propagators over a larger domain in the time-like region we might well improve the faithfulness of the representation and the agreement with full dynamical calculations of observables. The 3ccp fit can be used with confidence for practical reasons to qualitatively study QCD phenomena, but for quantitative accuracy the detailed fit should be examined. The knowledge of the exact location of the poles makes it a very enticing tool for the
Table 4.6: Pseudoscalar and vector meson decay constants from gap calculations, as well as from calculations using the gap BSE solution and gap or 3ccp propagators in the N,f integrals. The third column has the results from calculations using the gap solution while the fourth one has the decay constants calculated by using the gap BSA but the 3ccp propagators (instead the gap ones ) in the N,f integrals. The next two columns have the percentage difference between the last ones and the 3ccp: $\Delta f/f^{3ccp} = (f^1 - f^{3ccp})/f^{3ccp}$ or the gap: $\Delta f/f^{gap} = (f^1 - f^{gap})/f^{gap}$ calculated decays. $f^1$ is in GeV.

<table>
<thead>
<tr>
<th>meson</th>
<th>3ccp (GeV)</th>
<th>Gap (GeV)</th>
<th>3ccp in N,f($f^1$)</th>
<th>$\Delta f/f^{3ccp}$%</th>
<th>$\Delta f/f^{gap}$%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi(u\bar{u})$</td>
<td>0.131</td>
<td>0.131</td>
<td>0.140</td>
<td>-6.9</td>
<td>-6.9</td>
</tr>
<tr>
<td>$\rho(u\bar{u})$</td>
<td>0.240</td>
<td>0.207</td>
<td>0.235</td>
<td>+2.1</td>
<td>+13.5</td>
</tr>
<tr>
<td>$K(us)$</td>
<td>0.154</td>
<td>0.155</td>
<td>0.154</td>
<td>+0.6</td>
<td>+0.6</td>
</tr>
<tr>
<td>$K^*(us)$</td>
<td>0.218</td>
<td>0.241</td>
<td>0.240</td>
<td>-10.1</td>
<td>+0.4</td>
</tr>
<tr>
<td>$(s\bar{s})(fict.)$</td>
<td>0.184</td>
<td>0.182</td>
<td>0.190</td>
<td>-3.3</td>
<td>-4.4</td>
</tr>
<tr>
<td>$\phi(s\bar{s})$</td>
<td>0.279</td>
<td>0.259</td>
<td>0.276</td>
<td>+1.1</td>
<td>-6.6</td>
</tr>
<tr>
<td>$\eta(c\bar{c})$</td>
<td>0.362</td>
<td>0.387</td>
<td>0.372</td>
<td>-2.8</td>
<td>+3.9</td>
</tr>
<tr>
<td>$J/\psi(c\bar{c})$</td>
<td>0.340</td>
<td>0.415</td>
<td>0.368</td>
<td>-8.2</td>
<td>+11.3</td>
</tr>
<tr>
<td>$\eta_b(bb)$</td>
<td>0.547</td>
<td>0.692</td>
<td>0.647</td>
<td>-18.3</td>
<td>+6.5</td>
</tr>
<tr>
<td>$\Upsilon(bb)$</td>
<td>0.517</td>
<td>0.682</td>
<td>0.628</td>
<td>-21.5</td>
<td>+8.6</td>
</tr>
</tbody>
</table>
studies of excited states of mesons if one follows the approach in [67] to deal with the propagator singularities that would enter the BSE integration domain. From our studies we can estimate that the uncertainty of this approach to other calculated observables, after improving the c and b quark representations, will be between $1 - 15\%$. 
Chapter 5

**The constituent mass concept for heavy quark mesons**

The definition of a quark as heavy requires a comparison of its mass with the nonperturbative chiral symmetry breaking scale which is about 1 GeV ($\Lambda_\chi \sim 1 \text{ GeV}$) or with the scale $\Lambda_{QCD} \sim 0.2 \text{ GeV}$ that characterizes the distinction between perturbative and nonperturbative QCD. For quark masses significantly larger than these scales, nonperturbative dressing effects, or equivalently nonperturbative self-energy contributions, and relativistic effects are believed to be less important for physical observables. We examine the effectiveness of a constituent mass approximation for the heavy quarks in the DSE formalism, for light-heavy and heavy-heavy quark mesons by calculating their masses and electroweak decay constants.

5.1 Definitions of a heavy quark mass.

In the case of quarkonia (heavy-heavy quark) $Q\bar{Q}$ mesons, the heavy quarks with mass $m_Q$ have an effective coupling that is small; the strong interactions at the Compton scale, $\lambda_Q \sim 1/m_Q$, can be treated perturbatively. In this case the situation is quite simple, symmetric, and much like the positronium system. For the light-heavy mesons the situation is not that simple since the heavy quark is essentially in a strongly interacting medium of light quarks, anti-quarks and gluons. The size of such mesons is typically 1 $\text{fm} \sim 1/\Lambda_{QCD}$ and the momentum exchange between the heavy quark and light quark is of the order of $\Lambda_{QCD}$. In order to couple to the heavy quark, one has to use a hard (short distance) probe, but the gluons exchanged between
light and heavy degrees of freedom are soft. Therefore the light quarks largely are insensitive to the flavor and the spin orientation of the heavy quark; they only feel its color field. Effective heavy quark field theories and techniques have been developed to emphasize these aspects ([110],[111]).

An issue that has central significance for studies of heavy quark systems is the definition of the mass of a heavy quark. Since isolated quarks are never found, one has to introduce a specific mass definition for each different framework. The pole mass is the one we are going to use for our studies but there are others like the \( \overline{\text{MS}} \) mass, the Georgi-Politzer mass, the potential model (PM) mass and the heavy quark effective field theory (HQET) mass ([113],[114], [115]). The idea behind the estimation of a heavy quark pole mass, which is infrared finite and independent on the renormalization scheme, is that, since the running coupling is small at this scale, one can use perturbative QCD to calculate the quark propagator. Therefore the pole mass is a perturbative quantity. Since the quarks are confined, the exact propagator can not have a true mass pole and that fact by itself reveals the importance of nonperturbative infrared effects in the mass function of the quark. Confinement also creates ambiguity for a nonperturbative definition of a quark pole mass. The light-heavy quark meson has a pole mass that is a physical quantity which can, in principle, be used to extract the heavy quark pole mass by subtracting the binding energy which must be of the order of \( \Lambda_{QCD} \). But the definition of a binding energy for absolutely confined particles is itself ambiguous. Any so-determined pole mass will have then an uncertainty in the order of \( \Lambda_{QCD} \). The use of the pole mass concept for the studies of heavy quark systems is limited ([116], [117], [118]).
The concept of a constituent mass means a constant mass, independent of momenta and used within a calculation of a hadron and its properties. In our case we will compare results from such an assumption with results from the dressed quarks of our model.

5.2 Studying heavy quark systems.

A large amount of research work on heavy quark systems has been done within a Hamiltonian approach containing a static potential interaction. The quark mass is a parameter to be fitted to experimental data. One refers to this mass as a model-dependent constituent mass (see for example [119], [120]). In different potential models the range of the $c$ quark mass is $1.30 - 1.84 \, GeV$ while for the $b$ quark is $4.2 - 5.17 \, GeV$ ([121], [122]). Recent work has made a clear connection between a relativistic Hamiltonian and the QCD Lagrangian. However there remains no more than a qualitative relation between the various mass definitions within the currently used approaches ([123], [124], [125], and references therein).

Another approach is through the framework of the so-called instantaneous Bethe-Salpeter model. In this case the instantaneous BSE kernel and the use of constituent mass quark propagators allow explicit integration over time in the rest frame of the mesons. Such a model can give pseudoscalar and vector quarkonia meson spectra and decay constants in reasonable agreement to experimental data; however the number of parameters is considerably greater than the present approach. The lack of retardation effects is also a drawback. As you go to higher radial or angular excited states though the masses and the decay constants are not described in a satisfying way (see for example [126], [128], [129], [130], [131], [132] and references therein).
By using the Dyson-Schwinger equation formalism and a heavy-quark limit approach, which essentially begins with a constituent or pole mass heavy quark propagator, Kalinovski et al. in a series of works examined heavy quark meson observables ([133], [134]). For the light quark propagator they employed a parametrization fitted to light quark meson experimental data. The solution of the BSE is simplified through a parametrization of the BSA. In that way the calculations of other physical observables, like decay constants, form factors etc. is a relatively easy and straightforward task (for more details see [135], [136], [137], [138], [139]).

Heavy quark effective field theories (HQET) have been developed by appropriately manipulating the QCD Lagrangian with the ultimate goal of simplifying the description of nonperturbative physics. We stress that, unlike other effective theories, the purpose here is to describe heavy quark hadron observables, thus one can not completely remove the heavy quark degree of freedom. One can integrate out degrees of freedom that describe fluctuations around the mass shell. HQETs are also used in lattice QCD. In general, masses, decay constants, form factors and other physical quantities for mesons and baryons with heavy quarks are described with reasonable agreement to experimental data. For more details and a pedagogical introduction in the HQETs see for example ([142], [143],[144], [145]) and for more applications see ([148], [146], [149], [150],[151], [152], [153], [147]) and references therein. A detailed overview of the present state of experimental and theoretical studies of quarkonia can be found in reference [127].
5.3 MT effective kernel and heavy quark mesons.

The MT model was developed within the Dyson-Schwinger equations approach and the rainbow-ladder truncation to study the physics of DCSB and related phenomena, like the spectrum of light quark mesons of pions and Kaons. Calculations of other physical observables like decay constants, form factors, charge radii, of light mesons appear to be in very good agreement with experimental data. The QCD mechanisms responsible for the agreement are not so obvious. A more detailed and deeper understanding of QCD dynamics responsible for the light quark results is required.

An extension to heavy quark meson properties can inform on the physical content. Previous applications have been confined to light mesons because the solution of the DSE for the heavy quark propagator, is difficult in a numerical sense. Singularities that are near to, but not within, the domain of integration require careful numerical work. With use of the gluon momentum as the integration variable, we could extend the calculations to heavy quark quarkonia states [56]. The model gave us masses and decay constants that are in surprisingly good agreement with experimental data. The unequal heavy quark meson studies (i.e. the D, B mesons), reveal the limits of the model.

Finally it will be interesting to explore the effectiveness of a constituent mass approximation for the heavy quark propagators, especially for the meson masses and electroweak decay constants.
5.4 Meson masses.

In the constituent mass approximation the heavy (c or b) quark propagator simplifies to:

\[ S(p)^{-1} \sim i \not{p} + M_c \]  

(5.1)

where \( M_c \) is the constituent mass. The propagator amplitudes then reduce to the following expressions:

\[ \sigma_s(p^2) \sim \frac{M_c}{(p^2 + M_c^2)}, \quad \sigma_v(p^2) \sim \frac{1}{(p^2 + M_c^2)}. \]  

(5.2)

In essence we have more than one approximations involved in order to get the above expressions from the complete propagator. One is the constituent or pole mass approximation in the real part of the mass function \( (\text{Real}(M(p^2)) \sim M_c) \) and at the same time we ignore the imaginary part of this function \( (\text{Im}(M(p^2)) \sim 0) \) which from chapter 3 (see fig. 3.7 for the c quark) we found becomes more important at the peak of the parabolic region. So the complex quark mass function is replaced by a real constant and geometrically that means we replace the surface of the real part of the function (see fig. 3.6 for example for the c quark) by a flat surface in the complex plane. Unlike the ambiguous and problematic physical definition of the approximation a simple and precise mathematical description is possible by just referring to as a constant quark mass function approximation. The other approximation regards function A where we set \( \text{Real}(A(p^2)) \sim 1, \) and \( \text{Im}(A(p^2)) \sim 0. \) From fig. (3.4) for the function of the c quark propagator we see that it is generally over one in the area at the peak of the parabolic region and again the imaginary part of this function is quite important over there. The significance and the different role of the imaginary parts of these two functions in the solution of the BSE for the light-heavy and heavy-heavy
Figure 5.1: Quark mass functions in the real space-like axis, for the chiral limit, u/d, s, c and b quarks. Current quark masses are at $\mu = 19 \text{ GeV}$.

Quark systems will be revealed later in this chapter.

A plot used for justifying a constituent mass approximation for the heavy quarks is the one in fig. (5.1). From this plot one can readily see that there are very strong infrared dressing effects for chiral quarks and the u/d and s-quark making them much heavier in that infrared region. The dressing is relatively smaller for the much heavier c and b quarks and their mass functions are almost flat lines, therefore one can assume they can be well approximated by a constant. We should notice though that the plot is in logarithmic scale (both axes). The behavior of the mass amplitude and propagator amplitude $A(p^2)$ (real and imaginary parts) for the time-like or complex $p^2$, can be important influences upon the existence of a solution of the BSE.

The fitted constituent mass for the c quark was found to be $M_c^c \sim 2.0 \text{ GeV}$
and for the $b$ quark $M_b^0 \sim 5.3$ GeV. Besides the nonperturbative effects absorbed in the values of the parameters some error, because of the use of the light quark 3ccp representation, may be present but considering the analysis in chapter 4 we can safely assume that it is relatively very small and hence insignificant. The results of studies where we actually solve the gap equation for the light quark propagator in the parabolic region needed for the solution of the BSE verify the last statement. A direct comparison of the quark propagator amplitudes from the constituent mass approximation and the gap solution, in the real axis only is shown in figs. 5.2, 5.3. One can notice the stronger behavior of $\sigma_s$ over that of $\sigma_v$ near the mass-shell position because of the presence of the quark mass function in the numerator of $\sigma_s$. Far from the singularities, a better agreement is possible.

As the current quark mass increases, there is comparatively less dressing and the singularity of the quark propagators will be closer to the dominant region of the complex domain where they are needed for the meson bound state calculations. That is $q_i^2 \sim (\eta P)^2 \sim -Re\{M^2[(\eta P)^2]\} \sim -m_q^2(\mu)$ where $P^2 = -(M_{meson})^2$ and $M_{meson} \sim 2m_q$. The first pair of poles of the 3ccp representation for the $b$ quark produce the real part of the pole location on $q_+^2$ plane to be at $m_+^2 - m_1^2 = 27.977$ GeV$^2$ which is very close to the pole location of the constituent mass approximation $(M_\xi)^2 = 28.09$ GeV$^2$. This is also very close to the real part of the quark mass function near the peak of the parabolic region.

The results for the $c$ and $b$ quark mesons masses are presented in table (5.1).

All calculated masses appear to be in excellent agreement with experiment with a deviation of no more than 3.2 %. The reasons for this succesful description of the heavy quark meson masses are as explained in detail in chapter 4. We next examine
Figure 5.2: $c$ quark propagator amplitudes on the real axis and up to the mass-shell point in the time-like region, from the constituent mass approximation ($M_c = 2.0 \, \text{GeV}$) compared to the DSE solution.
Figure 5.3: b quark propagator amplitudes on the real axis and up to the mass-shell point in the time-like region, from the constituent mass approximation ($M_b = 5.3 \, GeV$) compared to the DSE solution.
Table 5.1: qQ and QQ pseudoscalar and vector meson masses using constituent mass approximations (noted as 1rp) for the c, b quark propagators. The pseudoscalar mesons $D(uc)$ and $B(ub)$ masses were used to determine the values of the constituent quark mass parameters. It was found $M_c \sim 2.0$ GeV and $M_b \sim 5.3$ GeV. The last column has the percentage differences between experimental and calculated hadron masses $\Delta m/m^{exp} = (m^{1rp} - m^{exp})/m^{exp}$. There are no experimental data available for the $B^*_c(cb)$ vector meson mass. The absolute relative percentage difference is from 0.13 – 3.26% and it appears there is no specific pattern for the differences in the calculated masses. Experimental data are from [80].

<table>
<thead>
<tr>
<th>meson</th>
<th>exp.(GeV)</th>
<th>1rp (GeV)</th>
<th>$\Delta m/m^{exp}$%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D(uc)$</td>
<td>$1.864 \pm 1.0 \times 10^{-3}$</td>
<td>1.852</td>
<td>$-0.6$</td>
</tr>
<tr>
<td>$D^*(uc)$</td>
<td>$2.007 \pm 4.0 \times 10^{-4}$</td>
<td>2.04</td>
<td>$+1.6$</td>
</tr>
<tr>
<td>$D_s(sc)$</td>
<td>$1.969 \pm 1.4 \times 10^{-3}$</td>
<td>1.975</td>
<td>$+0.3$</td>
</tr>
<tr>
<td>$D_s^*(sc)$</td>
<td>$2.112 \pm 6.0 \times 10^{-4}$</td>
<td>2.17</td>
<td>$+2.8$</td>
</tr>
<tr>
<td>$B(ub)$</td>
<td>$5.279 \pm 7.0 \times 10^{-4}$</td>
<td>5.254</td>
<td>$-0.5$</td>
</tr>
<tr>
<td>$B^*(ub)$</td>
<td>$5.325 \pm 6.0 \times 10^{-4}$</td>
<td>5.32</td>
<td>$-0.1$</td>
</tr>
<tr>
<td>$B_c(sb)$</td>
<td>$5.370 \pm 2.4 \times 10^{-3}$</td>
<td>5.38</td>
<td>$+0.2$</td>
</tr>
<tr>
<td>$B^*_c(sb)$</td>
<td>$5.413 \pm 1.7 \times 10^{-3}$</td>
<td>5.42</td>
<td>$+0.1$</td>
</tr>
<tr>
<td>$\eta_c(c\bar{c})$</td>
<td>$2.980 \pm 1.2 \times 10^{-3}$</td>
<td>3.025</td>
<td>$+1.5$</td>
</tr>
<tr>
<td>$J/\psi(c\bar{c})$</td>
<td>$3.097 \pm 1.1 \times 10^{-3}$</td>
<td>3.192</td>
<td>$+3.1$</td>
</tr>
<tr>
<td>$B_c(cb)$</td>
<td>$6.286 \pm 5.0 \times 10^{-6}$</td>
<td>6.36</td>
<td>$+1.2$</td>
</tr>
<tr>
<td>$B^*_c(cb)$</td>
<td>$6.440$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\eta_b(bb)$</td>
<td>$9.300 \pm 4.0 \times 10^{-2}$</td>
<td>9.603</td>
<td>$+3.3$</td>
</tr>
<tr>
<td>$\Upsilon(bb)$</td>
<td>$9.460 \pm 2.6 \times 10^{-4}$</td>
<td>9.645</td>
<td>$+2.0$</td>
</tr>
</tbody>
</table>
the quality of the calculated BSA. For the quarkonia we do this by comparing the Chebychev moments of the amplitudes.

Figures (5.4, 5.5) display the first Chebychev moment of the dominant invariant of the amplitude of the pseudoscalar quarkonia $\eta_c$, $\eta_b$. The effect of the constituent mass approximation for the heavy quark propagator is a faster fall off of the moments with momentum i.e. a larger meson in coordinate space. To see whether the difference in relative momentum dependence might be due to the different mass shell locations, we calculate the amplitude relative momentum dependence of the 3ccp and 1rp approximation evaluated at the mass shell of the full dynamical solution (kcp).
Figure 5.5: First Chebychev moment of the dominant invariant amplitude of the $b\bar{b}$ pseudoscalar meson by using the DSE propagator solution (kcp), the 3ccp propagator representation and the constituent mass approximation (1rp). In the parenthesis for each line we include the meson total momentum squared.
Table 5.2: $qQ$ and $QQ$ pseudoscalar and vector meson electroweak decay constants using constituent mass approximation for the $c,b$ propagators. The last column has the percentage differences between experimental and calculated hadron decays\[\Delta f/f_{\text{exp.}} = (f_{1\text{rp}} - f_{\text{exp.}})/f_{\text{exp.}}.\] Experimental data are from [80].

<table>
<thead>
<tr>
<th>meson</th>
<th>exp.(GeV)</th>
<th>1rp(GeV)</th>
<th>$\Delta f/f_{\text{exp.}}$%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D(uc)$</td>
<td>$0.223 \pm 1.9 \times 10^{-2}$</td>
<td>0.154</td>
<td>$-31.1$</td>
</tr>
<tr>
<td>$D'(uc)$</td>
<td>0.164</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_s(sc)$</td>
<td>$0.294 \pm 2.7 \times 10^{-2}$</td>
<td>0.197</td>
<td>$-33.0$</td>
</tr>
<tr>
<td>$D_s^*(sc)$</td>
<td>0.180</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B(ub)$</td>
<td>$0.176 \pm 4.9 \times 10^{-2}$</td>
<td>0.105</td>
<td>$-40.3$</td>
</tr>
<tr>
<td>$B^*(ub)$</td>
<td>0.182</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B_s(sb)$</td>
<td>0.144</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B_s^*(sb)$</td>
<td>0.20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\eta_c(\bar{c}\bar{c})$</td>
<td>0.340</td>
<td>0.239</td>
<td>$-29.7$</td>
</tr>
<tr>
<td>$J/\psi(\bar{c}\bar{c})$</td>
<td>$0.416 \pm 7.0 \times 10^{-3}$</td>
<td>0.198</td>
<td>$-52.4$</td>
</tr>
<tr>
<td>$B_c(cb)$</td>
<td>0.210</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B_s^0(cb)$</td>
<td>0.18</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\eta_b(bb)$</td>
<td>0.244</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Upsilon(bb)$</td>
<td>$0.700 \pm 2.4 \times 10^{-2}$</td>
<td>0.21</td>
<td>$-70.0$</td>
</tr>
</tbody>
</table>

We found there is no significant change and we can safely conclude that the amplitude differences are a true consequence of the different dynamics of the propagator approximations. The same pattern appears to hold true for the vector quarkonia. For the $qQ$ systems a similar analysis is not possible since a meson mass shell solution is not attained from the DSE propagator solutions.

5.5 Meson decay constants.

Using the constituent mass approximation we calculate the mesons electroweak decay constants. Table (5.2) contains these results.

Unfortunately there are not many experimental data for the electroweak decay constants of $qQ$ vector mesons but the available data for the other mesons are enough to get a clear picture of the pattern in the calculated decays. Overall it appears
that the calculated decay constants are much smaller than the experimental data and the situation is rapidly deteriorating as we go to heavier quark systems. The relative difference for the $\Upsilon(b\bar{b})$ meson is $-70.0\%$ and so it appears that, even for the relatively simpler quarkonia systems, the agreement is poor. Intuitively one would expect that the constituent mass approximation will get better for heavier quarks but the results for the electroweak decay constants indicate the opposite. To help understand how this comes about we repeat the analysis followed in the last chapter. Since we have no dynamical solution (kcp) for the $qQ$ meson systems, we will use the 3ccp model for the heavy quarks in the $N$ (normalization), $f$ (decay constant) integrals to contrast with the constituent mass approximation. For the quarkonia states the results are in Tables (5.3) and (5.4). For reasons of completeness we included the use of the 3ccp bound state amplitudes and propagators for the $c, b$ quarks. Different types of propagators ($1rp$ and 3ccp) were used in some of the calculations of $N$ (normalization), $f$ (decay constant) integrals to explore dynamical sensitivity and numerical accuracy.

The second column of Table 5.3 uses the constituent mass bound state amplitudes. We see that the the decay constant differs by $0.088\text{ GeV}$ if the $N, f$ integrals are computed using the dynamical (kcp) quark propagator. The corresponding difference for the full dynamical bound state (kcp) is $0.065\text{ GeV}$. These differences are very close and reveal the effect of the different dynamics of the propagators in the decay constant value. On the other hand the difference of the decays in the third row ($0.322\text{ GeV}$ and $0.239\text{ GeV}$) is $0.083\text{ GeV}$ and for the ones in the last row ($0.327\text{ GeV}$ and $0.387\text{ GeV}$) is $0.06\text{ GeV}$. This shows the influence of the bound state amplitudes on the decay constant. It appears that both $1rp$ propagators and $1rp$ BSA have about the same share of responsibility for the very small value of the $\eta_c$ decay constant.
Table 5.3: $\eta_c(c\bar{c})$ pseudoscalar, $J/\psi(c\bar{c})$ vector meson decay constants by using a BSA calculated with 1rp(constituent approximation), 3ccp(representation) or kcp(dynamical) for the $c,\bar{c}$ quark propagators and using 1rp, 3ccp or kcp propagators in the calculations in N and f integrals for each case indicated in the first column of the table where 1rp(x), 3ccp(x), kcp(x), x = $c, \bar{c}$ are for the type of propagator used for each quark. Here $\eta = 0.50$ but calculations were also done for $\eta = 0.45, 0.55$. The index in the BSA signifies the type of quark propagator used for the calculation of the meson amplitude. All calculated decay constants are in GeV.

<table>
<thead>
<tr>
<th>N, f integ.</th>
<th>$f_{\eta_c}^{exp.} = 0.340$ GeV</th>
<th>$f_{J/\psi}^{exp.} = 0.416$ GeV</th>
</tr>
</thead>
<tbody>
<tr>
<td>1rp(c)+1rp(\bar{c})</td>
<td>$0.239$</td>
<td>$0.319$</td>
</tr>
<tr>
<td>1rp(c)+3ccp(\bar{c})</td>
<td>$0.227$</td>
<td>$0.366$</td>
</tr>
<tr>
<td>3ccp(c)+1rp(\bar{c})</td>
<td>$0.227$</td>
<td>$0.366$</td>
</tr>
<tr>
<td>3ccp(c)+3ccp(\bar{c})</td>
<td>$0.326$</td>
<td>$0.362$</td>
</tr>
<tr>
<td>kcp(c)+kcp(\bar{c})</td>
<td>$0.327$</td>
<td>$0.378$</td>
</tr>
</tbody>
</table>

Also we see that the 3ccp representation of propagators captures the full dynamical solution very well. For the corresponding vector meson in the same table we have that the difference in the first and last values in the fifth column is 0.11 GeV and in the last column is 0.107 GeV. The difference in the first and last values of the decay in the third row of the table in the vector meson is 0.11 GeV and 0.107 GeV. Again we reach the same conclusion as in the pseudoscalar meson, only now the differences (about 0.11 GeV) resulting from the 1rp BSA or 1rp $c$ propagator are larger than before.

Moving to the $b$ quarkonia we see, in general, a deteriorating situation. For the pseudoscalar meson we have 0.201 GeV and 0.173 GeV difference due to the dynamics of the 1rp $b$ propagator (more than twice as much as in the pseudoscalar $c$ quarkonium) and for the differences due to the 1rp BSA we have 0.248 GeV and 0.248 GeV, so it appears now that the 1rp BSA is slightly more responsible for the very small decay.
Table 5.4: $\eta_b(b\bar{b})$ pseudoscalar, $T(b\bar{b})$ vector meson decay constants by using a BSA calculated with 1rp, 3ccp or kcp for the $b,\bar{b}$ quark propagators and using 1rp, 3ccp or kcp propagators in the calculations in N and f integrals for each case. Again we have $\eta = 0.50$ but calculations were also done for $\eta = 0.45,0.55$. Everything in this table has the same significance as in the c quarkonia table.

For the vector meson we have 0.240 GeV, 0.258 GeV and 0.215 GeV, 0.233 GeV for the corresponding cases. In this case we see there is a more balanced contribution form the 1rp propagator and BSA in the difference of the decay. Again we have a stronger influence in the numerics of the decay from these two factors than in the c vector case, the situation though appears to be the same for both pseudoscalar and vector b quarkonia. The 3ccp model lacks the success it had in the calculation of the c quarkonia and the reasons are well known and explained in chapter 4.

Overall we can conclude that the constituent quark mass approximation will give us smaller decay constants than the complete solution and the difference will increase as we go to heavier quarks. We found that the 1rp BSA and the 1rp propagator in the N,f integrals have about the same responsibility for getting so small decay constant values. It is very easy to understand the reason for the influence the 1rp propagator has in the N, f integrals if we just notice that the total momentum derivative in the N integral in this case will act only in the momentum $q^2_{\pm}$ since it is the only P dependent quantity. In the full propagator case though we will get eight additional terms from...
<table>
<thead>
<tr>
<th>meson</th>
<th>exp. (GeV)</th>
<th>$f^{1rp}(\text{GeV})$</th>
<th>$f^{3ccp}(\text{GeV})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D(uc)$</td>
<td>0.222</td>
<td>0.154</td>
<td>0.255 ± 0.010</td>
</tr>
<tr>
<td>$D^*(uc)$</td>
<td>0.164</td>
<td></td>
<td>0.288 ± 0.030</td>
</tr>
<tr>
<td>$D_s(sc)$</td>
<td>0.294</td>
<td>0.197</td>
<td>0.255 ± 0.005</td>
</tr>
<tr>
<td>$D^*_s(sc)$</td>
<td>0.180</td>
<td></td>
<td>0.326 ± 0.040</td>
</tr>
<tr>
<td>$B(ub)$</td>
<td>0.176</td>
<td>0.105</td>
<td>0.193 ± 0.005</td>
</tr>
<tr>
<td>$B^*(ub)$</td>
<td></td>
<td>0.182</td>
<td>0.549 ± 0.083</td>
</tr>
<tr>
<td>$B_s(sb)$</td>
<td>0.144</td>
<td></td>
<td>0.212 ± 0.002</td>
</tr>
<tr>
<td>$B^*_s(sb)$</td>
<td></td>
<td>0.20</td>
<td>0.425 ± 0.041</td>
</tr>
<tr>
<td>$\eta_c(\bar{c}\bar{c})$</td>
<td>0.340</td>
<td>0.239</td>
<td>0.326</td>
</tr>
<tr>
<td>$J/\psi(\bar{c}\bar{c})$</td>
<td>0.416</td>
<td>0.198</td>
<td>0.330</td>
</tr>
<tr>
<td>$B_c(cb)$</td>
<td></td>
<td>0.210</td>
<td>0.324 ± 0.004</td>
</tr>
<tr>
<td>$B^*_c(cb)$</td>
<td></td>
<td>0.18</td>
<td>0.328 ± 0.008</td>
</tr>
<tr>
<td>$\eta_b(bb)$</td>
<td></td>
<td>0.244</td>
<td>0.414</td>
</tr>
<tr>
<td>$\Upsilon(bb)$</td>
<td>0.700</td>
<td>0.210</td>
<td>0.381</td>
</tr>
</tbody>
</table>

Table 5.5: Pseudoscalar and vector mesons decay constants with constituent mass approximation for the c or b quark propagators in the BSE and 1rp or 3ccp in f and N integrals. Experimental data are from [80].

the action of the operator in the M and A functions (they have real and imaginary parts). So at the end not only will augment the influence of function M in the N integral by increasing the power of the denominator of the propagator amplitudes by one when it acts in the momenta $q^2_{\pm}$ but we will also have extra terms when it acts directly on the two functions (notice one of these terms will be $M.\partial_{P}M$). All these additional contributions from the extra terms are lost in the constituent mass approximation.

For the light-heavy systems the use of the heavy 3ccp model in the place of the 1rp propagator when calculating N,f gave us the results in Table (5.5). For reasons of completeness and comparison we include the 3ccp decay constant results for the quarkonia.
Before we start analyzing and discussing the data of the table, we can infer something about the approximation without even looking at the data. We found that the DSE solution for the heavy quark propagator failed to generate a bound state for the \(qQ\) mesons, but with the constituent mass approximation we did reach a bound state. Thus the constituent mass approximation does not reproduce the dynamics of the DSE dynamical propagator obtained with the effective MT interaction. The constituent mass propagator is rather what we would like to have from the MT model to reach physical meson states. This indicates that it is deficient for heavy quarks because it departs from a constituent-mass supporting behavior. Quarks are only found inside hadrons and cannot propagate macroscopically far from a hadron. Hence quarks cannot truly have a real pole mass. The fact that we obtain \(qQ\) bound states using a constituent mass approximation for the heavy quark, but not using a MT dynamical heavy quark propagator indicates that both are not very good representations of nature. It does not indicate that a constituent heavy quark mass approximation is generally better. It indicates that any dynamical model has inadequacies that are minor for quarkonia but more serious for \(qQ\) mesons. From the data in Table 5.5 we see that by use of the 3ccp representation for the heavy quark propagator gives us, decay constants that are much larger than the ones we get with the constituent mass propagator. The calculations for different values of \(\eta\) revealed a small dependence of the decay constants on that parameter, and for that we can blame the proximity of the singularities of the propagator. Nevertheless the decay constants now are closer to experiment and hence the constituent mass BS \(qQ\) meson amplitudes \((\text{unlike the } QQ \text{ ones})\) are reasonable for these \(qQ\) calculations. The explanation for this is rather simple. For the QQ systems we have essentially two propagators approximated by
constituent-like ones while for the qQ mesons there is only one approximate propagator. The same reasoning can be applied for the calculations of the \( N,f \) integrals.

The variation of functions \( M \) and \( A \) of the propagator along the real and imaginary \( p^2 \) axes near the mass shell are influenced mostly by the infrared behavior of the MT model. The derivatives of these functions will appear in the normalization integral and the latter are more sensitive probes for a larger area in the mass shell parabolic region in the complex plane. An approximation that was good for the meson mass is not necessarily good for a decay constant. The same is true for the approximations to the bound state amplitudes. The decay constant integrals show a sensitivity to self-energy corrections to the pole propagator and through them reveal the sensitivity to the MT model in the infrared region.

5.6 The extreme heavy quark limit.

There are relations [137] for the masses and decay constants of heavy quark mesons that have been extracted by using the heavy quark limit behavior of the propagator and a dimensional analysis of the involved equations. The approximation for the behavior of the propagator is based on the fundamental assumption that all the meson momentum is carried by the heavy quark alone. In this case the momentum of the quark will be:

\[
P_\mu = M_H v_\mu = (m_Q + E_H) v_\mu
\]

(5.3)

where \( P_\mu \) is the total momentum of the meson of mass \( M_H \) with a heavy quark of mass \( m_Q \), and energy \( E_H \). Then the heavy quark propagator can be written as ([137], [138]):

\[
S_Q(q + P) = \frac{1}{2} \frac{-i \gamma^\mu + 1}{q.v - E_H} + \mathcal{O}\left( \frac{|q|}{m_Q}, \frac{E_H}{m_Q} \right)
\]

(5.4)
where \( q \) is the relative quark momentum and the dimensionless velocity satisfies \( v^2 = -1 \). This approximation is considered to work satisfactorily well for \( qQ \) mesons as far as the binding energy of the system and the momentum scale defined as the width of the meson BSA are much smaller than the heavy quark mass. The width \( \langle q \rangle \) of the BSA gives a measure of how fast the amplitude decreases with momentum and a typical definition, which we actually used in chapter 4, is the momentum where the amplitude falls in half of its value at the origin. The definition of a binding energy as we explained at the beginning of the chapter is ambiguous. In any case we know that the difference between the meson mass and the heavy quark mass will be roughly of the order of \( \Lambda_{QCD} \sim 0.2 \, \text{GeV} \) and we assume this to be also the order of the binding energy. Therefore since a typical constituent mass for the \( c \) quark was found to be about 1.3 GeV and for the \( b \) about 4.6 GeV this was considered a reasonable approximation (see for example [137], [138]). The consequences of the other approximation are more complicated ones and essentially require the beforehand knowledge of the behavior of the complete solution of the BSE. In the studies for the \( qQ \) systems using a pole-mass heavy quark propagator it was found that the Chebychev moment of the dominant invariant amplitude it doesn’t decrease faster with the momentum as the \( Q \) quark mass increases but the quality of the result is unknown, hence the reliability of the approximation as well. At the end it was found that in this case the decay constant of the light-heavy mesons should behave like [137]:

\[
(5.5) \quad f_H \sim \frac{1}{\sqrt{M_H}} \quad 1/m_Q \sim 0
\]
Using the same analysis it was found for the so called in-hadron quark condensate:

\[ -<q\bar{q}>_H^\mu = \text{const.} + \mathcal{O}(1/M_H), \quad \text{if} \quad 1/M_H \sim 0 \]

and use of these in the pseudoscalar mass formula gives:

\[ M_H \sim \hat{m}_Q, \quad \text{if} \quad 1/\hat{m}_Q \sim 0 \]

where \( \hat{m}_Q \) is the renormalization-group-invariant current-quark mass for the heavy quark. This last result is something one should intuitively expect, as well as the fact that the mass difference between pseudoscalar and vector mesons should get smaller as \( \hat{m}_Q \to +\infty \), and they have been both numerically confirmed. Our studies of qQ mesons using a constituent mass approximation for the heavy quark propagator indicate the electroweak decay constants in the range of current mass up to the b quark (3.8 GeV) do not follow that behavior. Our calculated electroweak decay constants using constituent mass approximation are much smaller than the experiment and probably not reliable conclusion can be drawn.

In reference [155] the calculated masses of \( q_1q_2 \) mesons up to about 0.8 GeV current quark masses were fit by the form

\[ M_H^2 = M_0^2 + a_1 (m_1 + m_2) + a_2 (m_1 + m_2)^2 \]

where \( m_1, m_2 \) are the constituent current quark masses at \( \mu = 19 \text{ GeV} \), and the parameters of the relation fitted to the numerical data for the meson masses are 
\( M_0^{PS} = 0 \text{ GeV}, M_0^{V} = 0.75 \text{ GeV}, a_1^{PS} = 2.96 \text{ GeV}, a_1^{V} = 3.24 \text{ GeV} \), while for both vector and pseudoscalar mesons \( a_2 = 1.12 \text{ GeV} \). This expression and the masses of the quarkonia were used to determine the c and b current quark masses used in our calculations. They are \( m_c(19 \text{ GeV}) = 0.88 \text{ GeV}, \) and \( m_b(19 \text{ GeV}) = 3.8 \text{ GeV} \). From
the above expression it is obvious that as $m_2 \to +\infty$ the meson mass (for both vector and pseudoscalar) $M_H^2 \to a_2(m_2)^2$ and for the quarkonia $M_H^2 \to a_2(2m_Q)^2$ since $m_1 = m_2 = m_Q \to +\infty$. That relation was designed to have this behavior in the large quark limit.

Going back to the initial step of the analysis for the light-heavy quark mesons, it was assumed all the meson momentum is carried by the heavy quark and hence one can take $\eta = 1$. For our analysis we will try to find an approximated form by keeping $\eta$ general. Then we have for the denominator of the propagator

$$q_+^2 + M^2(q_+^2) = (\bar{q} + \eta P)^2 + M^2(q_+^2) = \bar{q}^2 - (\eta M_H)^2 + 2i\eta \sqrt{\bar{q}^2 M_H^2 v} + M^2(q_+^2) \sim (1 - \eta^2)M_H^2 + 2(i\eta |\bar{q}|v + E)M_H + \bar{q}^2 + E^2$$

(5.9)

where $v$ is now the $\cos$ of the angle between the four vectors $\bar{q}$ (the quark relative momentum and BSE integration variable), and the total meson momentum $P$ (with $P^2 = -M_H^2$). We also used the expression $M_H = M_Q + M_q - E \sim M_Q - E$ so $M_Q = M_H + E$, where $E$ is the binding energy of the system, $M_q$ and $M_Q$ are the constituent masses for the light and heavy quarks correspondingly. We assume $M_q << M_Q$, so the light quark mass can be ignored, and finally made the approximation $M(q_+^2) \sim M_Q$ for the whole parabolic BSE integration region. Notice that the constant coefficient of $M_H^2$ is $(1 - \eta^2)$ and this term does not exist if $\eta = 1$, dramatically changing the dependence of the denominator on $M_H$. So in this case if we write $P = iM_H u$ where $u$ is a unit 4-vector, we get for the propagator:

$$S_Q(q_+) \sim \frac{(1 + \eta \not{u})M_H + (E - i \not{q})}{(1 - \eta^2)M_H^2 + 2(i\eta |\bar{q}|v + E)M_H + \bar{q}^2 + E^2}$$

(5.10)

and at the end we can write:

$$S_Q(q_+) \sim \frac{1}{M_H} \frac{(1 + \eta \not{u})}{(1 - \eta^2) + 2(i\eta |\bar{q}|v + E)/M_H} + O\left(\frac{|\bar{q}|}{M_H}, \frac{E}{M_H}\right)$$

(5.11)
By replacing $E \rightarrow -E_H$, $u \rightarrow -iv$, $\bar{q} \rightarrow q$ to much the notation in ref. [137] and setting $M_H \sim M_Q$ we have

$$S_Q(q_+) \sim \frac{1}{M_Q} \frac{(1 - \eta \bar{v})}{(1 - \eta^2)} + \mathcal{O} \left( \frac{|q|}{M_Q} \frac{E_H}{M_Q} \right)$$

which gives 5.4 for $\eta = 1$

It is clear in this case that the heavy quark limit of the propagator has an $\eta$ dependence that significantly modifies its dependence on the heavy quark mass.

On the other hand relations in 5.5, 5.6, 5.7, obtained by dimensional arguments for $\eta = 1$, concern physical observables, so they should be independent of $\eta$. Therefore they should not hold true for the general case of 5.12 since that will introduce a different dependence on $M_Q$. They have also been checked by direct calculation for quarkonia (where $\eta = 0.5$) and found not to be true for heavy quarks [112]. In this case, $<q>/M_Q$, the measure of intrinsic size does not decrease rapidly and the neglect of higher terms in the expansion is not obviously justified. The qQ mesons should be a more favorable case, since $<q>$ should remain finite in the heavy quark but a dimensional analysis should consider the general case (5.12) rather than the special case of (5.4) for $\eta = 1$.

5.7 Role of heavy quark dressing.

For different constituent-like approximations of the real and imaginary parts of amplitudes $A$, $M$ in the heavy quark propagator, we solve the BSE equation and calculate the meson masses in order to better understand their importance in the studies of mesons. The studies involve the $D(uc)$, $\eta_c(\bar{c}c)$ and $B(ub)$, $\eta_b(\bar{b}b)$ meson systems. The resulting meson masses are presented in tables (5.6, 5.7, 5.8, 5.9) with their binding energies.
For comparison purposes we define a binding energy for the $qQ$ meson systems as

\begin{equation}
BE_{qQ} = M_H - M_q(-(\bar{\eta}M_H)^2) - M_Q(-(\eta M_H)^2)
\end{equation}

(5.13)

and for the QQ systems as

\begin{equation}
BE_{QQ} = M_H - 2M_Q(-\frac{M_H^2}{4}).
\end{equation}

(5.14)

For convenience we denote the $M$ and $A$ amplitudes evaluated at the peak of the mass shell parabolic integration domain by $M_u' = M_u(-(\bar{\eta}M_H)^2)$, $M_Q' = M_Q(-(\eta M_H)^2)$, and $A_u' = A_u(-(\bar{\eta}M_H)^2)$, $A_Q' = A_Q(-(\eta M_H)^2)$ with $Q = c, b$. Notice that for reasons of comparison, in Tables 5.6, 5.7, 5.8, and 5.9 we calculate binding energy this way even when a constituent mass approximation was used for the heavy quark propagators. In Table 5.10, we collect all data for heavy quark meson masses obtained using a constituent approximation for the heavy quark propagator, and here we display also the binding energies calculated by replacing the mass functions by the constituent masses. Any definition of a binding energy for quarks in QCD is ambiguous. The purpose of introducing such a concept is to remove the dominant quark mass scale from consideration so the approximations must be evaluated in terms of a quantity that is more sensitive to dynamics. Only the largest changes in "binding energy", in the Tables, provide any useful information on dynamics. Notice that the BE for $qQ$ mesons appear to be in general smaller than those of QQ mesons and that, as we increase the heavy quark mass, the BE increases. Intuitively one would anticipate a decreasing BE.

For the $D(uc)$ meson we observe that with the approximation $A \sim 1$ we can reach a mass shell that is close to the one from the constituent mass approximation (1rp).
<table>
<thead>
<tr>
<th>c-quark approx.</th>
<th>$M_H$(GeV)</th>
<th>$A_t$</th>
<th>$M_t^c$(GeV)</th>
<th>$M_c^t$(GeV)</th>
<th>BE(GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1rp(M=2.0, A=1.0)</td>
<td>1.852</td>
<td>1.338</td>
<td>1.978</td>
<td>0.63</td>
<td>-0.756</td>
</tr>
<tr>
<td>3ccp</td>
<td>2.236*</td>
<td>1.374</td>
<td>2.054</td>
<td>0.96</td>
<td>-0.778</td>
</tr>
<tr>
<td>kcp(M, A)</td>
<td>2.326*</td>
<td>1.436</td>
<td>2.177</td>
<td>0.66</td>
<td>-0.601</td>
</tr>
<tr>
<td>kcp(M=2.0, A)</td>
<td>2.074</td>
<td>1.395</td>
<td>2.097</td>
<td>0.63</td>
<td>-0.653</td>
</tr>
<tr>
<td>kcp(M=2.0,rA)</td>
<td>2.049</td>
<td>1.389</td>
<td>2.084</td>
<td>0.63</td>
<td>-0.665</td>
</tr>
<tr>
<td>kcp(M, A=1.0)</td>
<td>1.688</td>
<td>1.325</td>
<td>1.934</td>
<td>0.58</td>
<td>-0.826</td>
</tr>
<tr>
<td>kcp(rM,A=1.0)</td>
<td>1.688</td>
<td>1.325</td>
<td>1.934</td>
<td>0.58</td>
<td>-0.826</td>
</tr>
<tr>
<td>kcp(rM, A)</td>
<td>2.049</td>
<td>1.389</td>
<td>2.084</td>
<td>0.63</td>
<td>-0.665</td>
</tr>
<tr>
<td>kcp(M, rA)</td>
<td>2.049</td>
<td>1.389</td>
<td>2.084</td>
<td>0.63</td>
<td>-0.665</td>
</tr>
<tr>
<td>kcp(rM, rA)</td>
<td>2.025</td>
<td>1.384</td>
<td>2.072</td>
<td>0.62</td>
<td>-0.667</td>
</tr>
</tbody>
</table>

Table 5.6: $D$(uc) pseudoscalar meson masses ($M_H$) and binding energies(BE) as defined in Eq. (5.13) for different approximations to the A and M functions. Here kcp means a dynamical propagator and in the parenthesis is the approximations (if any) used for its amplitudes A and/or M. 1rp is for the constituent approximation of the heavy quark propagators. rA or rM means we keep only the real part of the corresponding amplitude. For the 1rp case $\eta = 0.78$, for 3ccp $\eta = 0.71$ and for all others $\eta = 0.80$. The two masses with the asterisk are an estimation based on the eigenvalue plot and they are considerate as approximate numbers.

<table>
<thead>
<tr>
<th>c(c) quark approx.</th>
<th>$M_H$(GeV)</th>
<th>$A_t$</th>
<th>$M_t^c$(GeV)</th>
<th>$M_c^t$(GeV)</th>
<th>BE(GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1rp(M=2.0, A=1.0)</td>
<td>3.025</td>
<td>1.356</td>
<td>2.014</td>
<td>-1.003</td>
<td></td>
</tr>
<tr>
<td>3ccp(M, A)</td>
<td>3.007</td>
<td>1.355</td>
<td>2.009</td>
<td>-1.011</td>
<td></td>
</tr>
<tr>
<td>kcp(M, A)</td>
<td>3.035</td>
<td>1.358</td>
<td>2.015</td>
<td>-0.995</td>
<td></td>
</tr>
<tr>
<td>kcp(M=2.0, A)</td>
<td>3.432</td>
<td>1.410</td>
<td>2.130</td>
<td>-0.828</td>
<td></td>
</tr>
<tr>
<td>kcp(M=2.0,rA)</td>
<td>3.432</td>
<td>1.410</td>
<td>2.130</td>
<td>-0.828</td>
<td></td>
</tr>
<tr>
<td>kcp(M, A=1.0)</td>
<td>2.405</td>
<td>1.302</td>
<td>1.870</td>
<td>-1.335</td>
<td></td>
</tr>
<tr>
<td>kcp(rM,A=1.0)</td>
<td>2.530</td>
<td>1.310</td>
<td>1.900</td>
<td>-1.270</td>
<td></td>
</tr>
<tr>
<td>kcp(rM, A)</td>
<td>3.336</td>
<td>1.397</td>
<td>2.101</td>
<td>-0.866</td>
<td></td>
</tr>
<tr>
<td>kcp(M, rA)</td>
<td>3.035</td>
<td>1.359</td>
<td>2.016</td>
<td>-0.997</td>
<td></td>
</tr>
<tr>
<td>kcp(rM, rA)</td>
<td>3.336</td>
<td>1.397</td>
<td>2.101</td>
<td>-0.866</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.7: $\eta_c$(c$\bar{c}$) pseudoscalar meson masses ($M_H$) and binding energies (BE) as defined in Eq. (5.14) for different approximations to the A and M functions. kcp means a dynamical propagator and in the parenthesis is the approximations (if any) used for its amplitudes A and/or M. 1rp is for the constituent approximation of the heavy quark propagators. rA or rM means we keep only the real part of the corresponding amplitude. A, M signifies full dynamical amplitude calculations. In all cases $\eta = 0.50$. 
### Table 5.8: B(ub) pseudoscalar meson masses ($M_H$) and binding energies (BE) as defined in Eq. (5.13) for different approximations to the A and M functions. Here kcp means a dynamical propagator and in the parenthesis is the approximations (if any) used for its amplitudes A and/or M. 1rp is for the constituent approximation of the heavy quark propagators. rA or rM means we keep only the real part of the corresponding amplitude. In all cases $\eta = 0.90$.

<table>
<thead>
<tr>
<th>b-quark approx.</th>
<th>$M_H$(GeV)</th>
<th>$A_b^t$</th>
<th>$M_b^t$(GeV)</th>
<th>$M_u^t$(GeV)</th>
<th>BE(GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1rp</td>
<td>5.254</td>
<td>1.134</td>
<td>5.376</td>
<td>0.75</td>
<td>-0.872</td>
</tr>
<tr>
<td>3ccp</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>kcp(M, A)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>kcp(M=5.3, A)</td>
<td>5.320</td>
<td>1.139</td>
<td>5.412</td>
<td>0.76</td>
<td>-0.852</td>
</tr>
<tr>
<td>kcp(M=5.3,rA)</td>
<td>5.316</td>
<td>1.138</td>
<td>5.408</td>
<td>0.76</td>
<td>-0.852</td>
</tr>
<tr>
<td>kcp(M ,A=1.0)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>kcp(rM,A=1.0)</td>
<td>5.348</td>
<td>1.139</td>
<td>5.425</td>
<td>0.77</td>
<td>-0.847</td>
</tr>
<tr>
<td>kcp(rM, A)</td>
<td>5.524</td>
<td>1.150</td>
<td>5.525</td>
<td>0.778</td>
<td>-0.779</td>
</tr>
<tr>
<td>kcp(M, rA)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>kcp(rM, rA)</td>
<td>5.511</td>
<td>1.149</td>
<td>5.517</td>
<td>0.778</td>
<td>-0.784</td>
</tr>
</tbody>
</table>

### Table 5.9: $\eta_b(b\bar{b})$ pseudoscalar meson masses ($M_H$) and binding energies (BE) as defined in Eq. (5.14) for different approximations to the A and M functions. kcp means a dynamical propagator and in the parenthesis is the approximations (if any) used for its amplitudes A and/or M. 1rp is for the constituent approximation of the heavy quark propagators. rA or rM means we keep only the real part of the corresponding amplitude. $\eta = 0.90$.

<table>
<thead>
<tr>
<th>b(b) quark approx.</th>
<th>$M_H$(GeV)</th>
<th>$A_b^t$</th>
<th>$M_b^t$(GeV)</th>
<th>BE(GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1rp</td>
<td>9.603</td>
<td>1.139</td>
<td>5.417</td>
<td>-1.231</td>
</tr>
<tr>
<td>3ccp</td>
<td>9.347</td>
<td>1.131</td>
<td>5.345</td>
<td>-1.343</td>
</tr>
<tr>
<td>kcp(M, A)</td>
<td>9.585</td>
<td>1.138</td>
<td>5.413</td>
<td>-1.241</td>
</tr>
<tr>
<td>kcp(M=5.3, A)</td>
<td>9.767</td>
<td>1.144</td>
<td>5.470</td>
<td>-1.173</td>
</tr>
<tr>
<td>kcp(M=5.3,rA)</td>
<td>9.767</td>
<td>1.144</td>
<td>5.470</td>
<td>-1.173</td>
</tr>
<tr>
<td>kcp(M ,A=1.0)</td>
<td>9.204</td>
<td>1.127</td>
<td>5.307</td>
<td>-1.410</td>
</tr>
<tr>
<td>kcp(rM,A=1.0)</td>
<td>9.592</td>
<td>1.139</td>
<td>5.416</td>
<td>-1.240</td>
</tr>
<tr>
<td>kcp(rM, A)</td>
<td>10.000</td>
<td>1.152</td>
<td>5.542</td>
<td>-1.084</td>
</tr>
<tr>
<td>kcp(M, rA)</td>
<td>9.515</td>
<td>1.136</td>
<td>5.393</td>
<td>-1.271</td>
</tr>
<tr>
<td>kcp(rM, rA)</td>
<td>9.515</td>
<td>1.151</td>
<td>5.533</td>
<td>-1.091</td>
</tr>
</tbody>
</table>
Table 5.10: $q\bar{Q}$ and $Q\bar{Q}$ pseudoscalar and vector meson masses using constituent mass approximations for the $c,b$ propagators with their BE. $BE^{kcp}$ is the binding energy calculated using (5.13, 5.14) while for the $BE^{trp}$ we use the $M^{c}_c = 2.0 \text{ GeV}$ and $M^{b}_b = 5.3 \text{ GeV}$ fitted constituent masses. All data are in GeV units.

However for the $B(ub)$ meson that is not possible unless we set $Im(M) \sim 0$, indicating the growing importance of the imaginary component of the heavy quark mass function. Setting $Im(A) \sim 0$ has different effects depending on the approximation for $M$, but in general $Im(A)$ appears to be of decreasing importance for increasing quark mass. The most important outcome of these studies is the role of $Im(M)$. The imaginary part of the mass amplitude appears to have an increasing importance and a different role for $q\bar{Q}$ and $Q\bar{Q}$ mesons.

From the studies so far, it is now very clear that Fig. (5.1) does not display all aspects of the quark propagator that influence the meson calculations. The variations of both real and imaginary parts of $A$ and $M$, in the parabolic mass shell region in the complex plane, especially near the mass shell point, are important. So for a
Figure 5.6: Variation regions of the real part of $c$ and $b$ quark mass functions in the complex plane compared to our fitted constituent masses and with the empirical ones.

first evaluation of whether constituent-like behavior is a good representation requires consultation with (5.6, 5.7, 5.8, 5.9).

For all calculations the heavy quark propagators cover the same parabolic areas in the complex plane determined by the product $(\eta P)^2$ and the plots correspond to the on-shell cases. Due to the Schartz reflection principle the real parts of functions $A$ and $M$ are even functions of $\text{Im}(p^2)$, while their imaginary parts are odd. In the $\text{Re}(M)$ figure we also indicate our fitted constituent $c$, $b$ quark masses (dashed lines) and a band for the empirical ones. We observe the fitted constituent masses to have a value very close to the value of $\text{Re}(M)$ at the mass-sell point and that they are larger than the maximum empirical mass by 0.17 GeV for the $c$ quark and by 0.3
Figure 5.7: Variation regions of the real part of $c$ and $b$ quark $A$ functions in the complex plane.
Figure 5.8: Variation regions of the imaginary part of $c$ and $b$ quark mass functions in the complex plane.
Figure 5.9: Variation regions of the imaginary part of $c$ and $b$ quark A functions in the complex plane.
GeV for the b. Since the empirical masses are determined by perturbative QCD, that difference is also a measure of the nonperturbative effects (as they represented by the phenomenological IR term of the MT model) that have been absorbed in our fitted masses. It appears that the nonperturbative dressing effects have increasing contribution as we go to the much heavier b quark. Thus contradicting our physical intuition that should be less. This is probably another indication that the MT model for heavy quarks is outside its domain of applicability. The variation of Re(M) in the complex domain is more for the b quark and that is because the width of the parabolic region is measured by $(\eta P)^2$ and therefore larger. The same is true for its imaginary part. On the other hand Re(A) for the c quark appears to vary fast along the real axis, especially near the mass shell point region. It has values much larger than one, and a small variation along the imaginary axis. For the b quark Re(A) is closer to one, varies slowly even at the mass shell region, and varies only slightly along the imaginary axis. The b quark Im(A) has a much smaller variation in the complex domain than that of the c quark.

Thus as current quark mass is raised from the c to the b quark, Re(M) does not become more flat (approaching a constant mass function behavior), and Im(M) does not decrease to approach zero. Re(A) does have the expected approach to one, and the Im(A) does approach zero. We can qualitatively understand this by inspecting the DSE Eq. (3.18, 3.19) and the propagator amplitudes (3.20). In the integral for $A'(p^2)$ we see the presence of $\sigma_v(q^2)$ while in the integrand for $B'(p^2)$ we have $\sigma_s(q^2)$. The last one has a mass function in the numerator that will keep providing a strong support for the kernel of the integral in (3.18) even as we go to heavier quarks while for the other equation (3.19) the strength of the kernel will diminish faster ($\sim 1/m_Q^2$).
therefore we will reach the expected limit behavior for $A$ much sooner.

5.8 An effective kernel with quark mass dependence.

Higher order diagrams in the expansion of the BSE kernel contain internal quark propagators which will introduce a quark mass dependence to it. The parameters of an effective kernel that is applicable over a wide range of quark mass can be expected to have an explicit quark mass dependence. The way the parameters should vary with the quark mass is not easy to determine. In ref. [156] it was noted that diagrams higher order than ladder-rainbow provide an attractive effect on mesons which decreases with increasing quark mass. Qualitative estimates of this effect for quarkonia were used to produce an $m_q$-dependent effective kernel (of rainbow-ladder format) whose role was to produce meson bound state quantities that left room for subsequent and explicit corrections from higher order kernel processes. The IR strength and range of such a core kernel was fitted to reproduce the expected behavior of eq.(9) in ref. [156] in the low quark mass region. With the MT-model form used as a template for the core kernel, the IR strength was fitted to light meson properties leaving room for higher order terms effects. On the other hand the IR strength of the MT effective kernel was fitted to light quark meson properties absorbing higher order effects. Since the MT-model parameters are fixed these effects will be still present in the heavy quarks region. The extrapolation of the core kernel to the heavy quark region provides an interesting point of comparison with the MT-model.

So far we had numerous indications that the MT effective kernel provides too much dressing for the heavy quarks. At the same time we believe it gives a BSE kernel that is too attractive and that cancels to some degree the overdressing effects in the quark mass. Therefore it is not so easy to determine the net effect of the MT interaction
kernel in the heavy quark region from the calculated heavy quark meson observables. We decided to investigate whether the core kernel of ref. [156], is beneficial for heavy quark mesons. The heavy quark region was not considered in ref. [156].

The expression for the core effective kernel of ref. [156] is:

\[
\frac{4\pi\alpha(k^2)}{k^2} = C(\omega, \hat{m}) \frac{(2\pi)^2 k^2}{\omega^2} e^{-\frac{\omega^2}{\hat{m}^2}} + \frac{2(2\pi)^2 \gamma_m F(k^2)}{\ln[\tau + (1 + \frac{k^2}{\Lambda_{QCD}^2})^2]}.
\]

The midpoint value of \(\omega\) in the minimal sensitivity region, from studies going as far as about the \(s\) quark mass, was found to have the following dependence on the renormalization-group-invariant quark current mass \(\hat{m}\):

\[
\omega(\hat{x}) = 0.38 + \frac{0.17}{1 + \hat{x}}, \quad \hat{x} = \frac{\hat{m}}{\hat{m}_0}, \quad \hat{m}_0 = 0.12 \text{ GeV},
\]

and in this case the product of the two core-model parameters, \(D\) and \(\omega\), will depend only on \(\hat{x}\):

\[
C(\hat{x}) = \omega D = C_0 + \frac{0.86}{1 + C_2 \hat{x} + C_3 \hat{x}^2}
\]

with \(C_0 = 0.11, C_2 = 0.885, C_3 = 0.474\). If we assume these relations are true for all current masses then we notice that from the chiral limit all the way to infinite quark mass, parameter \(\omega(\hat{x})\) varies from \(\omega(0) = 0.55 \text{ GeV}\) to \(\omega(+\infty) = 0.38 \text{ GeV}\) and \(C(\hat{x})\) from \(C(0) = 0.97 \text{ GeV}\) to \(C(+\infty) = 0.11 \text{ GeV}\) or for \(D(\hat{x})\) from \(D(0) = 1.763\) to \(D(+\infty) = 0.289\). So the infinite range of quark masses is mapped into a very small domain of the parameters \(\omega(\hat{x})\) and \(C(\hat{x})\). From the plot in fig. (5.10) we observe there is a very fast variation of \(C(\omega)\) for quark masses up to \(\hat{m} \sim 2 - 3 \text{ GeV}\) and then above that the parameters rapidly approach their limit values and they don’t vary much. The limiting kernel parameters \(C, \omega\) are a possible definition of a rainbow
Figure 5.10: Variation of parameter $C(\omega) = \omega D(\omega)$ vs. parameter $\omega$ of the core kernel of ref. [156] for $0 \leq \hat{m} < +\infty$. The arrow points in the direction of increasing $\hat{m}$. 
-ladder kernel for very massive quarks. As a qualitative comparison with the MT-model at high quark mass we present some meson results from use of the new effective kernel which we call a core model.

Since the $c$ quark mass of 0.88 GeV at scale $\mu = 19$ GeV corresponds to a $\hat{m}$ of about 1 GeV, we obtain $D = 0.308$ and $\hat{\omega} = 0.39$ GeV\(^1\) for this core kernel.

We get a $m_{\eta_c} = 2.93$ GeV from this core model compared to $m_{\eta_c} = 3.03$ GeV from the MT-model parameters $D = 0.93$ and $\omega = 0.4$ GeV. The change in the parameters ($\Delta D = -0.622$ and $\Delta \omega = -0.01$ GeV) brought a mere 100 MeV decrease in the $\eta_c$ mass or a 3.3% decrease. For the corresponding vector meson $J/\psi(c\bar{c})$ we go from 3.235 GeV to 3.05 GeV, a decrease by 185 MeV or a relative decrease by 5.71%.

For the $b$ meson studies we use the core model values ($D=0.289$, $\omega = 0.38$ GeV). The change in the parameters now is ($\Delta D = -0.641$ and $\Delta \omega = -0.2$ GeV) and they reduce the pseudoscalar $\eta_b$ meson mass from 9.585 GeV to 9.537 GeV a decrease by 48 MeV, i.e. 0.50%. Similarly the $b$ vector meson $\Upsilon(b\bar{b})$ mass in the core model is 9.612 GeV compared to 9.685 GeV in the MT model, a decrease by 73 MeV or 0.75%.

The results for the $c$- and $b$-quarkonia masses and their decay constants from the core model for comparison with the MT model results are collected in table (5.11). Summing up, the relative percentage changes in the parameters of the model and the resulting relative percentage changes in the quarkonia masses and decay constants, for the $c$ quarkonia are:

$$(\Delta D / D, \Delta \omega / \omega) = (-66.88, -2.5)\% \rightarrow (\Delta M / M, \Delta f / f)_{\eta_c} = (-3.3, -13.7)\%$$

$$(\Delta M / M, \Delta f / f)_{J/\psi} = (-5.7, -23.1)\%$$

\(^1\)Although we use an approximate relation to calculate $\hat{m}$ and there is some uncertainty, the parameters vary slowly in that mass area and we don’t have to worry much about the exact value of the quark mass corresponding to these parameter values.
<table>
<thead>
<tr>
<th>Quarkonia</th>
<th>((M_{\eta_c}^{\text{exp.}}, f_{\eta_c}^{\text{exp.}}) = (2.980, 0.340))</th>
<th>((M_{J/\psi}^{\text{exp.}}, f_{J/\psi}^{\text{exp.}}) = (3.097, 0.416))</th>
</tr>
</thead>
<tbody>
<tr>
<td>c</td>
<td>((D, \omega(\text{GeV})))</td>
<td>((M_{\eta_c}(\text{GeV}), f_{\eta_c}(\text{GeV})))</td>
</tr>
<tr>
<td></td>
<td>((0.93, 0.40))S</td>
<td>3.035</td>
</tr>
<tr>
<td></td>
<td>((0.308, 0.39))</td>
<td>2.934</td>
</tr>
<tr>
<td>b</td>
<td>((M_{\eta_b}^{\text{exp.}}, f_{\eta_b}^{\text{exp.}}) = (9.30, -))</td>
<td>((M_{\Upsilon}^{\text{exp.}}, f_{\Upsilon}^{\text{exp.}}) = (9.46, 0.700))</td>
</tr>
<tr>
<td></td>
<td>((D, \omega(\text{GeV})))</td>
<td>((M_{\eta_b}(\text{GeV}), f_{\eta_b}(\text{GeV})))</td>
</tr>
<tr>
<td></td>
<td>((0.93, 0.40))S</td>
<td>9.585</td>
</tr>
<tr>
<td></td>
<td>((0.289, 0.38))</td>
<td>9.537</td>
</tr>
</tbody>
</table>

Table 5.11: c and b Quarkonia masses and decay constants using the values \((D=0.308, \omega = 0.39 \text{ GeV})\) and \((D=0.289, \omega = 0.38 \text{ GeV})\) from the core model [156]. The letter S signifies the MT-kernel values of parameters D, \(\omega\). Experimental data are also in GeV.

while for the b systems:

\[
(\Delta D/D, \Delta \omega/\omega) = (-68.92, -5.0)\% \rightarrow (\Delta M/M, \Delta f/f)_{\eta_b} = (-0.50, -8.5)\%
\]

\[
\rightarrow (\Delta M/M, \Delta f/f)_{\Upsilon} = (-0.75, -13.6)\%.
\]

It is obvious that the decay constants, especially those of the vector mesons, are much more sensitive to the changes of the parameters. The vector meson masses also appear to be more sensitive than the pseudoscalar ones, but for the b quarkonia the mass difference indicate insensitivity to the dynamics of the model. On the other hand the decay constants show greater sensitivity. Since the vector mesons are more extended particles than the pseudoscalars they can be more sensitive to variations of the IR part of the interaction and that is confirmed by the last results. The values of the parameters we used for our MT-model calculations \((\omega = 0.4 \text{ GeV}, D = 0.93)\) were fitted to the quark condensate in the chiral limit and they should not agree with the \(\hat{m} = 0\) values of the core model. From the last results one may also consider the vector meson decay constants as a more appropriate candidate to probe a quark mass
dependence of the effective kernel.

The $c$ quarkonia hyperfine splitting in the core model is 0.116 GeV (with the MT model it is 0.2 GeV), and this compares well with the experimental value of 0.117 GeV. No such improvement is observed for the $b$ system. The core model produces a $b$ quarkonia hyperfine splitting of about 0.075 GeV. We obtained 0.073 GeV in the MT-model and both are almost half the experimental value.

5.9 Recovery of a $qQ$ meson mass shell.

Given that the core model of an effective BSE kernel is $m_q$-dependent, and that its use for $qQ$ mesons containing quarks of different masses is not defined, we decided to explore the point of view that the DSE kernel for the dressing of the heavy quark propagator could be too strong in the infrared. Hence we performed $qQ$ meson calculations in which the IR term of the MT-model kernel is removed. The BSE kernel for the interaction of the light and heavy quark though should be approximately unchanged since there is no significant change in the size of the $qQ$ meson compared to the size of the light quark mesons. Solving the DSE for the $b$ quark propagator with only the UV tail term of the MT kernel, where the full MT kernel was used for the calculation of the light quark propagator and the solution of the BSE, we were able to reach a mass shell for light-heavy mesons having a $b$ quark. The masses and decay constants appear in Table (5.12) These calculations represent a suppression of IR dressing of heavy quarks in mesons.

With this modification of the dressing of the $b$-quark we can find meson masses that are only about 7-12 % smaller than the experimental ones. For the only decay constant experimentally known there is a -24.4 % difference. A more realistic approach might be an IR momentum cut-off. This decreases the effective IR strength and agrees
with the indications from the core model. The present results and those in chapter 3 for the equal quark systems appear to provide a partial confirmation of the recent suggestion by Brodsky and Shrock of a universal maximum wavelength for quarks and gluons in mesons [157], [159], [158]. The existence of a maximum wavelength for quark and gluons would be due to confinement in mesons with size of the order of 1 fm. That in turn, through the Compton relation, will introduce a minimum quark and gluon momentum inside hadrons. Considering the analysis in chapter 3 it is possible that implementation of a universal minimum momentum for quarks and gluons in all hadrons could lead to consistent results. It may also indicate a possible scenario for the natural realization of this scale through the combined IR dynamics of the quark mass dependent kernel and the quark propagators in full QCD calculations. More investigation is required in that direction.

Table 5.12: qb q=u/d,s,c pseudoscalar meson masses and decay constants with their differences from experiment after eliminating the IR term of the MT kernel in the dressing of the b-quark. The full MT kernel was used for the light quark propagator and the solution of the BSE. All data are in GeV units.

<table>
<thead>
<tr>
<th>qb</th>
<th>η</th>
<th>$M_{H}^{exp.}$</th>
<th>$M_{H}^{UV}$</th>
<th>$\Delta M/M%$</th>
<th>$f_{H}^{exp.}$</th>
<th>$f_{H}^{UV}$</th>
<th>$\Delta f/f%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ub</td>
<td>0.90</td>
<td>5.279</td>
<td>4.658</td>
<td>-12.16</td>
<td>0.176</td>
<td>0.133</td>
<td>-24.4</td>
</tr>
<tr>
<td>sb</td>
<td>0.90</td>
<td>5.370</td>
<td>4.748</td>
<td>-11.59</td>
<td>-</td>
<td>0.164</td>
<td>-</td>
</tr>
<tr>
<td>cb</td>
<td>0.86</td>
<td>6.286</td>
<td>5.831</td>
<td>-7.238</td>
<td>-</td>
<td>0.453</td>
<td>-</td>
</tr>
</tbody>
</table>
Chapter 6

Remarks and Conclusions

The Maris-Tandy (MT) effective interaction provides light quark mass dressing consistent with dynamical chiral symmetry breaking and quark confinement. It provides a successful description of light quark meson physics and appears to capture some, if not most, of the important qualities of light quark QCD. It is therefore a very useful tool for a qualitative, and to a good degree quantitative analysis, of certain QCD aspects. The infrared component of this effective rainbow-ladder kernel is phenomenological because QCD is unsolved in such a non-perturbative domain. To help replace such phenomenology by specific mechanisms, it is necessary to first characterize its performance in new domains.

We extended the use of the MT interaction kernel to explore its effectiveness in the study of heavy quark meson properties. This is the first systematic study of the performance of the rainbow-ladder truncation of the DSEs for quarkonia and light-heavy quark mesons into the b-quark region involving comparison with the constituent mass concept.

It was found that the calculated observables for quarkonia over a large range of quark mass are in surprisingly good agreement with the experimental data. However the model failed to lead to a physical solution of the Bethe-Salpeter equation (BSE) for light-heavy $qQ$ systems, thus revealing its limits. The dressed heavy quark mass function and its imaginary part appear to be part of the problem. One can conclude that the DSE kernel that dresses light quarks is inappropriate for dressing
heavy quarks and that deficiency is revealed only in light-heavy ($qQ$) mesons but not in quarkonia studies. The deeper mathematical reasons for these two different outcomes lie in the combined dynamics of the two quark propagators. A quark mass dependence in an effective kernel will make it difficult to satisfy the Axial Vector Ward-Takahashi Identity (AV-WTI) that is off-diagonal in flavor. Without that constraint upon modeling the way forward is unclear.

Overall we find the meson masses to be less sensitive, and the meson decay constants to be more sensitive, to variations of the parameters of the interaction model. As the quark mass increases that sensitivity decreases very fast for the masses and somewhat slower for the decay constants. There is a possibility, that a more general, mass-dependent, kernel might produce solutions for all pseudoscalar and vector mesons. These studies, especially the weak agreement of the calculated decay constants with experimental values, inform future developments to improve the description of the physical properties of hadrons.

We extended the above analysis to distinguish the role of the infrared term and the ultraviolet term of the effective MT-interaction for equal quark mesons and we found that the infrared term becomes less important as we raise the quark mass and the UV term becomes more relevant. With only the UV term, a mass shell couldn’t be reached for the $s$ and $c$ quarkonia, but for the $b$ quarkonia we obtained physical observables very close to the ones from the full MT model calculations. The UV term appears to be mostly responsible for the physical properties of $b\bar{b}$ mesons. That makes physical sense since the smaller Compton size of heavy quarks suppresses infrared content such as long distance (low momentum) gluons within such quasi-particles. The small ultraviolet attraction is adequate to produce binding
for the heavy quarkonia. We then examined the light-heavy mesons in this light by suppressing the IR dressing for the b quark propagator, but not for the light quark or for the binding interaction. The corresponding bound state solution of the Bethe-Salpeter equation provided masses and decay constants for qb pseudoscalar mesons in reasonable agreement with experimental data. The full effective interaction was used for the Bethe-Salpeter bound state equation since the size of the meson was expected to be similar to that of the light quark systems. A more realistic future approach could be to separate out the infrared physics by a lower momentum cut-off or smaller IR suppression. Aspects of this study may be a partial confirmation of Brodsky and Shrock’s suggestion of a universal maximum wavelength of quarks and gluons in hadrons [157]. These results may also suggest a possible scenario for the natural realization for such a scale in QCD.

The constituent mass concept for heavy quarks was also explored within our work. Our findings on the extent to which a constituent mass approximation can adequately represent heavy quarks within mesons can be summarized as follows. For equal quark mesons (quarkonia), the self-energy dressing dynamics provided a substantial improvement in the decay constants. For $c\bar{c}$ and $b\bar{b}$ mesons, a constituent mass approximation yields very good mass results but leads to calculated electroweak decay constants that are too low by some 30-70 %. Use of dynamically dressed propagators removes almost all of this deficiency and the decay constants are within 20 % of experiment. This improvement provided by dynamical dressing of c- and b-quarks is persistent and systematic in the following sense. When the dressing is progressively introduced into all three stages (bound state solution, normalization loop integral,
and then the loop integral for evaluation of the decay constant), the final value always increases towards the experimental value. This indicates that a constituent mass approximation, even for b-quarks is inadequate. Small departures from a strictly constant mass function and renormalization function Z for quarks in the relevant region of the complex plane are magnified due to the very weak binding of the mesons in question. The results of the MT-model for heavy quark propagator dressing are not well summarized by a constituent mass approximation for the propagator.

However, our findings in the case of heavy-light mesons indicates that the role of self-energy dressing is a much more complicated topic. With fully dressed quark propagators, our model does not provide a physical bound state solution for $q_c$, $q_b$, $q = u/d, s$ or even $cb$ mesons. Such physical states are easily obtained with a constituent mass approximation, but again the indication is that the decay constants are too low.

We are thus confronted by evidence that the constituent mass approximation is not reliable even for b-quarks, and that our ladder-rainbow model kernel has deficiencies that are masked in QQ mesons but plainly evident in qQ mesons. Our investigation further suggest that the infrared sector of the model kernel should be suppressed in a way that is limited by the space-time size of the object under consideration. We find, through preliminary investigations that such a modification of the model kernel can explain why the QQ and qQ mesons partly respond differently to self energy dressing of the heavy quarks.

The size of the QQ mesons becomes smaller as quark mass is raised and the dressed quark quasi-particles themselves become smaller in size. The infrared component of the kernel relates to large distance gluons; such components should be less physically relevant for internal dynamics of heavy, small Compton size, particles. This is
supported by our finding that the heavier QQ quarkonia receive diminishing contributions from the infrared component of our model kernel; the UV component alone provides a very good description of the $b\bar{b}$ states. We note that this refers not only to the binding interaction but also to the quark self-energy dressing.

In contrast, the qQ mesons have a size that does not diminish significantly with increasing heavy quark mass. So the infrared sector of the binding interaction should remain relevant. However the self-energy dressing of the heavy quark should not receive strong contributions from large distance gluons. This is supported by our finding that a suppression of the infrared component of the b-quark dressing kernel allows a physical $q\bar{b}$, $q = u/d, s, c$ meson state even though the b quark is dressed. This suggest that one should, in the future, explore a systematic lower momentum cut-off, dictated by hadrons of quasi-particle size, for all quarks and gluons.

In all these studies the three complex conjugate pole (3ccp) representation for the dressed quark propagators proved to be a convenient devise to simplify the calculations for mesons, excluding the pion. Use of a larger domain of the time-like $p^2$ real axis, to fit the parameters for the $c$ and $b$ quarks, we expect to bring significant improvement in the quantitative results.
References


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