ABSOLUTE CONTINUITY AND ON THE RANGE OF A VECTOR MEASURE

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In this dissertation, we will study the following two problems:

1. Let $\Omega$ be a compact Hausdorff space with Borel $\sigma$-field $\Sigma$ and let $\mu$ and $\nu$ be regular Borel probabilities on $\Omega$. Then the following are equivalent:

   (a) $[C(\Omega) \hookrightarrow L^1(\mu)] \ll [C(\Omega) \hookrightarrow L^1(\nu)]$

   (b) $\mu \ll \nu$

   (c) $[B(\Sigma) \hookrightarrow L^1(\mu)] \ll [B(\Sigma) \hookrightarrow L^1(\nu)]$

where $B(\Sigma)$ is the Banach space of all bounded Borel measurable functions equipped with the supremum norm. We extend this result to vector-valued cases.

2. To which Banach spaces $X$ is it so that if $C$ is a countable subset of $X$ that lies in the range of a countably additive $X^{**}$-valued measure with the same $\sigma$-field domain, then there is an $X$-valued countably additive measure with a $\sigma$-field domain, whose range also contains $C$?

We show that if $X$ is $c_0$ or $X$ is a $C(K)$-space for a compact Hausdorff space $K$, then any countable subset $C$ of $X$ that lies in the range of an $X^{**}$-valued countably additive measure on a $\sigma$-field lies in the range of an $X$-valued countably additive measure on the same $\sigma$-field.
INTRODUCTION

The notion of vector measures is central to a study of Banach space-valued functions, to the representation and classification of linear operators, and the classification of Banach spaces. Much of the early interest in weak and weak* compactness was motivated by such considerations.

The notion of (absolute) continuity of one measure with respect to another has its roots in classical real variables. The absolute continuity of point functions defined on intervals goes back at least as far as Harnack in his study of integration. However, it was G. Vitali who established absolutely continuous functions as objects of great importance in a sequence of fundamental papers in the first decade of the twentieth century. A decisive step in the evolution of absolute continuity was taken with the famous theorem of S. Banach and M. Zarecki to the effect that a continuous function \( f : [0, 1] \to \mathbb{R} \) of bounded variation is absolutely continuous if and only if \( f(E) \) has Lebesgue measure zero whenever \( E \) does. The emergence of general measure theory was accompanied by the important role to be played in that theory by absolute continuity. Already by 1930, Nikodým had established the so-called Radon-Nikodým theorem, extending earlier work of J. Radon regarding Borel measures in Euclidean spaces. The establishment by A.N. Kolmogorov of the foundation of probability theory on a measure theoretic basis and the critical role played by the Radon-Nikodým theorem in understanding conditioning ensured absolute continuity of a permanent and central place in mathematical analysis.

The first and third chapters consist of notation, definitions and theorems that are fundamental for the main results in the second and last chapter, respectively. In the first chapter
we define the concept of absolute continuity of scalar-valued and vector-valued measures, and of operators. That leads us to the classical Radon-Nikodým Theorem, and the Radon-Nikodým Property. We also state some topological theorems that are at the heart of our proofs for the main results. We complete Chapter 1 by introducing the Bochner integral.

The second chapter is on absolute continuity of regular Borel measures on compact Hausdorff spaces. Our main results build on earlier work of C.P. Niculescu; the ideas of Niculescu (see [14]) were used, for example, to broach the subject of weakly compact operators on $C(K)$-spaces and their relationship to absolutely summing operators in ([6], Chapter 15). We prove that for two regular Borel probabilities $\mu$ and $\nu$ on a compact Hausdorff space $\Omega$, absolute continuity of measures $\mu \ll \nu$ is equivalent to absolute continuity of inclusion operators

$$[C(\Omega) \hookrightarrow L^1(\mu)] \ll [C(\Omega) \hookrightarrow L^1(\nu)].$$

We first generalize this theorem to vector-valued functions, i.e. we show that absolute continuity of measures $\mu \ll \nu$ is equivalent to absolute continuity of inclusion operators

$$[C(\Omega, X) \hookrightarrow L^1(\mu, X)] \ll [C(\Omega, X) \hookrightarrow L^1(\nu, X)].$$

Then we generalize the result to vector measures, i.e absolute continuity of vector measures $F \ll G$ is equivalent to absolute continuity of inclusion operators

$$[C(\Omega) \hookrightarrow L^1(F)] \ll [C(\Omega) \hookrightarrow L^1(G)].$$

The above results are also valid if we replace $C(\Omega)$ with $B(\Sigma)$, where $B(\Sigma)$ denotes the Banach space of all bounded Borel measurable functions equipped with the supremum norm.

In the third chapter we first discuss projections in Banach spaces and then state a sequence of theorems on complemented subspaces that will be useful in the proofs of the
results in Chapter 4.

The fourth chapter consists of a study of the range of a vector measure. The study of the range of a vector measure is important because of the vivid geometric properties of the range that have applications to optimal control theory. For example Liapounoff’s theorem (see [11]) in the finite dimensional case is intimately related to the 'bang-bang' principle (see [9] and [10]). Properties of the range of a vector measure also provide information about the vector measure itself, for instance the range of a vector measure determines its total variation (see [18]). In [21], Professor A. Sofi discusses the problem of deciding whether a sequence which lies inside the range of a vector measure actually lies inside the range of a vector measure with better properties. In particular we address the following question he proposes (Problem 6):

*To which Banach spaces* $X$ *is it so that if* $C$ *is a countable subset of* $X$ *that lies in the range of a countably additive* $X^{**}$-*valued measure with a* $\sigma$-*field domain, then there is a countably additive* $X$-*valued measure with the same* $\sigma$-*field domain, *whose range also contains* $C$ ?

Of course, if $X$ is complemented in $X^{**}$, then the answer is plain and easy. A natural first approach (with an eye to a counterexample), would be to consider $X$ to be the classical Banach space $c_0$. In fact, we show that if $X$ is $c_0$ or if $X$ is a $C(K)$-space for a compact Hausdorff space $K$, then any countable subset $C$ of $X$ that lies in the range of an $X^{**}$-valued countably additive measure on a $\sigma$-field lies in the range of an $X$-valued countably additive measure on the same $\sigma$-field. The first case can be extended to obtain the following: If $C$ is a countable subset of $c_0$ and $C$ lies in the range of a vector measure $F : \Sigma \to X$, where $X$ is a Banach space containing $c_0$, then $C$ lies in the range of a $c_0$-valued countably additive vector measure.

The "techniques" are all Banach space techniques. We never called on a change of domain.
We believe that this partial positive solution to the problem will provide the groundwork for further extensions.

Mienie de Kock

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CHAPTER 1

VECTOR MEASURES

1.1 Variation and semivariation

Let \((\Omega, \Sigma, \mu)\) be a measure space, i.e. \(\mu\) is an extended real-valued nonnegative countably additive measure defined on a \(\sigma\)-field \(\Sigma\) of subsets of a point set \(\Omega\). The triple \((\Omega, \Sigma, \mu)\) is called a finite measure space if it is a measure space and \(\mu(\Omega)\) is finite. A subset of a Banach space is called relatively norm (weakly) compact if its norm (weak) closure is norm (weakly) compact. We deal with finite measure spaces only. Let \(X\) be a Banach space with closed unit ball \(B_X\) and dual space \(X^*\).

Definition 1.1.1 A function \(F\) from a field \(\mathcal{F}\) of subsets of a set \(\Omega\) to a Banach space \(X\) is called a finitely additive vector measure if whenever \(E_1\) and \(E_2\) are disjoint members of \(\mathcal{F}\) then \(F(E_1 \cup E_2) = F(E_1) + F(E_2)\).

If in addition, \(F(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} F(E_n)\) in the norm topology of \(X\) for all sequences \((E_n)\) of pairwise disjoint members of \(\mathcal{F}\) such that \(\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}\), then \(F\) is termed a countably additive vector measure or simply, \(F\) is countably additive.

Definition 1.1.2 Let \(F : \mathcal{F} \to X\) be a vector measure. The variation of \(F\) is the extended nonnegative function \(|F|\) whose value on a set \(E \in \mathcal{F}\) is given by

\[
|F|(E) = \sup_{\pi} \sum_{A \in \pi} \|F(A)\|
\]
where the supremum is taken over all partitions $\pi$ of $E$ into a finite number of disjoint members of $\mathcal{F}$. If $|F|(\Omega) < \infty$, then $F$ will be called a measure of bounded variation. We prove the following useful property of the variation of a vector measure.

**Lemma 1.1.3** The variation is a monotone finitely additive function on $\mathcal{F}$.

**Proof** Let $F : \Sigma \to X$ be a vector measure. The monotonicity of $|F|$ is clear. We prove that $|F|$ is finitely additive. Indeed, let $E_1$ and $E_2$ be two disjoint sets in $\mathcal{F}$ and let $\pi$ be a partition of $E_1 \cup E_2$. We can write $\pi = \pi_1 \cup \pi_2$, where $\pi_1$ is a partition of $E_1$ and $\pi_2$ is a partition of $E_2$. Then we have that

$$\sum_{A \in \pi} \|F(A)\| = \sum_{A \in \pi_1 \cup \pi_2} \|F(A)\| = \sum_{A \in \pi_1} \|F(A)\| + \sum_{A \in \pi_2} \|F(A)\| \leq |F|(E_1) + |F|(E_2).$$

Since the partition $\pi$ was arbitrary, we conclude that

$$|F|(E_1 \cup E_2) \leq |F|(E_1) + |F|(E_2).$$

To prove that

$$(1.1) \quad |F|(E_1 \cup E_2) \geq |F|(E_1) + |F|(E_2),$$

we proceed by contradiction. Indeed, suppose that (1.1) does not hold and let

$$\epsilon = |F|(E_1) + |F|(E_2) - |F|(E_1 \cup E_2).$$

There exist $\pi_1$, a partition of $E_1$, such that

$$\sum_{A \in \pi_1} \|F(A)\| > |F|(E_1) - \frac{\epsilon}{2}.$$
and $\pi_2$, a partition of $E_2$, such that

$$\sum_{A \in \pi_2} \|F(A)\| > |F|(E_2) - \frac{\epsilon}{2}.$$ 

Take $\pi = \pi_1 \cup \pi_2$, a partition of $E_1 \cup E_2$, and we have that

$$\sum_{A \in \pi} \|F(A)\| = \sum_{A \in \pi_1} \|F(A)\| + \sum_{A \in \pi_2} \|F(A)\| > |F|(E_1) - \frac{\epsilon}{2} + |F|(E_2) - \frac{\epsilon}{2} = |F|(E_1 \cup E_2),$$

which leads us to our contradiction.

\[\square\]

**Definition 1.1.4** The semivariation of $F$ is the set function

$$\|F\|(A) = \sup \{|x^*F|(A) : x^* \in B_{X^*}\},$$

where $|x^*F|$ is the variation of the scalar measure $x^*F$.

If $\|F\|(E) < \infty$, then $F$ will be called a measure of bounded semivariation or a bounded vector measure (see [7] I.1.11).

We will need the following property of the semivariation of a vector measure.

**Lemma 1.1.5** 1. The semivariation is a monotone subadditive function on $\mathcal{F}$.

2. For each $E \in \mathcal{F}$ one has $\|F\|(E) \leq |F|(E)$.

**Proof** 1. The monotonicity is clear. The proof of the subadditivity of the semivariation of a vector measure, relies on the additivity of the variation of a vector measure. Let $E_1$
and $E_2$ be two disjoint sets in $\mathcal{F}$, and let $F : \Sigma \to X$ be a vector measure. Then by Lemma 1.1.3 we have that

$$\|F\|(E_1 \cup E_2) = \sup\{|x^*F|(E_1 \cup E_2) : x^* \in B_{X^*}\}$$

$$= \sup\{|x^*F|(E_1) + |x^*F|(E_2) : x^* \in B_{X^*}\}$$

$$\leq \sup\{|x^*F|(E_1) : x^* \in B_{X^*}\} + \sup\{|x^*F|(E_2) : x^* \in B_{X^*}\}$$

$$= \|F\|(E_1) + \|F\|(E_2).$$

2. It suffices to realize that

$$\{ |x^*F|(E) : x^* \in B_{X^*}\} = \sup_{\pi} \sum_{A \in \pi} |x^*F(A)|$$

$$\leq \sup_{\pi} \sum_{A \in \pi} \|x^*\| \|F(A)\|$$

$$\leq \sup_{\pi} \sum_{A \in \pi} \|F(A)\|$$

$$= |F|(E)$$

for $E \in \Sigma$.

The regularity of a vector measure is defined in terms of the semivariation as follows.

**Definition 1.1.6** Let $\Omega$ be a topological space with Borel $\sigma$-field $\Sigma$. Then we say that a vector measure $F : \Sigma \to X$ is regular if for any $E \in \Sigma$ and $\epsilon > 0$, there exist a compact set $K$ and an open set $O$ such that $\|F\|(O \setminus K) < \epsilon$, where $K \subset E \subset O$.

Notice that if $F$ is scalar-valued, then the above definition coincides with the classical scalar-valued definition of regularity.

**Example 1.1.7** Let $\mu$ be the Lebesgue measure.
1. Then $\mu$ is an example of a regular, scalar-valued measure.

2. Let $F : \Sigma \rightarrow L_1(\mu)$ be defined as $F(E) = \chi_E$ for $E \in \Sigma$. Then we have that

$$\|F\|(E) = \sup_{x^* \in B_{X^*}} |x^*F|(E) = \mu(E).$$

And so $F$ is an example of a regular, vector-valued measure.

1.2 Absolute continuity and the Radon-Nikodým Theorem

In this section we define absolute continuity for scalar-valued measures and for vector-measures. We also define absolute continuity of operators. Then we state the Radon-Nikodým Theorem for scalar-valued measures and vector-valued measure. We let $(\Omega, \Sigma)$ be a measurable space, so $\Sigma$ is a $\sigma$-field of subsets of the set $\Omega$.

**Definition 1.2.1** Let $\mu$ and $\nu$ be two scalar-valued countably additive measures on $(\Omega, \Sigma)$. We say that $\mu$ is absolutely continuous with respect to $\nu$, and we write $\mu \ll \nu$, if and only if, for all $E \in \Sigma$ we have that $\mu(E) = 0$ implies $\nu(E) = 0$.

For a proof of the following classical theorem see [19], p 276-278.

**Theorem 1.2.2** (Radon-Nikodým) Let $\nu$ be a $\sigma$-finite measure on $(\Omega, \Sigma)$. Let $\mu$ be a scalar-valued measure on $(\Omega, \Sigma)$ such that $\mu \ll \nu$. Then, there exists some $h \in L_1(\Omega, \Sigma, \nu)$ such that for every $E \in \Sigma$

$$\mu(E) = \int_E h d\nu.$$

The function $h$ obtained in the previous theorem is called the Radon-Nikodým derivative of $\mu$ with respect to $\nu$. It is sometimes denoted by $\frac{d\mu}{d\nu}$.
Definition 1.2.3 Let $\mathcal{F}$ be a field of subsets of $\Omega$, $F : \mathcal{F} \to X$ be a vector measure and $\mu$ be a finite nonnegative real-valued measure on $\mathcal{F}$. If $\lim_{\mu(E) \to 0} F(E) = 0$, then $F$ is called $\mu$-continuous and this is signified by $F \ll \mu$.

The following theorem by B.J. Pettis (see [7], p10) shows that countably additive measures defined on a $\sigma$-field share a common property with their scalar counterparts.

Theorem 1.2.4 (Pettis) Let $\Sigma$ be a $\sigma$-field, $F : \Sigma \to X$ be a countably additive vector measure and $\mu$ be a finite nonnegative real-valued measure on $\Sigma$. Then $F$ is $\mu$-continuous if and only if $F$ vanishes on sets of $\mu$-measure zero.

The notion of absolute continuity extends to vector measures in the following way:

Definition 1.2.5 We say that a vector measure $F : \Sigma \to X$ is absolutely continuous with respect to another vector measure $G : \Sigma \to Y$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \Sigma$, $\|G\|(E) < \delta$ implies $\|F\|(E) < \epsilon$. We denote it by $F \ll G$.

Definition 1.2.6 A Banach space $X$ has the Radon-Nikodým property with respect to $(\Omega, \Sigma, \mu)$ if for each $\mu$-continuous vector measure $G : \Sigma \to X$ of bounded variation, there exists $g \in L_1(\mu, X)$ such that

$$G(E) = \int_E gd\mu$$

for all $E \in \Sigma$. A Banach space $X$ has the Radon-Nikodým property if $X$ has the Radon-Nikodým property with respect to every finite measure space.

We can translate the concept of absolute continuity to operators. The following two theorems (involving absolute continuity of scalar-valued measures) motivate the definition
for absolute continuity of operators (see [6], page 309-310). We recall that if \( K \) is a compact Hausdorff space, then \( C(K)^* \) is the space of all regular Borel measures on \( K \).

**Theorem 1.2.7** The following two statements are equivalent:

1. A subset \( W \) of \( C(K)^* \) is (relatively) weakly compact.

2. There exists a positive measure \( \mu \in C(K)^* \) such that each \( \nu \in W \) is \( \mu \)-absolutely continuous and the corresponding densities form a (relatively) weakly compact subset of \( L_1(\mu) \).

**Theorem 1.2.8** Let \( T : C(K) \to Y \) be a bounded linear operator. Then the following are equivalent:

1. \( T \) is weakly compact.

2. For some (and then every) \( 1 \leq p \leq \infty \) there exists a probability measure \( \mu \in C(K)^* \) with the property that for each \( \epsilon > 0 \) we can find an \( N(\epsilon) > 0 \) such that, for all \( f \in C(K) \),

\[
\|Tf\| \leq N(\epsilon) \cdot \left( \int_K |f|^p \, d\mu \right)^{\frac{1}{p}} + \epsilon \cdot \|f\|.
\]

The above condition that characterizes weak compactness of operators, serves as motivation for the following definition.

**Definition 1.2.9** Let \( X, Y \) and \( Z \) be Banach spaces. Suppose \( T : X \to Y \) and \( S : X \to Z \) are bounded linear operators. We say that \( T \) is absolutely continuous with respect to \( S \), written \( T \ll S \), if given \( \epsilon > 0 \), there is a \( k > 0 \) so that for any \( x \in X \)

\[
\|Tx\| \leq k\|Sx\| + \epsilon\|x\|.
\]
The next two theorems are central to the theory of vector measures (see [7], page 14).

**Theorem 1.2.10** (Bartle-Dunford-Schwartz) Let $F$ be a countably additive vector measure defined on a $\sigma$-field $\Sigma$. Then there exists a nonnegative real-valued countably additive measure $\mu$ on $\Sigma$ such that $\mu(E) \to 0$ if and only if $\|F\|(E) \to 0$; in fact $\mu$ can be chosen so that $0 \leq \mu(E) \leq \|F\|(E)$ for all $E \in \Sigma$.

**Theorem 1.2.11** (Bartle-Dunford-Schwartz) Let $F : \Sigma \to X$ be a countably additive vector measure on a $\sigma$-field $\Sigma$. Then the range of $F$ is relatively weakly compact.

We have the following corollary from the above theorem:

**Corollary 1.2.12** Let $(\Omega, \Sigma)$ be a measurable space. Suppose $F : \Sigma \to X$ is a countably additive vector measure on the $\sigma$-field $\Sigma$. Let

$$\int dF : B(\Sigma) \to X$$

be the integration operator on the Banach space $B(\Sigma)$ of all bounded $\Sigma$-measurable scalar-valued functions on $\Omega$ equipped with the sup-norm

$$\|f\|_\infty = \sup\{|f(\omega)| : \omega \in \Omega\}.$$

Then $\int dF$ is a weakly compact linear operator.

**Proof** The following proof is a variation on the theme of integration-by-parts. Let $f \in B(\Sigma)$ with $\|f\| = 1$ and $f \geq 0$ and such that

$$f = \sum_{i=0}^{n} a_i \chi_{E_i},$$
for \(a_0, a_1, \ldots, a_n \geq 0\) and \(E_0, E_1, \ldots, E_n\) pairwise disjoint members of \(\Sigma\). Now

\[
a_0 \chi_\Omega + (a_0 - a_1) \chi_{\Omega \setminus E_0} = a_0 \chi_\Omega + a_1 \chi_{\Omega \setminus E_0} - a_0 \chi_{\Omega \setminus E_0} = a_0 \chi_{E_0} + a_1 \chi_{\Omega \setminus E_0}
\]

The Bartle-Dunford-Schwartz Theorem produces a finite non-negative real-valued measure \(\mu\) on \(\Sigma\) such that \(F \ll \mu\). We can actually take \(\mu\) to be of the form \(|x^*F|\) for certain \(x^*\) in \(X^*\). This is Rybakov’s theorem (see [7], page 267).

**Definition 1.2.13** A Rybakov control measure for \(F: \Sigma \to X\) is a measure \(\mu = |x^*F|\) for some \(x^* \in X^*\), such that \(\mu(E) = 0\) if and only if \(\|F\|(E) = 0\).

**Theorem 1.2.14** (Rybakov) Let \(F: \Sigma \to X\) be a countably additive vector measure. Then there is \(x^*\) in \(X^*\) such that \(F \ll |x^*F|\).

1.3 Topological tools

We recall the following well-known lemma from topology. It states that if every pair of disjoint closed sets in a topological space \(X\) can be separated by disjoint open sets, then each such pair can be separated by a continuous function.

**Lemma 1.3.1** (Urysohn’s Lemma) Let \(X\) be a normal space; let \(A\) and \(B\) be disjoint closed subsets of \(X\). Let \([a, b]\) be a closed interval in the real line. Then there exists a continuous map

\[
f: X \to [a, b]
\]
such that

\[ f(x) = \begin{cases} 
  a & \text{for all } x \in A \\
  b & \text{for all } x \in B. 
\end{cases} \]

One immediate consequence of the Urysohn lemma is the useful theorem called Tietze Extension Theorem. It deals with the problem of extending a continuous real-valued function that is defined on a subset of a space \( X \) to a continuous function defined on all of \( X \).

**Theorem 1.3.2** (Tietze Extension Theorem) Let \( X \) be a normal space; let \( A \) be a closed subset of \( X \).

1. Any continuous map from \( A \) into the closed interval \([a, b]\) of \( \mathbb{R} \) may be extended to a continuous map of all from \( X \) into \([a, b]\).
2. Any continuous map from \( A \) into \( \mathbb{R} \) may be extended to a continuous map of all from \( X \) into \( \mathbb{R} \).

The following theorem is due to N. Lusin and states that every measurable function is a continuous function on nearly all its domain:

**Theorem 1.3.3** (Lusin) For an interval \([a, b]\) and scalar-valued measure \( \mu \), let

\[ f : [a, b] \to \mathbb{C} \]

be a measurable function. Then given \( \epsilon > 0 \), there exists a compact subset \( E \subset [a, b] \), such that \( f|_E \) (\( f \) restricted to \( E \)) is continuous and

\[ \mu([a, b] \setminus E) < \epsilon. \]

**Remark 1.3.4** We can replace the interval \([0, 1]\) by any normal topological space.
1.4 The Bochner integral

This section is devoted to an examination of the Bochner integral. For a more complete discussion, see [7], II.2. The Bochner integral is a straightforward abstraction of the Lebesgue integral. Similar to the development of the Lebesgue integral, we will start with the definition of a simple function.

Definition 1.4.1 A function \( f : \Omega \to X \) is called simple if there exist \( x_1, x_2, \ldots, x_n \in X \) and \( E_1, E_2, \ldots, E_n \in \Sigma \) such that

\[
 f = \sum_{i=1}^{n} x_i \chi_{E_i},
\]

where

\[
 \chi_{E_i}(\omega) = \begin{cases} 
 1 & \text{if } \omega \in E_i \\
 0 & \text{if } \omega \notin E_i 
\end{cases}
\]

A function \( f : \Omega \to X \) is called \( \mu \)-measurable if there exists a sequence of simple functions \((f_n)\) with \( \lim_n \| f_n - f \| = 0 \) \( \mu \)-a.e.

Definition 1.4.2 A measurable function \( f : \Omega \to \mathbb{R} \) is integrable with respect to a vector measure \( F : \Sigma \to X \) if

1. \( f \) is \( x^*F \) integrable for every \( x^* \in X^* \) and

2. for every \( A \in \Sigma \) there exists an element of \( X \), denoted by \( \int_A f \, dF \), such that

\[
 x^* \int_A f \, dF = \int_A f \, dx^*F
\]

for every \( x^* \in X^* \).
Identifying two functions if the set where they differ has null $\|F\|$ semivariation, we obtain a linear space of classes of functions which, when endowed with the norm

$$\|f\|_{L^1(F)} = \sup \left\{ \int_{\Omega} |f| |d| x^* F| : x^* \in B_{X^*} \right\},$$

becomes a Banach space. We denote it by $L^1(F)$ (see [4]).

For a discussion on integration of a vector valued function with respect to a scalar valued measure (the Bochner integral), see [7], II.2.

**Definition 1.4.3** A $\mu$-measurable function $f : \Omega \to X$ is called Bochner integrable if there exists a sequence of simple functions $(f_n)$ such that $f = \lim_n f_n \mu$-a.e. and

$$\lim_n \int_{\Omega} \|f_n - f\| d\mu = 0.$$

In this case, $\int_E f d\mu$ is defined for each $E \in \Sigma$ by

$$\int_E f d\mu = \lim_n \int_E f_n d\mu,$$

where $\int_E f_n d\mu$ is defined in the obvious way. The following characterization holds for the Bochner integral.

**Theorem 1.4.4** A $\mu$-measurable function $f : \Omega \to X$ is Bochner integrable if and only if

$$\int_{\Omega} \|f\| d\mu < \infty.$$

The following property of the Bochner integral will be very useful.

**Theorem 1.4.5** (Dominated Convergence Theorem) Let $(\Omega, \Sigma, \mu)$ be a finite measurable space and $(f_n)$ be a sequence of Bochner integrable $X$-valued functions on $\Omega$. If $\lim_n f_n = f$ in $\mu$-measure, i.e.

$$\lim_n \mu \{ \omega \in \Omega : \|f_n - f\| \geq \epsilon \} = 0$$
for every $\epsilon > 0$ and if there exists a real-valued Lebesgue integrable function $g$ on $\Omega$ with

$$\|f_n\| \leq g \mu - a.e.,$$

then $f$ is Bochner-integrable and

$$\lim_n \int_E f_n d\mu = \int_E f d\mu$$

for each $E \in \Sigma$. In fact,

$$\lim_n \int_\Omega \|f - f_n\|d\mu = 0.$$

A proof of the following elementary facts about the Bochner integral can be found in [7], page 46.

**Theorem 1.4.6**  
1. $\lim_{\mu(E)\to 0} \int_E f d\mu = 0$;

2. $\|\int_E f d\mu\| \leq \int_E \|f\|d\mu$, for all $E \in \Sigma$;

3. if $(E_n)$ is a sequence of pairwise disjoint members of $\Sigma$ and $E = \bigcup_{n=1}^\infty E_n$, then

$$\int_E f d\mu = \sum_{n=1}^\infty \int_{E_n} f d\mu,$$

where the sum on the right is absolutely convergent;

4. if $F(E) = \int_E f d\mu$, then $F$ is of bounded variation and

$$|F|(E) = \int_E \|f\|d\mu$$

for all $E \in \Sigma$.

1.5 Bartle integral

There is a substantial theory of integration of vector-valued functions with respect to vector-valued measures. This type of integral is called the Bartle integral. Let $\lambda$ be a control
measure for $F : \Sigma \to X$. The Barle integral can be viewed as an operator

$$I_F : L_\infty(\lambda) \to X,$$

defined on a simple function

$$f = \sum_{i=1}^{N} x_i \chi_{E_i},$$
as

$$I_F(f) = \int f dF = \sum_{n=1}^{N} x_n F(E_i).$$

$I_F$ is a weak*-weak continuous operator when $L_\infty(\lambda)$ is considered as the dual of $L_1(\lambda)$. Given $f \in L_\infty(\lambda)$ the vector measure $fF$ with density $f$ with respect to $F$ is defined by

$$fF(E) = \int_{E} f dF = \int f \chi_{E} dF$$
for every $E \in \Sigma$.

For the variation of $fF$, we have

$$|fF|(E) = \int_{E} |f| d|F|.$$
CHAPTER 2

ABSOLUTE CONTINUITY OF MEASURES AND OPERATORS.

2.1 Introduction

In this chapter we characterize absolute continuity of measures and absolute continuity of operators. We start with two scalar-valued absolutely continuous Borel measures $\mu$ and $\nu$ in the first section and prove that it is equivalent to absolute continuity of the inclusion operators

$$[C(\Omega) \hookrightarrow L^1(\mu)] \ll [C(\Omega) \hookrightarrow L^1(\nu)]$$

and

$$[B(\Sigma) \hookrightarrow L^1(\mu)] \ll [B(\Sigma) \hookrightarrow L^1(\nu)].$$

The proof relies on the classical, scalar-valued version of the Radon-Nikodým theorem (Theorem 1.2.2) and Urysohn’s Lemma (Theorem 1.3.1).

We extend the result to the case where we have a vector-valued function $f : \Omega \to X$ in section 2.

In section 3 we extend the result to two absolutely continuous vector measures $F : \Sigma \to X$ and $G : \Sigma \to Y$. The function is scalar-valued. Since not all Banach spaces have the Radon-Nikodým property, we turn to Rybakov (see Theorem 3.2.3) in order to obtain a Radon-Nikodým derivative.

In section 4 we obtain the most general form where we have a vector valued function and vector-valued measures. We prove that absolute continuity of the vector measures $F \ll G$
is equivalent to

$$[C(\Omega, X) \hookrightarrow L^1(\mu, X)] \ll [C(\Omega, X) \hookrightarrow L^1(\nu, X)]$$

and

$$[B(\Sigma, X) \hookrightarrow L^1(\mu, X)] \ll [B(\Sigma, X) \hookrightarrow L^1(\nu, X)].$$

All the results in this chapter appear in [5].

2.2 Absolute continuity of scalar-valued functions and scalar-valued measures

**Theorem 2.2.1** Let $\Omega$ be a compact Hausdorff space with Borel $\sigma$-field $\Sigma$ and let $\mu$ and $\nu$ be regular Borel probabilities on $\Omega$. Then the following are equivalent:

(a) $[C(\Omega) \hookrightarrow L^1(\mu)] \ll [C(\Omega) \hookrightarrow L^1(\nu)]$

(b) $\mu \ll \nu$

(c) $[B(\Sigma) \hookrightarrow L^1(\mu)] \ll [B(\Sigma) \hookrightarrow L^1(\nu)],$

where $B(\Sigma)$ is the Banach space of all bounded Borel measurable functions equipped with the supremum norm.

**Proof**

(a) $\Rightarrow$ (b). Suppose $\mu$ is not absolutely continuous with respect to $\nu$. Then there exists $E \in \Sigma$ and $\epsilon > 0$ such that $\mu(E) = 3\epsilon > 0$ and $\nu(E) = 0$. Using the regularity of $\mu$ and $\nu$, choose a compact set $K \subset E$ so that $\mu(K) > 2\epsilon$ and a sequence of open sets $(O_n)$ such that $O_n \supseteq E$ and

$$\nu(O_n \setminus E) = \nu(O_n) \rightarrow 0$$
as $n \to \infty$. For the above $\epsilon > 0$ our assumption $(a)$ gives a $k > 0$ such that

$$\int_{\Omega} |f|d\mu \leq k \int_{\Omega} |f|d\nu + \epsilon \|f\|_{C(\Omega)}$$

for all $f \in C(\Omega)$. For each $n$, Urysohn’s Lemma (see Lemma 1.3.1) gives $f_n \in C(\Omega)$ such that

$$\chi K \leq f_n \leq \chi O_n.$$

Then

$$2\epsilon < \mu(K) \leq \int_{\Omega} f_n d\mu$$

$$\leq k \int_{\Omega} |f_n|d\nu + \epsilon \|f_n\|_{C(\Omega)}$$

$$\leq k \nu(O_n) + \epsilon$$

for all $n$; a contradiction.

$(b) \Rightarrow (c)$. Assume $\mu \ll \nu$. Then by the Radon-Nikodým theorem (see Theorem 1.2.2) there exists $h \in L^1(\nu)$ such that $0 \leq h < \infty$ (see Remark ??) and

$$\mu(E) = \int_E h d\nu$$

for all $E \in \Sigma$. Given $\epsilon > 0$ choose $k > 0$ such that

$$\int_{\{h > k\}} h d\nu < \epsilon.$$

Then, given $f \in B(\Sigma)$,

$$\|f\|_{L^1(\mu)} = \int_{\Omega} |f|d\mu = \int_{\Omega} |f|h d\nu$$

$$= \int_{\{h \leq k\}} |f|h d\nu + \int_{\{h > k\}} |f|h d\nu$$

$$\leq k \|f\|_{L^1(\nu)} + \epsilon \|f\|_{B(\Sigma)}.$$

$(c) \Rightarrow (a)$. This is clear.
2.3 Absolute continuity of vector-valued functions and scalar-valued measures.

We now generalize the above result to vector-valued functions.

**Theorem 2.3.1** Let $X \neq \{0\}$ be a Banach space, $\Omega$ a compact Hausdorff space with Borel $\sigma$-field $\Sigma$, and let $\mu$ and $\nu$ be regular Borel probabilities on $\Omega$. Then the following are equivalent:

(a) $[C(\Omega, X) \hookrightarrow L^1(\mu, X)] \preccurlyeq [C(\Omega, X) \hookrightarrow L^1(\nu, X)]$

(b) $\mu \preccurlyeq \nu$

(c) $[B(\Sigma, X) \hookrightarrow L^1(\mu, X)] \preccurlyeq [B(\Sigma, X) \hookrightarrow L^1(\nu, X)]$,

where $f \in B(\Sigma, X)$ if and only if there exists a sequence $(f_n)$ of $\Sigma$-simple $X$-valued functions so that $f$ is the uniform limit of $(f_n)$.

**Proof** (a) $\Rightarrow$ (b). In view of Theorem 2.2.1, it suffices to show that (a) implies

(a*) $[C(\Omega) \hookrightarrow L^1(\mu)] \preccurlyeq [C(\Omega) \hookrightarrow L^1(\nu)]$.

Pick $x_0 \in X$ with $\|x_0\| = 1$. Given $\epsilon > 0$, by (a) there is a $k > 0$ such that

(2.1) $\int_{\Omega} \|f\|d\mu \leq k \int_{\Omega} \|f\|d\nu + \epsilon \|f\|_{C(\Omega, X)}$

for all $f \in C(\Omega, X)$. For each $f \in C(\Omega)$, consider

$f x_0 \in C(\Omega, X)$

in (2.1), in order to get

$\int_{\Omega} |f|d\mu \leq k \int_{\Omega} |f|d\nu + \epsilon \|f\|_{C(\Omega)}$, 
which proves \((a^*)\).

\((b) \Rightarrow (c)\). The proof is similar to the proof of the previous theorem. In fact, assume 
\(\mu \ll \nu\), then there is an \(0 \leq h \in L^1(\nu)\) such that \(\mu(E) = \int_E h d\nu\) for all \(E \in \Sigma\). Given \(\epsilon > 0\) choose \(k > 0\) such that 
\[
\int_{\{h > k\}} h d\nu < \epsilon.
\]
Then given \(f \in B(\Sigma, X)\),
\[
\|f\|_{L^1(\mu, X)} = \int_{\Omega} \|f\| d\mu = \int_{\{h \leq k\}} \|f\| h d\nu + \int_{\{h > k\}} \|f\| h d\nu \leq k \|f\|_{L^1(\nu)} + \epsilon \|f\|_{B(\Sigma)}.
\]

\((c) \Rightarrow (a)\). This is clear.

\[\blacksquare\]

2.4 Absolute continuity of scalar-valued functions and vector-valued measures.

We can extend the result in the previous section to absolutely continuous vector measures.

\textbf{Theorem 2.4.1} Let \(\Omega\) be a compact Hausdorff space with Borel \(\sigma\)-field \(\Sigma\). Let \(X\) and \(Y\) be Banach spaces and let \(F : \Sigma \to X\) and \(G : \Sigma \to Y\) be countably additive regular vector measures. Then the following are equivalent:

\((a)\) \([C(\Omega) \hookrightarrow L^1(F)] \ll [C(\Omega) \hookrightarrow L^1(G)]\)

\((b)\) \(F \ll G\)

\((c)\) \([B(\Sigma) \hookrightarrow L^1(F)] \ll [B(\Sigma) \hookrightarrow L^1(G)]\),
where $B(\Sigma)$ is the Banach space of all bounded Borel measurable scalar-valued functions on $\Omega$.

**Proof** (a)$\Rightarrow$(b). Suppose $F \ll G$ is false. Then there is a sequence $(E_n)$ in $\Sigma$ and $\epsilon > 0$ such that

$$\|F\|(E_n) > 3\epsilon \quad \text{and} \quad \|G\|(E_n) < \frac{1}{2^n}$$

for all $n$. By the regularity of $F$ and $G$ there exist compact sets $(K_n)$ and open sets $(O_n)$ such that $K_n \subset E_n \subset O_n$ and

$$\|F\|(E_n \setminus K_n) < \epsilon \quad \text{and} \quad \|G\|(O_n \setminus E_n) < \frac{1}{2^n}$$

for all $n$. By the Urysohn lemma, we can choose $f_n \in C(\Omega)$ such that

$$\chi_{K_n} \leq f_n \leq \chi_{O_n}.$$ 

We claim that $\|f_n\|_{L^1(F)} > 2\epsilon$ and $\|f_n\|_{L^1(G)} < \frac{1}{n}$ for all $n$. Indeed, notice that

$$\|f_n\|_{L^1(F)} = \sup \left\{ \int_{\Omega} |f_n| d|x^*F| : x^* \in B_{X^*} \right\}$$

$$\geq \sup \{|x^*F|(K_n) : x^* \in B_{X^*}\}$$

$$= \|F\|(K_n)$$

$$\geq \|F\|(E_n) - \|F\|(E_n \setminus K_n)$$

$$> 3\epsilon - \epsilon$$

follows from the subadditivity of the semivariation. That $\|f_n\|_{L^1(G)} < \frac{1}{n}$ follows in a similar fashion. By (a) we have for the above $\epsilon > 0$ a constant $k > 0$ such that

$$\|f\|_{L^1(F)} \leq k\|f\|_{L^1(G)} + \epsilon\|f\|_{C(\Omega)}$$
for all $f \in C(\Omega)$. Then it follows from our claim that
\[
2\epsilon < k \cdot \frac{1}{n} + \epsilon
\]
for all $n$; a contradiction.

(b)$\Rightarrow$(c). Assume $F \ll G$. By the Rybakov theorem there is a $y_0^* \in Y^*$ such that
\[
\|y_0^*\| \leq 1 \text{ and } G \ll |y_0^* G|.
\]
Put
\[
\mu = |y_0^* G|.
\]
Since we assume $F \ll G$ we have $F \ll \mu$. Hence, given $\epsilon > 0$ there is $\delta > 0$ such that
\[
\|F\|(E) \leq \epsilon \text{ whenever } E \in \Sigma \text{ and } \mu(E) \leq \delta.
\]
Put
\[
k = \frac{\|F\|((\Omega))}{\delta}.
\]
Then, for each $x^* \in B_{X^*}$, since $|x^* F| \ll \mu$, writing $g_{x^*}$ for the Radon-Nikodým derivative of $|x^* F|$ with respect to $\mu$, we have
\[
\int_{\{g_{x^*} > k\}} g_{x^*} d\mu = |x^* F|(\{g_{x^*} > k\}) \leq \|F\|(\{g_{x^*} > k\}) \leq \epsilon
\]
because
\[
\begin{align*}
\mu(\{g_{x^*} > k\}) & \leq \frac{1}{k} \int_{\Omega} g_{x^*} d\mu \\
& = \frac{1}{k} |x^* F|((\Omega)) \\
& \leq \frac{\|F\|((\Omega))}{k} = \delta.
\end{align*}
\]
Hence, for any \( f \in B(\Sigma) \) and \( x^* \in B_{X^*} \), recalling that \( \mu = |y_0^*G| \) with \( \|y_0^*\| \leq 1 \), we have

\[
\int_{\Omega} |f| d|x^* F| = \int_{\Omega} |f| g_{x^*} d\mu \\
= \int_{\{g_{x^*} \leq k\}} |f| g_{x^*} d\mu + \int_{\{g_{x^*} > k\}} |f| g_{x^*} d\mu \\
\leq k \int_{\Omega} |f| d\mu + \|f\|_{B(\Sigma)} \int_{\{g_{x^*} > k\}} g_{x^*} d\mu \\
\leq k \|f\|_{L^1(G)} + \epsilon \|f\|_{B(\Sigma)}.
\]

Now, taking the supremum over \( x^* \in B_{X^*} \), we obtain

\[
\|f\|_{L^1(F)} \leq k \|f\|_{L^1(G)} + \epsilon \|f\|_{B(\Sigma)}
\]

for \( f \in B(\Sigma) \). This proves the implication.

\((c) \Rightarrow (a)\). This is obvious.

\[\blacksquare\]

**Remark.** It is interesting to observe that in Theorem 2.4.1 the implication \((a) \Rightarrow (c)\) can be proved directly if one applies Lusin’s theorem and Tietze’s extension theorem. Indeed, given \( f \in B(\Sigma) \), Lusin’s theorem gives a sequence \((K_n)\) of compact subsets of \( \Omega \) such that \( f|_{K_n} \) is continuous and

\[\eta(K_n^c) < \frac{1}{n},\]

\[\mu(K_n^c) < \frac{1}{n}\]

for all \( n \in \mathbb{N} \). Here

\[\eta = |x_0^* F|\]

and

\[\mu = |y_0^* G|\]
are Rybakov measures for $F$ and $G$, respectively. Then using Tietze’s extension theorem one obtains a sequence $(f_n)$ in $C(\Omega)$ such that

$$\|f_n\|_{C(\Omega)} \leq \|f\|_{B(\Sigma)}$$

and

$$\int_{\Omega} |f_n - f| d\eta \leq \frac{2}{n} \|f\|_{B(\Sigma)},$$

$$\int_{\Omega} |f_n - f| d\mu \leq \frac{2}{n} \|f\|_{B(\Sigma)}$$

hold for all $n$. Hence, in order to prove $(a) \Rightarrow (c)$, it suffices to check that

$$\|f_n\|_{L^1(F)} \to \|f\|_{L^1(F)}$$

and

$$\|f_n\|_{L^1(G)} \to \|f\|_{L^1(G)}$$

as $n \to \infty$. Given $\epsilon > 0$ let $\delta, k > 0$ be as in the proof of $(b) \Rightarrow (c)$ in Theorem 4.3.2. Also let $g_{x^*}$ be the Radon-Nikodym derivative of $|x^*F|$ with respect to $\eta$. Then

$$\int_{\Omega} |f_n - f| d|x^*F| = \int_{\{g_{x^*} \leq k\}} |f_n - f| g_{x^*} d\eta + \int_{\{g_{x^*} > k\}} |f_n - f| g_{x^*} d\eta$$

$$\leq k \int_{\Omega} |f_n - f| d\eta + 2 \|f\|_{B(\Sigma)} \int_{\{g_{x^*} > k\}} g_{x^*} d\eta$$

$$\leq 2 \left( \frac{k}{n} + \epsilon \right) \|f\|_{B(\Sigma)}$$

for all $x^* \in B_{X^*}$.

Now, taking the supremum over $x^*$, then taking the limit as $n \to \infty$, one gets

$$\limsup_{n \to \infty} \|f_n - f\|_{L^1(F)} \leq 2\epsilon \|f\|_{B(\Sigma)}.$$

Since $\epsilon > 0$ was arbitrary one has

$$\|f_n\|_{L^1(F)} - \|f\|_{L^1(F)} \leq \|f_n - f\|_{L^1(F)} \to 0$$
as $n \to \infty$.

If $F, G : \Sigma \to X$ are countably additive measures on the $\sigma$-field $\Sigma$, with $F \ll G$, then the range of $F$ is relatively compact if the range of $G$ satisfies this condition.
CHAPTER 3

PROJECTIONS

3.1 Projections on Banach spaces

Let $\mathcal{K}$ denote a scalar field. Then we let $c_0(\Gamma)$ consist of all the functions $f : \Gamma \to \mathcal{K}$, such that for every $\epsilon > 0$,

$$\{ \gamma \in \Gamma : |f(\gamma)| \geq \epsilon \}$$

is finite. Note that each $f \in c_0(\Gamma)$ has countable support and is bounded and that

$$\|f\|_{c_0(\Gamma)} = \|f\|_{\infty}.$$ 

If $\Gamma = \mathbb{N}$, we use the usual notation and write $c_0$, which consists of all sequences converging to zero. Likewise, $l_\infty$ consists of all bounded sequences of scalars.

We say that a set $A$ lies in the range of a $Y$-valued measure if there is a $\sigma$-field $\Sigma$ and a countably additive measure $F : \Sigma \to Y$ so that $A \subseteq F(\Sigma)$. We will denote the support of a function by $\text{supp}(f)$ where

$$\text{supp}(f) = \{ \gamma \in \Gamma : f(\gamma) \neq 0 \},$$

where $\Gamma$ is any index set.

**Definition 3.1.1** Let $X$ and $Y$ be Banach spaces. Then a map $f : X \to Y$ is

1. an embedding if $f$ is linear, injective and continuous.
2. an isomorphism if $f$ is linear, bijective, continuous and open (i.e. the inverse is continuous).

3. an isometric isomorphism if $f$ is an isomorphism that preserves distance.

**Definition 3.1.2** Let $X$ be a Banach space. A bounded linear operator $P$ on $X$ is a projection if $P^2 = P$.

**Definition 3.1.3** A Banach space $X$ is injective if every isomorphic embedding of it in an arbitrary Banach space $Y$ is the range of a bounded linear projection defined on $Y$.

**Definition 3.1.4** A topological space is zero-dimensional, if its topological dimension is zero, or equivalently, if it has a base consisting of clopen sets.

Note that every discrete space and the space of rational numbers are examples of zero-dimensional sets (see [1]).

**Definition 3.1.5** A Banach space is weakly compactly generated whenever it is the closed linear span of one of its weakly compact subsets.

The study of weakly compactly generated spaces was initiated by D. Amir and J. Lindenstrauss (see [2]), building on previous work by Lindenstrauss on reflexive spaces. All separable and reflexive spaces are weakly compactly generated.

Throughout the dissertation we use the same notation as in [7].

3.2 Complemented subspaces

**Definition 3.2.1** Let $X$ be a Banach space.
1. A closed subspace $M$ of a Banach space $X$ is complemented if there exists a closed subspace $N$ such that $X = M \oplus N$.

**Theorem 3.2.2** Let $X$ be a Banach space and $M$ a closed subspace of $X$. Then $M$ is complemented in $X$ if and only if $M$ is the range of a bounded linear projection $P$ on $X$.

**Theorem 3.2.3** (Rosenthal) Any weakly compact subset of $l_\infty$ is norm separable.

Sobczyk (see [20]) proved the following theorem that states that $c_0$ is complemented in every separable space in which it resides.

**Theorem 3.2.4** (Sobczyk) If $X$ is a separable Banach space and $Y \subseteq X$ is a closed subspace isometric to $c_0$, then there is a continuous linear projection $P$ from $X$ onto $Y$ with $\|P\| \leq 2$.

The proof of the following theorem can be found in [2].

**Theorem 3.2.5** (Amir and Lindenstrauss) Let $X$ be a weakly compactly generated Banach space. If $X_0$ is a separable subspace of $X$ and $Y_0$ is a separable subspace of $X^*$, then there is a projection $P : X \to X$ whose range is separable such that $X_0 \subseteq P(X)$ and $Y_0 \subseteq P^*(X^*)$.

A proof of the following two theorems can be found in [15] and [13].

**Theorem 3.2.6** (Miljutin) If $K$ is an uncountable compact metric space, then the space $C(K)$ is isomorphic to $C(\Delta)$, where $\Delta$ denotes the Cantor discontinuum.

**Theorem 3.2.7** (Pełczyński)
1. Let $K$ be a zero-dimensional compact metric space. If a separable Banach space $X$ contains a subspace $Y$ that is isometrically isomorphic to $C(K)$, then there are a subspace $Z$ of $Y$ and a projection $P : C(K) \to Z$ (onto $Z$) such that $Z$ is isometrically isomorphic to $C(K)$ and $\|P\| = 1$.

2. Let $K$ be a compact metric space. If a separable Banach space $X$ contains a subspace $Y$ that is isomorphic to $C(K)$, then there is a subspace $Z$ of $Y$ such that $Z$ is isomorphic to $C(K)$ and $Z$ is complemented in $X$. 
CHAPTER 4

THE RANGE OF A VECTOR MEASURE

4.1 Introduction

In this chapter we consider the question of when does a countable subset of a Banach space $X$ that lies in the range of an $X^{**}$-valued measure, lie in the range of an $X$-valued measure. Let $F : \Sigma \to X^{**}$ be a countably additive vector measure. When $X$ is complemented in $X^{**}$, we consider the projection $P : X^{**} \to X$ and compose it with the vector measure $F$ to obtain the countably additive $X$-valued vector measure

$$P \circ F : \Sigma \to X.$$

Therefore a natural approach will be to consider a space that is not complemented in its second dual. In the first section we consider the case where $X = c_0$. We prove that a countable subset of $c_0$ that lies in the range of an $l_\infty$-valued measure, lies in the range of a $c_0$-valued measure. We also extend the result to the case where $X = c_0(\Gamma)$.

In the third section we consider the case where $X = C(K)$. We start by letting $K$ be a zero-dimensional compact metric space. The proof of the theorem relies on a result by Pełczynski (see Theorem 3.2.7). A very useful result by Miljutin (see Theorems 3.2.6) and Pełczynski (see 3.2.7) allows us to also prove the result for a countable subset of $C(K)$, where $K$ is a compact metric space. A construction by Eilenberg allows us to generalize the result to the case where $X = C(K)$, with $K$ a compact Hausdorff space.
4.2 A countable subset of $c_0$ that lies in the range of an $l_\infty$-valued measure, lies in the range of a $c_0$-valued measure.

**Theorem 4.2.1** If $C$ is a countable subset of $c_0$ that lies in the range of an $l_\infty$-valued measure, then $C$ lies in the range of a $c_0$-valued measure.

**Proof** Let $F : \Sigma \to l_\infty$ be a countably additive measure defined on the $\sigma$-field $\Sigma$ such that $C \subseteq F(\Sigma)$. By Theorem 1.2.11 we have that $F(\Sigma)$ is a relatively weakly compact subset of $l_\infty$. By Theorem 3.2.3 we have that $F$ has a norm separable range. So $F(\Sigma)$ generates a separable closed linear subspace of $l_\infty$; we can enlarge our set by letting $X$ be the closed linear span (in $l_\infty$) of $F(\Sigma) \cup c_0$. In so doing, we obtain a separable Banach space $X$ that contains $C$, itself a subset of $c_0$. Now we are in a position to call on Theorem 3.2.4: The result is a bounded linear projection $P$ from $X$ onto $c_0$. We also have that

$$C \subseteq P \circ F(\Sigma)$$

and

$$P \circ F : \Sigma \to c_0$$

is a countably additive $c_0$-valued measure whose range contains $C$.

We can extend the result to $c_0(\Gamma)$.

**Theorem 4.2.2** If $C$ is a countable subset of $c_0(\Gamma)$ that lies in the range of an $l_\infty(\Gamma)$-valued measure, then $C$ lies in the range of a $c_0(\Gamma)$-valued measure.

**Proof** Let $F : \Sigma \to l_\infty(\Gamma)$ be a countably additive measure defined on a $\sigma$-field $\Sigma$ such
that $C \subseteq F(\Sigma)$ and let

$$\Gamma_0 = \bigcup_{n \in \mathbb{N}} \{supp(f_n)\}.$$ 

Then $\Gamma_0 \in \Gamma$ and the support of $C$ lies in $\Gamma_0$. Let $P : l_\infty(\Gamma) \to l_\infty(\Gamma_0)$ be defined by

$$P(f) = \begin{cases} f(\gamma) & \gamma \in \Gamma_0 \\ 0 & \gamma \notin \Gamma_0 \end{cases}$$

So

$$C \subseteq P \circ F(\Sigma) \subseteq l_\infty(\Gamma_0).$$

If we argue as in Theorem 4.2.1, we obtain a countably additive $c_0(\Gamma)$-valued measure whose range contains $C$.

\[\blacksquare\]

**Theorem 4.2.3** If $C$ is a countable subset of $c_0$ and $C \subseteq F(\Sigma)$, where $F : \Sigma \to X$, for $F$ a countably additive vector measure mapping from a $\sigma$-field $\Sigma$ to a Banach space $X$ containing $c_0$. Then there exists a countably additive vector measure $G : \Sigma \to c_0$ such that $C \subseteq G(\Sigma)$.

**Proof** As before let $C$ be a subset of the closed linear span of $W = F(\Sigma) \cup c_0$. Then $W$ is a weakly compactly generated set contained in $X$. There exists a separable set $S$ containing $c_0$, and we have that $C \subseteq S$ and a linear projection $P : X \to X$, with $\|P\| = 1$ and the range of $P$ is separable. By Theorem 3.2.4 there exists a linear projection

$$Q : S \to S$$

with $\|Q\| \leq 2$ so that $QX = c_0$. So we have the countably additive vector measure

$$Q \circ P \circ F(\Sigma)$$
that contains $C$.

The proof used for the above result hints strongly at possible variations. Here is one.

4.3 A countable subset of $C(K)$ that lies in the range of a $C(K)^{**}$-valued measure, lies in the range of a $C(K)$-valued measure

**Theorem 4.3.1** Let $C$ be a countable subset of $C(K)$, where $K$ is a zero-dimensional compact metric space. Suppose $C$ lies in the range of a $C(K)^{**}$-valued measure. Then $C$ lies in the range of a $C(K)$-valued measure.

**Proof** To get things set up properly, let

$$F : \Sigma \to C(K)^{**}$$

be a countably additive $C(K)^{**}$-valued measure with $C \subseteq F(\Sigma)$. Let $W$ be the closed linear span of $F(\Sigma) \cup C(K)$ in $C(K)^{**}$, as before, $F : \Sigma \to W$ is still countably additive and $F(\Sigma)$ contains $C$. But $W$ is also weakly compactly generated and contains $C(K)$. Hence, thanks to Theorem 3.2.5, we are led to a separable closed linear subspace $X$ of $W$ that contains $C(K)$ and is the range of a bounded linear projection $P$ acting on $W$.

$$P \circ F : \Sigma \to X$$

is countably additive and $C$ is contained in $P \circ F(\Sigma)$. We turn now to Theorem 3.2.7(1) to obtain a subspace $Y$ of $X$ that is isometrically isomorphic to $C(K)$, is contained in $C(K)$ and is complemented by a norm-one projection $Q$ acting on $X$, naturally, $Q|_Y = id_Y$ and $Q \circ P \circ F$ takes $\Sigma$ to $Y$ in a countably additive fashion. What is more,

$$Q(C) \subseteq Q \circ P \circ F(\Sigma).$$
Call

$$E : C(K) \rightarrow Z$$

the isometric isomorphism given as by Pełczynski and realize that

$$C = E^{-1} \circ Q(C) \subseteq E^{-1} \circ Q \circ P \circ F(C),$$

where

$$G = E^{-1} \circ Q \circ P \circ F$$

is a countably additive $C(K)$-valued measure.

We can extend the previous result to obtain the following:

**Theorem 4.3.2** Let $C$ be a countable subset of $C(K)$, where $K$ is a compact metric space, that lies in the range of a $C(K)^{**}$-valued measure, then $C$ lies in the range of a $C(K)$-valued measure.

**Proof** The issue is how to deal with uncountable, compact metric spaces $K$ that are not zero-dimensional. By Theorem 3.2.7(2) we have the following: if $K$ is such a compact metric space and $X$ is a separable Banach space containing $C(K)$, then there is a subspace $Y$ of $X$ that is isomorphic to $C(K)$, is contained in $C(K)$ and a bounded linear projection $Q : X \rightarrow X$ with $Q(X) = Y$. Now a close look at the proof of Theorem 4.3.1 will lead us to the conclusion.

Note: Of course the apt reader will realize that Miljutin’s theorem (see [13]) is involved in Pełczynski’s isomorphic variation of his zero-dimensional result.
In general we have the following

**Theorem 4.3.3** Let $C$ be a countable subset of $C(K)$, where $K$ is a compact Hausdorff space. If $C$ lies in the range of a $C(K)^{**}$-valued measure, then $C$ lies in the range of a $C(K)$-valued measure.

**Proof** First, we recall a construction due to Eilenberg (see []). If $k_1, k_2 \in K$, then we say $k_1 \sim k_2$ if $f(k_1) = f(k_2)$ for each $f \in C$; of course $\sim$ is an equivalence relation on $K$ and between equivalence classes $[k_1]$ and $[k_2]$, we can define a metric $d([k_1], [k_2])$ by

$$d([k_1], [k_2]) = \sum_{f_n \in C} \frac{|f_n(k_1) - f_n(k_2)|}{(\|f_n\| + 1)2^n}$$

remembering that

$$C = \{f_n : n \in \mathbb{N}\}$$

is countable. Each $f \in C$ “lifts” to an $\tilde{f} \in C(K_0)$, $K_0$ the metric space of equivalence classes; the map

$$q : K \to K_0$$

that takes $k$ to $[k]$ is a continuous surjection. So $K_0$ is a compact metric space and

$$q : K \to K_0$$

is a surjective continuous map; $q$ induces an isometric linear embedding

$$q^\circ : C(K_0) \to C(K),$$

where

$$q^\circ(\tilde{f})(k) = \tilde{f}([k]),$$

for any $\tilde{f} \in C(K_0)$. It is important to realize that if $\tilde{C}$ is the result of lifting members of $C$ in $C(K)$ to members of $C(K_0)$, then

$$q^\circ(\tilde{C}) = C.$$
Here is the setup.

\[ \tilde{C} \subseteq C(K_0) \subseteq C(K) \]

and

\[ C \subseteq F(\Sigma) \subseteq C(K)^{**} \]

for some countably additive

\[ F: \Sigma \rightarrow C(K)^{**} \]

with a \( \sigma \)-field domain \( \Sigma \). Because \( C(K_0)^{**} \) is isometrically isomorphic to a subspace of \( C(K)^{**} \); but \( C(K_0)^{**} \) is an injective Banach space so there is a bounded linear projection

\[ P: C(K)^{**} \rightarrow C(K)^{**} \]

on \( C(K)^{**} \) with range

\[ P(C(K)^{**}) = C(K_0)^{**}. \]

So we have a \( C(K_0)^{**} \) such that the countable set \( \tilde{C} \) lies in \( P \circ F(\Sigma) \). The previous theorem tells us there is a \( C(K_0) \)-valued countably additive measure \( G \) on \( \Sigma \) so that \( \tilde{C} \subseteq G(\Sigma) \).

Look at:

\[ C = q^\circ(\tilde{C}) \subseteq q^\circ(G(\Sigma)) \subseteq q^\circ(C(K_0)) \subseteq C(K). \]
BIBLIOGRAPHY


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