OPERATOR RANGES AND POROSITY

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by

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# TABLE OF CONTENTS

**ACKNOWLEDGEMENTS** ................................................. iv

**INTRODUCTION** ....................................................... 1

**CHAPTER 1**
- Porous Sets ......................................................... 3

**CHAPTER 2**
- Geometry of Porous Sets and Delta Nets .......................... 14

**CHAPTER 3**
- Nuclear and Compact Operators .................................... 21

**CHAPTER 4**
- Hamel Bases .......................................................... 36

**CHAPTER 5**
- Volterra Operator ..................................................... 42

**CHAPTER 6**
- Conclusion and Future Projects ..................................... 49

**BIBLIOGRAPHY** ........................................................... 52
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INTRODUCTION

In the first chapter, we will discuss porous and \( \sigma \)-porous sets within Banach spaces and, more generally, \( p \)-Banach spaces. This concept of porosity was first introduced by Dolženko in 1967 [8]. Porous sets have been described as being "nowhere dense with estimates" [3], as the notion of porosity is a more restrictive one than that of nowhere dense. It is known that, in \( \mathbb{R}^n \), the class of \( \sigma \)-porous sets is a proper subclass of the measure zero, first Baire category sets [19]. For this reason, the notion of a \( \sigma \)-porous set can be useful in sharpening results for sets of measure zero or of first Baire category. In the 1920’s, Banach knew that the set of all continuous real-valued functions on \((0, 1)\) which have a derivative at at least one point in \((0, 1)\) is of first category [2]. Later, in the 1990’s, this set was found to be \( \sigma \)-porous [1].

Porosity has been gaining importance as a field of study. Recent publications link porosity to hypercyclicity [3], convex minimization [13], applications to optimization [21] and more. In our studies, we focus on relationships between operator ranges and \( \sigma \)-porous sets. We also define our notion of a strongly \( \sigma \)-porous set. Our main result in this chapter is that if \( T \) is a continuous, linear operator between two \( p \)-Banach spaces, \( X \) and \( Y \), where the range of \( T \) is dense in \( Y \) but unequal to \( Y \), then the range of \( T \) is strongly \( \sigma \)-porous. This generalizes an earlier result of Olevskii [17] who proved it for Banach spaces. Our technique is very different from Olevskii’s; his proof relies heavily on the Hahn-Banach theorem which does not hold, in general, for \( p \)-Banach spaces.

The discussion of \( \sigma \)-porous sets is continued in the second chapter but this time the geometric aspects of these sets are explored. In addition, some connections between \( \delta \)-nets and
σ-porous sets are introduced.

In the third chapter, we are interested in characterizing nuclear operators and compact operators by their ranges. We begin by showing that if \( T : X \rightarrow Y \) is a continuous, injective linear operator between Banach spaces \( X \) and \( Y \) with \( X \) reflexive, then \( T(X) \) is contained in a countable union of nuclear sets if and only if \( T \) is a nuclear operator. We then give an analogous result regarding strongly nuclear sets and strongly nuclear operators. Also, in this chapter, we show that if \( T : X \rightarrow Y \) is a continuous, injective operator between Banach spaces \( X \) and \( Y \) and \( T(X) \) is contained in a countable union of compact sets, then \( T \) is a compact operator.

In the fourth chapter, we continue of our studies of operator ranges. Specifically, we show that if \( N \) is a non-closed algebraic complement of a non-empty closed subspace in an infinite-dimensional Banach space, then \( N \) is not the range of a continuous linear operator on a Banach space. An application to Hamel Bases is given also.

In the fifth chapter, we study some aspects of the classical Volterra operator, \( V \), in the space \( C(0,1) \). The Volterra operator is what is commonly referred to as the anti-derivative operator and is, of course, a very important operator in analysis. Volterra-type operators have been studied in many spaces. In a classical book by Kreǐn [10], the study of these operators is restricted to \( L_2(0,1) \), i.e. Hilbert space. We study these operators in \( C(0,1) \), the space of continuous functions on \([0,1]\). Let \( f : [0, 1] \rightarrow \mathbb{R} \) be bounded and integrable with \( f \) not identically zero. Our investigation will be focused on the orbit, \( \{ f, Vf, V^2f, V^3f, \ldots \} \), of \( f \) under \( V \). Specifically, we look at how the norms of three consecutive elements vary. The questions we pose can be studied in Hilbert space and other spaces, as well.
CHAPTER 1
Porous Sets

In this chapter, we discuss the notion of a porous set. For this, we begin with some definitions. Let \((X, d)\) be a metric space and \(P \subset X\).

**Definition 1.** Given \(y \in X\),

define

\[
B(y, \varepsilon) := \{x \in X : d(x, y) < \varepsilon\}.
\]

**Definition 2.** We say that \(P\) is porous if there exists a constant, \(0 < r < 1\), so that for each \(x \in P\) and each \(\varepsilon > 0\), there exists a \(y\) in \(X\) so that

\[
B(y, \frac{r}{2}\varepsilon) \subset B(x, \varepsilon)
\]

and

\[
B(y, \frac{r}{2}\varepsilon) \cap P = \emptyset.
\]

We call \(r\) a porosity constant for \(P\).

**Remark.** Some definitions of a porous set may have "for \(\varepsilon\) small enough" in place of "\(\varepsilon > 0\)" in the above definition. This is not crucial to the concept of porosity as, generally, one is interested in small values of epsilon.

**Definition 3.** The set \(P\) is strongly porous if \(P\) is porous if \(r\) can be made arbitrarily close to 1.

**Definition 4.** A \(\sigma\)-porous set is a countable union of porous sets.

**Definition 5.** A strongly \(\sigma\)-porous set is a countable union of strongly porous sets.
A simple example of a strongly porous set in $\mathbb{R}$, the set of real numbers, is any singleton set or, in fact, any set containing two numbers. It follows that any countable subset of $\mathbb{R}$ is a strongly $\sigma$-porous set and, thus, a $\sigma$-porous set.

Remark. We point out that here is one instance where our porosity definition and one which allows for "$\varepsilon$ small enough" differ slightly. For instance, if we were to use the latter, then any finite subset of $\mathbb{R}$ would be strongly porous.

Definition 6. For $1 \leq p < \infty$,

$$L_p(0,1) := \{ f : (0,1) \to \mathbb{R} : \int_0^1 |f(t)|^p dt < \infty \}$$

and

$$\|f\|_p := \left( \int_0^1 |f(t)|^p dt \right)^{\frac{1}{p}}.$$

Two functions, $f$ and $g$, in $L_p(0,1)$ are considered equivalent if they only differ on a set of Lebesgue measure zero.

We also have

$$L_\infty(0,1) := \{ f : (0,1) \to \mathbb{R} : f \text{ is measurable and bounded (up to a set of measure zero)} \}$$

where

$$\|f\|_\infty := \text{ess sup } |f| := \inf \{ a \in \mathbb{R} : \text{Lebesgue measure of } E_a \text{ is zero} \}$$

with

$$E_a := \{ t \in (0,1) : |f(t)| > a \}.$$

Or, equivalently,

$$\|f\|_\infty := \inf \{ c \geq 0 : |f(t)| \leq c \text{ for almost every } t \}.$$

As before, two functions, $f$ and $g$, are considered equivalent in $L_\infty(0,1)$ if they only differ on a set of Lebesgue measure zero.
We now present an example of a strongly $\sigma$-porous set which is not obviously so.

**Example.** Let $B_{L_1}(0,1)$ denote the bounded functions in $L_1(0,1)$. Then, $B_{L_1}(0,1)$ is strongly $\sigma$-porous.

**Proof.** Let

$$M_n = \{ f \in L_1(0,1) : \text{ess sup} |f| \leq n \},$$

so that

$$B_{L_1}(0,1) = \bigcup_n M_n.$$

Let $0 < r < 1$ and fix $n$.

We will show that $M_n$ is porous with porosity constant $r$. Let $f \in M_n$ and adjust $f$, if needed, so that

$$\sup |f| \leq n.$$

We can do this since both functions are equivalent in $L_1(0,1)$.

Let $\varepsilon > 0$. For any $N > n$,

we define $g_N$ by

$$g_N(x) = \begin{cases} N & \text{if } 0 \leq x \leq \frac{\varepsilon}{2N} \\ 0 & \text{if } \frac{\varepsilon}{2N} < x \leq 1 \end{cases},$$

so that $\| g_N \| = \frac{\varepsilon}{2}$.

Moreover,

$$f + g_N \geq N - n \text{ on } [0, \frac{\varepsilon}{2N}].$$

So, given $r$, choose $N$ large enough so that

$$\varepsilon \left( \frac{1}{2} - \frac{n}{N} \right) > \frac{\varepsilon r}{2}.$$
Now, if $h \in M_n$, adjust $h$ if needed, so that

$$\sup |h| \leq n.$$  

Thus, on $[0, r\varepsilon/2N]$, 

$$(f + g_N) - h \geq N - 2n$$

and so

$$\| (f + g_N) - h \| \geq (N - 2n) \frac{\varepsilon}{2N} = \varepsilon \left( \frac{1}{2} - \frac{n}{N} \right) > \frac{r\varepsilon}{2}.$$  

Therefore, $h$ is not in $B(f + g_N, r\varepsilon/2)$ and so

$$B(f + g_N, \frac{r\varepsilon}{2}) \cap M_n = \emptyset.$$  

In addition, it is clear that

$$B(f + g_N, \frac{r\varepsilon}{2}) \subset B(f, \varepsilon).$$

Thus, $M_n$ is strongly porous and this completes the proof. \qed

We need the following definition for our next example:

**Definition 7.** Let $A_0$ be the closed interval $[0, 1]$ in $\mathbb{R}$. Let $A_1$ be the set obtained from $A_0$ by deleting the ”middle third” $\left( \frac{1}{3}, \frac{2}{3} \right)$. Let $A_2$ be the set obtained from $A_1$ by deleting its ”middle thirds” $\left( \frac{1}{9}, \frac{2}{9} \right)$ and $\left( \frac{7}{9}, \frac{8}{9} \right)$.

In general, define $A_n$ by the equation

$$A_n = A_{n-1} - \bigcup_{k=0}^{3^{n-1}-1} \left( \frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right).$$

The intersection

$$\mathcal{C} = \bigcap_{n \in \mathbb{Z}_+} A_n$$

is called the Cantor set [16].

**Example.** The Cantor set is not strongly $\sigma$-porous.
Proof. Suppose not. Let \( C \) denote the Cantor set.

Then,

\[
C = \bigcup_n P_n
\]

where each \( P_n \) is porous with porosity constant \( r = r(P_n) \) near to 1.

Since \( C \) is uncountable, there exists an \( m \) so that \( P_m \) contains at least three elements of \( C \).

Choose \( x_1, x_2, x_3 \in P_m \) so that \( x_1 < x_2 < x_3 \). Let \( 1 > \varepsilon > \max\{x_3 - x_2, x_2 - x_1\} \). Then, clearly, \( x_1, x_3 \in B(x_2, \varepsilon) \) and so \( B(x_2, \varepsilon) \) cannot contain a "hole" of radius \( \frac{r}{2}\varepsilon \) with \( r \) near to 1. So, \( P_m \) is not strongly porous and we have a contradiction.

Before we discuss our main theorem for this chapter, we shall take a moment to explain why, in \( \mathbb{R}^n \), a \( \sigma \)-porous set has Lebesgue measure zero and is of first Baire category.

Let \( P \subset \mathbb{R}^n \) be \( \sigma \)-porous. So \( P = \bigcup_n P_n \) where each \( P_n \) is porous. That \( P \) is of first Baire category is a consequence of the fact that any porous set is nowhere dense. To show that \( P \) has Lebesgue measure zero, one must only show that each \( P_n \) has measure zero. This fact follows from a theorem of Lebesgue [23] regarding the density of sets which have positive measure. It states that sets of positive measure have density one almost everywhere.

Before we go on to the theorem, we need a few more definitions.

**Definition 8.** A \( p \)-norm \( |||x||| \), \( 0 < p \leq 1 \), on a vector space \( E \), over a field \( \mathbb{K} \), is characterized by:

- **(P1)** \( |||x||| \geq 0 \),
- **(P2)** if \( |||x||| = 0 \), then \( x = 0 \)-vector,
- **(P3)** \( |||\alpha x||| = |\alpha|^p |||x||| \), for all \( \alpha \in \mathbb{K} \),
- **(P4)** \( |||x + y||| \leq |||x||| + |||y||| \).

In the case that \( p = 1 \), we get the usual concept of a norm [14].
By $p$-Banach space, we mean a complete $p$-normed vector space. It is clear that a $p$-Banach space is a Banach space if and only if $p = 1$. Important examples of $p$-Banach spaces which are not Banach spaces are the $L_p$-spaces and $l_p$-spaces, $0 < p < 1$. In $L_p(0, 1)$, we have

$$|||f||| := \int_0^1 |f(t)|^p dt$$

and in $l_p(0, 1)$, we set

$$|||x||| := \sum_{i=1}^{\infty} |a_i|^p,$$

where $x = \{a_1, a_2, \ldots\}$.

**Theorem 1.** Let $T : X \to Y$ be a continuous linear operator between $p$-Banach spaces $X$ and $Y$ and denote the range of $T$ by $R(T)$. If $\overline{R(T)} = Y$ and $R(T) \neq Y$, then $R(T)$ is strongly $\sigma$-porous.

To prove Theorem 1, we will use the following theorem. It states that if the range is dense and not all of $Y$, then there is a $y \in Y$ which is hard to approximate by an element of the form $T(z)$, i.e. we cannot get close to $y$ without having a "huge" inverse.

**Theorem 2.** Let $T$ satisfy the hypotheses of Theorem 1. Let $\frac{1}{c} < 1$. Then, for every $N$, there exists $y \in Y \setminus R(T)$ of $p$-norm one so that if $|||T(z) - y||| < \frac{1}{c}$, then $|||z||| > N$.

**Proof.** Suppose not.

Then, there exists an $N$, so that for all $y \in Y \setminus R(T)$ of $p$-norm one with

$$|||T(z) - y||| < \frac{1}{c}$$

we have

$$|||z||| \leq N.$$
Let $y \in Y \setminus R(T)$ with $\||y|| = 1$.

Since $R(T) = Y$, we have that

$$B(y, \frac{1}{c}) \cap R(T) \neq \emptyset.$$ 

Let

$$T(z_1) \in [B(y, \frac{1}{c}) \cap R(T)].$$

Set

$$\alpha_1 = |||y - T(z_1)||| < \frac{1}{c}.$$ 

By assumption, $|||z_1||| \leq N$.

Now choose $z_2$ so that

$$||T(z_2) - \frac{y - T(z_1)}{\alpha_1^p}|| < \frac{1}{c}$$

and

$$||z_2|| \leq N.$$ 

Then,

$$||T(\alpha_1^{\frac{1}{p}} z_2 + z_1) - y|| = ||T(\alpha_1^{\frac{1}{p}} z_2) - (y - T(z_1))||$$

$$= \alpha_1 ||T(z_2) - \frac{y - T(z_1)}{\alpha_1^p}||$$

$$< \alpha_1 \frac{1}{c} < \frac{1}{c^2}.$$ 

Now, $||z_1||, ||z_2|| \leq N$ and $0 < \alpha_1 < \frac{1}{c}$ gives

$$||\alpha_1^{\frac{1}{p}} z_2 + z_1|| < \frac{1}{c} N + N.$$ 

Set

$$\alpha_2 = |||y - T(\alpha_1^{\frac{1}{p}} z_2 + z_1)||| < \frac{1}{c^2}.$$
Choose $z_3$ so that
$$|||T(z_3) - \frac{y - T(\alpha_1^\frac{1}{c} z_2 + z_1)}{\alpha_2^{\frac{1}{c}}}||| < \frac{1}{c}$$
onumber
and
$$|||z_3||| \leq N.$$ 

Then, we see that
$$|||y - T(\alpha_2^\frac{1}{c} z_3 + \alpha_1^\frac{1}{c} z_2 + z_1)||| < \frac{1}{c^3}.$$ 

Now, since $|||z_1|||, |||z_2|||, |||z_3||| \leq N$ with $0 < \alpha_2 < \frac{1}{c^2}$ and $0 < \alpha_1 < \frac{1}{c}$, we get
$$|||\alpha_2^\frac{1}{c} z_3 + \alpha_1^\frac{1}{c} z_2 + z_1||| < \frac{N}{c^2} + \frac{N}{c} + N.$$ 

Continuing in this manner, with $\alpha_0 = 1$, we get the series
$$\sum_{i=1}^{\infty} \alpha_{i-1}^{\frac{1}{c_i}} z_i.$$ 

As
$$\sum_{i=1}^{\infty} |||\alpha_{i-1}^{\frac{1}{c_i}} z_i||| < \sum_{i=1}^{\infty} \frac{1}{c_i} N < \infty,$$
we have that the series is absolutely convergent.

Thus,
$$\sum_{i=1}^{\infty} \alpha_{i-1}^{\frac{1}{c_i}} z_i = z,$$
for some $z$.

So,
$$T(\sum_{i=1}^{\infty} \alpha_{i-1}^{\frac{1}{c_i}} z_i) = T(z).$$

By the previous construction, we have that
$$T(\sum_{i=1}^{\infty} \alpha_{i-1}^{\frac{1}{c_i}} z_i) \to y.$$

Hence, $T(z) = y$. Since $y$ was chosen outside of $R(T)$, we have our contradiction and this completes the proof of Theorem 2.
We will now prove Theorem 1:

**Proof.** Let $\frac{1}{c} < 1$.

Let

$$M_n = \{T(x) : \|\|T(x)\|\| \geq \frac{\|\|x\|\|}{n}\},$$

so that

$$R(T) = \bigcup_n M_n.$$ 

Fix $n$. We show that $M_n$ is porous with porosity constant $\frac{1}{c}$.

Let $\varepsilon > 0$ and let $T(x_0) \in M_n$.

Choose $N$ so that

$$\left(\frac{N\varepsilon}{2} - \|\|x_0\|\|\right) > \|\|T(w)\|\| n,$$

for all $T(w) \in B(T(x_0), \varepsilon)$.

Let $y$ exist from Theorem 2.

Then,

$$(T(x_0) + \left(\frac{\varepsilon}{2}\right)^1 y) \in B(T(x_0), \varepsilon)$$

and since $y \in Y \setminus R(T)$, we have

$$(T(x_0) + \left(\frac{\varepsilon}{2}\right)^1 y) \in Y \setminus R(T).$$

Moreover,

$$B(T(x_0) + \left(\frac{\varepsilon}{2}\right)^1 y, \frac{1}{2c}\varepsilon) \subset B(T(x_0), \varepsilon).$$

We are left to show that

$$B(T(x_0) + \left(\frac{\varepsilon}{2}\right)^1 y, \frac{1}{2c}\varepsilon) \cap M_n = \emptyset.$$

By the density of $R(T)$, let

$$T(w) \in B(T(x_0) + \left(\frac{\varepsilon}{2}\right)^1 y, \frac{1}{2c}\varepsilon).$$
Then,

\[ \|T(w) - (T(x_0) + (\frac{\varepsilon}{2}) \frac{1}{c} y)\| < \frac{1}{2c} \varepsilon \Rightarrow \|T(w) - x_0\| < \frac{1}{2c} \varepsilon \]
\[ \Rightarrow \|T(2\frac{1}{c} w - 2\frac{1}{c} x_0) - \varepsilon \frac{1}{c} y\| < \frac{1}{c} \varepsilon \]
\[ \Rightarrow \|T((\frac{2}{\varepsilon}) \frac{1}{c} w - (\frac{2}{\varepsilon}) \frac{1}{c} x_0) - y\| < \frac{1}{c} \varepsilon \]
\[ \Rightarrow \|T(\frac{2}{\varepsilon}) \frac{1}{c} w - (\frac{2}{\varepsilon}) \frac{1}{c} x_0\| \geq N, \text{ by Theorem 2} \]
\[ \Rightarrow \|w - x_0\| \geq \frac{\varepsilon}{2} N \]
\[ \Rightarrow \|w\| + ||x_0|| \geq \frac{N \varepsilon}{2} \]
\[ \Rightarrow \|w\| \geq \frac{N \varepsilon}{2} - ||x_0|| \geq ||T(w)|| n. \]

Thus, \( T(w) \) is not an element of \( M_n \) and we have

\[ B(T(x_0) + \frac{\varepsilon}{2} y, \frac{1}{2c} \varepsilon) \bigcap M_n = \emptyset. \]

This proves the theorem.

\[ \square \]

**Example.**

Let \( B_{L_1}(0,1) \) denote the bounded functions in \( L_1(0,1) \). We already know this set is strongly \( \sigma \)-porous from a previous example where we verified the criteria using the definition.

Now, we will use the theorem: Let \( T : L_\infty(0,1) \rightarrow L_1(0,1) \) be the Identity operator. Since \( L_\infty(0,1) \) is \( B_{L_1}(0,1) \) equipped with the \( ess \) sup norm and \( B_{L_1}(0,1) \) is a dense proper subspace of \( L_1(0,1) \), it follows from the theorem that \( B_{L_1}(0,1) \) is strongly \( \sigma \)-porous.

This study of \( B_{L_1}(0,1) \) leads us to the following fact.

**Theorem 3.** There is no continuous linear operator which maps a separable Banach space onto \( B_{L_1}(0,1) \).
Proof. Suppose not. So there exists $T : X \to B_{L_1}(0, 1)$, onto, where $X$ is a separable Banach space.

For every $z \in X$, consider the element $Tz \in B_{L_1}(0, 1)$.

Let

$$V_n = \{ z \in X \mid ess \sup \frac{|Tz|}{\|z\|} \leq n \},$$

so that $V_n$ is closed $\forall n$ and

$$X = \bigcup_n V_n.$$

Now use Baire Category Theorem [20] to say that $V_n (= V_n)$ has interior points for some $n$.

So, $V_n$ contains a ball about $z_0$ for some $z_0$. Thus, $V_n - z_0$ contains a ball, $B(0, \delta)$, about zero.

Then, for all $z - z_0$ in $B(0, \delta)$ we have

$$ess \sup |T(z - z_0)| \leq ess \sup |Tz| + ess \sup |Tz_0|$$

$$\leq n \|z\| + n \|z_0\|$$

$$\leq n (\|z_0\| + \delta) + n \|z_0\|.$$

Thus, $T$ is bounded if we introduced the norm, $\|Tz\| = ess \sup |Tz|$. And so, with this norm, $T$ is also continuous. But $L_\infty(0, 1)$ is $B_{L_1}(0, 1)$ equipped with the $ess \sup$ norm and we would have the surjection

$$T : X \to L_\infty(0, 1)$$

This is an obvious contradiction as $X$ is separable and $L_\infty(0, 1)$ is not. \qed
We begin our discussion with a result regarding the geometry of porous sets.

**Theorem 4.** Let $S$ be a curve in $\mathbb{R}^2$ that is continuous and twice differentiable. Then, the following are equivalent:

1. If $T$ is a line tangent to $S$, then $S$ lies on one side of $T$.
2. $S$ is strongly porous.
3. Given $(x, y)$ on $S$ and $\delta > 0$, $\exists$ a circle, $C$, tangent to $S$ at $(x, y)$, with diameter $\delta$, whose interior does not intersect $S$.

We prove the theorem by showing the following equivalences:

$(1) \iff (2)$, and $(2) \iff (3)$.

**Proof.** We start by proving $(1) \Rightarrow (2)$:

Let $r$ arbitrarily close to 1 and let $(x, y)$ on $S$. Let $\varepsilon > 0$. Let $T$ be tangent to $S$ at $(x, y)$. By assumption, $S$ lies to one side of $T$. Let $(x_0, y_0)$ represent the normalized vector in direction perpendicular to $T$. Without loss, assume that it is $(x, y) + (x_0, y_0)$ which lies on the side of $T$ opposite from $S$.

Let

$$(x_1, y_1) := (x, y) + \frac{\varepsilon}{2}(x_0, y_0)$$

Then,

$$B((x_1, y_1), \frac{r}{2}\varepsilon) \subset B((x, y), \varepsilon)$$

and

$$B((x_1, y_1), \frac{r}{2}\varepsilon) \cap S = \emptyset.$$
Thus, $S$ is strongly porous.

We now prove $(2) \Rightarrow (1)$:

Suppose not. Let $r$ be arbitrarily close to 1. By assumption, there exists a line, $T$, tangent to $S$ at $(x, y)$ and there exists $(x_1, y_1), (x_2, y_2) \neq (x, y)$ on $S$ which lie on opposite sides of $T$. Choose $\varepsilon_1$ so that $(x_1, y_1)$ lies in the interior of the circle, $C_1$, tangent to $S$ at $(x, y)$. Similarly, choose $\varepsilon_2$ so that $(x_2, y_2)$ lies in the tangent circle, $C_2$. Let $\varepsilon > \max\{\varepsilon_1, \varepsilon_2\}$. As $S$ is strongly porous, there must exist a ball of radius $r\varepsilon$, contained in $B((x, y), \varepsilon)$, which does not intersect $S$. Since $r$ is arbitrarily close to 1, this ball must be the interior of some tangent circle, $C$. By the differentiability of $S$, $C$ must contain either $C_1$ or $C_2$ and we get a contradiction.

The equivalence of $(2) \Leftrightarrow (3)$ follows from the definition of strongly porous.

\[\square\]

**Remark.** We feel that there is a more general $\mathbb{R}^n$ version of this theorem that could be obtained by a similar proof.

We now move on to discuss $\delta$-nets. We shall start by defining a $\delta$-net.

Let $X$ be a metric space with metric $d$, and let $\delta > 0$.

**Definition 9.** We say that $N \subseteq X$ is a $\delta$-net for $X$ if for each $x \in X$, there exists an $y \in N$ so that $d(x, y) < \delta$.

Two obvious examples of $\delta$-nets for $X$ are $X$ itself or any dense subset of $X$. If $A \subset X$, we can also consider a $\delta$-net for $A$, whose elements may or may not contain elements from $A$. For $P \subset X$, a porous set, the following proposition shows a relationship
between the complement of \( P \) and a \( \delta \)-net for \( P \).

**Proposition 1.** Let \( X \) be a metric space and \( P \subset X \) be porous with porosity constant \( 0 < r < 1 \). Let \( \delta > 0 \). Then there exists a \( \delta \)-net, \( N \), for \( P \) with the following property:

If \( y \in N \), then \( B(y, \frac{\delta}{2}) \cap P = \emptyset \).

**Proof.** Let \( x \in P \). As \( P \) is porous, there is a "hole" of radius \( \frac{\delta}{2} \) within \( B(x, \delta) \). Choose one such hole and let \( y_x \) be its center so that

\[ B(y_x, \frac{\delta}{2}) \cap P = \emptyset. \]

As \( 0 < r < 1 \), it is clear that \( d(x, y_x) < \delta \). For each \( x \in P \), choose a corresponding \( y_x \).

Let

\[ N = \bigcup_{x \in P} \{ y_x \} \]

and the result is proven. \( \square \)

One thing the last result tells us is that given any \( \delta > 0 \) and any porous set, \( P \), \( P^c \) is a \( \delta \)-net for \( P \) (this can also be seen by noting that \( P^c \) is dense). This is untrue for a general subset of \( X \). For instance, consider \( X \) to be the real line and \( A \) to be closed interval \([0, 2]\).

If we let \( \delta = \frac{1}{2} \), then clearly the complement of \( A \) is not a \( \delta \)-net for \( A \).

Also, if \( B \subset X \) is a \( \delta \)-net for \( X \) and \( A \subset X \), then clearly, \( B \) is a \( \delta \)-net for \( A \). The converse is generally untrue, as seen in the case where

\[ X = [0, 2], \; \delta = \frac{1}{2}, \; A = [0, 1] \]

and

\[ B = \{0, \frac{1}{3}, \frac{2}{3}\}. \]

Here, \( B \) is a \( \delta \)-net for \( A \) but not for \( X \).

It can easily be shown that, in the case where \( A \) is dense in \( X \) and \( B \) is a \( \delta \)-net for \( A \), we
have that $B$ is a $(\delta + \varepsilon)$-net for $X$ and this is true for all $\varepsilon > 0$.

We now shift from looking at $\delta$-nets for subsets of $X$ to considering $\delta$-nets for $X$.

To continue, we will need the following definition.

**Definition 10.** Let $(X, d)$ be a metric space and $A \subset X$. Let $\varepsilon > 0$.

We say that $A$ is *separated by $\varepsilon$* if given any $x_1, x_2 \in A$ with $x_1 \neq x_2$, we have $d(x_1, x_2) \geq \varepsilon$.

We now consider this notion of separated nets and how it relates to complements of porous sets.

**Theorem 5.** Let $(X, d)$ be a metric space and $P \subset X$ be a porous subset with porosity constant $0 < r < 1$.

Then, for all $\delta > 0$:

1) $P^c$ is a delta net for $X$

and

2) $P^c$ contains a $\frac{3\delta}{2}$-net for $X$ separated by $\frac{\delta}{2}$.

**Proof.** We start with 1). Let $\delta > 0$. Since $P^c$ is dense in $X$, we have that $P^c$ is a $\delta$-net. To see that $P^c$ is dense, we let $x \in X$ and $\varepsilon > 0$. If $x \in P^c$, there is nothing to show and so we assume that $x \in P$. Since $P$ is porous, there exists a ball of size (diameter) $r\varepsilon$ which lies within $B(x, \varepsilon)$ and outside of $P$. Let $y$ be any element of the ball. Then, $y$ lies in $P^c$ and we have $d(x, y) < \delta$. That the density of $P^c$ implies $P^c$ is a $\delta$-net is obvious.

To show 2), we will only need to prove the following lemma:

**Lemma 1.** Let $\delta > 0$. Let $X$ be a metric space and $D$ a $\delta$-net for $X$.

Then, there exists $\tilde{D} \subseteq D$ so that $\tilde{D}$ is a $\frac{3\delta}{2}$-net for $X$ and if $x_1, x_2 \in \tilde{D}$ with $x_1 \neq x_2$, then $d(x_1, x_2) \geq \frac{\delta}{2}$.
Proof. We’ll construct \( \tilde{D} \) in the following manner. Let \( x \in D \). Allow \( x \in \tilde{D} \).

If
\[
D \cap [B(x, \frac{\delta}{2})]^C = \emptyset,
\]
then
\[
D \subseteq B(x, \frac{\delta}{2})
\]
and setting \( \tilde{D} = \{x\} \) completes the proof.

Otherwise, choose
\[
y \in D \cap [B(x, \frac{\delta}{2})]^C
\]
and allow \( y \in \tilde{D} \).

If
\[
D \cap [B(x, \frac{\delta}{2}) \cup B(y, \frac{\delta}{2})]^C = \emptyset,
\]
then setting \( \tilde{D} = \{x, y\} \) completes the proof.

Otherwise, choose
\[
w \in D \cap [B(x, \frac{\delta}{2}) \cup B(y, \frac{\delta}{2})]
\]
and allow \( w \in \tilde{D} \).

Continuing in this manner, possibly trans-finitely, \( \tilde{D} \) will be complete when
\[
D \cap \bigcup_{x \in \tilde{D}} (B(x, \frac{\delta}{2}))^C = \emptyset.
\]

So, \( \tilde{D} \subseteq D \) and, clearly, if \( x_1, x_2 \in \tilde{D} \) with \( x_1 \neq x_2 \), then \( d(x_1, x_2) \geq \frac{\delta}{2} \).

We are left to show that \( \tilde{D} \) is a \( \frac{3\delta}{2} \)-net.

Let \( x \in X \). We know there is a \( t \in D \) so that \( d(x, t) < \delta \).

Case 1: We have that \( t \in \tilde{D} \).

Then, \( d(x, t) < \delta < \frac{3\delta}{2} \) and we’re done.
Case 2: We have that $t$ is not an element of $\tilde{D}$.

Then, by the construction of $\tilde{D}$, there is an $s \in \tilde{D}$ so that $d(t, s) < \frac{\delta}{2}$.

Thus,

$$d(s, x) \leq d(s, t) + d(t, x) < \frac{\delta}{2} + \delta = \frac{3\delta}{2}.$$ 

This completes the proof of the theorem.

In the last result, we had to allow our sub-net to be a $\frac{3\delta}{2}$-net in order to guarantee we could separate it by $\frac{\delta}{2}$. This, of course, was within the complement of a porous set.

We now state a fact about general metric spaces and the existence of a particular kind of $\delta$-net; namely, one which is also separated by $\delta$.

**Theorem 6.** Let $(X, d)$ be a metric space and $\delta > 0$. Then, there exists a $\delta$-net for $X$ separated by at least $\delta$.

**Proof.** Let $\delta > 0$. We construct our $\delta$-net, $D$, in the following manner. Let $x \in X$ and allow $x \in D$. If $X \subseteq B(x, \delta)$ then setting $D = \{x\}$ completes the proof.

Otherwise, choose

$$y \in X \setminus B(x, \delta)$$

and allow $y \in D$. Note that $d(x, y) > \delta$.

Now, if

$$X \subseteq [B(x, \delta) \cup B(y, \delta)],$$

then setting $D = \{x, y\}$ completes the proof.

Otherwise, choose

$$w \in X \setminus [B(x, \delta) \cup B(y, \delta)]$$
and allow \( w \in D \).

Continuing in this manner, possibly trans-finitely, \( D \) will be complete when

\[
X \subseteq \bigcup_{x \in D} B(x, \delta).
\]

So, \( D \) is a \( \delta \)-net for \( X \) and, clearly, if \( x_1, x_2 \in D \) with \( x_1 \neq x_2 \), then \( d(x_1, x_2) \geq \delta \).

This proves our result.

To give an example, fix \( \delta > 0 \). Let \( \varepsilon > 0 \) and define \( X \subset \mathbb{R} \) to be the set

\[
X := \{0, \delta(1 + \varepsilon), 2\delta(1 + \varepsilon), 3\delta(1 + \varepsilon), \ldots\}.
\]

If we want \( D \subset X \) to be a \( \delta \)-net for \( X \), then we must have \( D = X \). Now, \( D \) is separated by \( \delta(1 + \varepsilon) \) and this shows that the \( \delta \) in the last theorem is sharp.

Another example is if we take \( X = [0, \frac{3\delta}{2}] \) on the real line. Then the subset \( \{\frac{\delta}{4}, \frac{3\delta}{2}\} \) is a \( \delta \)-net for \( X \) separated by \( \frac{5\delta}{4} \).
CHAPTER 3  
Nuclear and Compact Operators

We begin our discussion with nuclear operators.

For this, we will need the following definitions.

**Definition 11.** A subset $N \subset X$ is a **nuclear set** if

$$N \subset \{ \sum_j \alpha_j x_j : |\alpha_j| \leq 1 \forall j \},$$

for some subset $\{x_1, x_2, \ldots, x_n, \ldots\}$ of $X$ for which $\sum_j \|x_j\| < \infty$.

**Definition 12.** A subset $N \subset X$ is a **strongly nuclear set** if $N$ is a subset of a closed convex hull, i.e $\exists \{x_1, \ldots, x_n, \ldots\} \subset X$ with $\sum_j \|x_j\| < \infty$, so that

$$N \subset \{ \sum_j \alpha_j x_j : \sum_j |\alpha_j| \leq 1 \}.$$

**Definition 13.** An operator is a **nuclear operator** if it is an absolutely convergent sum of one-dimensional operators. That is,

$$T : X \longrightarrow Y \text{ with } T = \sum_j T_j, \sum_j \|T_j\| < \infty$$

and for each $j$, there is a $y_j$ in $Y$ so that

$$T_j(X) = \{ \beta y_j : \beta \in \mathbb{R} \}.$$

**Definition 14.** An operator is a **strongly nuclear operator** if it is nuclear with the additional condition that

$$\sum_j \sqrt{\|T_j\|} < \infty.$$
**Definition 15.** Let $X$ be a Banach space over $\mathbb{R}$. We denote by $X^*$ its continuous dual (the Banach space of all continuous linear operators from $X$ to $\mathbb{R}$). We can also form the double dual $X^{**}$, the continuous dual of $X^*$. There is a natural continuous linear operator 

$$J : X \to X^{**}$$

given by 

$$J(x)(f) = f(x)$$

for every $x \in X$ and $f \in X^*$. The operator $J$ maps $x$ to the functional on $X^*$ given by evaluation at $x$. We also know that $J$ is norm-preserving (i.e., $\|J(x)\| = \|x\|$) as a consequence of the Hahn-Banach theorem [22]. The space $X$ is called reflexive if $J$ is bijective.

**Theorem 7.** Let $T : X \to Y$ be a continuous, one-to-one linear operator between Banach spaces $X$ and $Y$, where $X$ is reflexive. Then, $T(X)$ is contained in a countable union of nuclear sets if and only if $T$ is a nuclear operator.

**Lemma 2.** Let $T : X \to Y$ be a continuous, one-to-one linear operator between Banach spaces $X$ and $Y$. Let $B_X$ denote the closed unit ball in $X$. If $T(B_X)$ is a nuclear set, then $T$ is a nuclear operator.

*Proof.* Let $x \in X$ be non-zero.

By hypothesis, 

$$T\left(\frac{x}{\|x\|}\right) = \sum_j \alpha_j x_j,$$
for some $\alpha_j$ and $x_j$, $j = 1, 2, \ldots$, with each $|\alpha_j| \leq 1$ and $\sum_j \| x_j \| < \infty$.

Define

$$T_1(x) = \| x \| \alpha_1 x_1, \quad T_2(x) = \| x \| \alpha_2 x_2, \ldots, T_n(x) = \| x \| \alpha_n x_n, \ldots$$

and so on.

In addition, when we define $T_n(0) = 0 \forall n$, we get

$$T = \sum_j T_j$$

and

$$\sum_j \| T_j \| = \sum_j |\alpha_j| \| x_j \| \leq \sum_j 1 \| x_j \| < \infty.$$ 

Also, it is clear that $T_j(X) = \text{Span}_\mathbb{R}\{x_j\}$. □

**Lemma 3.** The sum of two nuclear sets is nuclear.

**Proof.** Let $A$ and $B$ be nuclear subsets of $X$.

So,

$$A \subset \{ \sum_j \alpha_j x_j : |\alpha_j| \leq 1 \},$$

for some $\{x_1, \ldots, x_n, \ldots\} \subset X$ for which $\sum_j \| x_j \| < \infty$

and

$$B \subset \{ \sum_j \tilde{\alpha}_j \tilde{x}_j : |\tilde{\alpha}_j| \leq 1 \},$$

for some $\{\tilde{x}_1, \ldots, \tilde{x}_n, \ldots\} \subset X$ for which $\sum_j \| \tilde{x}_j \| < \infty$.

Let

$$\{y_1, \ldots, y_n, \ldots\} = \{ x_1, \ldots, x_n, \ldots, \tilde{x}_1, \ldots, \tilde{x}_n, \ldots\},$$

so that

$$A + B \subset \{ \sum_j \beta_j y_j : |\beta_j| \leq 1 \}$$

and

$$\sum_j \| y_j \| = \sum_j \| x_j \| + \sum_j \| \tilde{x}_j \| < \infty.$$
Thus, $A + B$ is nuclear. 

**Lemma 4.** Let $T : X \rightarrow Y$ be a continuous, one-to-one linear operator between Banach spaces $X$ and $Y$. Let $P$ be an epsilon ball about zero so that $T(P)$ is nuclear. Then, $T(B_X)$ is also nuclear.

**Proof.** First, let us assume that $P$ is the closed epsilon ball, with $\varepsilon > 0$. Otherwise we could set $\tilde{P} = \{ x : \| x \| \leq \frac{\varepsilon}{2} \}$ and proceed with the proof using the nuclear set $T(\tilde{P})$.

So, $P$ is closed and

$$T(P) \subset \{ \sum_j \beta_j y_j : |\beta_j| \leq 1 \forall j \},$$

for some subset $\{y_1, y_2, \ldots, y_n, \ldots\}$ of $Y$ for which $\sum_j \| y_j \| < \infty$.

Let $\tilde{y}_j = \frac{y_j}{\varepsilon}$ and let

$$\tilde{N} = \{ \sum_j \tilde{\beta} \tilde{y}_j : |\tilde{\beta}| \leq 1 \}.$$

We claim that $T(B_X)$ is a subset of $\tilde{N}$ and is, thus, nuclear. Let $y \in T(B_X)$.

Then, $y = T(x)$ for some $x \in B_X$.

Let $x(\varepsilon)$ denote the element on the boundary of $P$ so that $x = \alpha x(\varepsilon)$, for some scalar $\alpha$ where (of course) $|\alpha|\varepsilon \leq 1$.

Then,

$$T(x) = \alpha T(x(\varepsilon)) = \alpha \sum_j \beta_j y_j,$$

where $|\beta_j| \leq 1$,

$$= \sum_j \alpha \varepsilon \beta_j \frac{y_j}{\varepsilon},$$

$$= \sum_j \alpha \varepsilon \beta_j \tilde{y}_j,$$

$$\in \tilde{N},$$

since $|\alpha \varepsilon \beta_j| \leq 1$ for all $j$ and $\sum_j \| \tilde{y} \| \leq \frac{1}{\varepsilon} \sum_j \| y_j \| < \infty$. 

\qed
We start with the forward direction of the theorem.

**Proof.** Let $T$ satisfy the hypotheses of the theorem. By Lemma 2, it suffices to show that $T(B_X)$ is a nuclear set.

We have that

$$T(X) \subset \bigcup N_k,$$

where each $N_k$ is nuclear.

Define

$$N_{k,r} := \{ T(x) \in N_k : \| T(x) \| \geq \frac{\| x \|}{r} \}$$

and

$$P_{k,r} := \{ x : T(x) \in N_{k,r} \},$$

so that $B_X \subset \bigcup_{k,r} P_{k,r}$.

Note that each $N_{k,r}$ is closed. To see this, fix $k, r$ and let $(T(x_n))$ be a sequence in $N_{k,r}$ so that $T(x_n) \rightarrow y$ and assume that $y \neq 0$.

Since

$$\frac{\| x_n \|}{r} \leq \| T(x_n) \|,$$

we have

$$\| x_n \| \leq r \| T(x_n) \| \leq r \| y \|.$$

So, $(x_n)$ is bounded. Since $X$ is reflexive, we know that $(x_n)$ has a weakly convergent subsequence, $(x_{n_k})$, where $x_{n_k} \rightarrow z$ weakly (Theorem 4.41-B in [22]).

So,

$$T(x_{n_k}) \rightarrow T(z) \text{ weakly.}$$

But we also know that

$$T(x_n) \rightarrow y \text{ weakly.}$$
So, we must have $T(z) = y$.

Moreover,

$$
\| z \| \leq \lim \inf \| x_{n_k} \| \leq \lim \inf r \| T(x_{n_k}) \| = r \| y \| .
$$

Thus, $y \in N_{k,r}$ and so $N_{k,r}$ is closed.

Now, using Baire Category Theorem [20], there is a $k$ and an $r$ so that $P_{k,r}$ has non-empty interior.

Each $P_{k,r}$ is closed since $P_{k,r} = T^{-1}(N_{k,r})$, the inverse image of a closed set.

So, we get a $k$ and an $r$ so that $P_{k,r}$ contains an open ball.

Thus, the set $P_{k,r} - P_{k,r}$ contains an open ball about zero, say $P$, where

$$
T(P) \subset N_{k,r} - N_{k,r} \subset N_k - N_k.
$$

Since each $N_k$ is nuclear, the set $N_k - N_k$ is nuclear by Lemma 3 and so $T$ maps a neighborhood of zero into a nuclear set. So, by Lemma 4, $T$ maps $B_X$ into a nuclear set. By Lemma 2, $T$ is a nuclear operator.

This completes the forward direction of the theorem.

\[\square\]

We now prove the backwards direction of the theorem.

\textbf{Proof.} We now assume $T$ is a nuclear operator and must show that $T(X)$ is contained in a countable union of nuclear sets. We have that there exists a collection of operators $T_1, T_2, \ldots$ so that $T = \sum_j T_j$ and for each $j$ there is a $y_j$ so that $T_j(X) = \text{span}_\mathbb{R}\{y_j\}$.

Set $\tilde{y}_j = \frac{y_j}{\| y_j \|}$ and set $\alpha_j = \| T_j \|$.

Fix $n$. We will show that $T(B(0,n))$ is a nuclear set.

Fix $j$ and let $x \in B(0,n)$.

Since

$$
T_j(X) = \text{span}_\mathbb{R}\{y_j\} = \text{span}_\mathbb{R}\{n\alpha_j \tilde{y}_j\},
$$
we know there is a $\beta_j$ so that $T_j(x) = \beta_j n\alpha_j y_j$.

From $$\| T_j(x) \| \leq \| T_j \| \| x \|,$$
we get
$$|\beta_j| n \| T_j \| \leq \| T_j \| n$$
and thus $|\beta_j| \leq 1$.

So, we have
$$T(x) = \sum_j T_j(x) = \sum_j \beta_j n\alpha_j y_j \quad \text{with} \quad \| \beta_j || \leq 1.$$

Thus,
$$T(B(0,n)) \subset \{ \sum_j \beta_j n\alpha_j y_j : |\beta_j| \leq 1 \},$$
which is a nuclear set since
$$\sum_j \| n\alpha_j y_j \| = n \sum_j \| T_j \| = n \| T \| < \infty.$$ 

Now, since
$$T(X) = \bigcup_n T(B(0,n)),$$
we have that $T(X)$ is a countable union of nuclear sets, as desired.

We now show the same result for strongly nuclear operators:

**Theorem 8.** Let $T : X \to Y$ be a continuous, one-to-one linear operator between Banach spaces $X$ and $Y$, where $X$ is reflexive. Then, $T(X)$ is contained in a countable union of strongly nuclear sets if and only if $T$ is a strongly nuclear operator.

**Lemma 5.** Let $T : X \to Y$ be a continuous, one-to-one linear operator between Banach spaces $X$ and $Y$. Let $B_X$ denote the closed unit ball in $X$. If $T(B_X)$ is a strongly nuclear
set, then $T$ is a strongly nuclear operator.

**Proof.** Let $x \in X$ be non-zero.

By hypothesis,  
\[
T\left(\frac{x}{\|x\|}\right) = \sum_j \alpha_j x_j
\]

for some $\alpha_j$ and $x_j$, $j = 1, 2, \ldots$, with $\sum_j |\alpha_j| \leq 1$ and $\sum_j \|x_j\| < \infty$.

Define  
\[
T_1(x) = \|x\| \alpha_1 x_1, \quad T_2(x) = \|x\| \alpha_2 x_2, \quad \ldots, \quad T_n(x) = \|x\| \alpha_n x_n, \quad \ldots
\]

and when we define $T_n(0) = 0 \ \forall n$, we get $T = \sum_j T_j$.

Moreover,  
\[
\sum_j \sqrt{\|T_j\|} = \sum_j \sqrt{|\alpha_j| \sqrt{\|x_j\|}} \leq \sum_j \left(\frac{|\alpha_j|}{\sqrt{2}} + \frac{\|x_j\|}{\sqrt{2}}\right) < \sum_j |\alpha_j| + \sum_j \|x_j\| < \infty.
\]

And clearly, $T_j(X) = \text{Span}_\mathbb{R}\{x_j\}$. 

\[\Box\]

**Lemma 6.** The sum of two strongly nuclear sets is strongly nuclear.

**Proof.** Let $A$ and $B$ be strongly nuclear subsets of $X$.

So,  
\[
A \subset \left\{ \sum_j \alpha_j x_j : \sum_j |\alpha_j| \leq 1 \right\},
\]

for some \{x_1, \ldots, x_n, \ldots\} $\subset X$ for which $\sum_j \|x_j\| < \infty$.

and  
\[
B \subset \left\{ \sum_j \tilde{\alpha}_j \tilde{x}_j : \sum_j |\tilde{\alpha}_j| \leq 1 \right\},
\]

for some \{\tilde{x}_1, \ldots, \tilde{x}_n, \ldots\} $\subset X$ for which $\sum_j \|\tilde{x}_j\| < \infty$.

Let  
\[
\{y_1, \ldots, y_n, \ldots\} = \{2x_1, \ldots, 2x_n, \ldots, 2\tilde{x}_1, \ldots, 2\tilde{x}_n, \ldots\},
\]

so that
\[ A + B \subset \left\{ \sum_j \beta_j y_j : \sum_j |\beta_j| \leq 1 \right\}. \]

In addition,
\[ \sum_j \| y_j \| = 2 \sum_j \| x_j \| + 2 \sum_j \| \tilde{x}_j \| < \infty. \]

Thus, \( A + B \) is strongly nuclear.

\[ \square \]

**Lemma 7.** Let \( T : X \to Y \) be a continuous, one-to-one linear operator between Banach spaces \( X \) and \( Y \). Let \( P \) be an epsilon ball about zero so that \( T(P) \) is a strongly nuclear set. Then, \( T(B_X) \) is also strongly nuclear.

**Proof.** First, let us assume that \( P \) is the closed epsilon ball, with \( \varepsilon > 0 \). Otherwise we could set \( \tilde{P} = \{ x : \| x \| \leq \frac{\varepsilon}{2} \} \) and proceed with the proof using the strongly nuclear set \( T(\tilde{P}) \).

So, \( P \) is closed and
\[ T(P) \subset \left\{ \sum_j \beta_j y_j : \sum_j |\beta_j| \leq 1 \forall j \right\} \]
for some subset \( \{ y_1, y_2, \ldots, y_n, \ldots \} \) of \( Y \) for which \( \sum_j \| y_j \| < \infty \).

Let \( \tilde{y}_j = \frac{y_j}{\varepsilon} \) and let
\[ \tilde{N} = \left\{ \sum_j \tilde{\beta} \tilde{y}_j : \sum_j |\tilde{\beta}| \leq 1 \right\}. \]

We claim that \( T(B_X) \) is a subset of \( \tilde{N} \) and is, thus, strongly nuclear. Let \( y \in T(B_X) \). Then, \( y = T(x) \) for some \( x \in B_X \).

Let \( x(\varepsilon) \) denote the element on the boundary of \( P \) so that \( x = \alpha x(\varepsilon) \), for some scalar \( \alpha \) where (of course) \( |\alpha| \varepsilon \leq 1 \).
Then,

\[ T(x) = \alpha T(x(\varepsilon)) = \alpha \sum_j \beta_j y_j, \quad \text{where} \quad \sum_j |\beta_j| \leq 1 \]

\[ = \sum_j \alpha \varepsilon \beta_j \frac{y_j}{\varepsilon} \]

\[ = \sum_j \alpha \varepsilon \beta_j \tilde{y}_j \]

\[ \in \tilde{N}, \]

since \( \sum_j |\alpha \varepsilon \beta_j| \leq \sum_j |\beta_j| \leq 1 \) and \( \sum_j \| \tilde{y} \| \leq \frac{1}{\varepsilon} \sum_j \| y_j \| < \infty. \)

Thus, \( T(B_X) \subset \tilde{N} \) and we have that \( T(B_X) \) is a strongly nuclear set.

\[ \square \]

We start with the forward direction of the theorem.

Proof. Let \( T \) satisfy the hypotheses of the theorem. By Lemma 5, it suffices to show that \( T(B_X) \) is a strongly nuclear set.

We have that

\[ T(X) \subset \bigcup_k N_k, \]

where each \( N_k \) is strongly nuclear.

Define

\[ N_{k,r} := \{ T(x) \in N_k : \| T(x) \| \geq \frac{\| x \|}{r} \} \]

and

\[ P_{k,r} := \{ x : T(x) \in N_{k,r} \} \]

so that \( B_X \subset \bigcup_{k,r} P_{k,r}. \)

Note that each \( N_{k,r} \) is closed. To see this, fix \( k, r \) and let \( (T(x_n)) \) be a sequence in \( N_{k,r} \) so that \( T(x_n) \to y \) and assume \( y \neq 0. \)
Since
\[ \| x_n \|_r \leq \| T(x_n) \|, \]
we have
\[ \| x_n \| \leq r \| T(x_n) \| \leq r \| y \|. \]
So, \((x_n)\) is bounded. Since \(X\) is reflexive, we know that \((x_n)\) has a weakly convergent subsequence, \((x_{n_k})\), where \(x_{n_k} \rightharpoonup z\) weakly.
So,
\[ T(x_{n_k}) \rightharpoonup T(z) \text{ weakly.} \]
But we also know that
\[ T(x_n) \rightharpoonup y \text{ weakly.} \]
So, we must have \(T(z) = y\).
Moreover,
\[ \| z \| \leq \liminf \| x_{n_k} \| \leq \liminf r \| T(x_{n_k}) \| = r \| y \|. \]
Thus, \(y \in N_{k,r}\) and so \(N_{k,r}\) is closed.
Now, using Baire Category Theorem, there is a \(k\) and an \(r\) so that \(P_{k,r}\) has non-empty interior.
Each \(P_{k,r}\) is closed since \(P_{k,r} = T^{-1}(N_{k,r})\), the inverse image of a closed set. So, we get a \(k\) and an \(r\) so that \(P_{k,r}\) contains an open ball. Thus, the set \(P_{k,r} - P_{k,r}\) contains an open ball about zero, say \(P\), where
\[ T(P) \subset N_{k,r} - N_{k,r} \subset N_k - N_k. \]
Since each \(N_k\) is strongly nuclear, the set \(N_k - N_k\) is nuclear by Lemma 6 and so \(T\) maps a neighborhood of zero into a strongly nuclear set. So, by Lemma 7, \(T\) maps \(B_X\) into a strongly nuclear set. By Lemma 5, \(T\) is a strongly nuclear operator. 
\[ \square \]
We now prove the backwards direction.

**Proof.** We assume $T$ is strongly nuclear. Fix $n$ and let $x \in B(0,n)$. For each $j$, we know there is a $y_j$ so that $T_j(x) = \text{span}_\mathbb{R}\{y_j\}$.

Since $\sum_j \sqrt{\| T_j \|} < \infty$, set $\sum_j \sqrt{\| T_j \|} = L$.

Let
\[
\tilde{y}_j = \frac{n L \sqrt{\| T_j \|}}{\| y_j \|} y_j,
\]
so that
\[
T_j(x) = \text{span}_\mathbb{R}\{\tilde{y}_j\}
\]
and
\[
\sum_j \| \tilde{y}_j \| = n L \sum_j \sqrt{\| T_j \|} < \infty.
\]

For each $j$, there exists $\tilde{\beta}_j$ so that $T_j(x) = \tilde{\beta}_j \tilde{y}_j$.

Moreover, we know that
\[
\| T_j(x) \| \leq \| T_j \| \| x \|.
\]

From this, we get
\[
|\tilde{\beta}_j| n L \sqrt{\| T_j \|} \leq \| T_j \| n,
\]
which simplifies to
\[
|\beta_j| L \leq \sqrt{\| T_j \|}.
\]

Summing over $j$, we get
\[
L \sum_j |\tilde{\beta}_j| \leq \sum_j \sqrt{\| T_j \|},
\]
or
\[
\sum_j |\tilde{\beta}_j| \leq 1.
\]

Thus,
\[
T(x) = \sum_j T_j(x) \in \{ \beta_j \tilde{y}_j : \sum_j |\beta_j| \leq 1 \},
\]
a strongly nuclear set.

We now have that \( T(B(0, n)) \) is strongly nuclear for each \( n \) and since 

\[
T(X) = \bigcup_n T(B(0, n)),
\]

we have our desired result.

\( \square \)

We now discuss how a compact operator may be characterized by its range.

**Definition 16.** A collection \( \mathcal{A} \) of open subsets of a space \( X \) is called an open covering of \( X \) if the union of the elements of \( \mathcal{A} \) is equal to \( X \).

The space \( X \) is said to be compact if every open covering \( \mathcal{A} \) of \( X \) contains a finite subcollection that also covers \( X \).

If every sequence in a space \( X \) has a convergent subsequence, we say that \( Y \) is sequentially compact.

If \( X \) is a metric space, then it is well-known that compactness and sequential compactness are equivalent [16].

**Definition 17.** Let \( T : X \rightarrow Y \) be an operator between Banach spaces \( X \) and \( Y \). We say that \( T \) is a compact operator if \( T(B_X) \) is relatively compact, i.e. \( \overline{T(B_X)} \) is compact in the norm topology of \( Y \) [7].

**Theorem 9.** Let \( T : X \rightarrow Y \) be a continuous, one-to-one linear operator between Banach spaces \( X \) and \( Y \).

Then, \( T \) is a compact operator if and only if \( T(X) \) is contained in a countable union of compact sets.

We will start with the forward direction of the theorem.
Proof. So, we assume that $T$ is a compact operator. Let $B_X$ denote the closed unit ball in $X$. By hypothesis, $\overline{T(B_X)}$ is compact. Now, since

$$T(X) \subset \bigcup_{n=1}^{\infty} n \overline{T(B_X)},$$

we have that $T(X)$ is contained in a countable union of compact sets, as desired. \qed

We now prove the backwards direction of the theorem.

Proof. So, we assume that $T(X)$ is contained in a countable union of compact sets. We must show that $\overline{T(B_X)}$ is compact.

By hypothesis,

$$T(X) \subset \bigcup_{k=1}^{\infty} C_k$$

where each $C_k$ is compact in $Y$.

Define

$$C_{k,r} := \{T(x) \in C_k : \|T(x)\| > \frac{1}{r} \|x\|\}$$

and

$$P_{k,r} := \{x : T(x) \in C_{k,r}\}.$$

We see that

$$B_X \subset \bigcup_{k,r} P_{k,r}.$$

Now, using Baire Category Theorem, we know there is a $k$ and an $r$ so that $\overline{P_{k,r}}$ has nonempty interior. As each $P_{k,r}$ is closed, we conclude that there is a $k$ and an $r$ so that $P_{k,r}$ contains an open ball.

Thus, the set $P_{k,r} - P_{k,r}$ contains an open ball about zero, say $P$, where

$$T(P) \subset C_{k,r} - C_{k,r} \subset C_k - C_k.$$
Since each $C_k$ is compact, the set $C_k - C_k$ is compact and thus the operator $T$ maps a neighborhood of zero into a compact set. Thus, $T$ maps $B_X$ into a compact set. This gives us that $\overline{T(B_X)}$ is a closed subset of a compact set and thus, compact.

This proves the theorem. □

Remark. This proposition uses the fact that the sum of two compact sets is compact. To see this, show that the sum is sequentially compact:

Let $\{a_n + b_n\}$ be a sequence in $A + B$, where each of $A$ and $B$ is sequentially compact. Let $\{a_{n_k}\} \to a$ and $\{b_{n_k}\} \to b$ be convergent subsequences of $\{a_n\}$ and $\{b_n\}$, respectively.

Let $\varepsilon > 0$.

Choose $N_0$ so that $k > N_0$ implies

$$\|a_{n_k} - a\| < \frac{\varepsilon}{2}.$$ 

Choose $N_1$ so that $k > N_1$ implies

$$\|b_{n_k} - b\| < \frac{\varepsilon}{2}.$$ 

Let $N > max\{N_0, N_1\}$.

Then, $k > N$ implies

$$\|(a_{n_k} + b_{n_k}) - (a + b)\| < \varepsilon,$$

as desired.
CHAPTER 4
Hamel Bases

We begin this chapter by investigating a relationship between Hamel bases and operator ranges on a Banach space.

To start, we need some definitions.

**Definition 18.** A nonempty subset, $M$, of a linear space (vector space), $X$, is called a linear manifold in $X$ if $x + y$ is in $M$ whenever $x$ and $y$ are both in $M$ and if also $\alpha x$ is in $M$ whenever $x$ is in $M$ and $\alpha$ is any scalar.

On a historical note, the concept of a linear space (vector space) was introduced by G. Peano already in 1888. It is, in fact, the standard definition that could be used in any lecture today.

**Definition 19.** Let $X$ be a vector space and $U, V$ be subspaces of $X$. We say that $U$ and $V$ span $X$ and write, $X = U + V$, if every $x \in X$ can be expressed as a sum $x = u + v$ for some $u \in U$ and some $v \in V$.

If in addition, such a decomposition is unique for all $x \in X$, or equivalently if $U \cap V = \{0\}$, we say that $U$ and $V$ form a direct sum decomposition of $X$ and write

$$X = U \oplus V.$$ 

In such circumstances, we also say that $V$ is an algebraic complement of $U$. 
**Definition 20.** Let $X$ be a linear space with some nonzero elements. A set $H \subset X$ is called a *Hamel basis* of $X$ if:
1. $H$ is a linearly independent subset of $X$. 2. The linear manifold spanned by $H$ is all of $X$. [22]

Or, equivalently, a family $\{e_i\}_{i \in I}$ is called a *Hamel basis* of $X$ if every element $x \in X$ has a unique representation

$$x = \sum_{i \in I} \beta_i e_i$$

such that the number of non-zero coefficients, $\beta_i$ is finite.

Before we state our result regarding Hamel bases, we stop to make note of one of the workings of Hamel.

Using the well-ordering theorem, he showed that $\mathbb{R}$ has a basis when viewed as a linear space over the rational numbers [11].

Hausdorff then extended the proof to real and complex linear spaces [12].

We will use the following notation:

Let $B$ be a Hamel basis for an infinite-dimensional Banach space and let $b \in B$ be a non-zero element of $B$.

We define

$$B_{-1} := B \setminus \{b\}$$

In addition, by $\text{Span}(X)$, we mean finite linear combinations of elements from the set $X$.

**Theorem 10.** Suppose $\text{Span}(B_{-1})$ is dense in $\text{Span}(B)$. Then, $\text{Span}(B_{-1})$ is not the range of a continuous linear operator on a Banach space.
Proof. Suppose not. And so, we assume there is a continuous, linear operator $T$ and a Banach space $X$ so that

\[ T : X \rightarrow \text{Span}(B_{-1}) \]

is a surjection.

Define

\[ T_b : \text{Span}({b}) \oplus X \rightarrow \text{Span}({b}) \oplus \text{Span}(B_{-1}) \quad (\cong \text{Span}(B)) \]

by

\[ T_b(\alpha b, x) = \alpha b + T(x) . \]

Then, $T_b$ is a surjection between Banach spaces.

Let $\varepsilon > 0$ and let

\[ U = \{ \alpha b : |\alpha| < \frac{\varepsilon}{\|b\|} \} \]

and

\[ V = \{ x \in X : \|x\| < \varepsilon \} , \]

so that $U \oplus V$ is a neighborhood of zero in the domain of $T_b$.

By the Open Mapping Theorem [6], there is some $\delta > 0$ so that

\[ T_b(U \oplus V) \supseteq B(0, \delta) \]

in $\text{Span}(B)$.

Now fix $\alpha$ with $|\alpha| > \frac{2\varepsilon}{\|b\|}$.

Since $\text{Span}(B_{-1})$ is dense in $\text{Span}(B)$, there is an $x \in \text{Span}(B_{-1})$ so that $\|\alpha b - x\| < \delta$.

Thus,

\[ (\alpha b - x) \in B(0, \delta) \subseteq T_b(U \oplus V) . \]

So, there is some $u \in U$ and $v \in V$ so that

\[ T_b(u, v) = \alpha b - x . \]
As $\alpha b$ is not in $\text{Span}(B_{-1})$, we have

$$T_b(u) = \alpha b .$$

Thus, $\alpha b = u \in U$ and we have our contradiction.

In the previous result, we stated that if $B$ is a Hamel Basis for an infinite-dimensional Banach space and $0 \neq b \in B$, so that $\text{Span}(B \setminus \{b\})$ is dense in $\text{Span}(B)$, then $\text{Span}(B \setminus \{b\})$ is not the range of a continuous linear operator on a Banach space.

We now generalize the result.

**Theorem 11.** Let $\tilde{B}$ be an infinite-dimensional Banach space and let $C$ be any non-empty, closed subspace of $\tilde{B}$ whose algebraic complement, $N$, is non-closed. Then, $N$ is not the range of a continuous linear operator on a Banach space.

**Proof.** Suppose not. Then, there is a Banach space, $X$ and a continuous linear operator, $T$, so that $T : X \to N$ is a surjection.

Define

$$\tilde{T} : C \oplus X \to C \oplus N \approx \tilde{B}$$

by

$$\tilde{T}((c, x)) = c + T(x) ,$$

so that $\tilde{T}$ is a surjection between Banach spaces.

Let $\epsilon > 0$ and let $U$ and $V$ be $\epsilon$-neighborhoods of zero in $C$ and $X$, respectively. By the Open Mapping Theorem, there exists $\delta > 0$ so that

$$\tilde{T}(U \oplus V) \supset B(0, \delta)$$
in $C \oplus N \approx \tilde{B}$. Choose $\delta$ with $0 < \delta < 1$.

Case I: There exists $c \in C$ of norm one and an $n \in N$ so that

$$\| c - n \| < \delta .$$

Then, there is a $(c_0, x) \in U \oplus V$ so that

$$\tilde{T}((c_0, x_0))) = c - n$$

and $\| c_0 \| < \delta$, in particular.

Thus, we have

$$c_0 + T(x_0) = c - n .$$

Since $T$ is a surjection, there is an $x \in X$ so that $T(x) = n$ and we get

$$T(x_0 + x) = c - c_0 .$$

Since $C \cap N = \{0\}$, we must have

$$T(x + x_0) = 0 .$$

Thus, $c = c_0$ and we have our contradiction, since $\| c \| = 1$ and $\| c_0 \| < \delta < 1$.

Case II: If $c \in C$ is of norm one, then we have

$$\| c - n \| > \delta$$

for all $n \in N$.

Let $\tilde{b} \in \overline{N}$ and let $(n_j)$ be a sequence in $N$ so that $(n_j) \rightarrow \tilde{b}$.

Fix $c \in C$ and $n \in N$ so that $\tilde{b} = c + n$. Since $T$ is a surjection, there exists $x \in X$ so that $T(x) = n$ and for each $j$, there exists $x_j \in X$ so that $T(x_j) = n_j$.

Thus,

$$T(x_j) \rightarrow c + T(x) .$$
We claim that $c = 0$.

Suppose $c \neq 0$.

Then there exists $K$ so that $j > K$ implies

$$\| T(x_j) - (c + T(x)) \| < \frac{\delta}{2} \| c \|$$

and so

$$\| T\left(\frac{x_j - x}{\| c \|}\right) - \frac{c}{\| c \|} \| < \frac{\delta}{2},$$

which is a contradiction since $\frac{c}{\| c \|}$ is of norm one and

$$T\left(\frac{x_j - x}{\| c \|}\right) \in N.$$

Thus, $c = 0$. But then we have $\tilde{b} = 0 + n = n \in N$, from which we get $\overline{N} = N$, another contradiction as $N$ is non-closed.

Thus, our assumption that $Range(T) = N$ was wrong and this proves the theorem.

$\square$
CHAPnER 5
Volterra Operator

A general Volterra-type operator, \( K \), is defined by \( Kx = y \) where

\[
y(s) = \int_{a}^{s} k(s, t)x(t)dt.
\]

The kernel \( k(s, t) \) is assumed to be continuous in both variables in the triangular region described by \( a \leq t \leq s \leq b \) with \( k(s, t) = 0 \) for all \( (s, t) \in [a, b] \times [a, b] \) with \( t > s \) [22].

In this chapter, we will restrict our attention to the Volterra-type operator, \( V \), where the kernel \( k(s, t) \) is defined by

\[
k(s, t) = \begin{cases} 
0 & \text{if } t > s \\
1 & \text{if } t \leq s
\end{cases},
\]

also known as the anti-derivative operator. Below, we will call \( V \) the Volterra operator. Thus, with \( a = 0 \), we get

\[
V f(s) := \int_{0}^{s} f(t)dt.
\]

As mentioned previously, we will focus on orbits of the Volterra operator. The study of the geometric properties of orbits of operators in general and of the Volterra operator, in particular, is still in its beginning. Before we refer to a recent result regarding the Volterra operator, we need some definitions.

**Definition 21.** Let \( X \) be a Banach space and \( B(X) \) be the set of bounded linear operators from \( X \) into itself. If \( T \in B(X) \), then the orbit of a vector \( x \in X \) is the set \( Orb(T, x) := \{T^n x : n \geq 0\} \). A vector \( x \in X \) is called a hypercyclic vector for \( T \) if \( Orb(T, x) \) is norm dense in \( X \), i.e. for every \( y \in X \) there exists an increasing sequence \( (n_k)_{k \geq 1} \) so that \( \|T^{n_k}x - y\| \to 0 \) as \( k \to \infty \). \( T \) is called a hypercyclic operator if it has
a hypercyclic vector. A vector \( x \in X \) is said to be cyclic for an operator \( T \in B(X) \) if the linear span of \( Orb(T, x) \) is dense in \( X \). An operator \( T \in B(X) \) is cyclic if it has a cyclic vector.

A vector \( x \in X \) is said to be supercyclic for an operator \( T \in B(X) \) if the set
\[
\{ \lambda y : y \in Orb(T, x), \lambda \text{ a scalar} \}
\]

is dense in \( X \). An operator \( T \in B(X) \) is said to be supercyclic if it has a supercyclic vector. It is evident that hypercyclicity implies supercyclicity and this, in turn, implies cyclicity [5].

**Example.** Let \( L_p(0,1), 1 \leq p < \infty \), denote the Banach space of complex measurable functions, \( f \), for which
\[
||f||_{L_p} := \left( \int_0^1 |f(t)|^p dt \right)^{\frac{1}{p}} < \infty.
\]

Then, on any of these spaces, the Volterra operator, \( V \), is cyclic.

To see this, consider the function \( f(t) = 1 \) as the cyclic vector for \( V \).

Then,
\[
Orb(V, f) = \{ f, V f, V^2 f, \ldots \} = \{ 1, t, \frac{t^2}{2}, \ldots, \frac{t^n}{n!}, \ldots \}.
\]

From Lusin’s Theorem [23], we know that \( C(0,1) \) (the set of continuous functions on the interval \([0,1]\)) is dense in \( L_p(0,1), 1 \leq p < \infty \).

The Weierstrass Approximation Theorem states that every continuous function defined on an interval \([a,b]\) can be uniformly approximated, as closely as desired, by a polynomial function. From this, we see that the linear span of \( Orb(V, f) \) is dense. Weierstrass Approximation Theorem also shows that \( V \) is cyclic on \( C(0,1) \) with the same cyclic vector, \( f(t) = 1 \), as above.

The following result was proved independently by F. León-Saavedra and A. Piqueras-Lerena in 2003 [15] and then by E. Gallardo-Gutiérrez and A. Montes-Rodríguez in 2004 [9]. It says that the Volterra operator, although cyclic, is not supercyclic on any of the spaces
$L_p(0, 1) , 1 \leq p < \infty$.

We now move on to discuss our interest in the Volterra operator.

Let $f : [0, 1] \to \mathbb{R}$ be bounded and integrable with $f$ not identically zero. We will focus on the orbit, $\{f, Vf, V^2f, V^3f, \ldots\}$, of $V$. Specifically, we look at how the norms of three consecutive elements vary.

One motivation for such a study is the following: Let $f(t) = 1$. It is obvious that, in $C(0, 1)$, the norm of $V^n f$ is attained and is equal to $V^n f(1) = \frac{1}{n!}$. From this, it follows that the norms of the elements of the orbit decrease faster than exponential as $n$ approaches infinity.

It also follows that "the decrease of the decrease" of the norms slow down in the sense that

$$\frac{\|V^{n+1}f\|}{\|V^nf\|} \to 1 \text{ if } f(t) = 1 .$$

This happens for many other functions as well.

Here, we will study how small the ratio

$$\frac{\|Vf\|}{\|V^2f\|}$$

can be for the Volterra operator on $C(0, 1)$.

We have the following theorems:

**Theorem 12.** Let $f \geq 0$ on $[0, 1]$ be bounded and integrable but not identically zero.

Then,

$$\frac{\|Vf\|}{\|V^2f\|} \geq \frac{1}{2} .$$

**Proof.** Set $g := \frac{Vf}{\|Vf\|}$ so that $\|g\| = 1$. Since $g > 0$ on $[0, 1]$, we have that $Vg > 0$ and is increasing.

Thus,

$$\|Vg\| = Vg(1) := a \text{ with } 0 < a \leq 1 .$$
Let
\[
h(t) = \begin{cases} 
0 & \text{if } 0 \geq t < 1 - a \\
 t - (1 - a) & \text{if } 1 - a \leq t \leq 1 
\end{cases}.
\]

**Claim 1.** For $0 \leq t \leq 1$,
\[
h(t) \leq Vg(t).
\]

**Proof.** The claim is clearly true for $0 \leq t \leq 1 - a$.

Fix $t_0$ with $1 - a < t_0 \leq 1$.

Suppose
\[
h(t_0) > Vg(t_0).
\]

Then, by the mean value theorem, there exists a $t_1$ between $t_0$ and 1 so that
\[
(Vg)'(t_1) = \frac{a - Vg(t_0)}{1 - t_0} > \frac{a - h(t_0)}{1 - t_0} = 1,
\]
a contradiction, since $(Vg)' = g$ and $\|g\| = 1$.

Now,
\[
\|V^2g\| = \int_0^1 Vg(t)dt \geq \int_0^1 h(t)dt = \frac{1}{2}.
\]

Thus,
\[
\frac{\|g\|\|V^2g\|}{\|Vg\|^2} \geq \frac{\|g\|^2}{a^2} = \frac{1}{2}
\]
and so
\[
\frac{\|Vf\|\|V^3f\|}{\|V^2f\|^2} \geq \frac{1}{2}.
\]

\qed
To show that the constant $\frac{1}{2}$ from the last theorem is sharp, we consider the following example:

Let

$$g(t) := \begin{cases} 
0 & \text{if } 0 \leq t \leq \frac{1}{2} \\
1 & \text{if } \frac{1}{2} < t \leq 1
\end{cases}.$$

Then,

$$||g|| = 1 \quad ||Vg|| = \frac{1}{2}, \quad ||V^2g|| = \frac{1}{8}$$

and so

$$\frac{||g||||V^2g||}{||Vg||^2} = \frac{1}{2}.$$

Now choose $f$ so that $||Vf - g||$ is arbitrarily small. Then, we get

$$\frac{||Vf||||V^3f||}{||V^2f||^2}$$

arbitrarily close to $\frac{1}{2}$.

**Corollary 1.** Let $f \leq 0$ be bounded and integrable.

Then,

$$\frac{||Vf||}{||V^2f||} \geq \frac{1}{2}. $$

**Proof.** By changing signs, this follows from Theorem 12.

It is natural to ask if it is possible to find an $f$ for which

$$0 < \frac{||Vf||}{||V^2f||} < \frac{1}{2},$$

if we allow $f$ to change signs.
The following example shows that this is, indeed, possible.

*Example.*

Fix $0 < a \leq 1$

Let

$$g(t) = \begin{cases} 
-1 & \text{if } 0 \leq t \leq \frac{1-a}{2} \\
1 & \text{if } \frac{1-a}{2} < t \leq 1 
\end{cases}.$$  

Then,

$$Vg(t) = \begin{cases} 
-t & \text{if } 0 \leq t \leq \frac{1-a}{2} \\
-(t - (1-a)) & \text{if } \frac{1-a}{2} < t \leq 1 
\end{cases}.$$  

Notice that $\| Vg \|$ is the larger of $a$ and $\frac{1-a}{2}$.

Thus,

$$\| Vg \| = \begin{cases} 
\frac{1-a}{2} & \text{if } 0 < a < \frac{1}{3} \\
a & \text{if } a \geq \frac{1}{3} 
\end{cases}.$$  

Now, if

$$\frac{1}{2}a^2 < 2\frac{(1-a)^2}{4}$$

(i.e. $a^2 < (1-a)^2$), then $\| V^2g \| = \frac{(1-a)^2}{4}$ and we have

$$\frac{\| g \| \| V^2g \|}{\| Vg \|^2} = \frac{1}{4} \frac{(1-a)^2}{a^2} \geq \frac{1}{4}.$$  

Now, on the other hand, suppose that

$$\frac{1}{2}a^2 \geq 2\frac{(1-a)^2}{4}$$

(i.e. $a^2 \geq (1-a)^2$).

Then, we get that

$$\| V^2g \| = \frac{a^2}{2} - \frac{(1-a)^2}{4}$$

and so

$$\frac{\| g \| \| V^2g \|}{\| Vg \|^2} = \frac{1}{4} \frac{(a^2 - (1-a)^2)}{a^2} = \frac{1}{2} - \frac{1}{4} \frac{(1-a)^2}{a^2} \geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$
Note that \( a^2 \geq (1-a)^2 \) implies \( a \geq \frac{1}{2} \) and thus, \( \| Vg \| = a \).

This inequality is sharp as we achieve

\[
\frac{\| g \| \| V^2 g \|}{\| Vg \|^2} = \frac{1}{4}
\]

in the case \( a = \frac{1}{2} \).

Now, with \( a = \frac{1}{2} \) and \( g \) as above, choose \( f \) so that \( \| Vf - g \| \) is arbitrarily small.

Then, we get the quotient

\[
\frac{\| Vf \| \| V^3 f \|}{\| V^2 f \|^2}
\]

arbitrarily close to \( \frac{1}{4} \) and this, certainly, proves sharpness.
CHAPTER 6

Conclusion and Future Projects

When this process began, we knew that porosity, ranges of operators and orbits of operators in Banach spaces (and more general spaces) were topics in which, although there is a growing interest, have not been studied extensively. Our primary goal was to build a theory which addressed some aspects of these topics in a more systematic way. What has been done in this thesis is a start to build such a theory, in Banach spaces and also in the framework of more general spaces.

In Chapter 1, we studied the notion of porous and $\sigma$-porous sets within Banach spaces and, more generally, $p$-Banach spaces. Porosity has become important as a "smallness" concept. The fact that, in $\mathbb{R}^n$, $\sigma$-porous implies both Baire first category and measure zero is a powerful reason for this. This led us to study the connection between porosity and operator ranges. In particular, we accomplished to generalize a result of Olevskii’s (and his result was a strengthening of a theorem of Banach). Olevskii’s result essentially states that if $T$ is a continuous linear operator between Banach spaces $X$ and $Y$ where $T(X) = Y$ but $T(X) \neq Y$, then $T(X)$ is strongly $\sigma$-porous. Our theorem generalized this result to $p$-Banach spaces $X$ and $Y$ and our proof uses a technique different from Olevskii’s.

For the future, there are many things that can be done in this area as it is a relatively new field. My first project will be the following:

Our above mentioned theorem assumes that $X$ and $Y$ are $p$-Banach spaces, for the same value of $p$. I will study if the theorem still holds when $X$ is a $p_1$-Banach space and $Y$ is a $p_2$-Banach space where $p_1 \neq p_2$. I will study it for other classes of spaces as well.
In Chapter 2, we studied, in a geometric setting, strongly porous subsets of $\mathbb{R}^2$ and gave a new description for a large class of them. A future project here, as mentioned in Chapter 2, would be to attempt to generalize Theorem 4 to $\mathbb{R}^n$ and possibly infinite-dimensional spaces. We also gave results regarding $\delta$-nets, some linking $\delta$-nets to the notion to porosity. On a personal note, this is an area I potentially look forward to directing some undergraduate research in, as little background is required for these concepts.

In Chapter 3, our goal was to characterize some classes of operators by their ranges. We achieved this goal for nuclear operators, strongly nuclear operators and compact operators. Future works will involve seeking more operators for which we can do the same. In addition, for any of our results, we will study if it remains true in spaces other than Banach spaces, for instance in a $p$-Banach space.

In Chapter 4, we continued our studies of operator ranges. Specifically, we were able to show that if $N$ is a non-closed algebraic complement of a non-empty closed subspace in an infinite-dimensional Banach space, then $N$ is not the range of a continuous linear operator on a Banach space. We achieved providing an example of such a space, which cannot be the range of a continuous linear operator on a Banach space, but there is a long way to go to get a better understanding of which types of sub-manifolds can be ranges of operators.

In Chapter 5, our goal was to study some aspects of orbits of the classical Volterra operator, $V$, in the space $C(0,1)$. In particular we studied the variance in norms of three consecutive elements from the orbit $\{f, Vf, V^2f, V^3f, \ldots \}$. 
We wanted to minimize
\[ \frac{\|V f\|}{\|V^2 f\|} \]
for certain functions, \( f \), and achieved this goal. The next step will be to work with a general function, \( f \). For the future, we plan to investigate how the norms of the elements vary between \( n \) consecutive elements. Since \( V \) is an operator in many different spaces, it would be interesting to ask our same questions for \( V \) in spaces other than \( C(0,1) \).


