STOPPING TIMES RELATED TO TRADING STRATEGIES

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CHAPTER 1

Trading the Line Strategy in Discrete Time

1.1 Introduction

When one enters into a long (short) investment position the expectation is that the value of the asset will rise (fall). After making the entry decision the most critical decision deals with the setting of an exit position. When the possession of exact future information is ruled out, it will be difficult to scientifically justify the optimality of any emotionally derived exit decisions. Various trading strategies, such as “stop loss” and “trading the line” (also called “trailing stops”) strategy, have been devised to minimize losses (or lock in profits), if the market suddenly moves against an open position, [5], [10], [9], [12], [1], [18]. Although the literature on the use of the trailing stops strategy is voluminous, comparatively little is available in terms of its theoretical properties and analysis. This is partly due to the fact that the mathematical models quickly become intractable, even in the simplest price scenarios. In the discrete time settings, among a few exceptions are the papers [11], [10], where the trailing stops strategy is analyzed for two types of models for the underlying security, and [20] where the stop-loss strategy is studied.

One of the aims of this work is to provide a probabilistic framework by which an analytic analysis can be performed for the trailing stops strategy under various price dynamics. This, in part, is done by making a connection to the cumulative sum (cusum) procedure of quality control. Although the cusum procedure has been studied quite extensively, [8], [14], [15], its link with the trailing stops strategy has been mostly untapped. The recent article [17], provides some theoretical results that can be used to derive fundamental properties, in closed form, for the trailing stops strategy under various price dynamics for both long and
short positions.

Glynn and Iglehart [10] studied the trailing stops strategy when the price process is a geometric random walk generated by binomial or double exponential random variables, in the discrete case, and for the geometric Brownian motion for the continuous case. In this strategy for the long position, one sets an amount, \( h \), by which the log price needs to fall, compared to the past maximum, to trigger immediate selling of the investment. Should the price rise above the historic maximum, the new maximum becomes the current benchmark. There are several issues that require analysis. Among the basic issues are:

- The expected trading duration, once one enters into a short (long) position, called the expected first passage time, or average run length of the strategy. Symbolically, if \( N_h \) is the duration of the short position of the trailing stop strategy, obtain its expected duration, \( E(N_h) \).
- The variance of the trading duration of the strategy, \( Var(N_h) \).
- The expected level of the price at the exit time, called the stopped value, i.e., \( E(S_{N_h}) \).
- The variance of the stopped value, \( Var(S_{N_h}) \).
- The correlation between the stopped amount and the duration of the trading strategy, \( Corr(N_h, S_{N_h}) \).
- The sensitivity of the above parameters to the trailing stops triggering level, \( h \).
- The sensitivity of the above parameters to the various price dynamics.

Several criteria for the choice of the triggering constant, \( h \), are, for instance, presented in [10], therefore, we will not consider this aspect of the strategy. Historically, the continuous time settings have been used to study the properties of stopping times along with the stopped processes, and the results are often implicit in terms of the solutions of ordinary
differential equations, [7]. The second aim of this work is, to show that the discrete time analysis, in contrast to continuous time analysis, can provide exact and tractable closed form solutions, at least for certain types of price dynamics, which we specify in the paper.

In the next section we provide an analytic analysis of the trailing stops strategy under various price processes. We provide an exact correlation formula between the stopping time and the stopped amount. The formula depends on the expected stopping time and the variance of the stopping time. In particular, we are able to provide the average run length and the variance of the stopping time in closed form for some price dynamics, such as the binomial, trinomial, geometric and exponential models where the inter-observation duration could be chosen to be as small as practically possible. We should also point out that, for log normal processes, a closed form expression for the expected stopping time is still an open problem. In section three specific price scenarios are analyzed. The last section summarizes the results.

1.2 Analysis of Trading the Line Strategy in Discrete Time

Let \( Y_1, Y_2, \ldots \), be independent and identically distributed random variables representing the returns of the underlying security. Throughout we will assume that \( 0 < \text{Var}(Y_1) < \infty \). The random variables \( Y_i \) could be discrete or continuous. Let \( S_n = Y_1 + \cdots + Y_n \), and let \( P_n = P_0 e^{S_n}, n \geq 1, \) be the price process, where \( P_0 \) is a constant. The trading the line strategy can be defined in several ways. For instance, by using the log of the price process, \( S_n \), for the long position, we may define the stopping rule

\[
L_h := \inf \left\{ n \geq 1 : \max_{k \leq n} S_k - S_n \geq h \right\},
\]

where \( h > 0 \) is the specified amount by which the log-price has to drop to trigger sellout. The corresponding trailing stops strategy for the short position is

\[
N_h := \inf \left\{ n \geq 1 : S_n - \min_{k \leq n} S_k \geq h \right\},
\]
where $h > 0$ is the specified amount by which the log-price has to rise to trigger closing the position. Of course, the triggering constant, $h$, need not be the same for the two strategies.

Instead of using the log-price process, one may use the price process itself, $P_n = P_0 \exp(S_n)$, to define another stopping rule

$$T_h := \inf \left\{ n \geq 1 : \max_{k \leq n} P_k - P_n \geq h \right\}.$$ 

Let us denote by $M_n = \max_{k \leq n} S_n$. Then we can rewrite

$$\max_{k \leq n} P_k - P_n \geq h, \quad \text{as} \quad P_0 e^{M_n} - P_0 e^{S_n} \geq h.$$ 

Divide both sides by $P_0 e^{M_n}$ and then simplifying gives

$$M_n - S_n \geq \log \left( \left[ \frac{P_0 e^{M_n} - h}{P_0 e^{M_n}} \right]^{-1} \right).$$ 

Using the fact that $P_0 e^{M_n} = \max_{k \leq n} P_k$ and simplifying we finally get

$$M_n - S_n \geq \log \left( \frac{\max_{k \leq n} P_k}{\max_{k \leq n} P_k - h} \right).$$ 

So, we can rewrite the stopping time $T_h$ in the following equivalent form

$$T_h := \inf \left\{ n \geq 1 : \max_{k \leq n} S_k - S_n \geq \Lambda_n \right\},$$

where $\Lambda_n = \log(\max_{k \leq n} P_k/(\max_{k \leq n} P_k - h))$. Although this form resembles to $L_h$, since $\Lambda_n$ is random, mathematically it leads to a different and less tractable problem. Due to this reason, our main focus will be on the stopping rules $L_h$ and $N_h$, which are analytically more tractable. Furthermore, due to the fact that the stopping rules $N_h$ and $L_h$ are related, one does not need to study both separately. Indeed, by taking $Z_i = -Y_i$ in $N_h$ we get the stopping rule $L_h$. Hence, throughout, our primary focus will be on deriving results for $N_h$ only. The corresponding results for $L_h$ can be deduced by the transformation $Z_i = -Y_i$.

For the sake of comparison, we start off by considering a trading strategy, $\nu$, that is independent of the log price process, $\{S_n, n \geq 0\}$. The following result shows how the correlation of $S_\nu$ and $\nu$ is dependent on the basic statistics of the stopping rule.
Proposition 1.2.1. Let $Y_1, Y_2, \cdots$ be any sequence of independent and identically distributed random variables with $E(Y_i^2) < \infty$. If $\nu$ is any stopping rule independent of the log return process, $Y_1, Y_2, \cdots$, having positive finite variance, and $S_n = Y_1 + \cdots + Y_n$, then the correlation between $\nu$ and $S_\nu$ is,

$$\text{Corr}(\nu, S_\nu) = \frac{E(Y_1)}{\sqrt{E(Y_1^2) + \text{Var}(Y_1)E(\nu)/\text{Var}(\nu)}}.$$ 

The proof of this, and the following, proposition can be deduced by Wald’s equation or the theorem of total expectation [2]. We omit the simple details.

Proposition 1.2.2. Let $P_n, n \geq 1$, be the price process governed by geometric random walk, $P_n = P_0e^{S_n}$, where $S_n = Y_1 + \cdots + Y_n$, and let $Y_1, Y_2, \cdots$ be a sequence of independent and identically distributed random variables with moment generating function $\phi(\theta) = E(e^{\theta Y_1})$. If $\nu$ is any trading strategy that is independent of the log return process, $Y_1, Y_2, \cdots$, then the mean and the variance of the time discounted gain/loss are as follows.

$$E(P_0 - e^{-r\nu} P_\nu) = P_0 \left(1 - E(e^{-(r-s)\nu})\right),$$

$$\text{Var}(P_0 - e^{-r\nu} P_\nu) = P_0^2 \left\{ E\left(e^{-(2r-\alpha)\nu}\right) - \left(E\left(e^{-(r-s)\nu}\right)\right)^2 \right\},$$

where $s = \ln \phi(1)$, $\alpha = \ln \phi(2)$ and $r$ is the discounting rate.

The above two propositions show that the key statistics are $E(\nu)$, $\text{Var}(\nu)$ and the Laplace transform $E(e^{-sr})$. The aim of this paper is to consider the trading strategy, $N_h$, which is adapted to the filtration $\mathcal{F}_n = \sigma(Y_1, Y_2, \cdots, Y_n)$, $n \geq 1$. In other words, the decision to stop at time $n$ is purely determined by the information available up to time $n$, and not any information that would come in the future. Once again, the key statistics that are needed are the average amount of time the position will remain open, $E(N_h)$, the variance of the time the position will remain open, $\text{Var}(N_h)$, and the Laplace transform $E(e^{-sN_h})$, for appropriate values of $s$. In order to study the properties of the short position trailing stops, $N_h$, and the corresponding stopped log price, $S_{N_h}$, and the time discounted gain/loss
The following proposition explains how a double boundary crossing stopping rule, called the SPRT, plays an important role,

$$\tau_{a,h} := \inf \{ n \geq 1 : S_n \notin (a, h) \},$$

where $a < 0 < h$. The following result concerning $\tau_{a,h}$ is from [17].

**Proposition 1.2.3.** The stopping rules $N_h$ and $\tau_{0,h}$ are related through the following identity.

$$N_h = \tau_{0,h} + N_{h,\tau_{0,h}}I(S_{\tau_{0,h}} \leq 0),$$

where $N_{h,\tau_{0,h}}$ is an identical copy of $N_h$ which is independent of $S_{\tau_{0,h}}$ when $\tau_{0,h}$ is given. Here and elsewhere $I(A)$ is the indicator random variable taking value one when $A$ occurs and zero otherwise.

With the help of this link we easily deduce the Laplace transform of $N_h$ in terms of the Laplace transform of the SPRT. Therefore the statistics of $N_h$ are, in turn, linked to the statistics of the SPRT.

**Proposition 1.2.4.** For any $s > 0$, we have

$$E(e^{-sN_h}) = \frac{E(e^{-s\tau_{0,h}}) - E(e^{-s\tau_{0,h}}I(S_{\tau_{0,h}} \leq 0))}{1 - E(e^{-s\tau_{0,h}}I(S_{\tau_{0,h}} \leq 0))}.$$
• (A3). There exist functions $K(\theta), k(x), R(\theta), g(x)$, so that for $\theta = 0, \theta_1, \theta_2$,

$$H(x) := E(e^{\theta Y_1} I(Y_1 \leq x)) = K(\theta)k(x)e^{\theta x}, \quad x < 0,$$

$$G(x) := E(e^{\theta Y_1} I(Y_1 \geq x)) = R(\theta)g(x)e^{\theta x}, \quad x > 0.$$  

Here, in the notation $H(x), G(x)$, we suppress the dependence on $\theta$ and $s$.

As examples, the binomial, trinomial, two-sided geometric and double exponential models, for the log-returns, obey the above sets of conditions. Under the above mentioned assumptions, the following two theorems are the main results that give the probabilistic properties of the trailing stops strategy for the short position. Analogous results can be obtained, by replacing $Y_i$ by $-Y_i$, for the trailing stops strategy in the long position, we therefore omit the straight forward details.

**Theorem 1.2.1. (Nondegenerate case)** Under the above mentioned assumptions, when $K(\theta), R(\theta)$ are not constant functions (non-degenerate case), the short position trailing stops strategy, $N = N_h$, has the following Laplace transform, mean and variance.

$$E(e^{-sN}) = \frac{R(0)\{K(\theta_1) - K(\theta_2)\}}{\{K(\theta_1) - K(0)\}R(\theta_2)e^{\theta_2 h}} - \frac{\{K(\theta_2) - K(0)\}R(\theta_1)e^{\theta_1 h}}{\{K(\theta_2) - K(0)\}R(\theta_1)e^{\theta_1 h}}, \quad s > 0,$$

$$E(N) = \frac{1}{E(Y)} \left\{ h + \frac{R'(0)}{R(0)} - \frac{K'(0)\{R(0) - R(\theta^*)e^{\theta^* h}\}}{R(0)\{K(0) - K(\theta^*)\}} \right\}, \quad E(Y) \neq 0,$$

where $\theta_1(s) \to 0$ and $\theta_2(s) \to \theta^*$ as $s \to 0$. When $E(Y) = 0$, we get

$$E(N) = \frac{1}{Var(Y)} \left\{ h^2 + h \frac{2R'(0)K'(0) - R(0)K''(0)}{R(0)K'(0)} + \frac{K'(0)R''(0) - R'(0)K''(0)}{R(0)K'(0)} \right\}.$$  

Furthermore, when $\theta_1'(s) + \theta_2'(s) = 0$ in a neighborhood of zero, and $E(Y) \neq 0$, the variance simplifies to

$$Var(N) = \frac{1}{(E(Y))^2} \left\{ Var(Y)E(N) + 2(E(Y)E(N) - h) \frac{K'(0) + K'(\theta^*)}{K(0) - K(\theta^*)} \right.$$  

$$+ (E(Y)E(N) - h)^2 + \frac{K''(0)\{R(0) - R(\theta^*)e^{\theta^* h}\}}{R(0)\{K(0) - K(\theta^*)\}} - \frac{R''(0)}{R(0)}$$  

$$- 2 \frac{K'(\theta^*)R'(0) - K'(0)R'(\theta^*)e^{\theta^* h}}{R(0)\{K(0) - K(\theta^*)\}} + 4h \frac{K'(0)R(\theta^*)e^{\theta^* h}}{R(0)\{K(0) - K(\theta^*)\}} \right\}.$$
**Theorem 1.2.2. (Degenerate case)** When $Y$ is an integer valued random variable and $K(\theta), R(\theta)$ are constants, the trailing stops strategy for the short position, $N = N_h$, has the following Laplace transform, mean and variance

$$E(e^{-sN}) = \frac{e^{\theta_2} - e^{\theta_1}}{(1 - e^{\theta_1})e^{\theta_2(h+1)} - (1 - e^{\theta_2})e^{\theta_1(h+1)}}, \quad s > 0,$$

$$E(N) = \frac{1}{E(Y)} \left\{ h + \frac{1 - e^{\theta^* h}}{1 - e^{-\theta^*}} \right\}, \quad E(Y) \neq 0,$$

$$E(N) = \frac{h(h+1)}{Var(Y)}, \quad E(Y) = 0.$$

Furthermore, when $\theta_1'(s) + \theta_2'(s) = 0$ in a neighborhood of zero, where $\theta_1(s) \to 0$ and $\theta_2(s) \to \theta^*$ as $s \to 0$, and $E(Y) \neq 0$, the variance simplifies to

$$Var(N) = \frac{Var(Y)}{(E(Y))^3} \left\{ h + \frac{1 - e^{\theta^* h}}{1 - e^{-\theta^*}} + \frac{\{e^{\theta^* (h+1)} + 3\}\{e^{\theta^* h} - 1\}}{(E(Y))^2(1 - e^{-\theta^*})(e^{\theta^*} - 1)} + \frac{4he^{\theta^* (h+1)}}{(E(Y))^2(1 - e^{-\theta^*})}. \right.$$  

We should point out that the last term of the expression for $Var(N)$ in [17] has a minor typographical error, where $1 - (2h + 1)e^{\theta^*(h+1)}$ should be replaced by $-2he^{\theta^*(h+1)}$. Using these results, we may compute the exact expressions for the various statistics of the duration of the open position, $N_h$, and the stopped log price $S_{N_h}$. The following proposition gives the correlation.

**Proposition 1.2.5.** Let $Y_1, Y_2, \cdots$ be any sequence of independent and identically distributed random variables with $0 < E(Y_1^2) < \infty, E(Y_1) \neq 0$ and let $S_n = Y_1 + \cdots + Y_n$. If $\nu$ is any stopping rule adapted to the filtration $\sigma(Y_1, Y_2, \cdots, Y_n), n \geq 1$, having finite positive variance, then

$$Corr(\nu, S_\nu) = \frac{Var(S_\nu) + (E(Y_1))^2Var(\nu) - Var(Y_1)E(\nu)}{2E(Y_1)\sqrt{Var(\nu)Var(S_\nu)}}.$$

**Proof.** Let $\mu = E(Y_1)$ and $\sigma^2 = Var(Y_1)$. Wald’s equation applied to the martingale $(S_n - n\mu)^2 - n\sigma^2$ gives that

$$E(\nu S_\nu) = \frac{1}{2\mu} \left( E(S_\nu^2) + \mu^2 E(\nu^2) - \sigma^2 E(\nu) \right).$$

Using these results gives the desired expression for the correlation. \qed
In the next section we illustrate the above results with several examples.

1.3 Trailing Stop Strategy for Various Price Dynamics

In this section we provide some results under various price scenarios for the log return process. Only in the first two examples we were able to obtain $\text{Var}(S_{N_h})$ when $E(Y_1) \neq 0$. Therefore, by Proposition (1.2.5), for these two examples we were able to find closed form expressions for $\text{Corr}(N_h, S_{N_h})$. When $\mu = E(Y_1) = 0$, since $S_n^2 - n\sigma^2$ is a martingale, we have $\text{Var}(S_{N_h}) = E(S_{N_h}^2) = \sigma^2 E(N_h)$. In all the examples presented in the this section, $\text{Var}(S_{N_h})$ is computed this way whenever $E(Y_1) = 0$. It should be noted that when $Y_i$ are integer valued random variables, the constant $h > 0$ need only take positive integer values. This is because the stopping rule does not change if the non-integer value of $h$ is replaced by the least integer greater than or equal to $h$.

Example 1.3.1. (Binomial model) Consider the typical binomial model in which the log returns, $Y_i$, can take two possible values, $-1, 1$, with respective probabilities $q, p = 1 - q$. In this example we can obtain exact results. Cox, Ross, Rubinstein [6] used the binomial model to give a simplified approach to pricing options. Later Glynn and Iglehart [10] studied this model in the context of a long position of the trailing stops strategy with positive drift, $E(Y) = p - q > 0$. They computed $E(L_h), E(S_{L_h})$ and $\text{Var}(S_{L_h})$ when $L_h$ is the trailing stops strategy for the long position. To compute the correlation, $\text{Corr}(N_h, S_{N_h})$, we use the results of the last section.

For a given $s > 0$, we may solve $\phi(\theta) = e^s$, and get two solutions for $\theta$.

$$
e^{\theta_1} = e^s - \frac{e^{2s} - 4p(1-p)}{2p}, \quad e^{\theta_2} = e^s + \frac{e^{2s} - 4p(1-p)}{2p}.$$

Note that $\theta_1' + \theta_2' = 0$. When $p < \frac{1}{2}$, we see that $\theta_1 \rightarrow 0$ and $\theta_2 \rightarrow \theta^* = \ln(q/p)$. The Laplace transform is obtained from Theorem (1.2.2).

$$E(e^{-sN_h}) = \frac{e^{\theta_2} - e^{\theta_1}}{(1 - e^{\theta_1})e^{\theta_1(h+1)} - (1 - e^{\theta_2})e^{\theta_1(h+1)}}, \quad s > 0.$$
The Laplace transform completely characterizes the probability distribution of the duration $N_h$. In principle the probability distribution can be obtained by the standard inversion theory of Laplace transforms. The moments of the trading strategy can be obtained via differentiation. For instance,

$$E(N_h) = -\frac{d}{ds}E(e^{-sN_h})\bigg|_{s=0}.$$ 

Higher moments are obtained similarly by taking higher order derivatives of the Laplace transform and then inserting $s = 0$. After omitting the simple, but a substantial amount of algebra, the following moments of the trading strategy can be deduced from the above stated Laplace transform.

$$E(N_h) = \frac{h}{p-q} + \frac{q}{(p-q)^2}\{(q/p)^h - 1\}, \quad p \neq \frac{1}{2};$$

$$E(N_h) = h(h+1), \quad p = \frac{1}{2};$$

$$Var(N_h) = \frac{4p(1-p)}{(2p-1)^3}\left\{h + \frac{1 - ((1-p)/p)^h}{1 - (p/(1-p))}\right\} + \frac{4ph((1-p)/p)^{h+1}}{(2p-1)^3}$$

$$+ \frac{p(1-p)\{(1-p)/p\}^{h+1} + 3\{(1-p)/p\}^h - 1}{(2p-1)^4}, \quad p < \frac{1}{2};$$

$$Var(N_h) = \frac{h(h+1)}{3}(2h^2 + 2h - 1), \quad p \to \frac{1}{2}.$$ 

By Wald’s equation, we see that

$$E(S_{N_h}) = E(Y_1)E(N_h) = h - \frac{1-p}{2p-1}\{1 - ((1-p)/p)^h\}, \quad p \neq \frac{1}{2};$$

$$E(S_{N_h}) = 0, \quad p = \frac{1}{2};$$

$$Var(S_{N_h}) = \frac{1-p}{2p-1}\left\{1 - ((1-p)/p)^h + (1-p)\frac{((1-p)/p)^h - 1)^2}{2p-1}\right\}, \quad p \neq \frac{1}{2};$$

$$Var(S_{N_h}) = h(h+1) = E(N_h), \quad p = \frac{1}{2};$$

$$Corr(N_h, S_{N_h}) = \frac{Var(S_{N_h}) - E(N_h)Var(Y_1) + (E(Y_1))^2Var(N_h)}{2E(Y_1)\sqrt{Var(N_h)Var(S_{N_h})}},$$

$$Corr(N_h, S_{N_h}) = \frac{-(2h+1)}{\sqrt{3}\sqrt{2h(h+1)-1}}, \quad p \to \frac{1}{2}.$$
The last expression shows that the correlation between $N_h$ and $S_{N_h}$ is negative when $E(Y_1) = 0$, indicating that the longer the short position remains open, the lower the log price is going to be at the stopping time. In a short position the lower $S_{N_h}$ is the less the loss will be. The correlation of $N_h$ with $S_{N_h}$, as a function of the expected duration, $E(N_h)$, is shown in Figure 1.1 in semi-logarithmic scale. The vertical scale shows only four decimal place accuracy.

The curve going through the circled data points, $(E(N_h), Var(N_h))$, of Figure 1.2 shows that the variance $Var(N_h)$ seems to have a parabolic relationship with $E(N_h)$ regardless of the choice of $p$. This turns out to be not totally correct. More appropriately, when $p$ is
Figure 1.2: Mean-Variance Curve for Trailing Stop, Binomial Model.
near $\frac{1}{2}$, there is indeed a parabolic relationship, namely

$$\text{Var}(N_h) = \frac{1}{3} (2E(N_h))^2 - E(N_h), \quad p \to \frac{1}{2}.$$ 

However, for $p$ in the neighborhood of zero, there is another parabolic relationship between $\text{Var}(N_h)$ and $E(N_h)$,

$$\text{Var}(N_h) = (E(N_h))^2 + E(N_h) \{1 - 2h\} - 3h(h + 1), \quad p \to 0^+.$$ 

As the figure suggests, for local regions of $p \in (0, \frac{1}{2})$, a good parabolic relationship may be found between $E(N_h)$ and $\text{Var}(N_h)$. The header of each figure gives the best parabolic fit for that region of $p$.

**Example 1.3.2. (Trinomial model)** Consider the trinomial model where the log return, $Y_1$, can take three possible values, $-1, 1, 0$, with respective probabilities $q, p, 1 - p - q$. The trinomial model was studied by Boyle [3] in the context of option valuation. He used the model to price both European and American options. The trailing stops strategy can be analytically analyzed for the trinomial model as well. For a given $s > 0$, we may solve $\phi(\theta) = e^s$, and get two solutions for $\theta$.

$$e^{\theta_1} = \frac{(e^s + p + q - 1) - \sqrt{(e^s + p + q - 1)^2 - 4pq}}{2p},$$

$$e^{\theta_2} = \frac{(e^s + p + q - 1) + \sqrt{(e^s + p + q - 1)^2 - 4pq}}{2p}.$$ 

Note that $\theta'_1 + \theta'_2 = 0$. When $p < q$, we see that $\theta_1 \to 0$ and $\theta_2 \to \theta^* = \ln(q/p)$. Since both $K(\theta) = p$ and $R(\theta) = q$ are constant functions, and $Y$ is an integer valued random variable, Theorem (1.2.2) gives

$$E(e^{-sN_h}) = \frac{e^{\theta_2} - e^{\theta_1}}{(1 - e^{\theta_1})e^{\theta_2(h+1)} - (1 - e^{\theta_2})e^{\theta_1(h+1)}}, \quad s > 0,$$

$$E(N_h) = \frac{h}{p - q} - \frac{q}{(p - q)^2} \{1 - (q/p)^h\}, \quad p \neq q,$$

$$E(N_h) = \frac{h(h + 1)}{2p}, \quad p = q,$$
\[
\text{Var}(N_h) = \frac{p(1-p) + q(1-q) + 2pq \left\{ h + \frac{1 - (q/p)^h}{1 - (p/q)} \right\}}{(p-q)^3} + \frac{pq\{(q/p)^{h+1} + 3\{(q/p)^h - 1\}}{(p-q)^4} + \frac{4ph(q/p)^{h+1}}{(p-q)^3}, \quad p < q,
\]

\[
\text{Var}(N_h) = \frac{h(h+1)(h(h+1) + 1 - 3p)}{6p^2}, \quad p = q.
\]

In this example, \( \text{Var}(S_{N_h}) \) also follows from a simple modification of the argument of Glynn and Iglehart [10]. By Wald’s identity, we see that

\[
E(S_{N_h}) = E(Y_1)E(N_h) = h - \frac{q}{(p-q)}\{(1 - (q/p)^h)\}, \quad p \neq q,
\]

\[
E(S_{N_h}) = 0, \quad p = q,
\]

\[
\text{Var}(S_{N_h}) = \frac{1 - A}{A^2}, \quad A = \frac{q - p}{q - p + q((q/p)^h - 1)}, \quad p < q,
\]

\[
\text{Var}(S_{N_h}) = h(h+1) = 2pE(N_h), \quad p = q,
\]

\[
\text{Corr}(N_h, S_{N_h}) = \frac{\text{Var}(S_{N_h}) - \text{Var}(Y_1)E(N_h) + (E(Y_1))^2\text{Var}(N_h)}{2E(Y_1)\sqrt{\text{Var}(N_h)\text{Var}(S_{N_h})}}, \quad p < q,
\]

\[
\text{Corr}(N_h, S_{N_h}) = \frac{-2(h+1)}{\sqrt{6}\sqrt{h(h+1) + 1 - 3p}}, \quad p = q.
\]

For various choices of \( p, q \), Figure 1.3 shows that the variance \( \text{Var}(N_h) \) is approximately a parabolic function of \( E(N_h) \).

The following table compares the simulation results for the trinomial model with their corresponding theoretical values. It should be noted that the simulation was performed over a fixed time duration, \([0, 10,000]\). During this time period the trading strategy was implemented sequentially. This way the sample size of simulated values of \( N_h \) cannot be controlled. This feature is reflected in the last column of the following table. Also, the choice of \( p = 0.06 \) and \( p = 0.12 \) was made so that \( E(Y_1) = -0.56 + 0.06 = -\frac{1}{2} \) and \( E(Y_1) = 0.12 - 0.42 = -\frac{3}{10} \).
Figure 1.3: Mean-Variance Curve for Trailing Stop, Trinomial Model.
The amount of expected gain/loss, when $P_0 = 1$, is presented in the following table.

<table>
<thead>
<tr>
<th>Trinomial Model</th>
<th>$E(S_{Nh})$</th>
<th>$Var(S_{Nh})$</th>
<th>$E(N_h)$</th>
<th>$Var(N_h)$</th>
<th>Sample Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p = 0.06$, $q = 0.56$, $h = 1$</td>
<td>$-8.33$</td>
<td>$96.44$</td>
<td>$16.67$</td>
<td>$261.11$</td>
<td></td>
</tr>
<tr>
<td>Simulation</td>
<td>$-8.17$</td>
<td>$89.48$</td>
<td>$16.36$</td>
<td>$241.69$</td>
<td>$1222$</td>
</tr>
<tr>
<td>Error %</td>
<td>$-1.93$</td>
<td>$-7.22$</td>
<td>$-1.84$</td>
<td>$-7.44$</td>
<td></td>
</tr>
<tr>
<td>$p = 0.12$, $q = 0.42$, $h = 2$</td>
<td>$-13.75$</td>
<td>$263.81$</td>
<td>$45.83$</td>
<td>$1916$</td>
<td></td>
</tr>
<tr>
<td>Simulation</td>
<td>$-13.52$</td>
<td>$266.01$</td>
<td>$44.90$</td>
<td>$1930.9$</td>
<td>$1336$</td>
</tr>
<tr>
<td>Error %</td>
<td>$-1.69$</td>
<td>$0.83$</td>
<td>$-2.03$</td>
<td>$0.78$</td>
<td></td>
</tr>
</tbody>
</table>

Note that $E(Y_1) = p - q = -0.04$, which might suggest that a short position might be profitable. However, note that $E(e^{Y_1}) = 1.0399$ and $E(e^{Y_1}) > e^r$ if and only if $r < 0.0391$. Hence, it seems as if the short position trailing stops strategy requires that $E(e^{Y_1}) < e^r$ to remain profitable, and not just that $E(Y_1) < 0$. Needless to say that the corresponding conclusion goes for the long position of the trailing stops strategy, as well. Various correlations
are presented in the following table.

<table>
<thead>
<tr>
<th>$E(Y_1)$</th>
<th>Parameters</th>
<th>$h$</th>
<th>$Corr(N_h, S_{N_h})$</th>
<th>Sample Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>$p = 0.3, q = 0.2$</td>
<td>1</td>
<td>$-0.77$</td>
<td>5987</td>
</tr>
<tr>
<td>0.0</td>
<td>$p = 0.25, q = 0.25$</td>
<td>1</td>
<td>$-0.81$</td>
<td>5028</td>
</tr>
<tr>
<td>-0.1</td>
<td>$p = 0.2, q = 0.3$</td>
<td>1</td>
<td>$-0.87$</td>
<td>4100</td>
</tr>
<tr>
<td>0.1</td>
<td>$p = 0.3, q = 0.2$</td>
<td>2</td>
<td>$-0.71$</td>
<td>2332</td>
</tr>
<tr>
<td>0.0</td>
<td>$p = 0.25, q = 0.25$</td>
<td>2</td>
<td>$-0.81$</td>
<td>1641</td>
</tr>
<tr>
<td>-0.1</td>
<td>$p = 0.2, q = 0.3$</td>
<td>2</td>
<td>$-0.89$</td>
<td>1105</td>
</tr>
</tbody>
</table>

This shows that $Corr(N_h, S_{N_h}) < 0$ for all $E(Y_1)$ in a neighborhood of zero.

**Example 1.3.3. (Geometric model)** Consider the model in which the returns are governed by a two-sided geometric model.

$$P(Y = k) = \begin{cases} \frac{p_1 p_2}{1 - q_1 q_2} q_1^k & \text{if } k = 0, 1, 2, \ldots \\ \frac{p_1 p_2}{1 - q_1 q_2} q_2^{-k} & \text{if } k = -1, -2, \ldots \end{cases}$$

We may interpret this model as follows. The increment and decrements in log returns may be considered to be independent geometric random variables, $\xi_1, \xi_2$ with the net amount being $Y = \xi_1 - \xi_2$. It is obvious that $EY = (p_2 - p_1)/p_1 p_2$. It is not difficult to show that exact results for the trailing stops strategy can be obtained. We omit the straightforward details and just report the final results obtained from Theorem (1.2.1). The two roots of the equation, $\phi(\theta) = e^s$, are

$$e^{\theta_1} = 1 + q_1 q_2 - p_1 p_2 e^{-s} - \sqrt{(1 + q_1 q_2 - p_1 p_2 e^{-s})^2 - 4q_1 q_2}$$

$$e^{\theta_2} = 1 + q_1 q_2 - p_1 p_2 e^{-s} + \sqrt{(1 + q_1 q_2 - p_1 p_2 e^{-s})^2 - 4q_1 q_2}.$$
Note that \( \theta_1' + \theta_2' = 0 \). When \( q_1 < q_2 \), (i.e., \( E(Y) < 0 \)) and \( s \to 0 \), we see that \( \theta_1 \to 0 \) and 
\[ e^{\theta_2} \to \frac{q_2}{q_1} =: e^{\theta^*} \]. Therefore,

\[
E(e^{-sN_h}) = \frac{p_2(e^{\theta_1} - e^{\theta_2})(1 - q_1 e^{\theta_2})(1 - q_1 e^{\theta_1})}{p_1 (e^{\theta_2(h+1)}(1 - e^{\theta_1})(1 - q_1 e^{\theta_1})^2 - e^{\theta_1(h+1)}(1 - e^{\theta_2})(1 - q_1 e^{\theta_2})^2)}, \quad q_1 < q_2,
\]

\[
E(N_h) = \frac{q_2 p_1^2(q_2/q_1)^h}{p_2(p_1 - p_2)^2} - \frac{p_2(p_1 h + q_1)}{p_1 - p_2} - \frac{q_2 p_1^2}{(p_1 - p_2)^2}, \quad E(Y) \neq 0.
\]

\[
E(N_h) = \frac{p^2}{2q} \left( h^2 + h \left( 1 + \frac{4q}{p} \right) + \frac{2(2q + p)q}{p^2} \right), \quad p_1 = p_2 = p = 1 - q,
\]

\[
Var(N_h) = \frac{1}{(E(Y))^2} \left\{ Var(Y)E(N_h) + 2(E(Y)E(N_h) - h) \frac{q_2 p_1^2 + q_1 p_2^2}{p_1 p_2 (p_2 - p_1)} + (E(Y)E(N_h) - h)^2 + \frac{p_1 q_2 (1 + q_2)(p_2 - p_1(q_2/q_1)^h)}{p_1 p_2 (p_2 - p_1)} q_1(1 + q_1) \right\}
\]

\[ -2\frac{q_1^2 p_2^2 + q_2^2 p_1^2(q_2/q_1)^h}{p_1^2 p_2^2 (p_1 - p_2)} + 4h \frac{q_2^2 p_1 q_2(q_2/q_1)^h}{p_2^2 (p_2 - p_1)} \}, \quad q_1 < q_2.
\]

By Wald’s equation, applied to the martingales \( S_n - nE(Y_1) \) and \( (S_n - nE(Y_1))^2 - nVar(Y_1) \), we get

\[
E(S_{N_h}) = \frac{1}{p_1 p_2} \left\{ \frac{q_2 p_1^2(q_2/q_1)^h}{p_2 (p_2 - p_1)} + p_2(p_1 h + q_1) - \frac{q_2 p_1^2}{p_2 - p_1} \right\}, \quad q_1 \neq q_2,
\]

\[
E(S_{N_h}) = 0, \quad q_1 = q_2,
\]

\[
Var(S_{N_h}) = \frac{p^2 Var(Y_1)}{2q} \left( h^2 + h \left( 1 + \frac{4q}{p} \right) + \frac{2(2q + p)q}{p^2} \right), \quad p_1 = p_2 = p = 1 - q.
\]

Figure 1.4 shows that \( Var(N_h) \) can again be approximated by a parabolic function of \( E(N_h) \).

The following table provides some simulation results for the geometric model.

<table>
<thead>
<tr>
<th>Geometric Model</th>
<th>( E(S_{N_h}) )</th>
<th>( Var(S_{N_h}) )</th>
<th>( E(N_h) )</th>
<th>( Var(N_h) )</th>
<th>Sample Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_1 = \frac{1}{2}, \ p_2 = \frac{4}{9}, \ h = 1 )</td>
<td>-0.81</td>
<td>3.25</td>
<td>7.31</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simulation</td>
<td>-0.88</td>
<td>20.38</td>
<td>3.29</td>
<td>7.81</td>
<td>6069</td>
</tr>
<tr>
<td>Error %</td>
<td>7.79</td>
<td>1.38</td>
<td>6.84</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( p_1 = \frac{1}{2}, \ p_2 = \frac{4}{9}, \ h = 2 )</td>
<td>-1.375</td>
<td>5.5</td>
<td>22.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simulation</td>
<td>-1.42</td>
<td>34.68</td>
<td>5.53</td>
<td>21.06</td>
<td>3612</td>
</tr>
<tr>
<td>Error %</td>
<td>3.21</td>
<td>0.67</td>
<td>-7.43</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The amount of expected gain/loss, when \( P_0 = 1 \), is presented in the following table.

<table>
<thead>
<tr>
<th>( r )</th>
<th>( 1 - E(e^{-rN_h + S_h}) )</th>
<th>Std.Err</th>
<th>Sample Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>−2.93</td>
<td>0.32</td>
<td>4366</td>
</tr>
<tr>
<td>0.03</td>
<td>−2.84</td>
<td>0.23</td>
<td>4382</td>
</tr>
<tr>
<td>0.02</td>
<td>−4.22</td>
<td>0.98</td>
<td>4389</td>
</tr>
<tr>
<td>0.01</td>
<td>−6.09</td>
<td>2.01</td>
<td>4419</td>
</tr>
</tbody>
</table>

Note that \( E(Y_1) = −0.09 \), which might suggest that a short position might be profitable. However, note that \( E(e^{Y_1}) = 2.86 \) and \( E(e^{Y_1}) > e^r \) for all reasonable values of \( r \geq 0 \). Hence, the short position will always be a loosing proposition. Various correlations are presented.
in the following table.

<table>
<thead>
<tr>
<th>$E(Y_1)$</th>
<th>Parameters</th>
<th>$h$</th>
<th>$Corr(N_h, S_{N_h})$</th>
<th>Sample Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.83</td>
<td>$p_1 = 0.3$, $p_2 = 0.4$</td>
<td>1</td>
<td>-0.54</td>
<td>9635</td>
</tr>
<tr>
<td>0.00</td>
<td>$p_1 = 0.4$, $p_2 = 0.4$</td>
<td>1</td>
<td>-0.72</td>
<td>7453</td>
</tr>
<tr>
<td>-0.83</td>
<td>$p_1 = 0.4$, $p_2 = 0.3$</td>
<td>1</td>
<td>-0.80</td>
<td>6226</td>
</tr>
<tr>
<td>0.83</td>
<td>$p_1 = 0.3$, $p_2 = 0.4$</td>
<td>2</td>
<td>-0.57</td>
<td>7229</td>
</tr>
<tr>
<td>0.00</td>
<td>$p_1 = 0.4$, $p_2 = 0.4$</td>
<td>2</td>
<td>-0.74</td>
<td>4897</td>
</tr>
<tr>
<td>-0.83</td>
<td>$p_1 = 0.4$, $p_2 = 0.3$</td>
<td>2</td>
<td>-0.83</td>
<td>4023</td>
</tr>
</tbody>
</table>

This shows that $Corr(N_h, S_{N_h}) < 0$ for all $E(Y_1)$ in a neighborhood of zero.

**Example 1.3.4. (Double exponential model)** In this model we assume that the log returns follow a (not necessarily symmetric) double exponential model. More precisely, let the upward movement of the log return be governed by $U \sim Exp(\mu)$ and let the downward movement be governed by an independent $V \sim Exp(\lambda)$. The net effect is $Y = U - V$, whose density is

$$f_Y(y) = \begin{cases} \frac{\lambda \mu}{\lambda + \mu} e^{-\mu y}, & \text{if } y \geq 0, \\ \frac{\lambda \mu}{\lambda + \mu} e^{\lambda y}, & \text{if } y < 0. \end{cases}$$

As pointed out in the financial literature [4], [21], the log returns tend to have heavier tails than the tails of the corresponding normal density, which is the feature of this model. Once again our theoretical approach can be employed to get closed form expressions for the Laplace transform of $N_h$ along with its mean and variance. We once again omit the details, and just list the results. The roots of $\phi(\theta) = e^\theta$ are,

$$\theta_1 = \frac{(\mu - \lambda) - \sqrt{(\lambda + \mu)^2 - 4\lambda \mu e^{-\theta}}}{2}$$

$$\theta_2 = \frac{(\mu - \lambda) + \sqrt{(\lambda + \mu)^2 - 4\lambda \mu e^{-\theta}}}{2}.$$
Note that $\theta_1' + \theta_2' = 0$. When $\lambda < \mu$, (i.e., $E(Y) < 0$), as $s \to 0$, we see that $\theta_1 \to 0$ and $\theta_2 \to \theta^* = \mu - \lambda$. We have the following expressions for the mean, variance and Laplace transforms of $N_h$.

$$E(e^{-sN_h}) = \frac{\lambda(\theta_2 - \theta_1)(\mu - \theta_1)(\mu - \theta_2)}{\mu \{\theta_2 e^{\theta_2 h}(\lambda + \theta_1)(\mu - \theta_2) - \theta_1 e^{\theta_1 h}(\lambda + \theta_2)(\mu - \theta_1)\}}$$

$$E(N_h) = \frac{\lambda}{\rho - 1} \left\{ h + \frac{\rho}{\lambda} + \frac{\rho - e^{\theta_2 h}}{\lambda \rho(1 - \rho)} \right\}, \quad \rho = \frac{\lambda}{\mu} \neq 1,$$

$$E(N_h) = \frac{(h\mu + 2)^2}{2}, \quad \rho = \frac{\lambda}{\mu} = 1,$$

$$Var(N) = \frac{1}{(E(Y))^2} \left\{ Var(Y)E(N) + 2(E(Y)E(N) - h)(\mu^2 + \lambda^2) \frac{\lambda^2(\mu - \lambda)}{h\mu(\lambda - \mu)} + (E(Y)E(N) - h)^2 - 2 \frac{2\mu(\lambda - \mu e^{(\mu - \lambda)h})}{\lambda^3(\mu - \lambda)} - \frac{2}{\mu^2}
-2 \frac{(-\lambda^4 + \lambda^4 e^{(\mu - \lambda)h})}{\lambda^3\mu^2(\mu - \lambda)} + 4h \frac{\mu^2 e^{(\mu - \lambda)h}}{\lambda^2(\lambda - \mu)} \right\}, \quad \lambda < \mu.$$  

By using Wald’s equation for the martingales $S_n - nE(Y_1)$ and $E(S_n - nE(Y_1))^2 - nVar(Y_1)$, we get

$$E(S_{N_h}) = h + \frac{\rho}{\lambda} + \frac{\rho - e^{\theta_2 h}}{\lambda \rho(1 - \rho)}, \quad \rho = \frac{\lambda}{\mu} \neq 1,$$

$$E(S_{N_h}) = 0, \quad \rho = \frac{\lambda}{\mu} = 1,$$

$$Var(S_{N_h}) = \frac{(h\mu + 2)^2Var(Y_1)}{2} = \frac{(h\mu + 2)^2}{\mu^2}, \quad \rho = \frac{\lambda}{\mu} = 1.$$  

This model was studied by Iglehart and Stone [11] and later by Glynn and Iglehart [10]. By direct computations they obtained $E(L_h)$, and $E(S_{L_h})$, where $L_h$ is the trailing stops strategy in the long position. They also computed $Var(S_{L_h})$, however it seems that their expression may have an error. Figure 1.5 shows that $Var(N_h)$ again can be approximated by a parabolic function of $E(N_h)$. The following table provides some simulation results for the double exponential model.
Figure 1.5: Mean-Variance Curve for Trailing Stop, Double Exponential Model.
The amount of expected gain/loss, when $P_0 = 1$, is presented in the following table.

<table>
<thead>
<tr>
<th>Exponential Model</th>
<th>$E(S_{N_h})$</th>
<th>$Var(S_{N_h})$</th>
<th>$E(N_h)$</th>
<th>$Var(N_h)$</th>
<th>Sample Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu = 1.2, \lambda = 1.0, h = 2$</td>
<td>-1.91</td>
<td>11.45</td>
<td>100.35</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simulation</td>
<td>-2.12</td>
<td>33.95</td>
<td>11.60</td>
<td>105.23</td>
<td>1723</td>
</tr>
<tr>
<td>Error %</td>
<td>11.01</td>
<td>1.37</td>
<td>4.86</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu = 1.1, \lambda = 1.0, h = 3$</td>
<td>-1.42</td>
<td>15.67</td>
<td>181.84</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simulation</td>
<td>-1.62</td>
<td>43.97</td>
<td>16.07</td>
<td>199.29</td>
<td>1244</td>
</tr>
<tr>
<td>Error %</td>
<td>13.95</td>
<td>2.58</td>
<td>9.59</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\mu = 1.1, \lambda = 1.0, h = 1$

<table>
<thead>
<tr>
<th>$r$</th>
<th>$1 - E(e^{-rN_h+S_{N_h}})$</th>
<th>Std.Err</th>
<th>Sample Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>-6.12</td>
<td>0.96</td>
<td>3847</td>
</tr>
<tr>
<td>0.03</td>
<td>-4.28</td>
<td>0.41</td>
<td>3912</td>
</tr>
<tr>
<td>0.02</td>
<td>-5.65</td>
<td>0.64</td>
<td>3901</td>
</tr>
<tr>
<td>0.01</td>
<td>-5.76</td>
<td>0.78</td>
<td>3885</td>
</tr>
</tbody>
</table>

Note that $E(Y_1) = -0.09$, which might suggest that a short position might be profitable. However, note that $E(e^{Y_1}) = 5.5$ and $E(e^{Y_1}) > e^{r}$ for all reasonable values of $r \geq 0$. Hence, the short position will always be a loosing proposition. Various correlations are presented in the following table.

<table>
<thead>
<tr>
<th>$E(Y_1)$</th>
<th>Parameters</th>
<th>$h$</th>
<th>$Corr(N_h, S_{N_h})$</th>
<th>Sample Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1/11$</td>
<td>$\mu = 1.1, \lambda = 1.0$</td>
<td>1</td>
<td>-0.77</td>
<td>3047</td>
</tr>
<tr>
<td>0.0</td>
<td>$\mu = 1.0, \lambda = 1.0$</td>
<td>1</td>
<td>-0.73</td>
<td>4484</td>
</tr>
<tr>
<td>$1/11$</td>
<td>$\mu = 1.0, \lambda = 1.1$</td>
<td>1</td>
<td>-0.71</td>
<td>4682</td>
</tr>
<tr>
<td>$-1/11$</td>
<td>$\mu = 1.1, \lambda = 1.0$</td>
<td>2</td>
<td>-0.81</td>
<td>2091</td>
</tr>
<tr>
<td>0.0</td>
<td>$\mu = 1.0, \lambda = 1.0$</td>
<td>2</td>
<td>-0.79</td>
<td>2498</td>
</tr>
<tr>
<td>$1/11$</td>
<td>$\mu = 1.0, \lambda = 1.1$</td>
<td>2</td>
<td>-0.72</td>
<td>2703</td>
</tr>
</tbody>
</table>
This shows that $\text{Corr}(N_h, S_{N_h}) < 0$ for all $E(Y_i)$ in a neighborhood of zero.

**Example 1.3.5. (Log normal model)** Now we consider the case when the log returns are normally distributed. More precisely, let $Y_1, Y_2, \cdots \sim N(\mu, \sigma^2)$. Then $\phi(t) = \exp(\mu t + \frac{1}{2} \sigma^2 t^2)$, and $\phi(t) = e^s$ $(s > 0)$ has two solutions,

$$
\theta_1 = -\frac{\mu - \sqrt{\mu^2 + 2 \sigma^2 s}}{\sigma^2}, \quad \theta_2 = -\frac{\mu + \sqrt{\mu^2 + 2 \sigma^2 s}}{\sigma^2}.
$$

Once again note that $\theta_1 + \theta_2$ is a constant making $\theta'_1 + \theta'_2 = 0$. However, in this case $Y_1$ does not obey the conditions of Theorem 1.2.1. Exact expressions for $E(e^{-sN_h})$, $E(N_h)$, $\text{Var}(N_h)$, for the trailing stops strategy are still not known. Various approximations are, however, available. For instance, (for the case when $\mu \neq 0$), [17] gives that

$$
E(e^{-sN_h}) \approx \frac{\delta e^{\gamma h}}{\gamma \cosh(\delta h) + \gamma \sinh(\delta h)}
$$

where $\gamma = -\frac{\mu}{\sigma^2}$ and $\delta = \sqrt{\gamma^2 + \frac{2s}{\sigma^2}}$. When $\mu = 0$, we have

$$
E(e^{-sN_h}) \approx \frac{1}{\cosh(h(2s/\sigma^2)^{1/2})}.
$$

This Laplace transform turns out to be the exact Laplace transform of

$$
T := \inf\{t \geq 0 : W(t) - \min_{0 \leq s \leq t} W(s) \geq h\},
$$

where $W(t)$ is the Wiener process with drift parameter $\mu$ and volatility parameter $\sigma$. The question about the quality of approximation of the $E(N_h)$ has been discussed by several authors (cf. [8], [13] and [19]). Khan [13] also provides some refined approximations and the empirical evidence about the quality of such approximations is given in Example 3 (pp. 74-75) for normal and exponential distributions. The following table gives the simulation results for this model.
<table>
<thead>
<tr>
<th>Normal Model</th>
<th>( E(S_{Nh}) )</th>
<th>( \text{Var}(S_{Nh}) )</th>
<th>( E(N_h) )</th>
<th>( \text{Var}(N_h) )</th>
<th>Sample size</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu = 0, \sigma = 0.5, h = 1 )</td>
<td>0.02</td>
<td>2.41</td>
<td>9.97</td>
<td>68.81</td>
<td>2006</td>
</tr>
<tr>
<td>( \mu = 0, \sigma = 1.0, h = 1 )</td>
<td>0.06</td>
<td>4.73</td>
<td>4.63</td>
<td>14.62</td>
<td>4318</td>
</tr>
<tr>
<td>( \mu = 0, \sigma = 2.0, h = 1 )</td>
<td>0.08</td>
<td>12.09</td>
<td>3.02</td>
<td>5.75</td>
<td>6624</td>
</tr>
<tr>
<td>( \mu = -0.1, \sigma = 0.5, h = 1 )</td>
<td>-1.64</td>
<td>9.86</td>
<td>15.96</td>
<td>200.03</td>
<td>1253</td>
</tr>
<tr>
<td>( \mu = -0.1, \sigma = 1.0, h = 1 )</td>
<td>-0.50</td>
<td>7.06</td>
<td>5.48</td>
<td>21.18</td>
<td>3647</td>
</tr>
<tr>
<td>( \mu = -0.1, \sigma = 2.0, h = 1 )</td>
<td>-0.34</td>
<td>14.43</td>
<td>3.24</td>
<td>6.80</td>
<td>6167</td>
</tr>
<tr>
<td>( \mu = 0, \sigma = 0.5, h = 2 )</td>
<td>0.14</td>
<td>6.02</td>
<td>25.63</td>
<td>425.12</td>
<td>780</td>
</tr>
<tr>
<td>( \mu = 0, \sigma = 1.0, h = 2 )</td>
<td>0.09</td>
<td>9.38</td>
<td>9.80</td>
<td>68.20</td>
<td>2040</td>
</tr>
<tr>
<td>( \mu = 0, \sigma = 2.0, h = 2 )</td>
<td>-0.07</td>
<td>19.79</td>
<td>4.82</td>
<td>16.15</td>
<td>4144</td>
</tr>
<tr>
<td>( \mu = -0.1, \sigma = 0.5, h = 2 )</td>
<td>-5.73</td>
<td>64.83</td>
<td>57.93</td>
<td>2514.91</td>
<td>345</td>
</tr>
<tr>
<td>( \mu = -0.1, \sigma = 1.0, h = 2 )</td>
<td>-1.27</td>
<td>20.48</td>
<td>12.64</td>
<td>119.61</td>
<td>1577</td>
</tr>
<tr>
<td>( \mu = -0.1, \sigma = 2.0, h = 2 )</td>
<td>-0.65</td>
<td>24.66</td>
<td>5.21</td>
<td>19.57</td>
<td>3836</td>
</tr>
</tbody>
</table>

The amount of expected gain/loss, when \( P_0 = 1 \), is presented in the following table.

\[
\mu = -0.485, \quad \sigma = 1.0, \quad h = 1
\]

<table>
<thead>
<tr>
<th>( r )</th>
<th>( 1 - E(e^{-rN_h+S_{Nh}}) )</th>
<th>\text{Std.Err}</th>
<th>Sample Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>0.08</td>
<td>0.03</td>
<td>3685</td>
</tr>
<tr>
<td>0.03</td>
<td>0.05</td>
<td>0.03</td>
<td>3729</td>
</tr>
<tr>
<td>0.02</td>
<td>-0.003</td>
<td>0.04</td>
<td>3664</td>
</tr>
<tr>
<td>0.01</td>
<td>-0.03</td>
<td>0.04</td>
<td>3735</td>
</tr>
</tbody>
</table>

Note that \( E(Y_1) = -0.485 \), which might suggest that a short position might be profitable. However, note that \( E(e^{Y_1}) = e^{0.015} \) and \( E(e^{Y_1}) > e^r \) if and only if \( r < 0.015 \). Hence, the short position will be a loosing proposition when the risk free rate is low. Various
correlations are presented in the following table.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$h$</th>
<th>$\text{Corr}(N_h, S_{N_h})$</th>
<th>Sample Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu = -0.1$, $\sigma = 1.0$</td>
<td>1</td>
<td>-0.83</td>
<td>3604</td>
</tr>
<tr>
<td>$\mu = 0.0$, $\sigma = 1.0$</td>
<td>1</td>
<td>-0.79</td>
<td>4318</td>
</tr>
<tr>
<td>$\mu = 0.1$, $\sigma = 1.0$</td>
<td>1</td>
<td>-0.75</td>
<td>4908</td>
</tr>
<tr>
<td>$\mu = -0.1$, $\sigma = 1.0$</td>
<td>2</td>
<td>-0.86</td>
<td>1578</td>
</tr>
<tr>
<td>$\mu = 0.0$, $\sigma = 1.0$</td>
<td>2</td>
<td>-0.81</td>
<td>1990</td>
</tr>
<tr>
<td>$\mu = 0.1$, $\sigma = 1.0$</td>
<td>2</td>
<td>-0.75</td>
<td>2469</td>
</tr>
</tbody>
</table>

This shows that $\text{Corr}(N_h, S_{N_h}) < 0$ for all $E(Y_1)$ in a neighborhood of zero.

1.4 Conclusions

In this section we collect the salient features of this work.

- (1) The sign of the correlation between $S_\nu$ and $\nu$ is always controlled by the factor
  \[
  \frac{\text{Var}(S_\nu) + (E(Y_1))^2\text{Var}(\nu) - \text{Var}(Y_1)E(\nu)}{E(Y_1)}
  \]

  for any trading strategy that is adapted to the log returns filtration, $\sigma(Y_1, \ldots, Y_n)$, $n \geq 1$. As shown by the binomial and trinomial models, when $E(Y_1) < 0$, this amount is negative and rather close to negative one for the trading the line strategy, $\nu = N_h$.

  The same conclusions seem to hold for all the other models studied in this paper. To show that $E(S_{N_h})$ is a decreasing function of $h$, when $E(Y_1) < 0$, consider $\nu_2 = N_{h_2}$, and $\nu_1 = N_{h_1}$, when $h_2 > h_1$. Now, $S_n - nE(Y_1)$, being a martingale,

  \[
  E(S_{\nu_2} - S_{\nu_1} | \mathcal{F}_{\nu_1}) = E(S_{\nu_2} - S_{\nu_1} - (\nu_2 - \nu_1)E(Y_1) | \mathcal{F}_{\nu_1}) + E(\nu_2 - \nu_1 | \mathcal{F}_{\nu_1}) E(Y_1)
  \]

  \[
  (1.1) \quad \leq 0,
  \]

  where $\mathcal{F}_{\nu_1}$ is the information up to time $N_{h_1}$. The last inequality follows since $\nu_2 = N_{h_2} \geq N_{h_1} = \nu_1$ and $E(Y_1) < 0$. The inequality becomes strict if $N_{h_2} < N_{h_1}$ with
positive probability. Hence, $E(S_{N_h})$ is a nonincreasing function of the triggering constant $h$, whenever $E(Y_1) < 0$. However, as the last item (below) explains, this does not necessarily mean that we will gain more with $h_2$ versus $h_1$, when $E(Y_1) < 0$.

- (2) If $E(Y_1) = 0$, i.e., no drift in the log returns, then for any strategy, $\nu$, that is independent of the filtration of the log returns, $\text{Corr}(\nu, S_{\nu}) = 0$. However, as the binomial and trinomial examples showed, this is not so when $\nu = N_h$. For the models we presented, $\text{Corr}(N_h, S_{N_h}) < 0$, even when $E(Y_1) = 0$. This means that the longer we wait with the trailing stop strategy, $N_h$, the more we expect to gain, when $E(Y_1) \leq 0$.

- (3) Even when $E(Y_1) > 0$ and small, the correlation of $N_h, S_{N_h}$ did not change sign in the examples for which we were able to find a closed form expression for $\text{Corr}(N_h, S_{N_h})$ and as well as via simulation for the other examples. Therefore, we may conclude that $\text{Corr}(N_h, S_{N_h}) < 0$ for all values of $E(Y_1)$ in a neighborhood of zero (and not just at zero), since the correlation is a continuous function of $E(Y_1)$. When $E(Y_1) > 0$, by a similar argument as in (1.1), we should remark that $E(S_{N_h})$ becomes a nondecreasing function of $h$.

- (4) As the examples of the last section suggest, the variance of the duration that the position remains open, $\text{Var}(N_h)$, seems to have a parabolic relationship with the expected duration, $E(N_h)$. The coefficients of the parabola are functions of the trigger size $h$. Furthermore, the larger we pick the triggering constant, $h$, the larger is the expected duration and, in turn, the more unstable the strategy becomes.

- (5) The variance of $S_{N_h}$ also seems to have a parabolic relation with $E(S_{N_h})$. For instance, for both the binomial and trinomial models,

$$\text{Var}(S_{N_h}) = h - E(S_{N_h}) + (E(S_{N_h}) - h)^2.$$
Since $E(S_{N_h}) = E(Y_1)E(N_h)$, we also have

$$Var(S_{N_h}) = h - E(Y_1)E(N_h) + (E(Y_1)E(N_h) - h)^2.$$ 

This indicates that, when $E(Y_1) < 0$, one cannot hope to find an optimal $h$ that simultaneously minimizes the expected stopped amount, $E(S_{N_h})$, and also minimizes the variance, $Var(S_{N_h})$.

- (6) It seems reasonable to propose that the distribution of a constant multiple of $N_h$, when $E(Y) < 0$, is approximately a geometric random variable, at least when the assumptions (A1), (A2), and (A3) hold. This should be compared to the known approximation of $N_h$, [16], [17], which states that $N_h e^{-\theta h}$ is approximately an exponential random variable, as $h$ gets large.

- (7) If the price dynamics are assumed to follow geometric random walk, even when $E(Y_1) < 0$, any trading strategy in short position, that is independent of the filtration $\sigma(Y_1, \cdots, Y_n), n \geq 1$, will give time discounted gains to be negative provided $E(e^{Y_1}) > e^r$, where $r$ is the continuously compounding discount rate. That is, for those price dynamics in which $E(Y_1) < 0$ and $E(e^{Y_1}) > e^r$, there cannot exist a trading strategy, that is independent of the log returns process, and gives positive time discounted gains. In fact, this result remains valid for a wide class of adaptive strategies. A similar observation holds for the long position trading strategies.

- (8) For the trailing stops strategy, $N_h$, adapted to the filtration, $\sigma(Y_1, \cdots, Y_n), n \geq 1$, we observe the phenomenon of item (7) in our simulations. In fact, for the geometric model this observation takes even a more acute form. This is due to the fact that, on the one hand, the geometric random variable can take arbitrarily large values and, on the other hand, the price can never go below zero, regardless of how large a magnitude of the negative values of the double sided geometric random variable is. Since, in a
short position, by investing $1, one cannot hope to gain more than one dollar, however, there is no limit to how much one may lose. Hence, the short position is inherently more risky than the long position. Furthermore, an indicator of a possible gain is the criterion, $E(e^{Y_1}) < e^r$, where $r$ is the risk free rate, and not just that $E(Y_1) < 0$. 
CHAPTER 2

Trading the Line Strategy in Black-Scholes Model

2.1 Analysis of Trading the Line Strategy in Continuous Time

Let us start with formal definitions. Let us define the process $M(t)$ as

$$M(t) = \sup_{0 \leq s \leq t} P(s).$$

Let’s define the stopping time $\tau_c = \inf_{t \geq 0} \{M(t) - P(t) \geq c\}$. If a trader decides to sell the stock at time $\tau_c$ then we say that he/she follows the trading the line strategy with fixed value $c$.

From now on we will assume that $c$ is some fixed constant and $\tau_c$ will be denoted by $\tau$.

Let us define the hitting double barrier stopping time which will be used later on

$$\rho_{a,b} = \inf_{t \geq 0} \{P(t) \notin (a,b)\}.$$  

We will assume that the price process $P(t)$ is described by the following stochastic differential equation

$$dP(t) = \mu(P(t))dt + \sigma(P(t))dB(t), \ t \geq 0 \tag{2.1}$$

with $P(0) = 0$ a.s., $\{B(t), \ t \geq 0\}$ a standard Brownian motion process, and $\sigma(x) > 0$. It is also assumed that $\mu(x)$ and $\sigma(x)$ are measurable and define for $x$ in $[-c, \infty)$ and satisfy the conditions of the existence and uniqueness theorem for stochastic differential equations [23], more precisely $\exists$ a constant $K \ni \forall x, y \in [-c, \infty)$ the following inequalities hold

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq K|x - y|, \tag{2.2}$$

$$\mu^2(x) + \sigma^2(x) \leq K^2 (1 + x^2). \tag{2.3}$$
Note, that the most famous model in continuous time setting, Black-Scholes model \(^1\), is a particular type of the model we consider here. More precisely, in Black-Scholes model the price process is a geometric Brownian motion

\[
P(t) = P(0)e^{\mu t + \sigma B(t) - \sigma^2 t/2},
\]

where drift \(\mu\) and volatility \(\sigma^2\) are some given constants. Using Itô’s formula \(^2\) one can show that it is the solution of the following stochastic differential equation

\[
dP(t) = \mu P(t)dt + \sigma P(t)dB(t).
\]

In this case

\[
\mu(x) = \mu x, \quad \sigma(x) = \sigma x.
\]

We will be interested in finding Laplace transforms of stopping times \(\tau_v\) and stopped price processes \(P(\tau_v)\). These Laplace transforms will enable an investor to get a good picture of their distributions as long as other probabilistic characteristics. This, on the other hand, could help an investor to choose the “right” parameters \(c\). In the next three sections we will recall some theoretical results which will help us to find Laplace transforms.

2.2 Two Fundamental Lemmas about Hitting Double Barrier Times \(\rho_{a,b}\)

The next Lemmas will enable us to prove the theorems regarding Laplace transforms. Themselves, they are very helpful in pricing some exotic options.

**Lemma 2.2.1.** Let \(\{P(t), t \geq 0\}\) be a stochastic process taking values in a possibly infinite interval \(I\) satisfying (2.1), (2.2), and (2.3) with initial condition \(P(0) = x\) a.s. and \(\sigma(x) > 0\) both for \(x \in I\). Let \(a\) and \(b\) be some given constants such that

\[
a < b, \quad a, b \in I.
\]

\(^1\)This model has been proposed by Paul Samuelsen in 1965. See \(^3\) for detailed history.
Then

\[ \mathbf{P}(\mathbf{P}(\rho_{a,b}) = a) = q(a, b, x) = \int_x^b \Phi(z)dz / \int_a^b \Phi(z)dz, \]

\[ \mathbf{P}(\mathbf{P}(\rho_{a,b}) = b) = p(a, b, x) = \int_a^x \Phi(z)dz / \int_a^b \Phi(z)dz, \]

where

\[ \Phi(z) = \exp \left\{ -\int_a^z 2\gamma(u)du \right\} \]

and

\[ \gamma(u) = \mu(u)/\sigma^2(u). \]

**Proof.** Let us prove (2.5). Consider the function \( u(x) = \int_a^x \Phi(z)dz \). Then

\[ u'(x) = \frac{d}{dx} \left( \int_a^x \Phi(z)dz \right) = \Phi(x). \]

Therefore,

\[ \mu(x)u'(x) = \mu(x)\Phi(x). \]

Using (2.7)-(2.2) the second order derivative of \( u(x) \) is the following

\[ u''(x) = \frac{d^2}{dx^2} \left( \int_a^x \Phi(z)dz \right) = \Phi'(x) \]

\[ = \frac{d}{dx} \left( e^{-\int_a^x \gamma(u)du} \right) = \Phi(x) \frac{d}{dx} \left( -2 \int_a^x \gamma(u)du \right) \]

\[ = -2\Phi(x)\gamma(x) = -2\frac{\mu(x)}{\sigma^2(x)}\Phi(x). \]

Then

\[ \frac{1}{2}\sigma^2(x)u''(x) = \frac{1}{2}\sigma^2(x)(-2)\frac{\mu(x)}{\sigma^2(x)}\Phi(x) = -\mu(x)\Phi(x). \]

So,

\[ \frac{1}{2}\sigma^2(x)u''(x) + \mu(x)u'(x) = \mu(x)\Phi(x) - \mu(x)\Phi(x) = 0. \]

We get that the function \( u(x) \) satisfies the following equation

\[ \frac{1}{2}\sigma^2(x)u''(x) + \mu(x)u'(x) = 0. \]
Then the function \( \psi(x) = \frac{u(x) - u(b)}{u(a) - u(b)} \) satisfies the equation
\[
\frac{1}{2}\sigma^2(x)\psi''(x) + \mu(x)\psi'(x) = 0
\]
with
\[
\psi(a) = 1 \text{ and } \psi(b) = 0.
\]

Hence, by Itô’s formula we get
\[
\psi(P(\rho_{a,b})) - \psi(x) = \int_0^{\rho_{a,b}} \sigma(P(s))\psi'(P(s))dB(s)
+ \int_0^{\rho_{a,b}} \left\{ \frac{\sigma^2(P(s))}{2} \psi''(P(s)) + \mu(P(s))\psi'(P(s)) \right\} ds
= \int_0^{\rho_{a,b}} \sigma(P(s))\psi'(P(s))dB(s),
\]
and consequently,
\[
E[\psi(P(\rho_{a,b}))] = \psi(x).
\]

On the other hand
\[
E[\psi(P(\rho_{a,b}))] = \psi(a)P(P(\rho_{a,b}) = a) + \psi(b)P(P(\rho_{a,b}) = b).
\]

Then by (2.9), (2.10), and (2.11) we get
\[
\psi(x) = P(P(\rho_{a,b}) = a).
\]

Then using definition of \( \psi(x) \) and \( u(x) \) we deduce that
\[
P(P(\rho_{a,b}) = a) = \frac{u(x) - u(b)}{u(a) - u(b)} = \frac{\int_x^b \Phi(z)dz - \int_a^b \Phi(z)dz}{\int_a^b \Phi(z)dz - \int_a^b \Phi(z)dz}
= \frac{\int_x^b \Phi(z)dz}{\int_a^b \Phi(z)dz} = \frac{\int_x^b \Phi(z)dz}{\int_a^b \Phi(z)dz}.
\]

Equation (2.6) can be obtained by analogous considerations.

Let \( g \) and \( h \) be any two independent solutions of the following ordinary differential equation
\[
1 \frac{\sigma^2(x)f''(x) + a(x)f'(x)}{2} = \beta f(x).
\]
Lemma 2.2.2. Under the same conditions as in Lemma 1, the following three equations hold

\begin{align}
(2.13) \quad & E\left[e^{-\beta \rho_{a,b}} | P(\rho_{a,b}) = b \right] = u(a, b, x)/p(a, b, x), \\
(2.14) \quad & E\left[e^{-\beta \rho_{a,b}} | P(\rho_{a,b}) = a \right] = v(a, b, x)/q(a, b, x), \\
(2.15) \quad & E\left[e^{-\beta \rho_{a,b}} \right] = u(a, b, x) + v(a, b, x),
\end{align}

where

\begin{align}
(2.16) \quad & u(a, b, x) = \frac{g(a)h(x) - g(x)h(a)}{g(a)h(b) - g(b)h(a)} \quad \text{and} \quad v(a, b, x) = \frac{g(x)h(b) - g(b)h(x)}{g(a)h(b) - g(b)h(a)}.
\end{align}

Proof. Let us integrate (2.1) to get

\begin{equation}
P(t) = x + \int_0^t a(P(s))ds + \int_0^t \sigma(P(s))dB(s).
\end{equation}

Let \( f(x) \) be any solution of (2.12) and consider transformation \( Y(t) = e^{-\beta t}f(P(t)) \). Using Itô’s formula we get

\begin{equation}
dY(t) = e^{-\beta t}f'(P(t))\sigma(P(t))dB(t),
\end{equation}

with \( Y(0) = f(x) \) a.s. or

\begin{equation}
Y(t) - f(x) = \int_0^t e^{-\beta s}f'(P(s))\sigma(P(s))dB(s).
\end{equation}

Let us consider \( \xi_u = \rho_{a,b} \land u \) then from (2.17) we deduce

\begin{equation}
E[Y(t)] - f(x) = E[Y(t) - f(x)] = E\left[\int_0^t e^{-\beta s}f'(P(s))\sigma(P(s))dB(s)\right] = 0.
\end{equation}

Therefore,

\begin{equation}
f(x) = E[Y(\xi_u)] = E\left[e^{-\beta \xi_u}f(P(\xi_u))\right].
\end{equation}

As we mentioned before \( g \) and \( h \) are two independent solutions of (2.12). Lemma 1, total expectation theorem, and equation (2.18) yield two linear equations

\begin{align}
g(x) &= q(a, b, x)g(a)E\left[e^{-\beta \rho_{a,b}} | P(\rho_{a,b} = a) \right] + p(a, b, x)g(b)E\left[e^{-\beta \rho_{a,b}} | P(\rho_{a,b} = b) \right], \\
h(x) &= q(a, b, x)h(a)E\left[e^{-\beta \rho_{a,b}} | P(\rho_{a,b} = a) \right] + p(a, b, x)h(b)E\left[e^{-\beta \rho_{a,b}} | P(\rho_{a,b} = b) \right].
\end{align}
The first two equations (2.13) and (2.14) of the lemma follow easily by solving these two linear equations using $u$ and $v$ defined by (2.16). And the last one (2.15) then follows from total expectation theorem.

2.3 The Distribution of the Stopped Maximum Process $M(\tau)$

The next two theorems are due to John P. Lehoczky [25].

**Theorem 2.3.1.** Under the same conditions as in Lemma 1, $M(\tau)$ has the following distribution

$$
P(M(\tau) \geq x) = \exp \left\{ - \int_0^x \frac{\Phi(z)}{\Phi(u)} du \right\} \text{ for } x \geq 0,
$$

with $\Phi(x) = \exp \left\{ - \int_0^x 2\gamma(z)dz \right\}$, and $\gamma(z) = a(z)/\sigma^2(z)$.

**Proof.** To compute $P(M(\tau) \geq x)$, partition the interval $[0, x]$ into $n$ subintervals

$$
\{[s_{ni}, s_{ni+1}], \ 0 \leq i \leq n - 1\}
$$

with

$$
0 = s_{n0} < s_{n1} < \ldots < s_{nn} = x.
$$

Let

$$
m_n = \max_{0\leq i \leq n-1} (s_{ni+1} - s_{ni})
$$

and assume $m_n \to 0$ as $n \to \infty$. For $\{M(\tau) \geq x\}$ to occur, the $P(t)$ process must reach $x$. Consequently, if $M(\tau) \geq x$ and $M(t) = y$ for $0 \leq y < x$ and for some $t < \tau$, then the process must hit $y + dy$ before it hits $y - v$. As a discrete approximation to $P\{M(\tau) \geq x\}$ compute $P_n = P\left( \bigcap_{i=0}^{n-1} \{P(t) \text{ hits } s_{ni+1} \text{ before } s_{ni} - v\} \right)$. It will be shown that the limit as $n \to \infty$ of $P_n$ is independent of the particular sequence of partitions chosen, hence the limit is $P(M(\tau) \geq x)$. Using the Strong Markov property and time homogeneity

$$
P_n = \prod_{i=0}^{n-1} P(P(t) \text{ hits } s_{ni+1} \text{ before } s_{ni} - v|P(0) = s_{ni}).
$$
Using Lemma 1 the \( i \)th factor in the product is given by 
\[
p(s_{ni} - v, s_{ni+1}, s_{ni}).
\]
Then we get
\[
\lim_{n \to \infty} P_n = \exp \left\{ \lim_{n \to \infty} \sum_{i=0}^{n-1} \log(1 - q(s_{ni} - v, s_{ni+1}, s_{ni})) \right\}.
\]
As \( n \to \infty \), \( m_n \to 0 \) and \( \sum_{i=0}^{n-1} (s_{ni+1} - s_{ni})^k \to 0 \) for \( k \geq 2 \). This indicates that only the first term in the Taylor expansion has to be kept. Further, by the continuity of \( \Phi(z) \),
\[
\lim_{n \to \infty} \left\{ \int_{s_{ni}}^{s_{ni+1}} \Phi(z)dz / (s_{ni+1} - s_{ni}) - \Phi(s_{ni}) \right\} = 0.
\]
As \( n \to \infty \) the sum converges to the ordinary Riemann integral
\[
- \int_0^x \frac{\Phi(z)}{\int_{z-v}^{z} \Phi(u)du} dz.
\]
This limit is independent of the particular partition sequence chosen which completes the proof. It is possible that \( \tau \) can be infinite valued. Then \( M(\tau) = \infty \) and
\[
P(\tau = \infty) = \exp \left\{ - \int_0^\infty \frac{\Phi(z)}{\int_{z-v}^{z} \Phi(u)du} dz \right\}.
\]

2.4 The Bivariate Laplace Transform of \( P(\tau) \) and \( \tau \)

**Theorem 2.4.1.** Under the same conditions as in Lemma 1, \( M(\tau) \) and \( \tau \) have the following joint Laplace transform
\[
E \left[ e^{\alpha M(\tau) - \beta \tau} \right] = \int_0^\infty c(x) e^{\alpha x - \int_0^x b(z)dz} dx,
\]
where
\[
b(z) = \frac{g(z - v)h'(z) - h(z - v)g'(z)}{g(z - v)h(z) - g(z)h(z - v)}
\]
and
\[
c(x) = \frac{g(x)h'(x) - g'(x)h(x)}{g(x - v)h(x) - g(x)h(x - v)}.
\]
And \( f \) and \( g \) are two independent solutions of (2.12).
Proof. The joint Laplace transform of $M(\tau)$ and $\tau$ can be calculated by conditioning on $M(\tau)$.

$$
E \left[ e^{\alpha M(\tau) - \beta \tau} \right] = E \left[ e^{\alpha M(\tau)} E \left[ e^{-\beta \tau} | M(\tau) \right] \right]
$$

$$
= \int_0^\infty e^{\alpha x} E \left[ e^{-\beta \tau} | M(\tau) = x \right] f_{M(\tau)}(x) dx,
$$

where $f_{M(\tau)}(x)$ is the density of $M(\tau)$ derived from (2.19). We compute $E \left[ e^{-\beta \tau} | M(\tau) = x \right]$ using the discrete approximation technique of Theorem 1. Let us consider an arbitrary partition sequence $\{s_{ni}, 0 \leq i \leq n+1\}$ with

$$
0 = s_{n0} < \ldots < s_{nn} = x < s_{nn+1}
$$

and

$$
\epsilon_{nk} = s_{nk} - s_{nk-1}.
$$

We let $m_n = \max_{1 \leq k \leq n+1} \epsilon_{nk}$ and assume $m_n \to 0$ as $n \to \infty$. Define a sequence of stopping times

$$
\zeta_{nk} = \inf_{t>0} \{P(S_{nk-1} + t) - P(S_{nk-1}) = \epsilon_{nk} \text{ or } -v\},\ 1 \leq k \leq n + 1
$$

with $S_{nk} = \sum_{i=1}^k \zeta_{ni}$. Each $\zeta_{ni}$ is a.s. finite since $\sigma(x) > 0$. In this discrete formulation $\{M(\tau) = x\}$ is approximated by

$$
\left\{ \bigcap_{k=1}^n (P(S_{nk}) - P(S_{nk-1}) = \epsilon_{nk}) \bigcap (P(S_{nn+1}) - P(S_{nn}) = -v) \right\} = B_n
$$

and $E \left[ e^{-\beta \tau} | M(\tau) = x \right]$ by $E \left[ e^{-\beta S_{nn+1}} | B_n \right] = E \left[ \prod_{i=1}^{n+1} e^{-\beta \tau_{ni}} | B_n \right] = E_n$. The stopping time $\tau_{nk+1}$ is in the future of $S_{nk}$ for $0 \leq k \leq n$, and the $P$-process is time homogeneous, thus the strong Markov property can be applied to give

$$
E_n = \prod_{i=1}^n E \left[ e^{-\beta \tau_{ni}} | P(S_{ni}) = s_{ni-1}, P(S_{ni}) = s_{ni} \right]
\times E \left[ e^{-\beta \tau_{nn}} | P(S_{nn}) = x, P(S_{nn+1}) = x - v \right].
$$
Each of the \( n + 1 \) conditional expectations in (2.23) can be computed by using Lemma 2. We find
\[
E_n = \frac{\prod_{i=1}^{n} u(s_{ni-1} - v, s_{ni}, s_{ni-1}) v(x - v, s_{mn+1}, x)}{\prod_{i=1}^{n} p(s_{ni-1} - v, s_{ni}, s_{ni-1}) q(x - v, s_{mn+1}, x)}
\]
where \( p, q, u, \) and \( v \) are defined in Lemmas 1 and 2.

We now let \( n \to \infty \) and show \( \lim_{n \to \infty} E_n \) exists and is independent of the partition sequence chosen. We identify this limit as \( E \left[ e^{-\beta \tau} | M(\tau) = x \right] \). The limits are taken in the manner outlined in the proof of Theorem 1. We take logs, use \( m_n \to 0 \), and use the continuity of \( g'(x) \) and \( h'(x) \) to show
\[
\lim_{n \to \infty} \prod_{i=1}^{n} u(s_{ni-1} - a, s_{ni}, s_{ni-1}) = e^{\int_0^x b(z) dz}
\]
with
\[
b(z) = \frac{g(z - a)h'(z) - h(z - a)g'(z)}{g(z - a)h(z) - h(z - a)g(z)}.
\]
Moreover,
\[
\lim_{n \to \infty} \frac{\prod_{i=1}^{n} p(s_{ni-1} - a, s_{ni}, s_{ni-1}) q(x - a, s_{mn+1}, x)}{\epsilon_{nn+1}} = f_M(\tau)(x)
\]
where the limit of the product was calculated in the proof of Theorem 1. Finally,
\[
\lim_{n \to \infty} \frac{v(x - a, s_{nn+1}, x)}{\epsilon_{nn+1}} = c(x)
\]
where \( c(x) \) defined in (2.22).

Each of the limits (2.25)-(2.27) is independent of the particular sequence chosen, hence we have
\[
E \left[ e^{-\beta \tau} | M(\tau) = x \right] = \frac{e^{\int_0^x b(z) dz} c(x)}{f_M(\tau)(x)}.
\]
Substituting (2.28) into the original expression for \( E \left[ e^{\lambda M(\tau) - \beta \tau} \right] \) we get (2.20) and complete the derivation.

We comment that conditionally on \( \{ M(\tau) = x \} \), \( \tau \) is a.s. finite. Nevertheless \( \tau \) and \( M(\tau) \) can be infinite with the same positive probability, and \( P(\tau < \infty) \) can be calculated from (2.20) by setting \( \alpha = 0 \) and letting \( \beta \to 0 \). \( \square \)
2.5 Applications To Finance

Now we have all necessary instruments to study trading the line strategy with fixed value \( v \) (and fixed percentage \( p \)) in Black-Scholes model. Let us consider the case of trading the line strategy with fixed value first. We get the following proposition.

**Proposition 1.** Let us assume that the price process \( P(t) \) is given by (2.4) and a trader follows the trading the line strategy with fixed value \( v \). Then

\[
E\left[e^{\alpha M(\tau) - \beta \tau}\right] = e^{\alpha P(0)} \int_0^\infty c(x)e^{\alpha x - \int_0^x b(z)dz}dx,
\]

where

\[
b(z) = \frac{g(z-a)h'(z) - h(z-a)g'(z)}{g(z-a)h(z) - g(z)h(z-a)},
\]

\[
c(x) = \frac{g(x)h'(x) - g'(x)h(x)}{g(x-a)h(x) - g(x)h(x-a)},
\]

where

\[
g(x) = \frac{x}{\sqrt{\sigma^2}} \frac{\mu}{2} \frac{1}{2} + \sqrt{\frac{\mu^2}{\sigma^2} + 2\beta - \mu},
\]

\[
h(x) = \frac{x}{\sqrt{\sigma^2}} \frac{\mu}{2} \frac{1}{2} - \sqrt{\frac{\mu^2}{\sigma^2} + 2\beta - \mu}.
\]

**Proof.** Let us consider \( \mathcal{P}(t) = P(t) - P(0) \). Define \( \mathcal{M} = \sup_{t \geq 0} \{\mathcal{P}(t)\} \). Then \( \mathcal{P}(0) = 0 \) and \( \mathcal{M}(t) = M(t) - P(0) \). The stopping time \( \tau = \inf_{t \geq 0} \{\mathcal{M}(t) - \mathcal{P}(t) \geq a\} = \tau \). And

\[
E\left[e^{\alpha M(\tau) - \beta \tau}\right] = e^{\alpha P(0)} E\left[e^{\alpha \mathcal{M}(\tau) - \beta \tau}\right].
\]

The \( E\left[e^{\alpha \mathcal{M}(\tau) - \beta \tau}\right] \) is given by (2.20)-(2.22), where \( g \) and \( h \) are two independent solutions of (2.12). Using (2.4) we can rewrite (2.12) as

\[
\frac{1}{2} \sigma^2 x^2 f''(x) + \mu x f'(x) = \beta f(x).
\]

Let us solve this equation. After substitution \( z = \log(x) \) or \( x = e^z \) we get

\[
\frac{d^2 f}{dz^2} + \left(\frac{2\mu}{\sigma^2} - 1\right) \frac{df}{dz} + \left(\frac{-2\beta}{\sigma^2}\right) f(z) = 0.
\]
The solution of the last equation then is
\[ f(z) = c_1 e^{r_1 z} + c_2 e^{r_2 z}, \]
where \( c_1, c_2 \) are some constants, and
\[ r_1 = \frac{\mu}{\sigma^2} - \frac{1}{2} + \sqrt{\frac{\mu^2}{\sigma^2} + 2\beta - \mu}, \]
\[ r_2 = \frac{\mu}{\sigma^2} - \frac{1}{2} - \sqrt{\frac{\mu^2}{\sigma^2} + 2\beta - \mu}. \]

Then
\[ f(x) = c_1 x^{r_1} + c_2 x^{r_2}. \]

Therefore, two independent solutions of the equation (2.33) are \( g(x) = x^{r_1} \) and \( h(x) = x^{r_2} \).

Applying (2.32) with Theorem 2 we get (2.29)-(2.31).

**Corollary 2.5.1.** The gain of the trading the line strategy with fixed value \( v \) defined by
\[ G(v) = M(\tau) - v - P(0) \] (2.34)
has the following expectation and variance
\[ \mathbb{E}[G(v)] = P(0) \left( \int_0^\infty d(x)dx - 1 \right) + \int_0^\infty xd(x)dx - v, \]
\[ \text{Var}[G(v)] = P^2(0) \int_0^\infty d(x)dx + 2P(0) \int_0^\infty xd(x)dx + \int_0^\infty x^2d(x)dx - \left( P(0) \left( \int_0^\infty d(x)dx - 1 \right) + \int_0^\infty xd(x)dx - v \right)^2. \]

Where
\[ d(x) = c(x)e^{-\int_0^x b(z)dz}, \]
\( c(x) \) and \( b(x) \) are given by (2.30) and (2.31) with
\[ g(x) = x^{\frac{\mu}{\sigma^2} - \frac{1}{2} + \sqrt{\frac{\mu^2}{\sigma^2} + 2\beta - \mu}}, \]
\[ h(x) = x^{\frac{\mu}{\sigma^2} - \frac{1}{2} - \sqrt{\frac{\mu^2}{\sigma^2} + 2\beta - \mu}}. \]
Proof. Put $d(x) = c(x)e^{-\int_0^x b(z)dz}$ and $\beta = 0$. Then

$$
E\left[e^{\alpha M(\tau)}\right] = e^{\alpha P(0)} \int_0^\infty d(x)e^{\alpha x}dx
$$

Differentiating with respect to $\alpha$ we get

$$
\frac{d}{d\alpha} \left( e^{\alpha P(0)} \int_0^\infty d(x)e^{\alpha x}dx \right) = P(0)e^{\alpha P(0)} \int_0^\infty d(x)e^{\alpha x}dx + e^{\alpha P(0)} \int_0^\infty d(x)xe^{\alpha x}dx
$$

Then letting $\alpha = 0$ we deduce

$$
E[M(\tau)] = P(0) \int_0^\infty d(x)dx + \int_0^\infty xd(x)dx
$$

From where using (2.34) we get (2.35). Similarly we can find the second moment of $M(\tau)$.

$$
\frac{d^2}{d\alpha^2} \left( e^{\alpha P(0)} \int_0^\infty d(x)e^{\alpha x}dx \right) = P^2(0)e^{\alpha P(0)} \int_0^\infty d(x)e^{\alpha x}dx + 2P(0)e^{\alpha P(0)} \int_0^\infty d(x)xe^{\alpha x}dx + e^{\alpha P(0)} \int_0^\infty d(x)x^2e^{\alpha x}dx
$$

Then letting $\alpha = 0$ we deduce

$$
E[M^2(\tau)] = P^2(0) \int_0^\infty d(x)dx + 2P(0) \int_0^\infty xd(x)dx + \int_0^\infty x^2d(x)dx
$$

From where using (2.35) and definition of the variance we get (2.36).

Now we will consider the case of trading the line strategy with fixed percentage. We will assume that $P(t)$ is the logarithm of the price process described by the following equation

$$
P(t) = \mu t + \sigma B(t), \quad t \geq 0.
$$

Where $B(t)$ is the standard Brownian motion with $B(0) = 0$. And drift $\mu$, volatility $\sigma^2$ are some given constants. Note that $P(0) = 0$. When we consider logarithm prices, we deduce
"trailing stop" techniques with fixed percentage $l$ to "trailing stop" technique with fixed value $L = \log(l)$. This model was considered by Glynn and Iglehart [26]. We have included proofs here for the sake of completeness. To start the analysis we define the following processes:

$$M(t) = \sup\{P(s) : 0 \leq s \leq t\},$$

$$T(L) = \inf\{t \geq 0 : P(t) \leq M(t) - L\},$$

$$G(L) = P(T(L)) = M(T(L)) - L,$$

$$X(t) = \sigma B(t) - \mu t,$$

$$m(t) = \inf\{X(s) : 0 \leq s \leq t\},$$

and

$$D(t) = M(t) - P(t).$$

$D(t)$ denotes the drawdown at time $t$. Using the fact that Brownian motion, $\{B(t) : t \geq 0\}$ has the same distribution as $\{-B(t) : t \geq 0\}$, we can write

$$D(t) = \sup\{\mu s + \sigma B(s) : 0 \leq s \leq t\} - \mu t - \sigma B(t)$$

$$= \sup\{\mu s - \sigma B(s) : 0 \leq s \leq t\} - \mu t + \sigma B(t)$$

$$= \sup\{-X(s) : 0 \leq s \leq t + X(t)\} = X(t) - m(t),$$

where $\overset{D}{=} \sup\{\mu s - \sigma B(s) : 0 \leq s \leq t\} - \mu t + \sigma B(t)$

where $\overset{D}{=}$ denotes two processes with the same distribution. The process $Z(t) = X(t) - m(t), t \geq 0$ is the so called regulated Brownian motion [27]. Now let
\[ I(t) = \sup\{X(s)^- : 0 \leq s \leq t\}, \]

where \( x^- = \max\{0, -x\} \). The quantity \( I(t) \) represents the local time in the interval \([0, t]\) corresponding to \( Z = 0 \). In our security model \( I(t) \) also represents the amount the stop has been raised in the interval \([0, t]\), that is locked in profit. We also have the representation

\[ Z(t) = X(t) + I(t), \quad t \geq 0. \]

From the above we see that

\[ T(L) = \inf\{t \geq 0 : Z(t) \geq L\}. \]

Since \( L \) will remain a fixed positive number throughout this section, we will drop the \( L \) in \( T(L) \) and \( G(L) \). Since \( Z(t) \) is a Markov process, we can let \( P_z[\cdot] = P[\cdot | Z(0) = z] \) and \( E_z[\cdot] = E[\cdot | Z(0) = z] \). Our first task is to find the Laplace transform of \( T \). We derive the transform here as the method of proof will be used later.

**Theorem 2.5.1.** For \( \lambda > 0 \), let \( u \) be a twice differentiable function on \([0, L]\) satisfying the differential equation

\[ \frac{\sigma^2}{2} u'' - \mu u' - \lambda u = 0 \]

with \( \lambda > 0 \) and boundary conditions \( u'(0) = 0 \) and \( u(L) = 1 \). Then,

\[ u(x) = E_x[\exp(-\lambda T)], \quad 0 \leq x \leq L \]

is the Laplace transform of \( T \) for \( Z(0) = x \).

**Proof.** Let

\[ v(t, Z(t)) = \exp(-\lambda t)u(Z(t)). \]
Then using the product rule for differentiation and Itô’s formula [23] we obtain
\[
dv(t, Z(t)) = -\lambda \exp(-\lambda t)u(Z(t))dt + \exp(-\lambda t)[\sigma B(t) - \mu dt + dI(t)]u'(Z(t))
\]
\[
+ \exp(-\lambda t) \frac{\sigma^2}{2} du''(Z(t))
\]
or in integrated form
\[
v(t, Z(t)) - v(0, Z(0)) = \int_0^t \exp(-\lambda s) \left[ \frac{\sigma^2}{2} u''(Z(s)) - \mu u'(Z(s)) - \lambda u(Z(s)) \right] ds
\]
\[
+ \sigma \int_0^t \exp(-\lambda s) u'(Z(s)) dB(s) + \int_0^t \exp(-\lambda s) u'(Z(s)) dI(s).
\]
The first term is zero, since \((\sigma^2/2)u'' - \mu u' - \lambda u = 0\), and the last term is
\[
u'(0) \int_0^t \exp(-\lambda s) dI(s),
\]
since \(I\) increases only when \(Z(\cdot) = 0\). However, because \(u'(0) = 0\) this term also vanishes. We are left with
\[
(2.37) \quad v(t, Z(t)) - v(0, Z(0)) = \sigma \int_0^t \exp(-\lambda s) u'(Z(s)) dB(s).
\]
The integral on the right-hand side above is a martingale, since it is an Itô integral with respect to the standard Brownian motion, \(\{B(t) : t \geq 0\}\). We wish to apply the optional sampling theorem to this martingale. Since \(u\) is twice continuously differentiable on \([0, L]\), \(u'(Z(s))\) is bounded for \(0 \leq s \leq T\). Thus
\[
E_x \left[ \int_0^T u'(X(s))^2 ds \right] \leq K E_x \left[ \int_0^\infty \exp(-2\lambda s) ds \right] < \infty.
\]
Hence we can apply the optional sampling theorem to the right-hand side of (2.37) and conclude that
\[
E_x [v(T, Z(T))] = v(0, x) = u(x).
\]
So,

\[ u(x) = E_x [\exp(-\lambda T)u(Z(T))] \cdot \]

But \( Z(T) = L \) and \( u(L) = 1 \), thus

\[ u(x) = E_x [\exp(-\lambda T)], \ 0 \leq x \leq L. \]

\[ \square \]

We proceed now to solve the differential equation given in Theorem 3. A trial solution

of the form \( u(x) = \exp(rx) \) leads to the equation

\[
\left[ \frac{\sigma^2}{2} r^2 - \mu r - \lambda \right] u(x).
\]

Setting the bracketed term equal to 0 yields two distinct roots:

\[ r_1 = \frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2 + 2\lambda \sigma^2}{\sigma^2}} \]

and

\[ r_2 = \frac{\mu}{\sigma^2} - \sqrt{\frac{\mu^2 + 2\lambda \sigma^2}{\sigma^2}}. \]

Thus our solution \( u \) becomes

\[ u(x) = A \exp(r_1 x) + B \exp(r_2 x). \]

To fix the boundary conditions, note that

\[ u'(0) = r_1 A + r_2 B = 0, \]

\[ u'(L) = \exp(r_1 L) A + \exp(r_2 L) B = 1. \]

Solving this pair of simultaneous equations, we have

\[ A = \frac{r_2}{r_1 \exp(r_2 L) - r_2 \exp(r_1 L)} \]
and

\[ B = \frac{r_1}{r_1 \exp(r_2 L) - r_2 \exp(r_1 L)}. \]

Finally, the Laplace transform of \( T \) for initial state \( x = 0, u(0) \), is

\[ E_0 [\exp(-\lambda T)] = \frac{r_1 - r_2}{r_1 \exp(r_2 L) - r_2 \exp(r_1 L)}. \]

To compute \( E_0 [T] \) we can either differentiate that Laplace transform of \( T \) or use the fact that

\[ P(T) = M(T) - L = G(L). \]

Since \( \{P(t) : t \geq 0\} \) is assumed to be a \( (\mu, \sigma^2) \) Brownian motion, we have for every \( t > 0 \)

\[ E_0 [P(T \wedge t)] = \mu E_0 [T \wedge t] + \sigma E_0 [B(T \wedge t)] = \mu E_0 [T \wedge t]. \]

We let \( t \to \infty \), and note that by monotone convergence \( E_0 [T \wedge t] \to E_0 [T] \). Next use the inequalities

\[ E_0 [|P(T \wedge t)|] = E_0 [|X(T \wedge t)|] \leq E_0 [|Z(T \wedge t)|] + E_0 [I(T \wedge t)] \leq L + E_0 [M(T)] < \infty \]

plus dominated convergence theorem to conclude that

\[ \mu E_0 [T] = E_0 [G(L)]. \]

Finally, we have

\[ E_0 [T(L)] = \frac{L}{\mu \beta} [\exp(\beta) - 1] - \frac{L}{\mu}. \]

Next we seek the Laplace transform for \( M(\tau) \), which is equal to the gain \( G(L) \) except for a factor \(-L\). Note that for initial state \( Z(0) = 0 \), \( M(\tau) = I(\tau) \).

**Theorem 2.5.2.** For \( \lambda > 0 \), let \( u \) be a twice differentiable function on \([0, L]\) satisfying the differential equation

\[ \frac{\sigma}{2} u'' - \mu u' = 0 \]
with boundary conditions $u'(0) = 0$ and $u(L) = 1$. Then, for $0 \leq x \leq L$,

$$u(x) = E_x[\exp(-\lambda I(T))],$$

and thus

$$u(0) = E_0[\exp(-\lambda M(T))].$$

**Proof.** Let

$$v(t, Z(t)) = \exp(-\lambda I(t))u(Z(t)).$$

Then as in Theorem 3 we get

$$dv(t, Z(t)) = \exp(-\lambda I(t))(-\lambda dI(t))u(Z(t)) + \exp(-\lambda I(t))du(Z(t))$$

$$= -\lambda u(Z(t))\exp(-\lambda I(t))dI(t) + \exp(-\lambda I(t))\left[\sigma dB(t) - \mu dt + dI(t)u'(Z(t))\right]$$

$$+ \exp(-\lambda I(t))\frac{\sigma^2}{2} dtu''(Z(t))$$

or in integrated form

$$v(t, Z(t)) - v(0, Z(0)) = \int_0^t \exp(-\lambda I(s))\left[\frac{\sigma^2}{2}u''(Z(s) - \mu u'(Z(s)))\right] ds$$

$$+ \sigma \int_0^t \exp(-\lambda I(s))u'(Z(s))dB(s) + \int_0^t \exp(-\lambda I(s))[u'(Z(s)) - \lambda u(Z(s))]dI(s).$$

The first term is zero since $(\sigma^2/2)u'' - \mu u' = 0$, and the last term is zero since $I$ only increases when $Z(\cdot) = 0$ and $u'(0) - \lambda u(0) = 0$. This leaves

$$v(t, Z(t)) - v(0, Z(0)) = \sigma \int_0^t \exp(-\lambda I(s))u'(Z(s))dB(s),$$

a martingale. From here the argument follows that used in Theorem 3.

To solve the differential equation given in Theorem 4 we again try a solution of the form

$$u(x) = \exp(rx).$$

Then $\sigma^2 r^2 - \mu r = 0$, so $r = 0$ and $r = 2\mu/\sigma^2$. Therefore,

$$u(x) = A + B \exp(2\mu x/\sigma^2).$$
Since \( u'(0) - \lambda u(0) = 0 \) and \( u(L) = 1 \), we have

\[
2B\mu/\sigma^2 - \lambda(A + B) = 0
\]

and

\[
A + B \exp(2\mu L/\sigma^2) = 1.
\]

Solving for \( A \) and \( B \) yields

\[
A = \frac{2\mu/\sigma^2 - \lambda}{\lambda(\exp(2\mu L/\sigma^2) - 1) + 2\mu/\sigma^2}
\]

and

\[
B = \frac{\lambda}{\lambda(\exp(2\mu L/\sigma^2) - 1) + 2\mu/\sigma^2}.
\]

Hence,

\[
u(0) = E_0[\exp(-\lambda M(T))] = \frac{2\mu[\sigma^2(\exp(2\mu L/\sigma^2)) - 1]}{2\mu[\sigma^2(\exp(2\mu L/\sigma^2)) - 1]} + \lambda.
\]

The Laplace transform reveals that \( M(T) \) is exponentially distributed with parameter \( \beta/([\exp(\beta) - 1]L) \), where \( \beta = 2\mu L/\sigma^2 \). The mean and variance of the gain, \( G(L) = M(T) - L \), are given by

\[
E[G(L)] = L[(\exp(\beta) - 1)/\beta - 1]
\]

and

\[
\sigma^2[G(L)] = L^2[\exp(\beta) - 1]^2/\beta^2.
\]
CHAPTER 3

Trading the Line Strategy in Fractional Brownian Motion Model

3.1 Fractional Brownian Motion

We will assume that the logarithm of the price process is driven by fractional Brownian motion \([29]\) \(^1\) with zero drift and constant volatility \(\sigma^2\)

\[ P(t) = \sigma B^H(t), \quad t \geq 0, \]

\[ P(0) = 0, \quad 0 < H \leq 1. \]

Where \(B^H(t)\) is a fractional Brownian motion with Hurst exponent \(H\) \([28]\), i.e. a Gaussian process with continuous trajectories and with \(B^H(0) = 0, \quad \mathbb{E}[B^H(t)] = 0, \quad \mathbb{E}[(B^H(t))^2] = t^{2H}\), \(\mathbb{E}[\mathbb{C}_t \mathbb{C}_s[B^H(t), B^H(s)]] = \frac{1}{2}|t|^{2H} + |s|^{2H} - |t-s|^{2H}\). For the case \(H = \frac{1}{2}\) a fractional Brownian motion is a standard Brownian motion. For \(H = 1\) we get \(B^1(t) \overset{D}{=} \eta t\), where \(\eta \sim N(0, 1)\).

3.2 Trading the Line Strategy in the Markets Driven by FBM

Let us consider the case of trading the line strategy with fixed percentage \(p\). In this case the trader will be stopped out of the trade whenever the logarithmic price drops below its maximum by \(L = \ln(1 + \frac{p}{100})\) units. As in Black-Scholes model we define the following processes

\[ M(t) = \sup_{0 \leq s \leq t} P(s), \]

\[ D(t) = M(t) - P(t), \]

\(^1\)The process \(B^H(t)\) were first considered by A. N. Kolmogorov \([30]\) in 1940, where they were called Wiener helices. The name “fractional Brownian motion” was introduced in 1968, by B. Mandelbrot and J. van Ness \([29]\).
Theorem 3.2.1. Let $\tau = \inf_{t \geq 0} \{D(t) \geq L\}$. If

$$\int_0^t e^{-\lambda s^2} \sinh(x) dB^{\mathbb{H}}(s)$$

is defined in some sense and the following expectation exists and finite

$$\mathbb{E} \left[ \int_0^\tau e^{-\lambda s^2} \sinh(x) dB^{\mathbb{H}}(s) \right] = \gamma.$$ 

Then for $\lambda > 0$

$$\mathbb{E} \left[ \exp^{-\lambda \tau^2} \right] = \frac{1}{\cosh \left( \frac{\sqrt{2} \lambda L}{\sigma} \right)} + \frac{\gamma}{4 \lambda \cosh \left( \frac{\sqrt{2} \lambda L}{\sigma} \right)}.$$ 

Proof. Let $v(t, D(t)) = e^{-\lambda t} u(t, D(t))$, for some twice continuously differentiable on $[0, \infty) \times [0, L]$ function $u(t, x)$ satisfying the differential equation

$$(3.1) \quad -\lambda u(t, x) + u_t(t, x) + \sigma^2 \tau^2 \sinh(x) dx = 0$$

with boundary condition

$$(3.2) \quad u_x(t, 0) = 0.$$ 

Then by [32] we get

$$dv(t, D(t)) = -\lambda e^{-\lambda t} u(t, D(t)) + e^{-\lambda t} \left( u_t(t, D(t)) + \sigma^2 \tau^2 \sinh(x) dx \right) \, dt$$

$$-e^{-\lambda t} \sigma u_x(t, D(t)) dB^{\mathbb{H}}(t) + e^{-\lambda t} u_x(t, D(t)) dM(t),$$

or in integrated form

$$v(t, D(t)) - v(0, 0) = \int_0^t e^{-\lambda s} \left( -\lambda u(s, D(s)) + u_s(s, D(s)) + \sigma^2 \tau^2 \sinh(x) dx \right) \, ds$$

$$(3.3) \quad -\sigma \int_0^t e^{-\lambda s} u_x(s, D(s)) dB^{\mathbb{H}}(s) + \int_0^t e^{-\lambda s} u_x(s, D(s)) dM(s).$$

Since, $u(t, x)$ satisfies (3.1) the first integral in (3.3) equals to zero. We will show that

$$u(t, x) = c_1 \delta(x) e^{\lambda x} - c_2 \frac{\sqrt{2} \tau}{\sigma},$$
where
\[ \delta(x) = e^{\sqrt{\frac{\pi}{2}}x} + e^{-\sqrt{\frac{\pi}{2}}x}. \]

Therefore, by the assumption of the theorem the second integral is defined and has finite expectation. Let us consider the third integral

\[ \int_0^t e^{-\lambda s} u_x(s, D(s)) dM(s). \]

(3.4)

\( M(s) \) is increasing function. Therefore, it is a function of bounded variation on any finite interval \([0, t]\) and (3.4) is just a standard Lebesgue integral. Moreover, whenever logarithmic price process \( P(t) \) ties a new maximum,

\[ M(t) = P(t), \]

at the other points \( M(t) \) is flat. Let us introduce two sets. Set \( A_t = \{ 0 \leq s \leq t : M(t) = P(t) \} \) and set \( B_t = \{ [0, t] \setminus A_t \} \). Then,

\[ \int_0^t e^{-\lambda s} u_x(s, D(s)) dM(s) = \int_{A_t} e^{-\lambda s} u_x(s, D(s)) dM(s) + \int_{B_t} e^{-\lambda s} u_x(s, D(s)) dM(s). \]

Note, that over the set \( A_t \), by its definition, \( D(s) = 0 \). Over the set \( B_t \) the integrator \( dM(s) \) equals to zero. Therefore, the second integral on the right side equals to 0. So, on the right side of the last equation, in both integrals, we can replace \( u_x(s, D(s)) \) by \( u_x(s, 0) \). Finally, we get

\[ \int_0^t e^{-\lambda s} u_x(s, D(s)) dM(s) = \int_0^t e^{-\lambda s} u_x(s, 0) dM(s). \]

This integral equals to 0 by boundary condition (3.2). We have showed that the first and third integrals on the right side of (3.3) equal to 0. Therefore, we get

\[ v(t, D(t)) - v(0, 0) = -\sigma \int_0^t e^{-\lambda s} u_x(s, D(s)) dB^H(s). \]

Let us substitute time variable \( t \) by stopping time \( \tau \). Then we get

\[ v(\tau, D(\tau)) - v(0, 0) = -\sigma \int_0^\tau e^{-\lambda s} u_x(s, D(s)) dB^H(s). \]
Then we can deduce
\[ E[v(\tau, D(\tau)) - v(0, 0)] = \gamma. \]

Note, that \( D(\tau) = L \) and \( v(0, 0) = \exp^{-\lambda_0} u(0, D(0)) = u(0, 0) \). Therefore,
\[ E[v(\tau, D(\tau))] = u(0, 0) + \gamma. \]

Using definition of \( u(t, x) \) we can rewrite the last equation
\[ (3.5) \quad E\left[e^{-\lambda t} u(\tau, L)\right] = u(0, 0) + \gamma. \]

The general solution of (3.1) has the following form
\[
 u(t, x) = c_1 e^{-\frac{c_1^2 tH}{2t}} c_2 e^\frac{\sqrt{\pi} t}{\sqrt{H}} + c_1 e^{-\frac{c_1^2 tH}{2t}} c_3 e^\frac{\sqrt{\pi} t}{\sqrt{H}} e^{-\lambda t}.
\]

Where \( c_1, c_2, \) and \( c_3 \) are some constants. In order to satisfy (3.2) we set \( c_2 = c_3 = 1 \). Then we can rewrite \( u(t, x) \) in the following form
\[
 u(t, x) = c_1 \delta(x)e^{\lambda t - \frac{c_1^2 tH}{2t}},
\]
where
\[
 \delta(x) = e^{\frac{\sqrt{\pi} t}{\sqrt{H}}} + e^{-\frac{\sqrt{\pi} t}{\sqrt{H}}}.
\]

Using this formula we can derive from (3.5) the following
\[
 u(0, 0) + \gamma = E\left[e^{-\lambda \tau} u(\tau, L)\right] = E\left[e^{\lambda \tau} c_1 \delta(L) e^{\lambda \tau - \frac{c_1^2 \tau^2 H}{2\tau}}\right] = E\left[c_1 \delta(L) e^{-\frac{c_1^2 \tau^2 H}{2\tau}}\right].
\]

So, we get
\[
 u(0, 0) + \gamma = E\left[c_1 \delta(L) e^{-\frac{c_1^2 \tau^2 H}{2\tau}}\right].
\]

From where we derive
\[
 E\left[e^{-\frac{c_1^2 \tau^2 H}{2\tau}}\right] = \frac{u(0, 0) + \gamma}{c_1 \delta(L)}.
\]

Note, that \( u(0, 0) = c_1 \delta(0) \). Then we derive
\[
 E\left[e^{-\frac{c_1^2 \tau^2 H}{2\tau}}\right] = \frac{\delta(0)}{\delta(L)} + \frac{\gamma}{c_1 \delta(L)}.
\]
Now let us choose $c_1 = 2^H \lambda$ and use the definition of $\delta(x)$. Then we get

$$E \left[ \exp^{-\lambda \tau^{2H}} \right] = \frac{1}{\cosh \left( \frac{\sqrt{2} \lambda L}{\sigma} \right)} + \frac{\gamma}{4 \lambda \cosh \left( \frac{\sqrt{2} \lambda L}{\sigma} \right)}.$$ 

\[ \square \]

It is a well known fact that for regular Brownian motion (when $H = \frac{1}{2}$) the Laplace transform of the stopping time $\tau$ is

$$E \left[ \exp^{-\lambda \tau} \right] = \frac{1}{\cosh \left( \frac{\sqrt{2} \lambda L}{\sigma} \right)}.$$ 

Therefore, it is interesting to note that if $\gamma = 0$ then $\tau^{2H}$ has the same Laplace transform as the one in case of regular Brownian motion.
APPENDIX A

Literature and Results Review

There are many references to the trading the line strategy in the financial literature [1], [5], [9], [12], and [22]. However, there are only a few mathematical papers on this topic. The rigorous mathematical analysis of this strategy started with paper by Glynn and Iglehart in 1995 [26]. They studied several discrete time processes and Brownian motion with drift process only. It turns out that in discrete time, the theory can be extended to a very wide class of processes introduced by R.A. Khan and M. Kazim Khan [17] in 2004. This extension resulted in many applications introduced in the 1st chapter. In continuous time case, one can use results by John Lehoczky [25] to reduce the original problem to the boundary value problem for some partial differential equation. This analysis resulted in the Proposition 1 in the 2nd chapter. We have gone one step further trying to extend the result to the case of fractional Brownian motion. This produced the Theorem 3.0.3. There is a lack of the optional sampling theorem in the markets driven by fractional Brownian motion. This is a very tempting open problem in the theory of stochastic integrals at the moment. In the last theorem we assume that the following integral

\[ \int_0^t e^{-\lambda s^2t} \sinh(x) dB^H(s) \]

is defined in some sense and the following expectation exists and finite

\[ E \left[ \int_0^T e^{-\lambda s^2t} \sinh(x) dB^H(s) \right] = \gamma. \]

There appear to be several groups studying this type of stochastic integrals at the moment. Although there are some interesting discoveries [32] - [33], the question about the value of
the following expectation

\[ E \left[ \int_0^T e^{-\lambda s^2H} \sinh(x) dB^H(s) \right] \]

remains open. Therefore, we have denoted this expectation by \( \gamma \) in this dissertation.
APPENDIX B

Real Application Results

We used the trading the line strategy for the real time intraday trading. During the less than two month period (starting from January 9th of 2007 to February 23rd of 2007) we have made 52 trades. With a capital of $100,000 we virtually gained $6,780. This information is represented in the table below.

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<th># of Trades with Loss</th>
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In the table below you can see more detailed information about all 52 trades. The "Last Execution Date" column shows that date on which the trade took place. "P&L" stands for profit and loss. The positive numbers correspond to the profit due to the trade, the negative numbers correspond to the loss. "Security" column’s symbols consist of two parts. The first part (before the dot) is the corresponding stock ticker (by which the stock can be identified). The second part (after the dot) is the exchange’s symbol. "O" corresponds to NASDAQ, "N" corresponds to NYSE, and "A" corresponds to AMEX. "Type" shows that type of the traded securities, and "S" stands for a stock.
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