THE EFFECT OF A MODIFIED MOORE METHOD
ON CONCEPTUALIZATION OF PROOF
AMONG COLLEGE STUDENTS

A dissertation submitted to the
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by

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Research projects have long indicated incompetence in justification of mathematical arguments among students of high school as well as college level. This research sought to understand the mental proof schemes that students possess, and to determine if these schemes progress when the Moore method is used in teaching a mathematics course.

This study is significant in two major ways: First, it confirmed the mental schemes of proofs proposed by Harel & Sowder. Second, the study examined the effect of the Moore method on students’ learning and appreciation of proofs. Using a qualitative analysis design the study showed that the Moore method has positive affects on students’ conceptualization of mathematical proof, their self-confidence in their abilities, their appreciation of the relevance of proofs, and their ability to think autonomously. The Moore method allowed students to experience mathematics first hand. They built a coherent body of knowledge in which they created their own proofs.
In memory of

my mother Aminah

and my father Yousef
ACKNOWLEDGMENTS

This dissertation could not have been written without Drs. Genevieve Davis and Michael Mikusa who not only served as my advisers but also encouraged and challenged me throughout this research project. They and Dr. Ulrike Vorhauer patiently guided me through the process, never accepting less than my best efforts, and for that I thank them.

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CHAPTER I
INTRODUCTION
Focus and Rationale for the Study

Research projects have long indicated incompetence in justification of mathematical arguments among students of high school as well as college level (Lovell, 1971; Usiskin, 1987; Williams, 1980). This incompetence was also reported among pre-service elementary school teachers (Goetting, 1995; Martin & Harel, 1989). The incompetence in mathematical proofs is attributed to many misconceptions of what a mathematical proof is and what constitutes a valid justification of a general argument. Students tend to try to empirically justify a general statement by a series of particular instances (Goetting, 1995; Lovell, 1971; Martin & Harel, 1989). When a statement is proven deductively there is a tendency among students to further justify it empirically by using particular examples (Martin & Harel, 1989; Vinner, 1983). Research projects also identified the lack of appreciation of counter example to refute general statements among students (Galbraith, 1981). It was also found that for many students, proofs are confirmations to facts and intuitions already known to be true (Schoenfeld, 1985). This misconception may be largely attributed to the pedagogical approach to teaching proofs. It is often the case that students are presented with finished products of proofs and do not participate in the construction of knowledge (Alibert & Thomas, 1991). Teachers generally do not pay much attention to what constitutes proofs and mathematical
justifications in the minds of students; so rather than refining the conception of proofs in the minds of students, they impose on them methods of proofs and implication rules that are disconnected from their conceptions of proofs (Harel & Sowder, 1998).

A recommended approach by the National Council of Teachers of Mathematics (NCTM), in their *Principles and Standards for School Mathematics*, is to make mathematical justifications and constructing proof part of the students’ mathematical experience starting from preschool through grade 12. By the time students finish high school it is expected they are able to make mathematical conjectures and evaluate them as well as construct arguments and proofs (NCTM, 2000). This recommendation directly addresses Moore’s suggestion that one reason students have difficulties with proofs is due to experiences in their pre-college schooling that are typically restricted to traditional approaches frequently used in high school geometry (R. C. Moore, 1994).

Harel and Sowder (1998) provided the first attempt to map students’ proof schemes. In part of their “Proof Understanding, Production, and Appreciation” project (PUPA), Harel and Sowder have provided the following mapping of students’ proof schemes. They proposed three major categories of schemes: External Conviction Proof Schemes, Empirical Proof Schemes, and Analytical Proof Schemes. Each of the three major categories has subcategories. Under the External Conviction Scheme there are the Ritual, Authoritarian, and Symbolic schemes. The Empirical Scheme is divided into Inductive and Perceptual schemes. The Analytical Scheme has many sublevels. The two major sub-schemes of the Analytical Scheme are Transformational and Axiomatic Schemes, each of which has its own subcategories. The Axiomatic Schemes are Intuitive
Axiomatic, Structural, and Axiomatizing Scheme. The Transformational Scheme has three major categories: Internalized, Interiorized, and Restrictive Schemes. The Restrictive Scheme has three sub-schemes: Contextual, Generic, and Constructive.

Spatial Scheme is a sub-category of the Contextual Proof Scheme. Figure 1 is based on an illustration in Harel and Sowder (p. 245).

In their study, Harel and Sowder suggested that “the system of proof schemes described here must be validated by other researchers through multiple teaching experiments taught by various instructors in various institutions” (1998, p. 238). Therefore, one of the tools utilized in this study to gauge the change in students’ conceptualization of proof is the schematic classification provided by Harel and Sowder.

In addition, the teaching environment for this study is inquiry based. In particular, the approach utilized is modeled after the R. L. Moore method of teaching proof-oriented mathematics. His method is called the Moore method and sometimes it is called the Texan method. In a Moore method class students are given few definitions and axioms and are asked to prove from them a logical sequence of theorems which they would present in class. No notebooks are used, and the instructor gives some motivation but lectures only sparingly. R. L. Moore required that students create their own original solution without the assistance of textbooks or other individuals including classmates. Moore’s assessment of students’ performance was based on the theorem they proved that was either presented in class or submitted to him.

For the purpose of this research a modified version of the Moore method was used. Students were allowed to use extra resources and worked in pairs. Assessment in
Figure 1. Levels of students’ conceptual schema of mathematical proofs (Source: Harel & Sowder, 1998, p. 245)
the course was based on students’ presentations in class as a whole and to the instructor in his office as well in class quizzes. For the purpose of simplicity, this form of modified Moore method is referred to as the Moore method throughout this dissertation.

The targeted population for this study is undergraduate mathematics students and pre-service secondary mathematics education majors.

The study was conducted in an undergraduate “Numbers and Games” course in a state university in Ohio. The course is an upper division mathematics course. Typically, it is offered for mathematics majors as well as secondary mathematics education majors as an elective course to complete the mathematics requirement. Students enrolling in this class are expected to have had the calculus sequence, differential equations, linear algebra, fundamental concepts of algebra or abstract algebra, and fundamental concepts of geometry. The majority of students taking this class are in their senior year; however, some take it during their junior year.

The main goal of this study is to better understand the mental processes and the itineraries that promote students to become proof producers. Furthermore, the goal of this research is to understand the mental proof schemes that students possess to determine if these schemes progress to become more aligned with what is generally accepted among the community of mathematicians. This study investigates the construction of such schemes to see if they manifest themselves in the production and appreciation of proofs that are analytical as both inductive as well as deductive.
Statement of the Problem

Does the system of proof schemes of undergraduate mathematics students in an inquiry-based environment develop in a classroom where the Moore method is the pedagogical framework within which instruction takes place?

Research Questions

This research study is designed to answer the following questions:

1. What are students’ conceptions of proofs and what do they consider mathematical justification (Harel & Sowder, 1998)?
2. Do students’ experiences in the class taught using the Moore method (a description of the method is provided in Chapter 2) affect their conceptions of proofs?
3. Is the Moore method effective in promoting development of students’ proof schemes to higher schematic levels as defined by Harel and Sowder (1998)?

Major Goals of This Study

1. Document the progress college mathematics students make in their conceptions of mathematical proofs in a Moore method class.
2. Validate, expand or modify the mapping of the students’ proof schemes (Harel & Sowder, 1998).

Significance of the Study

This study is significant in two major ways: First, it is a follow-up to a study that provided the first attempt at providing mental schemes of proofs (Harel & Sowder, 1998), which would validate, expand, or modify the proposed classification of proof schemes.
Second, the study examines the effect of the Moore method on students’ learning and appreciation of proofs.

NCTM standards emphasize the importance of logical deduction and rigorous arguments. According to the NCTM standards, by the end of secondary school students should be able to rigorously deduce logically coherent arguments. By the time students enter college, students should be able to start with hypotheses and work their way to proving mathematical statements based on those hypotheses. The following quote from the NCTM standards highlights the value of producing and appreciating proofs. This study provides us with an insight into how the vision of NCTM could be achieved.

The NCTM standards for reasoning and proof emphasize the importance of developing mathematical reasoning. They recommended that instructional programs “should enable all students to

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof” (NCTM, 2000, p. 342).

Definition of Terms

The list of terms used throughout this dissertation is alphabetized below. Some of the terms are defined in terms of others in the list that may appear later. Those terms are italicized. Some knowledge of the constructivist theory of learning is assumed by the reader. A discussion of constructivism is included in chapter 2. The terms abstraction,
internalization, interiorization, and second level of interiorization and mental models

were defined by Battista (1995). The remaining terms were defined by Harel and Sowder (1998).

1. Abstraction: The process of creating new knowledge by making
   generalizations from particular instances (Battista, 1995).

2. Analytical proof scheme: Certainty is established through logical deductions.
   It consists of two categories, transformational schemes and axiomatic schemes
   (Harel & Sowder, 1998).

3. Authoritarian proof scheme: Certainty is established based on who presents
   the argument (Harel & Sowder, 1998). For example, a student at this level
   would accept a flawed argument if it is presented by an authority source such
   as a teacher.

4. Axiomatic proof scheme: Certainty is established through mathematical
   justifications that have originated from undefined terms and axioms that are
   accepted without proof (Harel & Sowder, 1998). A person possessing this
   scheme would be able to prove for instance that “every compact space is
   Lindelof.” The statement originated from the axioms and terms of topological
   spaces.

5. Axiomatizing proof scheme: Certainty is established by investigating the
   implications of varying a set of axioms. At this level the individual is able to
   axiomatize a certain field (Harel & Sowder, 1998). For example, students
understand that the axioms of topology do not require interpretation based on physical reality.


7. Constructive proof scheme: A restrictive proof scheme in which certainty is established by actual construction of objects. Individuals processing this scheme are not satisfied with the proof of the existence of such objects without the construction of those objects (Harel & Sowder, 1998). The projective plane, for instance, could not be physically constructed. It is an object living in $\mathbb{R}^4$; therefore, a person acting within this scheme would have difficulty proving facts about it.

8. Contextual proof scheme: A restrictive proof scheme in which certainty is established by interpreting mathematical assertions in a specific setting (Harel & Sowder, 1998). Individual possessing this scheme would deal with assertions about open sets in a topological space in the context of the real number with the usual topology. Hence open sets are open intervals; closed sets are closed intervals and so on.

9. Empirical proof scheme: Certainty is established based on inductive empirical or perceptual empirical proof schemes (Harel & Sowder, 1998). Someone at this level would consider the statement “for any odd integer $n$, $n^3 - n$ is always even” to be true by trying different numbers and plugging them in to prove the statement.
10. External conviction proof scheme: Certainty is established based on the *ritual*,
*authoritarian*, or *symbolic* proof schemes (Harel & Sowder, 1998).

11. Generic proof scheme: Restrictive proof scheme in which certainty is
established by assertions that are interpreted in general terms but the proof is
expressed in a particular context (Harel & Sowder, 1998). A person with this
proof scheme would state that a T2 topological space is T1 (the statement is
about any topological space) yet proves it by considering two points in the
real line and open sets are presented as open interval.

12. Inductive proof scheme: Certainty is established based on quantitative
evaluation of one or more specific cases. This scheme may be referred to as
proof by example (Harel & Sowder, 1998).

13. Interiorization: The process of decomposing, operating on, and using in novel
situations an *internalized* item or action (Battista, 1999a).

14. Interiorized proof scheme: An *internalized transformational* proof scheme
that has been reflected upon. At this stage the internalized method of proof
becomes the object on which the individual operates on and anticipates the
result of the operation such as performing comparisons between proof
schemes, communicating the proof schemes with others and determine when
the scheme could be utilized (Harel & Sowder, 1998). An example would be a
person looking at a mathematical claim and making a decision whether to use
inductive or deductive proof to either prove it or disprove it. The decision in
that case is based on mental manipulation and comparison of the two methods.
15. Internalization: The process of mentally re-presenting an abstracted item or mentally replaying an abstracted action in the absence of the item or the action (Battista, 1999a).

16. Internalized proof scheme: A transformational proof scheme that has been abstracted as a method of proof to establish certainty (Harel & Sowder, 1998).

17. Intuitive Axiomatic proof scheme: Axioms are accepted if they are self-evident or appeal to the individual’s intuition (Harel & Sowder, 1998). For example, students operating with this scheme can only understand “the union of any two open sets in a topological space is open” in the context of real plane with the usual topology.

18. Mental Model: A mental image or construction that represents an event a person has experienced or is experiencing (Johnson-Laird, 1983). Objects in the mental model are available for manipulation by the reasoner to observe the results of such manipulation. The construction of a mental model is controlled by the existing knowledge the person has and the meaning the person makes of the event being represented (Battista, 1995).

19. Perceptual proof scheme: Certainty is established based on perceptions and coordination of perceptions without the ability to transform or anticipate the outcome of transforming the perceived objects (Harel & Sowder, 1998).

20. Reflection: The process of the mind considering, for examination, its own thinking in order to detect patterns, look for commonalities and differences as well, and to solve problems (Battista, 1995).
21. Restrictive proof scheme: In the process of establishing certainty restrictions are presumed by individuals possessing the transformational proof scheme on the context of the conjecture, the generality of the justification, or the mode of justification. Restrictions may be presumed on one or more of the above. Restrictions are classified as contextual, generic, and constructive (Harel & Sowder, 1998).

22. Ritual proof scheme: Certainty is established based on the ritual of presentation of the argument (Harel & Sowder, 1998).

23. Second level of Interiorization: The process by which an interiorized item or action can be further abstracted, operated on and integrated into a new form, so that the item or action itself can be symbolized without having to re-present the entire action (Battista, 1999a).

24. Spatial proof scheme: A contextual restrictive transformational proof scheme in which the context is the individual’s own imaginative space, in other words, what the person is able to imagine (Harel & Sowder, 1998). One could imagine that the space $\mathbb{R}^3$ is made up of concentric spheres centered at the origin.

25. Structural proof scheme: Certainty is established by thinking of conjectures and theorems as representations of situations from different realizations understood to share a common structure. The common structure is characterized by a collection of axioms (Harel & Sowder, 1998). For example, one studies measure theory based on the axioms of metric spaces.
26. Symbolic proof scheme: Certainty is established based on the level of symbolic usage (Harel & Sowder, 1998). A different value is placed on the same proof based on the amount of narrative as opposed to the use of symbols.

27. Transformational proof scheme: Certainty is established by operating on objects and anticipating the results of the operation with specific goals driving those operations. Transformational schemes have three sub-schemes: *internalized proof scheme*, restrictive, and *interiorized proof scheme* (Harel & Sowder, 1998). For example, student transforms an annulus from being a planer ring into a cylinder in $\mathbb{R}^3$ for the purpose of creating a torus by identifying the points on the boundary of the cylinders. Or starting by a planer rectangle identifies the points of a pair of opposite sides to create a cylinder in $\mathbb{R}^3$ then identify the other pair of opposite sides to create a torus.

**Organization of the Study**

The intent of this study is designed to follow a small group of volunteers from the “Numbers and Games” course and investigate their proof schemes as they develop along the course of a Moore type inquiry based environment. See Appendix E for the class syllabus. This qualitative study is designed to have two to four participants. The goals of the study were announced at the beginning of the course, and the need for volunteers was communicated to the students taking the course. The class was not taught by the researcher; rather it was taught by a faculty member of the mathematics department. A form was distributed describing the study and volunteers were solicited (see Appendix
A). Once the volunteers were identified, they were interviewed by the researcher. One of the participants was interviewed twice and one was interviewed three times. The interviews were not structured yet all of them started with the questions, “What constitutes a proof in your mind?” and “Do you think you are constructing proofs in the class?” The intent of these interviews was to monitor and document the shifts in their proof schemes. All interviews were video taped. All students in the class were required to present their proof on a regular basis in the professor’s office. The researcher video taped volunteers in office presentations. The purpose of taping the office presentations was to study the volunteer’s ability to convince others of the correctness of his or her proof and how this may change that state of certainty he or she possessed.

The researcher took a constructivist, interpretive approach in analyzing students’ understanding of proof. To answer the research questions from a constructivist perspective, the design of this study is qualitative in nature. The purpose is to better understand the structure of proof schemes as proposed by Harel and Sowder (1998) within the context of this “Numbers and Games” course and the pedagogy adopted for it.

This chapter is devoted to defining the problem this study is designed to address. In chapter 2, a review of the literature is provided as well as a survey of the major research projects that addressed relevant issues. Chapter 3 is devoted to a detailed description of the design of the study and methodology for collecting and analyzing the data. The results are included in chapter 4. Chapter 5 is a synthesis of the findings and elaboration of how these findings fit into the existing body of research.
CHAPTER II
LITERATURE REVIEW

The intent of this research study is to gain a deep understanding of proof conceptualization, learning, and production. This chapter is designed to give the reader a perspective on the history of the development of proof. This history is briefly presented in the first section. At first glance the concept of proof appears to be obvious as to what it means and what its functionality is in the field of mathematics. The fact is that this meaning is far from obvious. The next two sections provide the reader with a brief overview of some of the different schools of thought concerning the meaning and functionality of proof. The goal of this study is to investigate the effectiveness of the Moore method in teaching and learning of proof. This chapter includes a section devoted to describing the Moore method and its history.

The theoretical foundation for this research study is anchored in the constructivist theory of learning. We first introduce Jean Piaget, one of the great thinkers who had a profound influence on the development of constructivist theory and philosophy. After discussing Jean Piaget, the theory itself is introduced. Finally, the chapter ends with some discussion of Van Hiele’s levels of learning geometric proof. Van Hiele’s theory adds another dimension to this study. Whereas Harel and Sowder (1998) proposed a framework to explain the conceptualization of proof, Van Hiele gave an explanation of
how the knowledge of proof develops. These two theories combined create a rich and comprehensive view of this important area in mathematical learning.

The sections in this chapter are presented in the following order: Jean Piaget and the Genetic Epistemology, Constructivist Theory of Learning, History of Proof, Meaning of Proof, Function of Mathematical Proof, Moore Method, and Van Hiele and the Obstacles to Learning Proofs.

Jean Piaget and Genetic Epistemology

Jean Piaget is one of the most influential figures in developmental psychology. The synergy of both training in biology as well as training in philosophy influenced his interest in the nature of knowledge and child development. Piaget created a general theory of how children acquire new knowledge. He called his theoretical framework “Genetic Epistemology.”

Over many decades Piaget was trying to understand the nature of knowledge and the biological factors that interact with knowing. Piaget’s professional goal was to answer the questions, “where does knowledge come from?” and “how does knowledge grow?” Piaget theorized that knowledge must be constructed and reconstructed; it cannot simply be verbally transmitted. Knowledge of the child’s world is constructed by acting on objects (Sigel & Cocking, 1977). Unlike many others who measured intelligence such as Simon, Binet, and Burt, Piaget was not interested in what errors children made but why they made reasoning errors. He realized that children’s reasoning is qualitatively different from adult reasoning and not simply less accurate (Wadsworth, 1996).
Piaget theorized that children think and reason differently at different periods in their lives. There is ample validation that everyone passes through a sequence of four qualitatively distinct stages. A person cannot skip stages or reorder them. There is some variability in the ages at which children attain each stage, but they pass through them in the exact same order.

The following are Piaget’s stages of cognitive development:

- **Sensorimotor stage** is the first stage which spans from birth to approximately two or two and a half years. At this stage the child’s intelligence and mental structures develop as a result of motor interactions with objects in his environment. The child cannot form mental representations of objects that are not in his immediate view.

- **Preoperational stage** is the second stage which starts around age two through seven. At this stage children can create mental representations of objects even if they are not directly in their view.

- **Concrete operational stage** starts at age 7 to around 11. The child develops mastery of classes, relations, numbers, and how to reason. Children at this stage are able to use deductive reasoning, demonstrate conservation of numbers, and differentiate their perspective from that of other people.

- **Formal operational (abstract thinking) stage** takes place around the age of 11 and up. This age could be described as the stage where the child achieves mastery of thought (Evans, 1973). Its most significant feature is the ability to think abstractly.
In a Piagetian sense, the ability of children to learn any cognitive content is always related to their stage of intellectual development. Children who are at a certain stage cannot be taught the concepts of a higher stage.

Another important principle of Piaget's stage theory is that there are genetic constraints inherent in the human organism. You can challenge a child to confront new ideas, but you cannot necessarily “teach” him out of one stage and into another. Moreover, a child cannot build new, increasingly complex schemes without interacting with his environment; nature and nurture are interlocked. The mind organizes reality and acts upon it. The learner must be active; he is not a vessel to be filled with facts (Brainerd, 1978).

Piaget's Genetic Epistemology is anchored on the principle that early stages of development establish primitive foundations on which increasingly complex intellectual processes are built (Lavatelli, 1973, p. 40).

Intelligence does not by any means appear at once derived from mental development, like a higher mechanism, and radically distinct from those which have preceded it. Intelligence presents, on the contrary, a remarkable continuity with the acquired or even inborn processes on which it depends and at the same time makes use of. (Piaget, 1963, p. 21)

Piaget theorized that the growth of intellect involves three fundamental processes: Assimilation, Accommodation, and Equilibrium.

- Assimilation involves the incorporation of new events into preexisting cognitive structures.
• Accommodation is the change of existing structures to incorporate new information.

• Equilibration involves the person striking a balance between himself and the environment, via assimilation and accommodation.

Intelligence is a form of adaptation; knowledge is constructed by each individual through the two complementary processes of assimilation and accommodation. Children interact with their physical and social environments; they organize information into groups of interrelated ideas called “schemes.” When children encounter something new, they must either assimilate it into an existing scheme or create an entirely new one to deal with it (Wadsworth, 1996).

When a child experiences a new event, disequilibrium sets in until he is able to assimilate and accommodate the new information and thus attain equilibrium. Many types of equilibrium take place between assimilation and accommodation which varies with the levels of development and the problems to be solved. For Piaget, equilibration is the major factor in explaining why some children advance more quickly in the development of logical intelligence than others. In his book, “To Understand Is To Invent,” Piaget (1972) explained the basic principle of active methods that can be expressed as follows:

To understand is to discover, or reconstruct by rediscovery and such conditions must be complied with if in the future individuals are to be formed who are capable of production and creativity and not simply repetition. (p. 20)
The teaching methodology that is being investigated in this research study is an example of a practical pedagogy that supports the above quote. The Moore method is based on the idea of having students find their own proofs (Wilder, 1976). This approach of teaching mathematical proof is inconsistent with Piaget’s theory of how the mind learns. The Moore method offers a different approach that is in line with Piaget’s theory of learning.

Constructivist Theory of Learning

“Knowledge is not passively received but built by the cognizing subject” (von Glasersfeld, 1995, p. 18). “The function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality” (p. 18). These two quotes sum up the constructivist view on the nature of knowledge, the learner, and learning. Knowledge is dynamic and constantly adjusted by the learner. Contrary to behaviorists, who believe that learning occurs as a result of conditioned responses to a set of stimuli, constructivists concern themselves with the processes that take place in the learner’s mind just as much as they do with the experiences the learner must encounter along the way to construct knowledge.

One cannot assume that “experiencing” implies automatically learning what is being experienced; furthermore, it is the learner who constructs a unique meaning for oneself out of that experience. My personal experience with English as a second language provides me with a plethora of examples of this adaptive process. One particular example that needs to be mentioned is my learning of English idioms. I remember the first time I heard, “I have my work cut out for me.” It was an ambiguous statement in which I could
interpret it as “things are going to be easier” or “things are going to be harder.” My initial interpretation was that the idiom meant “things are going to be easier.” In trying to make sense of the situation, I resorted to apply common logic. If something is cut out, it must be getting smaller; smaller work means an easier time ahead. That represented a reasonable and plausible meaning to me, yet that was not what the speaker meant. He had his own meaning of that phrase based on his own experiences. It was not until I faced a situation where my understanding of this phrase (the meaning that I constructed) was challenged and I had to adapt it so it would fit better in my overall understanding of the English language.

The mechanism in which learning occurs is explained by the constructivist theory through “abstraction,” the process that presupposes all new knowledge (Steffe & Cobb, 1988). Abstraction is the process of filing in memory mental items or acts that appear in the field of attention (Battista, 1999b). This filing process requires selection, coordination, unification, and registration in memory the items to be filed (Battista, 1999b). Piaget has differentiated two types of abstraction: empirical abstraction and reflective abstraction (von Glasersfeld, 1995). Whereas empirical abstraction occurs in response to sensory stimuli, reflective abstraction is an abstraction on already abstracted items. In reflective abstraction, the abstracted items are coordinated as “content” into “forms” or “structures” (Battista, 1999b; von Glasersfeld, 1991). Although abstraction explains learning in general, mathematical learning is explained through reflective abstraction.
Once an experience has been abstracted sufficiently so it could be played back, or “re-presented,” it is said to be “internalized” (Battista, 1999b). While re-presenting is merely playing back an experience or action, “reflection” is playing back an experience or action while considering its components or its results (Battista, 1999b). The first time I heard the expression “you stood me up,” I was able to abstract its meaning but I was not able to represent it; it was not internalized. I was trying to reproduce it but could not remember the exact wording. When I heard it a couple of times later, I internalized it as a unit or item in itself. I used the exact words to talk about things that happened in the past as well as the future without changing the tense in the verb “stand.” I was not able to reflect on it and look into its components. According to von Glasersfeld, for the mind to reflect it must “step out of the stream of direct experience, re-present a chunk of it, and to look at it as though it were direct experience, while remaining aware of the fact that it is not” (von Glasersfeld, 1991, p. 47).

Piaget theorized that perception is what triggers learning. Learning is cumulative. As we grow we abstract our experiences and create forms and structures. When an experience is perceived as familiar, it is “assimilated.” If it is perceived as unfamiliar, current structures must be altered, or new structures must be created to assimilate it. In a Piagetian sense, the alteration of existing structures to assimilate new items is “accommodation” (Battista, 1999b). The realization that an existing structure is not adequate to assimilate a new experience causes a breakdown in assimilation (perturbation). Hence, all accommodations are triggered by perturbation (Battista, 1999b).
The concept of schemes explains how the mind coordinates all the bits and pieces of abstracted ideas and makes them work together. Schemes are the structures in which an abstracted piece of knowledge resides and connects to other pieces. Battista (1999b, p. 13) explained that a scheme is “an organized sequence of actions or operations that has been abstracted from experience and can be applied in response to similar circumstances.” Von Glasersfeld (1995, p. 65) proposed that a scheme consists of three parts. These parts are “recognition of a certain situation;” “a specific activity associated with that situation;” and “the expectation that the activity produces a certain previously experienced result.” Although Battista’s view of schemes is structural and von Glasersfeld’s is operational, they both explained the assimilation and accommodation processes. In either view, if the person encounters a situation that represents a circumstance similar to one that resulted in the construction of an existing scheme (Battista, 1999b), or the situation is recognized as familiar by the person (von Glasersfeld, 1995) the situation will be assimilated; otherwise an existing scheme needs to be altered or a new one constructed to respond to the unfamiliar, unrecognized situation.

**History of Proof**

Proof as a concept developed over a few thousand years. Artifacts were discovered dating back to Ancient Babylonians and Chinese indicating that they had proved the Pythagorean Theorem. Tablets were found with inscriptions that dealt with ways of finding three positive integers a, b, and c that satisfy the equation $a^2 + b^2 = c^2$. In ancient Greece the well-known geometer mathematician Pythagoras (569-500 B.C.),
along with a community of mathematician that worked with him and are referred to as the
Pythagoreans, subscribed to the idea that mathematical statements must be verified by
rigorous proof. They also believed that a great body of mathematics (geometry being an
example) could be derived from a small number of postulates. This idea materialized
when less than 200 years later, Euclid (325 B.C – 265) from Alexandria established
formal definitions and axioms to create a body of mathematics in which he was able to
prove theorems. Needless to say that like Pythagorean, Euclid had a group of followers
that worked under his tutelage who proved theorems and wrote texts that were attributed
to Euclid himself (Reid, 2005).

During the middle ages there was not much advancement in the concept of
mathematical proof. Granted the Arabs in Baghdad contributed great works to the
discipline of mathematics, there was little done to contribute to the development of
mathematical proof. During that period, geometric representations were used to justify
algebraic methods with little emphasis on rigorous arguments

The 19th century witnessed a revival of the rigorous mathematics. There were a
plethora of contributors to the mathematical renaissance. Most notably is the famous
mathematician Fourier (1768-1830). He developed Fourier series and created the
expansion formula to expand arbitrary function into a geometric series. Evariste Galois
(1886-1823) and Cauchy (1789-1857) invented Group theory that became the basis for
modern algebra. Riemann (1826-1866) created differential geometry (Reid, 2005).
The great contributions by the mathematicians mentioned above as well a host of bright mathematicians ushered the field of mathematics and the concept of rigor and mathematical proof into the 20th century.

Meaning of Proof

Godino and Recio held a belief that formal logic, schools, mainstream mathematics, and science all had different meanings of proof. Maturana called this construct “domain of explanation” (Maturana, 1988; Reid, 2005).

The observer accepts or rejects a reformulation . . . as an explanation according to whether or not it satisfies an implicit or explicit criterion of acceptability. . . . If the criterion of acceptability applies, the reformulation . . . is accepted and becomes an explanation, the emotion or mood of the observer shifts from doubt to contentment, and he or she stops asking over and over again the same question. As a result, each . . . criterion for accepting explanatory reformulations . . . defines a domain of explanations. (Maturana, 1988, p. 28)

Others such as Fischbein and Kedem (1982) believed that “a formal proof of a mathematical statement confers on it the attribute of a priori universal validity” (Fischbein & Kedem, 1982). Reid called this definition of proof “Traditional” (Reid, 2005).

P. Davis (1972), Lakatos (1976), Tymoczko (1986), and Crowe (1988), among others, pointed out that proof cannot establish absolute truth because they are produced by fallible mathematicians. This school of thought sees proof to be part of a quasi-
empirical process; the function of proof is to clarify and make the detection of errors easier, but never complete (Reid, 2005).

Function of Mathematical Proof

The goal of producing a mathematical argument is to convince the audience that a statement is true. More importantly, the creator of the proof is convincing herself that a statement is true. Some believe that explanation should be the primary function in a mathematical class (Hanna, 1990; Hersh, 1993). For that reason those who believe in that premise believe that the exclusive focus on the logical structure of proof could hinder students’ abilities to create convincing arguments on their own (Alibert & Thomas, 1991). The view held by Hanna and Hersh is not universal. Proof may be used to organize and unify results that have been proved yet remained fragmented (de Villiers, 1990). Communication to convey and debate mathematical ideas is another purpose for proofs in the classroom (de Villiers, 1990; Knuth, 2002). De Villiers believed that proofs lead students to discover new models and theories. Some argue that proofs develop intuition (Pinto & Tall, 1999). They also help students become autonomous thinkers who are able to validate new mathematical propositions and conjectures (Yackel & Cobb, 1996).

Moore Method

R. L. Moore was a prominent mathematician whose works in mathematics made him one of the most influential American mathematicians in the 20th century. In his article published in the *American Mathematical Monthly*, Zitarelli (2001) chronicled the life of R. L. Moore as one key element of the revolution in the state of mathematics.
higher education and research that took place between the years 1890 and 1950. Moore is remembered for his contributions in the areas of topology and pedagogy. “Moore progeny has been called the most distinguished group of mathematicians in the United States taught by the same person” (Zitarelli, 2001, p. 622). His teaching method endures and has been adopted by many mathematics professors across the United States. R. L. Moore produced 50 doctoral students and has over 1,200 mathematical descendents. This distinguished group includes his Ph.D. students, the students of his students, and so on. The prominence of his “progeny” in the field of mathematics is well established as two of his students were former presidents of the American Mathematical Society; four of his students became presidents of the Mathematical Association of America; and three of them became members of the National Academy of Sciences (Wilder, 1976).

Moore’s intention during his teaching career was to develop research abilities among his students along with gaining knowledge about mathematics. The “Moore method,” also referred to as the “Texas method” (Dancis & Davidson, 1970), is based on the idea of having students find their own proofs of theorems and ultimately prove theorems they themselves suggested and conjectured (Wilder, 1976). Moore developed his own curriculum and syllabi with a unique style to develop mathematical thinking as its major goal. Starting with Trigonometry he taught courses using this unique style all the way through doctoral courses in mathematics. He determined that the class size had to be quite small to create the proper relationships between himself and his students (Dancis & Davidson, 1970). His courses demanded students’ deep participation and much effort; yet frequently, once a student took one of R. L. Moore courses he or she sought to take
the subsequent courses in the sequence taught by Moore. This is consistent with the findings of contemporary mathematical research and the NCTM (2000) recommendations that students develop mature mathematical habits of mind when they “make” mathematics rather than passively receive it.

The goal of the Moore method is to directly assist students in developing some of the abilities which characterize the working mathematician. In general the method is used to develop mathematical creativity and intuition, as well as skills in logical reasoning. The Moore method focuses on students’ abilities to invent techniques for solving problems and proving theorems; make conjectures and educated guesses; and to present a coherent argument. This approach is in concert with NCTM (2000) recommendations as stated in the *Principles and Standards for School Mathematics* which highlight the importance of students developing mathematical reasoning as a habit of mind. Although these abilities are being developed, students master a moderate amount of mathematical knowledge. An important emphasis, with the Moore method, is placed on creating new ideas by students (Dancis & Davidson, 1970). In this approach no formal lectures are given; students are provided with a short set of notes containing definitions, theorems, problems, and occasional examples. They are not, however, presented with proofs or solutions. Students’ responsibility is to work on proving the theorems and solve the problems included in the course notes. They present their work in class for evaluation, detailed critique, and suggestions by the instructor (Selden & Selden, 1987). When someone presents his or her work on the board students assume an active role in accepting or challenging the validity of the argument(s) being presented. In Moore
classes, individuals assumed ownership of the theorem they proved. Theorems bear the names of students who proved them throughout the semester.

Van Hiele and the Obstacles to Learning Proofs

One important theory that offers an explanation for students’ difficulty in writing proof at the college level is van Hiele’s levels of Geometric Thinking. Van Hiele postulated that there are five levels of Geometric Thinking and the last three levels also describe how proof develops. What follows is a brief description of these three levels.

- Abstraction: learners recognize relationships between types of shapes. They understand that all squares are rectangles, and that not all rectangles are squares. They can tell whether it is possible or not to have a rectangle that is, for example, also a rhombus. Students need to be comfortably at this level to be well prepared for a high school geometry course (Musser, Burger, & Peterson, 2003).

- Deduction: learners can construct complete and correct geometric proofs at a high school level. Learners should be exposed to deduction at a pre-high-school level.

- Rigor: learners understand how geometry proofs and concepts fit together to create the structure we call geometry. This is the level at which most college geometry courses (for math majors) are designed.

Van Hiele asserted one cannot get to the rigor level without achieving competence at the deduction level. Although the geometry courses are designed at the rigor level, in most cases students do not get enough experience functioning at the deduction level in high
school. If this transcends to other areas in mathematics, then that would be an explanation as to why students have difficulty learning proofs at the college level. For the purposes of this research; the data are analyzed in light of constructivist theory and Harel and Sowder’s (1998) model of mapping proof schemes.
CHAPTER III
RESEARCH DESIGN AND DATA COLLECTION

Selection of Research Design

One basic assumption of phenomenology is that a researcher cannot develop an understanding of a phenomenon apart from understanding people’s experience of or with that phenomenon. In this sense, a person’s understanding or reality is not “out there” in an objective or detached sense; rather it is tied to one’s consciousness of it. Phenomenologists refer to this idea as *intentionality of consciousness* (Schram, 2006). In this vein, the researcher enters the field of perception of the participants and sees how they experience and describe the phenomenon being studied. The researcher then looks for meaning of the participants’ experiences (Creswell, 1998). It is the intention of this research to better understand the mental processes that promote students to become proof producers; an intention that can be met with a phenomenological design.

This research set out to investigate students’ understanding of proof within a specific environment. It intended to attain deep understanding of students’ conceptualization and learning of proofs. In doing so, a qualitative design was used. This design is guided by the philosophical basis of all qualitative designs, phenomenology. In a phenomenological study, subjective experience rather than objective observation leads to deep understanding. The assumption underlying this design is that there are many ways of experiencing and interpreting an event and the meaning that participants construct
about the event is that person’s reality (McMillan & Wergin, 2006). The phenomenon at hand is students’ conceptualization and learning of proof. ‘A teaching experiment’ design was chosen for this study. This method is compatible with the teaching method utilized for teaching the course, the “Moore method.” Students were required to present their proofs to the teacher/researcher explaining their thinking process and rationalizing their thinking. The instructor/researcher was not a passive observer but rather a participant who assumed the role of an interviewer who is trying to understand students’ understanding as well as the role of a teacher.

Rationale

The shift in paradigm from behaviorism to constructivism made it necessary for mathematics education research to reconsider the relevance of quantitative research methodology. Much of the research in mathematics education strives to understand the mathematical constructions that students build. Although quantitative research methodology may answer some research questions, it does not assist in exploring students’ thinking and the deep ways in which mathematical schemes are constructed.

Description of Teaching Experiment Design

Leslie Steffe was one of the pioneers in developing a research method he called the “teaching experiment” (von Glasersfeld, 1995). As suggested in the following quote, Steffe’s teaching experiment is modeled after Piaget’s famous clinical method: “It was a hybrid of Piaget’s ‘clinical method’ of interviewing children and educational research” (von Glasersfeld, 1995, p. 17). The ultimate goal of the teaching experiment is to create a viable model that explains the way students construct a specific concept. Invariably some
of the abstractions would be a result of Piaget’s notion of reflective abstraction; hence the researcher would base his or her model on conjectures. In time the hypothetical models proposed by researchers will achieve “a high degree of plausibility and predictive usefulness” (von Glasersfeld, 1995, p.17). According to Steffe (1996, p. 1), “teaching experiment methodology involves bringing forth, sustaining, and modifying the mathematical schemes of students that an observer categorizes as being generalizable.”

As with all general theoretical constructs, it is difficult to apply them to specific situations, when the cognizing subject is not ourselves but a ‘subject’ we are observing. In practice there may be observable behavioral indications, on the basis of which levels of abstraction can be determined, but making the determination is not simple. One might say that assuming something as ‘given’ or not is exclusively the subject’s business. Hence, at best an observer can make educated guesses, taking into account—as does any experienced diagnostician—several indications collected over an extended period of observation (Steffe & von Glasersfeld, 1988, pp. 18-19).

What the researcher controls in this research design is the experiential field of students by posing tasks or problems to “guide student’s cognitive activity” (Battista, 1999b, p. 12) in order to focus on the targeted concepts and ideas. These tasks or problems are designed to guide students to reorganize their existing schemes of a certain idea to a more sophisticated scheme. The goal is not only to diagnose the present schemes, but to investigate the ways to take those schemes to higher levels of sophistication as well. In doing so, the researcher is striving to understand what is going on in the students’ minds and what it takes to create situations that would cause them to
reflect and reorganize their existing “organization of relevant ideas” (Battista, 1999b, p. 12). This process closely mirrors Piaget’s ideas of assimilation and accommodation (Piaget, 1971).

Teaching experiment-based research must have the four components (Battista, 1999b). The first component is the preliminary work. The second component is teaching, model building, and hypothesis testing. The third component is retrospective analysis, and the last component is scientific model building (Battista). In the preliminary stage, the researcher interviews students to identify the mathematical categories to which the students belong and identify their existing mathematical models (Battista). During the teaching experiment, the researcher should “formulate and test detailed ‘working models’ of students’ construction of specific areas of mathematical knowledge” (Battista, 1999b, p. 3). These working models are constantly challenged and modified based on observing how students react to the situations presented by the researcher. The researcher’s goal is to identify the “itineraries” that cause students to develop their mathematical knowledge within each mathematical category. Once the data is collected, the researcher reconstructs an historical analysis of the students’ actions and reactions during the teaching experiment by conducting a retrospective analysis of the data collected in the form of audio, video, written record of students, and researcher’s notes. At this stage the question in the researcher’s mind is “What was the student thinking so that his actions make sense from his perspective?” (p. 13). The scientific model building is the stage where the researcher uses mental processes as well as concepts particular to the topic of interest to
the research to explain specific students’ mathematical activities. Examples of mental processes are abstraction, accommodation, reflection, and schemes (Battista).

Methodological Considerations

Sample

An undergraduate class offered by the Department of Mathematical Sciences was chosen for this study. The department is part of a mid-size state university in the Midwest. The course’s title is “Numbers and Games.” The course usually targets pre-service high school mathematics teachers as well as students majoring in mathematics. The class was chosen because it uses a nontraditional teaching pedagogy. The instructional method used in this course was the Moore method (A brief overview of the Moore method was provided in Chapter II).

Students were asked to participate in the study on a voluntary basis. Four students were chosen for the study. Two of them, Shannon and Ryan, agreed to be interviewed one-on-one and observed as they presented their proofs to the professor in his office. The other two students, Chris and Zack, agreed only to being observed while presenting their work to the professor. The researcher sat in the classroom and took notes during all class sessions throughout the semester. Students were encouraged to participate since their involvement would potentially benefit them. For example, the special attention to their work as participants could increase their awareness of their own thinking and learning. It would also provide them with the opportunity of discussing their mathematical thinking with the other students which in turn could lead to a better performance in the class,
especially since the grading was based on their active involvement and the submitted paperwork.

Procedure

During the first week of class the researcher announced to the students that volunteers were needed to be part of a study aimed to understand students’ learning of mathematical proofs. It was explained to them that their participation was optional and their withdrawal from the study before the completion of the data collection, if they decided to participate, carried no penalties. They were encouraged to participate and that was done by listing a few of the many benefits from being participants in the study. For example, the class grade depended heavily on proof done outside of the class as well as class participation and presentation. Taking part of the study would provide them with the opportunity to reflect on their thinking and help them develop schematically in mathematical proof. After all, that is the purpose of a teaching experiment. A form was distributed that explained the nature of the research project, the benefit of participating in the project, and the time commitment needed of the volunteers. The form gave each student the choice of either checking yes if they wished to volunteer or no if they did not wish to volunteer. No names were required. Only those who consented to participate were asked to provide their names and email addresses as well as times in which they would be available for interviews. Students were also asked whether they objected to having a video camera in class (see Appendix E).
Two students out of eight checked the “yes” choice for “I would like to participate in the study.” All but one checked “no” for “I object to having a video camera in class.” Therefore, no camera was used in the classroom.

**Preliminary Work**

Interviews allow the researcher to probe more deeply with fewer participants and in ways that vary between participants. Interviews are more effective at exploring individuals’ meanings and allow the researcher to ascertain whether the participants’ meanings and understandings are clear. This is thought of as a type of internal validity. However, because the interview method of collecting data uses smaller and possibly unrepresentative samples, it is weak on generalizability (Vogt, 2007). For the purposes of this research, depth of understanding over generalizability was deemed more important to answer the research questions.

Students (participants) were interviewed, so their existing mathematical proof schemes could be identified and classified. All interviews started by posing the questions “What constitute a proof in your mind?” and “Do you think you are constructing proofs in the class?” From that point on interviews did not follow a predetermined structure. The schemes proposed by Harel and Sowder (1998) were used as the framework for this classification. Descriptions of the different schemes where discussed in chapter 1.

**Setting and Data Collection**

The course instructor is a topologist who is an academic descendent of R. L. Moore. He used a modified version of the Moore method. A brief lecturing to introduce the basic axioms and definitions took place. All mathematical statements to be
investigated and proved were assigned to students to prove and present in class (see Appendix U). The presenter had to convince the class of the validity of his or her proof. The rest of the class was required to challenge that validity. The researcher took notes of the unfolding events during class, focusing on the participants’ actions and reactions during class (see Appendix T). That was also the motivation for building hypothesis about the participants’ mental processes. Those hypotheses were tested during the weekly or biweekly interviews with the participants. The interviews were videotaped for the retrospective analysis. Each participant’s in-office presentation of the results they proved was videotaped. During these presentations, participants were required to present their results with a partner with which they worked on those results. Consent for video taping was obtained from all individuals involved. In addition, both students were twice interviewed one-on-one.

Validity and Reliability

To control the quality of a research study attention should be paid to the issues of reliability and validity. One should question if the issues of reliability and validity, as defined and measured in quantitative research, are consistent with the nature of qualitative, naturalistic investigations. To answer this question it is necessary to analyze the differences between the two methods and the paradigms within which each operates. Qualitative research aims to understand phenomena in context that is natural without modifications such as “real world setting [where] the researcher does not attempt to manipulate the phenomenon of interest” (Patton, 2002, p. 39). It is broadly defined to be “any kind of research that produces findings not arrived at by means of statistical
procedures or other means of quantification” (Strauss & Corbin, 1990, p. 17).
Researchers in a qualitative study, unlike quantitative researchers, seek to understand phenomena and extrapolate findings to similar situations (Hoepfl, 1997). They recognized that they are directly involved and play a role within the research. In qualitative research “the researcher is the instrument” (Patton, 2002, p. 14).

Credibility of a quantitative research study is measured by means of validity and reliability. Credibility and trustworthiness of qualitative research is decided on through the researcher’s ability and effort (Goafshani, 2003). Hence, unlike quantitative research, validity and reliability are not treated separately in qualitative analysis. Rather terms like credibility, transferability, and trustworthiness are used because they encompass both validity and reliability in the context of qualitative research.

One technique of establishing validity and credibility in a qualitative, naturalistic research is triangulation (Creswell, 2007). In this study, to triangulate I needed someone who was well versed in the area of mathematics as well as mathematics education. I selected a colleague who is a lecturer in mathematics and teaching at the same university where the participants were in attendance. She holds degrees in both mathematics as well as mathematics education. She has an extensive experience teaching mathematics at the pre-college as well as undergraduate level. Such background and credentials add yet another layer of confidence and trustworthiness of the quality of the outcome of coding.

After she verbally agreed to take part in the triangulation process, via email, one episode and the list of all the codes were sent to her. In addition the email contained the
coding of the sample episode. An explanation of how the code assignment was chosen was included in the email as well.

For the actual triangulation, she was presented with a sample of six episodes that were chosen randomly. These episodes featured Ryan, Chris, and Shannon. None of the episodes featured Zack. That was determined by the random choice process. For all episodes, with no exception, there was 100% agreement in coding for each person. In two instances, however, she had one extra code in addition to the codes on which we were in agreement. The extra codes did not interfere with the intention and integrity of the coding system. The results of this triangulation, which demonstrated a very high degree of agreement, raised the level of confidence and trustworthiness of the accuracy of assessing and determining the coding for the data.

Other techniques for establishing trustworthiness or research validity include prolonged engagement and persistent observation; external reflection of the work; member checking and providing a rich, thick description that allows the reader to enter the research context (Glesne, 2006). Such techniques were carefully and deliberately carried out in this study.
CHAPTER IV
DATA AND RESULTS

This study is designed to answer the following questions:

1. What are students’ conceptions of proofs and what do they consider mathematical justification (Harel & Sowder, 1998)?

2. Do students’ experiences in the class taught using the Moore method (a description of the method is provided in Chapter 2) affect their conceptions of proofs?

3. Is the Moore method effective in promoting development of students’ proof schemes to higher schematic levels as defined by Harel and Sowder (1998)?

In addition to answering the above questions the study sought to achieve two goals.

This research study is designed to document the progress college mathematics students make in their conceptions of mathematical proofs in a Moore method class, and to validate, expand, or modify the mapping of the students’ proof schemes (Harel & Sowder, 1998).

Four factors were considered to guide answering the above questions. They provided a standard against which the attainment of the stated goals is measured. These factors are:

1. The ability to carry on (produce) a proof of a mathematical statement.
2. Appreciation of a proof in terms of students’ judgment on the rigor of the argument.


4. Demonstration of self-confidence, by means of verbalization or behavior, in the ability of producing a proof, in particular self-perception as being autonomous thinkers and proof writers.

Four students volunteered to participate in this study. To maintain anonymity, each volunteer was assigned a pseudonym. The only female in the group was given the name Shannon. The three males were given the names Zach, Ryan, and Chris. They were observed and their performances and reactions to mathematical situations were carefully documented. The class in which participants were enrolled was called Numbers and Games. The mathematics that students experienced in this Numbers and Games course was not typical of what undergraduate students experience in standard mathematics courses. The difference was that it was not driven by the study and the analysis of functions or the nature and properties of mathematical structure such as manifolds and modules. Neither did it involve extensive mathematical calculations such as finding roots of functions as in the areas of numerical computation and numerical analysis. In such a course the mathematics deals with assigning values to games that satisfy certain conditions. When a value is assigned to a particular game it determines the strategies for
winning that game. Examples of elements of the generated set include 0, powers of \( \frac{1}{2} \), *, ↑ and ↓.

Shannon, Ryan, Zack, and Chris were selected based on their willingness to be part of the study. Participation was solicited and these four students expressed interest in making themselves available for data collection. The data were obtained from a variety of sources and video taped through observing them in the act of producing mathematics as well as occasional direct questioning. Students taking the course worked in groups or teams of two individuals. Shannon and Chris formed one team; Ryan and Zack formed another team. Members of each team worked on proving theorems together to submit for homework and to present the results in the professor’s office. Chris and Zack were not available at any time other than the time spent in office presentations. Shannon and Zack were available for one-on-one interviews with the researcher targeted to collect data that allowed the researcher to better understand their thought process in light of the research questions and goals. For one-on-one discussions, Shannon was available the most. Ryan was available but not as frequently as Shannon. Shannon was interviewed four times: twice in September, once in October, and once in late November. Ryan was interviewed twice, both taking place in September.

The function of this chapter is data reporting. The breakdown of this chapter is based on the month in which the data was collected. The course was offered during the fall semester. The semester started the end of August and ran through the first 10 days of December. The data was collected during the months of September, October, and
November and are reported accordingly. The reporting within each month was based on the chronology of events rather than by individual participant. The research questions are addressed as they unfold within the chronology of this report. In Table 1 dates, durations, and locations are laid out for each student. Throughout this chapter initials were used for the names of all four individuals taking part of this study. The letters C, J, R, S, Y, and Z represent Chris, Jack, Ryan, Shannon, Yaser, and Zack respectively. Yaser is the investigator in this study. Jack is the professor who taught the course. Chris, Ryan, Shannon, and Zack are the participants in the study.

September

In this month interviews and office visits took place during the period between the 15th and the 24th. Due to the fact that the observations of Zack and Chris were only during their office presentations, the data did not contain evidence that address some of the research questions and goals. Table 1 lays out the time line in which data and artifacts were collected.

Three weeks into the semester Shannon was asked about her conceptualization of proof. Episode 1 is an excerpt from an interview with Shannon where she responded to questions about what she thought a proof was. As it is demonstrated in Episode 1 she made reference to the two-column format as the standard way to writing proof. This is a textbook example of what Harel and Sowder (1998) defined as External Conviction proof scheme. The ritual of writing the proof in a specific form, in this case the two-column form, indicated that she was operating within the external conviction – ritual scheme.
Table 1

*Data Collection Timetable*

<table>
<thead>
<tr>
<th>Classroom Notes</th>
<th>August 25 - November 24</th>
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<tbody>
<tr>
<td>Forms</td>
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<tr>
<td>Request for Volunteers (Appendix E)</td>
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</tr>
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<td>Videotaping Consent Form (Appendix D)</td>
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<td>Audio taping Consent Form (Appendix B)</td>
<td>August 29</td>
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Presentations in Professor’s Office

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<td></td>
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<td></td>
<td>November 7</td>
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</tr>
<tr>
<td>Zack</td>
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<tr>
<td></td>
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</tr>
<tr>
<td></td>
<td>October 10</td>
<td>45 minutes</td>
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<tr>
<td></td>
<td>November 7</td>
<td>28 minutes</td>
</tr>
<tr>
<td>Shannon</td>
<td>September 19</td>
<td>30 minutes</td>
</tr>
<tr>
<td></td>
<td>October 10</td>
<td>40 minutes</td>
</tr>
<tr>
<td></td>
<td>November 7</td>
<td>20 minutes</td>
</tr>
<tr>
<td></td>
<td>November 21</td>
<td>30 minutes</td>
</tr>
<tr>
<td>Chris</td>
<td>September 19</td>
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</tr>
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</table>

*(table continues)*
Table 1 (continued)

*Data Collection Timetable*

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</thead>
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<tr>
<td></td>
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One-on-one Interviews with the researcher

<table>
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<td></td>
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<td>Shannon</td>
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<tr>
<td></td>
<td>November 14</td>
<td>25 minutes</td>
</tr>
</tbody>
</table>

Episode 1—Shannon’s first interview with the researcher (Appendix G):

**Y:** When you hear the word proof what comes to mind?

**S:** I think of Geometry.

**Y:** Geometry?

**S:** Yeah. The two column proofs.

**Y:** The two column proofs? Yeah, did you have geometry in school?

**S:** Yeah, I had one in high school and then a second one in college.

**Y:** . . . In the one in college did you do the two column thing?
S: We can do it however we want. So I, like, just made up my own style but every time in high school it was the two columns so I tried a model after that.

Y: Oh okay, so you still do the two-column!

S: Yeah, as much as I can.

Shannon did not appreciate the mathematical activities that took place the first three weeks as proofs. That stance is evident in Episode 2.

Episode 2—Shannon’s first interview with the researcher (Appendix G):

Y: Wouldn’t you consider what we are doing as a proofing right?

S: No.

Y: And why do you think that is so?

S: I’m sorry?

Y: Why do you think that is so? Why wouldn’t you consider . . . like those diagrams that we are doing, things like that? What . . . why wouldn’t you consider those as proofs?

S: I just don’t really see what they’re proving . . . I mean besides you know someone can win the game. That’s really all you get from it.

The course dealt with non-traditional mathematic material. Combinatorial Game Theory investigates strategies on how games, as defined in the axioms, are won. Shannon did not appreciate the relevance of the theorems being proved. In addition she had difficulties grasping the definitions behind the terms introduced in the course.

Episode 3—Shannon’s first interview with the researcher (Appendix G):
Y: If X is less than zero, and Y is greater than zero, then \( \{X|Y\} \). Do you know what that means?

S: No, I do know what it’s going to be zero. Other than that no.

Shannon’s obvious confusion and lack of understanding of the axioms and definitions made it difficult for her to produce proofs of her own. She did not have clear plans to navigate through the new mathematical context to produce proofs of theorems. Her frustration and confusion was obvious in her responses to the interview questions. In the following episode, Shannon’s lack of confidence in her ability to produce proofs is clear and evident.

Episode 4—Shannon’s first interview with the researcher (Appendix G):
Y: Yeah okay. Um, how about this table? We worked on last time?
S: I completely understood it when we were in class and then I got home to do it last night and, like I played out a game and I still couldn’t figure it out.

In Episode 5, Shannon carried out multiple steps. She could not finish the argument at hand. She was confused which is evident in her statement “I get confused.”

Episode 5—Shannon’s first interview with the researcher (Appendix G):
S: Then blue has 2 more, so that’s not a good move.

Y: And if, uh, blue cuts this then red has this cut and this cut, if he gets that, he has this cut or this cut.

S: Yeah, so the second one’s better.

Y: Okay, and that would be, what? What is the value of this one?
S: 2, 2 or 3.
Y: 2 or 3.

S: Negative 2 or 3. And don’t know how, this is where I get confused.

Shannon continued expressing her frustration with the constructs of this course. In the following episode Shannon kept expressing and verbalizing her frustration with how statements are proven in the new context. This in turn affected her confidence in her ability to prove general statements independently. Episodes 6 and 7 provide evidence to support this argument. In Episode 6, she said, “I personally don’t understand how they do it.” In Episode 7 she questioned the generalizability of a given proof.

Episode 6—Shannon’s first interview with the researcher (Appendix G):

S: It seems to me that this is a very complicated thing to do at the beginning because I personally don’t understand how they do it, how he did it. Because look here: if you go through the process, what did we do . . . we did two three, two times three right. When he did two times three he started by cutting one off and lies two by two.

Y: The two by two . . .

Earlier Shannon’s conceptualization of proof was consistent with the authoritarian scheme which is the least sophisticated scheme. As is evident in Episode 8, Shannon’s explicit articulation of what she thought a valid general argument was consistent with what Harel and Sowder (1998) described as “external conviction” conceptual scheme. This scheme is the second level of sophistication. As one sees in the following episode Shannon expressed lack of confidence that the proof was general and would only work
for one case. Although she said that the argument worked out mathematically, she showed little confidence in the generality of the argument.

Episode 7—Shannon’s first interview with the researcher (Appendix G):

S: Well, in this case it would work out mathematically. I don’t know how accurate it would be for the other ones.

Shannon demonstrated that she lacked the confidence in her self and her ability to be an autonomous proof writer.

Episode 8—Shannon’s first interview with the researcher (Appendix G):

S: Kind of, like I understand how to do it, it’s just whether or not I can think of it.

Y: Yeah.

S: You know, at the time because there are so many steps and then you have to carry all the numbers back and there are so many pictures, I just get confused.

Y: Right.

S: But like I understand like how you do it.

Y: You understand the process?

S: Yeah.

Her lack of self-confidence is evident in Episode 9.

Episode 9—Chris and Shannon’s first office presentation (Appendix J):

J: . . . you guys did in class for this course and play a game. You two against me.

S: You’re on my team we might win.

C: I don’t think so.
The following episode sheds brighter light on the way Shannon conceptualized proof. The one proof that she accepts was a proof by induction. That is another example of how the form is essential in deciding what a proof is and what it is not.

Episode 10—Shannon’s second interview with the researcher (Appendix K):

Y: Now, which, what stuff that we’re doing here you consider proof and what stuff you don’t consider proof?

S: I think the induction that I did there is proof.

Y: That’s proof, okay.

S: But, like I know everything else is proof, I just, like I know it is, I just don’t really think it is, you know?

Y: Yeah, that’s the, that’s what I’m asking.

S: Like everything that we play games for, like I don’t really, I see that it’s proof, but in my head it’s not really proof.

Y: And why is that? Can you put it in words?

S: Just because it’s not like the standard, you’re not really doing the two column and you’re not like, it’s not really concrete and I think proofs should be concrete and not abstract and that’s just really abstract and I don’t think of that as a proof.

Y: Okay.

Once again she demonstrated her authoritarian conceptualization of proof. She believed that these proofs are relevant because she was told that but she does not know
how they are relevant. The fact that it is not a two column proof kept her still from accepting the arguments made in the course to be mathematical proofs.

Episode 11—Shannon’s second interview with the researcher (Appendix K):

Y: Ryan, for example, doesn’t think it’s a proof because you can’t use it for anything else.

S: Well I’m sure you can and we just don’t know it yet.

Y: Yeah.

S: I’m sure that’s what he’s getting to.

Y: So if you can, you know, if you can use it for something else that will change your mind?

S: No, not really ’cause it’s still really abstract.

Y: Okay.

S: And I think proofs should be like really understandable and, like when we prove some of them you pick, you just pick a game, like off the top of your head and in proofs you can’t really, you don’t just pick what you want to use and pick what you don’t want to use, you know?

Y: Yeah, Yeah.

S: Like not in the kind of proofs that I think of.

Y: Why?

S: I really couldn’t see what they were doing on the board so I’ll just try it, like would you play it, like if left goes first then it would be this:
Figure 2. T3-07:59(a)

... then over that equals zero and then it would be ... it would be right turn

... Yeah, and then that would be this.

Figure 3. T3-08:14(a)

and that’s it.

Y: And that’s exactly what they did, right?

S: Yes, see I did it I guess.
Episode 11 is a clear example that Shannon could carry out a correct argument, yet did not have the confidence in her ability. That is evident in the conclusion of the episode when she ended the episode saying, “I did it I guess.”

Episode 12—Shannon’s second interview with the researcher (Appendix K):

Y: Okay so you did it and so is that a proof for you?
S: To me, no.
Y: No, why not?
S: ’Cause it’s, I don’t know what these represent, I don’t know what they are, it’s just not concrete enough.

Still Shannon did not appreciate the mathematical processes done in the class as representation of mathematical proofs. In the following episode she said, “I don’t look at it as a formal proof” indicating that the activities are not mathematically rigorous enough or did not fit with the two column format of proof with which she was familiar.

Episode 13—Shannon’s second interview with the researcher (Appendix K):

Y: Okay so it’s a fuzzy game which means?
S: Whoever goes first wins?
Y: Okay, is that a proof?
S: It’s not . . . I don’t look at it as a formal proof, I guess is what I am saying.
Y: Okay but it is a proof?
S: Yeah.
Y: Is that more concrete than the previous one?
S: Yeah, because there is only one way to do it, because you give me this game, so the only thing I am proving is this game. But if you give me this that could be any game and it’s very concrete because there’s only one game that you can use to prove it.

Later in September, Shannon continued to demonstrate lack of appreciation in terms of rigor of proofs of the unfamiliar context and theorems. In the above episode, Shannon produced a correct and complete argument; yet she did not consider what she wrote to be a formal proof. Ritual once again was the determining factor for her to classify if an argument was formal or not. That view of formality or lack thereof did not affect her validity of the proof. She indicated that the activity she carried out in this episode was an informal proof. She qualified her statement by emphasizing that a proof has a predictable set of steps and follows a linear process. Regarding her use of the word “concrete,” Shannon appeared to mean that there is only one logical path that when followed leads to the conclusion of the theorem. Conventional proofs have a certain logic that is usually followed. The logical path of a conventional proof starts with the hypotheses; then using logical tools one step is derived from the one before it along a path that leads to the conclusion stated in the statement of the theorem. The proofs constructed in the course do not adhere to that convention.

Shannon’s appreciation of the validity of the mathematical process to prove a theorem in this class did not change drastically; she demonstrated a stronger ability to carry out proofs as well as her confidence that she could do it.
Similar to Shannon, Ryan, in his first interview, did not appreciate the level of rigor in the type of proofs done in the course. In Episodes 14-16, Ryan expressed his conceptualization of what a mathematical proof is and why he did not consider the activities done in the course to be mathematical proofs. His view of what a proof should be is closest to what Harel and Sowder (1998) categorized as “perceptual proof scheme” and that is a subcategory of the empirical scheme. At this level the criteria of what is considered to be a mathematical proof is based on “perceptions and coordination of perceptions without the ability to transform or anticipate the outcome of transforming the perceived objects” (p. 255).

Episode 14—Ryan’s first interview with the researcher (Appendix H):

Y: Um the word proof . . . when you hear it what comes to your mind?

R: I would say it’s a mathematical statement that isn’t able to be contradicted.

That it’s true no matter what you use it for.

Y: Is it a mathematical activity?

R: Yeah

Episode 15—Ryan’s first interview with the researcher (Appendix H):

Y: Okay and um, is anything of what we’re doing here, playing those games, constitute a proof to you?

R: No, because it’s not, I mean, because I don’t really consider it, I guess it’s not really too defined; it’s too broad. A proof is usually very defined in what it’s about.
Episode 16—Ryan’s first interview with the researcher (Appendix H):

Y: So the game today, oh, not the game, the group work today.

R: Ah-ha.

Y: Was that a proof to you?

R: No, ‘cuz I consider it, I meant it’s too . . . I mean I guess it’s too broad of something like in mathematics that can be used for other things other than just we were doing in class.

Y: OK.

R: If you figure out the math behind it. I guess, no, I don’t . . .

Y: So you don’t consider it a proof?

R: No.

In the above three episodes Ryan defined proof as he saw it. He perceived what was done in the course up to that point as “too broad;” he defined a proof to be “very defined in what it’s about.” He was not able to transform or anticipate the outcome of transforming the mathematical objects; in this case the objects were games. That understanding is evident in his statement: “like in mathematics that can be used for other things other than just [what] we were doing in class.”

In the following episode it is clear how transferability was essential to Ryan to decide if a mathematical activity is a proof or not. He was willing to change his mind and classify the mathematical activities done in this class as mathematical proofs. There was no mention of altering or modifying those mathematical arguments which indicated that
the transferability, not the logical arguments, that is in question. All he needed was a perception that the results were transferable.

**Episode 17—Ryan’s first interview with the researcher (Appendix H):**

Y: But not a proof? How about this table here for the cuts?

![Figure 4. T1-54:45(a)](image)

Y: So if you can relate it to a different game, is that what you’re saying?

R: Yeah, Yeah.

Y: Somehow the results that you get here.

R: Right.

Y: Can be transferred to a different game?

R: Like for a zero, like a one, positive or it’s a negative.

Y: That would make it more of a proof of what it is now?
R: Ah-hum.

Y: So not knowing it transfers you are saying it is not a proof? Interruption . . .

Okay, so you were saying, if you were, um, since you don’t know if this is transferable to something else, it would be just a mathematical activity, right?

R: Uh, huh.

Y: Okay, so later in the semester if we were able to transfer it to something else, then you will change your mind.

R: Yes.

Y: Okay, cool. ’Cause it’s okay, you can change your mind. We’ll see, we might find that it is transferable, or not.

R: Uh, huh.

Episode 18 shows that Ryan had developed a high level of fluency in proving theorems. The same episode presents an example of Ryan’s and Zack’s mastery of the material. In these episodes, there was no evidence of lack of confidence. They worked as a team; and, as one can see, they took turns as they moved from one step to the other. One can conclude that they were functioning at the same level of mastery.

Episode 18—Ryan and Zack’s first in office presentation (Appendix F):

R: That will make this negative one half and it’ll be zero.

Z: It would be zero again?

R: Yeah and he will be next.

Z: Ok.
Figure 5. T1-05:28(a)

J: Ok, so I’ll take part of the arch, erase part of the arch for me.

Z: Ok, did you want the other side?

Figure 6. T1-05:38(a)

J: Oh, that’s good, good enough.

Z: So that makes . . .

R: one half, one half and one,

Z: One so we’ve got three halves and he’s only got three quarters so . . .
Figure 7. T1-05:48(a)

R: This should be negative one half because he’s got a one.

Figure 8. T1-05:58(a)

Z: Yeah he’s got a negative one half, that’s right. He’s got a one, and he’s got a negative one half, so we’re doing ok still.

R: Take one of these ’cause that makes it a zero game again.

J: Yeah it does.
Similar to Ryan and Zack, Chris developed fluency and ability to produce correct arguments early on in the semester. Episode 19 clearly shows that ability. In this episode, Chris projected self-confidence in the correctness of what he was doing. One can see that in his response to Shannon’s objection to his reasoning as being “dumb.” He defended his position with clear logical explanation. He showed no sign of hesitation or doubt in the correctness of his argument.

Episode 19—Shannon and Chris’s first office presentation (Appendix J):

C: We take the top right one.

S: But that would be dumb.

C: No. It gives us a zero game. He goes next. If he’s the first one to go on a zero game we win.
S: That just makes a negative number bigger.

J: Yeah.

C: Yeah but it makes negative one.

Shannon continued to show evidence that she was not confident in her ability.

Episode 20 is an excerpt from a presentation in the professor’s office in late September. It shows Shannon questioning her ability as she compared it to Chris’s and Jack’s.

Episode 20—Shannon and Chris’s first office presentation (Appendix J):

S: Now, why don’t you have to think about it?

J: ’Cause I’ve played this many times.

C: We’ve done it that many times.

The second interview with Ryan was a week after the first one. The material had developed its own lexicon different than any traditional mathematics symbolism. Frequently as the discussion of the subject matter grew, theorems were stated with heavy use of these symbols. Episode 21 shows an example of such symbolism and theorems.

Episode 21—Ryan’s second interview with the researcher (Appendix L):

Figure 10. T3-37:33(a)
R: This is a theorem.

Y: Yeah can you show me how you prove this theorem?

The theorem stated that “the sum of \{x\mid y\} and \{-x\mid -y\} is zero,” where \{x\mid y\} and \{-x\mid -y\} represent values assigned to specific games. Ryan proved two theorems during the interview. Episodes 22 and 23 present Ryan’s proofs of the first and second theorems respectively. The first theorem states that the value (*) is greater than \(-\frac{1}{2^n}\). The second theorem states that the sum of \{x\mid y\} and \{-x\mid -y\} is zero.

Episode 22—Ryan’s second interview with the researcher (Appendix L):

R: Star is greater than negative one over 2 to the n. That’s what I want to prove.

\[ \text{Figure 11. T3-29:39(a)} \]

Y: Ok, that’s star is greater than negative . . . Ok, so how are you going to do it?

R: It’s this
Figure 12. T3-29:51(a)

Plus B R to the n plus

Figure 13. T3-29:54(a)

Yeah I think. I will do right’s first move. Right has two moves, he can either take one here
Figure 14. T3-30:06(a)

or one here

Figure 15. T3-30:07(a)

so either way
**Figure 16.** T3-30:11

we’ll do this one first, so it’s L, R,

---

**Figure 17.** T3-30:25(a)

with this left B R to the n
Figure 18. T3-30:28

so this is right’s best move

Figure 19. T3-30:32(a)
Figure 20. T3-30:35

Blue’s best move is to take this whole thing.

Figure 21. T3-30:37(a)

Y: Yeah.

R: Then you’re left with LR and blue wins because R has no move.
Figure 22. T3-30:35

Y: So now what do you want to do?

R: So you’re left with L, R and blue wins because R then has no move.

Y: Okay, so R starts

R: Yep, R starts and blue wins. So R’s other move is to take this one here.

Figure 23. T3-31:25(a)

Left’s best move is to leave this
Figure 24. T3-31:26

and move here

Figure 25. T3-31:28(a)
Figure 26. T3-31:45

and R’s best move from here

Figure 27. T3-31:48(a)

. . . R has two moves, one move they’re going to lose if they go there
Figure 28. T3-31:53

R is going to lose on the next turn from here and on this one

Figure 29. T3-32:00(a)

the best move would be this
Figure 30. T3-32:01

So actually they are dead no matter what because we know the value of this is zero

Figure 31. T3-32:09(a)

and we know this is positive because it goes on the bottom.
Figure 32. T3-32:12

This is greater than zero

Figure 33. T3-32:22(a)

Y: Okay.

R: Yeah I had it right the first time.

Y: Yeah that’s greater than zero.

R: So we know blue is going to win no matter what.

Y: So R starts; blue wins.
R: Yeah.

Y: Which means that what?

Y: That start is greater than or equal to negative one half to the $n$. I am going to do a blue win, blue starts, and it’s going to have two moves again.

Figure 34. T3-33:04(a)

Blue is gonna have two moves. One is a very bad move; this is their bad move.

Figure 35. T3-33:08(a)
But they still win. This is their good move.

*Figure 36. T3-33:12(a)*

so let me do that one first. Red’s best move would be to take this

*Figure 37. T3-33:31(a)*

to take one of those. Blue’s best move would be to this
so you’re left with L, R

and we know this is zero
Figure 40. T3-34:30(a)

so R goes next; R loses.

Y: Okay,

R: So blue’s other move would be to take this

Figure 41. T3-34:39(a)

this is a bad move because actually this makes blue lose. R is next
Figure 42. T3-34:53(a)

and R goes there

Figure 43. T3-34:59(a)

and we know from this that blue goes next and blue will lose.
Figure 44. T3-35:07(a)

Y: Okay so there. Blue wins.

R: If blue wants to keep this around as long as possible because that will make sure they can win always. They would rather move in here first if possible than to take this.

Y: So this really doesn’t add up to your proof. It just shows you that it’s a bad move.

Figure 45. T3:35:42(a)
R: Yeah this is; that’s not a good move when what we’re trying to find out is when blue makes its’ best moves this thing works out.

One can see how unconventional the proof was. Nonetheless Ryan worked through it with relative ease. At his conclusion of the proof he was asked if he considered it to be a proof. His reply was, “Yeah you can use it; you can have any game here.” This response is consistent with his view in the first interview. In his mind, he did not use any particular game throughout the proof; therefore it was general enough to be a proof.

Episode 23—Ryan’s first interview with the researcher (Appendix L):

Figure 46. T3-37:33(a)

R: This is a theorem.

Y: Yeah can you show me how you prove this theorem?

R: Right okay. So blue and red both to start off have two moves.

Y: Ah-huh.

R: If you start off with blue, blue can either take this
Figure 47. T3-37:46(a)

make their best move from this one or this one

Figure 48. T3-37:47(a)

start with this one. So they make their best move from this one so they do an

X plus negative x and negative y . . . I think these are supposed to be flip
flopped, aren’t they?
Y: How did you figure that out? You are probably right but something told you that it has to be that way, what was it?

R: Because red is the next move and red . . .

Y: Right you’re correct because you want them to cancel out?
R: Yes, this is supposed to be negative y, this is negative x so red’s next move and their only move is this

\[ x + \frac{e}{x} = 0 \]

... (Figure 51. T3-38:53(a)

so it’s X plus negative x and that’s equal to zero .

\[ x + \frac{e}{x} = 0 \]

... (Figure 52. T3-39:00(a)

Y: Okay so similarly you can do the other.

R: Right.
Although Ryan considered both justifications for the two conjectures as Episode 24 shows, he considered the second proof to be more general than the first. Mathematically, both theorems have the same level of generality. The difference is that the proof of the second theorem lent itself to a more extensive use of symbols than did the first proof.

Episode 24—Ryan’s second interview with the researcher (Appendix L):

Y: Okay so similarly you can do the other.
R: Right.
Y: Is that more of a general proof than what you just did with the previous one?
R: Yes.
Y: It is.
R: I think in my opinion, yes.
Y: In your opinion okay which because
R: Because it’s like it doesn’t, that goes for any games, you just flip-flop them, it doesn’t involve two different games, and it’s the same game. It doesn’t involve two different games but the other one will probably involve two different games. This is only going to be one game but the exact opposite of the first game.
Y: And then is there any reference to what game in this one?
R: There’s nothing.
Y: Nothing.
R: It’s just a game with this value and a game with this value the opposite value.
Y: Okay so those could be any games?
R: Yes.

Y: While the other one there was no reference but
R: You had to have a game of negative one half to the N . . . this it doesn’t matter it can be, I mean you had to have a game of negative one half to the N somewhere in there.

Y: Right.
R: This, it doesn’t matter. I mean it could be one half, negative one half, three, negative three, and zero, star.

Y: I got you. In your opinion this is a more general one which qualified to being more of a proof?
R: Yes.

Summary

The four participants demonstrated varying degrees of abilities in proving conjectures and theorems in the context of combinatorial game theory. Ryan, Zack, and Chris evidenced good comfort level with the nature of the theory and the proof techniques in that context. Shannon was the least comfortable and at times was confused which is evident when she said, “It seems to me that this is a very complicated thing to do at the beginning because I personally don’t understand how they do it, how he did it.” While the other three were fairly confident in their abilities in proof writing in that context, Shannon demonstrated lack of self-confidence. For example, in one of the
interviews, she said, “Kind of, like I understand how to do it, it’s just whether or not I can think of it.”

Shannon’s proof scheme can be mostly classified as an external conviction proof scheme (Harel & Sowder, 1998). She presented evidence that her scheme encompasses elements of both subcategories of the external conviction proof schemes, namely ritual and authoritarian. Her emphasis on the importance of the two column format indicated that her conceptualization of what a proof is is based on the format and rituals of her former experiences with proof. In one of the interviews she said, “I think the induction that I did there is proof.” Induction to her was a ritual that she recognized. Hence that process to her was a mathematical proof. She also accepted the truth of a mathematical argument if she was told so. For example, in the same interview, when asked about the applicability of what was done in class she said, “But, like I know everything else is proof, I just, like I know it is, I just don’t really think it is, you know?” Meanwhile, when she said, “Well, in this case it would work out mathematically, I don’t know how accurate it would be for the other ones,” she is showing element of her proof schemes that are consistent with the empirical proof schemes. In particular it is consistent with perceptual empirical schemes.

Ryan, just as Shannon, did not appreciate what was done in the course up to that point to be mathematical proofs. He acknowledged that they are mathematical activities and would not be considered to be proofs unless the theorems prove to be applicable to something else. This phenomenon is consistent with the perceptual empirical scheme (Harel & Sowder, 1998). He had no problem with understanding the language of the
material. He showed strong comprehension of the definitions and the theorems in the course. He evidenced self-confidence in his ability to produce proofs autonomously.

As mentioned earlier in this chapter, Zach’s and Chris’s participation did not lend itself to provide enough data to consider two of the four factors in focus; namely, their proof scheme and appreciation of the process as a mathematical proof.

October

A presentation by Ryan and Zack in Jack’s office was the first data collection session in the month of October. During that presentation Zack had shown change in the way he approached proofs in this course. Rather than go through steps utilizing only definitions and simple routine procedures, he demonstrated an ability to make connections between concepts and theorems. It is difficult to determine what his proof schemes were earlier in the semester for lack of data about him in this particular area. During the same session Ryan also presented his proof. His proof relied heavily on using definitions to move from one line to the next. Zack contributed by pointing out a more concise way to reach that same result. He used already proved theorems to make his argument. This exchange is depicted in Episode 25 below. Zack’s statement was,

Can you say that this is star and this is star? And this is down so you already know it’s negative because star plus star is zero [already proven fact]. So you already know it’s negative because star plus star is zero.

It is an indication that he made connections between a particular situation and a theorem that has been proved earlier which indicate that he was functioning at a level beyond the empirical proof scheme as defined by Harel and Sowder (1989).
Episode 25—Ryan and Zack’s third in-office presentation (Appendix M):

Z: Can you say that this is star and this is star?

Figure 53. T3-56:50(a)

And this is down

Figure 54. T3-56:52(a)

so you already know it’s negative because star plus star is zero.

J: Yea right that’s a good thing to simplify to write it as equal to star plus down

plus star.
Both Ryan and Zack were fluent in writing their own proofs; both exuded self-confidence and did not demonstrate any hesitation during the process of recreating their proofs. This is apparent in the exchanges depicted in Episode 26.

Episode 26—Ryan and Zack’s second office presentation (Appendix M):

J: What about taking the first zero?

Z: That’s the bad move, I was thinking about that.

J: Yeah, that’s a bad one, why?

Z: Because then red would take the star.

J: Yeah, ok.

Z: And this is down too, even though it’s negative.

J: Yea right, right. Taking that zero is bad.

Z: Um, taking the star might be a good move.

J: Ok.

Ryan’s presentation indicated that he had moved beyond the empirical proof schemes as well. Episode 27 shows Ryan pointing out to Zack that he could use already established facts rather than working from definitions only.

Episode 27—Ryan and Zack’s third in-office presentation (Appendix M):

R: Can we do [inaudible] because that’s the same thing as this
and blue has the next move I think blue won that but go ahead

Z: Yea, I don’t know. Let’s go through it again. Okay blue moves next, blue’s best response is to take the star so you just get star plus star equals zero. And red goes next. Red’s next move is to take the zero here.
That will give you zero star bar zero plus star plus zero bar zero. Okay but this is star so all we actually have left is zero. So all we actually have left is this, left’s move take the zero or the star.

J: Yea the zero and the star are two possibilities.

Z: Star I believe is their best response.

J: And that’s not right because if you make this a star.

R: If you make it a zero game.

J: Whoever goes next wins.

Z: Right so we don’t want to make it star; we want to make it zero.

J: Yea general principle to remember is that you can make it zero; that’s a good move for you.

Z: Right and red goes next so they lose.

The exchange between Zack and Ryan demonstrated earlier in Episode 27 shows Ryan making connections between already proven theorems as Zack did. This is clear in this quote by Ryan: “Can we do [inaudible] because that’s the same thing as this and blue has the next move. I think blue won that but go ahead.” His statement “but go ahead” may be an indication that he was not yet fully confident in his thought process within what seemed to be a shift in his proof schemes. Zack’s response showed his confidence in his ability to verify Ryan’s claim, “Yea, I don’t know. Let’s go through it again.”

Similar to Ryan and Zack, Shannon’s comprehension of the subject matter had improved. This is evident in Episode 28 depicting an exchange between Shannon and Jack
Episode 28—Shannon and Chris’s second in-office presentation (Appendix N):

J: Ok, can you [Shannon] explain where all of this came from? From left played to this which equals this and you decided this was a bad move, so can you explain where this stuff came from? Show us, tell me where and how.

S: With this right’s turn?

J: Yea.

S: And right took the star?

J: Right to the star here, so, this game has been played in this position.

S: And the start is the same to zero. Zero bar zero.

J: Yea so that’s why he changed it here and how do you get this?

S: ’Cause it’s the only thing you can choose and zero and zero and zero plus zero is still zero.

Few steps later:

S: Okay. So you can do that [0+{0|0}] and that should be zero again. Right? No.

J: No.

S: That wouldn’t make me lose. I don’t want that.

J: It’s a star game not, its zero plus star. So it equals start.

S: But if left goes it will be zero plus zero.

The above episode shows that Shannon was a participant in the process of proving the theorem rather just an observer.

Chris continued to demonstrate good grasp of the material; he explained each step as he was writing his proof. He responded to Jack’s comments without hesitation and did
not seem intimidated by Jack’s questions and comments (see Episode 29 below). At a point when the material had shifted to be more abstract and symbolic, Chris demonstrated good understanding of the tenets of the material and carried out logical arguments using axioms, definitions, and theorems. He made the necessary abstraction to be able to utilize the skills he had in other mathematics courses to this course. The differences in the context seemed to be irrelevant to him. When Jack was prompting Shannon to justify her steps using theorems rather than strictly definitions, Chris responded promptly sighting an already proven theorem. That is an indication that he was functioning at a level higher than the empirical scheme level.

Episode 29—Shannon and Chris’s second in-office presentation (Appendix N):

S: ’Cause it’s the only thing you can choose and zero and zero and zero plus zero is still zero.

J: Right, but there’s a quicker way of saying

C: star plus star.

J: This is star plus star, the previous theorem is star plus star equals zero.

S: Yeah

Chris took over presenting his proof of the same theorem.

C: It’s basically just changing all your lefts to rights and your rights to lefts.

J: Yeah, that’s about it.

C: Ok, and this is right going first.

J: Yep.

C: So, right can choose that . . . (sound of chalk screeching on the board).
S: Ewww.

C: I don’t like that one.

J: It’s not the chalk’s fault.

C: it is

Figure 57. T3-1:16:45(a)

Ok, and left can go now, but that’s a bad move so left is going to choose from here which is there.

Figure 58. T3-1:16:56(a)
J: Did you see what he did? Do you see where that stuff came from? He sort of skipped a step.

S: Yeah, I see.

C: Worked it in my mind.

Unlike her attitude earlier in the semester, there was an emergence of a more confident and more participatory personality at that stage of the semester. She was willing to take the initiative and present her arguments. It is clear in the following episode when she said, “let me show how smart I am.”

Episode 30—Shannon and Chris’s second in-office presentation (Appendix N):

S: It’s up.

J: It’s up. And what do we know about up?

S: It’s positive.

J: Up is positive so from that you can say left will win at that point. Okay so as soon as you have a previous theorem that says who’s going to win then, then you stop okay? There’s another one like that.

S: Let me show you how smart I am.

Y: You just prove it, you’re smart!

Episode 31—Shannon and Chris’s second in-office presentation (Appendix N):

J: Did you see what he did? Do you see where that stuff came from? He sort of skipped a step.

S: Yeah, I see.
In Episode 31, she had no problem following Chris’s proof. Meanwhile in Episode 32, it is obvious that she still had doubt about her ability to prove conjectures. Her statement, “No, I don’t think I could do them,” is an explicit expression of her self-confidence at this point.

Episode 32—Shannon and Chris’s second in-office presentation (Appendix N):

S: There is still one more though . . .
C: He could make up five.
S: No, I don’t think I could do them.
C: Oh, sure you can.
S: I need those. You just erased it.
C: Go from memory.
S: No, because I will probably be horrible.

Episode 33—Shannon’s third interview with the researcher (Appendix O):

S: Because that’s the definition. That’s why we put them here and if we can’t choose, like, that’s sometimes why we have zero comma the star something because we don’t know which one is the best move.
Y: I see, okay.

Schematically, Shannon showed in Episode 34 that she was no longer functioning at the external conviction scheme. Her reference to definitions and deducting her own arguments autonomously indicates that she is no longer expecting proofs to follow certain forms. In Episode 33, she said, “Because that’s the definition. That’s why we put them
here and if we can’t choose, like, that’s sometimes why we have zero comma the star something because we don’t know which one is the best move.” As one can see, her justification was in a narrative form with which she was content. It was a novel expression of understanding and not something she heard from someone else with authority such as a professor or an author of a textbook. This assertion was the first sign that her schematic conception of proof had developed toward a more sophisticated conceptualization.

She continued to demonstrate improvement in creating correct arguments and justifying mathematical statements. She ended the following episode by concluding her argument by “and that’s why.” This is a clear sign that she had improved in two different areas: construction of proofs on her own and the confidence in her capability of defending them.

Episode 34—Shannon’s third interview with the researcher (Appendix O):

Y: Because you’re going to take the whole pile.

S: Right.

Y: All right.

S: So, this is a zero game because no matter what you do, no matter what side you’re on, no matter what you do, the second person can just wipe the rest of them out. [Inaudible] no matter what you’re just going to have one pile left and that’s bad.

Y: So you, no matter what you do you’re going to lose.
S: Yeah and that’s why.

At some point she reverted to accepting things because someone else said it was true. That assertion indicates that the transition from one proof scheme to another is not discrete but continuous. Episode 35 shows that she accepted a connection between nims and binary addition without “regrouping” without questioning the conceptual reason behind it. She trusted that it was true because someone else had told her so, in this case the professor. There was a noticeable difference between then and the early stages of the semester. As Episode 36 demonstrates, she did not hesitate to explore the possible justifications for the conceptual connection between the two areas.

Episode 35—Shannon’s third interview with the researcher (Appendix O):

S: It’s kind of complicated at first.

Y: Yeah. Now, have you ever thought of why this binary thing works with nims?

S: Not really. It’s really neat though.

Episode 36—Shannon’s third interview with the researcher (Appendix O):

Y: Have you thought of that? I mean, I’m not expecting you to give me answer because but did it occur to you . . .

S: I’m sure it’s not that hard to see
Kind of, like, if you do this game and the first player changes it to one, two, two, that’s going to do that.

and then it’s a star game ’cause you’re going to have only one in last column. Like I see how it corresponds, I can’t tell you exactly why but I see how it corresponds and then the next player would do this.
and then that would let that out and then it, you know, I see how it
 corresponds. I wouldn’t be able to prove it but I see how each move
 corresponds with changing it, changing this number down here and then
 copycat, like, whatever the first person does here or here, the other person can
do here or here too ’cause it’s the same number.

Once her willingness to accept the connection between nims and binary addition
was challenged and she was asked if she can prove it, she decided to take on the
challenge of showing why. This action is not a characteristic of the authoritarian proof
scheme. Her statement, “Like I see how it corresponds; I can’t tell you exactly why but I
see how it corresponds” is consistent with the perceptual empirical scheme (Harel &
Sowder, 1998). Elements of “inductive empirical” proof scheme were apparent in her
conceptualization of proofs and was evident is her statement, “‘Cause we’re given the,
what is that word? We’re given the idea and then you prove it to be true or not true and
. . .” She added that the proof must be based on “previous information.”
Shannon’s appreciation of the level of rigor did not change much (Episode 37). Her view is best explained in her statement, “I know they’re proving something but they’re not really proving anything in real life.” Meanwhile, she explicitly expressed her elevated confidence in her ability to produce proofs on her own toward the end of October, more than half way into the semester (Episode 37).

Episode 37—Shannon’s third interview with the researcher (Appendix O):

Y: Okay. How do you feel about proving things now? Do you feel more confident doing these than before?

S: Yeah but I still don’t see how we can apply it to things but, like, I can prove stuff using simple numbers now and see that a lot better than I did before.

Y: And those aren’t, these aren’t your mind proofs.

S: Like, I know they’re proving something but they’re not really proving anything in real life.

Y: Okay.

S: Like, I know they’re proofs, they’re just not the proofs that I’m used to.

Y: Right. And the reason why you know they’re proofs is because?

S: ’Cause we’re given the, what is that word? We’re given the idea and then you prove it to be true or not true and . . .

Y: And that is based on?

S: Previous information.

Y: Previous information that we either proved or defined?

S: Um-hmm.
Summary

Much information has been obtained during this month in relation to the four factors of interest to this study; proof schemes, ability to create proofs, self-confidence in the ability to do so, and the appreciation of rigor of the proofs. Shannon’s proof scheme had shifted from being consistent with the authoritarian proof scheme to have elements from the empirical proof scheme. Her transition or development of proof scheme did not appear to be discrete; rather it was transitional. She showed evidence that she held a scheme that had characteristics of both the authoritarian and empirical schemes. She also demonstrated better understanding of the subject matter. Her ability to create correct arguments improved as well. The areas of confidence and appreciation of the rigor showed less improvement. In comparison of “appreciation for rigor” and confidence, appreciation showed the least improvement. Chris possessed the most sophisticated conceptual schemes of proofs. The ease in which he operated in creating his proofs and work axiomatically indicates that he was functioning at the analytical level. Zack and Ryan both were functioning at a schematic level beyond the empirical proof schemes. Ryan’s conceptualization seemed to progress; meanwhile his self-confidence was still lagging behind. On the other hand, Zack appeared to have adequate confidence in his ability to investigate claims and make judgments on the truth values of those claims.

November

When data was collected in November, proof by induction was introduced. Students were provided with a list of theorems conducive to being proved using
induction. The following episode shows Zack proving a theorem using induction. He correctly started with the base case and expressed the assumption correctly. The theorem he had to prove was: “Every polygon with n sides can be triangulated where the sum of the interior angles of the triangles is \((n-2) \times 180\) degrees.”

Episode 38—Ryan and Zack’s fourth office presentation (Appendix P):

Z: Okay. All right. This is the one geometry \(n-2\) times a hundred and eighty. Am I right? Or the best case, we start with \(n = 3\). ’Cause \(n = 2\) isn’t a polygon. It’s just a line.

J: Yeah.

Z: You know, two vertices. That gives you triangle.

J: Okay, so that’s why it’s true for \(n = 3\), the base case.

Z: That’s just one times a hundred and eighty is one eighty for the induction. Assume it is true for \(n\). Show for \(n + 1\). Prove for \(V_{n+1}\). Okay, so if you have a vertex here \(V_1\) down to vertex \(V_n\)
You have some vertex up here $V_{n-1}$

have whatever in between these, doesn’t matter. So this is what you have
Figure 65. T4-55:02

if you add a point vertex, well, that should just be $V_{n}$. The way I’m proving it

Figure 66. T4-55:11(a)

Have a vertex $V_{n+1}$. That just makes a triangle.
Figure 67. T4-55:19(a)

J: Okay.

Z: Right? So by hypothesis we know that the angles up to $V_n$ well had to be just $V_n$’s; that’s why this was a $V_n$ minus one. By the hypothesis we know that the polygon up to $V$, up to $n$ sums to $n$ minus two times a hundred and eighty. Right?

J: Okay.

Z: And what we add it on was a triangle with $n$ plus first vertice.

J: Okay.

Z: So we just added on a hundred and eighty degree.

J: Right. Now we can factor out the one eighty and a hundred and eighty degrees times $n$ minus two plus one but that’s the same thing as . . .

Z: I can factor out one eighty . . . Hundred and eighty degrees times $n$ plus one.

That’s two.

J: Yeah.
Z: But that’s the same thing. A hundred and eighty degrees plus one.

J: Yeah.

Z: And that’s what we need I never put what we need off to the side somewhere well that’s what we need.

*Figure 68. T4-56:34(a)*

R: Yeah. Minus two.

Z: I had two minus [inaudible] yeah.

*Figure 69. T4-56:35(a)*
Zack comfortably finished the proof by induction. His choice of induction as the method to prove the theorem indicates that he anticipated the method of induction to be more effectively, in this case, that deduction. This choice and anticipation is consistent with what Harel and Sowder (1998) characterized as interiorized proof as scheme. They defined this scheme to be a proof scheme (method) that has been reflected upon. At this stage the internalized method of proof becomes the object on which an individual operates on and anticipates the result of the operation such as perform comparisons between proof schemes, communicate the proof schemes with others, and determine when the scheme could be utilized (Harel & Sowder, 1998).

After Zack had finished his presentation of the proof shown in Episode 39, Ryan presented a proof of another theorem.

Episode 39—Ryan and Zack’s fourth office presentation (Appendix P):

R: I’ve got some of the Fibonacci numbers, three or four?

J: Number four, okay.

R: And I don’t really know where to go from. I got most of it and I’m stuck at a point.

J: Okay. Well show us what you got.

R: So it’s \((F_{n+1})(F_{n-1}) - F_n^2 = (-1)^n\)

... And I, I was trying to solve it for this and I just thought what if I moved this to this side?

J: Okay.

R: So, I, n no plus one k+1, well let me start with base case first and
J: No, let’s skip the base case. That will be [inaudible]. The base case is \( n \) equal to what?

R: \( n \) equal to, it’d be \( f \) of two (\( F_2 \)), \( f \) of zero (\( F_0 \)) minus \( f \) of one squared (\( F_1^2 \)) equals negative one to the \( n \) \((-1)^n\)

J: I didn’t know we put \( f \) sub zero on the sheet.

R: But it’s one minus one so it’d be \( f \) sub zero.

J: Okay, what is \( f \) sub zero?

R: Zero.

J: Okay, yeah, use that, okay, well, that’s good.

R: So, we assume \( n \) equal to \( k \) and want to prove

J: Where?

R: \( n \) equals \( k \) plus one so I have \( k \) plus one plus one (\( F_{k+1+1} \))

J: Just put \( F_{k+2} \) Save space.

R: All right. \( F_{k+2} F_k \) equals \( F_{(k+1)}^2 \) plus one to the power (\( k+1 \)).

It is clear that Ryan evidenced no difficulty using the induction method. He understood the structure of proof by induction. He established the truth value for the base case. He stated the assumption correctly and wrote the induction step correctly as well.

J: Okay, that’s what you are trying to show.

R: Right. I’m trying, I then want to get rid of \( F \) of \( k \) plus tow (\( F_{k+2} \)).

J: Okay.

R: And you know \( F \) of \( k \) plus two (\( F_{k+2} \)) is equal to \( F \) of \( k \) plus \( F \) of \( k \) plus one.

J: Okay, all right. So you’re going to put that in there.
R: Right. So \((F_k + F_{k+1}) \times F_k\) is equal to.

J: That’s still what you’re trying to prove.

R: Right \((F_{k+2}^2) + (-1)^{k+1}\)

J: Trying to get those two things equal.

R: And then so, factor this so you have \((F_k)^2 + F_{k+1}F_k\) and then for, if, when I, if you subtract, \((F_{k+1}) + (F_{k-1})\) is equal to \((F_k)\) plus minus one to the k \((-1)^k\)

J: That one is squared.

R: Yea [Ryan added a square to \(F_k\)]. If you subtract this one [pointing at \((-1)^k\)] from both sides you get \((F_k)^2\) so that is why we will substitute in for \((F_k)^2\)

J: Ok.

R: \((F_{k+1}) + (F_{k-1}) (-1)^k\) equals \((F_k)(F_{k+1})\) and then you, I added this to the other side.

J: You added what to the other side?

R: The negative one to the \((-1)^k\)

J: Ok.

R: So I got, I also, so when this is the other side you have a factor of \(F_k\) plus one, so I got this and I’m stuck there.

J: Well replace some parentheses by something.

R: Is this equal to \(F_k\) plus one.

J: Yeah.

R: So you have \((F_{k+1})^2\) plus one is equal to \(F\) squared of \(F\) of \(k\) plus one, then you don’t have this term.
J: Why?

R: [Looking at the problem, thinking]

J: That’s sort of a trick; you can find what it’s equal to pretty easily.

R: That’s going to be zero.

All Ryan needed to overcome his obstacle that prevented him from finishing up his proof was a small hint from Jack. This occurrence is a clue that he had internalized the process and was focusing on the logical argument rather than the method itself.

J: Yeah you’re right, why?

R: You have . . .

Z: Oh, ok. Because this is always one term over and it would be negative.

J: One of those has to be plus one and the other one minus one because one of the exponents is even . . .

R: And one of them is odd. So, you just have F squared, k plus one is equal to f squared k plus one.

J: Yeah. That’s really good.

R: Yeah.

J: I want you to, I want to do it at class on Monday and but I would like to see you rewrite it because what I don’t like is to start with the equation you are trying to prove; you can’t keep going down on both sides and you sort of mixed in the way. Maybe a better way to do it would be to take this minus and this minus and then come out to be zero instead, and by just doing the same
steps you did. But write down this and then do the first step to change it to
would be like . . .

R: Minus.

J: Like that and then do the next step and then make it all zero at the bottom so
that you start with something that you

R: Okay, so you’re not equally something you’re equally zero.

J: Yea, start off with the left side and go through all your steps and then finally
get down you get to this, and you’ll say oh that equals zero, which is what
you’re trying to prove.

R: Okay.

J: So you start with one side while you take the equation and rewrite it so that
everything’s on one side and you prove that it is zero, then write it down.

As the above portion of Episode 40 shows, Ryan was able to discuss his proof and
communicate his reflections and anticipation of what his proof led him to. He identified
the gap in his proof and pinpointed the step where he faced a logical problem. This
presentation demonstrated that Ryan was functioning at the analytical level. In particular,
his behavior was consistent with a subcategory of the analytical proof scheme, the
interiorized proof scheme (Harel & Sowder, 1998).

The biggest change between the beginning of the semester and the end of it was
shown by Shannon. Her appreciation of the rigor of the new material’s proofs had
changed. Early in the semester she did not acknowledge the activities that took place to
be mathematical proofs. When she was asked about it later in November, as Episode 40
shows, she had a totally different opinion. She was specifically asked to reflect on the activities that took place earlier in the semester. The activities she refused to consider as being proofs were no longer the case. She had a different opinion than before and acknowledged that they were proofs. Her concern remained about the applicability of the conjectures being proved. The change in her view is evident that her appreciation of proof had increased. This represents a change for Shannon in this area.

Episode 40—Shannon’s fourth interview with the researcher (Appendix R):

Y: Uh-huh. Now do you think this is more of a proof than what we did with the hackinbush games at the beginning of the semester?

S: No.

Y: No?

S: No, they’re both proofs.

Y: No, they’re both proofs?

S: Yeah, I just don’t use this.

Y: But you don’t see the use of this one?

S: No.

Y: Not this one, but this particular problem?

S: No.

One of the characteristics of the interiorized proof scheme is that the individual is capable of communicating with others when a certain proof method could be utilized (Episode 42; Harel & Sowder, 1998). Episode 41 gives an example of Shannon being able to do exactly that.
Episode 41—Shannon’s fourth interview with the researcher (Appendix R):

Y:  But you see the use of proof by induction?

S:  Uh-huh.

Y:  Yeah, okay.

S:  I don’t like the induction with stuff that’s like not numbers; I like how we did,
like prove that this can be filled with 2 by 1 squares. Like, there’s no numbers
there, I don’t like that. Like I can show you, here look, I can fill it, however I
want, but you know

Y:  Yeah.

S:  I don’t, [inaudible] do any examples like that still I don’t see how you can do
that with induction.

Episode 42—Shannon’s fourth interview with the researcher (Appendix R):

S:  See like something like that is what I don’t understand. Like I understand
induction and like how it works and like it works for everything else.

Harel and Sowder (1998) classified the interiorized proof scheme to be an
internalized method of proof that becomes the object on which the individual operates. In
Episode 42, Shannon was reflecting on induction as being a method of proofs and when it
could be utilized. Episode 43 demonstrates another aspect of the interiorized proof
scheme which Shannon showed she possessed; that is, to perform comparisons between
proof schemes, communicate the proof schemes with others, and determine when the
scheme could be utilized (Harel & Sowder).
Episode 43—Shannon’s fourth interview with the researcher (Appendix R):

S: Okay, let’s say you can do. I understood that right now. [Writing on the board, trying out the solution] But even if you can prove that, like then you know you can add on another dimension, like it won’t work by doing induction like you can do this so that if you add on one more it’ll all cancel out, but it doesn’t.

Y: Is it, what the dimensions are? Is there a condition on the dimensions? How they would be?

S: I can look, all right [inaudible] it has to be 2 to the power n.

Y: Yeah, see, because that

S: So it should only [inaudible] 4 by 4, just got to figure it up.

Y: 4 by 4 or 8 by 8

S: Or 16 by 16

The next two episodes data suggest that Shannon possessed elements of another transformational proof scheme, namely, the restrictive scheme. Episodes 45 and 46 show that Shannon had two of the three subs schemes of the restrictive scheme. Individuals possessing restrictive scheme need to construct actual objects. This is evident when she said, “I like it with like real things, on a square equals whatever equals.”

Episode 44—Shannon’s fourth interview with the researcher (Appendix R):

Y: But how do you feel about induction now?

S: I like it with like real things, like on a square equals whatever equals

Y: Yeah, yeah.
S: Like is that the 1 plus 2 plus 3? Yeah 1 plus 2 plus 3 blah blah blah blah equals m squared. I like that. Like this is dumb.

Y: So this is not, what did you say; I like it when it’s with real things?

S: Well it’s easy, like stuff I’ve seen it with, you know. Because I’m not going to use this. I’m going to use like sometime I might need to know what this is. [inaudible] my everyday life, I might need to know what that is, well there’s 4 numbers and however it works.

Y: Uh-huh. Now do you think this is more of a proof then what we did with the hackinbush games at the beginning of the semester?

In this episode, Shannon demonstrated she had one of the two sub schemes of the restrictive scheme which is the constructive scheme. Constructive proof scheme is defined to be a scheme where certainty is established by constructing actual objects (Harel & Sowder, 1998). This is evident when she said “I like it with real things, like on a square equals whatever equals.”

Episode 45—Shannon’s fourth interview with the researcher (Appendix R):

Y: But you see the use of proof by induction?

S: Uh-huh.

Y: Yeah, okay.

S: I don’t like the induction with stuff that’s like not numbers; I like how we did, like prove that this can be filled with 2 by 1 squares. Like, there’s no numbers there, I don’t like that. Like I can show you, here look, I can fill it, however I want, but you know.
In this episode Shannon shows evidence of the contextual proof scheme. It is defined to be a scheme in which certainty is established by performing comparisons between schemes, communicating the proof method, and determining when the method could be utilized (Harel & Sowder, 1998). Shannon demonstrated this when she said, “I don’t like the induction with stuff that’s like not numbers.”

Summary

The semester ended after the first week of December. Hence the data collected in late November closely represented students’ learning during that semester. All four students were able to write their own proofs and think autonomously. They all demonstrated high confidence in their thinking process and the correctness of their proofs. No evidence in the data indicates otherwise. Earlier during the semester, it was either explicitly expressed or implicitly inferred that at least Ryan and Shannon did not have confidence in their proof writing. Although Shannon had the least confidence in her own abilities, Ryan and Zack demonstrated that as well to a lesser degree. Ryan’s proof scheme had shifted from empirical proof scheme to analytical proof scheme. Both Ryan and Zack showed evidence that they were functioning at the transformational proof scheme which is a subcategory of the analytical scheme. In particular, they were functioning within the interiorized scheme within the transformational scheme. The largest jump was made by Shannon. She started the semester holding the authoritarian proof scheme. She then progressed to the function at the empirical scheme in the middle of the semester. By the end of November, she had elements of contextual restrictive transformational analytical scheme, constructive restrictive transformational analytical
scheme, and interiorized transformational scheme. Table 2 presents a summary of the findings across participants and time.

Table 2

*Findings Across Participants and Time*

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<th>Ability to carry out a proof</th>
<th>Confident in the ability to produce proofs</th>
<th>Proof Scheme</th>
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<td></td>
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<td>NO</td>
<td>External conviction</td>
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<td>Empirical</td>
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<td>Able with assistance</td>
<td>Confident</td>
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<td>Confident</td>
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*(table continues)*
Table 2 (continued)

*Findings Across Participants and Time*

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</tr>
<tr>
<td>Chris</td>
<td>Not observed</td>
<td>Able without assistance</td>
<td>Confident</td>
<td>Analytical</td>
</tr>
</tbody>
</table>
CHAPTER V

RESULTS

Discussion

This study was conducted to attain a deeper understanding of the way students learn and write mathematical proofs, and to examine the effectiveness of the Moore method in developing students’ ability in writing formal proofs. A particular focus was to study how students use correct mathematical arguments to establish the truth of a mathematical conjecture. To achieve this goal, this study was designed to answer the research questions that were introduced in chapter 1. The questions for this study were:

1. What are students’ conceptions of proofs and what do they consider mathematical justification (Harel & Sowder, 1998)?

2. Do students’ experiences in the class taught using the Moore method (a description of the method is provided in Chapter 2) affect their conceptions of proofs?

3. Is the Moore method effective in promoting development of students’ proof schemes to higher schematic levels as defined by Harel and Sowder (1998)?

In this research, the researcher carefully studied four students as they were developing their proving abilities, their confidence in their own ability to write proofs, their appreciation of proofs as tools to establish mathematical truth, and their conceptual schemes of proofs. This developing phenomenon was studied in the context of the Moore
method, an inquiry based learning environment where students are responsible for participating in the development of the targeted mathematical theory rather than being passive recipients of the mathematical facts within that theory. The title of the course in which this Moore method environment was created was “Numbers and Games.” The theory students were aiming to develop was combinatorial game theory. Chapter 2 contains a detailed description of the Moore method.

Participants in the study had varying degrees of ability in producing mathematical proofs. The data showed that Chris demonstrated higher ability than all other participants. Zack and Ryan were close to one another in their mathematical abilities. Shannon was mathematically behind the other three participants. Chapter 4 provides detailed information about each participant.

Due to the nature of data collection and scheduling constraints, not all four factors were observed in all participants. Chris’s data did not provide enough evidence to make a conclusive judgment on the change in any of the factors through the duration of the semester. Shannon was the most accessible participant, which offered an ample amount of data to study changes across all four factors. Ryan was available to be interviewed one-on-one two times (a schedule of all data collection sessions is included in chapter 4). He was observed during his in-office presentations as well. Both interviews as well as the office presentations generated enough data to study changes in all four factors. Zack’s data lent itself to study some of the factors but not all four factors.
Data suggested that one may show characteristics of one or more scheme categories as defined by Harel and Sowder (1998) concurrently. This finding is consistent with the assertion by Harel and Sowder depicted in the following quote:

A given person may exhibit various proof schemes during one short time span, perhaps reflecting her or his familiarity for, and relative expertise in, the contexts, along with her or his sense of what sort of justification is appropriate in the setting of the work. (p. 277)

This finding was also confirmed by Housman and Porter (2003) who conducted a study that involved “above-average” college students. The participants in that study were 11 female students who performed well in college level mathematics courses. It was observed that 10 students exhibited characteristics of two or more proof schemes with 1 having four different schemes.

Shannon demonstrated elements of both the empirical and external proof scheme simultaneously (figure 70). Therefore, her development from the external scheme to the empirical scheme was not abrupt but rather it was a more gradual progression. The same was true in her move from the empirical to the analytical schemes (figure 70). No evidence was found that indicated an overlap between the external scheme and analytical scheme (figure 70). In the beginning of the semester Ryan’s and Zack’s behavior indicated that they both held the empirical proof scheme with evidence showing they held both the perceptual and inductive sub schemes.

In Ryan’s and Zack’s cases no transitional stage was evident as they moved from the empirical to the analytical stages (figure 70). It is not clear if that was due to the
Figure 70. Shannon, Ryan, and Zack's schematic development
nature of the collected data or it meant that a jump from one level to another is possible without going through a transitional stage. Shannon started the semester at the external conviction scheme and ended the semester functioning at the analytical level. When she held the external conviction scheme, she demonstrated elements of all three subcategories of that scheme: the ritual, the authoritarian, and the symbolic. She went into a transitional stage between the external and empirical schemes, and within the empirical scheme she held the inductive and the perceptual sub schemes. She then went through another transitional period, this time between the empirical and the analytical schemes. The analytical scheme has two sub schemes, axiomatic and transformational. No evidence was found to indicate that Shannon was functioning within the axiomatic sub scheme of the analytical scheme. Meanwhile, Shannon demonstrated that she held elements of the transformational analytical scheme (Harel & Sowder, 1998). Of the three sub schemes of the transformational scheme, the data showed that she held elements of the interiorized as well as the restrictive proof schemes. She also had elements of two of the three sub schemes of the restrictive scheme: the contextual and constructive (figure 71). The data concerning Ryan and Zack indicated that they initially held the empirical scheme with evidence of both sub schemes, inductive and perceptual. They did not go through a transitional period between the empirical and the transformational analytical scheme. There was no indication that they held the axiomatic analytical scheme or anything but the interiorized sub scheme within the transformational scheme. Finally, the data did not
show that Chris functioned at any of the external or the empirical schemes. These observations led to the conclusion that he held the analytical proof scheme. Not enough

Figure 71. Shannon’s schematic development (Harel & Sowder, 1998)
data was available to reveal where, within the analytical scheme, his conceptualization of proofs existed.

Shannon and Ryan were verbally asked if, in their opinion, they considered what they were doing to be mathematical proofs or, using their expression, mathematical activities. Ryan was asked the question once in September and once in October. He was not available for a one-on-one interview to address this question during the month of November. Shannon was asked the same question three times, in September, October, and November. Both Ryan and Shannon did not consider what they were doing to be mathematical proofs when asked in September and October. Shannon’s opinion changed in late November (figure 71). No conclusion can be made on what opinion Ryan would have held at the conclusion of the semester. In all cases the question was followed up with “Why do you think so?” This question served as a safeguard against giving an answer that meets the perceived interviewer’s expectations.

![Figure 72. Answer to the question “Are these mathematical proofs?”](image)

Proof writing ability by participants was categorized as: cannot prove, can prove with assistance, or can prove without assistance. Initially in September, Shannon fell
under the category “cannot prove.” Ryan and Zack fell under “can prove with assistance.” Chris fell under “can prove.” Later in October, Ryan, Chris, and Zack were able to generate their own proofs, hence fell under the category “can prove without assistance.” Shannon demonstrated that she “can prove with assistance.” During the month of November as the data suggested, all fell under the “can prove without assistance” category (figure 73).

Figure 73. Participants’ ability to write proofs

Finally, in Chris’s case changes in the confidence in his own ability to carry on a correct proof was difficult to detect. Nothing in the data suggested that he had a lack of confidence in his ability to prove theorems within the context of this course. Ryan and Zack started “somewhat confident” and ended the semester being “confident.” Shannon started the semester “not confident.” During the month of October she became “somewhat confident” and ended the semester “confident” (figure 74).
The above discussion led to important results. It provided answers to the research questions for this study introduced in chapter 1 and reiterated in the beginning of this chapter. What follows is the implications of the results of the study.

The study showed that the Moore method has positive affects on students’ conceptualization of mathematical proof, their self-confidence in their abilities, their appreciation of the relevance of proofs, and their ability to think autonomously. From the constructivist perspective these results are expected since knowledge in this course was constructed by the learners rather than being delivered to them. The inquiry-based learning that took place in the course allowed students to experience mathematics first hand. They built a coherent body of knowledge in which they appreciated the relevance of the different results that have been proven throughout the semester. The ability to create their own proofs led to students demonstrating higher levels of self-confidence.

The study achieved its goals; first, to document the progress college mathematics students make in their conception of mathematical proof when taught using Moore
method; and second, to validate the mapping of students’ proof scheme proposed by Harel and Sowder (1998).

The Moore method offers pedagogy that overcomes the invalid view among students that proofs are a confirmation of facts and intuitions already known to be true (Schoenfeld, 1985). Having students produce their own proofs is a departure from the traditional practice of presenting students, through lectures, with finished products of proofs (Alibert & Thomas, 1991). Instead, the Moore method emphasis is on creating new ideas by students (Dancis & Davidson, 1970). Therefore, the Moore method, as data indicated, promotes students’ appreciation for the use of, and the need for rigorous proofs to establish mathematical truth. Therefore, only after they were rigorously shown to be true that conjectures become theorems. This finding was evident in the development of the participants’ proof schemes. They all developed their schematic conceptualization of proof to the analytical proof scheme. This is the scheme where analytical arguments become essential for mathematical justifications. In the external conviction scheme and empirical scheme students accept the truth of an argument either by being told or by using examples that satisfy the argument.

The results of this study imply that the Moore method creates a culture of mathematical thinking and practice of rigor. Students are exposed to and participate in constructing meaning for mathematics. This construction has a lasting effect on students’ approach to mathematical thinking. It helps students develop constructive habits of mind. Passive receiving of knowledge and blind belief (external conviction scheme) in
mathematical truth diminishes as the culture of sense making develops throughout the semester.

Constructivist philosophy and theory provide an explanation of how individuals learn and build their knowledge. Although constructivism is a philosophy, there exist pedagogies that are consistent with the constructive view on the nature of learning. One such pedagogy is the Moore method. The constructivist view of learning is that “knowledge is not passively received but built by the cognizing subject” (von Glasersfeld, 1995, p. 18). The Moore method is based on this same concept. Telling mathematical facts and procedures is something to be avoided in this method. Unless it is necessary, lecturing is avoided in a Moore method classroom.

Students in the Moore method classroom are required to revise their proofs and re-present them to their classmates and teacher. This approach is compatible with the constructivist view that “the function of cognition is adaptive and serves the organization of the experiential world not the discovery of ontological reality” (von Glasersfeld, 1995, p. 18).

Piaget theorized that perception is what triggers learning. The mechanism in which learning occurs is through abstraction. As our experiences grow we abstract our experiences to form mental structures or schemes. Schemes are altered through assimilation and accommodation. When an experience is perceived to be familiar, it is “assimilated” into an individual’s own repertoire of knowing. If it is perceived as unfamiliar, current schemes must be altered or new schemes have to be created to assimilate it. This process is called “accommodation” (Piaget, 1971).
The data in this study provided many examples to demonstrate the concepts of assimilation and accommodation. When a new and unfamiliar proof situation was presented to Shannon and Ryan, they could not assimilate it into their existing scheme of what proof should be. Each created a new scheme which both of them called “mathematical activity.” When this scheme did not connect to other existing schemes their initial proof scheme was altered to accommodate the new type of proofs they encountered in this course.

This study documented that participants’ proofs were altered as their experiences grew throughout the semester. Shannon’s schemes had to be altered from external conviction to analytical proof scheme (Harel & Sowder, 1998). Similarly, Ryan and Zack experienced change in their proof schemes. They both initially had the empirical scheme and shifted to the analytical scheme (Harel & Sowder).

The results of this study reaffirm the NCTM (2000) recommendations that students develop deeper mathematical conceptualization when they construct their own understanding rather than being told a set of rules or procedures. A student will develop good habits of mind if given the opportunity to problem solve, make connections, and explore mathematics. The participants in this study provide examples that support such an assertion. The results indicate that mature mathematical thinking can be developed independent of the subject matter. That is also in concert with the NCTM recommendation that proof should be an integral part of every mathematics subject rather than be confined to geometry.
The Moore method emphasizes conceptual understanding and the development of one’s own ability to prove and conjecture and answer mathematical questions. In this method there is less emphasis on coverage as opposed to the traditional lecture based teaching method where students are overtaxed with an incoherent body of knowledge (Wiggins & McTighe, 1998). The researcher of this study proposed that using Moore method is one way to deal with deficiencies that are consistently revealed by the findings of the Third International Mathematical and Science Study (TIMSS, 2003). A synthesis of the finding indicated that the absence of central questions and ideas upon which organized inquiries and answers can be placed caused the weaker students to become confused (TIMSS, 2003; Wiggins & McTighe, 1998). In this context, so much of a wide content curriculum is simply passed over.

In a Moore method class, initially, progress is slow and so is the coverage of material in comparison to covered content in traditional teaching. This method, over the span of the semester, supported students to develop skills that helped them think autonomously. This autonomy is something they will carry on and transfer to other mathematical courses and subjects.

An environment where students work together and communicate their ideas to one another and to the instructor through presentations provides a living example to students that mathematics is not a solitary endeavor. Problem solving requires collaboration and exchange of ideas to reach solutions. The Moore method provides such an environment.
Figures 73 and 74 displayed earlier in this chapter show students’ ability to write proofs and their belief that they can do it are closely related but not the same. It is possible that each aspect develops at its own rate. Therefore, teachers need to be aware of this fact and not judge a student’s confidence in his or her ability to produce his or her own proof by solely observing the finished product of someone else’s proof.

**Implications for Further Research**

This research raised questions that need to be addressed by other studies. The following are some of the questions that need to be addressed:

- The data did not provide evidence that Ryan and Zack went through a transition period moving from one level to another. This is not enough though to conclude with certainty that a transitional state is not necessary. Further investigation about this phenomenon may yield a deeper understanding of this transition process.

- How would students’ conceptualization of proof change if the group size is different from the two students used in this study? Would students achieve similar results if they work alone? Would they experience development in any or all factors in a large group size?

- A natural extension to this study is to use quantitative methods to gage students’ development in the four factors by the Moore method as opposed to students learning by lecture method.
Limitation of the Study

It is difficult to implicitly gauge one’s confidence in proof writing; therefore this study relied heavily on participants’ verbalization that would give clues to their states of mind. Two of the participants were not available for one-on-one interviews which could have elicited explicit verbalizations reflecting their confidence in their own ability to write proofs.

The question of generalizability is an issue every qualitative study has to face. The goal of a study like this is to deeply understand students’ mental processes and the way they learn a specific concept. The goal of this study was to understand the nature of students’ learning and conceptualization of mathematical proofs. In particular, it was designed to more deeply understand how the Moore method helps students develop their conceptualization of proofs. It was not intending to make a generalization about how all students develop their conceptualization of such mathematical knowledge. By utilizing a qualitative design, this researcher was able to gain significant insight about the nature of mathematical thinking and learning.

Sampling for this study was pragmatic and convenient. The pool of participants was limited to the group of students taking the Combinatorial Game Theory course and being taught by the Moore method. Sample size was a constraint by the number of students willing to participate. One-on-one interviews were difficult to coordinate for logistical reasons. More information could have been obtained had the interviews been more frequent and involved all four participants.
As far as validity and reliability, they are as important to qualitative research as they are to quantitative research. As discussed in chapter 3, the measure of validity and reliability for qualitative research are trustworthiness and credibility. Triangulation is one way to establish credibility and trustworthiness. This research study used triangulation to establish its credibility and trustworthiness, as well as careful observation, external reflection of the work, member-checking and provide a rich description of the interactions between the researcher and the participants.
APPENDICES
APPENDIX A

RECRUITMENT
RECRUITMENT

The development of proof schemes of undergraduate mathematics students in a discovery learning environment

I want to do research on “The development of proof schemes of undergraduate mathematics students in a discovery learning environment” for my dissertation project. I would like you to take part in this project. If you decide to do this, you will be asked to be interviewed weekly for one hour and have your class presentations video/audio taped. Your interviews will be audio/video taped as well.

All data collected such as audio and video recordings as well as all samples of written work will be confidential. No one other than you and I have access to the information any information that would reveal your identity. Pseudonyms will be use in the dissertation to protect your identity. Any discussion concerning the data collected from you with other individuals will be done using pseudonyms.

If you take part in this project it will provide you with feedback on your learning progress in this course in particular on your learning of mathematical proofs. It should also provide you with extra opportunity for learning the material covered in this course. Taking part in this project is entirely up to you, and no one will hold it against you if you decide not to do it. If you do take part, you may stop at any time.

If you want to know more about this research project, please call me at 330 678-8631. The project has been approved by Kent State University. If you have questions about Kent State University's rules for research, please call Dr. Rathindra N. Bose, Vice President and Dean, Division of Research and Graduate Studies (Tel. 330.672.0700).

You will get a copy of this consent form.

Sincerely,
Yaser Dhaher, MSc, Med

B. CONSENT STATEMENT(S)

1. I agree to take part in this project. I know what I will have to do and that I can stop at any time.

Signature                                  Date
APPENDIX B

AUDIOTAPE CONSENT
AUDIOTAPE CONSENT FORM

I agree to audio taping at___________________________________________
on____________________________________________________________

Signature       Date

I have been told that I have the right to hear the audio tapes before they are used. I have decided that I:

_____want to hear the tapes       _____do not want to hear the tapes

Sign now below if you do not want to hear the tapes. If you want to hear the tapes, you will be asked to sign after hearing them.

Yaser Y. Dhaher and other researchers approved by Kent State University may / may not use the tapes made of me). The original tapes or copies may be used for:

_____this research project _____teacher education _____presentation at professional meetings

____________________________________________________________

Signature       Date

Address:
APPENDIX C

NOTE TAKING PROTOCOL
Notes will be taken during regular class sessions. The researcher will be sitting in class to record the professor / students’ interactions and presentations. The note taker will also write down any questions being formulated in his mind to be used during the regular interviews with the volunteers participating in the study. No names will be recorded; “T” refers to the professor and the letter “S” refers to a student. If more than one student needed to be indicated in an interaction a number will be assigned for each student such as “S1” and “S2” and so on. Only the researcher has access to the classroom notes. Notes will be dated. Any handouts given out by the professor will be included in the classroom notes.
APPENDIX D

VIDEOTAPE CONSENT
VIDEOTAPE CONSENT FORM

I agree to video taping
at________________________________________________________
on______________________________________________.

__________________________________________________________________

Signature       Date

I have been told that I have the right to see the video tapes before they are used. I
have decided that I:

_____want to see the tapes          _____do not want to see the tapes

Sign now below if you do not want to see the tapes. If you want to see the tapes,
you will be asked to sign after seeing them.

Yaser Y. Dhaher and other researchers approved by Kent State University may /
may not use the tapes made of me). The original tapes or copies may be used for:

_____this research project _____teacher education _____presentation at professiona

Signature                                      Date

Address:
APPENDIX E

PROJECT DESCRIPTION
This research project is aimed at better understanding students’ conceptualization of mathematical justification and proof construction. Dr. Neuzil and Mr. Dhaher are interested in studying the process of proof and mathematical justification’s evolution within a classroom environment based on discovery learning. Your participation in this project is needed. If you choose to participate, you will get the opportunity to have extra out-of-class contact time with Mr. Dhaher as well as with other participants in the study working on the different tasks assigned in class. What is required of you is to devote as little as 30 to 40 minutes a week (some weeks will be off weeks) to work on problems assigned in class individually or with other participants and respond to questions related to the way you think about those problems. The objective is not to evaluate you, but rather to understand how your ideas develop. Your participation is voluntary, and your termination of your participation if you decide to participate is voluntary as well. No penalties for not participating and no penalties for walking away from the study once it commences. All sessions will be video taped, but it will be strictly confidential. Your identity will not be revealed as pseudonyms will be used in all transcripts. If you are considering participating, a more detailed description of the study is available for your review. Please respond to the items below.

1. I would like to participate in the study. _____Yes _____ No

2. I object to having a video camera in class _____Yes _____ No

   If you answered “Yes” to item 1 above please continue. Otherwise, stop and please return this form to Mr. Dhaher.

3. Name: _____________________________

4. e-mail:_____________________________

5. Available times for weekly meeting (30-40 minutes)
Ryan and Zack first office visit September 15, 03

**Jack:** How about the other one?

**Zack:** $\{ \frac{1}{4} | \frac{1}{4} \}$

The other one would take away everything else and then there will just be one.

**Jack:** Right. That’s the best move

**Zack:** That’s the best way…so this is one half
Figure 77. T1-00:40

**Ryan**: So it is one and one half and one, the one is the best move

Figure 78. T1-00:48
Jack: Right. Ok. So, that’s the, that’s what you get for when blue moves, now to finish the problem he needs to figure out what happen when red moves.

Zack: It’s the easiest…

Jack: Yeah, you picked a hard one.

Zack: All right. If red goes first, pretty much the same either way. You’ll be left with a three quarters and a one on the side.

Jack: Yeah

Zack: So, its just one and three quarters, and the simplest would be one and a half.

{ | 1 ¾ }
**Jack**: There is not whole numbers between there, but there is one with a fraction denominator of a half.

**Zack**: Right. Ok.

**Jack**: So the value of this is one and a half.

So you do problems like that now….so, it is time to play an actual combat, let’s see we’re going to do…

**Jack**: you just erase it, just erased the value of this, can you remember what it was?
**Figure 82.** T1-02:01

**Ryan:** uh, 3 halves, one and one half so it is three halves. Oh no, that’s one.

**Figure 83.** T1-02:13

**Jack:** so the game consists of this arch and two sticks.
Figure 84. T1-02:46

**Jack:** ok, I’m giving you guys a choice. You can pick red, you can pick blue, or you can say you want to go to first or second. So you need to figure out the value.

**Zack:** Uh, this one more than … this is negative one eighth

Figure 85. T1-03:03

**Jack:** Right

**Zach + Ryan:** this one is three fourths

**Ryan:** it is a negative game.

**Zack:** ok yeah, because we’re only negative seven eighths,
he’s got one over here so it’s a positive.

**Ryan**: So, um it is a positive game

**Zack**: I want to be blue

Time marker: 3:17

**Jack**: You want to be blue? Ok…so, now I get to choose who goes first and second and I’m going to make you guys go first and I go second.

**Zack**: Ok

**Jack**: You’ve got to find good moves. I picked this game because there is some good moves and some bad moves here.

Time marker:3:30

Zach: That's it, were going to take, this one you just want me to erase it?

**Jack**: Ok

**Zack**: erase one blue
Ryan: That’s the best move.

Jack: Ok

Ryan: Ok, ummmm… Yeah, yeah

Time marker: 3:42

Jack: Why don’t you figure out the value now, just to check to see if you’re right? So you have a new value one eighth under that.

Zack: that would be negative fourth
Jack: So the value is now …

Ryan: Zero.

Jack: that is good for you or bad for you?

Time marker: 4:00

Ryan: Good for us because we have to go next.

Figure 89. T1-04:07

Jack: because you have to go next, ok. I just want to, just for practice see what happened if you had taken one apart of the arch instead.

Zack: Then our value would be…and this value would go…

Ryan: one fourth

Zack: down to one fourth
Figure 90. T1-04:20

Zack: this stays at one eighth

Ryan: One eighth.

Figure 91. T1-04:30

Jack: You would be way in the hole; it will be negative now so it would be a bad one.

Zack: Yeah

Jack: What if you took the blue one in the middle stick?

Ryan: That would be a negative one

Zack: so it would be a quarter lower, and it …
**Jack:** would still be a negative one.

**Ryan, Zack:** yeah

**Jack:** so actually it is the only one good move, and you guys found it. Ok.

Time marker: 4:43

**Jack:** So, I’ll take this red one here.

*Figure 92. T1-04:52*

*Figure 93. T1-04:58*

**Zack:** This is negative half
**Figure 94.** T1-05:08

**Ryan:** negative one fourth

**Zack:** negative one fourth here

**Figure 95.** T1-05:11

**Jack:** and that’s still one.

**Zack:** that still one
Ryan: That will make this negative one half and it'll be zero.

Zack: It would be zero again?

Ryan: yeah and he will be next

Zack: Ok.

Jack: Ok, so I’ll take part of the arch, erase part of the arch for me.

Zack: ok, did you want the other side?
**Figure 98.** T1-05:38

**Jack:** Oh, that’s good, good enough.

**Zack:** So that makes

**Ryan:** one half, one half and one,

**Zack:** one so we’ve got three halves and he’s only got three quarters so

**Figure 99.** T1-05:48

**Ryan:** this should be negative one half because he’s got a one
Zack: yeah he’s got a negative one half, that’s right

Zack: He’s got a one, and he’s got a negative one half, so we’re doing ok still.

Ryan: Take one of these cause that makes it a zero game again.

Jack: Yeah it does

Jack: I don’t think here its even zero now…

Ryan: It’s a zero, negative…
**Jack:** Oh yea, yea, it is a zero. Actually now you can use the copy cat strategy because these two match and they are opposite and these two match and they are opposites. That is not any fun any more, let’s try a different one. Let’s play this one

*Figure 102. T1-07:07.*

Ok, I will let you guys choose.

*Figure 103. T1-07:47*

**Ryan:** Seven eights

**Zack:** seven eights

**Ryan:** one fourth
**Zack**: Three eighths

**Ryan**: yea

**Zack**: and this one is

**Ryan**: negative three halves

**Zack**: negative three halves

*Figure 104. T1-07:31*

**Ryan**: BBR is three halves, so RRB is negative three halves. Twelve eighths, it is zero game right away

**Zack**: yea

*Figure 105. T1-07:43*
Jack: right

Zack: so we get to choose our color or choose who goes first?

Jack: right

Zack: we want you to go first

Jack: Okay I will choose well I got more blue sticks so I’ll choose blue, I’ll remove this one

Figure 106. T1-08:38

Zack/Ryan: this becomes three fourth

Ryan: that’s one and three eighths

Zack: yea, so that is a negative game

Jack: if I’ve been able to chant change it to zero or positive you guys would have been in trouble…

Zack: well we could take the top one lets take that so we can save this one

Ryan: but that gives him a positive game.

Jack: you don’t want to do that you want to change it to positive. That would change that stack to minus one.

Ryan: you could take one of those what will take it to one half. you could do one of these
because …

**Zack**: this would only go up by an eighth this one would go up by a forth so I’d say this is better.

**Jack**: Calculate what happens if you could take that middle one.

**Zack**: this one here?

**Jack**: yea, you guys are right, because you’re correct that is not a good move. What would happen if you took that thing?

**Zack**: If we took this and went up to a half
**Ryan**: right

**Zack**: so we got

**Ryan**: three fourths one half and three eights

**Zack**: yea, so we just change it to a positive game

**Jack**: yea, so that’s a bad move.

**Zack**: Yea, that’s a bad move,

but if we take this one it would only go up by and eighth…

![Image](image.jpg)

*Figure 109. T1-10:20*

**Ryan**: it is a zero game

**Zack**: so take this one
**Figure 110. T1-10:27**

*Ryan:* yup. That’s a zero game.

*Jack:* Take the blue one that is on the top on the left

*Ryan:* this one?
**Figure 112.** T1-10:48

**Jack:** yea

**Zack:**

**Figure 113.** T1-11:01

**Ryan:** I say take this one because it will go up by a fourth, .. a half, .. a half

**Zack:** yea, I would say so

**Ryan:**
Figure 114. T1-11:18

**Zack:** this will go up by a half if you take this one

Figure 115. T1-11:28

**Ryan:** you can’t take blue

**Zack:** oh, I am sorry, that is a half now
Figure 116. T1-11:40

**Jack**: anything you do lowers it.

**Ryan**: makes it a negative game.

**Jack**: yea

**Ryan**: make it a real negative game

**Jack**: yea

**Ryan**: Yea anything I do lowers it by a half gives us a negative half

even that blue one on top,

**Jack**: okay I’ll take the blue one on top.

**Zack**: Okay so now this is two
Figure 117. T1-12:27

**Ryan:** just take one of these, you don want touch that one

**Jack:** yea, if you take one of those on that right hand side

**Ryan:** we lose

**Jack:** change it to a positive game

**Zack:** right, so this is a one that

Figure 118. T1-12:41

**Jack:** I’ll take one of those reds on the top well there only one choice but not the single.

**Zack:** one of these two
Figure 119. T1-12:50

Jack: yea

Zack: Okey

Ryan: take this

Figure 120. T1-12:54

Zack: we still have a negative game

Jack: now it is zero and it’s obviously
you can use copy cat. So here’s what I want you guys to work on when you come back next week, I’m not
getting much response in class from these theorems like this

\{x|\}. Now it’s a game in which red has no play at all. It is blank, no play, it’s different then putting a zero
there. If there is a zero over there it means you can play and maybe there’s nothing left at that point after
you play, but this is before you play.

Zack: Right

Jack: So this is equal to .. the theorem is, it’s a zero game if x is negative, and it equals a smallest integer
greater than “X” if x is greater than or equal to zero. So that’s part of the simplicity theorem one and half
bar nothing (\{1 ½ | \})the answer is?

Ryan: two

Zack: oh, two

Jack: I 7/8 then, the answer still two. Take this thing (\{x|\}) which you don’t know anything about and you
want to prove equal n, so you want to show that minus n equal zero (\{x|\} -n=0). So the thing you do is to
get a game with these values and play it and prove that whoever starts was loses. Okay, so the meaning will
be, I would say and you’ve got to know you’ve got to say something about this being the smallest integer
greater than “X”. So this is a way to say that, “It’s greater than “X” but if you go one smaller it isn’t
greater than “X” (n-1 ≤ x<n). So that’s a way to say that. Okay?
Zack/Ryan: Okay.

Jack: Then here’s the game you play. This is sort of like what we did exactly this morning. You take this game and you add to it a game of value “-n” which is a hackenbush stick

![Board Drawing]

Figure 122. T1-16:35

And then you play this to show, um…, whoever starts loses. So, why don’t you guys think about that and you can ask me questions about this like after class or something. And tell me when you want to come back.

Ryan: It’s your schedule, I was thinking,

Jack: not before Friday.

Ryan: I was looking at this and saw next Friday?

Zack: Next Friday.

Ryan: That could work, I mean its up to you Zack. Do you have time right after class on Fridays? After this class?

Zack: Friday would be great for me. I have something on Friday. I think 1, I’m trying to.

Jack: Someone’s coming in at 1:30, so 12 looks good. Okay and just come up after then and I will plan this and you can write what those guys wrote on the board and then ask me questions about it on Wednesday or before it. Okay.
APPENDIX G

SHANNON, FIRST INTERVIEWS
First Interview with Shannon, September 17

Yaser: Well first I want to start by talking about your idea of proofs. What is a proof? When you hear the word proof what comes to mind?

Shannon: I think of Geometry.

Yaser: Geometry?

Shannon: Yea. The two column proofs.

Yaser: The two column proofs? Yea, did you have geometry in school?

Shannon: Yea, I had one in high school and then a second one in college.

Yaser: Oh really? And in the one in college you do the two column thing?

Shannon: We can do it however we want. So I, like, just made up my own style but every time in high school it was the two columns so I tried a model after that.

Yaser: Oh okay, so you still do the two-column?

Shannon: Yea, as much as I can.

Yaser: As much as you can?

Shannon: Yes.

Yaser: you’re in secondary education?

Shannon: Yes.

Yaser: In math?

Shannon: Yes, integrated math.

Yaser: Integrated math, that’s what they call them? So how many math classes do they require that you take? Do you know?

Shannon: A ton, gosh I would have to right them all out. There’s 3 calculus, probably 15-20 some. I don’t know, there was a ton of them.

Yaser: oh, and how many did you take so far?

Shannon: This is my last semester that I am going to be in math class.

Yaser: Oh really?
**Shannon**: I’m done with all, I am in three right now. So…

**Yaser**: Yea, okay. Does anything we are doing in this class resemble anything as a proof? I mean,

**Shannon**: Not really.

**Yaser**: No? I mean this, that would not be a, you wouldn’t consider what we are doing is a proofing right?

**Shannon**: No.

**Yaser**: And why do you think that is so?

**Shannon**: I’m sorry?

**Yaser**: Why do you think that is so? Why wouldn’t you consider like those diagrams that we are doing, things like that? What, Why wouldn’t you consider those as proofs?

**Shannon**: I just don’t really see what they’re proofing. I mean besides you know some one can win the game. That’s really all you get from it.

**Yaser**: You know some of those theorems. Have you tried some of those theorems that have talked about X and Y in general. Like if X is less than zero, and Y and Y is greater than zero, then \{X|Y\}. Do you know what I’m saying?

**Shannon**: No, I do know what it’s going to be zero. Other than that no.

**Yaser**: Okay, I have that, I have some questions here about some of the things we will do. Okay, what we did today for example, When you take a diagram and you break it down into these two numbers, okay? What do you think the thought process is, in your mind, what would you want to do?

**Shannon**: Play out the game, right?

**Yaser**: You would play the game out? Right? And then you know how at one point in class where they say you can stop now because.

**Shannon**: Okay now I don’t know what you asked me.

**Yaser**: Well okay, you know how you get, when you start with a problem with a game, you play the first step or two and then you stop and when they say, “okay well this equals the value is one-forth so stop” or “zero gain who ever stops loses.” Do you know how to do that or do you usually go all the way until the whole game is played?
Shannon: Like, I don’t have it memorized but if I had a paper in front of me or if I know its one-half or something.

Yaser: Yea.

Shannon: Like I can stop but I don’t have the five-eighths, or the four-fifths or anything memorized. I’d have to see the paper.

Yaser: Yeah well that’s, um, and can you, do you understand this idea of okay, well I don’t if this game is worth, so let me call it x, and then all add another x, and then I’ll subtract one and I’ll get a zero being.

Shannon: No

Yaser: No

Shannon: I never really understood equations.

Yaser: Alright um, see this notation?

Shannon: Uh, huh

Yaser: Do you understand this notation?

Shannon: That looks sort of familiar, yeah.

Yaser: Yeah. Do you understand what this means, x

Shannon: Yeah

Yaser: Yeah okay. Um, how about this table that we worked on last time?

Shannon: I completely understood it when we were in class and then I got home to do it last night and, like I played out a game and I still couldn’t figure it out. So,

Yaser: Okay, alright, so let’s work on one of those. So how would we do, let’s start to do this 1, 1-3

Shannon: Okay.

Yaser: Let’s remember what the game is. The game is what? Um, the cutting game, right? So um, if you have 1, (interruption), so 1 is uh, horizontal, vertical, left make vertical cuts, right make horizontal cuts. Left end vertical cuts is uh, let’s say. Right for red okay? And red make horizontal cuts. So we have, when you have horizontal 3 then you 1, 2, 3. Right?
Figure 123. T1-25:39

Shannon: Um, huh

Yaser: And left would be 1 and the border of course, right?

Shannon: Yeah.

Yaser:

Figure 124. T1-26:00

Okay, so let’s see how we would conclude the value of this one. Do you remember how we do it?

Shannon: Um, we would decide who’s going to go first and then you could play both of those game, right?

And then find the (inaudible)
Yaser: Yeah, okay, so let’s see. You do it like that right?

Figure 125. T1-26:29

Shannon: Yeah.

Yaser: And then what?

Shannon: Well if blue goes, then red,

Yaser: That would be, blue right?

Figure 126. T1-26:38

Shannon: Yeah.

Yaser: And that would be, red?
Figure 127. T1-26:41

Shannon: Yeah, if blue goes first, and red has 6 cuts to make,

Yaser: Okay so if blue goes first, then that is cut right?

Shannon: Um, huh

Yaser: Then what are we left with?

Shannon: 6 red cuts, so negative 6.

Yaser: We got 1, 2, 3, do you understand this notation when we did it in class?
**Shannon**: Honestly I can’t see the board when you right so I have to wait till it’s completely done, so that’s probably why I’m so confused.

**Yaser**: Oh,

**Shannon**: Because he stands like right there,

**Yaser**: Yeah

**Shannon**: And from now I can’t see the board.

**Yaser**: Yeah

**Shannon**: So I didn’t even see that notation before, but yeah I understand it.

**Yaser**: Yeah, so um maybe you could start, sitting somewhere else.

**Shannon**: I can’t because my group’s right there and they help me understand.

**Yaser**: Oh, okay.

**Shannon**: When I get lost.

**Yaser**: Alrighty. Okay, so now if uh, that if blue goes first.

**Shannon**: Uh, huh

**Yaser**: If that goes first what are the possibilities?

**Shannon**: Um, you can either cut one of the or the middle.

**Yaser**: Yes, so you would have? How would we write it down? Help me out because I’m so clumsy. Okay, um, okay let’s see, let me bring another color and then we’ll just um, highlight the cuts.

**Shannon**: Alright

**Yaser**: Alright, so you tell me. These are the, your cuts, so if uh, if red goes first, than what are the possibilities?

**Shannon**: You would have one here, or one here.

**Yaser**: So either end would have this with uh, with red.

**Shannon**: no it’s blue

**Yaser**: No it’s blue, alright. Or, and the rest would be this.
Figure 129. T1-28:51

**Shannon**: Yeah, and it would be just 2 and 3

Time marker: 29:03

**Yaser**: Yeah

**Shannon**: And now

**Yaser**: This goes like that, and then there’s (inaudible)

Figure 130. T1-29:00

**Shannon**: Yeah

**Yaser**: Okay, or was that the only possibility?
Shannon: No, there’s another cut right here.

Yaser: Cut in the middle, than we would have, 2, so let’s put a coma here then.

Figure 131. T1-29:21

Okay what’s the value for this one? For the blue, for blue?

Shannon: Negative 6

Yaser: Okay, negative 6 or positive 6?

Shannon: Negative, because it’s reds turn, and red has negative.

Yaser: Okay so, put negative there

Shannon:

Figure 132. T1-29:46
Yaser: alright, now let’s see what’s the value for this one?

Shannon: Um, red just went so its blues turn. SO it would be 2 either way, wouldn’t it?

Yaser: So if blue cuts this, then you’d have to go one, you’d have to go further right? Okay, is that enough to conclude about uh, red’s value?

Shannon: I guess not.

Yaser: Because you have two different moves, right?

Shannon: Um, huh

Yaser: Okay, now which is a better move for uh, for red?

Shannon: The second move, I think.

Yaser: Okay let’s see, if this, if that’s the move then how many uh, cuts blue has, 1, right?

Shannon: um, huh

Yaser: Or, either this or that right? If blue cuts this

Figure 133. T1-30:48

then red will cut this or that right?
Shannon: Um, huh

Yaser: So if blue cuts this then red cuts that, then blue would cut

Shannon: Then blue has 2 more, so that’s not a good move.

Yaser: And if, uh, blue cuts this then red has this cut and this cut, if he gets that, he has this cut or this cut.

Shannon: Yeah, so the second ones better.

Yaser: Okay, and that would be, what? What is the value of this one?

Shannon: 2, 2 or 3

Yaser: 2 or 3

Shannon: negative 2 or 3. And don’t know how, this is where I get confused.

Yaser: Okay, so let’s play this one. Okay?

Shannon: Okay

Yaser: So let’s play this one, and then let’s see. Uh so we decided, what did you decide? That this is a better

Shannon: Yes

Yaser: Move then this one?

Shannon: Yes

Yaser: Alright, let’s see, okay, starts with it,
Figure 135. T1-31:58

okay, now let’s see, uh,

**Shannon**: It’s blues turn right

**Yaser**: Yeah

**Shannon**: So I don’t get confused, okay. And it really doesn’t matter what he does.

Figure 136. T1-32:22

**Yaser**: Right

**Shannon**: and then red we’ll cut one of those

**Yaser**: So, if red cuts, yeah red should cut one of these.
Shannon: Um, huh

Yaser: Right?

Shannon: Um, huh

Yaser: And then uh, blue will cut that, and how many left? 1, 2, 3.

Shannon: Um, huh, so it would be

Yaser: 3 cuts

Shannon: Negative 3, because it’s red, and reds advantage.

Figure 137. T1-32:52

Yaser: So the values negative 4?

Shannon: Um, huh

Yaser: So here they came up with 3
Shannon: I don’t know

Yaser: So let’s go through with they did here what was it one and four

Shannon: oh, one and three

Yaser: Oh no one and three we did one and three, let’s go through this…one and three left cuts up/down.

One, two three

Shannon: One is vertical or here vertical.

Yaser: You look at the cut.
Shannon: I gotchya

Yaser: so we have one and three, let’s look at one and two because I took notes on this one.

Shannon: ok

Yaser: So you have one and two so he simplified it more, so one cuts two cuts.

Yaser: So lets do it again lets do…we know that he did two and three, so two would be two vertical so that would be like that.

Figure 140. T1-36:10

Shannon: Shall I color it in?

Yaser: yea

Shannon: They look the same now
Yaser: Okay so now if blue goes first,

Shannon: then he has the same choice, right?

Yaser: Yea it’s the same choice which means this,
Figure 143. T1-36:51

right.

Shannon:

Figure 144. T1-36:57

Yaser: and if red goes first what do we have?

Shannon:
**Figure 145. T1-37:11**

**Yaser:** Ok. And let’s see what’s the value for red here?

**Shannon:** Six

**Yaser:** because?

**Shannon:** Because .. oh no, four. Four four four.

**Yaser:** one, two, three, four.

**Shannon:** yea

**Yaser:** because, ah

**Shannon:** because he has four cuts, and blue .. Red doesn’t have any.

**Yaser:** Right so its four positive right

**Shannon:** yea

**Yaser:**
how about this one here

this is where blue went

**Shannon**: aha

**Yaser**: right then red could have…

**Shannon**: So blue is gonna do that, and red is gonna do that and red is gonna have two more. So this is negative two, right.
I think.

Yaser: Who’s going to win in this one

Shannon: Red

Yaser: Red? So it would be positive or negative

Shannon: Negative

Yaser: So what’s the simplest number between those?

Shannon: Zero

Yaser: Zero? Do we have zero?

Shannon: I’m sure that we do, yea. Zero so that means whoever starts loses.

Yaser: So whoever starts loses.

Time marker: 38:55

Shannon: It seems to me that this is a very complicated thing to do at the beginning because I personally don’t understand how they do it, how he did it. Because look here if you go through the process, what did we do…we did two three, two times three tight. When he did two times three he started by cutting one off and lies two by two. And

Yaser: the two by two…

Shannon: IS a zero game
Yaser: Is a zero game

Shannon: yea, its obvious I think!

Yaser: two by two is a zero game,

Shannon: yea

Yaser: whoever starts loses, right?

Shannon: Um Hum,

Yaser: ha ha! So how about then two times three right so what do you think can we use the step before. Great example here two times three can I think of it as two times two and two times one.

Shannon: Well, in this case it would work out mathematically. I don’t know how accurate it would be for the other ones.

Yaser: it would be the simplest number between the two? So what do you think three times three would be?

Shannon: That’s a zero.

Yaser: That’s a zero?

Shannon: That’s a zero, because that’s a square.

Yaser: okay and then four times four is a zero right?

Shannon: yea, the main diagonal will always be zero.

Yaser: So let’s see two times two is zero, two times one…Two times one is negative one because it’s a, do you notice a pattern here? So this is a two, this is a minus two?

Shannon: Yea like these are always going to that in an idea because there’s like the same game except opposite colors.

Yaser: Opposite colors.

Time marker: 41:26

Shannon: So really we only had to figure out half of the chart like the top of the track. And then you can just fill in those.
**Yaser**: Yea that reminds us of something else. Commutative property of addition or sub, multiplication right?

**Shannon**: Yes.

**Yaser**: Let’s see, this is two times three okay? Let’s see if we can understand what he does. Two times three he goes, two, and that’s what we did here. And then let’s see how he gets it. He got four. Got negative one, that’s what we got right? No, because negative two but this is negative one.

**Shannon**: we did not do this right cause…

**Yaser**: Because.

**Shannon**: There’s…

**Yaser**: This is a zero.

**Shannon**: So where do we go next. Okay yeah negative one

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*Figure 149. T1-42:27*

**Yaser**: this is negative one and four.

**Shannon**: It’ll still be zero.

**Yaser**: It will still be zero. Oh, so that would be an easier look, two times zero. Let’s see three times four. Ok, we will mimic that three times four.

**Shannon**: ah-ha

**Yaser**: Okay and lets how we’ll do that. Can you do that for me?
Shannon: Yea.

Yaser: Three times four.

Shannon:

Figure 150. T1-43:20

if blue goes first he would probably do this one right?

Yaser: yea

Shannon: So, you can try. This can go besides each other.

Figure 151. T1-43:49

Yaser: If you go like that, it will be, it will be three times two. So do we know what three times two is?
Shannon: zero

Yaser: Okay.

Shannon: Because that’s two times three so you just take the opposite and then its zero right?

Yaser: M-hum. Three times two, two times three, what was two times three?

Shannon: Zero.

Yaser: Three times two. Well this will be zero. Okay.

Shannon: See like do we have to do it just the best move or?

Yaser: The best move.

Shannon: Are we supposed to do all of them. SO we don’t have to do this one?

Yaser: Well actually you do both of them and then you pick the best?

Shannon: But if we know that this one is going to be better we don’t have to both of them?

Yaser: Right.

Shannon: Ok.

Yaser: I think but, don’t you think?

Shannon: I would imagine.

Yaser: And then if and then if red goes first…

Shannon: It really doesn’t matter what he does.
**Yaser:** Yea.

**Shannon:**

*Figure 153. T1-45:06*

So then what do we do over there? (inaudible) because this one will be cut off and then we would have one, two, three, four, five, six, seven, eight, nine, ten, eleven.

**Yaser:** Eleven.

**Shannon:** Eleven a-huh.

**Yaser:** Okay.

**Shannon:** Eleven would still be one.

*Figure 154. T1-45:29*
**Yaser:** Because?

**Shannon:** Cause you take this one on the right.

**Yaser:** Right and then what was that one? That was three times four?

**Shannon:** A-huh.

**Yaser:** Three times four. And four times, three times four. Is that one? Do they have one?

**Shannon:** M-hum.

**Yaser:** Okay, and so three times four would be one. What do you think four time three

**Shannon:** negative one

**Yaser:** will be negative one? The opposite? So lets see, so what we did here is three times four is one

—and then four times three is negative one.
Shannon: M-hum, positive 4, because all of them are positive.

Yaser: All ones here are positives and these are negative so two times four, so four times two.

Shannon: But that’s not what they got.

Yaser: But that’s not what they got
\textbf{Shannon}: But I don’t know how right this is.

\textbf{Yaser}: Okay. Now, now that you have thought about it a little bit more would you, would you consider working like you know within that, just four by four? And then we work on it again, and then we will do it again and see how we will come up with the numbers. They might have made a mistake here. But lets work on it again and then when we meet next time, the first thing we do is we just fill in those numbers.

Especially the ones,

\textbf{Shannon}: Okay so you just want me to go up to four times four?

\textbf{Yaser}: Yea.

\textbf{Shannon}: Okay.

\textbf{Yaser}: And then we will do it again.

\textbf{Shannon}: Okay.

\textbf{Yaser}: the other question I wanted to ask you is ah what we today, what we did today, do you, do you think you have a better understanding now after you worked with a group on how you come up with those values?
Shannon: Kind of, like I understand how to do it, its just whether or not I can think of it.

Yaser: Yea.

Shannon: You know, at the time because there are so many steps and then you have to carry all the numbers back and there are so many pictures, I just get confused.

Yaser: Right.

Shannon: But like I understand like how you do it.

Yaser: You understand the process?

Shannon: Yea.

Yaser: Okay, so you are going to have to hand in, he said if you’re not done then you hand in individual papers?

Shannon: I guess so.

Yaser: He said if you are done today you can hand in all four names or three names, whatever.

Shannon: Well, we got everything done but those two
So I don’t think it’s going to be that hard because it’s just going to be one stick.

**Yaser:** Yea.

**Shannon:** So, these ones were the hard ones
But I think we’ll be able to do the bottom two okay.

_Yaser_: Because once you remove this

then you have you know, because that, ALL that is connected to the ground.
**Shannon**: You know it's going to be positive.

**Yaser**: Right, so you just, you just go, removes that that's the best move.

**Shannon**: Yea.

**Yaser**: Right, and if you remove this

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then you know that “R” can remove that
Figure 166. T1-49:09

and then everything will disappear.

Shannon: H-hum. If we remove this one?

Figure 167. T1-49:18

this one doesn’t disappear right?
Figure 168. T1-49:20

Cause its still connecting? Is that correct?

Yaser: Well what you do if this is gone

Figure 169. T1-49:25

So let’s say R is gone and you don’t have it, you only have B. if you remove be and that’s not a good idea for B,
because when B removes, oh, Well it is a good for B I think, right?

**Shannon**: That just brings it to the zero again. But it can probably be something bigger than a zero, if he does something else.

**Yaser**: Cause yea, ah maybe.

**Shannon**: Because no matter what then he wants to try to get the higher number.

**Yaser**: You want to try to get the highest number yes.

**Shannon**: Cause if he takes this then its just going to be zero?
**Yaser:** If he takes this then, then he does nothing else for R to do. Right?

**Shannon:** Right, so

**Yaser:** So, who ever goes next will lose.

**Shannon:** So he would remove this one

![Image of a whiteboard with handwriting and diagrams]

*Figure 172. T1-50:11*

because that will get rid of an R and the R will remove that and then he’ll move that and he’ll move that and then he’ll have a positive one.

**Yaser:** Yea, but how bout this R?

![Image of a whiteboard with handwriting and diagrams]

*Figure 173. T1-50:19*
Shannon: But he’s not that number.

Yaser: Oh yea. If this is gone

Figure 174. T1-50:21

yes, that’s right. Okay can you do the diagram and then give me a copy of your diagram.

Shannon: Of the whole thing?

Yaser: Yea, this game, this number two.

Shannon: yea, I’ve started writing already, its pretty messy though.

Yaser: So next time we’ll talk about this, how we show it.

Shannon: Okay.
APPENDIX H

RYAN, FIRST INTERVIEW
First Interview with Ryan September 17

Yaser: Okay first of all I need to ask you about the math classes that you had to take for this program,

Ryan: Ah-ha

Yaser: what did you take?

Ryan: I’ve taken all the calcs. Calc 1, calc 2, calc 3, fundamental concepts of algebra and geometry, history of math, discrete math, linear algebra, fundamental concepts of algebra, fundamental concepts of geometry, I am taking numbers and games and ordinary deferential equations right now. And of course I have taken college algebra and trig, so I have only one other math class left to go.

Yaser: Do you know what it is?

Ryan: I haven’t yet decided on it

Yaser: Okay so you can do that later

Ryan: Yes

Yaser: And um the word proof when you hear it what comes to your mind?

Ryan: I would say it’s a mathematical statement that isn’t able to be contradicted. That it’s true no matter what you use it for.

Yaser: Okay and um, is anything of what we’re doing here, playing those games, does that constitute a proof to you?

Ryan: No because its not I mean because I don’t really consider it, I guess it’s not really too defined it’s too broad. A proof is usually, very defined in what it’s about.

Yaser: So the game today, oh, not the game, the group work today

Ryan: Ah-ha

Yaser: was that a proof to you?

Ryan: no, cus I consider it, I meant its too … I mean I guess its too broad of something like in mathematics that can be used for other things other than just we were doing in class.
Yaser: Ok

Ryan: if you figure out the math behind it. I guess, no, I don’t …

Yaser: So you don’t consider it a proof?

Ryan: No

Yaser: it’s a mathematical activity?

Ryan: yea

Yaser: but not a proof? How about this table here

*Figure 175. T1-54:45*

for the cuts?

Ryan: Sigh, I guess in a way (in audible) but it is also mathematical activity that can be played in, I mean you can use it in a different game if you can find the right formula.

Yaser: So if you can relate it to a different game, is that what you’re saying?

Ryan: yea, yea

Yaser: some how the results that you get here

Ryan: Right

Yaser: can be transferred to a different game?

Ryan: like for a zero, like a one, positive or it’s a negative

Yaser: that would make it more of a proof of what it is now?
Ryan: Ah-hum

Yaser: So not knowing it transfers you are saying it is not a proof? Interruption ….

Yaser: Okay, so you were saying, if you were, um, since you don’t know if this is transferable to something else, it would be just a mathematical activity right?

Ryan: Uh, huh

Yaser: Okay, so later in the semester if we were able to transfer it to something else, then you will change your mind.

Ryan: yes

Yaser: Okay, cool. ‘Cause it’s okay, you can change your mind. We’ll see, we might find that it is transferable, or not.

Ryan: Uh, huh

Yaser: Okay, um, Where is it? You know those theorems that he gave you guys?

Ryan: Right um,

Yaser: You know when,

Ryan: Yeah

Yaser: The two guys went on the board

Ryan: Right

Yaser: and then they did $r$ to the $n$

Ryan: Right

Yaser: Has a value of minus $n$, and then you use that, does that kind of thing constitute a proof to you?

Ryan: Yeah, because it’s um, the proof, it doesn’t, I mean it can be because, it can use for anything, it doesn’t have to be used for one specific thing, anything that comes in line with what the proof is.

Yaser: Right, so if I understand you correctly, although this is a particular game, $r$ to $n$, we know what it is, it’s a hackensack

Ryan: Hackinbush,

Yaser: Hackinbush, whatever I’m getting, but what is that game?
Ryan: The game doesn’t have a specific, it’s um, quantity. It’s just uh, general game.

Yaser: Right, so because of that you can use it for,

Ryan: Any hacking bush game, or . . .

Yaser: Or cuts game or whatever

Ryan: Right.

Yaser: Okay, so that would be in line with what you consider a proof more than the table for example.

Ryan: Yes.

Yaser: All right. I want you to explain to me cause I really, I struggled with filling out the table. Like, for example, have you filled out your table?

Ryan: Um, not all of it.

Yaser: Not all, but, if I ask you to do 2-4.

Ryan: Right.

Yaser: Would you be able to explain to me (inaudible)… There you go. All righty, so let’s do 2-4 and if you need…

Ryan: This is…

Yaser: Don’t need colors?

Throw that away. So right, right only has really one move which is this one cut that way which would give you blue would then have one, two, three, four, five, six.

So, using kind of a simplicity theorem here, so six, right’s best move would give a positive six. Blue’s best move, blue actually has two moves. They can take this one and you have a one, have this.
Figure 179. T2-04:30.

or and have this

Figure 180. T2-04:30.

Or they can cut it down the middle and get two, two by twos
Both, we know a two by two is a zero game. So that, those would both be zero.

but this is a negative one
and this is what? Two by three as in (inaudible), two by three as, it's a one game

So this is zero but I guess it doesn’t really matter 'cause they’re both going to be zero. Since they’re both zero, zero goes there. So this is going to be a positive one game. One is the simplest number between, whole number between zero and six and you just keep going.

Yaser: Okay. So, if, this is blue's move, right?

Ryan: Yeah.

Yaser: So, blue moves here
then what happens?

**Ryan**: Red can go there

or here
Which this would left

done and this last but red would have, red would or blue would have one, it’d have one, two cuts left and a …

Yaser: So red would have one cut left.

Ryan: Red would have one cut left and blue would have two cuts left which would make it, that’s a one game.

Yaser: Okay, so the difference is one.
Ryan: Right.

Yaser: In favor of?

Ryan: Blue.

Yaser: Blue and this is uh?

**Figure 189.** T2-06:25

Ryan: Negative one cause blue doesn’t have any cuts and red has one cut.

Yaser: So that’s one in favor of red.

Ryan: Right.

Yaser: So.

Ryan: The two together make it a zero game.

Yaser: A zero game. All right and then you’re looking for the simpler number between zero.

Ryan: Zero and one sixth.

Yaser: Zero and one sixth.

Ryan: Which is one.

Yaser: Okay? All right. Do you have any questions?

Ryan: No.

Yaser: All right. All righty. Well, thank you. Can we fix like, fix this time like every week. Is that good for you?
Ryan: That’s good for me. Even if, I mean if you want to do it a little earlier, that’s even better.

Yaser: Okay. Yeah, like at 2?

Ryan: That’s good.

Yaser: Thank you.

Ryan: No proble
Ryan and Zack office visit Sept, 19

Jack: Okay. So.

Zack: Okay. The theorem says X and nothing

equals next smallest integer. Greater than X, that was, if X is greater than or equal to zero cause you gave us the case where it was smaller than zero.

Jack: Yeah, the answer is zero.

Zack and Ryan: Zero.

Zack: Ah-um, for the proof, we had, you gave us the start that it’s supposed to equal n, so, minus n
Figure 191. T2-10:31

It equals zero where we are given the hypothesis that, \( n - 1 \), plus or equal to \( X \), less than \( n \).

Figure 192. T2-10:45

**Jack**: Right. Okay.

**Zack**: And, what we did was change this
to a red hackenbush stick n length

I have to prove that equals zero.

**Jack**: Right. Okay. So.

**Ryan**: So, here’s two cases for it.

**Zack**: You want to do one color?
Ryan: Yeah.

Zack: What color do you want to do?

Ryan: Doesn’t matter. Whatever you want.

Zack: So the blue goes first. You get $X$

cause that’s the best mode plus $R$ to the $n$ and that equals $X$ minus $n$ but we know that from our hypothesis that $n$ is greater than $X$. So $X$ minus $n$ is going to be a negative number.

Jack: Okay.
Zack: So…

Jack: yeah, negative, yeah, that’s right.

Zack: So, I know that’s a negative number. Red wins

Figure 197. T2-11:53

Jack: Yeah, so blue starts, red wins, okay?

Ryan: And then, if red goes… your left with

Figure 198. T2-12:06

I think it’s what, X? It’s…

Zack: Well, the X is gone. It’s R to the n-1.
Ryan: That’s right

Zack: cause you have to take this move, you no move anywhere else.

Jack: Now the other game is still left so you got two games sitting on the table and you’re playing the sum of two games.

Ryan: Right.

Jack: So, so that one is still there untouched. The x bar nothing ({x| }).

Ryan: Right. Oh, so X is here, right?

Jack: Yeah. X bar nothing ({x| })is still there. Not just the X but… the whole game is still there. Okay.

Ryan: Plus R, N minus one.

Jack: Yeah.

Ryan: So then, blues only move would be to take this, to take their best move which is X plus R to the n-1, which is really X plus n minus one.

Jack: it would really be better if you wrote an equals sign instead of the arrow there 'cause nobody’s made a move there. It’s just what equals.

Zack: Yeah.

Ryan: And this is going to be.

Jack: Now, wait, will you need to what you just wrote X.

Ryan: Plus n minus one.

Jack: Okay, well that’s not correct because see, it’s red. A red stack so it’s negative.

Ryan: Right.

Zack: its minus

Ryan: So it’s X minus n minus one.

Jack: No, that’s still not right. It’s the negative of the quantity. Yeah, there you go. Quantity, then minus one.

Ryan: Yeah. Which is, oh, this is going to be greater than zero. So blue wins.

Jack: Greater than or equal to zero.
Ryan: Greater than or equal to zero

Figure 199. T2-13:56

Then blue wins.

Jack: Yeah, well that’s right.

Ryan: Cause it’s a positive gain.

Jack: Well, it’s positive. If it’s positive blue wins.

Ryan: Right.

Jack: But if it’s greater than or equal to, also it could be equal to zero. What if it’s equal to zero? Then who wins?

Ryan: Well, red’s the next move, so.

Jack: Yeah, right. Right.

Ryan: Blue wins.

Time Marker: 14:20

Jack: That’s good. That was so much fun let’s do another one. In between there
That should be the answer. Prove that, prove that $X \bar{Y}$ equals one half with these conditions. Okay, so, let’s write down a hypothesis in terms of inequalities. $X$ is going to be greater than or equal to zero and less than one half. And $Y$ is bigger than that but less than or equal to one.

So here’s the inequalities that go with this picture there. Then I want you guys to show that $X \bar{Y}$ equals one half,
okay. Well what do you do? You set up a game like this. X bar Y minus one half.

**Ryan:** Right.

**Jack:** And play it. Show it to be zero

This minus a half equals zero when this is true. Okay, so the game we’re going to play is, like this game is not really don’t really know anything about what the game is except just these equalities. So it’s like an unknown game
Zack: Then minus the red blue hackenbush.

Jack: Yeah. Well, here’s the game to play

Ryan: Okay.

Jack: Okay, so pick red or blue and show that, you know, whoever you going to do can’t win if you’re opponent makes its best move.

Ryan: Right.
Jack: Okay, so.

Ryan: So, first of all we’ll have Y take their best move from this.

Jack: Okay, so you’re doing red.

Ryan: Right, red’s, red’s best move from this.

*Figure 206. T2-15:50*

be left with X.

Jack: No, see, if you’re doing red.

Ryan: Oh, you’re, then it’d be Y.

Jack: It’d be Y.

Ryan: Plus R. Y plus this stack
and then blue only has…

**Jack:** Now, see that’s not true because this $Y$ represents the value of some game you don’t know what it is.

**Ryan:** That’s true.

**Jack:** But so, so you shouldn’t do another move now. Well, what can you do instead?

**Ryan:** Well you know this is negative one half.

**Jack:** Yeah, right.

**Ryan:** And this is some value.
Figure 209. T2-16:38

Jack: Yeah. Some number.

Ryan: Some number n.

Figure 210. T2-16:42

Jack: Well, its Y.

Ryan: So this is Y.

Jack: Just write Y there.

Ryan: This is just Y
Jack: This is the current value of the position. Now you can, now you got to look at that hypothesis and tell me who’s going to win.

Ryan: Well, it, Y’s less than one, less than or equal to one.

Jack: Yeah.

Ryan: And greater than zero.

Jack: Okay, now neither of those is irrelevant right now.

Ryan: Right. This is one half and this is either greater or then equal to, it’s either greater than or equal to one half
Jack: Well, no. You’re almost right but look at the hypothesis. It’s strictly bigger.

Ryan: Yeah, I know.

Jack: Okay, so write that down.

Ryan:

Figure 213. T2-17:29

Jack: Okay, now it’s time for this. It’s strictly greater. Hypothesis just this Y is greater than one half.

Ryan: So now this is going to be a zero. Zero or positive

Figure 214. T2-17:39

Jack: I agree with the positive but it can’t be equal to zero.
Ryan: Oh, you’re right. Okay, since it’s not, okay good.

Jack: Okay. So from that inequality you just wrote, what do you know about the quantity Y minus one half.

Ryan: This is positive so it’s blue that’s going to win.

Jack: Yeah, right, right. Okay, well write that in.

Ryan:

Figure 215. T2-18:12

Jack: It’s a positive because …

Ryan: Y is

Jack: this quantity here is positive from this inequality.

Ryan: Right.

Jack: Now, blue, red has another move, opening move. It could do that hackenbush.

Ryan: Right, red could go
Figure 216. T2-18:37

**Jack:** That stays untouched.

**Ryan:** And this is gone.

**Jack:** That’s gone so, now you can’t use any of the inequalities yet.

**Ryan:** Right. So, it’s blue’s move

Figure 217. T2-18:44

and blue would leave.
Figure 218. T2-18:46

**Jack:** Okay. So that’s, so that’s the, so you’re playing that game and you don’t know anything about what the game is, so…

**Ryan:** So it’s just X.

Figure 219. T2-18:56

**Jack:** That game has the value X that you’re in.

**Ryan:** And you know, from the inequality it’s greater than or equal to zero.
Figure 220. T2-19:05

**Jack**: Right, now you know it’s greater than or equal to zero so you can tell me who wins.

**Ryan**: Blue’s going to win cause it’s a positive game.

**Jack**: Right. It’s positive so blue wins positive.

**Ryan**: Right.

**Jack**: But it could equal zero. So who would win if equaled zero?

**Ryan**: If it’s equal to zero, red would win. No, yeah, cause it’s, no, it’s blue, reds move.

Figure 221. T2-19:37

**Jack**: It’s reds move so blue would win.
Ryan: SO blue would in if it’s zero and if it’s greater than zero blue wins because it’s a positive game.

Jack: Right. Okay, so, when you reach position X and X is greater than or equal to zero.

Ryan: Right.

Jack: Okay.

Time marker: 19:45

Zack: I’ll do blue …

Figure 222. T2-19:50

Okay. Blue could take the hackebush or they could go here, I’ll do the hackenbush first.

Jack: Right. Okay.

Zack: I plus R over here, okay?
Jack: Yeah.

Zack: Ok. Now. Negative one but we don’t know enough from this…

Jack: Right, right. We can’t use inequalities till you get just one of those numbers.

Zack: Okay, now red can have two moves from here and.

Jack: Okay.

Zack: They could take the hackinbush in which case \{X|Y\}

Jack: Still no, still can’t tell who wins.
Zack: Right. Or they come down here. Do their best move.

Jack: Yeah and

Zack: Best move would be Y plus R.

Figure 225. T2-20:54

Jack: Now there, for now you should be able to tell who is going to win.

Zack: Right. R is negative one, so you have

Jack: So write what that equals then.

Zack: Okay. So you have equals Y minus one.

Figure 226. T2-21:00
Jack: Yeah, okay so what do you know about Y minus one from the hypothesis.

Zack: Y minus one. Y is less than or equal to one. So that’s going to be negative or zero.

Jack: Okay, so write that in, less or equal to zero. Okay, so…

Zack: And blue goes next. So if it’s zero, red wins.

Jack: Yeah.

Zack: Or it’s negative in which case red wins.

Jack: Right, so either case red wins then.

Zack: red wins

Jack: Okay, so if blue starts that way then red has a winning response. Just out of, so that, so that proves, that proves it. Just out of curiosity, what if red had taken to the ranch?

Zack: What if red had taken this up here

Jack: Yeah, what would be the result then?

Zack: Um, let’s see, blue would have to go under best move, would be X
and blue would win because it (inaudible) up here right?

**Jack**: Yeah, yeah.

**Zack**: So, but they do have a…

**Jack**: So would that?

**Zack**: That would be a bad move.

**Jack**: That would have been a move for red?

**Zack**: Right.

**Jack**: But assuming both players make their best move whoever starts loses. Did you do both cases?

**Zack**: I didn’t do both cases. I didn’t do the hackenbush stick. So, the other case is, they take the best move here right off the bat, in which case we have Y.

**Ryan**: Wouldn’t it be x cause blue’s going? If blue goes?

**Zack**: Oh yeah because blue’s best move would be X. X minus one half.
Jack: Yeah.

Zack: So you know that X is bigger than or equal to zero but it’s still less than one half. So X minus one half, that’d be a negative number. This would be strictly negative. So that’s less than zero, you know red wins.
Zack: Yeah.

Jack: Okay. Suppose, we’ll do the same thing. Instead of having numbers zero and one here so that, let’s just put two consecutive whole numbers in $n + 1$

![Figure 231](T2-23:23)

So this is the general situation where there’s no whole number between the two. So we’ll have $n$ less than $X$ and it’s, let’s make this positive. Positive $X$ is bigger or equal to $n$ and it’s less than $n$ plus one half

![Figure 232](T2-23:44)

and that’s less than, less than $Y$. That’s less than or equal to $n$ plus one
So here’s the picture. This is n plus one half in between. n plus a half. So you have X and Y between two consecutive whole numbers so that forces there to be no whole number between the two. But you do have a fraction with a denominator between the two, so, X is bigger than or equal to n and Y is less than or equal to n plus one. But the number n plus a half is in between so then the answer is one you need to prove that the answer is n plus half.

So all you need to show, the quantity, take this equation and move this end to the other side and the one half, you’ll have minus n minus a half. So, you play. Here’s the, here’s the minus one half, this and this
What should I put in for minus $n$?

**Zack:** hacksenbush $R$ to the $n$.

**Jack:** See that?

**Ryan:** Yeah.

**Jack:**

Okay, so it’s sort of the same thing but a little more general. Okay? Why don’t you start?

**Zack:** Okay.
Zack: did you do red last time.

Ryan: Yes.

Jack: Now red has three possible alternatives, slightly more complicated.

Zack: Red can start by making the best move here.

Figure 236. T2-25:09

Jack: Yeah.

Zack: In which case, reduced to Y minus one half, minus
Figure 237. T2-25:33

**Jack:** Okay. And what can we conclude about that?

**Zack:** Y minus one half… Can we conclude that that’s greater,

**Ryan:** no,

**Zack:** Y minus one half… is, no it wouldn’t be the same (inaudible) plus one half (inaudible).

**Jack:** Well, you haven’t gotten everything written there yet. So you’ve got, oh yeah you do have everything. Okay, now, here is something that involves all the stuff you got written down.

**Zack:** Okay.

**Jack:** There some of the wrong signs. Let’s write that, something plus a half is less than one. Now can you change this inequality
so that it has this in it.

Zack: Okay.

Jack: How would you do it algebraically?

Zack: All right, you just, take the, just have n is less than one half or Y minus one half.

Jack: Okay, now you’re still not there cause I want something that has this whole thing in it.

Zack: Right. So you have to subtract the n for the other side.

Jack: Yeah.
Zack: So. Zero less than $Y$ minus one half minus $n$ ($0 < Y - 1/2 - n$).

Jack: Okay, now you’re getting close.

Zack: Yeah, cause now we know that, that this greater than zero.

Jack: Right.

Zack: Which makes it positive the blue wins.

Jack: Okay, so… You see this, see this inequality, this one right here is just set up perfectly for what you need there.

Zack: Right.

Jack: Okay.

Zack: All right. Another move for red
Jack: That’s one of those moves.

Zack: is to take this hackenbush stick

Jack: Okay. So now you got the unknown game still sitting there.

Zack: All right and blue would move and their only move is to, you know, the best move over here.

Jack: Right, right.

Zack: So you get X plus, oh let’s just make it X minus n.
Jack: Yeah, right, right.

Zack: We know that X is bigger or equal to n.

Jack: So that quantity there, you’ve got in this raw is what?

Zack: Is positive or zero.

Jack: Okay, write that in, greater or equal to zero.

Zack: If it’s zero red goes next and loses. If it’s bigger, blue wins.

Jack: Yeah.
Zack: So, blue wins either way

Figure 245. T2-27:56

Jack: Right. Okay. Now blue has, red has one more opening move.

Zack: Right. They can take one of these sticks out here.

Jack: Yeah.

Zack: $\overline{X \lor Y} (\{X\lor Y\})$ plus RB plus R to the n minus one

Figure 246. T2-28:17

Blue has two moves from here
Jack: Which one do you think would be the best move?

Zack: I don’t really know cause don’t know what that game is.

Jack: yeah.

Zack: Is there a way to know?

Jack: Well, the one to try first is if you do the unknown game cause then you’ll be able to calculate right away.

Zack: Okay.
Jack: So, I think that’s the one to try first and see if that is a good move for blue.

Zack: All right. So, we get \( X - \frac{1}{2} - n - 1 \)

Figure 249. T2-28:57

So, we’re better off to make this minus \( n \) plus one

Figure 250. T2-29:20

And a plus one over here. So, \( X - \frac{1}{2} - (n+1) \)

Jack: That’s a little more tricky.

Zack: Yeah. (Inaudible). I don’t know where this falls in that.

Ryan: Can’t, can’t you write \( x - N \), plus one half?
Zack: minus n,

Figure 251. T2-29:44

Oh.

Jack: That’s a good idea, combine those two.

Zack: Yeah, that’s… All right

Figure 252. T2-29:50

Jack: you forgot something about that.

Zack: Okay. So, X minus N plus one half
Ryan: In this game you already know…

Zack: Well, we know, here we got n plus one half right here

and we know that n plus one half is bigger than X. So X minus that, we’ve got to be…

Jack: Yeah, but you, but you changed, see the, you shouldn’t have parenthesis around n plus a half.


Ryan: In this game you know that X minus n is greater than, or equal to zero but with the one half.

Jack: Yeah, that’s, he’s got the right idea. So with the one half it’s what?
**Ryan**: It’s going to be positive.

**Jack**: Yeah, right. Here’s what you got, you’ve got X minus n is bigger or equal to zero.

**Ryan**: this game.

**Jack**: But you really want someone with a one half in it. How would you get that from this (n-1)≥ 0.

**Zack**: Well, you know that n is less than, less than n plus one and a half. I don’t know if that helps at all.

**Jack**: Yeah, I think that’s right. Well, you want to know something about quantity X minus N plus a half.

So if I just stick a one half in here, what should I put on the other side.

**Zack**: Zero plus on half?

**Jack**: Yeah. Okay. So or just one half. So this quantity you’ve got here, trying to work with now is bigger than or equal to one half. So what can you say about who wins now?

**Zack**: It’s positive

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*Figure 255. T2-31:33*

**Jack**: Yeah, it’s definitely positive. It’s bigger than one half is positive.

**Zack**: All right. So that makes sense
Figure 256.  T2-31:41

**Ryan:** So that’s a bad move or that’s a good move?

**Jack:** It’s a good move red started, blue wins.

**Zack:** That’s a good move.

**Ryan:** That’s a good move for blue.

**Jack:** Yeah. So that takes care of reds three possible moves. Looks like blue’s going to be easier because there is only two possible moves.

**Ryan:** All right. Can I erase this or do you want me to leave it up.

**Jack:** I think we’re done unless you guys have any questions about it.

**Ryan:** Nope. So blue has two moves. They can either take this or make the best move here.

**Jack:** yea.

**Ryan:** Start with making their best moves there?

**Jack:** Probably the easiest one to work with.

**Ryan:** So it’s X plus R to then
Figure 257. T2-32:23

Just take this, X minus one half, minus n

Figure 258. T2-32:32

**Jack**: Okay.

**Ryan**: So from this equation X less than n plus one half
you get, one half, or $X$ minus one half, minus $N$ is less than zero.

**Jack**: Right. So red wins.

**Ryan**: Right.

**Jack**: So if blue starts that way negative game and red wins.

**Ryan**: So our, the other move for blue would be to take this

Figure 260. T2-33:05

So you’re left with this. (Inaudible) plus $R$, plus $R$ to the $n$
Then red has really three moves.

**Jack:** Yeah, but you need to find a good one so the red’s not (inaudible).

**Ryan:** I think the best move for red would be probably take Y

**Jack:** Yeah. I believe so, at least you can calculate if you do that.

**Ryan:** Right. So red, y plus, or, minus one, minus n

and then from this.
**Jack:** Right.

**Ryan:** Would get $y$ minus $n$, minus one is greater than or equal to zero

![Handwritten equation](image)

*Figure 263, T2-34:02*

**Jack:** Right.

**Ryan:** If it’s zero, blue goes next. So blue loses.

**Jack:** Right.

**Ryan:** And if it’s negative red wins.

**Jack:** Yeah, so either way red wins.

**Ryan:** Red wins

**Jack:** Okay, so that’s it. Okay, that’s good. You have any questions about this?

**Ryan:** No.
APPENDIX J

SHANNON AND CHRIS, FIRST PRESENTATION
Shannon and Chris office visit Sept, 19

Jack: … you guys did in class for this course and play a game. You two against me.

Shannon: your on my team we might win.

Chris: I don’t think so.

Jack:

Figure 264. T2-34:40

Shannon: who is gonna start?

Jack: That’s to be determined. You get your choice

Figure 265. T2-34:49

276
Jack:

Figure 266. T2-34:59

Shannon: Uh-oh.

Jack: Okay, so here’s the game. Two sticks and this house shaped thing and you guys get your choice. You get, you can choose either which color you’re going to be and if you choose a color, then I’ll choose who goes first or second and if you choose first or second then I’ll choose the colors. Okay, now you guys got to figure out…

Figure 267. T2-35:27

Chris: And how long do we have to think about this?
**Jack:** What?

**Shannon:** lets let him choose the color.

**Jack:** So, somebody start by putting down the values of those things…

**Chris:**

![Image](image.png)

*Figure 268. T2-35:40*

**Jack:** Okay, now let’s have Shannon do the house because she’s not confident about that stuff.

**Chris:** Have fun.

**Jack:** Okay, so, for each stick of that house, put a number by it. Each number, it’s supposed to be the value of when that is gone.

**Shannon:** I did not memorize those sticks.

**Jack:** Well you have them, I put them right there.

**Shannon:**
**Jack**: Now, that’s not right.

**Shannon**: It’d be the other way. It’d be this way

---

**Jack**: Yeah, when that’s gone…

**Shannon**: 

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*Figure 269. T2-36:09*

*Figure 270. T2-36:18*
same for that one

**Figure 271. T2-36:30**

**Figure 272. T2-36:33**

**Jack:** When that’s gone… Right. Okay. Now the next job is to fill in the right numbers there and tell me what it equals.

**Shannon:**
Figure 273. T2-36:52

**Jack**: Okay, so put a one in that whole house.

**Shannon**: 

Figure 274. T2-37:00

**Jack**: Okay. Now you guys should be able to tell me which one you want to be, assuming you guys want to win. That’s your job is to win.

**Shannon**: Right. Okay, we want to be blue.

**Jack**: Okay, why? You’re right.

**Shannon**: Because it’s, with the other are still less than one.
Chris: Positive.

Shannon: Yeah, it’s the positive quantity.

Jack: Specifically what is it?

Chris: You want me to figure it out too?

Jack: Yeah.

Chris: one eighth

Shannon: Yeah, these are one eighth.

Jack: What?

Shannon: one eighth

Jack: Okay, the one eighth is definitely positive. So you want to be what?

Shannon: Blue.

Jack: Okay, so I’m red. Actually, I don’t because I’ve played this game so much that I can’t remember from time to time. Okay, so, since you choose the colors I’m going to choose who goes first and I choose you guys.

Shannon: Okay.

Jack: So make a move.

Shannon: Shoot, that one

Figure 275. T2-38:00
Chris: Are you sure about that.

Shannon: No, but I’m guessing cause (inaudible).

Jack: Okay, see if her guess is right. If that’s, if you take that blue one what will the new value of that stick be?

Shannon: (Inaudible).

Chris: (Inaudible) so (inaudible).

Jack: Okay, so write that down. Write that down below the minus three fourths.

Shannon: [she writes -1 below the game]

Jack: Now, and now what will the new total value be?

Shannon: Negative.

Jack: So is that good or bad?

Shannon: That’s bad.

Jack: That’s bad for you guys.

Shannon: Don’t let me talk anymore.

Jack: So that’s not a good move.

Chris: I think we want the one on the top right.

Jack: Okay.

Chris: (Inaudible) negative one fourth. That would give us.

Shannon: I don’t (inaudible).

Chris: That would give us a zero which means he goes next, and he would loses.

Shannon: Well, the house is the only other feasible option cause that would be a dumb move cause that would just make it a bigger negative number.

Jack: Right. So that, that one the top of the right is a good choice. Trying to figure out what’ll happen, what’ll it be if you take part of the house?

Shannon: Like what?

Jack: What would the new value be then?
Shannon: Total?

Jack: Yeah. What would the house change to?

Shannon: Minus one, I think.

Jack: No. How’d you get that?

Shannon: Cause if you take blue then the one fourth is left, that’s…

Jack: Oh yeah, okay, you’re right. I’m sorry you’re right. Yeah, you’re right.

Shannon: Okay.

Jack: I was thinking the, I was thinking the red was … Okay. So what would the new value be then?

Shannon: Negative something.

Jack: Negative something right?

Shannon: Yeah.

Jack: Okay. So that’s bad.

Shannon: Well, if we’re supposed to win.

Chris: We take the top right one.

Shannon: But that would be dumb.

Chris: No. It gives us a zero game. He goes next. If he’s the first one to go on a zero game we win.

Shannon: That just makes a negative number bigger.

Jack: Yeah.

Chris: Yeah but it makes negative one.

Jack: Well you can have, well in hackenbush. When blue plays, the number always goes down. The total value always goes down and if red plays the value always go up. So what would total, so the value right now is what?

Shannon: right now?

Jack: Yeah.

Chris: it’s one eighth.

Jack: If you do what Chris says what would the new total value be?
Shannon: that would be … zero

Jack: It would be what?

Shannon: Zero.

Jack: What?

Shannon: Zero.

Jack: Yeah.

Shannon: Okay.

Jack: Is that good or bad for you?

Shannon: That’s good cause.

Jack: Yeah. Well, what you said previously if you take this one

Figure 276. T2-41:15

the total value would be negative which is much worse than zero and if you take this
one the total value would be negative which is still much worse than zero. So, if you could keep the positive that would be fine but you don’t have what blue does have. So, make you’re move.

**Shannon**

**Jack**: I’ll take the top one off the middle
Shannon: Now, why don’t you have to think about it?

Jack: Cause I’ve played this many times.

Chris: We’ve done it that many times.

Shannon: Do you want me to rewrite the values up here or?

Jack: Yeah, that’s good. See where it stands.

Shannon:

Chris: That’s a one forth.
**Jack:** It was zero. I played and the value decreased, which is pretty much the way it goes in hackenbush.

Okay.

**Chris:** anyone you take from the right.

**Shannon:** the same thing, can I do that?

**Jack:** Okay,

**Shannon**

![Figure 281. T2-42:18]

**Jack:** now I’ll take part of the roof.

**Shannon:** Do you care which one.

**Jack:** They’re the same. So…

**Shannon:** ok,
Chris: we want to take the halves.

Shannon: no we dont

Chris: Take half the house and then we have the copycat plus an extra blue.

Shannon: Oh, okay. I understand now.

Chris: Right?

Shannon: But no matter what we do it’s going to be zero isn’t it?

Jack: I think you’ll, what’ll happen if you take the far or far left over there?

Chris: far left would be bad.

Jack: What would your total value be then?

Chris: Negative one half?

Jack: Yeah.

Chris: There…

Jack: Well that would be, so that’s a bad move.

Chris: So, we get zeros no matter what.

Shannon: Yeah.

Jack: Yeah.

Chris: So you can raise any of the three on red.
**Shannon:**

*Figure 283. T2-43:40*

**Jack:** lets see. Okay, I'll take that other piece of the roof.

**Shannon:**

*Figure 284. T2-43:57*

**Jack:** What's the total value now?

**Shannon:** half.

**Jack:** half?

**Chris:** So, go ahead and take the B on top.
**Jack:** Now, it’s not an interesting game anymore and we take turns and you’ll go last. So I give up on that one. But I have another one. This one here four sticks

![Image](image1)

*Figure 285. T2-44:42*

**Shannon:** Aren’t games supposed to be fun?

**Jack:** Aren’t you having a good time? (laughter)

![Image](image2)

*Figure 286. T2-44:48*

**Chris:** It’s kind of fun

**Shannon:** you have to beat the teacher.

**Jack:**
Figure 287. T2-44:56

Figure 288. T2-45:06

ok, the same deal you guys get to pick what color you want to be or if you don’t want to pick a color you can pick who goes first. So write down the values again.

**Shannon**: Just the three of these, you don’t have a value for two of those

**Jack**: which one? We dont have this one here?

**Shannon**: Do we have that one?

**Jack**: Why don’t you figure it out. Write down a bracket thing for that.

**Shannon**: 
Jack: Okay, so… what is blue’s best more?

Shannon:

Jack: Okay, and what’s left is a value

Shannon:

three fourths

Jack: three fourths, okay.

Shannon: And then, I’m being retarded, I know. I can’t do that in my head.
Jack: So, well, is there a whole number between them?

Shannon: No

Jack: How about a number with denominator two? No, how about a number with denominator four? No. How about a number with denominator 8? Okay, that’s the answer. Some number with denominator 8.

Shannon: seven eighth? Okay.

Jack: Those numbers are six eighths and eight eighths, so seven eighths is in between.

Shannon

Figure 291. T2-46:40

Jack: Okay, so that’s seven eighths. Then that one’s over there.

Chris: one fourth!

Shannon:
Figure 292. T2-46:50

three eighths?

Figure 293. T2-47:06

Jack: Right, right. Now what? I did copy wrong, this is supposed to be an R
There.

Chris: negative three halves.

Shannon:

Jack: See how I got that?

Shannon: Yeah, okay. Right there, just (inaudible)

Jack: It's a negative up somewhere. Okay, so, what's your choice?

Chris: Twelve eighths?
Shannon: How do you do that in your head?

Chris: negative twelve eights

Jack: show here how to write it down all in eighths.

Shannon: I feel dumb if I have to do that. I should be able to do it in my head

Jack: Okay, so what’s the total?

Shannon: we want to be blue

Chris: No. No we don’t

Shannon: yes we do, why not?

Jack: What’d you get for the total?

Shannon: I don’t know. You just don’t want to go first.

Chris: Yes.

Shannon: Okay, we don’t care what color we get. We don’t want to go first.

Jack: Don’t want to go first? Okay. So, ah, I’ll be red again. And so I have to go first. Lets see…I will take, I don’t want that one, I want this one here

Figure 296. T2-48:52
**Figure 297. T2-48:58**

**Jack**: So you’re blue again, your turn.

**Shannon**: Shut up.

**Jack**: So what’s the total value? Okay, if you want to do something and not change it, but you don’t want to change it to negative. So, Shawnna, why don’t you try one and see what the new value would be. Pick one and tell me what the new value would be after you move.

**Shannon**: pick a good one or just pick any one?

**Jack**: Pick any one, well not one at the bottom, that would be too easy.

**Shannon**: wouldn’t this make it a zero?
Jack: Right, I think that’s right. What’s the new value for that stick when its gone?

Shannon: Um, six eighths?

Jack: Six eighths, okay, so write that down below and tell, so the new total is…

Shannon: Should I erase this? Is that how (inaudible). ‘Cause it would be zero (inaudible).

Chris: yeah, it would be zero-what if we get rid of this?

Jack: So that is a good move.

Chris: Yeah. That one.

Jack: So Chris, why don’t you pick another one and see if you can find another good move. Like how ‘bout the top on the right?

Chris: On the right?

Jack: Yeah.

Chris: I think this would be a better one then that. If we took this one on the right, then this would be negative 2.

Jack: Yeah. Would that be okay, or not?

Chris: Sixteen… we get killed on that one.

Jack: Okay, how ‘bout if you took that one over that the other one you said?

Chris: If I took that one
then that’s just gone. And it would still be negative.

**Jack**: Yeah.

**Chris**: that is our best move.

**Jack**: So that’s the only good move. Okay, so. Make your move.

**Shannon**: 

Hey, we can get rid of the eighths now.

**Chris**: Yeah.
Shannon: Can I?

Chris: Sure.

Shannon: I don’t like those

Figure 301. T2-51:51

Jack: Ah, that last one was bad.

Shannon: Is that wrong? Did I do it wrong? Okay! Got cha

Figure 302. T2-51:57

Jack: Okay, so…

Shannon: If we’re going to win, you might as well just give up now.
**Jack**: If your fractions wrong, then maybe I’ll win.

**Shannon**, Chris: Laughts

**Jack**: I will take the top one on the second from the left.

**Shannon**: Second from the left? This one?

**Jack**: Yeah.

**Shannon**: Did we take the top left one? Wanna do that?

**Chris**: Sure.

**Shannon**:

![Image of fraction calculations]

*Figure 303. T2-53:06*
Figure 304. T2-53:10

**Jack:** Okay, I’ll take one of those top reds.

**Shannon:**

Figure 305. T2-53:20

**Chris:** I thought you can’t reduce fractions.

**Shannon:** Shut up. He’s making fun of me.

**Chris:** if we take anything but the far left, it would be a zero game.

**Jack:** What would the value be? If you did that one? The new value, total value?

**Shannon:** Total? Uh… zero?
Jack: Yeah. So that’s a, that is a (inaudible).

Shannon:

Figure 306. T2-54:07

Jack: Okay, so I don’t want to leave my two reds over there alone, so I took one of the other ones.

Shannon: Like that

Figure 307. T2-54:12

Jack: Yeah.

Shannon
**Chris**: Now just be erase to the one with blue red.

**Shannon**: This whole one?

**Chris**: Hum-hu

**Shannon**: Now let’s understand more of this. Can’t erase it so you guys. Okay, so that was good. Any questions about these kind of things? Okay. Let’s talk about the simplicity theorem. Draw a picture first. Suppose
we’re trying to find the value of $x \bar{y}$ ($\{x|y\}$). Here’s what I’m going to tell you about it. He’s zero, and one. Phone rings …

**Jack:** … Okay, so here’s zero and a one on a number line and $x$ and $y$ are both between zero and 1.

*Figure 310. T2-55:55*

So that means the answer is not a whole number for this. Simplest number is, if you look for whole numbers, there aren’t any, in this case. Okay, so but what if the number one-half is in between? Then the answer should be one-half.

*Figure 311. T2-56:08*
Okay, so I want to let you, want you guys to prove that if this is the case, then the answer is one-half. So here’s the theorem and the inequalities. So you have $x$, greater than or equal to zero, and it’s less than one half, and that’s less than $y$.

*Figure 312. T2-56:25*

see, that says one-half is between. But zero and one are outside the range.

*Figure 313. T2-56:33*

Okay, then the conclusion is, under these conditions, which is the same as this picture. The value of $x$ bar $y$ is the number one-half.
Figure 314. T2-56:46

That will be the sum of this number. If you wanted to try to prove that, and so what you do is prove that \( \bar{x} - \frac{1}{2} \) is zero.

Figure 315. T2-57:00

That’s why every proof is like this is going to go. You set up a game that has these values, and prove it’s a zero game by showing whoever starts loses, no matter what they do. Okay, so, it’s the unknown game.
Figure 316. T2-57:20

You really don’t know anything about this game except just what these numbers are, and let’s review what that means, this $x \bar{y}$, I’m not telling you anything about what the game is, except that this is the value of the game after right’s best move.

Figure 317. T2-57:44

The number $y$ is the value then. Okay, so you want to play a game with this thing and then minus one-half. Can you tell me a game I can put, I can do that, that has a value of minus one-half. Think hackenbush.

**Shannon**: Rd Blue

**Jack**: yea,
Shannon: When you say that, do you want us to go top to bottom or bottom to top?

Jack: I don’t care as long as you tell me which. Okay so, from this game, left has two opening plays

Left can take the top of the hackenbush stick, and leave the other game alone. See, when you’re playing the sum of two games, you pick one of them, and leave the other one alone. So, the left plays the hackenbush thing and takes the blue, and this will still be just like it is, or left could play in this game, and leave a value from this game of what? Left play, this will stay the same, and the value of what’s left on this part will be the x, because that’s the val-because what this means is, here’s the value of left’s best move. Okay? So…
Chris: You want us to play out both?

Jack: Yeah, well, you can do left, and she’ll do right.

Chris: So we need to play out both of these, right?

Jack: Yeah, so you have two opening moves by left, and finish that one up and she’ll do one of them.

Chris: See, the x plus, red blue, the other one is going to be…?

Jack: Well, let’s finish this one first.

Chris: That one? Okay.

Jack: Okay, so now you can tell me what the value of that is,
Figure 322. T2-59:40

Jack: now you don’t even need to make a, you shouldn’t make another move for red, because you can, red can play on this but this is some game now, this is just the value of the position of this game and you really know what would come next in this one. So leave the x, but you can calculate anyway. You’ve got x plus, what’s the value of that hackenbush thing?

Chris: One-half.

Jack: Okay, so write that down.

Chris: this is negative one-half.

Jack: Yeah, right.

Chris:
Jack: Okay, solve, the total value is $x - \frac{1}{2}$.

Chris:

Jack: And now, what can you tell me about that? The quantity $x - \frac{1}{2}$. What’s given about that?

Chris: Supposedly that equals zero.

Jack: No, that’s not right.

Chris: Okay, $x$ is less than $\frac{1}{2}$.

Jack: Okay, so what do you know about $x - \frac{1}{2}$?
Chris: It’s a negative number.

Jack: You see why that’s true?

Shannon: Mm-hmm.

Jack: Okay, so write down that’s less than zero,

Chris:

Figure 325. T2-1:00:31

Jack: and then you can say that’s the value of the current position, and who wins?

Chris: it’s a negative then red wins.

Jack: Right, right or red right either one.

Chris:
**Figure 326. T2-1:00:45**

**Jack:** or, or, so if left plays in this unknown game, will end up in a positive of value $x-1/2$ and that’s negative and red wins. But, what if left makes the other move? Leaves that game alone, plays in the hackenbush?

**Chris:**

**Figure 327. T2-1:01:01**

**Jack:** Ok, now you cant us (inaudible) yet because

**Chris:** this is then
Figure 328. T2-1:01:07

**Jack:** the unknown game still there like it was before

**Chris:** the red has two moves from here

**Jack:** ok

**Chris:** either take that

---

Figure 329. T2-1:01:12

or take that

**Jack:** Okay, it is always best to do the y one first because then you’re able to calculate from that point

**Chris:** ok, so if red takes that
**Figure 330. T2-1:01:22**

**Jack**: do, do an arrow with mark on it

**Chris**: Ok

**Figure 331. T2-1:01:27**

**Jack**: and then

**Chris**: and then red need to take that
we will get that

**Jack**: you made a mistake, what mistake did you make?

**Shannon**: it should be

**Chris**: negative one half

**Shannon**: yea y

**Jack**: yea, it would be y. Ok, now

**Chris**: negative one half
Figure 334. T2-1:01:42

this has to be

**Jack**: na-ah, that’s not negative one half

**Shannon**: that’s negative one

**Chris**: negative one

Figure 335. T2-1:01:45

**Jack**: son the current value is? Total is?

**Chris**: this total is gonna be negative because y has to be less than 1.

**Jack**: Ok, but … you are right the total value though is y-1
Chris:

![Image of a chalkboard with mathematical equations]

Figure 336. T2-1:01:59

Jack: that’s what that equals, and that negative. Do you see why that is negative?

Chris:

![Image of a hand writing on a chalkboard]

Figure 337. T2-1:02:03

Shannon: yea

Jack: Why

Shannon: I was… because its …

Jack: where does it …
**Shannon**: because $y$ is less than 1

**Jack**: yea right, because $y$ is less than 1. Ok, who wins?

**Chris**: Red wins. The other move is to take this, so we’re still stuck with this

![Equation](image)

**Figure 338. T2-1:02:20**

**Jack**: Actually you could stop there because you know that if left starts, and right makes the best move right is gonna win, If left starts this way, right will win. If left starts this way, right has a good move.

…………..... Interuption

**Jack**: Now you’re actually much better at this than a lot of people who’ve been in the office so you should be, you should gain confidence from that.

**Yaser**: She doesn’t have much confidence though

**Shannon**: Not with this stuff

**Jack**: Now you should have some more because you did really well on this.

**Shannon**: But there’s no way I could have ever done this by myself without seeing you do it (inaudible)

**Jack**: There were some people that I had to really explain how these inequalities worked and this but you understood that right away.

**Shannon**: No I didn’t, I had to think about it, that’s why I got confused when the red one up there …

**Jack**: Stop telling me that because I know you did.

**Shannon**: No I didn’t at first cause I do know so that is good
(Laughter)

**Jack:** Well you picked it up fast so there you go, okay so I want you guys to come back again next week and do another problem like this.

**Jack:** we wont do this again cause you are fine in that. That’s what I wan you to do, suppose you have, well it’s like the situation we just did, but instead of having zero and a one, just have instead of those two consecutive numbers, have just any two consecutive numbers and then plus one. Then you have x and y in between there and so there’s no, can’t be any whole number between x and y because these are two consecutive whole numbers

![Image of a whiteboard with mathematical notation](image)

*Figure 339. T3-1:21*

Well let’s assume that the number n+half is in between that
and then that would be the answer by the simplicity theorem. Well I’d like to see if you could prove that.

So the situation would be is if you have n, bigger than or equal to zero, and that’s less then x, less than equal to x, and that’s less than n+ a half and that’s less than y, and that’s less than n+1, so it’s just like what you just did, that’s what you just did when n was equal to zero, you have zero less than x, less than a half, less than y, less than one.

**Shannon**: wouldn’t it be the exact same thing except you have n?

**Jack**: Right. It is the same thing and the conclusion is...

X bar y is the number n+ a half, so what you want to prove is, x bar y minus n minus a half is equal to zero.

Okay, so you want to set up a game with these values, just like you were saying. This is like what you just did, r, and b; tell me again with that value.

**Shannon**: R to the n

**Jack**: 
Figure 341. T3-2:27

Yeah, okay so it can be slightly more complicated because like red has three open moves, but this and this and this, okay so, but you got to come back next week and see if you can do that one.

Shannon: did you write this down?

Jack: Here’s another one like it…

Shannon: I am not gonna remember that if I don’t write it down

Chris: go ahead writ down.

Shannon: Ok

Jack: Now suppose you have this with a blank there, so red has no move at all
Figure 342. T3-3:21

Shannon: ah-hu

Jack: then the answer is you take the smallest whole number, which is greater than x, that’s the simplest number greater than x, so we’ll say if

Figure 343. T3-3:46

yeah…Okay so this says that n is bigger than x, but the next smallest whole number isn’t bigger than x, so that would force this to be the smallest whole number bigger than x and this equals n, okay, so, figure out the right game to play with this
so much (inaudible) for this one here. So, you guys can come back. What do you think?

**Shannon**: I can be here same time.

**Chris**: Next week same time

**Jack**: Okay, next week on Friday at was it 1:30?

**Shannon & Chris**: Um hum

**Jack**: Okay.
APPENDIX K

SHANNON, SECOND INTERVIEW
Shannon, One on one interview. Sept. 24, 2003

Yaser: Now, which, what stuff that we’re doing here that you consider proof and what stuff that you don’t consider proof?

Shannon: I think the induction that I did there is proof

Yaser: That’s proof, okay

Shannon: But, like I know everything else is proof, I just, like I know it is, I just don’t really think it is, you know?

Yaser: Yeah, that’s the, that’s what I’m asking

Shannon: Like everything that we play games for, like I don’t really, I see that it’s proof, but in my head it’s not really proof.

Yaser: And why is that? Can you put it in words?

Shannon: Just because it’s not like the standard, you’re not really doing the two column and your not like, it’s not really concrete and I think proofs should be concrete and not abstract and that’s just really abstract and I don’t of that as a proof.

Yaser: Okay

Shannon: Does that make sense?

Yaser: Yeah it does because, yeah I mean it does, cause Ryan for example doesn’t think its proof because you can’t use it for anything else.

Shannon: Well I’m sure you can and we just don’t know it yet

Yaser: Yeah

Shannon: I’m sure that’s what he’s getting to

Yaser: So if you can, you know, if you can use it for something else that will change your mind?

Shannon: No, not really cause it’s still really abstract

Yaser: Okay
Shannon: And I think proofs should be like really understandable and, like when we prove some of them you pick, you just pick a game, like off the top of your head and in proofs you can’t really, you don’t just pick what you want to use and pick what you don’t want to use, you know?

Yaser: Yeah, yeah

Shannon: Like not in the kind of proofs that I think of.

Yaser: Yeah I see, so it shouldn’t be up to you to pick something…

Shannon: Yeah it should be concrete

Yaser: Everybody should pick the same thing.

Shannon: Yeah

Yaser: Alright, okay there’s a proof that they had, somebody did and I want to see if you (inaudible) which is x … we have x bar y plus minus y bar minus x equals zero

Now if you want to attempt this okay and then if you don’t remember how we’re going to go through the how they proved it and see if it makes sense. So how would you start this?

Shannon: I really couldn’t see what they were doing on the board so I’ll just try it, like would you play it, like if left goes first then it would be this

Figure 345. T3-07:59

…then over that equals zero and then it would be … it would be right turn, yea and then that would be this
and that’s it.

**Yaser:** And that’s exactly what they did, alright

**Shannon:** Yes, see I did it I guessed

**Yaser:** Okay so you did it and so is that a proof for you?

**Shannon:** To me, no

**Yaser:** No, why not?

**Shannon:** Cause it’s, I don’t know what these represent, I don’t know what they are, it’s just not concrete enough

**Yaser:** Okay, now that’s, you did it if left started, right?

**Shannon:** Yeah, if right goes first then it’ll be the opposite

**Yaser:** Well red also can go a different way, cause…

**Shannon:** We could do it

**Yaser:** Yeah we could do that, right?

**Shannon:** Okay then…I always forget about both ways, I do it one way and I’m happy, so if, what don’t I do, left?

**Yaser:** Right

**Shannon:** Right?
Yaser: Yeah

Shannon: Okay…that was a parenthesis,

And then it would be left turn and then it would still be x plus minus x that equals zero?

Yaser: That’s zero, now if I say, similarly same happens when left starts right?

Shannon: Yap, it would still be the same thing you would still be adding something to it opposite, either way

Yaser: So that makes it a zero game which means whoever starts

Shannon: loses

Yaser: whoever starts loses, alright well that’s what’s given as a proof, that’s how they proofed it in the classroom.

Shannon: and I understand it

Yaser: You understand but you don’t think it’s a proof…it’s not proof enough.

Shannon: yea

Yaser: Okay how about, do you think…because I want to get a chance to have you talk about what your going to present so that would give you the practice. But did you think about this, the map
Shannon: Wasn’t there one more line

Yaser: Um yea that’s one more line, did you think about that?

Shannon: I don’t remember what were we supposed to do?

Yaser: We’re supposed to find the value of this, so

Shannon: It’s fuzzy

Yaser: Is it fuzzy

Shannon: Left goes here
if left goes first

Yaser: Ah-ha

Shannon: then right can’t go anywhere, so left wins. And if right goes first

and left here right cant win. Whoever goes first wins?

Yaser: Okay so it’s a fuzzy game which means?

Shannon: Whoever goes first wins?

Yaser: Okay is that a proof
**Shannon**: Its not … I don’t look at it as a formal proof I guess is what I am saying

**Yaser**: Okay but it is a proof?

**Shannon**: Yea

**Yaser**: Is that more concrete than the previous one?

**Shannon**: Yea because there is only one way to do it, because you give me this game, so the only thing I am proving is this game. But if you give me this that could be any game and its very concrete because there’s only one game that you can use to prove it.

**Yaser**: Alright. Now you’re presenting, what is it that you are presenting on Friday.

**Shannon**: In office hours

**Yaser**: Yea

**Shannon**: How the simplest number is something

**Yaser**: What number…oh number 26

**Shannon**: Oh I did that one today is that what you mean

**Yaser**: You did that one today. Okay now do you remember what you did…so during office hours what are you going to present then?

**Shannon**: We’re going to do that one

**Yaser**: Your going to do this one

**Shannon**: I think so because that’s where the assignment is on Friday

**Yaser**: So you want to try this

**Shannon**: Sure I tried to do it and I didn’t have anybody with me and I didn’t understand it.

**Yaser**: Okay so how about write it down and let’s see if we understand what it says or try to explain to me what it says.

**Shannon**: Ok, I don’t even know if I wrote it down right.

**Yaser**: Do you remember what number it was?

**Shannon**: I don’t think it is on there, I think it was just something he pulled.
**Yaser**: He pulled, ok.

**Shannon**: this is kinda redundant, actually these equal signs shouldn’t be there

![Image](image1.jpg)

*Figure 352. T3-14:00*

Ok then

![Image](image2.jpg)

*Figure 353.T3-14:12, ok I guess we are supposed to prove that and I think he started this*
Yaser: Ok so he wants you do this one. Ok so tell me what the statement of the theorem of um what is this.

Shannon: If X is between two numbers then on the either side of the halves, then it is the half number that is the simplest number.

Yaser: Ok

Shannon: Does that make sense? That’s how I say it in my head I am sorry.

Yaser: So if there is a whole number plus one half, between X and Y then the simplest number is that whole number plus one half. Does that make sense to you?

Shannon: Yea, as long as X and Y are between one whole number, like they have to be, they have to be 3 point something something. One can’t be three something the other be four something.

Yaser: Ok so they are crunched between two consecutive integers.

Shannon: Yes

Yaser: Alright, ok and he wants you to show this
that this is the case right? And why do you think he wants you to start with this one?

**Shannon:** Cause he wants us to show that this
that this equals plus one half. And when you bring this over

Figure 357. T3-15:52

Yaser: Right, ok so it is simple algebra,

Shannon: yea

Yaser: so how are you going to come up with a game?

Shannon: Well this is any game so we don’t know what it is and this is negative so it has to be R something
Figure 359. T3-16:27

and then negative one half is BR

Figure 360. T3-16:32
and that is assuming we are using Hackenbush because really you could be using any of the games that we have done.

**Yaser:** Right ok so as long as the value is that. Ok, so that is a good start. Now what do you think the next step would be then?

**Shannon:** Um I guess just playing it?

**Yaser:** you want to show whoever starts … what is a zero game?

**Shannon:** whoever starts loses.

**Yaser:** ok

**Shannon:** Can I erase this?

**Yaser:** yea, sure.

**Shannon:** So if, if left starts he could do this or this and if he do this it will be X plus R^N plus BR
and then red goes and

**Yaser:** Can you compute this?

**Shannon:** No cause we don’t know where X is. Right?

**Yaser:** Yea we don’t know where X is that’s right.

**Shannon:** So, if R goes, see I don’t know what a better move would be. If it was that one or that one.

**Yaser:** Well do you have to know which one is the better? Or can you figure that out later?

**Shannon:** Well I assume we want to do the best one right?

**Yaser:** Right

**Shannon:** Or no, we just want to do the best one for blue so it doesn’t really matter what R deos.

**Yaser:** But you want to do the best one for R as well because R wants to win.

**Shannon:** Yea, I don’t know. Is that not the best one?

**Yaser:** No, which one is best for R?

**Shannon:** I am not sure.

**Yaser:** Cause if R takes this

what’s left for blue? This is gone so one less move from blue right? If R takes one of

these
Figure 363. T3-19:04

R leaves an extra move for blue.

**Shannon**: Yea so that is the best move.

**Yaser**: So that would be the best move, ok.

**Shannon**: I really don’t know what x is.

**Yaser**: Have you done something like this?

**Shannon**: I don’t think so. I have no idea what x is.

**Yaser**: But what is X, X is blue right?

**Shannon**: ok

**Yaser**: So what is next?

**Shannon**: oh so

**Yaser**: What can blue do?

**Shannon**: Oh just take that away then its R to the n
Figure 364. T3-19:35

so blue started blue lest.

**Yaser**: Because R can go next but blue can’t.

**Shannon**: Blue’s turn, he cant do anything. Oh, no, its R’s turn then R to n-l

Figure 365. T3-20:01

**Yaser**: And blue has nothing to do

**Shannon**: Ok,
Figure 366. T3-20:04

And the other thing he can do … is it blue starts still… he can do that

Figure 367. T3-20:24

Ok and red’s turn, his best thing would be to do this
because he is grounded here

\begin{figure}
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\includegraphics[width=0.8\textwidth]{figure368.png}
\caption{T3-20:31}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure369.png}
\caption{T3-20:33}
\end{figure}

\textbf{Yaser}: That’s Red’s turn?

\textbf{Shannon}: Yea. And this is blue and blue can’t go so blue loses.

\textbf{Yaser}: Now what happens to X and Y’s?

\textbf{Shannon}: I changed the Y, cause this would be Y
but I don’t like this so I put R instead.

**Yaser:** Ok, so that is your Y. Ok so blue goes, blue has nowhere to go. So blue starts blue loses, now let’s see if red starts.

**Shannon:** If red starts um same thing you would want to do this first

Blue would go and he has one thing to do.

**Yaser:** Ok can you put Y on top of there so I can remember what it is? **Shannon:** And then red goes and the he is going to do something and it doesn’t really matter.
Yaser: Uh huh.

Shannon: And then

Yaser: Then blue can not go anywhere.

Shannon: Wait a minute.

Yaser: Right?

Shannon: It shouldn’t do that though.

Yaser: So what is the problem?

Shannon: I don’t know what is the problem? If red goes first, he does

Figure 372. T3-22:31

so its reds move.

Yaser: So,

Shannon: I don’t know why is going to happen that way. Do you?

Yaser: Okay, so you want to show that if R starts

Shannon: R loses

Yaser: R loses, R starts, then blue takes this
right?

**Shannon**: Um, huh

**Yaser**: If R starts, R will do this strategy

**Figure 374. T3-23:29**

**Shannon**: Right

**Yaser**: Right, which would leave him with

**Shannon**: With Y

**Yaser**: With Y
Shannon: Which is R,

Yaser: Yeah so Y, then blue goes next, blue plays this one, that’s the only way, right?

Shannon: And then blues out, so we’re doing something wrong.

Yaser: So R goes does whatever it does, um, R to the n is negative n right? R to the n is negative n,

Shannon: Yes

Yaser: And this is

Shannon: 1/2, negative 1/2

Yaser: This is negative 1/2

Shannon: Uh, huh

Yaser: Well the first part is, I guess it has no problem, but why is that true?

Shannon: Well I think it’s kind of obvious when R is going to win because he’s grounded twice, and all he’s gotta do is get rid of this and he’s screwed. And blue is screwed, because he’s grounded and then he has this many moves and there is nothing blue can do about it. I don’t, see how blue could win? I don’t know.

Yaser: You know why? Let’s right the um, let’s right the inequality again. What is in the middle. In the middle in n+1/2

Shannon: Yeah

Yaser: And what’s here?

Shannon: Y

Yaser: And what’s here?

Shannon: X

Yaser: Okay, when you played this
okay? When you played this, $x-n-1/2$, uh, this is negative right

\[ x < n+\frac{1}{2} \]

\[ x - n+\frac{1}{2} < 0 \]

**Figure 376. T3-25:38**

**Shannon:** No it should equal 0. According to what he had. Because there is an $n$ here and $n+1$ here, the zero is not in the picture at all.

**Yaser:** But doesn’t this make sense?
Shannon: It would actually be greater than or equal to zero, wouldn’t it be.

Yaser: Well you’re going to subtract from here, from both sides. So we want to show that this is less then 0, does this make sense?

Shannon: No, where are you getting 0?

Yaser: Just think of this, this is a number, and this is a number here, and then forget about the Y
Shannon: Okay

Yaser: Right, now if you subtract,

Shannon: Yeah, okay I see it not

Yaser: From both sides

Shannon: Uh, huh

Yaser: Alright, while this one, this one will be the opposite, will be if you subtract n+, n+1/2 from both sides you’re going to have the 0, um, Y-

Shannon: n-1/2

Yaser: (n-1/2)
Does this make sense to you? I mean is that a different strategy?

**Shannon:** I don’t see where it’s going.

**Yaser:** How about if you play this? What would that be, wouldn’t it be X bar nothing?

Right?

**Shannon:** Why?

**Yaser:** Because you know this means “X” and “Y” has no move.

**Shannon:** Yes he does, Cause there is a negative.
**Yaser**: No, this part here.

**Shannon**: Just this part, yea,

*Figure 382. T3-27:38*

**Yaser**: And this one would be plus “r” to the “n” plus BR

*Figure 383. T3-27:50*

So what does negative mean? A negative gain? What does that mean? If you want to play this game, okay?

**Shannon**: it means red is going to win.

**Yaser**: Negative means right is going to win. So whoever starts red will win.
This one, how do you write this one?

**Shannon**: Blue’s gonna win?

**Yaser**: This here, you are going to have “Y” right

**Shannon**: Yea and then…

**Yaser**: Can you write it down so I can?

**Shannon**: Yea it’ll be the same thing
Yaser: So what do you think?

Shannon: I still don’t see why what we were doing was wrong. Cause no matter what. R is going to, blue’s going to win this one no matter what.

Yaser: Ah-ha

Shannon: I don’t see that though. Cause if blue goes first then you can’t do anything here. So the only thing is there
And then it’s red’s turn and red is going to go when it’s blue’s turn, and blue’s has nothing to do. So, blue is still just going not win.

Yaser: Loses.

Shannon: but this is greater than zero so blue is gonna win here.
APPENDIX L

RYAN, SECOND INTERVIEW
Second one-on-one interview with Ryan September 24

**Ryan:** star is greater than negative one over 2 to the n

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**Yaser:** Ok, that’s star is greater than negative… Ok, so how are you going to do it?

**Ryan:** It’s this

---

*Figure 388. T3-29:39*

That’s what I want to prove.

**Yaser:** Ok, that’s star is greater than negative… Ok, so how are you going to do it?

**Ryan:** It’s this

---

*Figure 389. T3-29:51*
Plus B R to the n plus

Figure 390. T3-29:54

yea I think. I will do right’s first move. Right has two moves, he can either take

Figure 391. T3-30:06

one here or one here
so either way

we’ll do this one first, so its L, R,
Figure 394. T3-30:25

with this left B R to the n

Figure 395. T3-30:28

so this is right’s best move
Figure 396. T3-30:32

Blue’s best move is to take this

Figure 397. T3-30:35

Blues best move is to take this
this whole thing.

_Yaser_: Yeah

_Ryan_: then your left with LR

and blue wins because R has no move
Yaser: so now what do you want to do?

Ryan: So your left with L, R and blue wins because R then has no move

Yaser: Okay, so R starts

Ryan: Yap, R starts and blue wins. So R’s other move is to take this one here

Left’s best move is to leave this
and move here
Figure 404. T3-31:45

and R’s best move from here

Figure 405. T3-31:48

…R has two moves, one move they’re going to lose if they go there
R is going to lose on the next turn from here and on this one.

the best move would be this
So actually they are dead no matter what because we know the value of this is zero.

and we know this is positive because it goes on the bottom.
This is greater than zero

**Yaser**: Okay

**Ryan**: Yea I had it right the first time

**Yaser**: Yea that’s greater than zero

**Ryan**: So we know blue is going to win no matter what

**Yaser**: So R starts blue wins

**Ryan**: Yea
**Yaser:** Which means that what?

**Yaser:** That start is greater than or equal to negative one half to the n. I am going to do a blue win, blue starts and its going to have two moves again

*Figure 412. T3-33:04*

Blue is gonna have to moves. One is very bad move this is their bad move

*Figure 413. T3-33:08*

But they still win this is their good move
so let me do that one first. Red’s best move would be to take this

Figure 415. T3-33:31

to take one of those. Blue’s best move would be to this
so your left with L, R

and we know this is zero
so R goes next R loses.

Yaser: Okay,

Ryan: So blues other move would be to take this

this is a bad move because actually this makes blue lose R is next
and R goes there

and we know from this that blue goes next and blue will lose
Yaser: Okay so there blue wins

Ryan: If blue wants to keep this around as long as possible because that will make sure they can win always. They would rather move in here first if possible than to take this

Yaser: So this really doesn’t add up to your proof

it just shows you that it’s a bad move

Ryan: Yea this is that’s not a good move when what we’re trying to find out is when blue makes its best moves this thing works out
Yaser: Now is that a proof

Ryan: Yea you can use it you can have any games to here. You can find any star game and put that for this and you can find any one negative one half to the negative N for this and work the games out. And see that blue will always win

Yaser: Okay so in other words although these are specific games what you’re saying all the games that resemble…

Ryan: Star no matter what game you have with a star and a negative one to the two N if you put them in for that its still going to work out the same.

Yaser: Okay so although it looks like a specific case but because it worked here

Ryan: Right. You can get it to work for other cases

Yaser: Not because of…because nothing here in this is related to those specific games

Ryan: Right

Yaser: is that what you’re saying?

Ryan: Yea

Yaser: So would that be, remember this

Figure 424. T3-37:33

Ryan: This is a theorem
Yaser: Yea can you show me how you do this theorem?

Ryan: Right okay so blue and red both to start off have two moves

Yaser: Ah-hu

Ryan: if you start off with blue, blue can either take this

Figure 425. T3-37:46
make their best move from this one or this one

Figure 426. T3-37:47
start with this one. So they make their best move from this one so they do an X plus negative x and negative y … I think these are supposed to be flipped flopped
aren’t they?

**Yaser**: How did you figure that out, you are probably right but something told you that it has to be that way, what was it.

**Ryan**: Because red is the next move and red…

**Yaser**: Right your correct because you want them to cancel out?

**Ryan**: Yes, this is supposed to be negative y, this is negative x so reds next move and their only move is this
so its X plus negative x and that’s equal to zero

Figure 429. T3-39.00

**Yaser:** Okay so similarly you can do the other

**Ryan:** Right

**Yaser:** Is that more of a general proof then what you just did with the previous one?

**Ryan:** Yes

**Yaser:** It is

**Ryan:** I think in my opinion yes

**Yaser:** In your opinion okay which because

**Ryan:** Because it’s like it doesn’t that goes for any games, you just flip-flop them, it doesn’t involve two different games, and it’s the same game. It’s doesn’t involve two different games but the other one will probably involve two different games. This is only going to be one game but the exact opposite of the first game.

**Yaser:** And then is there any reference to what game in this one

**Ryan:** There’s nothing

**Yaser:** Nothing

**Ryan:** Its just a game with this value and a game with this value the opposite value

**Yaser:** Okay so those could be any games?
**Ryan:** Yes

**Yaser:** While the other one there was no reference but

**Ryan:** You had to have a game of negative one half to the N...this it doesn’t matter it can be, I mean you had to have a game of negative one half to the N somewhere in there

**Yaser:** Right

**Ryan:** This it doesn’t matter I mean it could be one half, negative one half, three, negative three, and zero, star

**Yaser:** I got you, in your opinion this is a more general one which qualified to being more of a proof?

**Ryan:** Yes
**Jack**: zero bar zero, as you go along you’ll need to know what up or down

**Ryan**: So both right and left have two moves

**Zack**: I don’t care go ahead and pick one
**Ryan**: I’ll do…

**Jack**: They have pretty much the same

**Ryan**, **Zack**: Yea

**Ryan**: So do this one, left

*Figure 432. T3-41:40*

, take up first

*Figure 433. T3-41:49*

**Jack**: Put an arrow there on the other side and then label it left so we can keep track of who started.

**Ryan**: So up is actually zero bar star plus.


Jack: Okay so zero bar zero

Figure 434. T3-42:20

, and label your “l” your horizontal looking letter “l” to keep track and then you have to find the new move for right.

Ryan: Ah, so right is going to want to take one of the zero here, to make it a zero here

Figure 435. T3-42:37

Jack: Okay write that down, I don't think that is going to be write so write that down and we’ll see why its wrong. I think its wrong. Okay go ahead and write it down.

Ryan: So we still have this
Figure 436. T3-42:54
That’s a good one. Plus zero

Figure 437. T3-42:59

**Jack:** Okay now you can just erase the zero.

**Ryan:** Right
**Figure 438. T3-43:04**

**Jack:** And what’s left is,

**Ryan:** so left is going win because they are going to have to take the zero here

**Figure 439. T3-43:10**

**Jack:** Yea if right makes that move

**Zack:** that’s positive

**Jack:** so that’s a bad move for right.

**Ryan:** So the other move for right…

**Jack:** So, you are still looking for the winning move for right,
**Ryan:** would be take star, that’s star plus zero bar zero

---

*Figure 440. T3-43:35*

okay. And.

**Zack:** put the arrow on the other side

**Ryan:** Last move.

**Jack:** You could play it out some more but I think its ...

**Ryan:** It’s going to be a down game.

**Jack:** No, no. You can simplify this.

**Ryan:** Its going to be star plus zero.

**Jack:** No.,

**Ryan:** Its going to be star because if left moves they are going to take this
Jack: Okay but simplify it before left moves.

Zack: What’s this?

Ryan: Ah that’s just star.

Jack: okay so you have star plus star but that’s an equals. Don’t put an arrow there put an equal sign instead.

Ryan: Right
Jack: Okay and that equals, what’s the third one we have?

Ryan: Zero.

Jack: yea, Star plus star equals zero.

Ryan: So whoever goes next is going to lose and left is next.

Jack: Yea whoever plays it to that point right played it to that point

Ryan: so its left’s move

Jack: yea, and right wins. So if left starts by playing in this game to that
then that’s a loss for left now you gotta show

![Image](image1)

`Figure 445. T3-44:48`

if left does that.

**Ryan:** So there other move is zero up down plus zero

![Image](image2)

`Figure 446. T3-45:06`

**Jack:** Put an arrow going into that again and label it.

**Ryan:**
Figure 447. T3-45:10

So right’s move would be the only move is this and that’s equal to

Figure 448. T-45:26

star plus star equal zero
Figure 449. T3-45:34

Jack:

Now you can look at just this

Figure 450. T3-45:36

and tell me who wins.

Ryan: Its negative so.

Jack: Yes down is negative so

Ryan: red wins.

Jack: Yeah. So left’s two moves red had a winning response. Okay. How about if right start.
Zack: Ok. Let's start the same equation

If red starts, well I will write it out should have left this up there but, up bar down plus zero zero

now right has two moves
and if they go over here they take it down first and they get down plus star

**Figure 454. T3-46:39**

did we do that one yet?

**Jack:** No

**Zack:** Alright, so they get down, plus zero, zero,
Figure 455. T3-46:48

so we can play that one out then, and

**Jack**: Now you got to find the winning move for left.

**Zack**: Right, so left, left only has one move

Figure 456. T3-46:58

**Jack**: 2 moves, because each of those things represents a position in the game.

**Zack**: Yeah, because this is star zero,
Figure 457. T3-47:11

okay, plus zero, zero, now, left

Figure 458. T3-47:21

can go to star or zero, probably the best move here is star

Jack: Yeah

Zack: Because, that would be, that would give you, star plus star

Jack: Yeah, same thing (inaudible)

Zack: Zero
right next so it’s a zero game, so left will lose

Jack: ok

Zack: Um, plus other move to stars, or rights other move to star, is … 2, one is to take a zero, so you’ll just be left with, up down

and then, blue goes next
blue would take the up,

Jack: Um huh

Zack: If up that’s positive then, blue wins of course
Jack: Ok thats good. Lets do another one like this. This and its going to work out the same but (inaudible) Ok go ahead.

Zack: So you want me to do right or left?

Ryan: It doesn’t matter.

Zack: Ok alright. I’ll do…

Jack: Let’s go with left.

Zack: Ok alright. Left goes first, ah … let me right it out, zero, down plus zero, zero should
that equal zero. So left has two moves again they both to zero.

**Jack**: Right they both move to zero but left is different so

**Zack**: right yeah.

**Jack**: left had different move.

**Zack**: So I’d say we’re over here first and you get zero over here plus star,

![Image](image.png)

*Figure 465. T3-49:49*

and that equals star, and so whoever goes next wins,

**Jack**: Right

**Zack**: and then it goes back so red wins

**Jack**: Red wins.

**Zack**: Left starts there, left starts um over here you get … this stays the same. Red goes next they take it down, down’s negative so red wins.

**Jack**: Ok. That was quick how about Mr. Right?

**Ryan**: Red has two moves so if take it down first your left with down plus zero bar zero this equals to star bar zero plus zero bar zero
Left’s best move would be to take the star

**Jack:** Yeah it’s going to work out the same as you did

**Ryan:** Star plus zero bar zero, which is equal to star plus star which is zero

and this was blue’s move to this. And red goes next

**Zack:** so red loses.

**Ryan:** Another move would be to take the zero so your left with zero bar down
and then its left’s turn and they make it zero, they make it zero

so red is next
left wins

**Jack**: Ok let’s do one more, let’s do that longer one number 38

**Ryan**: You want to go first or second?

**Zack**: I’ll go second

**Jack**: See if you could use that from were you where, seeing at this point here it was, I erased it but it was  

**Ryan**: down
Jack: down plus star okay, well that would be the negative of this so it would be zero bar zero star well that would be the negative of this so it would be zero bar, zero star, so you can say that this that you’ve got here is (inaudible). And who’s turn was it do you remember?

Ryan: It was blues turn or left

Jack: So left could in this game make it zero, it’s the same thing it might be easier that what you did it might go quicker. Ok prove this one.

Ryan: prove this one here?

Figure 472. T3-54:05

Jack: Yes, yea

Ryan:
So I’m going to go yea, zero star bar zero equals to

that opposite of up is down, so this goes down, and the opposite of star is just star. Need to prove that
equal to zero and this is equal to zero star bar zero plus star bar zero plus zero bar zero

So left has four moves and right has three moves.

**Jack:** yea

**Ryan:** I’ll do left first

**Jack:** Ok

**Ryan:** first I’ll take the zero from this one
so you’re left with star bar zero plus star bar zero plus zero bar zero

**Jack**: You have made a mistake and that is when left plays

**Ryan**: ah yes

**Jack**: in this game picks one of these two and replaces it either by the zero or where the star is so it should be the zero.

**Ryan**: Yea, just this
Jack: yea. And you got to do it again the whole thing just like this but with a star instead of the zero.

Ryan: Right, so your left with zero so rights next move would be take, I think their best move is take the zero here, they went next. It would be zero plus stars bar zero plus zero and we know this is down

so it’s negative so red wins.

Jack: Right. Ok that’s good.

Ryan: Another move for left need to do star so its star plus star bar zero plus zero bar zero then red’s move
Um

**Zack:** can you say that this is star and this is star

And this is down
so you already know its negative because star plus star is zero.

**Jack:** Yea right that’s a good thing to simplify to write it as equal to star plus down plus star.

**Ryan:**

**Jack:** then the star plus star is

**Ryan:** this equal to down, so reds wins.
Jack: yeah, so you see you have this…well it’s still fairly complicated as a game because this is zero bar zero. But the trick is what you just noticed its actually equals to this game plus the zero game. And you can erase the zero games because they don’t affect the outcome.

Ryan: Ok. Red has one more thing two more moves

Jack: yea

Ryan: the next move would be this star so your left with zero star bar zero plus star plus this is star. So they are just left with this

Figure 485. T3-58.07

and its rights turn
Jack: it's right's turn yeah

Ryan: and they take the zero so right would win.

Jack: right

Ryan: Because it's left's turn next…

Jack: If it's your turn and you make it zero that's a good move for you,

Ryan: there other move, the next move would make this zero so you're left with

I don't know the value of this, so you can really do anything
**Jack:** You can’t simplify anything down so its, right turn

**Ryan:** Rights best move will be to take this so its zero plus down and that’s negative so right wins

![Figure 488. T3-59:13](image)

**Jack:** Yea, so that takes care of the left’s four moves

**Zack:** Right has three moves first thing they can do is take the zero here and your left with down plus star we don’t know that yet so better to just write it out. So you’re left with

**Ryan:** Can we do (inaudible) because that’s the same thing as this

![Figure 489. T3-59:55](image)

and blue has the next move I think blue won that but go ahead
Zack: Yea I don’t know lets go through it again. Okay blue moves next, blue’s best response is to take the star so you just get star plus star equals zero. And red goes next. Reds next move is to take the zero here

Figure 490. T3-1:00:34

that will give you zero star bar zero plus star plus zero bar zero. Okay but this is star so all we actually have left is zero. So all we actually have left is this, lefts move take the zero or the star

Jack: Yea the zero and the star are two possibilities

Zack: Star I believe is their best response.

Jack: And that’s not right because if you make this a star

Ryan: if you make it a zero game

Jack: whoever goes next wins

Zack: Right so we don’t want to make it star we want to make it zero

Jack: Yea general principle to remember is that you can make it zero that’s a good move for you.

Zack: Right and red goes next so they lose
Jack: so left wins. Okay one more

Zack: Alright their only other move is to take the zero over here so you have this

Zack: Ok, still don’t know the values of that, so blue, or left goes next and that takes zero, ok star here, could take a star here
Jack: What about taking the first zero?

Zack: That’s the bad move, I was thinking about that.

Jack: Yeah, that’s a bad one, why?

Zack: Because then red would take the star.

Jack: Yeah, ok.

Zack: And this is down too, even though it’s negative.

Jack: yea right, right. Taking that zero is bad.

Zack: Um, taking the star might be a good move.

Jack: Ok.

Zack: So you have star plus star bar zero and red would go, I don’t really know what star plus down so red would go and red would take zero.

Jack: Now that would be bad, but red has another move.

Ryan: And the star.

Zack: On the star. Stars are actually zero-zero, ok. So, red’s other move would be take the zero.

Jack: Yeah, and leave.

Zack: And leave.

Ryan: the down.
Zack: the down. Yeah, ok and then it would be negative and red would win.

Jack: Ok, so what does that mean? Red started and red wins.

Zack: That means that that was bad for blue. Bad for left.

Jack: That was actually a bad move for blue by taking that.

Zack: It took zero, right?

Ryan: Took the star from there

Jack: It took the star. So we’re still searching for the good move for left.

Zack: That has to be it right there then.

Jack: Right.

Zack: So, let’s see how we do at this one.

Jack: Now we’re trying to prove that this is a winning move for left, but it looks like right has two moves to this point, so you have to show that right can’t win either way.

Zack: Ok, so right can go to zero here, which you do with zero plus star, so that goes to star and blue goes next and wins

Jack: Ok, so red can’t win by doing that.

Zack: Ok, the other move from red is to take the zero over here and you just get, zero star bar zero and then blue goes next, blue will make it zero and its red turn so red loses.
**Jack:** Ok, so this is actually more complicated because you played this and then you got down to a place where you had to go through two choices for red, so that those weren’t winning moves. Ok, well that’s good, you guys got this down, I think.
APPENDIX N

SHANNON AND CHRIS, SECOND PRESENTATION
Shannon and Chris Office visit 10/10/03

Jack: Definition up and down and one theorem is that

equals star, so this equation is the same as, that minus star equals zero

and this is zero bar zero, that’s the definition of star. When we take the negative of star, it’s just itself, so this is –Just copy this every time- minus star is the same as plus star, so this is plus zero bar zero
You need to prove that this is zero game?

So you got to prove that whoever starts loses? Ok, so you get to pick left or right to start.

**Chris**: Ok, so it left goes first, left has two options. First option is taking this
Figure 499. T3-1:06:19

so we have up

Figure 500. T3-1:06:28

this is zero-zero again.

**Jack:** Put a horizontal line going into that and label it with who did that.

**Chris:**
Ok. And so right has…

**Jack**: Did you see that, what’s going on? Ok, now that was a move by right, so put an arrow there

**Chris**: 

**Jack**: and right has just blundered, why?

**Shannon**: because …

**Chris**: Because that’s a positive game.

**Jack**: Right, but why would blue win?
**Shannon**: It’s positive.

**Jack**: Right it’s positive.

**Chris**: So this

*Figure 503. T3-1:07:07*

is this

*Figure 504. T3-1:07:11*

**Jack**: So at this point, playing this game is bad for right, so it’s a bad move for right.

**Chris**: Right would play here and right can choose star and leave this
Figure 505. T3-1:07:24

**Jack:** do you see where that’s starting from?

**Shannon:** No

**Jack:** Ok, lets, why don’t you right equals here whatever it is. He just replaced up by what it equals.

**Chris:**

Figure 506. T3-1:07:43

**Jack:** Now the important thing to remember is when you’re doing this stuff is that everything written here represents a game. So this represents some game and this represents a different game. But if its right’s
turn, right can play in either part, so right can play here or right can play there. You guys just figured out that playing here is a bad move, so right’s good move is going to have to be in this one which equals this.

**Chris**: No matter what’s chosen, you get zero no matter what. Left can go here or here which is zero, right’s here or here which is zero, so this equals this.

**Jack**: Ok, can you explain where all of this came from? From left played to this which equals this and you decided this was a bad move, so can you explain where this stuff came from? Show us, tell me where and how.

**Shannon**: With this right’s turn?

**Jack**: Yea.

**Shannon**: And right took the star?

**Jack**: Right to the star here, so, this game has been played in this position.

**Shannon**: And the start is the same to zero. Zero bar zero.

**Jack**: Yea so that’s why he changed it here and how do you get this?

**Shannon**: Cause it’s the only thing you can choose and zero and zero and zero plus zero is still zero.

**Jack**: Right, but there’s a quicker way of saying

**Chris**: star plus star.

**Jack**: This is star plus star, the previous theorem is star plus star equals zero.

**Shannon**: yeah

**Jack**: Okay so

**Chris**: left has another move

**Jack**: Left can now

**Chris**: left cannot choose that
which will be up bar zero plus zero

*Jack*: Okay

*Chris*: so rights move. Right’s only move there
**Jack**: Okay, see that? Now this isn’t strictly correct but that’s only rights only move, because that only represents some game also.

**Chris**: Okay.

**Jack**: of value zero. But that equals zero. The theorem is that you know the game plus a game of zero value. It doesn’t, the outcome isn’t going to be the same as this. So if you want to know the outcome of the game you can just erase this part. So you’re right. In that sense there is only one move. Okay, now it’s your turn.

**Shannon**: So right is gonna to first?

**Jack**: Right, so you will start with this and do right’s moves.

**Chris**: you get to use the right chalk?

**Shannon**: okay. So you can do that [0+{0|0}] and that should be zero again. Right? No.

**Jack**: No.

**Shannon**: That wouldn’t make me lose. I don’t want that.

**Jack**: It’s a star game not, its zero plus star. So it equals start.

**Shannon**: but if left goes it will be zero plus zero
Jack: Yeah.

Shannon: Okay. Do I have to play this one the other way or do I just have play this.

Jack: Well no because you found a winning move

Shannon: Okay

Jack: So that was good. Now right has another opening move.

Shannon: We could do that one? And then left will take that
which is $\{0\}$. Do you want me to write the zero every time or can I just write that, if that’s zero.

Jack: Well you cannot write it because the outcome is the same with or without the zero.

Shannon: So left wins

Jack: Let’s see, who’s turn is it? Left’s turn, so left wins, star game. Okay now, you can tell me after left made this move to this
you can tell me who is going to win from that. Cause this equals… [{0|*}]

Shannon: It’s up.

Jack: It’s up. And what do we know about up?

Shannon: It’s positive

Jack: Up is positive so from that you can say left will win at that point. Okay so as soon as you have a previous theorem that says who’s going to win then, then you stop okay? There’s another one like that.

Shannon: let me show you how smart I am.

*Dh: You just prove it that you’re smart.

Shannon: Na-ah

Jack: Show me that you’re smart with up and down.

Shannon: Not that one.

Chris: What three, four steps total and then you were done?

Shannon: that all there was though.

Jack: Another one of the theorems here is up bar down is star
Figure 515. T3-1:12:55

okay so why don’t you do left?

Figure 516. T3-1:13:08

**Shannon**: Does that mean I am gonna use blue chalk?

**Chris**: No go for green this time.

**Shannon**: Okay so left can go up plus that \{0|0\}
and then right will have to choose zero

Shannon: I don’t know! does it matter? I only want to find good moves for both, not right.

Jack: Well you’re doing a right starting, so you are trying to show that right can’t win. I’m sorry you’re doing left starting so you want to show me that the left can’t win no matter what they do. So right has a winning response to whatever left does. Okay so this is a bad move for right, Why?

Shannon: because that will make a positive.
**Jack**: Yea that will make a positive. And then left will win. So that’s a bad move.

**Shannon**:  

*Figure 519. T3-1:14:11*

**Chris**: No, that was fine.

**Shannon**: I know I’m rewriting it, so I can think, that was up?

**Jack**: Yes that was up.

**Shannon**: So then right, if that’s a bad move then they are gonna do

*Figure 520. T3-1:14:29*

**Jack**: Yea.
Shannon: And that’s also star plus star.

Figure 521. T3-1:14:34

Jack: Yea, so write that down because that’s the easiest way to get what we want. So that equals, what does that equal. So, instead of taking a move, what’s that equals.

Shannon: It equals zero.

Jack: So, whose turn is it? Right loses.

Shannon: Right, Right?

Jack: cause it’s a zero game now.

Shannon: Yea but its left’s turn.

Jack: Okay so right wins.

Shannon:
Left goes and then

**Jack**: So you took this one.

**Shannon**: Ok, so I am gonna take this one
Figure 524. T3-1:15:24

Figure 525. T3-1:15:46

**Jack**: Yeah. Ok, you could of determined the winner at an earlier point than what you did.

**Shannon**: Right here
Figure 526. T3-1:15:56

but I don’t like that. I like to see this, I want to get rid of the ups and downs. That’s the only way I see it.

Jack: You have to learn to love up and down.

Shannon: No!!

Jack: Ok, so…

Shannon: It’s your turn.

Jack: It’s Mr. Wright’s turn.

Chris: It’s basically just changing all your lefts to rights and your rights to lefts.

Jack: Yeah, that’s about it.

Chris: Ok, and this is right going first.

Jack: Yep

Chris: So, right can choose that… (sound of chalk screeching on the board)

Shannon: Ewww

Chris: I don’t like that one.

Jack: It’s not the chalk’s fault.

Chris: it is
Figure 527. T3-1:16:45

Ok, and left can go now, but that’s a bad move so left is going to choose from here which is there

Figure 528. T3-1:16:56

Jack: Did you see what he did? Do you see where that stuff came from? He sort of skipped a step.

Shannon: Yeah, I see.

Chris: Worked it in my mind. Ok, so the other choice is picking here
yeah, so the left choice is that

If left chose that, its going to turn into that
right’s choice is then star, and that could just do that but…

Jack: yea, at that point you could say whos going to win because we have a theorem. Up says down is negative. Ok. You will be happy to know I only have one more left like this I want you to do.

Shannon: That makes me happy?

Jack: Because there is only one more.

Shannon: There is still one more though…

Chris: He could make up five.
Shannon: No, I don’t think I could do them.

Chris: Oh, sure you can.

Shannon: I need those. You just erased it.

Chris: Go from memory.

Shannon: No, because I will probably be horrible.

Jack: What I want to do is the theorem that says…This

Figure 533. T3-1:18:43

game left has two choices. One choice is better than the other. Equals up plus star. Ok, so, you’re job is
first to make it something that is equal to zero. Ok now…

Shannon:
Figure 534. T3-1:19:07

**Jack:** Ok

**Shannon:** is minus up down?

**Jack:** What?

**Shannon:** Is minus up down?

**Jack:** Yes.

**Shannon:**

Figure 535. T3-1:19:24

is that right?
**Jack:** Now you should replace each of these two left things by what they equal, so you will be able to see the moves.

**Shannon:**

![Math equations on a board](image)

*Figure 536. T3-1:19:57*

**Jack:** Ok, well you have to choose. What do you want to play? Left or right?

**Shannon:** Let’s go right. I have less moves.

**Chris:** See, you’re smart, you at least knew that.

**Jack:** She did the work to figure it out though. The game, the negatives stuff.

**Shannon:** Alright, we’ll start here
Jack: Can you tell me before you do the next move why it would be bad for left to take this...

Shannon: Because right, (inaudible) I can’t do it in my head, its just not working.

Jack: Well remember… what that …

Shannon: It would be right’s turn and right would choose this

Jack: Don’t think of that, what does this equal
Shannon: That would be, down.

Jack: So you would have a zero plus down plus zero which would be… good for who?

Shannon: right.

Jack: Why? Because down is right’s favorite thing in this game. Ok so that would be a bad move for left

Shannon:
Jack: and that’s a good move because it’s zero. So you find a good move so this is one good for left. So left had three moves possible here so you got to find a good move.

Shannon: Ah-see I can’t do it…

Jack: If you can’t figure it out just try one and see how it works out. See if it works out for left or not. So which one are you taking? So then what will we have?

Shannon:
**Jack:** Now you can tell me though who wins the star game?

**Shannon:** I don’t want that…

**Jack:** You don’t want that right will win.

**Shannon:**

*Figure 543. T3-1:22:31*

**Jack:** Ok, so doing this was not good.

**Shannon:**

*Figure 544. T3-1:22:47*
**Jack:** that was left’s good move. Ok. Now right’s last chance.

**Shannon:** It probably not going to be zero … it has not been zero before, it probably its gonna be star

**Jack:** If you play the zero the value would be this plus that is down which is good or bad for left?

**Shannon:** Bad…

**Jack:** Bad for left, that is not a good move for left …

**Shannon:** Does it matter which star is it?

**Jack:** Well, the star would be… it wouldn’t make a difference what the star is but the other thing would be left because it’s still there. The other thing is still there so if you left it this it would be star plus up, if left at this it would be star plus this whole thing?

**Shannon:**

*Figure 545. T3-1:24:11*

**Jack:** I’m sorry, this is star plus down.

**Shannon:**
It’s not the right move, so
Jack: Zero plus star, ok? Now, if right did this

it’s a star. Now, one more thing to consider this star
represents a position in the game, so right can plan that also because it equals zero bar zero. So, you have to put another line here showing what would happen if right can’t. If right does that

**Shannon:**

*Figure 551. T3-1:25:23*

**Jack:** It’s probably still good for left because has, still has two choices left. It’s somewhat more complicated than the previous theorems you had because when right starts this way, left has this response, and left starts, if right starts this way right has this response which is a good one, but right has two choices
and you’re trying to show that right cannot win no matter what so you’ve got to go through both choices and show that you can’t win for right. Ok. Now it’s time for Mr. left (Inaudible).

**Chris**: (Inaudible) again.

**Shannon**: You’re going to be up for a while?

**Chris**: Na, not too long. Ok, left’s first move can be here

\[ Figure 552. \ T3-1:26:17 \]

so left does not want star because it would be a down game.

**Jack**: Yeah, but you’ve got to go through every, it won’t be down it would be down plus star if he takes the zero.

**Chris**: Well, if left took this start
I’ve have down plus down plus star.

**Jack:** Ok, let’s write that down.

**Chris:** that wouldn’t be good though.

**Jack:** write down what you said. Now it would be…

**Chris:** It would be right’s move.

**Jack:** Now let’s see, the first down where did it come from?

**Chris:** It’s left of that
**Jack:** Ok, it should be just star. Because that’s…

**Chris:** These just cancel each other

*Figure 555. T3-1:26:56*

**Jack:** No. This one stays the same and this one stays the same but right here you are thinking what is left is this and that

*Figure 556. T3-1:27:05*

**Chris:** Oh yeah… its just a star,
Jack: But it’s just the star. That’s one of the two things that left can move to.

Chris: Star plus star. Star plus star. Am I able to do this star plus star
Jack: Yeah, that would be a good thing to do. Can you see what he is gonna do?

Chris: Zero plus down, that’s down, right wins already

Jack: You see what he did?

Shannon: Yeah.

Jack: Ok. So left takes that first star.

Chris: So this other move here is to
Jack: Yeah, take the zero from there.

Chris:

Jack: Ok, now it’s down plus star but left, but right should have a winning response.

Chris: So it right takes this
what is left is down.

Jack: Yeah

Chris: right wins

Jack: right wins, ok.

Chris: right can take that

Jack: Ok, before you do another step lets see if Shannon can figure out what simplification could be made

Shannon: star plus star is zero
**Jack:** yea. That’s the thing to do

**Chris:** so that’s gone, right’s good move in here is zero

![Figure 565. T3-1:28:34](image)

left loses

**Jack:** no, right made it zero

**Chris:** yea

**Jack:** so left lose, ok

**Chris:** so left’s other move is here

**Jack:** yea

**Chris:**
Now right is definitely wanna take this because that leaves down again

**Jack**: yea

**Chris**: so right takes that, all is left is down

---

*Dh: she still wants nims, next visit*

**Jack**: well lets do nims now. Do you have time?
**Shannon**: I don’t care, as long as you don’t mess me up, and give me what we did in class. Don’t give me something hard here.

**Chris**: give here ten piles

**Jack**: do you have your notes with all these things written down?

**Shannon**: yes.

**Jack**: Ok

**Chris**: you don’t have them memorized?

**Shannon**: No.

**Jack**: Those are srota random now, on Monday ill show you …

**Shannon**: Math is not about memorization

**Jack**: 

![Image of handwritten numbers](image)

*Figure 568. T3-1:30:27*

**Shannon**: Now see he is gonna make it hard now

**Jack**: yea, I don’t want to put one two three, that’s too easy

**Shannon**: See, that’s exactly what I just said

**Jack**: 
Figure 569. T3-1:30:36

ok lets play me against you.

**Shannon**: oh, no, I didn’t say that (laugh)

**Jack**: Do you want to be first or second?

**Shannon**: can I think about it?

**Jack**: sure, I want you to figure it out and think out loud.

**Shannon**: I don’t want to think out loud.

**Chris**: it’s a number three game

**Shannon**: shut up, ok 245 what’s 245, …

**Chris**: go for 1 6 7, one, six, seen is zero

**Shannon**: I know, so now I am figuring out what’s left

**Chris**: 2,4,5, four and five is one

**Shannon**: yes it is

**Chris**: so you have 2 and one

**Jack**: one, four, five is zero
Figure 570. T4-00:02

, ok. Well I am gonna change Six to a three

Figure 571. T4-00:14

Shannon:
Jack: So I’m going to change, going to change the four to a two.

Chris: Oh, five down to a three
Jack: Ok, now not going to be any fun anymore because why?

Shannon: Now it’s just going to be you do something, I do the same thing.

Jack: Yeah right. Now we call that, we call that copycat, so whatever I do to one of these piles, if I do something to three, this three, your going to do the same thing to the other three and so on. Ok. Want to play again?

Shannon: Sure. I like the game.

Chris: Hmm? You like what?

Shannon: I like beating him, its fun.

Jack:
Figure 575. T4-01:37

Ok, so want do you want to do, do you wanna go first or second.

Shannon: First

Figure 576. T4-01:59

Jack: Change the four to a three?

Shannon: it says over here in my notes

Chris: Three,five six is zero?

Shannon: I don’t know that’s what I did in my homework and it worked.

Jack: Ok.
**Shannon**: Is it going to work or is that not right?

**Jack**: Um, well this is zero, so is this, I don’t remember the list, is it three five six, is that a zero?

**Shannon**: Yes.

**Jack**: Ok. So…

**Shannon**: see he doesn’t even know what the list is.

**Jack**: I’m going to erase one of these threes

![Image of a whiteboard with handwritten numbers: 123456, 22254, 12356]

*Figure 577. T4-02:39*

**Chris**: I knew all last game without looking at a list.

**Shannon**: Does that just mean (inaudible)
Figure 578. T4-02:56

Chris: No.

Shannon: Why not?

Chris: No.

Shannon: Why not?

Chris: Oh yeah, that’ll work, ok, (inaudible) I was thinking if you took the six down to a five, you have one, two, three as a zero (inaudible) Yeah, I didn’t see the other one.

Jack: One, two, three and then five, five, so that would be zero plus zero after you change this to a five, but this is zero plus zero also. So, I’m going to take one, take one of these away, so one, three, five, six. Ok.
Shannon: You’re not going to beat me (inaudible)

Jack: No, but I’m trying to make a move that you have to figure something out to make it real easy. Let’s see…

Chris: You’re the teacher (inaudible)

Jack: If I change this to a one or a two, what would you do?
Shannon: I’d change the six to a one or a two (inaudible)

Jack: Ok, then I’m not going to do that then. How about if I change this to a four

Chris: Change the three to a two.

Shannon: What was there before? A five?

Chris: Yeah.

Shannon: Ok, I was like I thought it was a three and like you can’t do that.

Chris: Just check yours; I think two, four, six in on that list.
Shannon:

Figure 583. T4-04:27

Chris: See.

Jack: I’m going to change this to a five

Shannon:

Figure 584. T4-04:33
Jack: Ok. Now the only thing I can do is to give you a way to change it two one, two, three or two equals so I’m going to… (inaudible)

Jack: OK, you know about binary numbers?

Shannon: A little bit I’m not really good at them, but I can try.

Jack: OK, um, in decimal numbers, this is the one’s place, the ten’s place, hundred’s and so on, so this is the one’s place that actually ten to the zero, this is ten to the one, ten squared, ten cubed and like that, binary is two to the zero, two to the one, two squared, two cubed and so this is the one’s place, two’s place, four’s place, eight’s place, so this equals…
Shannon: eleven

Jack: Right. Now that equals eleven, now what if I said I wanted it to equal five, what would you put down

Shannon: zero one, zero one

Jack: Now what if I said it wanted to equal twelve?

Shannon: one one zero zero

Jack: Yeah. (inaudible) Ok, lets put it down, two, a four and six, in binary can you do that?
Shannon:

Figure 588. T4-06:41

Jack: Now put it in a column nicely line up because we’re going to do something like addition next.

Shannon: Oh I know how to add this.

Jack: Yeah, but we’re not going to do that, we’re going to do something like addition. Ok, binary addition, but I’s gonna teach you something easier than that, instead of, you do something in each column, but do binary addition, but forget to carry it, ok, two four six, now do the same thing with, um, one two three over here, put one two three and then (inaudible).

Shannon:
Figure 589. T4-07:51

Jack: Yeah. Ok, now do it for…

Chris: One six seven?

Jack: Which one?

Chris: One six seven?

Jack: No, let’s get a better one than that. How about three five six?

Shannon:

Figure 590. T4-08:28

Jack: What do you think the theorem is?
Shannon: are all these zero games?

Chris: yea

Jack: I took those right out of your notes.

Shannon: Can you do binary addition without carrying you get a zero game. I don’t know how to word it.

Jack: Yeah, ok. Suppose we did one two three four five seven

Figure 591. T4-09:04
Do all those.

Shannon:

Figure 592. T4-09:14
Jack: Well not its not.

Shannon: It’s a zero.

Jack: I don’t think it’s a zero position.

Shannon: Actually no, it looks like (inaudible)

Jack: Yeah, gotta start at the top and see what it equals.

Shannon:

Figure 593. T4-09:38

is that right?

Jack: Yea, and the answer is?

Shannon:
Chris: so the answers six.

Jack: Yeah, so the answers six, now can you change on of these numbers, can you decrease one of these numbers to make this answer to come out to be all zeros.

Shannon: I don’t know

Chris: You can take four down to a two.

Jack: Ok, Sh can you think of a different one to make a different way to make this to all zeros at the bottom by decreasing one of these numbers? Now don’t look at that, just look at this.

Shannon: But I’m, I need this.

Jack: No you don’t, you don’t…

Shannon: shut up Chris

Chris: You don’t want to do any of that.

Shannon: One to three.

Jack: Do what?

Shannon: I don’t know (inaudible)

Jack: Which things, which things need to be changed to make these to zeros.

Chris: All the columns have three.

Shannon: Yeah
**Jack:** Which ones?

**Shannon:** that, and that and that

**Jack:** Now see, I want you to decrease it. So if you change this to a one or a zero it’ll, if you change it to a zero that’ll make it come out, if you change this one to a zero that will make that one come out to be zero, but it won’t, this one won’t change

**Shannon:** So it’ll be either this one or this one
Figure 597. T4-11:20

Chris: No

Jack: Which ones?

Shannon: This one.

Chris: That one or…

Shannon: This one. That one

Figure 598. T4-11:30

Jack: Ok, so how would you change this one to make it come out to be zero at the bottom?

Shannon: Which one, the four?
Jack: The five.

Shannon: The five? I don’t think I could, can I?

Jack: Well let’s do this one

Shannon: I don’t think I could though because when I decrease, do I decrease by one or by whatever I want?

Jack: You can decrease by whatever you want?

Shannon: Oh.

Jack: So tell me what the binary would be here to change this, the new binary?

Shannon:
Figure 600. T4-11:57

Jack: So, the play is to change this to a one

Figure 601. T4-11:59

Shannon: Yes.

Jack: And then this is the zero position
Ok, but I’m going to give you one where you can’t look in your notes.

**Chris:** This is easier.

**Jack:** Lets take some big piles, a pile of twelve, a pile seven, a pile of six and a pile of five. Heres twelve, here’s seven, here’s six, here’s five

ok (inaudible). Lets see what it is, its not the sum, its just.

**Shannon:** What will this be called?

**Chris:** Bad binary addition.
Jack: This is what you can call forgetful addition, so when you forget. So now, when, this is not a zero position, but you can make it a zero position by changing what? By changing that to a zero.

Chris: The twelve would be changed to a…

Jack: No you change this to a zero, so what’s this new number?

Chris: Yeah, that’s what I just said …

Shannon: It got changed to a four.

Jack:

Yeah, so the winning play is to change this to a four.

Shannon: don’t second guess me.

Chris: Yeah.

Jack: What if this was, let’s see what if this was, let’s do this one twelve and let’s make this one thirteen which will be that

![Image of a person writing on a chalkboard with a mathematical problem]

Figure 604. T4-13:44

Shannon: So that’ll be
Jack: Is it zero now already?

Shannon: Yeah.

Jack: Ok, now I don’t want to do that.

Shannon: Don’t make it easy on me.

Jack: Let’s change this to, oh, this was six, so this was a one. But this what it was, then I changed it to a zero. Ok, so what if we start with this be a nine, so this will be 101

Can you find the lead play now?
**Shannon:**

![Image of handwritten notes]

*Figure 607. T4-14:32*

**Jack:** You got to change, you got to change all three of these down here without changing the right hand column.

**Chris:** You have two choices.

**Shannon:** Why can’t I just…Does that work?

**Jack:** Yeah, now the nine changes to?

**Shannon:** Six. Oh, that’s seven, no.

**Jack**

Yeah seven, ok that’s what it was
Chris can you find a different winning play?

Chris: You can take the twelve down to a two or you can take the thirteen down to a three.

Jack: Ok, so this next theorem coming in class, you can, you don’t have to memorize all that, those random numbers, you can figure out how to win a nim game by doing this forgetful addition. There is a couple of theorems involving that. ok, so we have to some stuff for those people who like nims.

Chris: Do you still like nims?

Shannon: I don’t dislike it as much as a like it up and down and star and everything.

Jack: Ok, so, you guys are good, unless you have questions for me
APPENDIX O

SHANNON, THIRD INTERVIEW
Third one-on-one interview with Shannon October 15

**Shannon**: We started out, um, two people were playing and we could just go that

![Figure 609. T4-16:05](image1)

and their best move was the same as the other person’s best move

![Figure 610. T4-16:10](image2)

so you can’t give the value of um, positive or negative because they both have the same moves so its pretty much just star or zero?
Yaser: Right.

Shannon: And you had to figure out the best move to do from here or, um, after you figured that out you had to figure out if you were going to go first what would you want n to be so that you could win?

Yaser: Oh, ok and then how did you figure out what n is

Shannon: There’s a theorem, like after we talked about it a little bit, you want it to be the smallest number that’s not already in here, so here to turn this into…

Yaser: So sort of like, oh what are they called, the simplicity theorem?

Shannon: Kind of. You would want this to be star three
and then it would be a star game and the first person would win.

**Yaser:** Ok. I think I got it. So what you do is you want to give a value to this and the value is…

**Shannon:** This, this would be star.

**Yaser:** The value of this game is a star.

**Shannon:** Hm-hmm.

**Yaser:** And a star means whoever starts wins?

**Shannon:** Wins. Yeah. Like the value of this, just this, is zero because it’s the copycat, like if one person goes, if the first person goes and he picks like star one, then the second person goes and its zero so then the first player loses.

**Yaser:** Right.

**Shannon:** So this is a zero no matter what and then you have to figure out what you have to add on to it to make it not be zero anymore, cause if you added on like a one, um, and say the first player, um, took the one and kept this the same without rewriting everything
it would just be this and it would be a zero again, then the second player…

Yaser: And the second player…

Shannon: First player.

Yaser: Second player will have a zero game?

Shannon: Yeah. And then if the first person takes like one of these that’s zero

so the second player loses.

Yaser: Is it the second player loses or the first player? Cause the second…
Shannon: This…

Yaser: Yeah the second loses.

Shannon: This should still be a zero game.

Yaser: The second loses because here the first is done, the second will take this one, the second players turn, nothing to do so loses.

Shannon: Yeah, so you have to figure out, ’cause that means it’s a zero game and then you have to figure out how to make it a star game and there’s only n right end and in this case it’s three

Yaser: So the value of this game is three?

Shannon: The value of this game with the three for the n is star. Anything else is zero. Yes.

Yaser: Ok, I’m going to try to understand this. Let’s say you have, let’s not put stars.

Shannon: Ok.

Yaser: Put one two four and lets take one two four. Ok same situation, um, I meant three. One two three. Ok, now what’s the value of this one.

Shannon: Zero.

Yaser: So this is zero. That means whoever starts loses. You start, you lose. Ok. Now what is that
Figure 616. T4-20:19

**Shannon:** Plus another game.

**Yaser:** Plus another game?

**Shannon:** Mm-hmm.

**Yaser:** So this is hackenbush

**Shannon:** No this is nims

**Yaser:** Oh, another pile.

**Shannon:** Yeah, just a pile.

**Yaser:** Ok, so, you want to pick a pile so you change this from a zero game to…

**Shannon:** A star game.

**Yaser:** A star game which means whoever starts wins. Ok and the idea is what that number should be?

**Shannon:** Yes.

**Yaser:** So if you were to design a game, if you were to design a game, you want to design it in such a way that you know the outcome in advance? Is that the point?

**Shannon:** Yes.

**Yaser:** Ok, so if you follow the same thing, what should go here? The smallest number that is not here?

**Shannon:** it would be four.

**Yaser:** So it’d be four
Figure 617. T4-21:12

So you put four, ok, so, um, so the game, actually this translates into one two three four?

Shannon: No, because like if you go first, we'll do all your options, you can either have one plus four
and then I can change that to a zero game by doing one plus one and then you would lose.

_Yaser_: Ok.

_Shannon_: If you did two plus four I can change that to a zero game and you lose.

If you took three plus four I can change that to a zero game and you lose. And if you took this, like let’s say you took the whole thing away, um…

_Yaser_: Ok, the four (inaudible)

_Shannon_: You took the four, then I…
Yaser: Now it’s the opposite, now you’re starting so you have those options and then you lose.

Shannon: Correct.

Yaser: So on this one, so ok by adding the four I’m winning, I’m starting, I’m winning. Without the four I’m starting, I’m… So from this point on, these are the options for you.

Shannon: Well these, I can either do one or two or three and no matter what I pick, you can change that to a zero game, but wiping out the whole pile and then I would go next and I would lose.

Yaser: Oh, but you don’t have to wipe the whole pile, right?

Shannon: No, but it would be silly if you didn’t, because if I changed it to a three, let’s just say I changed it to a three and you changed it to a two, then I would wipe out the whole pile.

Yaser: Ok, I see.

Shannon: So, that’s pretty much what we did today.

Yaser: So if he were to play the game with you, if you were going to go to the office, how would he formulate the question? He would say, uh…

Shannon: Well he really wouldn’t be able to because he already told us what number to pick, like if it was one two three five six seven eight nine ten, we’d know that we’d have to pick four, so there’s really no…

Yaser: Oh, so and those don’t have to be identical

Figure 620. T4-23:58
Shannon: No, these two have to identical

Yaser: This is one game?

Shannon: Yes.

Yaser: Because this is not our game the game is, oh I see, the game is one two three and what we’re doing is right the first left move, the first guy’s best moves, the second guy’s best moves, is that what it is?

Shannon: Well like, we started out talking about if it like in Nims, no matter who goes first it can’t be positive or negative because they can do the same thing. Where in Hackenbush, red can’t take blue sticks and blues can’t take red sticks.

Yaser: Yes.

Shannon: So, we kind of change this…

Yaser: Oh, so the notation now changes on us too?

Shannon: Yeah, like this when you find, this is the simplest number notation, right?

Yaser: Ok, yeah.

Shannon: This is the simplest number.

Yaser: Right.

Shannon: And either, we don’t know, its kind of like choosing between up and something else before we knew what the values were. We don’t know if this is the best move
Figure 622. T4-25:03

if this is the best move

Figure 623. T4-25:04

or if this is the best move

**Yaser**: Ok.

**Shannon**: So this can actually be a game full of a hundred numbers and we’ve narrowed it down to these three could be the best move for left and also these three could be the best move for right. So when you pick, cause like, if you have.

**Yaser**: Oh I see and…
Shannon: If you have zero and one and it’s left’s turn, left is going to zero.

Yaser: Yeah.

Shannon: So same thing here. He’s going to choose one of these. It’s like finding the simplest.

Yaser: Right, and that’s, that’s why it’s like that (inaudible) because whoever starts should pick the best move and it’s usually there’s only one best move or not one, the best move. So…

Shannon: Here let’s just, let’s just play this game. I’m not sure if I’m explaining it right. If we just…

Yaser: Let me tell you what I understood.

Shannon: Okay.

Yaser: Well this here, we don’t have left and right, right? Okay. Start with this game. Let’s say we start with this game.

Shannon: That’s not really the game though.

Yaser: Not the game but let’s just start with a big, big game. We simplify it to the best, if you start you’re best thing to do is to simplify it to one, two, three, or one through four, right? But because whoever starts can do it, that’s why you put identical on both sides right. Is it, am I making sense?

Shannon: We don’t know the original game.

Yaser: We don’t know the original game but we know that the original game simplifies for this.

Shannon: No.

Yaser: No?

Shannon: We know that from the original game one of these is our best move.

Yaser: Okay. What (inaudible).

Shannon: Because that’s the definition. That’s why we put them here and if we can’t choose, like, that’s sometimes why we have zero comma the star something because we don’t know which one is the best move.

Yaser: I see, okay.

Shannon: It’s the same thing here. One of these is the best move.

Yaser: All right.
**Shannon**: So, if you go first you can choose any one of these so we end up having three best moves.

**Yaser**: I see, okay.

**Shannon**: If you go first you can choose the one, you can choose the two, you can choose the four. Kind of like here. You either have to choose the zero or the star and the other one disappears.

**Yaser**: Okay.

**Shannon**: And then I can go and I’m going to turn it into a zero games no matter what

![Image of a blackboard with numbers and arrows]

*Figure 624. T4-27:19*

**Yaser**: Because you’re going to take the whole pile.

**Shannon**: Right.

**Yaser**: All right.

**Shannon**: So, this is a zero game because no matter what you do, no matter what side you’re on, no matter what you do the second person can just wipe the rest of them out. (Inaudible) no matter what you’re just going to have one pile left and that’s bad.

**Yaser**: So you, no matter what you do you’re going to lose.

**Shannon**: Yeah and that’s why.

**Yaser**: And these are your options, these are your best options?

**Shannon**: Yes.

**Yaser**: So, other options will be even worse.
Shannon: Yes.

Yaser: Okay.

Shannon: So, and he said this game is no fun. So to make it fun we’re going to add another game to it.

Yaser: A nim game.

Shannon: Yeah, another nim game and assuming you go first, what is the best number that you would want there and in this case it would be three.

Yaser: I see and then this way if you go first, you’re going to win.

Shannon: Yes.

Yaser: Whether playing the pile or playing whatever game that has those choices.

Shannon: Um-hmm.

Yaser: I see. Okay. Well, now I understand it.

Shannon: It’s kind of complicated at first.

Yaser: Yeah. Now, have you ever thought of why this binary thing works with nims?

Shannon: Not really. It’s really neat though.

Yaser: I think it’s neat but I’m trying to think why does it work and how is it, you know, how is it related to this. Have you thought of that? I mean, I’m not expecting you to give me answer because but did it occur to you…

Shannon: I’m sure it’s not that hard to see
Kind of, like, if you do this game and the first player changes it to one, two, two, that’s going to do that

and then it’s a star game cause you’re going to have only one in last column. Like I see how it corresponds. I can’t tell you exactly why but I see how it corresponds and then the next player would do this.
and then that would let that out and then it, you know, I see how it corresponds. I wouldn’t be able to prove it but I see how each move corresponds with changing it, changing this number down here and then copycat, like, whatever the first person does here or here, the other person can do here or here too cause it’s the same number.

**Yaser:** Yeah, so sort of like adding without carrying is like… How about this, I think, I think, I have a suggestion, no not a suggestion but maybe an idea that makes it clearer. We don’t carry when we add because we’re actually, the sequence is removing things right? But the sequence, like, every time we remove, so, the numbers get close, get smaller, not larger. So when these…

**Shannon:** So in that case would we just subtract if it weren’t (inaudible)?

**Yaser:** Yeah, why wouldn’t you? Yeah, why wouldn’t you subtract? But we’re not, aah, actually subtracting the numbers. We’re just reducing each one. Does this make sense? I’m not (inaudible).

**Shannon:**

All right, before I lose it.

**Yaser:** Okay.

**Shannon:**
Shoot, I think I lost it. Oh, yeah I did. You said reducing and that clicked something into my head and I
don’t remember what it was, what it was now. Nope it’s gone.

Yaser: Nope?

Shannon: It’s gone.

Yaser: It’s gone. Well think about it because I’m interested too.

*She: He gave us some theorems today that relate (inaudible).

Yaser: (Inaudible) going to make, we need to make copies of them. Oh these are like, numbered on…

Shannon: That’s pretty much and this is on there. So you put the numbers on the column, blah, blah, blah,

(inaudible), blah, blah, blah, theorem.

Yaser: Some part of the number has the none zero (inaudible) sum. Then one of the numbers can be
decreased to give a (inaudible) sum of zero.

Shannon: Which is the same thing as saying… If this isn’t… And the same thing. If this is zero then you
can reduce it to make it not zero. And, there’s no proof on here. Never mind.

Yaser: No proofs and then that’s what we’re currently doing. Did you do the frog thing? Did you play

with him the frog thing? Or not yet?

Shannon: No we do that in the office on Friday (inaudible).

Yaser: Oh, you’re doing that. Okay.
Shannon: But I did, I did fill this out so I know the values of all the games and all I have to do is…

Yaser: Just present it to them.

Shannon: This actually comes out as three stars, two zeros, and two ups which reduces to up up star.

Yaser: Okay.

Shannon: Cause we, we were, me and Chris worked that out by ourselves but it reduces to up up star.

Yaser: the whole game?

Shannon: Yeah.

Yaser: Which is a zero up?

Shannon: Yeah.

Yaser: And zero up means? Up is the larger, larger than any…

Shannon: Zero. If we play.

Yaser: Up is smaller than any positive number? Is that what we did…

Shannon: Right. If right goes first, up is that and then left goes, which leaves a zero. So, right moves next. So, that means we want to be left which makes sense because in my head, like, up up star, up plus up plus was starred. If star is kind of zero, then if you add up to it twice it's going to be positive right? That makes sense to me.

Yaser: Yeah, cause up is the, it’s, up is positive isn’t it?

Shannon: Yep.

Yaser: It’s the smallest.

Shannon: It’s the smallest positive.

Yaser: Positive. So, up…

Shannon: Up is always going to win this. So we want to be on Friday. Can you remind me that?

Yaser: Okay. There’s another thing I wanted to ask you about. Oh, and then I’ll let you go. Ask Shawna, it says here ask Shawna, what do I want to ask you? Okay, one, two, three, four, five, seven.

Shannon: is this one a nim game?
**Yaser:** It’s a nim game, yeah. One, two, three, four, five, seven. Okay, and you want to, you want to find the value for this one.

**Shannon:**

*Figure 629. T4-36:19*

**Yaser:** So you don’t need to know which one’s zero. So you simplify it that way. Okay.

**Shannon:**

*Figure 630. T4-36:26*

The value is, now is that… Is six the value?

**Yaser:** The value is four and two yes…
Shannon: Is this, does that mean the value? Okay, so value equals six and then, so that means, whoever goes first can win and their best move would be to change that to a one. So, one, two, three, four, five, one.

Figure 631. T4-37.06

Yaser: Okay.

Shannon: And that’s a zero (inaudible).

Yaser: Well, the… All right, somebody did this on, in, on the board. Is that the one you did on the board and what were they showing?

Shannon: I did this.

Yaser: Okay, this is how it was done on the board, now, so I don’t know why I said “ask”.

Shannon: Okay.

Yaser: (Inaudible) more notes. It was reduced to one, two, five.

Shannon:
It was reduced to one, two, five because the three, four and seven cancel.

**Yaser**: Three, four, seven is a zero.

**Shannon**: Is it?

**Yaser**: Okay.

**Shannon**: So then I would get one, two, five, into a zero game. We can do that and you can do that by changing the five to the three
and that would be another good move cause you could change this to a zero and this to a one. There’s a couple of good moves.

**Yaser:** Okay. How do you feel about proving things now? Do you feel more confident doing these than before?

**Shannon:** Yeah but I still don’t see how we can apply it to things but, like, I can prove stuff using simple numbers now and see that a lot better than I did before.

**Yaser:** And those aren’t, these aren’t your mind proofs.

**Shannon:** Like, I know they’re proving something but they’re not really proving anything in real life.

**Yaser:** Okay.

**Shannon:** Like, I know they’re proofs, they’re just not the proofs that I’m used to.

**Yaser:** Right. And the reason why you know they’re proofs is because?

**Shannon:** Cause we’re given the, what is that word? We’re given the idea and then you prove it to be true or not true and…

**Yaser:** And that is based on?

**Shannon:** Previous information.

**Yaser:** Previous information that we either proved or defined?

**Shannon:** Um-hmm.
Office Visit Ryan and Zack 11-7

**Zack:** So that makes sense.

**Jack:** Looks like you changed the four to a two.

**Zack:** Yes I did. This is two squared. That’s what, that’s what I was just trying to (inaudible).

**Jack:** Well, but you don’t want to say just two to the n, you want it to say two the n by two to the, two to the n plus one

![Figure 634. T4-39-54](image)

by two to the n plus one. That’s what it is.

**Zack:** Okay.

**Jack:** So you have this many squares all together.

**Zack:** Okay. So, you have that many squares all together and you have
Jack: Then you can divide it into four like you said.

Zack: Divide it into four

Ryan: two to the n

Zack: so you have four times two to the ns, right? Four two to the ns plus ten

Ryan: Yeah.

Zack: No! We just have four two to the ns.

Jack: Yeah, right.
Zack: That’s it. So…

Jack: Okay, so you divided it into four equal pieces and they’re all two to the n by two to the n, and then what?

Ryan: And then you keep doing each piece.

Jack: Well, no. Well, what do you do with that one? Now that you’ve got it divided into four pieces, what do you do with that, with those four pieces?

Zack: You do,

Ryan: you take each piece by itself and divide it into two to the n minus one. You take, as you, you broke these down

![Image of a chalkboard with a grid drawn on it.]

Figure 637. T4-41:08

piece down so then you have.

Jack: Yeah, but there’s a step you did in between.

Zack: Now this one is defined
Figure 638. T4-41:18

**Jack:** You didn’t, you didn’t tell me about something that you did.

**Zack:** we took the thing in the middle.

**Jack:** You took that thing in the middle right?

**Ryan:** And say you put the thing in the middle for each square

Figure 639. T4-41:38

**Jack:** Okay. Now, instead of saying it that way. Before you, before you do this you say assume what?

**Zack:** Assume true for n then…

**Jack:** For n, okay, meaning what?
Ryan: Two to the n,

Zack: two to the n, meaning that we know that two to the n by two to the n…

Ryan: Can be.

Zack: Can be cut up into triod.

Jack: Yeah. Assume we know how to do

Ryan: You just take.

Jack: Two to the n by two to the n minus one squared.

Zack: I’m on to the whole thought over here (inaudible) makes any sense. This is, can we just break this up into

Figure 640. T4-42:34

Jack: Break it up, you take the big one too. Two to the n plus one by two to the n plus one…

Zack: Which is the same thing as two to the n times two
I don’t know if this works but one, two, two to the $n$

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**Figure 641. T4-42:45**

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**Figure 642. T4-42:51**

**Jack:** Yeah.

**Zack:** And then you can.

**Ryan:** Make sure it’s four to the $n$ isn’t it?

**Jack:** Right.

**Zack:** But I don’t know if that helps us still
Jack: Well, the, I don’t think the equation helps any but what you said helps. You break it into four pieces and each piece is two to the n by two to the n and you assume you know how to do what?

Ryan: Two to the n,

Zack: we know how to do two to the n.

Jack: With one square missing.

Zack: and Ryan: Right.

Jack: Okay. So what do you do then?

Zack: Two to the n minus one.

Jack: All right. So, so then what do you do with the four pieces?

Ryan: Well, on each four pieces, you have one square missing.

Jack: How do you get that? But you gotta do something to get that?

Ryan: Cause you take the middle part to the
and you have to do this.

**Jack**: Okay, why do you do that? That’s the right thing to do but why?

**Zack**: Because that makes, that takes one out of every one of your…

**Ryan**: Right. You already have one missing from one square so you want to make the other three squares have one missing.

**Jack**: Okay. So you’ve got one solved. You got one more thing to tell me is what about that three you take out?

**Zack**: So…

**Jack**: Why didn’t you pick that shape? That sort of gives it away. Why did you pick three like that? Why didn’t you pick three in a row?

**Zack**: Pick this three
Well, because that takes one out of every one of your squares.

**Jack:** Okay, suppose you could do it with three in a row. Like, take one. This shape. You can do it like that, why wouldn’t you want to? Why would you want to do this shape instead of?

**Ryan:** Because you already have one square missing from one of the squares.

**Jack:** Okay.

**Zack:** And there’s no way you can take one square on the other ones into (inaudible) for ones you already know.

**Jack:** Okay. Suppose I said well, I want to take this one, this one, this one, this one out. I took out three squares and made it into a, the right kind of shape. Why wouldn’t you do that?

**Zack:** Well, cause then you can’t find it from the stuff you already know. Well, I got this using a corner.

**Jack:** Okay, the corner. Let’s make it a one to four corners. Suppose you, this one is missing over already and I say, well, take those other three corners out and then you can cover those.

**Zack:** You could cover them up yeah.

**Jack:** Why wouldn’t, why is that? Why would this be a bad way to do it?

**Ryan:** Cause it makes… I mean, I don’t think you would be able to do it then.

**Jack:** Why? You’re right. Let’s look at this one. One corner down here where, look at this part here. This one has been removed.
Ryan: Right.

Jack: So you can do what you said put one of these things here.

Ryan: But then it doesn’t work. Then you won’t be able to…

Zack: Yeah, flip that over.

Ryan: You wouldn’t be able to…

Jack: Oh yeah, would have to flip it over right? Go through it like that instead of all the, okay, and then let’s see, you could one there and then one…

Ryan: Would have to be this way.

Zack: Yeah. (Inaudible) what you did there is you split by twos.

Jack: Yeah, okay so if we do it like this and then like this.

Ryan: Right, right.

Jack: And like that.

Ryan: Right.

Jack: So, why isn’t that any good?

Zack: It is, the only problem with it is you can’t start with anything bigger than and then split it up.

Ryan: Right, as, if you expand it more you’re not, you wouldn’t be able to, you couldn’t do it that way if you expanded farther up. If you would make it like higher than.

Jack: Okay, why?

Ryan: Cause you wouldn’t be left with four squares of two to the n. You wouldn’t be able to get four squares, two to the n.

Jack: If I had it, see this is eight by eight. So if I had a sixteen by sixteen I could just divide it into four pieces?

Zack: But we didn’t do it this way cause still we had to use right triangles.

Jack: Right, yeah. Use these. Well that’s what I did here right? I used.

Ryan: Yeah, but you split them… we started by, you did one here, one here, and one here and…

Jack: Right.
Zack: And we couldn’t do that.

Jack: Why not? Why not?

Zack: Cause you had to use a right triod and there was only one square missing.

Jack: Okay. Well, see, what we’re talking about is we have this eight by eight. We already know how to
do four by four.

Ryan: Right.

Jack: So you take eight by eight with one, with this one missing.

Ryan: Right.

Jack: Divide it into four pieces, with smaller size and take out the four corners and then cover everything
that’s left.

Zack: But then you still got four corners and we can only take off one.

Jack: Yeah, right, right. So, why, so then, now, can you tell me why, so, could I do it like, what if I did it
like this? What if I did, instead of taking out that one I took out these two? That’s still bad cause why?

Zack: Well, because you still, you didn’t use the trimino. You took out four squares and you can only take
out one.

Jack: Okay. Well, take, well. See, that’s what you said to do. You said there’s one taken out and you take
out three of the middle. Okay, well (inaudible).

Zack: Well, you covered three in the middle.

Ryan: You covered three, you know.

Jack: Cover three in the middle.

Ryan: Right.

Jack: Okay, that’s it. So you got to pick out three of them to do so that they can cover, be covered by one
of the things that you’re using.

Ryan: Right.

Jack: If I take them out of the corners like that I can’t use it, a trimino to cover this plus this plus this.

Ryan: Right.
Jack: If they’re not even connected to each other. And if I did the same thing out here, if I took out these two and this one.

Ryan: They’re not (inaudible).

Jack: Yeah, okay. Okay, so if you have two to n by two to n with one corner missing, you make it two to n by two to n four pieces

![Image](image1.png)

Figure 646. T4-50:46

and one of them will, let’s say this top one

![Image](image2.png)

Figure 647. T4-50:52

has the original corner missing.
Ryan: Right.

Jack:

Figure 648. T4-50:58

Okay. So, if you know how, if you already know how to do it for a two n by two n, you can cover this

Figure 649. T4-51:05

Ryan: Right and so you want to.

Jack: (Inaudible) three, you do what?

Ryan: Want to cover up.

Zack: That’s where you, that’s where you put your trimino.
Jack: Yeah, you put in.

Ryan: you wanna cover these three.

Zack: Cover one up.

Jack: You put in a triomino that takes the one corner out of each one of these

Ryan: Right.

Jack: There has to be three of them that can be covered by a trimino and so that’s what you do. Okay, so if you know how to do it for two to n by two to n, and to do the two n plus one problem, you divide it into…

Ryan: Into four pieces.

Jack: Four of this kind of problem.

Ryan: Right.

Jack: And the trick is to put in one trimino that…

Ryan: Makes it.

Jack: Will cover original problems.

Zack: Okay.

Jack: And so that’s the induction. You take this, you take one of this size
Figure 651. T4-51:56

and reduce it to four problems like this and the trick is to put one thing in the middle like that to make it four problems like this. Okay?

Ryan: Alright.

Jack: So, you guys really had it. Just…

Zack: It’s just hard when, (inaudible) write it out.

Jack: Yeah.

Zack: I did number two. It’s pretty easy and fifteen.

Jack: Okay. Let’s do fifteen.

Zack: Okay. Alright. This is the one geometry n-2 times a hundred and eighty. Am I right? Or the best case, we start with n equals three. Cause n equals two isn’t a polygon. It’s just a line
Jack: Yeah.

Zack: You know, two vertices. That gives you triangle.

Jack: You know how to do it for a triangle?

Zack: How to prove it?

Jack: Yeah.

Zack: No. I didn’t think it that we had to.

Ryan: It’s three minus two which is, it’s one and you already have the one eighty.

Jack: Why? But why is one eighty?

Zack: I think he meant how to prove that it?

Jack: Why is it one eighty okay?

Ryan: I’ve done it before.

Jack: Only the, yeah, that one. Like Greek letters for angles. Okay, so I’m going to draw a line parallel to this, through this vertex parallel to opposite side.

Ryan: Oh, okay.

Jack: Okay, now fill in stuff to prove that, that alpha plus beta plus gamma is one eighty.

Zack: Okay. All we have to use is that this.

Jack: Right.
**Zack:** The same as alpha.

**Jack:** Right. That’s alpha.

**Zack:** That’s …

**Ryan:** And uh, that is a straight line.

**Jack:** And that’s not alpha.

**Zack:** That’s …

**Ryan:** Gamma.

**Jack:** Gamma. Okay, so there you go.

**Zack:** Alpha plus beta plus gamma equals one eighty.

**Jack:** Yeah.

**Ryan:** That was easy. That was super easy.

**Zack:** I remember doing that before but.

**Jack:** Okay, so that’s why it’s true for n = 3, the base case.

**Zack:**

That’s just one times a hundred and eighty is one eighty for the induction. Assume it is true for n. Show for n plus one. Prove for V sub n plus one. Okay, so if you have a vertex here V sub one down to vertex V sub n

*Figure 653. T4-54.50*
You have some vertex up here $V_{n-1}$

Figure 654. T4-54:57

have whatever in between these, doesn’t matter. So this is what you have

Figure 655. T4-55:02

if you add a point vertex, well, that should just be $V_n$. The way I’m proving it
Have a vertex \( V_{n+1} \). That just makes a triangle

*Figure 656. T4-55:11*

*Figure 657. T4-55:19*

**Jack:** Okay.

**Zack:** Right? So by hypothesis we know that the angles up to \( v_n \) probably have two \( v_n \)'s that's why this was a minus one. By the hypothesis we know that the polygon up to \( V \), up to \( n \) sums to \( n \) minus two times a hundred and eighty. Right?

**Jack:** Okay.

**Zack:** And what we add it on was a triangle with \( n \) plus first vertice.
**Jack:** Okay.

**Zack:** So we just added on a hundred and eighty degree.

**Jack:** Right. Now we can factor out the one eighty and a hundred and eighty degrees times n minus two plus one but that’s the same thing as…

**Zack:** I can factor out one eighty ...Hundred and eighty degrees times n plus one. That’s two.

**Jack:** Yeah.

**Zack:** but that’s the same thing. A hundred and eighty degrees plus one

**Jack:** yeah

**Zack:** that’s why we, I never put that we, off to the side somewhere where that’s what we need.

**Ryan:** Yeah. Minus two.

**Zack:** I had two minus (inaudible) yeah.

**Jack:** Okay. What other ones have you guys worked on?

**Zack:** Do you want me to do number two right now? I can do number two. I’m doing that in class tomorrow. I think I need a (inaudible).

**Ryan:** I’ve gotten part of the (inaudible) number, I think it was three or four.

**Jack:** Number four, okay.

**Ryan:** And I don’t really know where to go from. I got most of it and I’m stuck at a point.
**Jack:** Okay. Well show us what you got

**Ryan:** So it’s

… and I, I was trying to solve it for this and I just thought what if I moved this to this side?

**Jack:** Okay.

**Ryan:** So, I, and then this thing and two plus one.

**Jack:** Okay.

**Ryan:** We assume, also the base case first.

**Jack:** No, let’s skip the base case. That will be (inaudible). The base case is n equal to what?

**Ryan:** N equal to, it’d be f of two, f of zero minus f of one squared minus C(inaudible)

**Jack:** I didn’t know we put f sub zero on the sheet.

**Ryan:** But it’s one minus one so it’d be f sub zero.

**Jack:** Okay, what is f sub zero?

**Ryan:** Zero.

**Jack:** Okay, yeah, use that, okay, well, that’s good.

**Ryan:** So, we assume n equal to K and want to prove

**Jack:** Where?

**Ryan:** N equals K plus one so I have one plus two plus one plus
**Figure 660. T4-58:57**

**Jack:** Just put p plus two. Save space.

**Ryan:** All right. FK equals FK plus one, plus (inaudible) K plus one.

**Jack:** Okay, now did (inaudible) show.

**Ryan:** Right. I’m trying, I then want to get rid of F of K plus two.

**Jack:** Okay.

**Ryan:** And you know F of K plus two is equal to F of K plus F of K plus one.

**Jack:** Okay, all right. So you’re going to put that in there.

**Ryan:** Right. So FK plus F of K plus one. F of K is equal to.

**Jack:** That, that’s still what you’re trying to prove.

**Ryan:** Right.

**Jack:** Trying to get those two things equal.

**Ryan:** And then so, factor this is so you have F squared of K plus F of K F of K plus one and then for, if, when I, if you subtract,

**Jack:** I forgot about this one.

**Ryan:** FK plus one, FK plus FK minus one is equal to F of K plus minus one to the K

**Jack:** That one is squared.
Ryan: If you subtract this one from both sides you get $F(k)^2$ so that is why we will substitute in for $F(k)^2$.

Jack: Ok.

Ryan: $F(k) + 1 + F(k) - 1 - (-1)^{k+1} + F(k) + F(k)^2$ and then you, I added this to the other side,

Jack: You added what to the other side?

Ryan: the negative one to the $k$

Jack: Ok.

Ryan: So I got, I also, so when this is the other side you have a factor of $F(k)^2$, so I got this and I’m stuck there.

Jack: Well replace some parentheses by something.

Ryan: Is this equal to $F(k)^2$.

Jack: Yeah.

Ryan: So you have $F(k)^2$ plus one is equal to $F(k)^2$, they you don’t have this term

Jack: Why

Ryan: (Looking at the problem, thinking)

Jack: That’s sort of a trick, you can find what it’s equal to pretty easily.

Ryan: That’s going to be zero.

Jack: Yeah you’re right, why?

Ryan: you have…. 

Zack: oh, ok. Because this is always one term over and it would be negative.

Jack: One of those has to be plus one and the other one minus one because one of the exponents is even…

Ryan: And one of them is odd. So, you just have $F(k)^2$, $k$ plus one is equal to $f$ squared $k$ plus one.

Jack: Yeah. That’s really good.

Ryan: Yeah.
Jack: I want you to, I want to do it at class on Monday and but I would like to see you rewrite it because what I don’t like is to start with the equation you are trying to prove you can’t keep going down on both sides and you sort of mixed in the way. Maybe a better way to do it would be to take this minus and this minus and then come out to be zero instead, and by just doing the same steps you did. But write down this and then do the first step to change it to would be like…

Ryan: Minus

Jack: Like that and then do the next step and then make it all zero at the bottom so that you start with something that you

Ryan: Okay, so your not equally something your equally zero

Jack: Yea, start off with the left side and go through all your steps and then finally get down you get to this, and you’ll say oh that equals zero, which is what you’re trying to prove.

Ryan: Okay

Jack: So you start with one side while you take the equation and rewrite it so that everything’s on one side and you prove that it is zero, then write it down and go through your steps and get it equal to zero. So that’s real good, okay so very good.

Ryan: Do you want to do another one?

Zack: I was just looking at seven

Jack: Seven which is that one

Zack: That’s the dominoes one. Do you want that Ryan?

Ryan: I wrote it in my notes. I just rewrite it equal to zero.

Zack: Alright, Um, We found a formula to D to the zero right? I am gonna define recursively D sub n equals D sub n minus one plus D sub n minus two

Jack: Does that look familiar

Ryan: and Zack It’s the Fibonacci

Jack: So what your trying to explain is why is it like that, why does DN equal those two terms

Zack: I have a feeling it is related to Fibonacci sequence. It shouldn’t be that far
Zack: when I, when I, I don’t think this is right

Two by N equals D to the N

Jack: Yeah. That’s the board and DN is a number of ways to put those things on that board.

Zack: Right. So assume this for N, is that right?

Jack: Okay then what about N plus one?

Zack: And then two times N plus one equals D to the end plus one. And the only way I can do this though which I don’t know if that makes any sense or not. Say this one …… yeah but I don’t know where to go from here. The only way I can’t do it is to use this which we don’t know. That is what we are trying to prove.

Jack: Why do you think that thing on top is true, where did you get that?

Zack: We got that from adding the two numbers before and finding that they are…

Jack: Okay but how did you decide it was that one on the top was true, how did you arrive at that?

Ryan: Because we just kept going on like we went to D and five and D and 6

Jack: So you did the pictures?

Ryan: Yea

Jack: So you saw the pattern there

Ryan: Right

Jack: That’s the key to figure out why that’s always true.

Figure 661. T4-1:08:09
Well this is supposed to be two by n

**Ryan:** Okay

**Jack:** Well if your going to cover this you could put one of those dominos right here cover this up

![Image of a blackboard with dominoes](image.png)

*Figure 662. T4-1:08:20*

How many ways are there to cover what is left?

**Zack:** Uh,

**Ryan:** N minus three and N minus two. D N…

**Jack:** Nah N minus one. N minus one.

**Jack:** What’s left is two by…

**Zack:** n minus one.

**Jack:** That’s the induction thing is to take the 2 by n born and put a domino right there, the reason being there is then you get the board of the next longer side and you are assuming what that is. So, continue in the train of thought. Ok?
Shannon: You have to do the basis before you can do the induction.

Chris: Yeah, you have to prove n…

Shannon: Wrote this down.

Chris: Are you sure you want the parentheses?

Shannon: Oh, it’s divided by.

Chris: That’s for n equal one. You get one equals one.

Shannon: Did I write this one right?

Chris: Yes, yes.

Shannon: Oh, let me look at number one. Ok, let me see.

Chris: where is the two put back.

Shannon: (rights on the board)

Chris: this is even, and now it’s odd.

Shannon: Yes, yes, here we go.

Chris: Proof that n equals one. You put one in here you get one, you put one in here you get one. It’s the base.

Shannon: It’s one. This is one. (Chris erasing the board) What are you doing? Two times one minus one is equal to one.

Chris: one equals one squared. It’s true.

Shannon: Ok, thank you. I’m going to start with this. Is it ok? Two plus three plus five minus one plus 2n minus one plus 2N plus one

Chris: You’re supposed to make a claim first.

Shannon: I am.
Chris: That’s not the claim.

Shannon: Why isn’t it?

Chris: Isn’t this our claim?

Figure 664. T4-1:12

Shannon: this is N squared plus 2N plus one equals N plus one squared

Figure 665. T4-1:13

Shannon: a we just did the first one.

Jack: The first one is which one?

Shannon: It’s number two.
Jack: Ok, so that’s good. What else have you been working on?

Chris: Cute.

Shannon: Lots of stuff. What do you want us to start with? We worked on six of them.

Jack: Let’s do them in order then.

Chris: Let’s do the catch one we just tried.

Shannon: Well, he said let’s go in order.

Chris: OK.

Shannon: We worked on six of them and we really didn’t (inaudible).

Jack: do it in whatever order you want.

Shannon: We tried number three and we got it down to

Chris: What was number three?

Shannon: The stamp one. We couldn’t solve it but we got an arrow down, we know that n for any number has to be a linear combination somehow that.

Figure 666. T4-1:13:52

Jack: Right.

Shannon: And…

Jack: Four n bigger than eleven.

Shannon: We were thinking that we could do something like this…
Chris: We just don’t know how to do a combination; because you know either n mod four has to be zero or n mod five has to be zero or the third thing is n have to be a combination of four and five.

Jack: Yeah.

Chris & Shannon: but we don’t know how to write that

Jack: Combination of four and five?

Chris: Yeah, a combination of four and five.

Shannon: Yeah, we dug out everything that we’ve ever taken in college and that is a toughy.

Chris: You know a way to write that mod four mod five…?

Shannon: We tried doing it like four plus five, that didn’t work and then we tried n mod four and n mod five and that didn’t work.

Chris: What we ended up with at the end was n minus a mod four that was plus a

Shannon: mod five

Chris: mod five

Shannon: Equals zero.

Chris: that equals zero

Shannon: But we didn’t know what to do about a

Chris: So we look at here this mod four equals zero it had to be this mod five equal zero, but it was more complicated when we started doing it.

Jack: that’s a good approach

Shannon: Like, can we do it like that and just figure out what to do with A, we added an extra variable.

Jack: What is A?

Shannon: The remainder.

Chris: a, a was the number necessary for this to be zero and made this new number zero based on mod 4. (silence) Cause we, we randomly chose thirty-seven. That’s the number we were using. The problem was thirty-seven divided by four when getting a remainder, close numbers, four times nine which gave us one.
One mod five is one, not zero. But we realized if we use, what we, you didn’t write that down?
(Inaudible). I know I have this down somewhere.

**Shannon:** Our logic behind it is, if…

**Chris:** Twelve and twenty-five. Okay.

---

**Shannon:** I don’t remember

**Chris:** You were the one that randomly said thirty-seven so based on thirty-seven we need to make this twenty-five. So we’re using five times five.

**Jack:** Which number are you working on?

**Chris:** This is number three.

**Shannon:** Three.

**Jack:** I mean, no, I mean which, which…

**Chris:** Oh, you said the base that works for every number so we randomly selected thirty-seven.

**Jack:** Okay.

**Shannon:** We did a proof for twelve. That’s easy.

**Chris:** Yeah, we proofed for twelve and that was simple. But for thirty-seven we also claim that four times three. So with N being thirty-seven we needed A to equal twenty-five. If A equaled twenty-five then this worked.
Jack: Okay.

Chris: But it just seemed like we were adding more headache to ourselves when we were solving it. Cause we know we can do this for every number. We...

Jack: Try it for thirty-nine. What would you, tell me what you would do for thirty-nine. That might be easier.

Chris: Thirty-nine?

Jack: Yeah.

Chris: We would need A to equal thirty-five and then that would give four over here.

Jack: Right.

Chris: But other than guessing and checking in my mind, I’m not sure how to write that.

Jack: Okay, so…

Chris: we tried the division algorithm

Jack: Well, I think that’s, that’s right

Chris: then we made a mistake trying to use the division algorithm.

Jack: Yeah no.

Chris: Or we can (inaudible).

Jack: I think the division algorithm you can use that. Take a number and divide it by five. Thirty-eight.

Chris: Thirty-eight divided by five.

Jack: Write down what you get.

Chris: Thirty-eight divided by five. We’re getting a remainder of three.

Jack: Okay, so write down equals quotient times divisor plus remainder.

Chris: Okay. This is what we’re, we’re tried…

Jack: Now you’ve got the, you’ve got seven fives.

Chris: Using the, using the… it would be like this

Jack: And if you had a three (inaudible) you’d be good.

Chris: Like this.
Jack: No, don’t do that. Don’t do that.

Chris: That’s division algorithm. So…

Jack: So, you see, you got to quotient the seven with a remained of three.

Chris: We need this to be divisible by four.

Jack: Yeah.

Chris: We need this to be.

Jack: Okay, so.

Shannon: So when you do take the seven down one and add it.

Jack: Right.

Chris: (Inaudible) the seven down.

Shannon: Actually you probably want to take it (inaudible).

Chris: take the seven down to six that will make it ten.

Shannon: Then add six.

Jack: No.

Chris: No, wait.

Shannon: No

Chris: Oh yeah this gives us eight. Yeah, brings that to five.
Shannon: Yeah.

Jack: Okay, so you did the division algorithm and then sort of and saw what the remainder was.

Chris: we messed with it so we got this mod 4


Chris: Twelve time five sixty plus three. So if you knock this down to eleven.

Jack: You get what?

Chris: You get eight so that will work.

Jack: Okay. Okay. Who…

Chris: that’s our number. We got five times something.

Jack: Well, put a q for quotient.

Chris: Quotient plus the remainder.

Jack: Okay. Now, if R is four then you’re good.

Chris: Yeah.

Jack: What if R is, what if R is three?

Chris: R must be, (inaudible) and zero.

Jack: What if R is four we are fine. What if R is zero.

Chris: Then we’ll, we’re fine, we’re set.

Shannon: cant you use zero mod four.

Chris: Zero mod four is zero.

Shannon: Well, I mean, could you just say that is about R? Would that make the same logic?

Jack: If R is zero you’re good. If R is four you’re good. What are the other possibilities for R? No, don’t, I don’t like that.

Shannon: Why not?

Jack: Well if it is zero cause. Cause the only thing it could be, since it’s a remainder, you’re dividing by five. Why, you’re right there’s two possibilities, zero and four. Then it’s zero minus four, okay. Now let’s talk about the other possibilities.
Shannon: The ones that don’t (inaudible).

Jack: One, two, and three. What if the remainder is three? How would you adjust it?

Chris: We would add a five cause that would give us something divisible by four.

Jack: Okay. Write that down. So really, (inaudible) put a three in place of the R. That’s the case we’re considering.

Chris: Okay.

Jack: And then how would you adjust everything?

Chris: We would then add five and subtract one.

Jack: Okay. Okay, now, now. Instead of having three, put a two there.

Chris: Right.

Jack: And let them, let Shannon: tell us what to do. So N is five q plus two.

Shannon: And we just subtracted one from three so we should get two?

Jack: Okay. And what do you get? Plus?

Shannon: Ten.

Jack: Ten. Which is good or bad?

Shannon: Good cause it’s divisible by four.

Jack: Right. Okay. Okay, now you’ve figured out what to do if the remainder is three. What to do if the remainder is two now.

Chris: And one should be the same.

Jack: What if the remainder is one?

Chris: Give us sixteen.

Shannon: Oh, that was the one.

Jack: Now, when you’re subtracting, when you’re working with counting things. Like you’re counting how many stamps to use, better not count any negative numbers. Don’t tell me I got to use the numbers, five cents stamps I got to use is minus one. When you’re subtracting you got to worry about that.

Chris: Okay.
**Jack:** Could that q, could q minus three be negative?

**Shannon:** Is that why it doesn’t work for less than eleven cents?

**Jack:** Yeah, right. But…

**Shannon:** So with our entry of the (inaudible) we wouldn’t have to worry about that?

**Jack:** Right, right. You guys didn’t use inductions at all?

**Shannon:** But we still got it so you should definitely give us credit.

*Figure 669. T4-1:24:02*

**Jack:** this is good.

**Chris:** We thought about it.

**Shannon:** The next one we worked on…

**Chris:** And I am not happy with that Fibonacci one, the one you said you couldn’t solve.

**Shannon:** We’re not going to worry about that one. Do number eleven.

**Chris:** Brian got it.

**Jack:** Yeah.

**Chris:** grrr

**Shannon:** Yeah, Bryan was the one that told me the final (inaudible).

**Chris:** For what, this one?

**Shannon:** Yeah, you’re diagonals.
Chris: Oh, the diagonal lines.

Shannon: This is number eleven.

Jack: Okay.

Chris: the sides equals four. The diagonals equals one.

Jack: Yeah. Okay, now, so you got to come up with a formula. The number of sides is N. How many diagonals will (inaudible)?

Chris: We claimed there was N minus three.

Jack: Yeah.

Chris: The number of diagonals plus N minus three and we proved it for, you know, I proved it for four through seven and it (inaudible). So I claimed if I added one diagonal…

Shannon: You want to one M.

Jack: So four is like you’re base case.

Chris: Yeah.

Jack: Cause you have no diagonals and you have only three.

Chris: Yeah, well (inaudible).

Jack: So that could be your base case also? Yeah. Three sides, subtract three you have zero diagonals which is right.

Chris: So this was our claim the only option.

Jack: Okay, so you got the base case. Now assume it’s true that the polygon has n sides and so, let’s draw a polygon with n plus one sides. Well the biggest one I can draw is eight so let’s make (inaudible).

Shannon: that is the biggest one you can draw?

Jack: Yeah, I’ll make it look half way decent.
Chris: Basically if you use this as your starting point every time you add another side, you’re adding a vertex. So that’s one of the diagonals.

Jack: Okay. Depends on what you say starting point.

Shannon: when you start here and you draw all of them with two we already have that’s why we subtract two.

Jack on the phone, Shannon and Chris discussing the problem among themselves.

Shannon: Ok. No matter where you start it’s going to be the same thing. So let’s just say we start here.

Jack: Yeah.

Shannon: You can draw diagonals to all vertices.

Jack: Yeah.

Shannon: But three of them. You can’t draw one to here cause it’s already there.

Jack: And you got the minus three.

Shannon: Yeah.

Jack: Okay. Here’s… tell me about that.

Shannon: He’s writing this out cause we want to look at this before you start yelling.
Jack: So you got, if you have eight, you got five by doing this (inaudible). How do you know you can’t get more if you don’t all start at the same place? Suppose I say, well, I want to start like this. Okay. Draw some more diagonals.

Chris: you mean without crossing them at all?

Jack: Nope. But just start drawing some more diagonals?

Chris: So you’re done?

Jack: No. You can draw more than two.
**Chris**: Oh yeah (inaudible) I go like that.

**Jack**: Yeah.

**Shannon**: How…

**Jack**: You got more than that too.

**Chris**: And then that here.

**Jack**: Okay. How many you got?

**Chris**: one, two, three, four, five.

**Jack**: Got five different one. Now how do you know you can’t do, see I started by it, so you, so they’d all start at the same point. So there’s lots of ways to draw diagonals and they don’t have to all start at the same place. How do you know you can’t do it with some really big polygon like twenty size? You can’t start some funny way and get more than seventeen?

**Shannon**: Can we erase this stuff? Can we do number fifteen?

**Chris**: what is fifteen?

**Shannon**: Um, the one hundred and eighty degrees (inaudible).

**Jack**: Let’s talk about this one some more.

**Shannon**: But it’s right here though. Cause, we did this right? You can maybe try go all the, into N minus three triangles.

**Jack**: That’s true.
Shannon: And then the formula for the triangles is (inaudible) blah, blah, blah, and then you multiply it by (inaudible). So that’s the (inaudible). And I just, I…

Jack: So let’s come back to that one. I have, I have…

Shannon: Times one eighty is.

Jack: Yeah.

*Dh but the blah blah blah is not part of the inductive proof

Chris: Its Shannon’s way

Shannon: Well, it’s there. You just have to (inaudible) where one eighty comes from…

Jack: I erased all the diagonal I had and we want to talk about whether you can get more than N-3 We are supposed to the find the maximum number so we start drawing diagonals somehow and we have to start somewhere, so suppose we drew this first one
so tell me what you see (inaudible) polygon? Her I have N sides. So I started, I drew a diagonal so tell me what do you see.

**Shannon:** A polygon with N time N+1 divided by two

**Jack:** Well not divided by two but you got the right idea I got two polygons I guess did divide it into two equal ones. This first diagonal could be any place. You almost have it

**Shannon:** Like you can, like you do it recursively so you can draw one diagonal, so you can draw three diagonals!

**Jack:** Well, yeah, this divide it into polygons let say this one has, this could have been like this. Let say these two polygons have p sides for one of them and q sides for the other one.

**Shannon:** You can keep dividing them down until you get

**Jack:** No I don’t want to do that, don’t want to do that

**Shannon:** What do you do? If you keep divide it up you get something that has ……… number of sides

**Jack:** Yes that gives you a lot, so you have to figure out how many ways you can divide it into triangles, right? That assuming what you trying to prove. Number of way you can divide it into triangles is, yeah so, how many diagonal does this one have?
Shannon: P minus three

Jack: How many diagonals does this one have

Shannon: Q minus three

Jack: Now what she just do? She just did it by induction she said start with N sides take one diagonal, it could be any one, it divides it into smaller polygons, by induction assumption know how many diagonal those have. That is the second form of induction because neither one of them have n minus one sides. So lets assume that this is true for any polygon who’s sides is less than N. That what she did, she assumes that ........ so this is the total number of diagonals or did I make a mistake?

Shannon: Yeah there is one more

Jack: Why?

Shannon: Because you already have a diagonal

Jack: Yeah that one, that one too, it is a diagonal. Ok. So the total number of diagonals for the whole thing is P+Q+5 you got to figure out why is that equal to N minus three.

In here P is

Shannon: Four

Jack: Four, and q is

Chris: six
Jack: It doesn’t work out.

Shannon: take the absolute value?

Jack: (inaudible) it’s plus 2

Shannon: and then that would work

Jack: Here’s what’s Shannon got

Chris: now you want to get that equal to this. Now I know N minus 2 minus 5 equals that (n-3) and you get n-3=n-3.

Jack: right.

Shannon: Can we do the 180 now the exact same way and just say times everything by 180 and we’re done. We’ll get credit for (inaudible)

Jack: No, you can’t do it like that, because

Shannon: you can go first I don’t care.

Jack: A polygon could look like this

![Image of a polygon](image)

*Figure 676. T5-00:01:04*

No, you can’t just start at one place and do all these diagonals like that, because it’s not a regular polygon in that problem.

Chris: (inaudible)
**Jack:** But what you want to do is, see you were just saying to just divide the whole thing up and then count these things and then times 180 and you’re done. That’s not using the induction again. You guys want to try and avoid that at all costs. So what you do is you, there is a hint, there’s someplace in here you can find 2 consecutive vertices to join and make a triangle. So see this one is consecutive to this one but you can’t (inaudible) because this thing sticks in the way. But here you can find this triangle, so that’s an induction kind of thing. Because you have taken (inaudible) take 2 and replace it by 1, so this thing over here compacts as one fewer side.

………..

**Jack:** There they are. Why don’t you guys work on that one. The object of this game is to remove all the coins.

![Image of coins](Figure 677. T5-00:04:03)

**Shannon:** Okay

**Jack:** And you get to remove one that’s heads, now when you do that, you flip any adjacent coins. Okay, so if I take this one, it was here and this isn’t touching so you don’t flip that one, you just flip this one. Okay? Now if I take this one, I have to take this, I’ve got to flip this and now that’s the place I have to take it and if I take this one, I flip these two and I can take this one and I take this one and flip this and then it’s done. Okay?
**Shannon:** do you always go down to nothing or (inaudible)?

**Jack:** No if you do it wrong, then you won’t get nothing.

**Shannon:** Okay

**Jack:** Okay, so why don’t you do it **Shannon:** and start with this one.

![Figure 678. T5-00:04:28](image.png)

Okay, now what?

**Shannon:** we don’t want to remove that one. Because then that one will flip over and then you’ll have no way of taking those.

**Jack:** Right.

**Shannon:** So you’re screwed no matter what.

**Jack:** Yeah, right. With this one, I think this is the same thing that I started with before, so you can’t always do it. If you do it the wrong way (inaudible) either. let’s see, if you have this
Can you do it?

**Chris:** No, no wait

**Jack:** if you take this one then you’ll have 2 tails left like bad, you’ll have all tails. If you do this one, then they’re split, so you need to know when to turn this one over. Okay so if you start with, 2 heads like this, you can’t do it at all. Okay, what if I had another one like this?

**Chris:** Yeah

**Shannon:** you can do it

**Chris:** *** explains how the game should be played.

**Jack:** but you start the same way and you take the middle one and

**Shannon:** You can’t take anymore.

**Jack:** Yeah, you can’t take anymore. What if you start here like this?

**Shannon:** Then you can only take one more or two more, you can win that way, yeah

**Jack:** you can with that way. So if there’s 3, you can win, you just have to do it right. Now if there’s 2, you can’t win. Okay, now here’s why it’s an induction problem, if you, here we have n coins, if you take one from the end, you have a string of n-1 coins so that like induction. If you take one from the someplace that’s not on the end, like this one, you could use induction again because it’s this game, plus this game and each one is shorter. Okay. So think about that.
**Shannon**: So we’re trying to prove what?

**Jack**: Trying to prove, trying to figure out when you can win this solitaire game.

**Shannon**: Okay

**Chris**: And does it matter how many? You said a string this long, what if these are all head inside does that matter? What if all tails inside, does it matter when we are trying to do the induction?

**Jack**: Well, it won’t matter if do the induction assumption right. If you assume something about shorter strings. Now if you assume something about the number of heads and tails then it might make a difference.

**Shannon**: So we manipulate …

**Jack**: So just say shorter strings, if the string is shorter the total number of coins is shorter.

**Chris**: (inaudible) the string you had before worked, but this string won’t work (inaudible).

**Jack**: Right, right. What if you take this one.

*A* (inaudible)

**Jack**: (inaudible) then you’re screwed again because you’re going to get an isolated coin. So anyways 6 head down, yes 6 heads. What if you start with 5?

**Shannon**: (inaudible)

**Chris**: so it matters whether it’s heads or tails

**Jack**: right it matters that you have heads and tails.

**Chris**: If you arrange these differently I could win with 6, but then you have to flip some of these over and have some of them as tails.

**Jack**: Right. Yeah, show me why you can win. Well actually what I was thinking was if it was all tails, but one on the end, you could win, because you just go down the line.
APPENDIX R

SHANNON, FOURTH INTERVIEW
Shannon interview 4, Nov 14

Yaser: Okay, okay, now this is what you showed. You showed that if you have a number \( n \) that is greater than 11, right?

Shannon: Yeah

Yaser: You guys said that \( n \) equals, write \( n \) as 5 times a number, whatever that is \( m \), plus remainder, if you remember, and the remainder was either 0, 1, 2, or 3 right?

Shannon: Or 4

Yaser: Or 4

Shannon: If your using five.

Yaser: Right, yeah. And then you found out that in any case you can take one out of here, right? Because if you take one out of here, 5 times 1 is 5 plus do you remember that?

Shannon: Yeah

Yaser: So let’s say the remainder, let’s say this was 5 times 7 and the remainder was 2. Just take one of those

Shannon: That would be 7

Yaser: Now 5 plus 2 is 7, no so you take 2, yeah

Shannon: if this is 6, this gotta be 7. If this is that would be 12, yeah that will work.

Yaser: Yeah, so it works. And if it’s 0, then you take, if the remainder is 0, then that’s it, because you know it would be

Shannon: Cause it’s 0 mod four

Yaser: Yeah and if it’s 1, we did one right? No

Shannon: We did 2

Yaser: Okay, this was one, then
Shannon: I can’t, I’m not really sure how to right a shortcut 6 that would be 6, if it is 5 that would be 11, if this 4 that would be 16.

Yaser: And then that would work right? And if the remainder was 3

Shannon: (Shannon writes 6 and 8 on the board),

Yaser: so that would work. And if the remainder was 4, then that’s it.

Shannon: Yeah

Yaser: Yeah, so on the side you would say, okay on the side, well we just said if the remainder is this or this or that, okay. Than I can write it in this form. Right? So if you want to do the, let’s do the

Shannon: See I don’t understand why that just didn’t prove it?

Yaser: It did prove it.

Shannon: Okay

Yaser: But I was, let’s write it as an induction.

Shannon: Okay

Yaser: So what would be your base? Your base would be

Shannon: 12

Yaser: 12 and it works? Right?

Shannon: Yeah

Yaser: Then what would you assume?

Shannon: Assume it is true up to n or up to k whichever you want use.

Yaser: True up to k. Now what does that mean? Just remember what that means here.

Shannon: That means ……

Yaser: k can be written as five times something plus r, now take that and move to the next step. What would be the next step? Uh-huh.

Shannon: (rights k+1 = (?) 5(m) + r

Yaser: Right. Okay and these are not the same numbers right? You just call it, so let’s start. How would you go from there?
Shannon: I don’t know

Yaser: Now notice that k plus 1 can be obtained from k, right? So start with, just write k plus 1, right, okay. Now what does this mean here? That there is some remainder right? What could r be? R could be 0

Shannon: Yeah

Yaser: If r is 0, then what would you have? You would have 5n plus what?

Shannon: One

Yaser: One. Could that be changed into this form \([4x+5y]\)?

Shannon: (inaudible)

Yaser: and if r is 1, what would you have? 1 plus 1

Shannon: Okay, I see what we’re going to do and how it’s going to look like that, but

Yaser: It’s always going to look like that and we can

Shannon: but I don’t see that we are showing anything.

Yaser: But we

Shannon: owed that, if it looks like this, it’s always going to write it down as 5 times something plus 4 times something.

Shannon: But doesn’t that mean that any numbers can be written in linear combinations?

Yaser: Yeah, that’s what we’re going to

Shannon: And not four and five, like 3 and 4.

Yaser: Oh, I see what you’re saying. That means you want to show that n equals 3x plus 4y?

Shannon: Yeah. Like I don’t know if that’s true, but I mean it’s like the same thing, at least that’s what I see it.

Yaser: Let’s check. What would you think the base would be for this one?

Shannon: I don’t know

Yaser: That’s a conjecture, right?

Shannon: 12
Yaser: 12, is that the smaller one? 12. What would you write here? 3 times

Shannon: 3 times 1, oh we’re doing (inaudible) yeah

Yaser: 4 times 3?

Shannon: Uh-huh

Yaser: That would be 12 plus 3, that’s 15.

Shannon: Okay, then make a 4 and a 0.

Yaser: 4 and 0. Okay. Can you write 11? Let’s check. 3 times what, plus 4 times what

Shannon: Yeah, 8 plus 3

Yaser: Okay, like that? Can you do 10?

Shannon: 6 plus 4

Yaser: 3 times … 2 and 1?

Shannon: Uh-huh and 9 can be 3 times 3. 8 would be 4 times 2.

Yaser: 8 would be 4 times 2.

Shannon: And then if they’re both 1’s. 3 times 1 (inaudible).

Yaser: 6?

Shannon: I think that’s the cut off point. No, because you’d have 3 times 2

Yaser: Plus 0?

Shannon: Yeah

Yaser: 5?

Shannon: That’s the cut off.

Yaser: 3 times, plus 4 times, if you put 0 here, and put 1 here, then you can’t get the 5. If you put 0, can’t get the 5. So the 5 is the cut off point?

Shannon: Uh-huh

Yaser: Because … solve … the base is 6. So what would be your theorem then?

Shannon: You just erased everything, I was just going to read it to you, everything was in order.

Yaser: No. Just write the theorem then. What is your theorem? For all n
Shannon: n greater than five

Yaser: Uh-huh.

Shannon: And \( n = 3x + 4y \)

Yaser: Okay, now how do we do it, by induction? What’s your base?

Shannon: Base is, like \( n=6 \)

Yaser: Yeah

Shannon: the base is … like 3 time 2 plus 0 time y.

Yaser: Okay, so it would be … you can write it down as … so what would be the assumption?

Shannon: (writes on the board) Assume that it true for \( k \)

Yaser: which means \( k \) could be written as,

Shannon: (writes on the board) \( 3(x) + 2(y) \)

Yaser: right. Now how could you rewrite this in terms of remainder? So you would use, take that to the next step.

Shannon: (writes on the board)

Yaser: Or do you want to the larger number plus the remainder?

Shannon: No, I don’t. Because this way I have less remainders, I have less remainder that 0, and the 2. And that way I have 0, and 4 (inaudible) this way.

Yaser: Okay

Shannon: That way I don’t have check the case (inaudible)

Yaser: Right. Now are we sure that this will work? Let’s see if the remainder is 0, we’re done, right?

Shannon: Uh-huh

Yaser: The 0 times, if the remainder is 1, then you take 1 of these, 3 times 1 and 1, right?

Shannon: Uh-huh

Yaser: Let’s (inaudible) check, for example if the remainder is 0, we’re done right? If the remainder is 1

Shannon: Well what are we looking for?

Yaser: We’re looking for … how you’re going to make this a multiple of 4.
Shannon: Oh you need another number then

Yaser: Yeah, so whatever … let’s say we have 5 in here. It doesn’t matter.

Shannon: Okay

Yaser: Then what do you need to do in order to make it look this way? Yeah and if the remainder is 2?

Shannon: (Writes on the board) (inaudible)

Yaser: Okay and if the remainder is? And that’s it, right?

Shannon: Yeah

Yaser: Okay. So we know that these are equivalent right?

Shannon: Uh-huh

Yaser: So let’s … now what would you …

Shannon: (writs on the board)

Yaser: how would you take it to the next step? So you assume it’s true for the base step, you want to show it’s true for k plus 1, right?

Shannon: Uh-huh.

Yaser: That’s what you want to show right?

Shannon: Uh-huh. I mean will this work, cause now we’ve just done the same thing as we were for four and five will it work no matter what the linear combination we pick? I mean could we do 17x plus 2y?

Yaser: I don’t know

Shannon: Because that’s pretty much what we’re proving like if we can get her, get the basis.

Yaser: get the basis.

Shannon: then all the inductions are going to be doing the exact same thing.

Yaser: Right.

Shannon: So what would knew the basis its going to be true

Yaser: Yes, if you’ve got the bases, once you get to this that it’s always true, it’s always possible; you know you get to this point.

Shannon: Then we don’t have to inductions its gonna be the exact same thing (inaudible)
Yaser: Yeah, but do you see now how this is sort of like a little bit different in writing, it’s like an induction form now, right? Because you took … this is what you did, you and Chris, what you did is this. You showed the most important part of the induction step.

Shannon: Uh-huh

Yaser: Because after that, what is that? You know that this is 3x plus r plus 1,

Shannon: (writs on the board what was said

Yaser: right, now why is that possible?

Shannon: Because of this

Yaser: Yeah, because this is the remainder, if it’s 0 then this will be 1 and you Shannon: owed that if it’s 1 then you can tweak it.

Shannon: But what if this is 2?

Yaser: If it’s 2, then that would be 3

Shannon: Oh this is just going to go up? Okay

Yaser: Yeah and if it’s, that’s it.

Shannon: (inaudible)

Yaser: Yes, but do you see the induction now better?

Shannon: Yes

Yaser: Because you start with your base.

Shannon: But wait, on my own if I got to this, but that was like I didn’t know what to do

Yaser: Oh you got to this point?

Shannon: Yeah

Yaser: You know why? Because you kept it in this form. You kept thinking of it in this form

Shannon: Oh, but I had r’s

Yaser: You had the remainders; you had this product plus a remainder?

Shannon: Yeah

Yaser: Okay
**Shannon**: (Inaudible)

**Yaser**: Okay, yeah just add one and then just think of this one as your new remainder, but you showed in your previous one that it’s always okay, no matter what the remainder is.

**Shannon**: I proved the coin thing, I think

**Yaser**: You did?

**Shannon**: That we’re going to do in the office today

**Yaser**: But how do you feel about induction now?

**Shannon**: I like it with like real things, like on a square equals whatever equals

**Yaser**: Yeah, yeah

**Shannon**: Like is that the 1 plus 2 plus 3? Yeah 1 plus 2 plus 3 blah blah blah blah equals m squared. I like that. Like this is dumb.

**Yaser**: So this is not, what did you say; I like it when it’s with real things?

**Shannon**: Well it’s easy, like stuff I’ve seen it with, you know. Because I’m not going to use this. I’m going to use like sometime I might need to know what this is. (inaudible) my everyday life, I might need to know what that is, well there’s 4 numbers and however it works.

**Yaser**: uh-huh. Now do you think this is more of a proof then what we did with the hackinbush games at the beginning of the semester?

**Shannon**: No

**Yaser**: No?

**Shannon**: No, they’re both proofs.

**Yaser**: No, they’re both proofs?

**Shannon**: Yeah, I just don’t use this

**Yaser**: But you don’t see the use of this one?

**Shannon**: No

**Yaser**: Not this one, but this particular problem?

**Shannon**: No
**Yaser:** But you see the use of proof by induction?

**Shannon:** Uh-huh

**Yaser:** Yeah, okay.

**Shannon:** I don’t like the induction with stuff that’s like not numbers; I like how we did, like prove that this can be filled with 2 by 1 squares. Like, there’s no numbers there, I don’t like that. Like I can show you, here look, I can fill it, however I want, but you know

**Yaser:** Yeah

**Shannon:** I don’t, (inaudible) do any examples like that still I don’t see how you can do that with induction.

**Yaser:** Yeah I think Ryan did something in the office, I walked in late, so I don’t remember what the theorem was, where you take 3, the letter L, 3 squares I think and then, I’m not really sure what theorem it was, but if you take those 3

**Shannon:** oh I remember that

**Yaser:** Then the rest can be, … I don’t remember what the

**Shannon:** triminals, that’s what they call them, triminals

**Yaser:** Okay, did you do that one?

**Shannon:** No, we did not solve that one.

**Yaser:** Actually they, Ryan worked on that problem

**Shannon:** (Writes on the board) we can do that, and we can do that and we can do that, and these two left I don’t know what do with them.

**Yaser:** Okay, so you can keep doing that until you

**Shannon:** Until all the squares are done.

**Yaser:** Yeah. So what they did, the way they used induction is they took this whole thing and then they showed that they can cut it into smaller, so they’re base was a little harder, it was like a 4 by 4

**Shannon:** Uh-huh

**Yaser:** And they showed that it works in 4 by 4, then it’ll work for the next step.
Shannon: See like something like that is what I don’t understand. Like I understand induction and like how it works and like it works for everything else. Like if you have a 4 by 4, um it doesn’t work. So say we have

Yaser: That was 2 by 2.

Shannon: Yeah, sorry

Yaser: 4 by 4.

Shannon: this is 3 by 3 does is this work?

Yaser: So let’s go 4 by 4.

Shannon: Okay, let say you can do. I understood that right now. (writing on the board, trying out the solution) But even if you can prove that, like then you know you can add on another dimension, like it won’t work by doing induction like you can do this so that if you add on one more it’ll all cancel out, but it doesn’t.

Yaser: Is it, what the dimensions are? Is there a condition on the dimensions? How they would be?

Shannon: I can look, alright (inaudible) it has to be 2 to the power n.

Yaser: Yeah, see, because that

Shannon: So it should only (inaudible) 4 by 4, just got to figure it up.

Yaser: 4 by 4 or 8 by 8

Shannon: Or 16 by 16

Yaser: So adding just one row would not be. You’re next step would be from going from 2 to n, to 2 to the n plus 1.

Shannon: do the same thin to n greater than or equal to 1. So we could deal with (inaudible). Start at the right corner and cut out

Yaser: Yeah, so that would work

Shannon: So then with that you could be able to go do a 4 by 4.

Yaser: Yes.
**Shannon:** But then when you do that (inaudible) numbers and math, like he won’t let you do pictures. So you have to have numbers and I don’t see any numbers. That’s my problem.

**Yaser:** Yeah, well the thing is that with induction, you don’t sometimes have to write, do the picture. It’ll say this is your base, base is 2 to the 1, assume it works for 2 to the k, right?

**Shannon:** Yeah

**Yaser:** We want to show that it works 2 to the k plus 1 and you want to relate this 2 to the k plus 1, you want to relate it somehow to the k.

**Shannon:** But we don’t have an equation, we don’t have anything for the 2 to the k. All we have for 2 to the k is pictures.

**Yaser:** Yeah, but the pictures will help you find the relationship

**Shannon:** So how do you do that, I mean there’s no equation, I don’t have any idea how you would do that.

**Yaser:** Um…

**Shannon:** So could you give me an equation or something, I would be able to do it just fine.

**Yaser:** Okay

**Shannon:** But there’s nothing there for me to relate it to.

**Yaser:** Let’s look at the jump from 2 to the k and let’s take 2 to the 1 and 2 to the 2, okay?

**Shannon:** Uh-huh.

**Yaser:** (inaudible) so 2 by 2 and this one would be 4 by 4, right?
APPENDIX S

SHANNON AND CHRIS, PRESENTATION
\textbf{Shannon and Chris: November 21}

\textbf{Jack}: now you can with that, that’s right but you gotta start at the right place

\textbf{Shannon}: middle, (Chris removes the coin in the middle) that makes two groups of odd. While if you start at the end actually (inaudible)

\textbf{Shannon}: yeah

\textbf{Chris}: because it is still odd

\textbf{Jack}: is there any place you can’t if you make a mistake on your first move

\textbf{Chris}: if we take either these, we take this flip that flip this

\texttt{Figure 680. T5-00:36:45}

now its even

\textbf{Jack}: ok

\textbf{Chris}: the same with this one. That will keep it

\textbf{Jack}: I don’t think you guys told me were you want

\textbf{Chris}: no because you have like this the first time.

\textbf{Jack}: how is it right now?

\textbf{Chris}: you just put it like this

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Jack: OK
Chris: if we take that, we flip that we still have odd. If we take that we still have odd.
Shannon: (inaudible) erasing the board
Jack: so the worst case scenario if there is only one coin and there is an odd number of heads which means that coin is got to be heads
Chris: ah ha
Shannon: ok we are going to assume,… we basically did for five for the base case
Jack: five! Why five?
Shannon: I don’t know, we picked five it looked like a fine number
Jack: (laugh)
Shannon: so assume … Oh I now I did not do one because I am going to do the second form. So assume, ok, base is five ok, assume to
Jack: I don’t know how you picked five
Shannon: assume true for zero up until k, (0,k]
Jack: ok
Shannon: where k is five and that you base case and then we assuming
Jack: k is the number of what
Shannon: k is points, the number of points
Jack: ok
Chris: and the number of heads inside must be odd
Shannon: right
Jack: ok
Shannon: should I write that down?
Jack: no that’s fine
Shannon: Um, so
Jack: k points with an odd number of heads inside
Shannon: so pile ups of k+1 points with odd heads you gonna have somewhere in the middle your gonna have a head H and you are gonna have more coins and its gonna be even heads here and even heads here because if the is odd and take one in the middle so there even on both sides.

Jack: so you look at the ones that are heads and you find the middle one

Shannon: anywhere, it does not really matters, does it?

Chris: yeah just anywhere

Shannon: wait, anywhere between tails, there we go it has to be tail here and tail here

Jack: well

Shannon: because the way I am doing it

Chris: or it could be heads on both sides

Jack: what if it is like this, what if its all heads, you cant

Chris: (inaudible)

Shannon: ok, at least you have to have the same thing

Chris: yeah it has to be the same on each side of it

Shannon: so find a head that has the same thing on the side of it

Chris: you can also take this and you flip here and still have the same ..

Jack: I think your gonna lose if you take this one

Chris: if I take this?

Jack: no this middle one

Chris: the middle one?

Jack: that’s the middle one, right?

Shannon: we’ll lose

Chris: yeah we’ll lose because it will leave that

Jack: both even

Chris: both even
**Jack:** your ok if you take the middle one there are tails on each side like she said because if you take this you change both of these two

**Chris:** two heads

**Jack:** two heads

---

**Chris:** but still there is one you can’t take which is that one. Which is with in, she is not talking about dead center, but with in

**Shannon:** you want to find out so you and, Um, so when you take this there is going to be odd heads here and odd heads here.

**Jack:** right. How do you decide, in general how do you decide which one you pick? How do you know there is one? If there is 20 coins or something

**Shannon:** because ….. if there is even right now over here

**Jack:** yeah

**Shannon:** and this is and H, let me start with a T. If this is a T,

**Jack:** if there is even here and even here and these are both T’s then you change both of them to odd.

**Shannon:** right

**Jack:** so that’s good

**Shannon:** if they both even, then this the one that is a head
**Jack**: this is a head

**Shannon**: and its even right now and this is one of them so when you flip this it is going to become a tail, then .. I don’t know this is going to be odd

**Jack**: ok, so you gotta look for coin with an even number on both sides

**Chris**: this one

**Jack**: lets look at the other case what if there is an even number on both sides and flip two and one of the ones you flipped is heads and the other is tails. So lets draw a picture like this. This is the one you flip, you have heads hear tails here HHT and this is …..H H T an even number of heads and this is an even number of heads.

**Shannon**: so when you flip it this one is going to become a tail and even minus one is always and odd and when you flip this …. This is a tail? Right? This will heads and even plus one is always odd. So you always …..

**Jack**: right

**Shannon**: and after we do that we have two smaller piles does not matter how much smaller but we already proved its true from zero to K ..

**Jack**: you assume it is true for smaller piles

**Shannon**: right and we have that. And now we have two smaller piles we already proved we can win this and we can win this.

**Jack**: right

**Shannon**: and we going do like numbers of games, a check plus a check we are going to win

**Jack**: ok, so

**Shannon**: by the second form of induction because your are assuming anything less than k is true

**Jack**: so tell me how to find a coin

**Shannon**: you did not ask me how to find it, I just proved it that is up to you

**Jack**: well part of your proof is to find a coin

**Chris**: you just said how to find it
**Jack**: how to find which coin you turn and what you do you look for and even number of heads on both sides, OK suppose I am skeptical suppose I say how do you know there is one, say you have a hundred coin, you cant find that one

**Chris**: the you take the end one

**Shannon**: well our conjecture said that there is and odd amount of heads, so if you take one there always be and even amount of both sides

**Jack**: No, there could be and odd number, it could be three on each side and you turn the seventh one. See like this one has an odd number on both side

**Chris**: so we would not take that one

**Jack**: we would not take that one ok?, so you claim is there gotta be one that has an even number on both sides

**Chris**: it would be that one

**Jack**: ok, how do you know there is always such a thing?

**Shannon**: is this like the one game where we break it piles that are not even?

**Chris**: I don’t think so

**Shannon**: because that’s what you just did, if you take one the piles are even it does not work but if you take this one the piles are not even so it does work

**Jack**: sort of like that but here you gotta tell me why there always gonna be that coin. You almost got it

**Shannon**: you have this many coins \((k+1)\)

**Jack**: there is an odd number of heads in there when you break it up there will be

**Chris**: even plus and odd is always and odd, right?

**Jack**: yeah

**Shannon**: like you have to ask?

**Chris**: so if just divide this pile into two sections anywhere we always going to have even and an odd

**Jack**: right if you have the string

**Shannon**: before you flip over
Chris: before you flip anything

Shannon: yes

Chris: so that is basically what we are trying to prove, that there is a spot where we can take

Jack: so here are the coins where you cannot have even on one side and odd number on the other side because then you have odd plus odd. See, if you have this heads then the even number on one side and an odd number on the other you are going to have even plus odd plus one which is what?

Chris: even

Shannon: laugh, I am sorry, your right even

Jack: do you get that?

Shannon: yes

Jack: you have an even number plus and odd number you get and odd, then you add one you get and even. Ok, so you cant have that no matter heads or tail. Your hoping it comes out like this (even H even). What if it comes out like this (odd H odd) you don’t want to remove this one because that will (inaudible) up

Chris: but it has to be within a string that you do have

Jack: which one

Shannon: the end one, because if you always pick the end one, there is zero over here so you win that game and in this game was even before and you flip this so you either add one or subtract one

Jack: right, right, so it looks …

Shannon: if worst come to worst you just pick out the end one

Jack: ok, do you mean this one?
There is five heads

**Chris**: we can take that one, can we?

**Jack**: Ok Could you always ... You are trying to convince me that there is always a way to find one, and then she says his one, is that right

**Shannon**: if cant find (inaudible)

**Jack**: can you remover … how about his one can you remove that?

**Chris**: yes you can

**Jack**: supposes you have this and not heads on the end, which one would you pick?

**Shannon**: I am sorry, I am like

**Chris**: it would be this one

**Jack**: you couldn’t take that one

**Shannon**: yeah that one

**Jack**: but still you got to describe to me which you would choose

**Shannon**: Ok, ok, you have and odd amount of heads, therefore there is center. You take the center, you check the center, if the center doesn’t work you take the end

**Jack**: ok, ok, so here’s

**Chris**: no, because you flip one over
Shannon: but the end would work

Jack: so here the center, maybe you take this one, but it does not work because there is no even number on both sides, so now what?

Shannon: oh no that one would work

Jack: ah-ha, it would work, right. Ok lets modify it lets change this to and H and this to an H.

Shannon: you have T at the end

Jack: yeah T here

Shannon: on the other side

Jack: here we have a T, now the center does not work, now odd number of both sides. So now what do you do?

Shannon: we take this one

Jack: why

Shannon: yeah, because you make this into heads and that an odd and you take that one

Chris: (inaudible)

Shannon: oh yeah

Jack: yeah but you gotta give me a way I can always use to find it without..

Shannon: this will change into a tail and you have one, two, three four five and that works

Jack: what if there is more tails here

Shannon: That’s ok you still gonna have one..

Jack: right, right so your method is to look at the middle one, if there is and odd, and even number of heads on both sides do that one. If the middle one has odd number on both sides you ..

Shannon: do the farthest head over

Jack: yeah

Yaser: cant she go to the next one, to the right or to the left

Jack: this

Yaser: you start from the middle, if does not work, move one to the next
**Jack:** you could do that also, that would be another way. Look at the middle one if that does not work look at the one next to it

**Yaser:** that should work

**Jack:** now, now, if this is odd, there is odd number here then pick this one instead. That will take one from the odd and that makes it even so that would work. But here method is good

**Shannon:** so I still prove this by induction?

**Jack:** yes, you prove first that is you could find a heads with and even number on both sides and that one will break it into two

**Shannon:** the one we already know?

**Jack:** yeah, if the center one does not work, then your rule says go to the one farthest to the end

**Shannon:** wouldn’t that be like, this is not part of it but my personal question is if the center one does not work, would any of them work?

**Jack:** No.

**Chris:** the center one didn’t work

**Shannon:** I am saying if the center does not work would any of them work

**Jack:** this one does not work here because this will be an isolated string all of tails

**Chris:** yeah

**Shannon:** mmm not really

**Jack:** yeah because if you remove this one, and this one becomes a tails

**Shannon:** oh I though you were pointing to the first one.

**Jack:** no, the first one would be ok

**Shannon:** right

**Jack:** it would be like what you said before, but flipping this one would not work

**Shannon:** Ok

**Jack:** you will have add number in here so

**Chris:** (inaudible)
Jack: ok, so guys your good

Figure 683. T5-00:52:46

Jack: Here’s one I like and would like to see one do. Remember I showed you tape these two together and make a cut

Shannon: (inaudible) give us something else mine, it did not make a square

Chris: Laugh

Shannon: laugh

Chris: I wanted to do that

Jack: ok, do it. We have ten minutes before my meeting

Shannon: (inaudible) and I look really dump

Using paper Shannon and Chris: each cutting two square with different sizes.

Jack: you want to cut the square with strait lines and reassembling them to make one big square, and the trick is to start the reassembling before you do the cut, past them together like that, go ahead do it.

Jack: here is the tape, Shannon side she messed it up before

Shannon: I taped it OK!

Jack: well somebody tape then

Shannon: my square is not strait

Inaudible
Shannon: this is yours right?

Chris: Oh my gosh how did you cut that

Shannon: Shut up Ok

Yaser: it is not a true square but

Shannon: use this side

Jack: now, take the big square and measure how far away across … We got rulers

Shannon: we don’t need ruler, its this far

Jack: and then mark off the distance like A

Shannon: mark here

Jack: and then cut from here…. Draw a straight line

Chris: and Shannon did the marking and made the cuts

Jack: and this will be one corner of the square obviously, that’s a right angle.

Shannon: how do you know it’s a right angle?

Jack: if this side is B then …. Um.. this plus this 90 degrees so all around is 180 if this is 90

Shannon: how do we get that

Jack: because this angle here is the same as this one

Shannon: right

Jack: by .. because these to line are perpendicular essentially and then this is a right angle so this plus this equals this plus this

Shannon: ok, Ok

Jack: but that’s a good question. …. Now you reassemble these ones into a square, part of the reassembly has already been done

Shannon: (inaudible) I don’t know

Jack: it gonna be better if you start here first because you know that the right angle. Now what I want someone do in class is how about if we have three of them, three squares.

Chris: lets do that
**Jack**: you want to tape these two together and these two together?

**Chris**: yeah

**Jack**: that’s not it

**Shannon**: maybe we should bring another one here

**Jack**: that is not the way either. Tape it all together to start

**Shannon**: in the middle

**Jack**: that is not going to work either

**Chris**: I never figure it out

**Shannon**: then we defiantly we should use those two ones to start with

**Chris**: if we tape two of them … we can tape two of them together like that and then from that we can get that

**Jack**: ok try it

**Chris**: that’s what gotta do

**Shannon**: no, no, no …

**Chris**: yes, yes, ….

**Shannon**: it is not gonna work

**Chris**: yes it does

**Shannon**: no, I don’t now how to, I just say its not gonna work

**Jack**: so what did you do for the last one

**Shannon**: well these two .. so we have three different triangles to start out. Then we use two that are next to each other

**Jack**: ok, that’s what he is doing now
Shannon: right, but there is no way you can cut another triangle and put it on there

Jack: your gonna see

Shannon: No! this is why we should use the one on the end

Chris: using the ruler to make lines

Shannon: will you stop moving

Chris: sorry, thank you thank you

Jack: ok, let Shannon assemble it

Shannon: no, don’t let Shannon assemble it

Jack: yeah

Chris: ok Shannon, assemble

Shannon: (tapping the pieces of paper together) where is the other … oh this way

Jack: now Shannon what do you think you should do next?

Shannon: tape!

Jack: right! Now tape all that together, after your done tapping then what?

Shannon: cut that

Jack: No

Yaser: think induction
Shannon: tape it

Jack: ok, now what?

Yaser: think induction

Chris: we just did this

Shannon: it would be easier to tape it on the back

Chris: if you want

Shannon: it is easier to tape it on the back

Jack: ok, let Shannon tell us what to do next

Shannon: we finish tapping and find

Jack: ok we finish tapping it and then what

Shannon: this over here, you cut the diagonals and you do it again

Jack: right, see now this, this is induction

Shannon: um hu

Jack: because if you can do it for n squares and you want to know how to it for n+1

Chris: add one

Jack: do it for n and assemble and tape it together stick on that next one

Chris: and do it again

Jack: and then cut again

Chris: that’s easy, the easiest on that list

Jack: it is easy but its kinda cleaver because its hard to think of

Shannon: did we do the whole thing now or we just proved …. Did we pove the loop invariant
APPENDIX T

CLASS NOTES
January 25, 03

Games

Discovery learning:
The class will determine the direction of the

Combinatorial Game Theory
1. Two players (team being a player)
2. Complete information (Chess)
3. No Chance Elements (Chess) [card, Monopoly NO]
4. Must end (Monopoly could go on \textit{et cetera})
5. No tiles
Hockenbush
left (B)  
right (R)

Your turn erase a stick of your color also erase anything molang connected to the ground.

You lose if you can't play.

(C) leg for example is not a good thing.

Teams:

R
3 students discussed, 2 watching

B
3 student discussed,

Game B
every stick is connected to the ground by their own color.

9 blue sticks, 7 red sticks

Best strategy is not to erase opponent's sticks / Blue wins.
Game C

Identical figures with opposite colors.
Both player wins if it does away with the first one. "Copy cat" strategies.

Game C: if 2nd player has a winning strategy $G = 0$ (G is a zero game).

*If* L (blue) always has winning strategy then we say $G$ is positive.

No matter who starts, $G < 0$ if when right has winning strategy always.

Hackenbush
If every stick is connected to the ground by its own color then count sticks to conclude ($G = 0$, $G > 0$, $G < 0$)

value $= -4 + 3 = -1$
Positive

\[ G^2 = 1 \]

\[ G = x \]

Red will win if Red plays its best.

\[ x - 1 < 0 \]

\[ x - 1 < 0 \quad 0 < x < 1 \]

Theorem: \( x = k \)

Proof: play \( x + x - 1 \)

Show this is zero

Show this is a zero game.

Next term: Someone homework

Show who ever starts loses.
If \( x - x - 1 = 0 \) then \( x = \frac{1}{2} \).

Proof: To prove \( G = \frac{1}{4} \), play \( G + G + G + G = 1 \) and prove that in \( E \), \( G + G + G + G = 1 = 0 \).

Or show \( G + G - \frac{1}{2} = 0 \).

\( 2G = \frac{1}{2} \), \( G = \frac{1}{4} \).

Homework: Play this game.
show $x + \frac{1}{4} = 1$

$x + \frac{1}{4} = 1 = 0$

OR $x + x - 1 - \frac{1}{2} = 0$

OR $x - \frac{1}{2} - \frac{1}{4} = 0$

show $x + x - 3 = 0$

OR some other combinations

show $\frac{3}{2}$

show $\frac{1}{8}$
Count games

3 - 2 > 0, positive means left wins

1 - 4 = -3 < 0, Red

Zero 2nd player wins, starting always loses

Zero game 2nd player wins, "copy cat" strategy

Definition: Sum of two games C + H

Class + Checkers

Your turn, pick one of the games to move in that game.

Sum of the two games
Negative of a game (Hackenbush)

\[ \frac{5}{3} + \frac{3}{5} = 0 \]

In Hackenbush, negative of G is the same picture, but reverse color.

[First will lose if second used copycat strategy]

\( G > 0 \)

\( 1 > G > 0 \)

Prove \( \frac{G}{3} < 1 \)

Play \( G-1 \) and prove \( G < 0 \)
5. Then \[ \frac{1}{2} = \frac{B}{B} = \frac{3}{4} \]

\[ x + x - 3 = 0 \]

Red unexpected first (602)

Count \# blue \# reds, whoever gets next loses.

\( \begin{array}{c}
1 \\
\end{array} \)

\( \begin{array}{c}
\text{Red loses.}
\end{array} \)
Blue goes first.

Student (G)

Other proof

Student (D) Red first

\[ x = 1 - \frac{1}{2} = 0 \]

Blue first

Red wins

Red loses

1 - 1 - \frac{1}{2} = \frac{1}{2} Red wins

Blue wins
\[ \text{Red loser} \]

\[
\begin{array}{c}
\text{Red} \rightarrow \text{Red loser} \\
\text{Red} \rightarrow \text{Blue}
\end{array}
\]

\[ \text{Blue will win} \quad \text{Position (Red loses)} \]

\[
\begin{array}{c}
\text{Red} \rightarrow \text{Red loser} \\
\text{Red} \rightarrow \text{Blue}
\end{array}
\]

\[ \text{First game} \]

**Definition:**

\[ G = \frac{\text{Value of What blue can move to}}{\text{Value of What red can move to}} \]

**Example:**

\[
\begin{array}{c}
\text{Red} \leftrightarrow \begin{cases}
0 & 1^2 = \frac{1}{2} \\
\text{shown before}
\end{cases}
\end{array}
\]

\[
\begin{array}{c}
\text{Blue} = \begin{cases}
1 & 2^2 = 3/2 \\
\text{shown before}
\end{cases}
\end{array}
\]

\[
\begin{array}{c}
\text{Red} = \begin{cases}
0 & \frac{1}{2} \times 3^2 = 0 \times 3^2 = \frac{1}{4}
\end{cases}
\end{array}
\]

For next line Theorem 14.
Theorem XX.6

\[ B \quad B \]
\[ \frac{3}{4} \]

\[ x + \frac{1}{4} - 1 = 0 \]

\[ B \quad \rightarrow \quad B \quad R \quad B \quad \rightarrow \quad R \quad R \quad R \quad \rightarrow \quad R \quad R \quad R \quad \rightarrow \quad B \quad R \quad R \quad \rightarrow \quad B \quad R \quad \rightarrow \quad B \quad R \quad \]

\[ \frac{1}{2} \quad \frac{1}{4} \quad 1 \]
\[ -\frac{1}{4} \]

Teacher: When you get to a point where you know all the numbers, you don't need to play the game all the way.

Student A: Or \( B \rightarrow B \rightarrow R \rightarrow B \rightarrow B \rightarrow B \) blue loses

Teacher: Any reasonable move? (S) I don't think so

Student A: Or \( B \rightarrow R \rightarrow B \) blue loses

\( -\frac{3}{4} \)

Blue has three opening moves: all blue wins.
Theorem 7: 

\[ B \times B = \frac{3}{8} \]

\[ \frac{2}{8} + \frac{3}{8} - \frac{3}{4} = 0 \]

\[ B \rightarrow R \rightarrow B \rightarrow B \rightarrow R \rightarrow B \rightarrow \text{red loses} \]

\( \frac{1}{2} - \frac{3}{4} \) since B loses again.

next B loses.

R wins.

\[ B \rightarrow R \rightarrow B \rightarrow R \rightarrow B \rightarrow \text{red wins} \]

\[ \frac{1}{2} - \frac{3}{4} \]

\[ B \rightarrow R \rightarrow B \rightarrow R \rightarrow B \rightarrow \text{red wins} \]

\[ \frac{1}{2} - \frac{3}{4} \]
Student (a)

BB, RR, RR  
BB  

Two possible moves

OK

\[ \frac{1}{2} \cdot \frac{1}{4} - \frac{3}{4} = 0 \]
Red goes next
Red loses

or

\[ \frac{1}{4} \cdot \frac{1}{2} - \frac{1}{2} > 0 \]
Red wins

(all blues connected to ground)

Student (k)

Game I going to play \( x - \frac{3}{2} + \frac{1}{2} = 0 \)

Prior we showed

\( \Box \) to solve the equation for \( x \), \( \Box \) \( x = 1 \)

I don't understand, \( \Box \) work more and present it next time.
\[ G = \frac{\text{values left}}{\text{can play to}} \quad \text{values right}^2 \]

\[
\frac{1}{2} = \frac{B}{B} = 0.113^2
\]

\[
\frac{1}{4} = \frac{B}{B} = 0.1 \frac{1}{2}, 13 = 0.1123
\]

Best move
Out two bad moves

\[
\frac{B}{B} = 0.1 \frac{1}{2}, 13 = 0.1123
\]

\[
\frac{3}{2} = \frac{B}{B} = 0.1123
\]

- I am going to mislead you
- What does that look like?
  - \( x \) is \( \frac{1}{2} \) of \( y \)
  - \( x \) is \( \frac{1}{2} \) of \( y \)
  - \( x \) is \( \frac{1}{2} \) of \( y \)
  - \( x \) is \( \frac{1}{2} \) of \( y \)
  - \( x \) is \( \frac{1}{2} \) of \( y \)

Next time she will show what \( \frac{B}{B} = 1 \)

\[
B \sqrt{B} = 0.1123
\]
Theorem: If \( x < 0, y > 0 \)

\[ x \cdot y^2 = \]

\((1)\) anybody got that?

\((2)\) I can explain it

\((3)\) what is your answer

\((4)\) answer is zero

\((5)\) write that down

\((6)\) \( 0 = 0 \)

\((7)\) why

\((8)\) explains by words

\((9)\) write down, make a diagram

Teacher leads student to set the following

\[ x \cdot y^2 \to x < 0 \text{ left dot} \]

\[ y > 0 \text{ right dot} \]

that is it. That's the proof.

Students surprised.
For example: \[ \sum_{i=1}^{n} 1/i^2 = \pi^2/6 \]

**Theorem:**

If \( b > 0 \) then left wins when right starts. If \( b < 0 \) then right wins when left starts.

**Theorem:** If \( 0 < 1 < y \) then \( \sum_{i=1}^{y} i = 1 \)

**Proof:** \[ \sum_{i=1}^{y} i = 1 \]
15. Theorem: \( \frac{R}{B} = 1 \)

\( \frac{1}{2} < x < \frac{3}{2} \) because

want to show \( x = 1 \) (or \( x-1 = 0 \))

- Red wins
  - \( \frac{1}{2} = 1 = -\frac{1}{2} \)
  - \( R \rightarrow B B R \rightarrow R \) Blue wins
  - \( 2 - 1 = 1 \)
  - \( R \rightarrow R B R \rightarrow R \) Blue wins
  - \( R \rightarrow B R \rightarrow R \) Blue wins
  - \( \frac{1}{2} \)

Teacher:

Who is going to win?

Student: Red. Why because

\( \frac{1}{2} \)

Show that Red always wins by playing it out. I.e.

- 1. Show that Blue has no winning move no matter how Blue starts, Red has at least one move to respond
- 2. Show that Red has at least winning if move what wins
It is not

| \( \text{avg} = \frac{\frac{1}{2} 16 + \frac{1}{2} 32}{2} = 3 \) |

for \( B \times B \)

last time

Alex did the proof of the first part last time.

\[ \text{neg} \text{pos} 3 = 0 \]

Theorem: \( x \times y^{2} = \) simplest number that is strictly between \( x \) and \( y \)

Simplest number: \( 0 \) is the simplest of all.

\( \pm 1 \) is the next simplest number.

\( \pm 2 \) is the next simplest number.

\( \pm 3 \) is the next simplest number.

\( \pm 4 \) is the next simplest number.

\( \pm 5 \) is the next simplest number.

\( \pm 6 \) is the next simplest number.

\( \pm 7 \) is the next simplest number.

\( \pm 8 \) is the next simplest number.

\[ \frac{R}{B} = \frac{1}{2} \times 16 \]

\[ \frac{R}{B} = \frac{1}{2} \times 32 \times 3 \]

After the whole numbers, you look for the fractions with denominator 2.

Then fractions with denominator 4.

fraction with denominator 8.

Why?
Class will play new game: "Ski Jumps"

Left

Right

Use checker board.

Jump this way only

Blue loses
Blue (Red wins no matter what)

1. \[ R \xrightarrow{B} R \xrightarrow{R} B \xrightarrow{R} \text{Red wins} \]

2. \[ R \xrightarrow{B} B \xrightarrow{R} R \xrightarrow{R} \text{Red wins} \]

Teacher: you need a good one for red, do you have one? They both are good.

Student 2

Red had only have to show that red had a winning move

\[ R \xrightarrow{B} B \xrightarrow{R} R \xrightarrow{R} \text{Red wins} \]

\[ \frac{1}{2} - 1 = -\frac{1}{2} \]

This would be a losing move for red.

Student 3

If \[ \frac{R \xrightarrow{B} B}{R \xrightarrow{R} B} \]

\[ \frac{1}{2} = \frac{1}{2} \]

Blue would win.
Notes: $G = 0$ means whoever starts loses.

Suppose whoever starts $G$ loses, then if you have any other game $H$

\[ H + G = H \] (outcome)

Simple numbers:

\[ \pm 1 \]

\[ \pm \frac{1}{2} \]

Fractions with denominator 2:

Fractions with denominator 4:

Fractions with denominator 8:

Theorem: If $x < y$, then $G = \{ x | y \} = 0$

Theorem: If $G = \{ x | y \}$ and zero is not between $x$ and $y$, then some positive integer is when the answer is least such integer.

\[ 0 < n - 1 < x < n < y \]

Then $x | y = n$

How to prove this? Set up the right game.

Proof: play $\{ x | y \} - n$ show whoever starts loses.

\[ x | y \cup \emptyset (\emptyset) \] (this is a $x | y - \emptyset$)
\[3x1y^2 + R^2 \rightarrow x + R\]

value of biases
best move

Someone will need to show the full proof.

1. Compute Hacker bush value
   a) Using Simplicity theorem
   b) Play game with

\[R \times R \times R\]

sticks of height 4

\[x \times x \times 1 = \text{zero game}\]

\[x^2 \leftarrow \text{positive}\]

\[\frac{R^2}{8} = \text{Simplicity theorem}\]

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 \\
1 & 1 & 2 & 1 \\
1 & 2 & 1 & 1 \\
\end{bmatrix}
\]

\[\frac{e}{18} = \frac{s}{18} = \frac{2}{18}\]

\[\frac{2}{4} = \frac{1}{4} \cdot \frac{2}{2} = \frac{1}{2}\]

\[\frac{2}{4} - \frac{1}{2} = \frac{1}{2}\]

---

Choose a color or move

Error: Choose a color (Red)

Value of the game is negative
Student: Theorem
If \( x \perp y \mid z = n \)
if \( 0 \leq n-1 < n \) in \( y \) show \( x \perp y \mid z = n \) will show

\[ S \perp Y^3 + R^n = 0 \rightarrow x \rightarrow x + R^n = x - n < 0 \rightarrow \text{Red wins} \]

Did you understand?

\[ \text{Start with } R \rightarrow y + R^n = y - n > 0 \rightarrow \text{Blue wins} \]

\[ \text{or } R \rightarrow x + y + R^{n-1} \] 

"T" you don't have that yet

\[ \text{(s) yeah, } S \perp Y^3 + R^{n-1} = x + R^{n-1} \]

"T" rig \( n-1 \leq x \)
moved \( n-1 \) to the other side \( 0 < -n+1+1 \) that means somebody wins

"T" explains to class
\[ G = \{ x \mid y^3 \} \text{ that is all you know} \]

Play \( S \perp Y^3 + R \) 

If \( s \) \( G + R = 0 \)

\[ G - n = 0 \text{ (} R^n \text{ has value } -n \) \]

If \( \text{red} \)
\[ \frac{R}{G} \to x \perp y^3 + R^{n-1} \]

or \( R \to y + R^n = y - n > 0 \)

because hypothesis \( n \leq y \rightarrow y - n > 0 \)
with \( \sum x_1 y_3 + R^2 \frac{R}{1-R} \sum x_1 y_3 + R^{n-1} \)

\[ B_0 \ x + R^{n-1} > 0 \]

Ask in the interview

New game

Left

Blue

Right

Red

Grid:

<p>| | | | | |</p>
<table>
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7x4

Cut along your color

No line left
to cut you

Red goes first / Red loses

Value of cells gone:

\[ \begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 0 & 1 & 2 & 3 \\
2 & -1 & 0 & 0 & 0 \\
3 & -2 & 0 & 0 & 0 \\
4 & -3 & 0 & 0 & 0 \\
\end{array} \]

Left cuts \( \uparrow \)
Right cuts \( \rightarrow \)

\[ \sum \frac{1}{i+1} \left| \begin{array}{cc}
-1 & c \\
-2 & 2 \\
\end{array} \right| = 0 \]
2x3

\[ \begin{array}{c|cc} 
  & 3 & 5 \\
\hline 
1 & 4 & 2 \\
0 & 3 & 1 \\
\end{array} \]

\[ (1, 2, 3) = 0 \]

2x4

\[ \begin{array}{c|cc|c} 
  & 4 & 5 & 6 \\
\hline 
1 & 2 & 3 & 1 \\
4 & 5 & 6 & 7 \\
\end{array} \]

\[ \sum_{\text{dike}} 6 = 3 \]

2x3

negative is

3x2

Show how the diagonal \( \mathbf{g} \) is filled in the table. Due Friday.
\[ B = n \rightarrow \text{stack of sticks} \quad \text{all of } n \text{ blue} \]

\[
\begin{array}{cccc}
R_E & B & B & B \\
\frac{1}{2} & R_E & B & B \\
B & B & B & B \\
\frac{3}{2} & B & B & B \\
\frac{5}{4} & B & B & B \\
\frac{3}{2} & B & B & B \\
\end{array}
\]

\[
\begin{array}{cccc}
R_E & B & B & B \\
\frac{1}{2} & \frac{3}{2} & B & B \\
\frac{5}{4} & \frac{3}{2} & B & B \\
\end{array}
\]

\[
B_{\frac{1}{2}} \rightarrow R_E \\
B_{\frac{1}{2}} \rightarrow \frac{3}{2} \\
B_{\frac{1}{2}} \rightarrow \frac{5}{4} \\
\]

\[
\frac{1}{4} B_{\frac{1}{2}} \rightarrow \frac{1}{2} B_{\frac{1}{2}} \\
\left(\frac{1}{2} B_{\frac{1}{2}}\right)^2 = \frac{1}{4} B_{\frac{1}{2}}^2 = \frac{1}{2} \\
\]

\[
\frac{3}{2} B_{\frac{1}{2}} \rightarrow \frac{3}{2} B_{\frac{1}{2}}^2 = 2 \\
\frac{1}{2} B_{\frac{1}{2}} \rightarrow \frac{1}{2} B_{\frac{1}{2}}^2 = 1 \\
\]

\[
\]
1. Definitions  Let $G$ be a game.

(a) $G = 0$ means whoever starts loses.
(b) $G > 0$ means Left wins no matter who starts.
(c) $G < 0$ means Right wins no matter who starts.
(d) $G \parallel 0$ (G is fuzzy) if whoever starts wins.
(e) Let $G$ be any game. The negative of $G$ is the same game as $G$, but with the moves of Left and Right interchanged. For example, if $G$ is a Hackenbush game, you can draw $-G$ as the same picture but with the colors reversed.

2. Theorem  \[
\begin{array}{c}
\text{R} \\
\text{B}
\end{array}
\]  $= 1/2$

3. Theorem  \[
\begin{array}{c}
\text{R} \\
\text{B}
\end{array}
\]  $= 1/4$

4. Theorem  \[
\begin{array}{c}
\text{R} \\
\text{B}
\end{array}
\]  $= 1/8$

5. Theorem  \[
\begin{array}{c}
\text{B} \\
\text{B}
\end{array}
\]  $= 3/2$

6. Theorem  \[
\begin{array}{c}
\text{R} \\
\text{B}
\end{array}
\]  $= 3/4$

7. Theorem  \[
\begin{array}{c}
\text{R} \\
\text{B}
\end{array}
\]  $= 3/8$
8. Theorem \[ R \]
   \[ B = \frac{5}{8} \]

9. Theorem \[ R \]
   \[ B \]
   \[ R \]
   \[ B = \frac{5}{4} \]

10. Theorem \[ B \]
    \[ B \]
    \[ B \]
    \[ R \]
    \[ B = \frac{5}{2} \]

11. Theorem \[ B \]
    \[ B \]
    \[ B \]
    \[ R \]
    \[ B = \frac{7}{4} \]

12. Theorem \[ B \]
    \[ B \]
    \[ B \]
    \[ R \]
    \[ B = \frac{7}{8} \]

13. Definition Let \( G \) be any game. We write \( G = \{ x \mid y \} \) where \( x \) is the value of Left's best move and \( y \) is the value of Right's best move. So
    \[ R \]
    \[ B = \{0|1\} \]

14. Theorem
    If \( G = \{ x \mid y \} \) with \( x < 0 < y \) then the value of \( G \) is always the same. What is the value of \( G \)?
15. Theorem

\[ R = 1 \]

16. Theorem If \( G \geq 0 \) then Left wins when Right starts.
   If \( G \leq 0 \) then Right wins when Left starts.

17. Theorem The Meaning of Zero Suppose \( G \) is a zero game (whoever starts loses.)

(a) If Left wins \( H \) when Left starts then Left wins \( H + G \) when Left starts \( H + G \).

(b) If Left wins \( H \) when Right starts then Left wins \( H + G \) when Right starts \( H + G \).

(c) If Right wins \( H \) when Right starts then Right wins \( H + G \) when Right starts \( H + G \).

(d) If Right wins \( H \) when Left starts then Right wins \( H + G \) when Left starts \( H + G \).

Summary: If \( G = 0 \) then \( H + G = H \) (as games)

18. Theorem Simplicity Thm, second case
Suppose \( G = \{ x \mid y \} \) with \( 0 \leq x < y \) If there is an integer between \( x \) and \( y \) then \( G \) equals the least such integer.
I.E. Suppose \( n \) is an integer with \( 0 \leq n - 1 \leq x < n \leq y \). Then the value of \( G \) is \( n \).

19. Question
What is the analogous theorem for \( G = \{ x \mid y \} \) with \( x < y \leq 0 \)?
Ask during interview 9/8/03

R has no
The number on the top is the horizontal size of the rectangle and the number of the side is the vertical size of the rectangle. **Left** makes vertical cuts and **Right** make horizontal cuts.

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Ryan Klodnick
Zack Wiles
Steve Freckel
R + S + 2

9/17/03

23
\[ \begin{align*}
-6 & \rightarrow -3 \\
& \rightarrow -4
\end{align*} \]

\[ \begin{align*}
\{ \text{grid} \} & \rightarrow \{ \text{grid} \} \\
& \rightarrow \{ \text{grid} \} = 0
\end{align*} \]

So, 9/17/03

Interview
\[ \binom{0}{1} [6] = 1 \]

Interview

R, 9/17/03

[Diagram of a grid with numbers]
Theorem (6.17, 7.17)

\[ \text{If } G = 0 \text{ then } H + G = H \]

- \[ G > 0 \text{ then left wins if right starts} \]
- \[ G < 0 \text{ right wins if left starts} \]

Frog Game

Left \[ \begin{array}{c} L \end{array} \]

Right \[ \begin{array}{c} R \end{array} \]

Value of this move is \( (SL) \)

- \[ 2 \text{ moves for } R \]
- \[ 0 \text{ moves for } L \]

Left-Starts Left Loses

\[ \left\{ x^1 y^3 \right\} \]

\[ \frac{SN < 1}{SN \geq 0} \]

\[ \text{left} \]

\[ \text{right beats} \]

\[ \frac{\text{left}}{\text{right beats}} \]

\[ \frac{1}{2} \]

\[ \frac{1}{2} \]

\[ 0 \]

\[ 1 \]

\[ 0 \]

\[ 1 \]

\[ \frac{1}{2} \]

\[ \frac{1}{2} \]

\[ 0 \]

\[ 0 \]

\[ \frac{1}{2} \]

\[ \frac{1}{2} \]

\[ \text{the value} \]

\[ \frac{SN < y}{\frac{1}{2} \leq 0} \]

Simplifies \( x > 0 \)
\[ \frac{R_k}{R_{k+1}} = \frac{1}{32} \]

\[ R_k = \frac{1}{64}, \quad R_{k+1} = \frac{1}{2^n} \]

We know \( x = 2010^3 \)

\[ x + 0 - 1 \cdot x + x = 0 \]

**Theorem**

\( x \) is less than every positive number and \( x \) is greater than every negative number.

**Proof:**

\[ \begin{array}{cccccccc}
   \frac{1}{2^n} & \frac{1}{2^n} & \frac{1}{2^n} & \frac{x}{2^n} & \frac{x}{2^n} & \frac{x}{2^n} & \frac{x}{2^n} & \frac{x}{2^n} \\
               &              & \mid                &                   &                   &                   & \mid                   \\
   -1 & \frac{-1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{x}{2^n} & \frac{x}{2^n} \\
   \frac{1}{2^n} & \frac{1}{2^n} & \mid & \frac{1}{2^n} & \frac{1}{2^n} & \mid & \frac{1}{2^n} & \frac{1}{2^n} \\
  \end{array} \]

**Idea of the proof:**

1. Show that \( x < \frac{1}{2^n} - x \)

Think of a game and play it:

- \( \frac{1}{2^n} + R^n \) show it is negative (Red wins)

2. Show \( x > -\frac{1}{2^n} - x \) (i.e., \( x + \frac{1}{2^n} > 0 \))

- \( \frac{1}{2^n} + B \) show it's positive (Blue wins)
20. Theorem
Suppose \( G = \{ x | \} \) (This means \( \text{Right} \) has no play at all. Then
(a) If \( x < 0 \) then \( G = 0 \)
(b) If \( 0 \leq x \) then \( G \) is the least integer greater than \( x \)

21. Question
What is the analogous theorem if \( G = \{ x | \} \)

22. Theorem  \( \text{Simplicity Thm, third case} \)
If \( G = \{ x | y \} \) with \( 0 \leq x < y \) If there is no integer between \( x \) and \( y \) but there is a fraction with denominator 2 between \( x \) and \( y \) then \( G \) equals that fraction. I.E. Suppose \( n \) is an integer with \( 0 \leq n \leq x < n + 1/2 \leq y \leq n + 1 \). Then the value of \( G \) is \( n + 1/2 \).

23. Theorem
The negative of \( G = \{ x | y \} \) is
\[ \{ -y | -x \} \]
To Prove this, show
\[ \{ x | y \} + \{ -y | -x \} = 0 \]

24. Definitions
\[ \{ 0 | 0 \} = * \]
Note: In the Frog Game, \( L \square R = * \)

25. Theorem
\( * + * = 0 \)

26. Theorem
\[ \frac{R^n}{B} = 1/2^n \]

27. Theorem
\( * \) is less than every positive number and greater than every negative number. To show this, it suffices to prove \( * < 1/2^n \) and \(-1/2^n < * \) for every positive integer \( n \).
Theorem 16: \( G > 0 \)

17: If \( G = 0 \) then \( H + G = H \)
20: If \( 0 < x \) then \( \lceil x \rceil^2 = \text{smallest int} > x \)
23: If \( G = \{x | y \} \) then \( -G = \{ -y \} - x^2 \)

Play: \( \{ x | y \} + \{ -y \} - x^2 = 0 \)
26: \( x + x = 0 \)

\( x \) is not zero but \( x \) is a twin

27: Every negative number < \( x \) < every positive number.

\( \leftrightarrow \)

\( -\frac{1}{2} \) \hspace{1cm} 0 \hspace{1cm} \frac{1}{2} \rightarrow 2 \)

\( \text{negative} \) \hspace{1cm} \text{positive} \hspace{1cm} \text{ask} \)

\( x + x = 0 \)

\( x \) is not zero but \( x \) is a twin

Exercise

12: Indent

\[ B \leftrightarrow B \rightarrow \{011\} = \frac{1}{2} \]

(\text{specific game})

imperical

Keble use the frog game

\[ \begin{bmatrix} 1 \leftrightarrow R \leftrightarrow B \leftrightarrow R \end{bmatrix} \]

\[ \begin{bmatrix} 1 \leftrightarrow B \leftrightarrow B \rightarrow \{1B\} \rightarrow \{0\} \end{bmatrix} \]
G \xrightarrow{\text{you}} 0

Good move because whoever goes next loses.

\[ 3 \times 9 \xrightarrow{\text{two moves}} 0 \]

First player loses.

King Arthur:
Round Table - Knight should be seated next to:
Left - Seated lady on the left
Right - Seated lady on the right

Rules:
L L L illegal
R R R R illegal

Even 16 chairs translate

\[ 1 \]

16 places:
\[ 1 - - - R 0 \] 0 3 = 10
Homework for Today

Find the value of this game (the one because blue will win).

\[ V_3 = \frac{e}{3} \]
\[ 0 = \frac{e}{3} \]
\[ R = \frac{e}{3} \]

\[ V_2 = \frac{e}{2} \]
\[ 0 = \frac{e}{2} \]
\[ R = \frac{e}{2} \]

\[ V_1 = \frac{e}{1} \]
\[ 0 = \frac{e}{1} \]
\[ R = \frac{e}{1} \]

\[ \sum_{i=1}^{3} \frac{1}{2} b_i^2 = 1 \]
\[ \sum_{i=4}^{4} \frac{1}{2} b_i^2 = \frac{3}{2} \]

\[ \sum_{i=1}^{4} \frac{1}{2} b_i^2 = \frac{3}{2} \]
A18: proof by induction

Student:

\[ \ast + \ast = 0 \]

Left starts:

\[ LR + LR \rightarrow LR + LR \rightarrow \text{wins} \]

Right starts:

\[ \ast \rightarrow LR + LR \rightarrow \text{left wins} \]

\[ \ast < \frac{1}{2^n} \text{ for all } n \]

\[ n \ast < \ast + n \ast < \ast - \frac{1}{2} \ast \]

\[ \ast \text{ is unbroken} \]

\[ 0, 1, 1, 2, 3, \ldots \]

\[ 0, \ast, 1, 1, 3, \ldots \]

Proof:

\[ 0, \ast, 1, 1, 3, \ldots = 0 \]

\[ 1, 0, *_{1/2} + \frac{B}{R} = 0 \]

\[ 0, \ast, 1, 1, 3, \ldots = 0 \]
20. **Theorem**
Suppose $G = \{ x \mid \}$ (This means Right has no play at all. Then
(a) If $x < 0$ then $G = 0$
(b) If $0 \leq x$ then $G$ is the least integer greater than $x$

21. **Question**
What is the analogous theorem if $G = \{ x \mid \}$

22. **Theorem**  *Simplicity Thm, third case*
If $G = \{ x \mid y \}$ with $0 \leq x < y$ If there is no integer between $x$
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$y$ then $G$ equals that fraction. I.E. Suppose $n$ is an integer with
$0 \leq n \leq x < n + 1/2 \leq y \leq n + 1$. Then the value of $G$ is $n + 1/2$.

23. **Theorem**
The negative of $G = \{ x \mid y \}$ is
\[ \{ -y \mid -x \} \]
To Prove this, show
\[ \{ x \mid y \} + \{ -y \mid -x \} = 0 \]

24. **Definitions**
\[ \{ 0 \mid 0 \} = * \]
**Note:** In the Frog Game, $L \Box R = *$

25. **Theorem**
\[ * + * = 0 \]

26. **Theorem**
\[ B^n = 1/2^n \]

27. **Theorem**
\[ * \text{ is less than every positive number and greater than every negative number. To show this, it suffices to prove } * < 1/2^n \text{ and } -1/2^n < * \text{ for every positive integer } n. \]

28. **Theorem**
If $x > 0$ then $\{ * \mid x \} = ?$
If $x < 0$ then $\{ x \mid * \} = ?$
Theorem 36: 

\[ 0 < x < \] every positive \( x \)

Theorem 37: \( x < 0 \) every negative \( x \) \( < 0 \)

Theorem 38: \( \sqrt{0, 10^2} = x \)

39. (a) \( \sqrt{10^2} = x \)
   (b) \( \sqrt{0, 10^2} = x \)
   (c) \( \sqrt{10^2} = x \)

Student: (Proof of 36)

\[ u + R^n < 0 \] (Teach Hackerbush stick

is not good.)

Student \[ u + R^n \] (Show what is that number

\[ S. R^n = u - \frac{1}{2} \to \frac{1}{2} < \frac{1}{2} \]

\[ \to -\frac{1}{2} \]

Want to show

\[ \sqrt{R^2 + 1 < 0} \]

Red wins

\[ \sqrt{R^2 + 1} \]

Starts \[ R^n \]

Starts \[ 10^2 + R \to R^n \]

Starts \[ S. \text{that is negative because} x \]

T. Show that on the side

\[ \frac{1}{2} \]

S. We know that \( x \) is closer

to negative than any number

T. Write \[ x < 4 \to x < \frac{1}{2} \]
So Red has no best move (right)

\[ 0 < \beta < \frac{1}{2} \quad \text{for all } n \]

**Question:** \( B = \uparrow \) is that true \& or false

This question is the same as

\[ \beta - \tau = 0 \quad ? \quad \text{show these are true} \]

\[ \beta + \downarrow = 0 \quad ? \]

\( k \) in stands for \( \text{NIM heap of } n \) objects

\( \text{NIM is played as a sum of heaps} \)

\[ \ast 2 + \ast 3 + \ast 4 \rightarrow \ast 2 + \ast 3 + \ast 1 = 0 \quad \text{next player loses} \]

\[ \ast 1 + \ast 4 + \ast 5 = 0 \]

\[ \ast 1 = \frac{1}{2} \ast \rightarrow \]

\( \ast \)

\[ \text{fuzzy if player wins} \]

\[ \ast 2 = \ast \quad \text{asking this question is the same as asking} \]

\[ \ast 2 - \ast \geq 0 \]

\[ \ast 2 + \ast = 0 \]
\[ *1 + *2 + *3 \]

\[ \text{remove 1 whole pile} \quad 2nd \quad \text{two equal} \]

\[ \rightarrow \text{reduce} \quad *2 \quad \frac{1}{2} \quad *1 \quad \frac{1}{2} \quad *1 \]

\[ \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \]

\[ \rightarrow *1 \times *2 + *2 \quad \frac{1}{2} \quad *2 + *2 \]

So the second play can always change into two equal parts.

---

**Tues:** 10:00
11:00
12:00

**Wed:** 12:00
2:30
3:00

**Thurs:**

**Fri:** 12:00
1:00

10/10/03

Ryan

Shawney
Polite Hackenbush: a player cannot erase the other players' sticks.

Blue

Red

Ask: How to find the value of this game ($r + 0$)

$f + x$

$r + 0$ (Check $B$)

$f + x = g[0;A]0^3$
2. 1st leaves 3 piles and two equal
and take whole pile, the unequal pile
3. 1st: $1 + 2 + x_5$
   2nd: $1 + x_2 + x_3$

4. 1st: $1 + x_3 \cdot 5$
   2nd: $1 + x_3 \cdot 2$

5. 1st: $1 + x_4 + x_3 \cdot 2$
   2nd: $1 + x_2 + x_3$

Next time

Your position

1232
345
67
89

Show

West here

Find

The value

6) 1 3 5 7 2
   7) 2 3 4 6 7
   8) 1 3 4 6 7
   9) 1 2 3 4 5 7 (shawn)

13 2 4 5 7
Thm: \[ 501 \uparrow^2 = \uparrow + \uparrow + \uparrow \]

34. Thm: \[ \star + \star = \star x \{1 \times 3\} \]
\[ \circ = \star + \star = \{\star x 1 \times 3\} = 0 \]

Q. 
\[ \star + \uparrow = \{\uparrow \times 1 \times 3\} \]
\[ \text{if not} \]
\[ \geq \]

Student: \[ 1 \uparrow \] 67 N.M.

1. \[ 1 \text{st} \] take the whole pile
2. \[ 2 \text{nd} \] make 2 piles equal

2. \[ 1 \text{st} \] leave 3 piles of 2 piles being equal
3. \[ 2 \text{nd} \] take whole unequal piles \[ \rightarrow 0 \]

3. \[ 1 \text{st} \] change to \[ 1 + 2 + 2 + \uparrow \]
4. \[ 2 \text{nd} \] \[ + 0 + 2 + 3 = 0 \]

4. \[ 1 \text{st} \] change to \[ 1 + 3 + 7 \]
5. \[ 2 \text{nd} \] \[ 1 + 3 + 2 = 0 \]

5. \[ 1 \text{st} \] change to

6. \[ 1 \text{st} \] \[ 1 + 4 \\uparrow \]
7. \[ 2 \text{nd} \] \[ 1 + 4 = 0 \]

8. \[ 1 \text{st} \] change to \[ 1 + 5 + 2 \]
9. \[ 2 \text{nd} \] \[ 1 + 5 + 4 = 0 \]
10. \[ 1 \text{st} \] change to \[ 1 + 6 + 4 \]
11. \[ 2 \text{nd} \] \[ 1 + 5 + 4 \rightarrow 0 \]
- Students (Ryan x Student) 3 4

1) 1st 3 4 1 5 4
   2nd 1 2 3

2) 1st 3 4 2
   2nd 1 2 3

3) 1st 3 4 5
   2nd 1 4 5

4) 1st 3 4 6
   2nd 2 4 6

Students 3 5 6

5) 1st 3 5 4
   2nd 1 5 4

6) 3 5 4

7) 3 5 2
   3 2 1

8) 3 5 1
   8 2 1

-
Students: \[\begin{array}{c}
123 \\
145 \\
167 \\
246 \\
257 \\
347 \\
356
\end{array}\]

\[2 \div 87 = 0\]

\[\sqrt{4.67} = 2.167 = 0\]

\[45 \text{ if 2nd player makes 44 when looped last} \]

Notation:
\[\frac{1}{4} + 5 + 6 + 7 = 0 + 4 + \frac{4}{5} \rightarrow \]

Students: \[2 \quad 3 \quad 5 \quad 7 = 0 + 3 = \_\]

\[1 + 345 + 6 + 7 = 0 + 345 = 31 = 2\]

\[\text{Chang 8+5 one it is *'x' game}\]
13 \times 6 \times 6 = 3 \times 11

Binary fraction sum:
5 = 1.01
6 = 1.10
\hline
0.0

3 \quad 0.11
4 \quad 1.00
7 \quad 1.11
\hline
0.00

If parity sum is zero, then its a zero position.

9.74 \quad 9 \quad 1.001 \quad 9.35

7 \quad 0.111
4 \quad 0.100
\hline
1.000

If class go first or me
8: go first
If you want to decrease one number

to get 000

which one do you think should be reduce?
8: The nine
7: why
6: Because they whispered
5: theynine
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Who wins this game of Frogs?
What is the best opening move?

\[
\begin{array}{cccc}
\text{LL} & \square & \text{RR} \\
\text{L} & \square & \text{L} & \text{R} \\
\text{L} & \text{L} & \text{L} & \text{R} \\
\text{L} & \text{R} & \text{L} & \text{R} \\
\text{R} & \text{L} & \text{L} & \text{R} \\
\text{L} & \text{R} & \text{L} & \text{R} \\
\text{R} & \text{L} & \text{L} & \text{R} \\
\text{L} & \text{R} & \text{L} & \text{R} \\
\text{L} & \text{L} & \text{L} & \text{R} \\
\end{array}
\]

Pertinent facts:

1. every negative number $\downarrow < 0 < \uparrow < \text{every positive number}$

2. $\uparrow + * = \{0, * \mid 0\}$ which is fuzzy

3. $\uparrow + \uparrow + * = \{0 \mid \uparrow\}$
Next office visit play NIMs.

10/17/03

Nim's addition

\[
\begin{array}{c}
12 \\
+ 9 \\
\hline
1001 \\
1001 \\
\hline
101 \\
\end{array}
\]

*12 + 9 * ? = 0

\[
\begin{array}{c}
12 \\
\times 5 \\
\hline
1000 \\
0000 \\
\end{array}
\]

Frog game (Teacher vs. Class)

Left →

\[
\begin{array}{ccc}
L & L & R \\
L & L & R \\
L & R & L \\
R & L & R \\
\end{array}
\]

→ Right

Don't

* in fuzzy

1st player wins.

Class night

Class 1st

\[
\begin{array}{c}
L & L & R & R \\
L & L & R & R \\
L & R & L & R \\
R & L & R & L \\
\end{array}
\]

or

\[
\begin{array}{c}
L & L & R & R \\
L & L & R & R \\
L & R & L & R \\
R & L & R & L \\
\end{array}
\]

1st
Red Poker NIM

clar 0 grade

Poker NIM is same as NIM with reversible moves added.

Survivor NIM (one pile)

take either 1, 2, or 3 objects
start with 21

if you get to a point with

*  
*  
*  

-> 

winning move - your position
One pile take 1, 2 or 3
Stack of 4 is a draw game 2nd player wins.

Stack  
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New game

\[ 5 = 2^2 + 3 + 0 \]

Do same \[ \succeq 1 \]

Largest value that is not with set \([\text{Mex}]\)

TEST 5 = \( \times 1 \)

\[ 5 - \times 1 = 0 \]

\[ 5 + 1 = 0 \]

\( \times \times \) + \( \times \times \) = 0

\[ 5 \rightarrow 3 \]

\[ 3 \rightarrow 1 \]

\[ 1 \rightarrow 1 \]

1st loser

1st move

\[ 5 \rightarrow 4 \]

2nd move

\[ 4 \rightarrow 1 \]

3rd move

\[ 1 \rightarrow 1 \]
\[ R + \downarrow \text{ who wins} \]

Blue starts Blue wins done before so it is not zero.

Blue wins

Red wins

Blue starts -> Blue wins
Red starts -> Blue wins

T: is this zero? S: no -
T: is it fuzzy? S: no -
T: is it true or not? S: Oh, it is up to because blue wins

Theorem: \[ R + \downarrow > 0 \]

So what is \[ \frac{e^R}{B} > 1 \] why because add A to both sides
### The Knight Game

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**Names**

[Mex 3 = 1, *0*]

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**10/2 #03**

---

raw_text: "Explain!!"
The Queen Game

Using the MEX theorem, fill in the nimbers.

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Sentence with $n$ as a variable

**Example**

The sum of the first $n$ positive integers is $\frac{n(n+1)}{2}$.

**Proposition**

Suppose $P(n)$ is a statement with variable $n$ (true or false).

To prove $P(n)$ is true

I) **Base**: true for $n=1$

II) **Prove**: if it is true for $n$ then it must be true for $n+1$

$P(n)$ \rightarrow $P(n+1)$

or $P(n+1)$ can be derived from $P(n)$

**Example**

Sum of first $n$ positive integers is $\frac{n(n+1)}{2}$

**Proof**

I) **Base**: $n=1$

$\text{Sum} = 1$

$$\frac{n(n+1)}{2} = \frac{(1)(2)}{2} = 1$$

For $n=1$, sum = formula

II) **Assume true for $n$**

i.e., $1+2+\ldots+n = \frac{n(n+1)}{2}$
Theorem: If \( G_1 = G_2 \) then \( H + G_1 = H + G_2 \)

\( G_1 = G_2 \) means \( G_1 - G_2 = 0 \)

To prove that \( H + G_1 - (H + G_2) = 0 \)

\( n \) (Math induction worksheet)

The sum of the first \( n \) Fibonacci numbers is \( F_{n+2} - 1 \)

\[ f_1 + f_2 + \ldots + f_n = F_{n+2} - 1 \]

\[ \sum_{k=1}^{n} f_k = F_{n+2} - 1 \]

Proof: \( f_1 = 1 \), \( f_2 = 1 \), \( f_n = f_{n-1} + f_{n-2} \) (or \( f_{n+2} = f_{n+1} + f_n \))

So \( f_3 = 2 \), \( f_4 = 3 \), \( f_5 = 5 \), \( f_6 = 8 \), \( f_7 = 13 \), ...

1. One number to add \( f_1 = 1 \)

\[ \text{left side: } f_1 = 1 \]
\[ \text{right side: } f_2 - 1 = 2 - 1 = 1 \] \( \checkmark \) done for the base step
7. **Problem** Let $D_n$ be the number of ways a $2 \times n$ chessboard can be covered by regular dominoes. Find $D_1$, $D_2$, $D_3$, $D_4$, and $D_5$. Guess a formula for $D_n$ and prove your answer by induction.

8. **Theorem** The sum of the first $n$ Fibonacci numbers is $f_{n+2} - 1$

9. **Theorem**

(a) In the Frog Game what is the value of $\square^m R \square^p L \square^q$ (This is after the jump)
Prove this by induction on $n = m + q$.

(b) Find the value of $\square^m L \square^p R \square^q$ with $m > 0$ and $q > 0$ (This is before the jump).

10. **Question** Suppose one person plays Solitaire Cuts with no red or blue lines, the player can cut vertically or horizontally. If the player starts with an $m \times n$ grid, how many cuts does it take to cut the grid into single squares? Prove your answer by induction (second form).

11. **Question** What’s the maximum number of non-intersecting diagonals that can be drawn inside a regular polygon with $n$ sides? Prove your answer by induction.

12. **Question** A solitaire game is played with $n$ coins in a row, some with heads showing and some tails. A move in the game consists of removing one coin that is heads up and flipping the adjacent coins, if any. You win by removing all coins. This game can’t always be won, it depends on the number of coins that are heads up at the start. Figure out when the game can be won and prove your answer by induction.

13. **Theorem** Given $N \geq 2$ squares of different sizes, they can be cut and reassemble (without overlap) to make a single square. The cuts should be ‘straight’ cuts.

14. **Theorem** A triomino is like a domino but consists of 3 squares instead of two. There are two different triominoes, the straight triomino and the right triomino (L-shaped). Suppose you have a large pile of right triominoes and the squares in these right triominoes are the same size as the squares in a chess board.
Prove: If you have a $2^n$ by $2^n$ chess board ($n \geq 1$) with one corner square removed then it can be completely covered with right triominoes.

15. **Theorem** The sum of the interior angles of an $n$-sided polygon is $(n-2) \cdot 180^\circ$.
You can use the following fact from geometry: For any $n$-sided polygon there is a line segment joining two vertices that divides the polygon into a triangle an an $(n-1)$-sided polygon.
Your turn - slide of your piece on its row
no jumping

one row game

[\[ LR \]]

zero position (2nd wins)

\[ LR \]

\[ LR \]

1st loses (2nd wins)

If there are spaces \( a \) between the first person wins
If no spaces \( a \) between the second person wins

Pancake Game - not impartial
(Blackboard)
- left and right have different moves
- values and numbers, \( +, \times, \uparrow, \downarrow \)

Find \( a \):

Hackenbush stick = \( \frac{1}{3} \)

(2) Find the value of the game on the opposite side of this page.

[\[ LR \]]

[\[ LR \]]

[\[ LR \]]

\[ LR \]

[\( x = 0.109 \)]

[\( x_2 = \approx 0.49 \)]

[\( x_2 = \approx 0.61 \approx 0.61 \)]
13) on math induction

Base two squares:

Oct 31, 03

Queen ←↑ only

Race ←↑ same as adding numbers

11/2/03
Friday/November
12:30 Showers / Monday 1:30

Protest: Reagan / Monday 2:15

North Watts Game
Shawn: Find a haken bush with worth $\frac{1}{3}$

\[
\frac{1}{3} = 0.101 \ldots
\]

T: Can you show me that this game

Sh: 1

B

R

B

R

B

Value \( \frac{1}{2} - \frac{1}{4} \), \( \frac{1}{2} > 0 \)

Class picks - blue
Teacher picks to go second.

B

R

B

R

B

Value = 0

Better move is to go to the blue higher up. The value is true.
Theorem: Suppose \( G \) is a switch game \( G = \mathcal{S}_x \times \mathcal{S}_y \) with \( x > y \). If \( z \) is a number, i.e. \( z = \frac{a+1}{b} \), \( a < z < b \).

Then:
1. If \( z > y \) then \( z > \ell_y \).
2. If \( x > y > z \) then \( z < \ell_y \).
3. If \( x > z > y \) then \( z - y \) is fuzzy.
Number + switch. (Both players are better off doing a switch.)

Theorem 2:

$x + (y + z)$  

$x \geq y, z \in \{a, b\}$

$L \rightarrow a + (x + y)$  (play game first)

$L \rightarrow z + x$  (play switch first) ← better, why?

Dominating:

\[
\begin{array}{ccc}
0 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
\end{array}
\]

defect by $\phi^2$

changes values to negative

\[
\left( \begin{array}{c}
0 \\
1 \\
1 \\
\end{array} \right) \rightarrow (0, 0, 0) \rightarrow (0, 0, 0)
\]

if round

Homework: Find values.
NIM

1. Facts
The value of a NIM heap of \( n \) objects is \(*n\).
If the value is a sum of NIM heaps is zero, the position is called a P-position (Previous player wins).
If the value is a sum of NIM heaps is not zero, the position is called a N-position (Next player wins). So an N-position is fuzzy.
\(*n + *n = 0\) or \(*n + *n\) is a P-position
if \( n \neq k \) then \(*n + *k\) is an N-position since the next player can play to two equal piles.
If there are 3 unequal piles and it's your turn, you will lose if you erase one of the piles.
If there are 3 piles, and it's your turn, you will lose if you equalize two of the piles.

\(*1 + *2 + *3 = 0\) since the first player has to erase a pile of equalize two of them.
So \(*1 + *2 + *3\) is a P-position.
If \( 1 \leq x < y \leq 3 \) and \( z > 3 \) then \(*x + *y + *z\) is an N-position
since the next player can play to \(*1 + *2 + *3\)
The following are all equal to zero, and so are P-positions

(a) \(*1 + *2 + *3 = 0\)
(b) \(*1 + *4 + *5 = 0\)
(c) \(*1 + *6 + *7 = 0\)
(d) \(*2 + *4 + *6 = 0\)
(e) \(*2 + *5 + *7 = 0\)
(f) \(*3 + *4 + *7 = 0\)
(g) \(*3 + *5 + *6 = 0\)

2. Definition
To do the NIM sum of binary numbers, put the numbers in a column. If there are an even number of 1's in a column, the answer for that column is 0. If there are an odd number of 1's in the column, the answer for that column is 1.

3. Fact
If you change some of the bits in a binary number and the left most change is a change from 1 to 0 then the new number is less than the original number.
14. **Problem TripleCross, a version of Tic-Tac-Toe**
   
   Put a row of \( n \) boxes on the blackboard. When it's your turn you put an \( X \) in a box. If you complete a row of three \( X \)'s then you win. Since this is an impartial game, it can be analyzed by using NIM heaps and NIM addition, but how?

15. **Problem Rip Van Winkle's Game**

   Starting with one or more heaps, each player can:

   (a) Remove one or two beans from a heap or,

   (b) Remove one or two beans from a heap and make two heaps from what's left.

   If there is one heap at the start then player One can win by dividing it into two equal heaps so that the copy-cat strategy applies. If there are two heaps of size \( n \) and \( n+1 \) or size \( n \) and \( n+2 \) two then player One can do the same thing. What if there are two heaps with size difference three or more?

xx
Grundy's Game

Show the sets for that are used to find the MEX for $!10$ to $!19$.

<table>
<thead>
<tr>
<th>stack</th>
<th>$!0$</th>
<th>$!1$</th>
<th>$!2$</th>
<th>$!3$</th>
<th>$!4$</th>
<th>$!5$</th>
<th>$!6$</th>
<th>$!7$</th>
<th>$!8$</th>
<th>$!9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NIM</td>
<td>*0</td>
<td>*0</td>
<td>*0</td>
<td>*1</td>
<td>*0</td>
<td>*2</td>
<td>*1</td>
<td>*0</td>
<td>*2</td>
<td>*1</td>
</tr>
<tr>
<td>equiv</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>stack</th>
<th>$!10$</th>
<th>$!11$</th>
<th>$!12$</th>
<th>$!13$</th>
<th>$!14$</th>
<th>$!15$</th>
<th>$!16$</th>
<th>$!17$</th>
<th>$!18$</th>
<th>$!19$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NIM</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>equiv</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$!3 \rightarrow !1 + !2$

$!4 \rightarrow \{ !3 + !1, !4 \}
\{ !1 + !0 \}$

$!5 \rightarrow \{ !1 + !4, !2 + !3 \}
\{ !0, !0 + !1 \}$
Shanna vs. 2x11\textsuperscript{2}\\
\[ 2x11^2 \xrightarrow{L} \star \rightarrow \text{right wins} \]
\[ R \xrightarrow{1} \text{left wins} \]

who even goes first loser = it's a zero gain
so I can make it into 0 = a draw

\[ \exists x^2 = 0 \quad \text{if } x > 0 \]
\[ \exists x^2 = 0 \quad \text{if } x < 0 \]

\underline{Student}

**Theorem 1 on switches**

\[ \star + 3x^2y^3 = \begin{cases} \star & +x \quad \star + y^3 \end{cases} \]
\[ \star + 3x^2y^3 \quad \Rightarrow \star + x \quad \star + y^3 = 0 \]
\[ \star + 3x^2y^3 \quad \Rightarrow \star - y \quad \star - x^3 = 0 \]
\[ \L_1 \implies 0 + 3x^2y^3 \quad \Rightarrow \star - y \quad \star - x^3 \]
\[ \L_2 \implies 2010^3 + x + 3 \quad \star - y \quad \star - x^3 \]

\[ \L_3 \implies 2010^3 + x + 3 \quad \star - y \quad \star - x^3 \]

\[ \L_4 \implies 2010^3 + x + 3 \quad \star - y \quad \star - x^3 \]

\[ \text{Bad move} \quad \text{Both lose} \]

\[ \L_2 \quad 2010^3 + x + 3 \quad \star - y \quad \star - x^3 \]

\[ \L_3 \quad 2010^3 + x + 3 \quad \star - y \quad \star - x^3 \]

\[ \L_4 \quad 2010^3 + x + 3 \quad \star - y \quad \star - x^3 \]

\[ \star + x \quad \star + y^3 \]

\[ \L_2 \implies \star + x \quad \star + y^3 \]

\[ \L_3 \implies \star + x \quad \star + y^3 \]

\[ \L_4 \implies \star + x \quad \star + y^3 \]

\[ \text{Left wins} \]

\[ \text{Right wins} \]
10. **Theorem**

*Both players prefer switches to Star, unless the switch has temperature zero.*

Suppose \( \{ x \mid y \} \)

(a) If the temperature of the switch is greater than zero then both players prefer the switch to Star.

(b) If the temperature of the switch is zero then both players, when asked which they prefer, say, "whatever."

11. **Theorem**  *Star plus a switch again*

Suppose \( \{ x \mid y \} \) is a switch with \( x \geq y \). Then \( * + \{ x \mid y \} = \{ x * \mid y * \} \)  \( \star \star \ = \ x + \star \)

12. **Question**

What are \( \{ * \mid 1 \} \) and \( \{ -1 \mid * \} \)? Generalize this to a theorem.

13. **Question**  *Star plus a lukewarm switch*

What is \( * + \{ z \mid z \} \)?

14. **Theorem**

If \( x \geq y \). Then \( * + \{ x * \mid y \} = \{ x \mid y * \} \) and \( * + \{ x \mid y * \} = \{ x * \mid y \} \)
Rules: This is a subtraction game. You can remove a prime number of objects from the heap. For this game, 1 is considered to be prime.

Show the sets for that are used to find the MEX for !8 to !13.

<table>
<thead>
<tr>
<th>stack</th>
<th>!0</th>
<th>!1</th>
<th>!2</th>
<th>!3</th>
<th>!4</th>
<th>!5</th>
<th>!6</th>
<th>!7</th>
<th>!8</th>
<th>!9</th>
</tr>
</thead>
<tbody>
<tr>
<td>NIM equiv</td>
<td>*0</td>
<td>*1</td>
<td>*2</td>
<td>*3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>stack</td>
<td>!10</td>
<td>!11</td>
<td>!12</td>
<td>!13</td>
<td>!14</td>
<td>!15</td>
<td>!16</td>
<td>!17</td>
<td>!18</td>
<td>!19</td>
</tr>
</tbody>
</table>
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Show the sets for that are used to find the MEX for 8 to 13.

<table>
<thead>
<tr>
<th>stack</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>NIM equiv</td>
<td>*0</td>
<td>*1</td>
<td>*2</td>
<td>*3</td>
<td>*0</td>
<td>*4</td>
<td>+2</td>
<td>+3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>stack</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
</tr>
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<td></td>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Game of three stacks:

```
\[ \begin{array}{ccc}
  & +2 & +3 \\
+2 & \times 1 & +2 \\
\end{array} \]
```

\[ \rightarrow 2 \times 2 \]

Copy Cat.
APPENDIX U

PROBLEM SET
PROBLEM SET

1. Definitions: Let \( G \) be a game.
   
a. \( G = 0 \) means whoever starts loses.
   
b. \( G > 0 \) means Left wins no matter who starts
   
c. \( G < 0 \) means Right wins no matter who starts.
   
d. \( G \parallel 0 \) (\( G \) is fuzzy if whoever starts wins.
   
e. Let \( G \) be any game. The negative of \( G \) is the same game as \( G \), but with the
   moves of left and Right interchanged. For example, if \( G \) is a Hackenbush
   game, you can draw \(-G\) as the same picture but with the colors reversed.

A Hackenbush game consists of one or more colored sticks (blue or red) the sticks
are either connected to the ground or else to other sticks. Each player (team) has a color
assigned to him/her (it). Who has the turn to play removes one stick of his/her (its) color
and with it all other sticks connected to it and above it. So if the game consists of a stack
three stick high, with blue connected to the ground and the two sticks above it are to reds,
then if blue removes his stick then he must remove the two red sticks as well. The player
losses when it is his/her turn and have no sticks left to remove. Examples of Hackenbush
games are
Figure 685. Examples of Hackenbush games

Notation: when there is no ambiguity, sticks will not be included in the diagram of the game. Instead the letters R and B will be used in vertical formation to indicate a stack of sticks with the indicated letters. The letter at the bottom of the stack is assumed to represent the stick connected to the ground. The following is an example that illustrates that notation

The above game is represented as follows

R
B
1. Theorem
\[ R \]
\[ = \frac{1}{2} \]
\[ B \]

2. Theorem
\[ R \]
\[ R = \frac{1}{4} \]
\[ B \]

3. Theorem
\[ R \]
\[ B = \frac{5}{8} \]
\[ R \]
\[ B \]
4. Theorem
R
R = 5/4
B
B

5. Theorem
R
R = 1/8
R
B

6. Theorem
R
B = 5/2
B
B
7. Theorem
\[ R = \frac{7}{4} \]

8. Theorem
\[ R = \frac{3}{2} \]

9. Theorem
\[ R = \frac{3}{4} \]
10. Theorem
B
B = 7/8
R
B

11. Theorem
B
R = 3/8
R
B

Definition: Let G be any game. We write G = {x | y} where x is the value of Left’s best move and y is the value of Right’s best move.

12. If G = {x | y} with x < 0 < y then the value of G is always the same. What is the value of G?
13. Theorem:

\[ R \]

\[ B \]

\[ = 1 \]

14. Theorem

\[ R \]

\[ = \{0|1\} \]

\[ B \]

15. Theorem: If \( G \geq 0 \) then Left wins when Right starts. If \( G \leq 0 \) then Right wins when Left starts.

Notation: Let \( G \) and \( H \) be two games, then \( G + H \) (or \( H + G \)) is game consist of the two games where players can move in either game when it is his/her turn. They can only move in one game at a time.

1. Theorem: (The meaning of Zero) suppose \( G \) is a zero game (whoever starts loses.)

   f. If \textbf{Left} wins \( H \) when \textbf{Left} starts then \textbf{Left} wins \( H + G \) when \textbf{Left} starts \( H + G \).

   g. If \textbf{Left} wins \( H \) when \textbf{Right} starts then \textbf{Left} wins \( H + G \) when \textbf{Right} starts \( H + G \).

   h. If \textbf{Right} wins \( H \) when \textbf{Right} starts then \textbf{Right} wins \( H + G \) when \textbf{Right} starts \( H + G \).
i. If **Right** wins $H$ when **Left** starts then **Right** wins $H + G$ when **Left** starts $H + G$.

**Summary:** If $G = 0$ then $H + G = H$ (as games).

16. **Theorem:** (Simplicity Theorem-Second case).
Suppose $G = \{x \mid y\}$ with $x < 0 < y$. If there is an integer between $x$ and $y$ then $G$
eq equals the least such integer. I.e. suppose $n$ is an integer with $0 \leq n - 1 \leq x < n \leq y$.

Then the value of $G$ is $n$.

17. What is the analogous theorem for $G = \{x \mid y\}$ with $0 \leq x < y$?
18. **Theorem:** Suppose $G = \{x \mid \}$ (This means **Right** has no play at all). Then
   
   j. If $x < 0$ then $G = 0$

k. If $0 \leq x$ then $G$ is the least integer greater than $x$.

19. What is the analogous theorem if $G = \{x \mid \}$?
20. **Theorem:** Simplicity Theorem, Third case
If $G = \{x \mid y\}$ with $0 \leq x < y$. If there is no integer between $x$ and $y$ but there is a
   
   fraction with denominator 2 between $x$ and $y$ then $G$ equals that fraction. I. E.
   
   Suppose $n$ is an integer with $0 \leq n \leq x < n + \frac{1}{2} \leq y \leq n + 1$. Then the value of $G$ is $n + \frac{1}{2}$.

21. **Theorem:** The negative of $G = \{x \mid y\}$ is $\{-y \mid -x\}$. To prove this, show
   
   \{$x \mid y\} + \{-y \mid -x\} = 0$

**Definitions** $\{0 \mid 0\} = *$

**Note:** the frog game

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>R</td>
</tr>
</tbody>
</table>

is a * game

22. $* + * = 0$
23. Theorem: * is less than every positive number and greater than every negative number. To show this, it suffices to prove \( * < \frac{1}{2^n} \) and \( \frac{1}{2^n} < * \) for every positive integer \( n \).

24. Theorem

\[ R^n = \frac{1}{2^n} \]

B

25. If \( x > 0 \) then \( \{ * \mid x \} = ? \)
   If \( x < 0 \) then \( \{ x \mid * \} = ? \)

26. \( \{ * \mid * \} = ? \)

27. Theorem: \( \{0, * \mid 1\} = \frac{1}{2} \)

28. Theorem: \( \{0, * \mid \frac{1}{2}\} = \frac{1}{4} \)

29. Theorem: \( \{0, * \mid \frac{1}{4}\} = \frac{1}{8} \)

30. Theorem: \( * + 1 = \{ 1 \mid 1 \} \)
   Hint: Play \( * + 1 - \{ 1 \mid 1 \} = \{ 0 \mid 0 \} + \{ 0 \mid 2 \} + \{ -1 \mid -1 \} \)
31. If \( x \) is any number \( \ast + x = \{x \mid x\} \). Hint: Since \( x \) is a number, \( x = \{y\mid z\} \) with \( x < y < z \). Then play a game similar to the preceding theorem.

32. Theorem: \( \uparrow = \{0\mid \ast\} \) and \( \downarrow = -\uparrow = \{\ast\mid 0\} \)

33. Theorem: \( 0 < \uparrow < \) every positive number.

34. Theorem: every negative number \( < \uparrow < 0 \).

35. Theorem: \( \{0, \ast\mid 0\} = \uparrow + \ast \)

36. Theorem:
   
a) \( \{\uparrow\mid 0\} = \ast \)
   
b) \( \{0\mid \downarrow\} = \ast \)
   
c) \( \{\uparrow\mid \uparrow\} = \ast \)
APPENDIX V

GRADING SCHEME
Grading in the class:

**C**: Do all the homework, group projects, and office visits. A few in-class presentations. Good questions and conjectures will help your grade to a B.

**B**: Same as “C” with more presentations and/or presentations of some harder or non-routine problems and some problems that require more independent thinking. Presentations of some harder things may be done outside of class.

**A**: Same as “C” with more presentations and presentations of some harder or non-routine problems and some problems that require more independent thinking. Helping with the direction the class will take is a plus.

Solutions to problems should be the student’s own solutions or the student’s and his/her partner for group work.
REFERENCES


Battista, M. T. (1999b). *Notes on constructivist view of learning and teaching mathematics*. Unpublished manuscript, Kent State University, Kent, OH.


