MULTIFRACTAL MODELS AND SIMULATIONS
OF THE U.S. TERM STRUCTURE

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CHAPTER 1
INTRODUCTION

Interest rates form major driving factors in finance. A change in interest rates usually has a substantial impact on an investor’s rate of return. When an investor requires higher (lower) rates of return from his investment given that all else are the same, his willingness to pay for the investment is lower (higher). In corporate finance, interest rates are part of the cost of capital impacting a firm’s investment selections. A firm’s projects may initially appear to be very attractive and profitable. However, if these projects are very sensitive to changes in interest rates, e.g., projects which heavily rely on substantial debt financing, the firm may find that such projects are unattractive after taking account of the interest rate risks involved.

Interest rates are also crucial in the field of derivatives. An investor can trade his interest rate risk in derivatives markets via interest rate options. Interest rate options currently show high trading volumes, implying that they are popular among investors. However, to fairly price interest rate options so as to price the interest rate risk, investors need to understand the term structure of interest rates.

The very first interest rate model, known as the one-factor affine model, or simple linear model with intercept, was introduced by Vasicek (1977).\textsuperscript{1} Since then, interest rate

\textsuperscript{1}Dr. Oldrich Alfons Vasicek who first introduced the dynamic equilibrium models for the term structure was named 2004 IAFE/Sungard Financial Engineer of the Year. IAFE = the International Association of Financial Engineers.
modeling has grown rapidly. Several models, ranging from very simple to very complex in terms of techniques employed, have been proposed. Unfortunately, only a few of them really provide tractability and are simply enough to earn popularity among practitioners and academicians. A common trait of the more popular models is the assumed underlying stochastic process and the number of factors needed as inputs. According to the number of inputs, one- or two-factor models of interest rate are preferred, as long as they can fairly explain the term structure of interest rate.

Perhaps, more importantly, most of these popular models assume Geometric Brownian Motion (GBM) for their price diffusion processes. This is not a coincidence, since it is imperative for transparency and scientific logic that the proposed models are tractable. In addition, quite sophisticated investors prefer the GBM, as they may already be familiar with it from the usual options pricing formulas. However, there might be a substantial price to pay for this familiarity and convenience. Accumulating evidence in the academic financial literature already indicates that the GBM based models cannot explain several features of the empirical financial data. In other words, empirical financial data do not strictly adhere to the GBM. Already in 1970 Roll studied the distribution of the Treasury bill yield changes using the stability index or Lipschitz-α and found that "With a large degree of confidence, most of the distributions of interest rate changes have α significantly lower than 2 and are thus nonnormal. Indeed, the upper limit of the simulation range suggests that most of the α’s are significantly lower than 1.5" (p. 73).

The characterizing feature of the GBM is that it scales the instantaneous volatility of price processes according to the square root of the maturity or time horizon. This scaling
characteristic can greatly affect actual derivatives prices when the underlying asset prices do not adhere to such time-frequency scaling and thus not to the GBM.

The Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model was introduced by Bollerslev (1986) to explain the time-varying volatility issue, although it does not time-scale the volatility properly. This model earned the Nobel Memorial Prize in Economics in 2003. The main drawback of the GARCH model is that its users experience serious difficulties with identifying the proper order (= number of autoregressive time lags). Practically, the GARCH (1, 1) is now very common in time series modeling, mainly due to its simplicity. However, the GARCH models cannot capture important characteristics of empirical financial data such as time-and-frequency scaling and long-range dependence, or "Long Memory". An interesting question is then what kind of models can more closely preserve the market pricing characteristics than the GBM and GARCH models? This study proposes the Multifractal Model of Asset Return (MMAR) as a viable and empirically more satisfying substitute.

The objective of this dissertation is to introduce and analyze a new process that is expected to replace the GBM and to surpass the GARCH process in the area of empirical interest rate modeling. Because the MMAR is a combination of the monofractal model and multiplicative probability measures, it has most, if not all, desired properties found in almost all empirical economic and financial data series. Those properties consist of time-and-frequency scaling, long memory, nonstationarity with stationary increments, thick-tail distribution, volatility clustering, and consistency with martingale (= fair market) pricing, i.e., it does not allow time arbitrage.
The fractional Brownian Motion (fBM), a predecessor of the MMAR, was originally introduced by Mandelbrot and Van Ness (1968). Its successor and improvement, the MMAR, was further developed by Mandelbrot, Fisher, and Calvet (1997) and Calvet and Fisher (2002).

The MMAR has already shown high potential as a substitute for the ARCH family models. This dissertation implements the MMAR for the first time as the underlying process of the U.S. Treasury interest rate series for various maturities. Supported by the Cash Flow theory of Los (2003), each interest rate series for a particular maturity will have its own characteristic underlying MMAR process. Thus, it is highly possible that one might eventually be able to devise a system of price diffusion models that empirically identifies the complete term structure of US Treasury interest rates, while tractability is maintained, although that has not been accomplished in this dissertation. This dissertation introduces and analyzes the MMAR for its use in term structure modeling, and compares its performance with that of the affine GBM and GARCH models.

The Monte Carlo simulations are the main tool used in the model performance measurement. It should be recognized that this study does not attempt to price any bonds. Rather, the dissertation focuses on the analysis of simulated fractal price diffusion processes of the interest rates with the expectation that this analysis will have a large influence in the further development of interest rate modeling.

The dissertation is structured as follows. Chapter 2 reviews the literature about interest modeling, GARCH time series modeling, and the fractality hypotheses of this study. Chapter 3 provides details of the empirical data and the MMAR methodology used. This chapter presents a detailed methodology of how to detect multifractality, to identify
the crucial parameters of the return series, to reconstruct the series using the identified parameters, to simulate the return process, to measure the model’s overall performance, and to analyze the higher moments of the simulated results. Chapter 4 presents the empirical results while Chapter 5 discusses them in detail. Finally, Chapter 6 concludes this study and summarizes the effectiveness of the MMAR. This dissertation is the first to use the MMAR to identify and simulate the nodal market pricing of the U.S. term structure. We need such models of the interest rate market so that we can use them to assess risk, analyze investments, or insure against catastrophic losses.

Our main findings are that the interest rate series from instantaneous to 10-year maturities are nonlinear. They also exhibit the system of Long Memory where the underlying processes follow the Multifractal fractional Brownian Motion (Multifractal fBM). In addition, the resulted model simulations suggest that the MMAR is superior to the GBM and GARCH(1,1) processes in explaining the time-scaling characteristics of the interest rate series implying that any existing interest rate models that are based on the GBM and GARCH(1,1) should be discarded. Most importantly, the overall findings raise the need of a better dynamic equilibrium model of the U.S. term structure under the multivariate Multifractal FBM framework proposed at the end of this dissertation.
Chapter 2

Literature Review: U.S. Term Structure of Interest Rates Modeling and its Evolution

One commonly finds three main term structure theories in most financial textbooks: theories based on the respective concepts of (1) unbiased expectations, (2) liquidity premia, and (3) market segmentation. Each of these theories is briefly reviewed below. In addition to the above theories, Los (2003) proposes the Cash Flow theory explaining the term structure in terms of cash flows with various degrees of market persistence.

2.1 Theories of the Term Structure of Interest Rates

2.1.1 Unbiased Expectations Theory

According to the unbiased expectations theory of the term structure of interest rates, at a given point in time the yield curve reflects the markets’ current expectation of future short-term rates. Specifically, the unbiased expectations theory assumes that current long-term interest rate are geometric averages of current and expected future short-term interest rates. In discrete time, as follows;

\[(1 + R_N)^N = (1 + R_1)(1 + E\{r_2\})...(1 + E\{r_N\})\]  

(2.1)
therefore:

\[ R_N = [(1 + R_1)(1 + E\{r_2\})...(1 + E\{r_N\})]^{1/N} - 1 \]  \hspace{1cm} (2.2)

where

\[ R_N = \text{Actual N-period rate today} \]
\[ N = \text{Term to maturity} \]
\[ R_1 = \text{Actual 1-year rate today} \]
\[ E\{r_t\} = \text{Expected one-year rates for years 2, 3, 4, \ldots, N in the future} \]

2.1.2 Liquidity Premium Theory

The liquidity premium theory can be viewed as an extension of the unbiased expectations theory. Its concept is that investors will hold long-term maturities only if they are offered at a premium to compensate for future uncertainty in a security’s value, which increases with an asset’s maturity. The theory states that long-term rates are equal to geometric averages of current and expected short-term rates with additional liquidity risk premia, which are assumed to increase with the maturity of the security.

The liquidity premium theory might be mathematically represented as:

\[ R_N = [(1 + R_1)(1 + E(r_2) + L_2)...(1 + E(r_N) + L_N)]^{1/N} - 1 \]  \hspace{1cm} (2.3)

where \( L_t \) = liquidity premium for period \( t \), and usually it is assumed that \( L_2 < L_3 < \ldots L_N \)

2.1.3 Market Segmentation Theory

The market segmentation theory suggests that individual and institutional investors have specific maturity preferences. To persuade them to hold securities with maturities other
than their preferred ones requires higher interest rates. Accordingly, the theory does not consider securities with different maturities as perfect substitutes.

Rather, investors have preferred investment horizons controlled by the nature of the assets and liabilities they hold. Thus, interest rates are determined by distinct supply and demand conditions within particular maturity segment. The major assumption is that investors and borrowers are generally unwilling to shift from one maturity sector to another without sufficient compensation in the form of an interest rate risk premium. The market segmentation theory suggests inefficiency in the general equilibrium of the partially non-communicating interest rate markets, ranked by maturity, which is not completely eliminated by inter-market arbitrage.

The market segmentation theory might be mathematically represented as:

$$R_N = [(1 + R_1)(1 + E\{r_2\} + L_2 + \varepsilon_2)...(1 + E\{r_N\} + L_N + \varepsilon_N)]^{1/N} - 1$$ (2.4)

where $\varepsilon_t = \text{market inefficiency cost for period } t$.

The market segmentation theory encompasses the liquidity premium theory, which encompasses the rational expectation theory. However, none of these contemporaneous theories states anything about the dynamics of each of the interest rate processes. In contrast, this dissertation focuses on the dynamics of the nodal markets of the term structure and how it characterizes each of the interest rate models.

2.1.4 Cash Flow Theory

To make a connection between the static categorization of interest rate markets and their dynamics, we need to use the concept of changing cash flows over time.
The cash flow concept is a very important financial concept. Both small and large businesses will not survive if they are facing insufficient cash flows from assets. A lack of cash flows in operations reflects an illiquidity problem within the company.

In the past, the level of liquidity was discussed in the terms of two extremes only; there was either liquidity or illiquidity. Measuring the degree of liquidity along a whole range of values is a difficult task. However, with the introduction of fractality by Mandelbrot and Van Ness (1968) and the progress of the studies of the turbulence of fluid flows, it is now possible to quite precisely measure the degree of persistence (= "illiquidity") of cash flows. Still, no one has yet found which level of persistence leads to either turbulence (under super-liquidity) or to catastrophes or discontinuities (under sub-liquidity).

But extensive advanced research is currently on its way, and this particular cash flow concept can be applied directly to the term structure of interest rates. In the Treasury bill markets investors might prefer to trade on their own investment horizons. Some institutional investors like insurance and pension funds are likely to invest in long-term investments, while small investors usually prefer short-term investments. When the two types of investors coexist in the same market, there can be a high pressure on the cash flow streams that cause volatility within and across investment channel of particular maturities. This empirically observed phenomenon is totally opposite to the assumed-to-be perfect financial markets in which all cash flows move into the same direction in a particular investment channel and at the same speed. Thus, the perfect markets can only explain the simplest shapes of the term structure and the parallel up-and down, one-factor, shifts of it. But this excludes the changing curvature and deformation found in empirical term structures.
Although a fluctuation of cash flow streams within a particular maturity is, for most of us, unobservable, regardless the observable orderflow used by Brandt and Kavajecz (2004), a reasonable proxy is the change of interest rates for the same maturity. Thus for the theory of cash flows it is imperative that each maturity of Treasury bill rate has its own interest rate model. To study the term structure across the eight different maturity "nodes," as is customary for the US Treasuries, one should expect to have eight different interest rate models, or a system of interest rate models, where each model is distinguishable by its own characteristic degree of liquidity and/or degree of inefficiency. With such a system of price diffusion models for each interest rate maturity, the investigation of co-movements of higher moments across the maturities may become possible.

Brandt and Kavajecz (2004) provide empirical strong evidence supporting the possibility of implementing such a cash flow theory in the U.S. Treasury markets. They found that orderflow imbalances (excess buying or selling pressure) can explain up to 26% of the day-to-day variation in yields on days without major macroeconomic announcements and that the impact of the imbalances could last longer than two weeks, implying the existence of long memory in the Treasury markets.

Before the concept of liquidity levels and cash flow streams are combined to devise a new kind of interest rate model, the basic ideas of conventional interest rate modeling, asset price diffusion processes, and time-and-frequency scaling properties of such processes need to be properly defined.
2.2 Interest Rate Modeling

In the past three decades, several interest rate models have been proposed. Some of them are seemingly successful and have been further developed; some failed from the very beginning. The very first interest rate model was introduced by Vasicek (1977). The model is also known as a one-factor affine model, since it is driven by only the short term interest rate, or a linear model with an intercept. Other popular affine models include the two-factor model of Longstaff and Schwartz (1992) and the one-factor model of Hull and White (1993). In particular, the two-factor model followed the introduction of the extended Vasicek and extended Cox, Ingersoll, and Ross (CIR) one-factor affine models. Although the two-factor affine model can much better explain the shapes and movements of the term structure of interest rates than the one-factor affine model, there remained considerable room for improvement.

The next generation of affine models, the three-factor affine models, was proposed by Balduzzi et al. (1996). Based on the principal components analysis of the covariance matrix of interest rates, Brandt and Kavanecz (2004) suggest that more than one, but not many more than three factors, can sufficiently capture the shape and day-to-day variation of yield curve.

Apart from the class of affine models, there exist three other major classes of interest rate models: (1) the Heath, Jarrow, and Morton (HJM) model, (2) market models, and (3) a group of idiosyncratic models. As an extension of the Ho and Lee (1986) model and proposed by Heath et al. (1992), the HJM model depends on the evolution of forward rates. This model is widely used because of its flexible framework. Several of the market models
have been recently introduced by Brace \textit{et al.} (1997), Jamshidian (1997), and Musiela and Rutkowski (1997). Each of these models has its own peculiar assumptions.

The idiosyncratic models are those models that cannot be classified into the first three classes. This idiosyncratic class includes the Consol, Price Kernel, Positive, Non-linear, and GARCH models. These models have attracted much attention of both academicians and practitioners because of their novelty and empirical promise. However, these models often lack of tractability and identifiability, issues that quickly become major obstacles when one implements such models for bond valuation. In fact, lack of tractability and of unambiguous identifiability appears to be a major issue of all nonlinear model classes. James and Webber (2000) provide an extensive collection of the interest rate models in all of the classes mentioned above.

To provide the proper context for this dissertation, the following subsection provides some details of the affine models and GARCH-based models as they appear in the interest rate modeling literature.

2.2.1 Affine Models

An interest rate model is considered an affine model when a pure discounted bond price recovered from the model has the following exponential affine term:

$$B_t(X_t, \tau) = e^{a(\tau)+b(\tau)X_t} \quad (2.5)$$

where $B_t$ is a bond price function of the state variable $X_t$, and an investment horizon or time to maturity $\tau$ is $T - t$. The exponent on the right hand side of the equation indicates
clearly an affine function of the state variable. Then, spot rates can be written as

\[ r_t(\tau) = -\frac{a(\tau)}{\tau} - \frac{b(\tau)'}{\tau}X_t \]  

(2.6)

In fact, Duffie and Kan (1996) proves that the state variable \( X_t \) process must be in the form of

\[ dX_t = (MX_t + m)dt + \Sigma \Lambda_t dz_t \]  

(2.7)

where \( dz_t \) follows the Wiener Brownian Motion. \( M, \Sigma \) are constant matrices. \( \Lambda_t \) is a diagonal matrix with a particular form, \( m \) is a vector of constants. Understanding the process of the state variables is very useful because it allows one to classify all available affine models. Dai and Singleton (2000) suggested that the affine models should be classified according to the number of state variables and how many of those variables are present in the volatility matrix.

The affine models are very popular ranging from the simple one-factor model of Vasicek (1977), the mean-reverting short rate dynamic model of Cox et al. (1985), the extended Vasicek model of Hull and White (1990, 1993), the two-factor models of Longstaff and Schwartz (1992), the two-factor dynamic mean model of Sorensen (1994), and to the three-factor model of Balduzzi et al. (1996). For example, the classification of one-factor models can begin with the Vasicek model where the Brownian Motion has the following simple form of

\[ dX_{1,t} = -k_{1,1}X_{1,t}dt + dz_{1,t} \]  

(2.8)

where \( k \) is a vector, and the other variables are as mentioned earlier.
Unlike the Vasicek model, the CIR model simply requires more restrictions on the drift vector and the Wiener Brownian Motion such that there is a mean reversion and non-negative value from the result. The conventional process of the CIR is

\[ dX_{1,t} = k_{1,1} (\theta_1 - X_{1,t}) dt + \sqrt{X_{1,t}} dz_{1,t} \]  \hspace{1cm} (2.9)

The two-factor affine models can be extended easily by assigning one more set of state variables and simply choosing the framework of either Vasicek or CIR. The same concept can be applied to general \( n \)-factor affine models with some variations in the restrictions on the Wiener Brownian Motion.

Prior research suggests that the simplest and, not surprisingly, most popular affine models are the one-factor models. Chapman and Pearson (2001) argue that the source of their popularity derives from empirical studies using a principal component analysis showing that approximately 90 per cent of the variation of the term structure can be attributed to the first principal component. In other words, 90 percent of the variation in interest rates can be explained by parallel shifts in the level of the whole yield curve suggesting that any point on the term structure can be used as a proxy for the whole yield curve. Generally, the intercept of the term structure, or the instantaneous short rate of interest, is then chosen to represent that one factor. Thus, the short, or instantaneous, interest rate can be expressed mathematically as the following linear combination of state vector elements:

\[ r_t = f + u_1 X_{1,t} + \ldots + u_n X_{n,t} \]  \hspace{1cm} (2.10)
The same principle component analysis is also applied to the two-factor models such as those of Brennan and Schwartz (1979), Chen and Scott (1992), and Longstaff and Schwartz (1992).

Recently, Kennedy (1997), Goldstein (2000), and Santa-Clara and Sornette (2001) introduced the so-called random field or stochastic string model. This model attempts to incorporate an infinite-dimensional Gaussian shock for each point of the forward curve implying that each point of the yield curve has its own particular affine model. However, in practice, the observations of the yield curve are limited to eight or nine well-defined maturity "nodes." Thus, the empirical shocks are finite in number. Moreover, there is an obvious problem of reconciling the infinitely dimensional theoretical model with the finite number of empirical observations available.

The most recent development in this affine class of models belongs to Johannes (2004), who attempts to introduce jumps into the conventional affine diffusion model. This model successfully replicates the thick-tail behavior found in interest rate series. However, a priori knowledge of the new information is required to identify the model correctly with the proper jumps. This defeats the whole idea of scientific model identification so that future behavior can be predicted.

The crucial advantages of the affine models include their tractability and availability of numerical solutions. The tractability comes from the nice properties of the stationary and independent Wiener shocks of the Brownian Motion. Unfortunately, although these affine models are very tractable, they do not properly identify the behavior of the real empirical term structure.
2.2.2 Limitation of the Affine Models

Los (1989, 2003) and Rebonato and Cooper (1996) argue that the prejudicial statistical pitfalls of the principal component analysis that is used to identify the factors for the models creates a serious model identification problem. As generally accepted, the percentage of variation attributable to a particular component depends on the number of components that are prejudicially retained from the covariance matrix. Thus, both the size of the covariance matrix, or the number of maturity segments, as well as how many of these segments are considered significant, are crucial since they determine the percentages of variation decomposition. The components that are left out might be very important parts in pricing interest rate options. In fact, this difficulty applies to all more-than-one-factor affine models.

Once first introduced, the simple Vasicek model appears to fit all of the term structure shapes including monotonically increasing, monotonically decreasing, and hump-shaped. However, the one-factor model only allows for the parallel shift of the term structure (Schlögl & Sommer, 1998). In addition, Bakshi and Chen (1996) and Rogers (1996) comment that with its assumed Gaussian distribution, the model erroneously produces a negative interest rate that affects the modeling of nominal interest rates and the pricing of interest rate derivatives.

Cox et al. (1985) attempt to solve the problem by incorporating a reflecting boundary for the diffusion process. The resulting price process remains the same, an exponentially affine pricing process, even after their adjustment, although the new horizon dependent deterministic functions are different. The need for a multi-factor model then arose to overcome some of these pitfalls. Schrögl and Sommer (1998) argue that while the two-
factor models allow for the twists or slopes of the term structure, they still do not generate the correct third and higher order dynamic distribution moments. Kappi (1997) uses the statistical method of maximum likelihood (which can be shown to be equivalent to the scientifically deficient principal components analysis) to estimate the two-factor models and found that they can fit the level and the slope, but still miss the curvature of the term structure. With one additional factor to explain the curvature, the three-factor models perform better than the two-factor models. However, together with Los (1989), Brandt and Kavajecz (2004) also state that the three factors of those three-factor models are typically not uniquely identified, even though modelers commonly think of them as the level, slope, and curvature of the term structure.

Thus, while Duffie and Kan (1994) and Dai and Singleton (2000) correctly point out that a price process that is exponentially linear in the short rate price process with drift and variance components in the one or two factors are features of the general class of affine models, these are incorrect from a scientific identification or modeling perspective.

Johannes (2004) has recently tested both nonparametric one-factor models and the two- and three-factor models of Andersen and Lund (1997, 1998). The results indicate that none of these models are able to capture the ARCH and non-Gaussian long-term dependence ("scaling") features of the empirical data. Johannes (2004) argues that a multi-factor model might be able to match the nonlinearity of the observed data by tremendously increasing the variance in the model. However, the cost of the modification is so severe that the simulated interest rate path becomes completely irregular.

The affine models fail correct empirical identification by the simple fact that the information structure of the model is generated by Fickian Brownian Motion that does not
allow any singularities or exceptional, unpredictable surprises, i.e., the features that are empirically normally found in the financial markets. Thus, better underlying information processes are scientifically required to identify the empirical terms structure.

2.2.3 GARCH Process Models

Bollerslev (1986) introduced GARCH \((p, q)\) process that generalized the Nobel Memorial Prize winning ARCH process of Engle (1982). The GARCH \((p, q)\) process has been found to be very useful in modeling the time-varying price volatility of underlying assets, i.e., they are non-stationary in the weak sense and been accepted widely among academicians and practitioners.

Although GARCH can be used in any higher order of Moving Average and Autoregressive, in practice only GARCH \((1, 1)\) is widely used. Andersen and Lund (1997), Dai and Singleton (2000) use GARCH to approximate a density of short-rate series and then estimate their underlying processes. Without pricing any interest rate options, Brenner et al. (1996) also use GARCH to model the short-rate series of the 3-month Treasury bill and its volatility. Their results suggest that GARCH model can capture stochastic volatility and varying serial correlation of the short-rate series.

Longstaff and Schwartz (1992) also once use GARCH to capture the underlying process of short rate in their two-factor model, but later found that model is inadequate. Fisher et al. (1997) argue that a finite or weak memory in a discrete time of GARCH process prevents it from replicate the long memory time-and-frequency scaling patterns found in empirical financial market data.
Bali (2003), Heston and Nandi (2003), and Saltoglu (2003) have incorporated GARCH in their modified multi-factor interest rate models to capture either the short-rate process or its volatilities. Their results show that GARCH helps improve the predictability of the multi-factor models, although Saltoglu (2003) reports that GARCH model is slightly outperformed by other nonparametric methods such as (nonlinear) Artificial Neural Networks and Kernel smoothing.

The limitations of the GARCH model, such as its finite short memory, still urge researchers to develop an even better model, and one of the currently most promising models is the Multifractal Model of Asset Returns (MMAR) implemented and thoroughly tested in this dissertation.

2.3 Long Memory in Finance

Long Memory (LM), or global dependence, is prevalent in many disciplines of research including finance. Table 2.1 provides a collection of Long memory investigation in the financial price or return series as they appeared in the high quality finance journals (highly regarded as "A" level journals) in the past 10 years. All studies in the table are categorized according to the time series investigated. Notably, most of studies have used stock market data. Foreign exchange rate studies form the first runner-up. But, there are a few empirical studies using futures and interest rate series. There are two interesting observations from Table 2.1. First, none of this work of long memory uses options series. Second, many studies have been done in the year 2004. This avalanche of recent publications suggests that the topic has rapidly gained interest from the financial community only very recently.
<table>
<thead>
<tr>
<th>Author</th>
<th>Year</th>
<th>Stock</th>
<th>Term Structure</th>
<th>Futures</th>
<th>Cash</th>
<th>Forex</th>
</tr>
</thead>
<tbody>
<tr>
<td>Karuppiah &amp; Los</td>
<td>2005</td>
<td></td>
<td></td>
<td></td>
<td>X</td>
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<tr>
<td>Gil-Alana</td>
<td>2004</td>
<td></td>
<td></td>
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<td>X</td>
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<tr>
<td>Lillo &amp; Farmer</td>
<td>2004</td>
<td>X</td>
<td></td>
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<tr>
<td>McCarthy et al.</td>
<td>2004</td>
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<tr>
<td>Morana &amp; Beltratti</td>
<td>2004</td>
<td></td>
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<tr>
<td>Matteo et al.</td>
<td>2004</td>
<td>X</td>
<td>X</td>
<td>X</td>
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<tr>
<td>Mulligan</td>
<td>2004</td>
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<td>Mulligan &amp; Lombardo</td>
<td>2004</td>
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<tr>
<td>Corazza &amp; Malliaris</td>
<td>2002</td>
<td></td>
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<td>X</td>
<td>X</td>
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<tr>
<td>Henry &amp; Olekalns</td>
<td>2002</td>
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<tr>
<td>Cheung &amp; Lai</td>
<td>2001</td>
<td></td>
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<tr>
<td>Lee et al.</td>
<td>2001</td>
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<tr>
<td>Crato &amp; Ray</td>
<td>2000</td>
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<tr>
<td>Grau-Carles</td>
<td>2000</td>
<td>X</td>
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<tr>
<td>Lien &amp; Tse</td>
<td>1999</td>
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<td>Opong et al.</td>
<td>1999</td>
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<td>Hauser</td>
<td>1998</td>
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<td>Barkoulas &amp; Baum</td>
<td>1998</td>
<td>X</td>
<td>X</td>
<td>X</td>
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<td>Barkoulas et al.</td>
<td>1997</td>
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<td>Jacobsen</td>
<td>1996</td>
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<tr>
<td>Cheung &amp; Lai</td>
<td>1995</td>
<td>X</td>
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<td>Evertsz</td>
<td>1995a,b</td>
<td>X</td>
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<td>X</td>
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<tr>
<td>Evertsz &amp; Berkner</td>
<td>1995</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
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</tbody>
</table>

Table 2.1: Studies of Long Memory Published in "A" Level Finance Journals Since 1994. (Source: Jamdee and Los, Working Paper, 2005)
Calvet and Fisher (2002) show that the compounding process between the mono-fractal fractional Brownian Motion (mono-fractal fBM) and trading time can be used to construct the MMAR and thus to eliminate arbitrage opportunities in the mono-fractal fBM by making it consistent with martingale theory. But, since the time-frequency scaling properties of the monofractal fBM are transferred to the multifractal process via the compounding procedure, the investigation of the LM effects on financial asset prices following the monofractal fBM remains essential for the further development of multifractal modeling in finance.


Figure 2.1 by Jamdee and Los (2005) shows the impact of LM on the European Call options valuation with various degrees of persistence. The LM options with long-term maturity as well as high degree of persistence have very similar characteristics to the ESOs. The simulated results clearly support that the values of LM options can be greatly deviated from those suggested by the Black-Scholes options formula.
Figure 2.1: This picture shows various prices of the fractal Brownian Motion at-the-money call options. The underlying stock price, $S_0$, is $40. The strike price, $X$, is $40$. The risk-free rate, $r$, is 6% where the volatility, $\sigma$, is assumed to be 0.3. The expiration time, $\tau$, varies over 1 day, 30 days, 90 days, 180 days, 1 year, 2 years, 5 years, 10 years, and 20 years for various degrees of persistence, $H$, between 0 and 1. Source: Jamdee and Los (Working Paper, 2005).
2.4 Research Questions

As mentioned above, there exist clear gaps in the financial literature: 1) lack of applications of multifractality in interest rate modeling literature and 2) an inconsistency between the theoretical modeling of the term structure and the model identification from empirical data. The first gap can be attributed to the lack of multifractal analysis of fundamental interest rate markets. Multifractal analysis is very new to finance modelers, both practitioners and academicians, and only a small group of them has been working on this topic. The lack of proper theoretical models or processes to explain the empirical data might cause the second gap to exist. While most interest rate models exhibit a trade-off between simplicity (tractability) and unrealistic and intractable restrictions, the MMAR appears to overcome this trade-off.

The MMAR is a very appropriate model for interest rate modeling given the results by Brandt and Kavajecz (2004), who found that the orderflow imbalance (pressure) and liquidity can explain the variation of the term structure of the US Treasury interest rate while the effect of pressure is permanent for at least two weeks. Their findings provide partially empirical support for the cash flow hypothesis of Los (2003) to explain the term structure. As discussed above, the MMAR allows us to incorporate the degree of liquidity (the long memory effect) and the period of pressure effect through the Hölder-Hurst exponent and time investment horizon (scales), respectively. Other support for the MMAR is provided by Calvet and Fisher (2002) comparing the MMAR with other models and arguing that the MMAR is the only continuous-time model that preserves the long memory effect while showing volatility clustering (like in GARCH models) and martingale properties, thereby excluding arbitrage opportunities. Table 2.2 adapted from Mandelbrot, Fisher, and Calvet
<table>
<thead>
<tr>
<th>Models</th>
<th>Volatility Clustering, Martingale Pricing</th>
<th>Volatility Clustering, Arbitrage Opportunities</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>MMAR</td>
<td>FBM</td>
<td>Long Memory</td>
<td>Scale Consistency</td>
</tr>
<tr>
<td>FIGARCH</td>
<td>ARFIMA</td>
<td>Long Memory</td>
<td>Scale Inconsistency</td>
</tr>
<tr>
<td>GARCH</td>
<td>ARMA</td>
<td>Short Memory</td>
<td>Scale Inconsistency</td>
</tr>
</tbody>
</table>


(1997) and Calvet and Fisher (2002) suggests that the MMAR is superior to other time series models in any aspects needed for modeling financial time series.

Beside the MMAR, only Fractional Integrated (FI) GARCH and GARCH can exhibit the volatility clustering in return series, while having the martingale property. FBM, ARFIMA, and ARMA all exhibit correlation in return series without the martingale property. The non-martingale properties provide arbitrage opportunities which have been subject to many arguments in financial environments that assume arbitrage-free pricing such as the derivatives pricing areas. The MMAR as a continuous time model has an advantage over its discrete model candidates, because the MMAR can be used with high frequency (e.g., tick-by-tick) data. More and more high frequency data are now available in finance. The cost of data storage has dropped significantly while finance articles studying the microstructure and very short-horizon investments emerge exponentially and gain more interest from the professional financial community.
Since the MMAR is still at the edge of the current frontier of financial knowledge, 1) the empirical identification of the MMAR is still unsettled and 2) most of the test results have been based on the Monte Carlo simulations. Thus this dissertation has as its objective to be the first to thoroughly analyze the Treasury term structure in multifractal terms. It is hoped that this first step of multifractal analysis of interest rate will substantiate future research in this area. Particularly, with all the caveats of the MMAR and without attempting to price any financial assets, this dissertation aims to answer the following questions.

1. Are interest rate series nonlinear and do they exhibit time- and frequency-scaling?

2. Can the MMAR be used to synthesize the interest rate series of each maturity and reproduce the scaling, thick-tail, and long-term dependence properties?

3. Is the MMAR-based model superior to the affine BM-based and GARCH-based models and how can such "superiority of performance" be measured?

4. Does the current MMAR represent a possible model for the cash flow theory proposed by Los (2003)?
Chapter 3

Data and Methodology

3.1 Interest Rate Data

The empirical U.S. Treasury interest rate series are obtained from the Federal Reserve Bank of St. Louis website (H.15). This dissertation uses the daily computational interest rate series, called Treasury constant maturity, with 3-month, 6-month, 1-year, 2-year, 3-year, 5-year, 7-year, and 10-year maturities.\(^1\) In addition, the Fed Funds rate, which is normally used as an instantaneous rate for affine interest rate models, is analyzed.

The interest series in this study cover the period of April 13, 1987 – December 31, 2002. This approximate 15-year period provides 4096 daily observations. The length of observations is chosen such that the study is applicable to the processing capacity of a simple personal computer. Moreover, Los (2003) suggested that the business cycle can last approximately 10 to 12 years. Thus, the 15-year period is expected to cover at least one whole business cycle. Each interest rate series for any maturity has the same length of observations. Like McCarthy et al. (2004), missing values were replaced with the rate from the previous day. The same length of period is used to calculate the partition function (sample sum), scaling function, Legendre multifractal spectrum, and homogeneous or monofractal Hölder-Hurst exponent, where the first five moments are used or \(q = \{1, 2, 3, 4, 5\}\).

\(^1\)In addition, the secondary market rates are analyzed, and the results suggest that these rates do not exhibit the long memory effect. The degrees of persistent \(H\) are slightly below 0.5.
3.2 The MMAR: Detection, Identification, Synthesis, and Simulation

The main purpose of this dissertation is to investigate the behavior of the eight interest rate ("node") series of the U.S. Treasury term structure and their coherent system as well as the Fed Funds rate through the MMARs. The first five moments of the nine simulated interest rate paths are then analyzed thoroughly. The technique used in this study has been initially proposed by Calvet et al. (1997), Fisher et al. (1997), Mandelbrot et al. (1997), Calvet and Fisher (2002) and empirically tested with the foreign exchange rate, the U.S. stock index, and the individual U.S. stocks. In addition, Fillol (2003) has successfully applied the MMAR to the French Stock Index. Regarding the significance of the fBM process in the MMAR, McCarthy et al. (2004) and Mulligan (2004) recently estimate the Hölder-Hurst exponents of the term structure of U.S. treasury rates and hi-tech stocks, respectively, and find that these exponents of the studied series are not equal to 0.5. Hosking (1981) elaborately shows the link between the fBM and ARFIMA model in time series and first applies it to the field outside economics and finance. Unlike McCarthy et al. (2004) who use the Haar Wavelet to measure the degree of persistent or (Global) Hölder-Hurst exponents of the Treasury rate series across the maturities, this dissertation examines the multifractal process of the Treasury rate series, identify the parameters of their corresponding MMARs, and compare the performance of these multifractal models with the GBM and GARCH(1,1) processes.

The procedure of MMAR is straightforward and practical. It begins with detecting the time-and-frequency scaling properties of the time series of interest rates, estimating some particular parameters, synthesizing the multiplicative measure that possesses the multifractality properties, and simulating the compounded process, the fBM process in the multifractal trading time. Calvet and Fisher (2002) argue that the MMAR is a hybrid between
jump diffusion and the GBM models such that the MMAR contains all necessary characteristics found in the observed empirical data.

The MMAR is the third generation of models developed according to the proposals of Mandelbrot and Hudson (2004). The model is based on the proper combination of the multifractality and monofractal Brownian Motions. The scaling property is one of the main key components that generate thick-tail and long-term dependence phenomena in time series, as explained by Mandelbrot (2001a, 2001b, 2001c, 2001d). Mandelbrot argues that both phenomena are common in nature (Mandelbrot, 1982). A clean example is a stock price which is invariant under scaling or multiscaling. If one invests in a stock market for a long period of time, the average returns from that investment might be around seven percent. However, at some points in time there could be a very short period within the investment horizon that the rate of return is far above its average. In addition, a current stock price might be nonlinearly influenced by its past prices. Thus, the MMAR, which possesses such properties, makes at least intuitive sense.

Mandelbrot et al. (1997), Calvet and Fisher (2002) defined the MMAR as a stochastic process

\[ X(t) = \ln P(t) - \ln P(0) \quad (3.1) \]

where \( P(t); \ 0 \leq t \leq T \) is the price series while \( t \) represents conventional clock time. Thus, \( \{X(t)\} \) is a multifractal process that has the following properties:

Assumption 1: \( X(t) \) is a compound process, where a Brownian Motion operates on multifractal trading time:

\[ X(t) \equiv B_H[\theta(t)] \quad (3.2) \]
where $B_H(t)$ is a fractional Brownian Motion (fBM) and $H$ is a Hölder-Hurst exponent with its range between 0 and 1, with trading time $\theta(t)$. The $\theta(t)$ process deforms conventional clock time into trading time, while the $B_H(t)$ process operates the return process.

**Assumption 2**: Trading time $\theta(t)$ is the cumulative distribution function (c.d.f.) of a multifractal measure defined on the time axis $[0, T]$. It reflects the actual news dissemination process that affects the volume of trades on the trading floors.

**Assumption 3**: $B_H(t)$ and $\theta(t)$ are independent. The news process/trading time $\theta(t)$ is independent of the return generation process $B_H(t)$.

The principle here is that one can transform a monoscaling process like the monofractal BM into a multiscaling process by properly mapping the 1 dimension (1D) domain $t \rightarrow \theta(t)$ where the trading time possesses all multifractal properties that are expected to pass on to the 1D fBM. Figure 3.1 and 3.2 provide a non-technical description of the multifractal nature of trading time $\theta(t)$ and how it affects the actual price process (Mandelbrot & Hudson, 2004).

The deformation or warping of time is a simple reflection of the observation that financial market trading happens in clusters, since news events arrive in clusters. Consequently, interest rate volatility clusters too.

### 3.2.1 Detection of Scaling Properties of the Interest Rates and Their Multifractal Spectra

With the goal of tractability, the detection of the multifractality of the interest rates begins with the use of sample partition function

$$S_\delta(T, q) = \sum_{i=1}^{n} \left| X_{[i-\delta]} - X_{[(i-1)\cdot \delta]} \right|^q$$

(3.3)
Figure 3.1: Compounding Process of Multifractal Models. This diagram provides the Baby theorem, a non-technical description of the multifractal models generated by compounding the Brownian Motion (BM) with the trading time. At top right is the mother generator which is a BM in conventional clock time. On top of the picture is the increment plot of the BM. At bottom right the father generator deforms the conventional clock time into the trading time. Surrounding the father generator are the slow and fast plots of the trading time indicating different pattern of clustering. Adopting the father’s trait - trading time, the mother generates a multifractal baby shown at top left. Its increment plot is on top obviously different from the increment plot of BM from the mother’s side. Source: Mandelbrot and Hudson (2004).
Figure 3.2: The Three Dimensions of the Multifractal Process. This three dimension plot depicts how the deformation of the clock time into the trading time happens. The left wall represents a picture of Brownian Motion (BM) while the jagged path moving along the diagonal is the trading time. The resulting process is on the right wall indicating the multifractal price process of financial markets.
where $\lfloor \cdot \rfloor$ is the integer part operator, and $\delta$ is the time increment.

By the definition of multifractality, a stochastic process $X(t)$ is multifractal if it has stationary increments and satisfies

$$E\{|X(t)|^q\} = c(q)t^{\tau(q)+1}$$  \hspace{1cm} (3.4)

for all $t \in B, q \in Q$, where $E\{\cdot\}$ is the expectation operator and $c(q)$ is called a prefactor. $B$ and $Q$ are positive real numbers where $0 \in B$ and $[0, 1] \subseteq Q$. In other words, the movements of the process $X(t)$ are clearly scaled with scaling exponent $\tau(q)$. The moments, $q$, can theoretically be negative. According to Mandelbrot et al. (1997) negative moments are not occurring in financial data, but others (Los & Yalamova, 2003) found differently for some empirical stock market prices.

Thus, by logarithmic transformation, the partition function can be rewritten as

$$\log(S_\delta(T, q)) \approx \tau(q) \log(\delta) + \log[c(q)] + \log(T)$$ \hspace{1cm} (3.5)

Clearly, the logarithm of the partition function is linear. This equation also implies that if any function has scaling properties, the logarithmic plot of the partition functions against the time increment should be approximately linear. With various moments, $q$, and incremental time, $\delta$, one can identify the scaling function, $\tau(q)$, using various simple projection methods.
Then, using Legendre transform of the identified scaling function, $\hat{\tau}(q)$ can identify the multifractal spectrum from the identified scaling function as

$$\hat{f}(\alpha) = \min_q [q\alpha - \hat{\tau}(q)] \tag{3.6}$$

where $\alpha$ is a localized Hölder-Hurst exponent.

### 3.2.2 Identified Values of the Homogeneous Hölder-Hurst Exponent and of the Parameters of Measures

Given that an interest rate series is scaling, Mandelbrot et al. (1997) and Fisher and Calvet (2002) have proven that the following identities hold.

$$\hat{\tau}_X \left( \frac{1}{H} \right) = \tau_{\theta}(1) = 0 \tag{3.7}$$

$$\tau_X(q) = \tau_P(q) = \tau_{\theta}(Hq) \tag{3.8}$$

$$f_X(\alpha) = f_P(\alpha) = f_{\theta}(\frac{\alpha}{H}) \tag{3.9}$$

The first identity allows the identification of the Hölder-Hurst exponent $\hat{H}$ of the log price series. This identification can be done by finding the partition function plot at a particular moment $q$ that is approximately flat and parallel to the x-axis. In other words, the partition function plot of a particular $q$ and $\delta$ that has a zero slope can be used to solve backward for the Hölder-Hurst Exponent.

The second identity suggests that the log price and price processes $X(t)$ and $P(t)$, respectively, have the same scaling function, while the scaling function of the trading time
Figure 3.3: The Multifractal Analysis of the GBM. The dotted-dashed line is the theoretical multifractal spectrum of trading time \( \theta(t) \) of the GBM series. The dashed line indicates the theoretical multifractal spectrum of the GBM \( X(t) \) series itself. The solid line shows the identified multifractal spectrum of the \( X(t) \) series using Wavelets. This picture clearly shows the relationship between the multifractal spectrum of the price process and its trading time according to the third identity. Source: Los (2003).

has the same shape, but is shifted by the factor of \( Hq \). Figure 3.3 illustrates the relationship between the multispectrum of the price process and that of its trading time.

The third identity indicates that the multifractal spectra of the processes \( X(t) \) and \( P(t) \) computed by using Legendre transformation of the scaling function are the same. Again, the multifractal spectrum of the trading time has the same shape, but shifted by the factor of \( \frac{q}{H} \). This last identity is very important for constructing the MMAR and will be used rigorously in next section.
In addition to this identification procedure of Mandelbrot et al. (1997), Gonçalves and Riedi (2000) and Los (2003) have shown that the Wavelet Multiresolution Analysis (MRA) can be used to approximate both partition functions, scaling functions, and the multifractal spectrum of the log price series. This is not a coincidence because Calvet et al. (1997) and Mandelbrot et al. (1997) had already shown that the multifractal spectrum, $f(\alpha)$, can be viewed as

1. the limit of a renormalized histogram of a coarse Hölder-Hurst exponent,

2. the fractal dimension of the set of instants with Hölder-Hurst exponent $\alpha$, and

3. the limit of the log distribution by Large Deviation Theory.

In this dissertation, Wavelet MRA is also used to identify the mono-fractal Hölder-Hurst Exponent from the multifractal spectrum. The results are then compared with the initially proposed identification by Calvet and Fisher (2002).

3.2.3 Synthesizing the Lognormal Measures

Multiplicative cascade probability measures are a key for constructing the MMAR. They have all ideal scaling properties that will be transferred to the simulated fBM having the desired long-term dependence when the two are compounded. The cascade probability measures normally are used to generate positive mutifractality using iteration techniques. The value of the multiplicative measure, $\mu_{k,b_k}$ built on a single mass $M_0^0$, after $k$ iterations at interval $B_k$, is given by

$$
\mu_{k,b_k} = M_{b_k}^k \cdot M_{b_{k-1}}^{k-1} \cdots M_{b_1}^1 \cdot M_0^0
$$

(3.10)
for any dyadic (= by a factor of 2) partitioning of the intervals. The initial probability mass $M_0$ is "divided" up by multiplying by a cascade of probability measures to result in a final empirically realistic distributional measure $\mu_{k,b_k}$.

To ensure our understanding of the measure, consider this numerical example. Let the uniform probability measure $\mu_0$ on the unit interval $[0, 1]$ has the mass of one and two positive numbers $M_0 = 0.6$ and $M_1 = 0.4$ adding up to one. In the first step of the cascade generating, a measure $\mu_1$ is generated by uniformly spreading (multiplication) the mass $M_0$ on the left sub-interval $[0, 0.5]$, and the mass $M_1$ on the right sub-interval $[0.5, 1]$. Thus, after the first step, the two new measures are

$$\mu_1[0, 0.5] = 0.6 = M_0, \quad \text{and}$$

$$\mu_1[0.5, 1] = 0.4 = M_1$$

In the second step, the two existing intervals are then sub-divided further. Each of the two existing intervals is divided into two more sub-intervals with an equal length. Thus, there should be four intervals with the same length altogether after this process. The fraction of the masses of the two existing intervals is then re-distributed into those new sub-intervals in the same fashion done in the first step. Thus, after the second iteration,
there exist four measures as follow;

\[
\begin{align*}
\mu_2[0, 0.25] &= 0.6 \times 0.6 = 0.36 = M_0 \times M_0 \\
\mu_2[0.25, 0.5] &= 0.6 \times 0.4 = 0.24 = M_0 \times M_1 \\
\mu_2[0.5, 0.75] &= 0.4 \times 0.6 = 0.24 = M_1 \times M_0 \\
\mu_2[0.75, 1] &= 0.4 \times 0.4 = 0.16 = M_1 \times M_1 \\
\text{Total sum} &= 1
\end{align*}
\]

(3.13) (3.14) (3.15) (3.16) (3.17)

After a certain number of iterations, there should exist an infinite sequence of measure \( \mu_k \) that weakly converges to the binomial measure \( \mu \). The binomial measure is a continuous but singular probability measure that has no density and no point mass.

There are several extensions of these multiplicative cascade probability measures. One can introduce the stochastic probability measure by using any known distribution, e.g., normal distribution. Empirically, one can ascertain, by examining their multifractal spectra, that most financial time series, including interest rates, exhibit multifractal spectra with humped shapes, while their trading time is the cumulative distribution function (c.d.f.) of a multiplicative random probability measure with the simple lognormal masses (c.f. Bailli \textit{et al.}, 1996; Bollerslev, 1986; Calvet \textit{et al.}, 1997; Calvet and Fisher, 2002; Mandelbrot \textit{et al.}, 1997).

Calvet \textit{et al.} (1997) proved that the multifractal spectrum function of trading time with lognormally distributed masses is

\[
f_\theta(\alpha) = 1 - \frac{(\alpha - \lambda)^2}{4(\lambda - 1)}
\]

(3.18)
where $f$ is hump-shaped and symmetric around its maximum or most probable Hölder-Hurst exponent, $\hat{\alpha}_0 = \lambda$.\textsuperscript{2}

The first two moments of the lognormal distribution are as follows:

\[
\hat{\lambda} = \frac{\hat{\alpha}_0}{H}, \text{ and } \quad \hat{\sigma}^2 = \frac{2(\hat{\lambda} - 1)}{\log b} \tag{3.19} \]

where $b = \text{any dyadic number.}\textsuperscript{3}$ This suggested closed form solution is very useful for the synthesis of the measure that leads to obtaining trading time in the MMAR.

\textbf{Proof.} The first moment is obtained from the third identity in previous section that

\[
f_X(\alpha) = f_\theta\left(\frac{\alpha}{H}\right) \tag{3.21} \]

Thus, for the log price spectrum there exist the relations

\[
\hat{\alpha}_0 = \hat{\lambda}H \tag{3.22} \]
\[
\hat{\lambda} = \frac{\hat{\alpha}_0}{H} \tag{3.23} \]

\textbf{Proof.} To build a multifractal lognormal spectrum, consider a random variable $M$ such as $-\log_b M \sim N(\hat{\lambda}, \hat{\sigma}^2)$ where $E\{M\} = \frac{1}{b}$.

\textsuperscript{2}See proof in Calvet et al. (1997)
\textsuperscript{3}Calvet and Fisher (2002) suggested a different variance equation, which is incorrect. In this dissertation, the correct variance equation is provided. Also, in this dissertation, $b = 2$ unless otherwise stated.
Let $u$ be a random variable satisfies $E(u) = \frac{1}{b}$ and the following function

$$v = -\log_b(u) \sim N(\lambda, \sigma^2)$$  \hspace{2cm} (3.24)

Therefore,

$$\log(u) \sim N[-(\log b)\lambda, (\log b^2)\sigma^2]$$  \hspace{2cm} (3.25)

Let $y \sim N(m, s^2)$ and $x = e^y$. $x$ is lognormally distributed and has the first two moments

$$E(x) = e^{m + \frac{s^2}{2}}$$  \hspace{2cm} (3.26)
$$V(x) = e^{2m + s^2}(e^{s^2} - 1)$$  \hspace{2cm} (3.27)

Then, replace the variable $x$ by $u$ and get

$$E(u) = \left(\frac{1}{b}\right)^\lambda e^{\frac{[(\log b)\alpha]^2}{2}} = \frac{1}{b}$$  \hspace{2cm} (3.28)
$$\log b = 2\frac{\lambda - 1}{\sigma^2}$$  \hspace{2cm} (3.29)

Thus,

$$\hat{\sigma}^2 = \frac{2(\lambda - 1)}{\log b}$$  \hspace{2cm} (3.30)

Let the multifractal spectrum of trading time $f_\theta(\alpha)$ be a second order polynomial function

$$f_\theta(\alpha) = 1 - \frac{1}{2\ln b} \left(\frac{\alpha - \lambda}{\sigma}\right)^2$$  \hspace{2cm} (3.31)
Substitute $\widehat{\sigma}$ and obtain

$$f_{\widehat{\theta}}(\alpha) = 1 - \frac{(\alpha - \widehat{\lambda})^2}{4(\widehat{\lambda} - 1)} \quad (3.32)$$

With the identified mean and variance, the multiplicative lognormal probability measure path can be synthesized for a length of $2^K$ which should be greater than the desired length of simulated interest rate series. For each step of the construction of the multifractal lognormal probability measure path, we draw the masses $M$ where $-\log_b M \sim N(\widehat{\lambda}, \widehat{\sigma}^2)$.

According to the MMAR architect, the key parameter is $\widehat{\lambda}$. There are two methods that can be used to identify $\widehat{\lambda}$. The first method uses the relation $\widehat{\lambda} = \widehat{\alpha}_m$ and obtains the first moment $\widehat{\lambda}$ directly from the multifractal spectrum of the log price process $f_X(\alpha)$ (Fillol, 2003).

Calvet and Fisher (2002) proposed the second method by fitting the $f_X(t)$ for the $\widehat{\lambda}$. Let the most probable exponent $\alpha_0 = \lambda H$, then the log-price series $X(t)$ has the following quadratic multifractal spectrum

$$f_X(\alpha) = 1 - \frac{(\alpha - \alpha_0)^2}{4H(\alpha_0 - H)} \quad (3.33)$$

Thus, if the Hölder-Hurst exponent $H$ is known, the only free parameter $\alpha_0$ can be obtained by a simple fitting process. The identified $\widehat{\alpha}_0$ then leads to the calculation of $\widehat{\lambda}$ and $\widehat{\sigma}^2$ using the equation 3.19 and 3.20. After a proper synthesis of the multiplicative lognormal probability measure path, the next step is to compute the c.d.f. of the synthesized measures. The resulted function is then called the trading time, $\theta(t)$. Appendix B discusses the probability measure in greater details.
3.2.4 Simulation of the GBM, MMAR, and GARCH(1,1)

This dissertation presents three Monte Carlo simulations including those of the Geometric Brownian motion (GBM), the MMAR, and the GARCH (1, 1) models.

Simulation of the GBM

The simulation of the GBM is straightforward, following the simple diffusion process. The standard deviation over the discrete time interval of the log price series are obtained and multiplied by an independent standard Gaussian.

Simulation of the MMAR

The simulation of the MMAR can be conducted by compounding a simulated fBM process with corresponding trading time.\(^4\)

Calvet and Fisher (2002) and Fillol (2003) obtain a simulated fBM by computing the cumulative sum of the simulated fractional Gaussian Noise (fGN) with respect to the identified mono-fractal Hölder-Hurst exponent $\tilde{H}$. The authors do not explicitly suggest the particular simulation technique of the fBM process and the MMAR used in their studies. Nevertheless, the overall scheme of the MMAR construction is as follow.

1. A finite-stage synthesis of a multifractal probability measure is obtained following the preceding section. The length of synthesized measure path must be corresponding with the desired length of the MMAR. For example, under the dyadic base of 2, if a simulated MMAR of length $T$ is required, the length of measures will have a minimum integer number of stages $k$ such that $2^k \geq T$. At each stage $k$, independent

\(^4\)More detail of the fBM simulation is provided in the appendix.
log normal multipliers or masses that have distribution with respective to those mean and variance obtained from the empirical series are drawn. Once all \( k \) stages finish, the cumulative sum of the measures, a discrete approximation to the quadratic variation of a multifractal path, provides the accumulated trading time \( \theta(t) \).

2. A discretized path of an fBM \( B_H(t) \) is then generated with corresponding identified Hölder-Hurst exponent \( \hat{H} \) for a length of \( T \). There are many techniques for generating the fBM. They may have different algorithms, but their results should be very similar. Let \( B_H(t) \) be a stochastic process with continuous sample paths and such that

\[
B_H(t) = \text{Gaussian} \quad (3.34)
\]
\[
B_H(0) = 0 \quad (3.35)
\]
\[
E[B_H(t) - B_H(s)] = 0 \quad (3.36)
\]
\[
Cov[B_H(t), B_H(s)] = \frac{\sigma^2}{2} (|t|^{2H} - |t - s|^{2H} + |s|^{2H}) \quad (3.37)
\]

for any Hölder-Hurst exponent \( H \in (0, 1) \) and \( \sigma^2 \) is a variance scaling parameter. Then \( B_H(t) \) is called fractional Brownian Motion (fBM). Essentially, this definition is the same as for the standard Brownian Motion except that the covariance term is different. When \( H = 0.5 \), this definition yields standard Brownian motion as a special case, but in general \( H \neq 0.5 \), so the increments \( B_H(t) - B_H(s) \) are not independent anymore.

The resulting stochastic process from calculating the first differences of fBM \( B_H(t) - B_H(t-1) \) is called fractional Gaussian Noise (fGN) with parameter \( H \). The covariance
at lag $k$ of fGN follows from the above covariance definition:

$$
\gamma(k) = \text{Cov}[B_H(t) - B_H(t - 1), B_H(t + k) - B_H(t + k - 1)] \\
= \frac{\sigma^2}{2}(|k + 1|^{2H} - 2|k|^{2H} + |k - 1|^{2H}), \quad \text{and} \tag{3.39}
$$

for $0.5 < H < 1$, the process is persistent, while for $0 < H < 0.5$, the process is anti-persistent.

Thus, one can simulate either the fBM directly or the fGN and compute its cumulative sum to obtain the fBM. The very straight forward algorithm to simulate the fBM directly is called the Cholesky method. The main concept is to construct a covariance matrix with respect to the Hölder-Hurst exponent and then multiplying the matrix by a vector of random variables. The following discusses the Cholesky simulation algorithm in some details.

The Cholesky method uses the Cholesky decomposition of the covariance matrix. Implicitly, the covariance matrix $\Gamma(n)$ can be written as $L(n)L(n)'$, where $L(n)$ is a $(n + 1) \times (n + 1)$ lower triangular. Let element $(i, j)$ of $L(n)$ be $l_{ij}$ for $i, j = 0, ..., n$, $L(n)$ is said to be lower triangular if $l_{ij} = 0$ for $j > i$. This decomposition exists if and only if $\Gamma(n)$ is a symmetric and positive definite matrix. Define $\gamma(\cdot)$ as the covariance function of the zero mean process, get

$$
\gamma(k) := E(x_n x_{n-1}) \tag{3.40}
$$

for $n, k = 0, 1, 2, ...$. Let $\gamma(0) = 1$ and also let $\Gamma(n) = (\gamma(i - j))_{i,j=0,...,n}$, the elements of $L(n)$ can be calculated by noting that element $(i, j)$ of $L(n)L(n)'$ and should be
equal for \( j \leq i \), thus

\[
\gamma(i - j) = \sum_{k=0}^{j} l_{ik} l_{jk}, \quad j \leq i. \tag{3.41}
\]

When \( i = j = 0 \), the result is \( \gamma(0) = l_{00}^2 \). When \( i = 1 \) and \( j = 0 \), the result is \( \gamma(1) = l_{10} l_{00} \). For \( i = 1 \) and \( j = 1 \), \( \gamma(0) = l_{10}^2 + l_{11}^2 \). These few values of \( i \) and \( j \) have already determined \( l_{10} \) and \( l_{00} \). Since the element of the matrix does not depend on the number of \( n \), \( L(n + 1) \) can be calculated from \( L(n) \) by adding a row and some zeros to make \( L(n+1) \) become lower triangular. The additional row is determined by

\[
l_{n+1,0} = \frac{\gamma(n+1)}{l_{00}} \tag{3.42}
\]

\[
l_{n+1,j} = \frac{1}{l_{jj}} \left[ \gamma(n+1 - j) - \sum_{k=0}^{j-1} l_{n+1,k} l_{jk} \right], \quad 0 < j \leq n \tag{3.43}
\]

\[
l_{n+1,n+1}^2 = \gamma(0) - \sum_{k=0}^{n} l_{n+1,k}^2 \tag{3.44}
\]

The formula still keep \( L(n) \)'s diagonal elements strictly positive. When \( \Gamma(n+1) \) is a positive definite matrix, the non-negativity of \( l_{n+1,n+1}^2 \) is warranted. Thus, the matrix \( L(n+1) \) is real. Let \( V(n) = (V_i)_{i=0,\ldots,n} \) an \((n+1)\)-column vector of i.i.d standard normal random variables, \( V(n + 1) \) is constructed from \( V(n) \) by padding a standard normal variable. Thus, the simulation \( x(n) = L(n)V(n) \) can be done recursively. For every \( n \geq 0 \), \( x(n) \) has zero mean and covariance matrix in matrix form.

\[
\text{Cov}(L(n)V(n)) = L(n)\text{Cov}(V(n))L(n)' = L(n)L(n)' = \Gamma(n) \tag{3.45}
\]
In sum, once $L(n + 1)$ is computed, $x_{(n+1)}$ can be simulated as follows;

$$x_{(n+1)} = \sum_{k=0}^{n+1} l_{n+1,k} V_k$$  \hfill (3.46)

Further details of the fBM simulation techniques are provided in the Appendix.

3. An interpolation is used for compounding the fBM with the trading time resulting in the values of the MMAR path $B_H[\theta(t)]$ at the synthesized values from the path $\theta(t)$. Since the fBm path is discrete, the interpolation generates the continuous path of the simulated MMAR.

**Simulation of the GARCH(1,1)**

The third simulation is of the GARCH (1, 1) process. As known, the GARCH($p$, $q$) process is a generalized ARCH($q$) process with conditional mean and variance. The Matlab routine is flexible enough that any high order of ($p$, $q$) can be used. But this dissertation only implements GARCH(1,1), and the conditional mean and variance equations of the GARCH(1,1) are\(^5\)

$$x_t = c + \varepsilon_t$$  \hfill (3.47)

$$\sigma_t^2 = k + a\sigma_{t-1}^2 + b\varepsilon_{t-1}^2,$$  \hfill (3.48)

\(^5\)The GARCH(1,1) process with low order is very common. The MMAR technique in this dissertation is very primitive because the compounded process of the MMAR is the simplest version of the multifractal models. Thus, the simulated GARCH (1,1) should provide an adequate benchmark for the comparative study among the models here.
In the conditional mean model, the returns $x_t$ consist of a simple constant $C$ plus an uncorrelated, white noise disturbance $\varepsilon_t$. This model is often sufficient to describe the conditional mean in a financial return series.

Although empirical financial return series typically exhibit only mild short-term correlations, the squared returns or volatility of return series often indicate significant correlations. This implies correlation in the variance process and is an indication that the data is a candidate for GARCH modeling. Although a simple model, the GARCH(1,1) has several advantages. It represents a parsimonious model that requires a modeler to estimate only four parameters ($c$, $k$, $a$, and $b$). These GARCH parameters are measured by the conventional statistical maximum likelihood method.

After the three models are simulated, the resulting processes from all simulations are then compared with each other.

3.3 Performance Measurement of the GBM, MMAR, and GARCH (1, 1)

This dissertation attempts to measure performance of the MMAR using visualization. When possible, the large amount of data generated from the analysis will be transform into visible results. The wavelet-based identified distribution plot is used to examine the entire distribution while a scalogram is used to demonstrate how effectively the MMAR performs over time and scales.

3.3.1 Scaling Properties

The partition functions of the simulated interest rate series by both the MMAR and GARCH (1, 1) for each maturity are calculated using the methods previously discussed. The better
model should provide the scaling exponents or the slopes of partition functions for the first five moments across maturities that are very close to those of the empirical nodal interest rate series.

3.3.2 Standard Deviations of the Wavelet Resonance Coefficient Matrixes

Since an average of the 1,000 simulated series is not so useful, the best candidate from each model simulation needs to be identified for future performance comparison among the models. This study employs the method of Wavelet Transformation generating the matrix of the resonance wavelet coefficients, which are comparable to Fourier resonance coefficients, that represents the correlation between the studied time series and the wavelet used in the process to identify which model performs most similarly to the empirical processes according to their time-frequency profiles (c.f. Los, 2003 for further details).

This dissertation implements the continuous wavelet transform defined as follows;

\[ C(scale, position) = \int_{-\infty}^{\infty} f(t) \Psi(scale, position, t) dt \]  \hspace{1cm} (3.49)

or mathematically,

\[ C_{a,b} = \int f(t) \frac{1}{\sqrt{a}} \Psi\left(\frac{t - b}{a}\right) dt \]  \hspace{1cm} (3.50)

where \( a, b, \) and \( t \) are scale, position, and time, respectively. \( C_{a,b} \) is the continuous wavelet resonance coefficient at scale \( a \) and position \( b \), and \( \Psi(\cdot) \) is a wavelet function. In other words, by definition, the continuous wavelet transform (CWT) is defined as the sum over all time of the time series multiplied by scaled, shifted versions of the wavelet function.
Because studied interest rate series have many irregularities, this study uses the Gaussian and Morlet wavelets in its wavelet MRA analysis.

While the Gaussian wavelet is defined as a normal Gaussian function, it can be expanded into a family of Gaussian by taking derivative of order \( p \)th on the Gaussian function, which is

\[
\Psi(x) = C_p e^{-x^2}
\]  

(3.51)

where \( p \) is the parameter of this wavelet family, and \( C_p \) is a constant such that

\[
\left\| f^{(p)} \right\|^2 = 1
\]  

(3.52)

where \( f^{(p)} \) is the \( p \)th derivative of \( \Psi \). This study simply uses the first-order Gaussian wavelet function

\[
\Psi(a, b, t) = \frac{a}{2\pi} (t - b) e^{-\frac{(t-b)^2}{2a^2}}
\]  

(3.53)

and the Morlet wavelet function

\[
\Psi(x) = C e^{-\frac{x^2}{2}} \cos(5x)
\]  

(3.54)

where \( C \) is a constant used for normalization.

By calculating the standard deviation of the simulated wavelet resonance coefficient matrixes relative to the empirical wavelet resonance coefficient matrixes, we should be able to identify the closest simulated series in terms of time and scale. The best candidates from each competing models should have the smallest standard deviations among their
counterparts in the same simulated process. This standard deviation of matrix is common in physics and can be computed as follow:

1. Compute the wavelet coefficient matrix of the empirical series and keep it as a base matrix, $\mathbf{X}$. Since this study examines up to 64 scales, the resulting matrix is of the order $n \times m = 64 \times 4096$).

2. Compute the wavelet resonance coefficient matrix, $X_i, i = 1, ..., 1000$ of all 1,000 simulated series for each process. Thus, there will be 1,000 of $64 \times 4096$ matrixes for each process.

3. Calculate the element-pair-wise standard deviation between the base matrix and each simulated matrix as follows:

$$STDM_i = \left[ \frac{1}{n \times m - 1} \sum_{i=1}^{m} \left( \sum_{i=1}^{n} (X_i - \overline{X})^2 \right) \right]^\frac{1}{2} \tag{3.55}$$

where the decision rule used in the comparison is that the smaller the $STDM$, the better the simulated series is.

3.3.3 Investigation of Higher Moments over Time and Scale

Since the MMAR process itself is not stationary, but only its increments are stationary, the time varying distribution of simulated interest rate series is of crucial importance. The distributions are expected to change over time depending on several factors such as the Federal Reserve policy regime and additional exogenous factors.

The graphical comparison of the entire distribution is conducted using the wavelet-based identification technique. Nevertheless, this graphical comparison merely indicates a snap
shot analysis. While the overall results of the entire distribution are initially discussed, the focal interest is on the correlation of the second moment, or variance, across investment horizons or maturities. This second moment warrants more attention because the variance (volatility-squared) is conventionally used in pricing financial derivatives.

To provide a complete analysis, this dissertation implements a crucial time-scale analysis using Wavelet Multi-Resolution analysis. Suggested by Los (2003), a tool called "scalogram" is used to visually analyze the time series over time and scales. The scalogram analysis begins with the wavelet transformation and finishes with the color-coded plot of obtained wavelet resonance coefficients over the time-scale axis. This study takes full advantages of Wavelet MRA in analyzing the second moment or variance of the studied interest rate series, because the scalogram analysis practically reveals the energy or risk or variance of the time series as it dissipate over time and scale.

Figure 3.4: Scalogram Performance Measurement. These two plots provide an example of how scalogram can detect the variation of time series characteristics. On the left hand side is the plot of simulated time series that possesses all possible characteristics e.g. jumps, clustering, slow wave movement, and changes in frequency. The scalogram plot on the right hand side is able to detect all different characteristics and simply shows them in colorful picture over time and scales.
Figure 3.4 illustrates how effectively a scalogram can detect all possible characteristics such as jumps, clustering, changes in frequency of a simulated series over time and scales and in colorful plot that also indicates the level of energy or variance.

In this dissertation the scalograms will effectively measure the performance of each simulated models as those results are relatively compared with the scalograms of the empirical interest rate series. The knowledge of the MMAR and its estimated variance gained from this study should eventually lead to a new venue of derivatives (asset) pricing research, in particular, of interest rate options.
Chapter 4

Empirical Results

The empirical analysis begins with the plot of the US term structure across nine maturities. Figure 4.1 shows the time series plot of the eight Treasury rate and Fed Funds rate series altogether. This plot suggests various shapes of the term structures over time. First, consider only the constant maturity rates. During the early of year 1987 the US Treasury term structure exhibits an upward-slope. The slope of the term structure becomes less and possibly flats out in the first quarter of year 1989. The flat slope with a potential of humped shape of the term structure can be seen over the period between years 1989 and 1990.

After year 1991, the slope of the term structure is upward and maintains its shape until the end of year 1994 when the slope starts to decline. However, the slope does not flatten out in year 1995 as it did in year 1989, but rather becomes more positive over the 2-year period. The slope, then, becomes flat again in the end of year 1997 and stays flat throughout a first few months of year 1998. The slope slightly increases in the second quarter of year 1998 and remains at the similar level until the last quarter of year 2000. Once again, the term structure indicates the flat slope with a potential hump, which lasts until the end of year 2000. The term structure starts to exhibit an upward slope and remains in this very similar shape until the end of year 2002 which is the end of studied period.

The Fed Funds rate behaves differently during the studied period. In late 1980s, the Fed Funds rate does not conform with the constant maturity term structure. During the
Figure 4.1: The Term Structure of Constant Maturity US Treasury and Fed Funds Rates. The US Treasury rates for eight maturities and the Fed Funds rate are plotted over time beginning from April 21, 1987 to December 31, 2002. For example, when the 10-year line is above the 3-month line, the term structure is upward-sloping. Vice versa, it is downward sloping. FF stands for Fed Funds.
period between 1987 and 1988 the Fed Funds rate, which is an overnight borrowing rate, is higher than the 3- and 6-month constant maturity Treasury rates. During 1989-1990, the Fed Funds rate even lies above the 10-year Treasury rate. However, after the beginning of year 1991, the Fed Funds rate becomes more conforming with the whole term structure. Obviously, the plot reflects the interventions in the Fed Funds market by Federal Reserve. Moreover, the Fed Funds plot contains more spikes and jumps than the Treasury rates do. This may be because financial institutions like commercial banks have to determine their reserves at the end of the two-week reserve settlement periods. Thus, they can end up with very high interest rate borrowing transactions in order to meet the bank reserve requirements.

Figure 4.2 provides the movement of each nodal Treasury rate as well as the Fed Funds rate over the studied period. This plot clearly provides strong evidence that the Treasury rate series are highly correlated although not perfectly correlated. Most of the time the level of short-maturity interest rate is less than those of the long-maturity rates. More interestingly, the short-rate exhibits a possibly persistent behavior, exemplified by periods of long tendencies, up, down, or flat, while the long-rate series show a movement which is more randomly. The last plot belongs to the Fed Funds rate, and it shows a cluster of fast movements.

Figure 4.3 plots the natural logarithm first difference of all nine return series with their corresponding monofractal Hölder-Hurst exponents calculated by the MMAR method. Since the Hölder-Hurst exponent is a measure of persistence, the fact that the short-rate series are more persistent \((H > 0.5)\) than the long-rate is \((H = 0.5)\) clearly observable, except for the Fed Funds rate that has \(H\) below 0.5.
Figure 4.2: Time Series Plot of Individual Maturity US Treasury and Fed Funds Rates. Each interest rate series is plotted over the studied period between April 21, 1987 and December 21, 2002. The top time series plot shows the longest maturity, 10-year, Treasury rate while the bottom shows the shortest maturity, Fed Funds rate.
Figure 4.3: The Time Series Plot of Individual Return Series of the US Treasury and Fed Funds Rates (First Differences of the Logarithm of the Original Series). The left column indicates the maturity for each plot starting from the longest, 10-year maturity, to the shortest, Fed Funds, rates. The right two columns next to the plots provide the degree of persistence or monofractal Hölder-Hurst Exponent identified by the Power Spectrum (Slope of Power Spectrum) method for each return series across studied maturities and its corresponding trading time, respectively.
Unlike the 3-month interest rate series, the longest maturity, 10 year, explicitly shows the mean reversion in its return series over the studied period and hardly indicates any singularities (or jumps). Apparently, this behavior is similar to that of Geometric Brownian motion, supported by the calculated Hölder-Hurst exponent of 0.5. Once moving to the shorter maturity rates of return, the calculated Hölder-Hurst exponent monotonically increases from 0.5 to 0.57, except the jump of Hölder-Hurst exponent at the 6-month return series to $H = 0.59$. The higher the Hölder-Hurst exponent, the more persistent the interest rate series. Thus, the Hölder-Hurst exponents of the nine return series, except the Fed Funds rate, suggest that the shorter-maturities are more persistent than the longer-maturity return series. Moreover, the plots themselves reveal a very stunning picture of the singularities or jumps and volatility clustering (a small change tends to be followed by small changes and vice versa) in short maturity Treasury interest rate series, e.g., the time series plot of the 3-month and Fed Funds return series.

A possible behavioral explanation for the observed persistence may be that the Federal Reserve auctions the Treasury papers in auctions with only 40 primary dealers. In contrast, the Fed Funds market is an open market with hundreds of commercial banks.

4.1 Time-Scale Analyses of Empirical Treasury Rates

The time-scale analysis begins with the longest maturity Treasury rate series. Figure 4.4 provides the analytical profile of the 10-year Treasury rates including the a) partition function $S_q(T, \delta)$ b) scaling function $\tau(q)$ c) multifractal spectrum $f(\alpha)$, and d) the quadratic fitting of the multifractal spectrum. Panel (a) in the figure exhibits the partition functions, $S_q(T, \delta)$, plotted against the based 2 logarithm of investment horizon, $\log_2 \delta$, covering the
first five integer moments of the distributions including the zeroth moment. The partition
functions are adjusted such that they have the same intercept on y-axis for comparison
purpose. The plot starts from the zeroth moment $q = 0$ (the lowest line) with a unit
increment and ends at the fifth moment $q = 5$ (the highest line). From the discussion of
the partition function in previous chapter, the plot of each moment that is nearly straight
suggests that such a particular moment exists and vice versa. This time-scale analysis is
not restricted to the integer moments. Fractal analysis even allows us to investigate the
fractional moments. However, this study focuses on the first few integer moments that are
common in the financial literature.

Noticeably, the partition functions of first three moments are nearly straight with some
increasing variation when the investment horizon ($\delta$) becomes longer and eventually reaches
the maximum investment horizon of $2^8 = 256$ business days or about 1 year in this study.
This commonly known issue can affect the accuracy of the analysis and the MMAR para-
meters. To sustain the accuracy this study uses the $2^6 = 64$ business days or the investment
horizon of one quarter to obtain the MMAR parameters for all maturities unless otherwise
stated. Although using only the $2^6 = 64$ day investment horizon, the plot of fourth and fifth
moments still indicate several humps. Although these humps might affects the analysis of
higher moments, the plots overall can still be used to calculate the scaling function, $\tau(q)$.

Panel (b) provides the scaling function $\tau(q)$ plotted against the first five moments,$q = 1, 2, 3, 4$ and 5. The scaling function is obtained from the slopes of the partition
functions in panel (a). The 10-year series scaling function is concave upward and deviates
from the straight line with a $H = 0.5$ slope which is the scaling function slope of the GBM.
By definition, its value always is equal to -1 at the zeroth moment. Evidently, when the
Figure 4.4: The Time-Scale Analytical Profile of 10-Year US Treasury Rate: April 1987 - December 2002. Panel (a) shows the partition function plots for the first five integer moments including zero. The function intercepts are normalized for comparison purpose. The highest line is for the fifth moment while the lowest line is for the zeroth moment. Panel (b) shows the scaling function plot of the Treasury rate across the five moments. Panel (c) provides the Multifractal Spectrum obtained from the Legendre Transformation of the Scaling function. Panel (d) shows that the Multifractal spectrum can be fitted by the quadratic function that leads to the detection of the MMAR’s parameters.
moments are less than 2, the scaling function of the 10-year rate is slightly higher than that of the GBM. When the moments are greater, the scaling function becomes lower than that of the GBM. The Hölder-Hurst exponent of the Brownian Motion of 0.5 is directly obtained from the scaling function using the relationship shown in the previous chapter indicating the near GBM of the 10-year series. Its slight concavity also suggests that there exists some multifractality in the studied interest rate series.

As shown, panel (c) indicates the quadratic-like multifractal spectrum, \( f(\alpha) \), of the 10-year rate series obtained from Legendre transformation of its scaling function. The multifractal spectrum is simply the entire distribution of the Hölder-Hurst exponents of the series with a humped shape and maximum of unity by default. A complete, parabola-like multifractal spectrum containing negative slopes can be obtained using a different technique (Los, 2003). However, this study only emphasizes the positive real moments that play an important role in finance.

Since the quadratic multifractal spectrum of the Treasury rate also implies the quadratic multifractal spectrum of its trading time, panel (d) assures us that the multifractal spectrum of 10-year Treasury rate is almost perfectly quadratic. The fitted line lies almost exactly on top of the empirical multifractal spectrum. The fitted line slightly deviates when the Hölder-Hurst exponents are less than 0.4. The consequence of this issue will be revealed in the studies of MMAR performance in a later section. Using the multifractal spectrum of the rate series and the closed form solution of the lognormally distributed multifractal spectrum of trading time in the previous chapter, the most probable Hölder-Hurst exponent of its trading time can be recovered. Then, using the properties in discussing in the preceding chapter and the most probable Hölder-Hurst exponent of the trading time of 0.56 helps to
identify the first two moments of lognormally distributed measures, which are \( \hat{\lambda} = 1.11 \) and \( \hat{\sigma} = 0.22 \), respectively. These two moments will be used later in the synthesis of measures and simulation of the MMARs. Notice that the trading time shows more persistence than the Brownian Motion of the interest rates.

The analytical profile of 7-year rate appears in figure 4.5. The partition function in panel (a) is very similar to those of 10-year rate series. The fourth and fifth moments are not so smooth even for the short-time investment horizon, \( 2 < \hat{\delta} < 2^5 = 32 \). Also, all five functions indicate humps at about \( 2^7 = 128 \) day investment horizon with an increasing variation in all of them. Panel (b) shows a slightly concave upward sloping scaling function that suggests a multifractal Hölder-Hurst exponent \( > 0.52 \). The curvature is obvious for higher moments. This again suggests that the series itself is multifractal. Its multifractal spectrum in panel (c) is quadratic-like, except where the slope are high or \( \alpha < 0.4 \). Evidently, the fitting line in panel (d) deviates even more from the empirical series at high moments. The effect will be obvious with the simulated results of the MMAR. The most probable Hölder-Hurst exponent is 0.57 that leads to \( \hat{\lambda} = 1.1 \) and \( \hat{\sigma} = 0.21 \) for its first two moments of the lognormally distributed spectrum of the trading time, respectively.

Moving to the profile of 5-year rate as shown in figure 4.6, in panel (a) the partition function of the fourth and fifth moments are even more troublesome. There appears to be several peaks over the entire investment horizon. However, this issue does not affect the estimate of the monofractal Hölder-Hurst exponent. As suggested by the scaling function in panel (b), only the second and its neighbor moments matter when calculating the monofractal Hölder-Hurst exponent which turns out to be 0.54.
Figure 4.5: The Time-Scale Analytical Profile of 7-Year US Treasury Rate: April 1987 - December 2002. Panel (a) shows the partition function plots for the first five integer moments including zero. The function intercepts are normalized for comparison purpose. The highest line is for the fifth moment while the lowest line is for the zeroth moment. Panel (b) shows the scaling function plot of the Treasury rate across the five moments. Panel (c) provides the Multifractal Spectrum obtained from the Legendre Transformation of the Scaling function. Panel (d) shows that the Multifractal spectrum can be fitted by the quadratic function that leads to the detection of the MMAR's parameters.
Figure 4.6: The Time-Scale Analytical Profile of 5-Year US Treasury Rate: April 1987 - December 2002. Panel (a) shows the partition function plots for the first five integer moments including zero. The function intercepts are normalized for comparison purpose. The highest line is for the fifth moment while the lowest line is for the zeroth moment. Panel (b) shows the scaling function plot of the Treasury rate across the five moments. Panel (c) provides the Multifractal Spectrum obtained from the Legendre Transformation of the Scaling function. Panel (d) shows that the Multifractal spectrum can be fitted by the quadratic function that leads to the detection of the MMAR's parameters.
The multifractal spectrum in panel (c) is similar to quadratic function. However, the empirical spectrum now deviates much more from the quadratic function in panel (d), implying that the assumption of a quadratic spectrum of its trading time is becoming less corroborated. The deviation becomes substantial for the small Lipschitz $\alpha$’s. This can partially be attributed to the considerable non-smoothness of the partition function of higher moments. Changing the resolution or the scales in this analysis might possibly improve the estimate of multifractal spectrum.

The fitting process reveals the most probable Hölder-Hurst exponent of the trading time which is 0.59 while its first two moments of the measure distribution are $\hat{\lambda} = 1.1$ and $\hat{\sigma} = 0.2$, respectively.

Figure 4.7 provides the profile of the 3-year interest rate. There is again an increasing variation in all partition functions shown in panel (a). The variation is greater in the third, fourth, and fifth moments. As appeared in previous analysis, the fourth and fifth moments have several peaks even in the short investment horizon, $2 < \hat{\delta} < 2^4 = 16$.

In addition, the partition functions seem to change their slopes where $\delta \approx 2^7 = 128$. The concave scaling function is in panel (b). The scaling function is equal to zero when $q$ is near two. The function is not a straight line, in support of multifractality. The Legendre transformation yields the quadratic multifractal spectrum in panel (c) where the maximum is always one. Panel (d) shows the fitting process and identification of the MMAR parameters.

The empirical spectrum now almost perfectly fits the quadratic function implying the spectrum of its trading time has a quadratic shape. The first two moments of the lognormal distribution are used for synthesizing the measure are $\hat{\lambda} = 1.11$ and $\hat{\sigma} = 0.23$, respectively.
Figure 4.7: The Time-Scale Analytical Profile of 3-Year US Treasury Rate: April 1987 - December 2002. Panel (a) shows the partition function plots for the first five integer moments including zero. The function intercepts are normalized for comparison purpose. The highest line is for the fifth moment while the lowest line is for the zeroth moment. Panel (b) shows the scaling function plot of the Treasury rate across the five moments. Panel (c) provides the Multifractal Spectrum obtained from the Legendre Transformation of the Scaling function. Panel (d) shows that the Multifractal spectrum can be fitted by the quadratic function that leads to the detection of the MMAR's parameters.
Figure 4.8: The Time-Scale Analytical Profile of 2-Year US Treasury Rate: April 1987 - December 2002. Panel (a) shows the partition function plots for the first five integer moments including zero. The function intercepts are normalized for comparison purpose. The highest line is for the fifth moment while the lowest line is for the zeroth moment. Panel (b) shows the scaling function plot of the Treasury rate across the five moments. Panel (c) provides the Multifractal Spectrum obtained from the Legendre Transformation of the Scaling function. Panel (d) shows that the Multifractal spectrum can be fitted by the quadratic function that leads to the detection of the MMAR's parameters.
Next analysis is on the 2-year rate. Figure 4.8 indicates all four analyses as before. The top two partition functions in panel (a) are not smooth; however, they are rather straight over the investment horizon or scales. The variation toward the end of the scale seems to be smaller than those of longer maturity rate series. The peaks appearing on the partition functions of the fourth and fifth moments are rather small. Panel (b) shows the concave upward scaling function where its value is equal to zero when the moment is close to 2.

The resulting monofractal Hölder-Hurst exponent is 0.55. Panel (c) suggests the quadratic multifractal spectrum, and it really is almost perfectly fitted by the quadratic function as shown in panel (d). The fitting spectrum lies almost exactly along the empirical spectrum of the 2-year rate. The most probable trading time Hölder-Hurst exponent that provides this great fitting is of 0.62. The first two moments of the lognormal distribution that drives its trading time are $\hat{\lambda} = 1.13$ and $\hat{\sigma} = 0.26$, respectively.

Figure 4.9 presents the analytical profile of the 1-year maturity rate of the term structure. Partition function plots in panel (a) are much alike those shown above. The fourth and fifth moments still carry several humps, particularly when the investment horizon is about $2^3 = 8$ to $32 = 2^5$ days. However, the overall plots are straight. There are no clear humps at the $2^7 = 128$ investment horizon as before. The variation toward the end of investment horizon of the partition function seems rather small. This suggests that a longer than $2^6 = 64$ day investment horizon could have been used in the time-scale analysis of this particular rate.

Panel (b) indicates a curve or the scaling function plotted against the first five moments. The curvature suggests again the multifractality of the distribution of the studied series. The deviation from the monofractal Gaussian distribution is more pronounced for the higher moments. The multifractal spectrum is shown in panel (c) where the calculated Hölder-
Figure 4.9: The Time-Scale Analytical Profile of 1-Year US Treasury Rate: April 1987 - December 2002. Panel (a) shows the partition function plots for the first five integer moments including zero. The function intercepts are normalized for comparison purpose. The highest line is for the fifth moment while the lowest line is for the zeroth moment. Panel (b) shows the scaling function plot of the Treasury rate across the five moments. Panel (c) provides the Multifractal Spectrum obtained from the Legendre Transformation of the Scaling function. Panel (d) shows that the Multifractal spectrum can be fitted again by the quadratic function that leads to the detection of the MMAR’s parameters.
Hurst exponent from the scaling function is 0.57. The quadratic-like spectrum is obtained from the Legendre transformation of the scaling function in panel (b). To ensure that the multifractal spectrum is quadratic, panel (d) shows again the quadratic fitting process. The fitted quadratic function plot matches the empirical spectrum almost exactly. This fitting also provides the most probable Hölder-Hurst exponent of the trading time multifractal spectrum which is 0.64. Using the theorem from the previous chapter, the first two moments of the lognormally distributed measures are \( \tilde{\lambda} = 1.13 \) and \( \tilde{\sigma} = 0.25 \), respectively.

The time-scale analysis of the 6-month Treasury rate appears in figure 4.10. Panel (a) shows plots of partition functions of first five integer moments against the log of investment horizon. All plots are very close to straight line over the studies investment horizon. Thus, the first five moments seem to be finite, although the increasing variation is found in the fourth and fifth moment partition function beginning at the four day investment horizon. Moreover, the deep valley is obvious at the \( 2^4 = 16 \) day investment horizon. The scaling function plotted over the first five moments appears in panel (b). The scaling function appears to be zero when the moment is close to two. This yields the monofractal Hölder-Hurst exponent of 0.59. Thus, the pricing series possesses a clear long memory. The curvature of the scaling function is obvious particularly for the moments higher than two indicating the departure from the monofractal Gaussian distribution. Panel (c) plots the multifractal spectrum of the Treasury rate, and its shape appears to be quadratic, as expected. The fitting process in panel (d) assures the quadratic shape of the empirical spectrum. However, the fitted spectrum does not lay exactly on top of the empirical spectrum. In fact, the fitted spectrum appears to be marginally higher than the empirical spectrum when Hölder-Hurst exponent is below 0.55. The opposite is true when the Holder exponent
is greater than 0.55. This small deviation certainly affects the MMAR parameters and then the simulation performance of the MMAR. The most probable Hölder-Hurst exponent of the trading time obtained from the quadratic multifractal spectrum is of 0.67 where the resulted first two moments of the lognormally distributed measures are $\lambda_1 = 1.132$ and $\sigma = 0.265$, respectively.

![Graphs showing partition function, scaling function, multifractal spectrum, and empirical fitted multifractal spectrums.]

Figure 4.10: The Time-Scale Analytical Profile of 6-Month US Treasury Rate: April 1987 - December 2002. Panel (a) shows the partition function plots for the first five integer moments including zero. The function intercepts are normalized for comparison purpose. The highest line is for the fifth moment while the lowest line is for the zeroth moment. Panel (b) shows the scaling function plot of the Treasury rate across the five moments. Panel (c) provides the Multifractal Spectrum obtained from the Legendre Transformation of the Scaling function. Panel (d) shows that the Multifractal spectrum can be fitted properly by the quadratic function that leads to the detection of the MMAR’s parameters.
Moving to the 3-month maturity profile, figure 4.11 provides detailed time-scale analysis of the 3-month Treasury rate. Partition function plots are in panel (a). The plots of first three moments are rather smooth and straight while the fourth and fifth moment plots are so volatile especially between the $2^2 = 4$ to $8 = 2^3$ day investment horizons. These peaks and valleys question the existence of the higher moments over the very short investment horizon. Also, there exists an increasing variation beginning at the $2^6 = 32$ days investment

![Figure 4.11: The Time-Scale Analytical Profile of 3-Month US Treasury Rate: April 1987 - December 2002. Panel (a) shows the partition function plots for the first five integer moments including zero. The function intercepts are normalized for comparison purpose. The highest line is for the fifth moment while the lowest line is for the zeroth moment. Panel (b) shows the scaling function plot of the Treasury rate across the five moments. Panel (c) provides the Multifractal Spectrum obtained from the Legendre Transformation of the Scaling function. Panel (d) shows that the Multifractal spectrum can be fitted properly by the quadratic function that leads to the detection of the MMAR’s parameters.](image-url)
horizon for all partition function plots where the variation is so pronounced for the fourth and fifth moments.

However, the magnitudes of variation seem to be quite small relative to the variation found in the other Treasury rate series with longer maturities. Panel (b) exhibits the scaling function plot across the moments. Clearly, the plot is not straight and deviates from the 0.5 instantaneous scale of the GBM. When the moment is approximately below two, the scaling function seems to be lower than what suggested by the GBM. The opposite is true when moments are greater than two. The scaling function value appears to be zero when the moment is close to two as expected. The calculated pricing Hölder-Hurst exponent is 0.57 indicating the 3-month interest rate series is persistent.

Panel (c) shows the concave and upward sloping multifractal spectrum of the 3-month Treasury rate series. When fitted, the quadratic function lies closely to the empirical spectrum, and the resulting most probable Hölder-Hurst exponent of the trading time obtained from the fitting process is 0.66. With both identified pricing Hölder-Hurst exponent and the most probable trading time Hölder-Hurst exponent found, the first two moments of the lognormally distributed measure which will be used in a simulation process of the MMAR are $\hat{\lambda} = 1.153$ and $\hat{\sigma} = 0.306$, respectively.

Figure 4.12 provides the multifractal analysis of Fed Funds Rate. Panel (a) shows the partition function which is totally different from those partition functions of the Treasury rate series. If considering up to the scale $\delta = 2^5$, none of the partition function plots of the first five moments indicates a straight-line shape, except that of the first moment. The partition functions began to look like a straight line where $\delta = [2^5, 2^8]$. With this unique characteristic, the computation of its scaling function requires a large investment horizon
Figure 4.12: The Time-Scale Analytical Profile of Fed Funds Rate: April 1987 - December 2002. Panel (a) shows the partition function plots for the first five integer moments including zero. The function intercepts are normalized for comparison purpose. The highest line is for the fifth moment while the lowest line is for the zeroth moment. Panel (b) shows the scaling function plot of the Fed Funds rate across the five moments. Panel (c) provides the Multifractal Spectrum obtained from the Legendre Transformation of the Scaling function. Panel (d) shows that the Multifractal spectrum can be fitted properly by the quadratic function that leads to the detection of the MMAR’s parameters.
which is $\delta = 2^8$, and panel (b) shows the corresponding scaling function of the Fed Funds rate providing the pricing Hölder-Hurst exponent $H = 0.48$ suggesting some degree of anti-persistence. The multifractal spectrum is shown in panel (c) where it can be fitted with the quadratic function shown in panel (d) with the first two moments of lognormal distribution of $\hat{\lambda} = 1.183$ and $\hat{\sigma} = 0.365$.

Notice that in all preceding cases, the trading time $\theta(t)$ is much more persistent than the neutral GBM, and that the compounding with the Brownian Motion $B_H(t)$ makes the interest rate pricing series less persistent than the trading time, but still more persistent than the GBM, except the Fed Funds rate. The anti-persistence of the Fed Funds rate may be related to the fact that all transactions in the Fed Funds market are based on overnight repurchase agreements, i.e., which are immediately reversed after one night.

4.2 Parameters of Competing Models: MMAR and GARCH(1,1)

Table 4.1 provides all necessary MMAR parameters for each maturity. In other words, here is the collection of the time series models for each of the nine nodal interest rates in its simplest parametrization. Being able to handle the nonlinearity of the time series and preserving the correct scaling properties of the empirical data, the MMAR requires only 4 parameters including the Hölder-Hurst exponent $H$, the most probable Hölder-Hurst exponent of trading time $\alpha_0$, and the first two moments $\hat{\lambda}$ and $\hat{\sigma}^2$ of the lognormal distribution of the multiplicative measures which are used for synthesizing the trading time having a quadratic multifractal spectrum. The list of MMAR’s parameters begins from those series with the longest maturity, 10-year, to the shortest maturity, Fed Funds rate.
<table>
<thead>
<tr>
<th>Maturity</th>
<th>Pricing Hurst ($H$)</th>
<th>Trading Time Alpha Zero ($\alpha_0$)</th>
<th>Mean ($\hat{\lambda}$)</th>
<th>Variance ($\hat{\sigma}^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10-Year</td>
<td>0.504</td>
<td>0.56</td>
<td>1.110</td>
<td>0.221</td>
</tr>
<tr>
<td>7-Year</td>
<td>0.516</td>
<td>0.57</td>
<td>1.104</td>
<td>0.208</td>
</tr>
<tr>
<td>5-Year</td>
<td>0.537</td>
<td>0.59</td>
<td>1.099</td>
<td>0.199</td>
</tr>
<tr>
<td>3-Year</td>
<td>0.548</td>
<td>0.61</td>
<td>1.113</td>
<td>0.226</td>
</tr>
<tr>
<td>2-Year</td>
<td>0.548</td>
<td>0.62</td>
<td>1.132</td>
<td>0.264</td>
</tr>
<tr>
<td>1-Year</td>
<td>0.568</td>
<td>0.64</td>
<td>1.127</td>
<td>0.253</td>
</tr>
<tr>
<td>6-Month</td>
<td>0.592</td>
<td>0.67</td>
<td>1.132</td>
<td>0.265</td>
</tr>
<tr>
<td>3-Month</td>
<td>0.572</td>
<td>0.66</td>
<td>1.153</td>
<td>0.306</td>
</tr>
<tr>
<td>Fed Funds</td>
<td>0.482</td>
<td>0.57</td>
<td>1.183</td>
<td>0.365</td>
</tr>
</tbody>
</table>

Table 4.1: The Parameters of Treasury Rate MMARs. Hurst is the monofractal Hurst Exponent of the empirical interest rate process. Alpha zero is the most probable Hurst exponent of the trading time’s multifractal spectrum. Mean and variance are the first two moments of the lognormally distributed multinomial measures, respectively.

The information contained in the table reveals rather striking facts. The degree of persistence or long memory $H$ is almost neutral for the 10-year interest rate series where $H = 0.5$ represents the GBM. However, when the maturity becomes shorter, the $H$ systematically and gradually increases to reach 0.59 for the 6-month maturity rate and decreases slightly to 0.57 for the 3-month rate. However, this does not apply to the Fed Funds rate where $H < 0.5$.

This persistence in interest rates is attributed to the most probable Hölder-Hurst exponents $\alpha_0$ of multifractal spectrum of trading time. In other words, while the pricing process may be a theoretical, neutral GBM, the observed persistence is caused by the clustering of the actual trading, which is induced by the clustering of news events (Mandelbrot & Hudson, 2004). The 10-year rate has the least $\alpha_0$ of 0.56 while the 6-month rate shows the highest value of 0.67. The value decreases slightly to 0.66 for the 3-month series and
dramatically drops to 0.57 for the Fed Funds rate. The first moments of the lognormally distributed measures $\lambda$ do not show any systematic characteristic. Evidently, $\lambda$’s across maturities are greater than one as expected, and their maximum is 1.183 which is for the Fed Funs rate series. Interestingly, the variances $\hat{\sigma}^2$ of those log normal probability measures seem to indicate a possible system. The shorter maturity Treasury rate series seems to have higher variance of lognormally distributed measures or more dispersion than those of interest rates with longer maturities.

Overall, all studied series support long memory in the interest rate, except the 10-year rate that is very close to the GBM and the Fed Funds rate which is anti-persistent. Moreover, the results indicate the non-Gaussian distribution, the existence of higher moments, nonlinearity, and time-and-frequency scaling properties in the studied Treasury series. These findings provide overwhelming empirical evidence falsifying the popular neutral GBM models in the term structure literature which assumes an Hölder-Hurst exponent $H = 0.5$. As a part of MMAR’s performance measurement relatively to those of GBM and GARCH(1,1) processes, the parameters of the GARCH(1,1) models for all studied Treasury rate and the Fed Funds rate series are shown in table 4.2, while those of the GBM simply are the first two moments of the studied return series.¹

The constants of mean equations $c$ for all series, except those of the 6-month, 3-month maturities, and Fed Funds rate are negative and very close to zero. Moving to the variance equation, the constants of variance equation $k$ are positive and very close to zero across all maturities. The coefficients of the GARCH effect $a$ in the variance equation are slightly

¹MMAR and GBM codes are provided in the appendix. GARCH routine in Matlab is used to parametrize and simulate GARCH(1,1). A summary of GARCH parametrization and simulation routines is also shown in the appendix.
Table 4.2: The Parameters of GARCH(1,1). c is the coefficient of the constant term in mean equation. k is the coefficient of the constant term in the variance equation. a is the coefficient of the GARCH effect in the variance equation. b is the coefficient of the ARCH effect in the variance equation.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>c</th>
<th>k</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>10-Year</td>
<td>-0.0002587</td>
<td>0.0000012</td>
<td>0.9434</td>
<td>0.0441</td>
</tr>
<tr>
<td>7-Year</td>
<td>-0.000262</td>
<td>0.0000011</td>
<td>0.9444</td>
<td>0.0454</td>
</tr>
<tr>
<td>5-Year</td>
<td>-0.0002326</td>
<td>0.0000015</td>
<td>0.9372</td>
<td>0.0515</td>
</tr>
<tr>
<td>3-Year</td>
<td>-0.0002262</td>
<td>0.0000013</td>
<td>0.9367</td>
<td>0.0563</td>
</tr>
<tr>
<td>2-Year</td>
<td>-0.0001589</td>
<td>0.0000015</td>
<td>0.9285</td>
<td>0.0643</td>
</tr>
<tr>
<td>1-Year</td>
<td>-0.000026</td>
<td>0.0000012</td>
<td>0.9273</td>
<td>0.0694</td>
</tr>
<tr>
<td>6-Month</td>
<td>0.0000067</td>
<td>0.0000014</td>
<td>0.9069</td>
<td>0.091</td>
</tr>
<tr>
<td>3-Month</td>
<td>0.0002927</td>
<td>0.0000038</td>
<td>0.8362</td>
<td>0.1638</td>
</tr>
<tr>
<td>Fed Funds</td>
<td>0.0031</td>
<td>0.000627</td>
<td>0.2906</td>
<td>0.638</td>
</tr>
</tbody>
</table>

below one, except the Fed Funds series that is close to 0.3, and gradually decrease when the maturities become shorter. These large magnitudes of the coefficients are expected because prior literature commonly agrees that volatility of financial time series has very significant Long Memory effect. The opposite is true for the coefficients of the ARCH effect b of the variance term. In all, the parametrization of GARCH(1,1) suggests that the constant terms are very close to zero and provides again clear support to those results of MMAR across maturities that there is a possible systematic relationship between the interest rates of the constant maturity Treasury term structure.

4.2.1 ALTERNATIVE HÖLDER-HURST EXPONENT IDENTIFICATION AND IMPROVEMENTS

Obviously, Hölder-Hurst exponent identification is very crucial for the MMAR where its identification can be done independently at the initial stage of the MMAR process. Prior literature, both inside and outside finance, suggests several techniques of Hölder-Hurst expo-
ment identification. Those identification techniques typically can be categorized into three
groups; parametric, semi-parametric, and non-parametric. Nevertheless, their common
major issue in the Hölder-Hurst exponent identification is their accuracy. Following the
five step process of analysis discussed in Los (2003), this dissertation also implements an
alternative Hölder-Hurst exponent identification technique, the Continuous Wavelet Multi-
Resolution Analysis (CWMRA).

This dissertation uses Morlet(6) wavelet and $2^6 = 64$ day scale $a$ in a wavelet trans-
formation process for all Treasury rate series. The scale $a = 2^8 = 256$ is used for the
Fed Funds rate series. The resulting wavelet resonance coefficients are used to construct
the partition functions that, later, lead to the scaling exponent functions. The Legendre
transformation helps create the multifractal spectra where their modes reveal the global
Hölder-Hurst exponents, $H_{WL}$, for each corresponding studied series. The resulted $H_{WL}$
estimated by Morlet(6) wavelet and those $H_{MCF}$ obtained using Mandelbrot et al. (1997)'s
technique are shown in Table 4.3. Unlike previous results by power spectrum methods and
those published in related literature, the WMRA results indicate that all Treasury rates
in this study are either GBM or slightly anti-persistent ($0.47 \leq H \leq 0.5$). Particularly,
only the 2-Year maturity Treasury series has a range of $H_{WL}$ that reaches 0.5 while the
Fed Funds series indicates the lowest $H_{WL} = 0.45$. Obviously, some resulting
multifractal spectra have a flat slope at the top resulting in multiple values of Hölder-Hurst exponents
for the corresponding series. Only the Treasury rates of 3-year, 1-year, 6-month maturities,
and the Fed Funds suggest a single Hölder-Hurst exponent each.

Like many other Hölder-Hurst identification techniques, the resulting Hölder-Hurst
exponents not only vary a bit over time period, but also over the range of scales, $a$, or
Table 4.3: Continuous Wavelet Hurst Exponents vs. Power Spectrum Hurst Exponents. $H_{WL}$ is the wavelet Hurst exponent while the $H_{MCF}$ is the Hurst exponents estimated by the Power Spectrum Method suggested by Mandelbrot et al. (1997).

<table>
<thead>
<tr>
<th>Maturity</th>
<th>$H_{WL}$</th>
<th>$H_{MCF}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10-Year</td>
<td>0.47-0.48</td>
<td>0.50</td>
</tr>
<tr>
<td>7-Year</td>
<td>0.46-0.47</td>
<td>0.52</td>
</tr>
<tr>
<td>5-Year</td>
<td>0.47-0.48</td>
<td>0.54</td>
</tr>
<tr>
<td>3-Year</td>
<td>0.49</td>
<td>0.55</td>
</tr>
<tr>
<td>2-Year</td>
<td>0.49-0.50</td>
<td>0.55</td>
</tr>
<tr>
<td>1-Year</td>
<td>0.49</td>
<td>0.57</td>
</tr>
<tr>
<td>6-Month</td>
<td>0.49</td>
<td>0.59</td>
</tr>
<tr>
<td>3-Month</td>
<td>0.48-0.49</td>
<td>0.57</td>
</tr>
<tr>
<td>Fed Funds</td>
<td>0.45</td>
<td>0.48</td>
</tr>
</tbody>
</table>

investment horizons, $\delta$, in the wavelet and power spectrum methods, respectively. Prior research simply chooses the range of $\delta$ in such a way that visually straight plots of related partition function result. This dissertation initially introduces an alternative method that attempts to quantify the rule of thumb by introducing the noise-data ratio that provides more systematic measure of $\delta$ to obtain an accurate Hölder-Hurst exponent for any financial time series. As previously observed, the partition functions of all studied Treasury rate series show a monotone increasing variation when the investment horizon increases. Since Hölder-Hurst exponent is obtained from the projection of those partition functions, the error will occur if the projection method fails. In other words, the Treasury partition functions have already shown the sign of increasing variance over investment horizon. Thus, the simple Ordinary Least Square (OLS) projection could result in incorrect Hölder-Hurst exponent values. There are two alternatives for this issue. One is to employ state-of-art projection techniques. The other is to measure where the OLS is optimal. The noise-data
ratio method falls into the latter alternative. Using the fact that $R^2$ of the OLS is a global measure of bivariate goodness-of-fit, one can define the noise-data ratio as a function of the moments, $q$, and investment horizon, $\delta$, as follows

$$\text{Noise-Data Ratio} = f(q, \delta)$$

(4.1)

$$= 1 - R^2(q, \delta)$$

(4.2)

Table 4.4 reports the noise-data ratios of the 3-month Treasury rate series in panel (a) and their corresponding scaling exponents in panel (b) for fractional moments 1.5 to 2.5 with an increment of 0.1 over the scales ranging from 4 to 256 with a power of two increment. Each element of the panel a in the table indicates a magnitude of noise that cannot be explained by the OLS. By averaging the elements for each column or $\delta$, the averaged noise-data ratio for each particular $\delta$ is obtained revealing how much the noisy issue of the projection method exists for each particular scale. The higher the averaged noise-data ratio, the noisier the projection results are. The first observation from panel (a) is that the averaged noise-data ratio is only 25% for the 4 day investment horizon. The value increases to around 34% – 46% for $\delta = [8, \ldots, 256]$. The averaged noise-data ratio then jumps to about 91% when $\delta$ is greater than $2^{10} = 1024$. Thus, for the 3-month Treasury rate series, the very high $\delta$’s does not provide any good Hölder-Hurst exponent identification.

As previously discussed, the Hölder-Hurst exponent is calculated from the relationship in equation 3.8. Thus, panel (b) provides the corresponding scaling exponents, and the

$^2$The resulting noise-data ratios of $\delta > 2^9$ are computed but not reported.
(a) Noise-Data Ratio for moments $q \in [1.5, 2.5]$ where $\delta \in [2^2, 2^8]$  
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
\textbf{(q)} & 4 & 8 & 16 & 32 & 64 & 128 & 256 \\
\hline
1.5 & 0.09 & 0.08 & 0.04 & 0.07 & 0.30 & 0.77 & \textbf{1.00} \\
1.6 & 0.15 & 0.16 & 0.08 & 0.14 & 0.62 & \textbf{1.00} & 0.73 \\
1.7 & 0.32 & 0.36 & 0.20 & 0.33 & \textbf{0.99} & 0.79 & 0.47 \\
1.8 & 0.79 & 0.76 & 0.51 & 0.79 & 0.84 & 0.52 & 0.33 \\
1.9 & \textbf{0.83} & \textbf{1.00} & \textbf{0.97} & \textbf{0.95} & 0.54 & 0.36 & 0.25 \\
2 & 0.31 & 0.78 & 0.89 & 0.60 & 0.35 & 0.27 & 0.20 \\
2.1 & 0.13 & 0.56 & 0.63 & 0.40 & 0.26 & 0.22 & 0.17 \\
2.2 & 0.07 & 0.43 & 0.47 & 0.30 & 0.20 & 0.19 & 0.15 \\
2.3 & 0.05 & 0.35 & 0.39 & 0.25 & 0.17 & 0.17 & 0.14 \\
2.4 & 0.03 & 0.30 & 0.34 & 0.23 & 0.15 & 0.15 & 0.13 \\
2.5 & 0.03 & 0.27 & 0.31 & 0.22 & 0.14 & 0.15 & 0.12 \\
\hline
\textbf{Average} & 0.25 & 0.46 & 0.44 & 0.39 & 0.42 & 0.42 & 0.34 \\
\hline
\end{tabular}

(b) Scaling Exponents $\tau(q)$ for moments $q \in [1.5, 2.5]$ where $\delta \in [2^2, 2^8]$  
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
\textbf{(q)} & 4 & 8 & 16 & 32 & 64 & 128 & 256 \\
\hline
1.5 & -0.18 & -0.21 & -0.22 & -0.18 & -0.12 & -0.06 & \textbf{0.01} \\
1.6 & -0.13 & -0.16 & -0.17 & -0.13 & -0.07 & \textbf{0.01} & 0.08 \\
1.7 & -0.08 & -0.11 & -0.12 & -0.08 & \textbf{-0.02} & 0.06 & 0.14 \\
1.8 & -0.03 & -0.05 & -0.07 & -0.04 & 0.04 & 0.12 & 0.20 \\
1.9 & \textbf{0.03} & \textbf{0.01} & \textbf{-0.02} & \textbf{0.02} & 0.09 & 0.17 & 0.25 \\
2 & 0.08 & 0.06 & 0.04 & 0.07 & 0.14 & 0.22 & 0.31 \\
2.1 & 0.13 & 0.12 & 0.09 & 0.11 & 0.19 & 0.27 & 0.36 \\
2.2 & 0.17 & 0.17 & 0.14 & 0.15 & 0.23 & 0.32 & 0.42 \\
2.3 & 0.22 & 0.22 & 0.18 & 0.20 & 0.27 & 0.37 & 0.47 \\
2.4 & 0.27 & 0.28 & 0.23 & 0.24 & 0.31 & 0.41 & 0.52 \\
2.5 & 0.32 & 0.33 & 0.28 & 0.28 & 0.35 & 0.45 & 0.57 \\
\hline
\end{tabular}

Table 4.4: Noise-Data Ratios of the 3-month Treasury Rate Series and their Corresponding Scaling Exponents. Panel (a) reports the noise-data ratios for moments 1.5 to 2.5 with an increment of 0.1 over the scales ranging from 4 to 256 with a power of 2 increment. Panel (b) reports the scaling exponents of the relative moments and scales. The bold values suggest the moments that should be used to calculate the Hurst exponent for a particular scale.
bold faced values suggest the approximate moments needed to be used in Hölder-Hurst calculations for each particular investment horizon. These moments turn out to be the same as those detected in the panel (a), as expected. Thus, by simply examining the noise-data ratios, one can identify the moments needed for the Hurst exponent calculation from the maximum value of the noise data ratio in each column. The bold faced numbers in the panel (a) suggests that for $\delta = [4, 8, 16, 32]$, the Hölder-Hurst exponents are very similar, or about $\frac{1}{19}$, and very close to 0.5 or the GBM. When $\delta$ increases, the needed moments also increase. The Hölder-Hurst exponent is highest at $\delta = 256$. Overall, the study of noise-data ratios suggest that the Hölder-Hurst exponent of the 3-month Treasury rate series is between 0.5 and 0.67 over the range of $\delta$ between 4 and 256 where the projection method can explain the data up to about 60%. The result strongly suggests that the 3-month rate series is persistent and indicates strong Long Memory effect when the investment horizon is between $64 = 2^6$ and $2^8 = 256$ days.

For the sake of simplicity and reasonably comparative tests among the competing models (to avoid the subjective issue), this dissertation uses the $2^6 = 64$ day investment horizon in all MMARs for all Treasury rate series for the rest of the studies.

4.3 Comparative Analyses of Simulated Results

Using simulations to gauge the performance of models is always subject to critique in terms of the length of simulated samples and the quality of the pseudo-randomness used to generate the simulations.

This dissertation has attempted to implement variance reduction methods. Since the functional form of the MMAR is not monotone, most of the common variance reduction
techniques are not applicable. The Quasi-Monte Carlo method also has a very limited use in this context. The fundamental idea of the Quasi-Monte Carlo simulation is to generate a vector of deterministic (non-randomness) values that work well with the functional form. Unfortunately, the technique does not support the comparative test based on simulation in this dissertation, because the Quasi-Monte Carlo can only generate one non-random vector.

The second generated vector will be identical to the first vector while all the simulations in this dissertation require 1,000 independently generated vectors. The length of simulated samples is also crucial for the simulated MMAR. The preliminary results show that the longer the simulated samples, the better the MMAR can preserve the scaling exponent found in the empirical data. Using 100,000 replications, Calvet and Fisher (2002) found similar results in their studies.

With these caveats in mind, the following analysis compares the elegant, but traceable MMARs with the GBM and GARCH(1,1) processes using the Monte Carlo simulation technique. The comparative test of their performance provides insightful information that will be very beneficial to the interest modeling in near future. The test begins with 1,000 simulations of each process using its corresponding parameters.

First, the performance of MMAR is tested by investigating how effectively it can preserve the Long Memory found in the empirical series. The monofractal Hölder-Hurst exponent for each simulated MMAR is calculated for each maturity. The resulting Hölder-Hurst exponents are averaged for each maturity and then reported in table 4.5. As expected, the average Hölder-Hurst exponents for all Treasury maturities are identical to those Hölder-Hurst exponents fed into the simulation algorithm, except the 10-year rate where the Hölder-Hurst exponent is 0.01 larger than the corresponding input. More precisely, the averaging
Table 4.5: Averaged Hurst Exponents of 1,000 Simulated MMARs. The average Hurst (MMAR) column provides the average of Hurst exponents from the 1,000 simulated MMARs for each maturity. The Input Hurst (Empirical) column shows the detected Hurst exponents from the empirical data for each maturity.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Average Hurst (MMAR)</th>
<th>Input Hurst (Empirical)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10-Year</td>
<td>0.51</td>
<td>0.50</td>
</tr>
<tr>
<td>7-Year</td>
<td>0.52</td>
<td>0.52</td>
</tr>
<tr>
<td>5-Year</td>
<td>0.54</td>
<td>0.54</td>
</tr>
<tr>
<td>3-Year</td>
<td>0.55</td>
<td>0.55</td>
</tr>
<tr>
<td>2-Year</td>
<td>0.55</td>
<td>0.55</td>
</tr>
<tr>
<td>1-Year</td>
<td>0.57</td>
<td>0.57</td>
</tr>
<tr>
<td>6-Month</td>
<td>0.59</td>
<td>0.59</td>
</tr>
<tr>
<td>3-Month</td>
<td>0.57</td>
<td>0.57</td>
</tr>
<tr>
<td>Fed Funds</td>
<td>0.45</td>
<td>0.48</td>
</tr>
</tbody>
</table>

The simulated Hurst exponent of the 10-year rate is 0.506 while the empirical input is 0.504. The difference is only 0.4%. Notably, the average of the simulated MMARs of the Fed Funds rate has a quite different degree of anti-persistence found in the empirical series. It appears that it is more difficult to comparatively simulate anti-persistent series than persistent series. In sum, the advantage of these results is that they provide numerical support that the MMAR can effectively preserve the monofractal Hölder-Hurst exponent for studied Treasury rate series across maturities, although it is having some difficulty with the simulation of the anti-persistent Fed Funds series.

Moving to another interesting comparative test, table 4.6 presents the identified scaling exponents or slopes of the partition functions for the first five integer moments of the empirical Treasury rates and the mean scaling exponents of those simulated MMAR, GBM, and GARCH(1,1) series with the maturities longer than two years while tables 4.7 and 4.8
<table>
<thead>
<tr>
<th>Moment (q)</th>
<th>10-Year Mean $\tau(q)$</th>
<th>7-Year Mean $\tau(q)$</th>
<th>5-Year Mean $\tau(q)$</th>
<th>3-Year Mean $\tau(q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sim. Emp.</td>
<td>Sim. MMAR</td>
<td>Sim. Cholesky</td>
<td>Sim. GARCH</td>
</tr>
<tr>
<td>1</td>
<td>-0.47</td>
<td>-0.45</td>
<td>-0.50</td>
<td>-0.50</td>
</tr>
<tr>
<td></td>
<td>[-0.54, -0.37]</td>
<td>[-0.58, -0.41]</td>
<td>[-0.58, -0.41]</td>
<td>[-0.55, -0.36]</td>
</tr>
<tr>
<td>2</td>
<td>0.01</td>
<td>0.01</td>
<td>0.00</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>[-0.20, 0.23]</td>
<td>[-0.16, 0.16]</td>
<td>[-0.17, 0.18]</td>
<td>[-0.18, 0.21]</td>
</tr>
<tr>
<td>3</td>
<td>0.44</td>
<td>0.41</td>
<td>0.49</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td>[-0.30, 0.88]</td>
<td>[0.23, 0.75]</td>
<td>[0.20, 0.82]</td>
<td>[-0.15, 0.80]</td>
</tr>
<tr>
<td>4</td>
<td>0.81</td>
<td>0.74</td>
<td>0.98</td>
<td>0.98</td>
</tr>
<tr>
<td></td>
<td>[-0.08, 1.48]</td>
<td>[0.60, 1.35]</td>
<td>[0.52, 1.54]</td>
<td>[-0.30, 1.40]</td>
</tr>
<tr>
<td>5</td>
<td>1.13</td>
<td>1.04</td>
<td>1.47</td>
<td>1.44</td>
</tr>
<tr>
<td></td>
<td>[-0.17, 2.02]</td>
<td>[0.94, 1.97]</td>
<td>[0.76, 2.25]</td>
<td>[-0.49, 1.98]</td>
</tr>
</tbody>
</table>

Table 4.6: The Comparison Between Scaling Exponents of Empirical Treasury Rates and Mean Scaling Exponents of Corresponding Simulated Processes. The results are for Treasury rates with longer than 2-year maturities. The first left column under each maturity provides the scaling exponents for the first five integer moments. The next three columns provide the average scaling exponents from 1,000 simulations for the MMAR, GBM, and GARCH(1,1) processes, respectively. The minimum and maximum values of the exponents are reported in parentheses. Emp. means 'Empirical' while Sim. means 'Simulated'.
<table>
<thead>
<tr>
<th>Moment ((q))</th>
<th>2-Year Mean (\tau(q))</th>
<th>1-Year Mean (\tau(q))</th>
<th>6-Month Mean (\tau(q))</th>
<th>3-Month Mean (\tau(q))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sim. MMAR</td>
<td>Sim. Cholesky</td>
<td>Sim. GARCH</td>
<td>Sim. MMAR</td>
</tr>
<tr>
<td></td>
<td>Emp. Series</td>
<td>Sim. GBM</td>
<td>(1,1)</td>
<td>Emp. Series</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Sim. GBM</td>
</tr>
<tr>
<td>1</td>
<td>-0.41</td>
<td>-0.40</td>
<td>-0.50</td>
<td>-0.40</td>
</tr>
<tr>
<td></td>
<td>[-0.48, -0.31]</td>
<td>[-0.58, -0.41]</td>
<td>[-0.61, -0.42]</td>
<td>[-0.46, -0.29]</td>
</tr>
<tr>
<td>2</td>
<td>0.09</td>
<td>0.09</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>[-0.10, 0.33]</td>
<td>[-0.16, 0.16]</td>
<td>[-0.25, 0.19]</td>
<td>[-0.18, 0.32]</td>
</tr>
<tr>
<td>3</td>
<td>0.50</td>
<td>0.49</td>
<td>0.51</td>
<td>0.47</td>
</tr>
<tr>
<td></td>
<td>[0.01, 1.01]</td>
<td>[0.25, 0.74]</td>
<td>[0.02, 0.89]</td>
<td>[0.07, 1.00]</td>
</tr>
<tr>
<td>4</td>
<td>0.83</td>
<td>0.81</td>
<td>1.00</td>
<td>0.93</td>
</tr>
<tr>
<td></td>
<td>[-0.11, 1.69]</td>
<td>[0.64, 1.32]</td>
<td>[0.20, 1.63]</td>
<td>[0.05, 1.58]</td>
</tr>
<tr>
<td>5</td>
<td>1.13</td>
<td>1.10</td>
<td>1.49</td>
<td>1.36</td>
</tr>
<tr>
<td></td>
<td>[-0.22, 2.32]</td>
<td>[1.02, 1.94]</td>
<td>[0.35, 2.34]</td>
<td>[0.03, 2.15]</td>
</tr>
<tr>
<td></td>
<td>[-0.24, 2.42]</td>
<td>[1.03, 1.99]</td>
<td>[0.09, 2.62]</td>
<td>[-0.39, 2.42]</td>
</tr>
<tr>
<td></td>
<td>[-0.15, -0.25]</td>
<td>[-0.16, 0.15]</td>
<td>[-0.25, 0.32]</td>
<td>[-0.13, 0.37]</td>
</tr>
<tr>
<td>1</td>
<td>0.17</td>
<td>0.16</td>
<td>0.01</td>
<td>0.14</td>
</tr>
<tr>
<td></td>
<td>[-0.05, 0.43]</td>
<td>[-0.15, 0.15]</td>
<td>[-0.25, 0.32]</td>
<td>[-0.13, 0.37]</td>
</tr>
<tr>
<td>2</td>
<td>0.61</td>
<td>0.56</td>
<td>0.52</td>
<td>0.45</td>
</tr>
<tr>
<td></td>
<td>[0.01, 1.11]</td>
<td>[0.25, 0.76]</td>
<td>[-0.07, 1.13]</td>
<td>[0.06, 1.08]</td>
</tr>
<tr>
<td>3</td>
<td>0.95</td>
<td>0.88</td>
<td>1.02</td>
<td>0.86</td>
</tr>
<tr>
<td></td>
<td>[-0.04, 1.79]</td>
<td>[0.64, 1.38]</td>
<td>[0.02, 1.90]</td>
<td>[0.81, 0.83]</td>
</tr>
<tr>
<td>4</td>
<td>1.25</td>
<td>1.17</td>
<td>1.51</td>
<td>1.25</td>
</tr>
<tr>
<td></td>
<td>[-0.15, 2.42]</td>
<td>[1.03, 1.99]</td>
<td>[0.09, 2.62]</td>
<td>[1.06, 1.10]</td>
</tr>
<tr>
<td>5</td>
<td>[-0.39, 2.42]</td>
<td>[1.03, 1.96]</td>
<td>[0.02, 1.58]</td>
<td>[1.04, 1.50]</td>
</tr>
</tbody>
</table>

Table 4.7: The Comparison Between Scaling Exponents of Empirical Treasury Rates and Mean Scaling Exponents of Corresponding Simulated Processes. The results are for Treasury rates with less than 3-year maturities. The first left column under each maturity provides the scaling exponents for the first five integer moments. The next three columns provide the average scaling exponents from 1,000 simulations for the MMAR, GBM, and GARCH(1,1) processes, respectively. The minimum and maximum values of the exponents are reported in parentheses. Emp. means 'Empirical' while Sim. means 'Simulated'.
<table>
<thead>
<tr>
<th>Sim.</th>
<th>MMAR</th>
<th>Sim.</th>
<th>GARCH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Emp.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Series</td>
<td>Cholesky</td>
<td>GBM</td>
<td>(1,1)</td>
</tr>
<tr>
<td>-0.46</td>
<td>-0.5</td>
<td>-0.51</td>
<td>-0.41</td>
</tr>
<tr>
<td></td>
<td>[-0.62, -0.34]</td>
<td>[-0.64, -0.37]</td>
<td>[-0.59, -0.26]</td>
</tr>
<tr>
<td>-0.03</td>
<td>-0.07</td>
<td>-0.02</td>
<td>0.06</td>
</tr>
<tr>
<td></td>
<td>[-0.41, 0.23]</td>
<td>[-0.29, 0.22]</td>
<td>[-0.34, 0.31]</td>
</tr>
<tr>
<td>0.3</td>
<td>0.29</td>
<td>0.46</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td>[-0.53, 0.82]</td>
<td>[0.05, 0.84]</td>
<td>[-0.45, 0.83]</td>
</tr>
<tr>
<td>0.56</td>
<td>0.58</td>
<td>0.92</td>
<td>0.59</td>
</tr>
<tr>
<td></td>
<td>[-0.76, 1.31]</td>
<td>[0.38, 1.45]</td>
<td>[-0.71, 1.35]</td>
</tr>
<tr>
<td>0.79</td>
<td>0.84</td>
<td>1.37</td>
<td>0.78</td>
</tr>
<tr>
<td></td>
<td>[-1.09, 1.78]</td>
<td>[0.68, 2.07]</td>
<td>[-0.98, 1.84]</td>
</tr>
</tbody>
</table>

Table 4.8: The Comparison Between Scaling Exponents of Empirical Fed Funds Rates and Mean Scaling Exponents of Corresponding Simulated Processes. The results are for Fed Funds rates. The first left column provides the scaling exponents for the first five integer moments. The next three columns provide the average scaling exponents from 1,000 simulations for the MMAR, GBM, and GARCH(1,1) processes, respectively. The minimum and maximum values of the exponents are reported in parentheses. Emp. means 'Empirical' while Sim. means 'Simulated'.

Present the scaling exponents for those Treasury series with the maturities less than three years and the Fed Funds rate series, respectively. The mean scaling exponent of simulated series is calculated by averaging those scaling exponents of the 1,000 simulated series for each process. The mean scaling exponents of simulated GBM series for the maturities are almost exactly the same. Their scaling exponents increase approximately by 0.5 for each moment order. Although the results are not surprising, they ensure that the scaling analysis and simulation methods in this study are reliable and mutually compatible. In addition, the scaling functions of the simulated GBM for the studied maturities obviously deviate from those of empirical Treasury rate series.
Moving to the simulated GARCH(1,1), their scaling exponents for the first two moments are almost constantly at \(-0.5\) and 0, respectively. The scaling exponents of higher moments show some minor deviations from those of simulated GBM series as well as the empirical series. In fact, the average of the identified slopes of the simulated GARCH(1,1) series seems to follow those of the GBM for all studied moments. Thus, the results again suggest that the GARCH(1,1) process does not well represent the empirical Treasury rate processes. In particular, the scaling exponents of the fourth and fifth moments seem to be higher than those of empirical series for most maturities.

Only the mean scaling exponents of the simulated MMAR represent the scaling exponents of empirical series well. The MMARs preserve the scaling properties quite accurately for the low moments across maturities, while for most of the times the mean scaling exponents of the fourth and fifth exponents are lower than those of empirical series across maturities, except the case of the Fed Funds rate. Remarkably, the mean scaling exponents of the simulated MMARs almost perfectly match those of empirical series for 1-year, 2-year, and 3-month maturity rates. More interestingly, the mean estimated slopes of MMARs for the second moment across all studied maturities are very accurate and far superior to those of simulated GBM and GARCH(1,1), except for the Fed Funds series where the mean estimated slope of GBM is closer to that of the empirical series. Both simulated GBM and GARCH(1,1) processes can closely preserve the estimated slope of second moment for the 10-year maturity only. This finding is important since this confirms that the MMARs can provide a significant contribution to volatility modeling and to the proper pricing of fixed income options.
In tables 4.6, 4.7, and 4.8 the minimum and maximum of the 1,000 simulated scaling exponents for each process across maturities are also reported in parentheses. Another measure of performance is to measure the range (maximum - minimum) of the simulated scaling exponents. Beginning with the simulated GBM series, their ranges grow very slowly from the first to fifth moments, and the similar results appear across all maturities. The range is very narrow between 0.14 – 1.02. However, this is not true for the Fed Funds rate where the range is significantly wider.

Like the GBM process, the range of scaling exponents of the 1,000 simulated GARCH(1,1) series gradually increases over the studied moments. For the first moment, the range is the narrowest among the five moments and very close to that of the simulated GBM across maturities. However, the ranges become wider on average for higher moments relative to those of simulated GBM series. Another interesting observation is that the range of the GARCH(1,1) process seems to expand when the maturities become shorter. Although both simulated GBM and GARCH(1,1) series exhibit a narrow range of uncertainty over the five moments, it is significant that their identified slopes of partition functions are far different from those of empirical series for most of maturities. This caveat should be kept in mind while investigating this range measurement. Thus, both GBM and GARCH(1,1) seems to converge to the wrong scaling functions which are different from those of the empirical series, in contrast to the MMAR which converges to the correct scaling functions.

Like the simulated GBM and GARCH(1,1), the simulated MMARs yield similar results suggesting the monotone increasing range over the studied moments. Nevertheless, the range is significantly larger than those of both simulated GBM and GARCH(1,1) series, except those of the 3-year, 1-year, and 6-month maturities where the range is narrower than
that of GARCH(1,1). The difference of ranges among the three processes is not so obvious for the first two moments. Particularly, the range of the first moment across maturities is very similar to those of GBM and GARCH(1,1) processes although the maximum and minimum values are quite different. However, the difference becomes more pronounced for the third to the fifth moments in which the range expands more quickly, particularly when the moments increase. Also, the ranges of the simulated GARCH(1,1) series seem to expand rapidly and become close to those of the MMARs when the maturities become shorter. An exception occurs in the case of the Fed Funds rate where the range of the GARCH(1,1) for the higher moments turns to be very close to that of the MMAR. This casts doubt also on the performance of the GARCH(1,1) model as a correct model for the process of the instantaneous rate.

Although the shortcoming of the MMARs relative to the other two processes appears to be its wider ranges of the simulated scaling exponents, i.e., the greater uncertainty regarding these exponents, their average of identified slopes for each moment is very accurate relative to the empirical series.

4.4 Visualized Distributions: Snapshot vs. Time and Scale

In this section, the best candidate of 1,000 simulated series from each process is identified using the Wavelet transformation method as discussed in chapter 3. The performance measurement of the MMAR and comparative models begins with the investigation of how effectively the models preserve the entire time-frequency distribution found in the empirical series.
4.4.1 Frequency Distributions

Figures 4.13, 4.14, and 4.15 provide the snapshot distribution comparisons between those of the empirical interest rate series and their corresponding simulated series.

![Wavelet-Based Density Functions: Empirical vs. Simulated Processes](image)

Figure 4.13: The Comparison of the Density Functions Identified by the Wavelet Transformation. The solid line indicates the density function of the empirical series. The dashed-dotted line indicates the density function of the simulated GBM. The dotted line indicates the density function of the simulated MMAR. The dashed line indicates the density function of the simulated GARCH (1,1). Panel (a) shows the comparison of the densities for those series with 10-Year maturity. Panel (b) shows the comparison of the densities for those series with 7-Year maturity. Panel (c) shows the comparison of those densities from those series with 5-Year maturity. Panel (d) shows the comparison of those densities from those series with 3-Year maturity.

As seen from the panel a of figure 4.13, the solid line exhibits the wavelet compiled density function (distribution) of the 10-year Treasury return series. With a high peak around mean, thick tails, and a small hump at each side of the tails, the empirical distribution clearly deviates from the Gaussian distribution shown in a dotted-dashed line while the
dashed line show the compiled distribution of the corresponding simulated GARCH(1,1) series. Although very similar to each other, neither the simulated GBM nor GARCH(1,1) distributions is close to the empirical distribution. Only the dotted line indicating the compiled distribution of the corresponding simulated MMAR has a shape that is similar to the empirical distribution. But the peak of the simulated MMAR distribution is too high. The MMAR does get the kurtosis around the central tendency correct, but deviates in the moderate tails.

Panel b provides the comparison of those compiled densities with the density of the 7-year maturity. The overall result is similar to that of the 10-year maturity series, except that the empirical series now deviates more from the Gaussian and shows an obvious skewness. The simulated MMAR distribution clearly outperforms its counterparts having the similar level of peakedness and width relative to the empirical distribution. Moreover, GARCH(1,1) compiled distribution now slightly departs from the simulated GBM distribution having a higher peak.

The comparison of the distribution for those series with the 5-year maturity is shown in panel c, and the results are as expected. Although the MMAR seems to outperform the other models, it only captures the peakedness. The width of the MMAR distribution is too narrow. The simulated GBM and GARCH(1,1) distributions still are similar to each other, but very different from the empirical distribution, particularly for their peaks and tails showing no sign of thickness.

Panel d shows the comparison for those distributions with the 3-year maturity. Apparently, the shape of the empirical distribution is obviously different from those of the GBM and the GARCH(1,1). Unlike its competing processes, the MMAR yields the compiled dis-
distribution which is very similar to the empirical distribution, except that it is skewed, having a much thicker left tail. Although slightly skewed, the simulated MMAR distribution traces the right side or the positive return of the empirical distribution quite accurately.

![Wavelet-Based Density Functions](image)

Figure 4.14: The Comparison of the Density Functions Identified by the Wavelet Transformation. The solid line indicates the density function of the empirical series. The dashed-dotted line indicates the density function of the simulated GBM. The dotted line indicates the density function of the simulated MMAR. The dashed line indicates the density function of the simulated GARCH (1,1). Panel (a) shows the comparison of the densities for those series with 2-Year maturity. Panel (b) shows the comparison of the densities for those series with 1-Year maturity. Panel (c) shows the comparison of those densities from those series with 6-Month maturity. Panel (d) shows the comparison of those densities from those series with 3-Month maturity.

Figure 4.14 shows the compiled distribution comparison for those Treasury series with maturities between 2-year and 3-month. In panel a, the result of the comparison for the series with the 2-year maturity suggests that the MMAR is a better model to capture the empirical distribution than the GBM or the GARCH(1,1). The shape of simulated
MMAR distribution is almost exactly matched with that of the empirical distribution while the simulated GARCH(1,1) distribution seems to be very similar to the GBM, having too smooth and too thin tails.

Panel b presents the comparison of the distributions for those series with the 1-year maturity. The empirical distribution clearly deviates from the Gaussian. Only the simulated MMAR yields a similar compiled distribution. The simulated GBM distribution is slightly different from that of the GBM and is closer to the empirical distribution by having noticeably higher peak.

Panel c provides the comparison of the distributions for the 6-month maturity series and its corresponding processes. Unlike the MMAR, both the GBM and GARCH(1,1) do not yield shapes of distributions that are close to that of the empirical. Although maintaining the correct constant variance, they cannot trace the higher peak and thicker tails of the empirical distribution. In contrast, the simulated MMAR series shows a very high peak and thick tails found in the empirical series. Unfortunately, the peak is too high while the left tail is too thick relative to the estimated empirical distribution.

Panel d provides the plot of distributions for the 3-month maturity series, and the overall result is quite similar to those of the longer maturity series except that the peak of the empirical distribution is significantly higher relative to other empirical distributions with the longer maturities that usually fall in the range of 0.06 – 0.08. Again, only the MMAR shows superiority in preserving the shape of empirical distribution.

Figure 4.15 shows the comparison of the compiled distributions for the Fed Funds rate and its corresponding simulated series. Although the identification of the MMAR of the Fed Funds series faces some difficulties, its compiled distribution of the MMAR is very similar
Figure 4.15: The Comparison of the Density Functions Identified by the Wavelet Transformation. The solid line indicates the density function of the empirical series. The dashed-dotted line indicates the density function of the simulated GBM. The dotted line indicates the density function of the simulated MMAR. The dashed line indicates the density function of the simulated GARCH (1,1).
to the empirical distribution and superior to those of the GBM and GARCH(1,1). Like the analysis of the 1-year maturity rate, the identified distribution of the GARCH(1,1) has a high peak, but is still insufficient to replicate the peak of empirical distribution.

The figures suggest that the peakedness of the empirical distributions seems to be slightly higher when the maturities become longer. Also, the width looks narrower when the maturities increase. Moreover, the plots make the scaling distribution preservation of MMARs more visible. Nevertheless, these plots are only a snapshot of the time-varying distributions. Thus, to thoroughly investigate the MMAR’s performance overtime and scale, the Wavelet Multi-Resolution Analysis (MRA) is implemented.

4.4.2 Scalograms: Time-Scale (Frequency) Distributions

The continuous wavelet method of time-scale analysis of the studied series results in a scalogram that is a graph with time on the horizontal axis and different scales on the vertical axis. It traces the second moment, variance, or energy in the time series over a certain time period of 4,096 days and over scales 1 – 64 except the scales of the Fed Funds that range from 1 – 256. The absolute values (magnitudes or sizes) of the wavelet resonance coefficients, or localized variance, generated by the transformation in a certain scale at a given time, are represented by the color whose corresponds to the 128-color bar ranging from red (smallest variance) to reddish-pink (largest variance). Analogy to the R-squared of the regression projection, the reddish-pink color on the scalogram representing the largest magnitude of the wavelet resonance coefficients indicates the highest correlation between the mother wavelet and the interest rate series at a particular time and scale.
Thus, a significant advantage of the scalograms is that the graphic conveys not only when (over time), but also where (across scales or frequencies) an interesting event has taken place. This dissertation uses the scalogram analysis to compare the overall patterns of the localized variance found in the empirical data with its corresponding simulated processes.

Figures 4.16 - 4.24 show the scalograms of all studied Treasury and Fed Funds rates as well as those corresponding simulated series by the MMAR, GBM, and GARCH(1,1). For all figures in this analysis, the top left scalogram shows the localized variance of the empirical series at particular time moment on certain scales (or investment horizons). The top right scalogram plot belongs to the simulated GBM series. The lower right plot provides the scalogram of the simulated GARCH(1,1) series, while the lower left scalogram plot shows the local variance analysis of the simulated MMAR.

The scalogram analysis of the 10-year maturity series and its corresponding simulated series reveals only singularities that can be seen as vertical lines that run through many scales. As indicated by the identified H"older-Hurst exponent $H \approx 0.5$, the scalogram of the empirical series is similar to that of the GBM. However, the magnitudes of the wavelet resonance coefficients of the GBM are larger whose the resulting scalogram seems to be close to white noise (as it should be). Likewise, the scalogram of simulated GARCH(1,1) series suggests prevalent singularities and looks similar to that of the GBM. Only the scalogram of the simulated MMAR appears to be quite similar to that of the empirical series. But it is not a perfect match. Although its singularities occur approximately at the similar time interval found in the empirical series, the magnitude of the variance seems to be too large at the high scale, or low frequency.
Figure 4.16: Scalograms of the 10-Year Treasury Return Series and Its Corresponding Simulated Processes. The upper left plot shows the scalogram of the empirical rate over the 4,096-day period and across the 1-64 scales. The magnitude of wavelet coefficients are colorized with a 128-color scheme. The wavelet used in the transformation process is the Morlet(6). The upper right scalogram is calculated from the simulated GBM having the empirical mean and variance. The lower right plot shows the scalogram of the simulated GARCH(1,1) return series while the lower left plot shows the scalogram of the simulated MMAR series.
Figure 4.17: Scalograms of the 7-Year Treasury Return Series and Its Corresponding Simulated Processes. The upper left plot shows the scalogram of the empirical rate over the 4,096-day period and across the 1-64 scales. The magnitude of wavelet coefficients are colorized with a 128-color scheme. The wavelet used in the transformation process is the Morlet(6). The upper right scalogram is calculated from the simulated GBM having the empirical mean and variance. The lower right plot shows the scalogram of the simulated GARCH(1,1) return series while the lower left plot shows the scalogram of the simulated MMAR series.
Figure 4.18: Scalograms of the 5-Year Treasury Return Series and Its Corresponding Simulated Processes. The upper left plot shows the scalogram of the empirical rate over the 4,096-day period and across the 1-64 scales. The magnitude of wavelet coefficients are colorized with a 128-color scheme. The wavelet used in the transformation process is the Morlet(6). The upper right scalogram is calculated from the simulated GBM having the empirical mean and variance. The lower right plot shows the scalogram of the simulated GARCH(1,1) return series while the lower left plot shows the scalogram of the simulated MMAR series.
Figure 4.19: Scalograms of the 3-Year Treasury Return Series and Its Corresponding Simulated Processes. The upper left plot shows the scalogram of the empirical rate over the 4,096-day period and across the 1-64 scales. The magnitude of wavelet coefficients are colorized with a 128-color scheme. The wavelet used in the transformation process is the Morlet(6). The upper right scalogram is calculated from the simulated GBM having the empirical mean and variance. The lower right plot shows the scalogram of the simulated GARCH(1,1) return series while the lower left plot shows the scalogram of the simulated MMAR series.
The scalogram profiles of the 7-year, 5-year, and 3-year Treasury return series shown in figures 4.17, 4.18, and 4.19, respectively, reveal similar facts found in the analysis of the 10-year interest rate series. An interesting fact is that the scalogram of the empirical series shows more intermittence of the red areas when the maturities become shorter, a gradual departure from the GBM. The simulated GARCH(1,1) scalogram is slightly closer to the empirical scalogram than the GBM when the maturities decrease. However, the very large variances, indicated by pink and blue areas, appear too frequently. Only the scalogram of simulated MMAR seems to provide the similar patterns of colors found in the empirical series and to be consistently accurate across these maturities although the red areas seem to appear over longer time intervals on the scalogram of the 7- and 3-year series.

Figures 4.20 - 4.23 provide the scalogram profiles for those empirical and simulated series with the 2-year, 1-year, 6-month, and 3-month maturities, respectively. The scalograms of the simulated GBM series are still white noise. The differences are obvious for the scalograms of the empirical series, where the red area appears more often over a time period relative to the longer maturity Treasury return series. The shorter the maturity, the more red areas appear on the scalograms. In addition, when the maturity becomes shorter, on average the magnitudes of the wavelet resonance coefficients seem to be smaller in all scales and at certain time. The scalograms of simulated GARCH(1,1) also indicate the improvement, as more intermittent red areas begin to appear when the maturities decrease. However, the color or the size of the wavelet resonance coefficients is still different and larger than those of their corresponding empirical series. Moreover, the simulated GARCH(1,1) processes do not locate a cluster of singularities over time period correctly. As expected, the scalogram of simulated MMAR shows superiority over those of simulated GBM and
GARCH(1,1) by having scalograms that have a similar pattern of colors and clusters found in the empirical return series for these maturities. Nevertheless, the simulated MMAR scalograms show too many yellow breaks over time period and misses to locate some clusters of singularities found in the scalogram of the empirical series.

Figure 4.20: Scalograms of the 2-Year Treasury Return Series and Its Corresponding Simulated Processes. The upper left plot shows the scalogram of the empirical rate over the 4,096-day period and across the 1-64 scales. The magnitude of wavelet coefficients are colorized with a 128-color scheme. The wavelet used in the transformation process is the Morlet(6). The upper right scalogram is calculated from the simulated GBM having the empirical mean and variance. The lower right plot shows the scalogram of the simulated GARCH(1,1) return series while the lower left plot shows the scalogram of the simulated MMAR series.

The scalogram analysis of the Fed Funds rate is shown in figure 4.24 and requires careful investigation and interpretation. The Fed Funds return series behaves very differ-
Figure 4.21: Scalograms of the 1-Year Treasury Return Series and Its Corresponding Simulated Processes. The upper left plot shows the scalogram of the empirical rate over the 4,096-day period and across the 1-64 scales. The magnitude of wavelet coefficients are colorized with a 128-color scheme. The wavelet used in the transformation process is the Morlet(6). The upper right scalogram is calculated from the simulated GBM having the empirical mean and variance. The lower right plot shows the scalogram of the simulated GARCH(1,1) return series while the lower left plot shows the scalogram of the simulated MMAR series.
Figure 4.22: Scalograms of the 6-Month Treasury Return Series and Its Corresponding Simulated Processes. The upper left plot shows the scalogram of the empirical rate over the 4,096-day period and across the 1-64 scales. The magnitude of wavelet coefficients are colorized with a 128-color scheme. The wavelet used in the transformation process is the Morlet(6). The upper right scalogram is calculated from the simulated GBM having the empirical mean and variance. The lower right plot shows the scalogram of the simulated GARCH(1,1) return series while the lower left plot shows the scalogram of the simulated MMAR series.
Figure 4.23: Scalograms of the 3-Month Treasury Return Series and Its Corresponding Simulated Processes. The upper left plot shows the scalogram of the empirical rate over the 4,096-day period and across the 1-64 scales. The magnitude of wavelet coefficients are colorized with a 128-color scheme. The wavelet used in the transformation process is the Morlet(6). The upper right scalogram is calculated from the simulated GBM having the empirical mean and variance. The lower right plot shows the scalogram of the simulated GARCH(1,1) return series while the lower left plot shows the scalogram of the simulated MMAR series.
ently from those constant maturity Treasury return series as appeared in the preceding
time-scale analysis and identification of the corresponding MMAR. Its unique partition
functions require the analysis over the scale $\delta = 2^8 = 256$ reflecting the use of long scale
analysis of $\alpha = 2^8$ or 256 on the scalogram calculation. Its resulting scalogram reveals new
patterns of color and behaviors. The plot hardly indicates any clear singularities except
some yellow background in high scales while their wavelet resonance coefficients seem to be
very small as shown by large red areas.

Figure 4.24: Scalograms of the Fed Funds Return Series and Its Corresponding Simulated Processes.
The upper left plot shows the scalogram of the empirical rate over the 4,096-day period and across
the 1-64 scales. The magnitude of wavelet coefficients are colorized with a 128-color scheme. The
wavelet used in the transformation process is the Morlet(6). The upper right scalogram is calculated
from the simulated GBM having the empirical mean and variance. The lower right plot shows the
scalogram of the simulated GARCH(1,1) return series while the lower left plot shows the scalogram
of the simulated MMAR series.
More interesting is that the empirical scalogram suggests some periodicity and cyclicity. The horizon line at the scale \( a = 207 \) over the period of \( 500 - 2,900 \) days represents the periodicity, while the horizon dashed line at the scale \( a = 170 \) running almost across the entire time period indicates cyclicity.

The uniqueness of the Fed Funds return series also affects the resulting scalograms of its corresponding simulated processes. Unlike the resulting scalograms of those Treasury return series where the simulated MMAR is always superior to the GBM and GARCH(1,1), the result of the Fed Funds return appears differently. The scalogram of the simulated MMAR no longer replicates that of the empirical series. The background of the MMAR scalogram shows different color suggesting too large magnitudes of the wavelet resonance coefficients. In contrast, the simulated GARCH(1,1) scalogram seems to provide a better pattern of colors and on average seems to have smaller wavelet resonance coefficients than those of the MMAR. This result might be attributed to an inaccurate identification of the MMAR in the preceding analysis due to the collapse of the partition functions.

The above scalogram analysis of Treasury and Fed Funds return series across nine maturities assures the superiority of the MMAR regarding the preservation of the scaling properties of the empirical series over the traditional GBM process and GARCH(1,1) through the investigation of the corresponding simulated processes in this dissertation, except in the case of the Fed Funds return series. This unique feature of the MMAR may lead to further improvements in volatility modeling in the very near future.

Theoretically, there exists a relationship among all Treasury term structure nodes. Nevertheless, the MMARs, by construction, are not designed to capture and reserve such an interaction. Thus, the question whether the univariate MMAR provides the coherent term
structure suggested by the cash flow theory requires some empirical tests. Figure 4.25 provides a preliminary answer for this question. The top panel presents the empirical term structure over the studied time period. One of 1,000 simulated series for each maturity is selected and altogether plotted in bottom panel. It is clear that the design of the current MMAR as a univariate model does not provide a feature that provides a coherent relationship among the nodal maturities. Thus, the resulting term "structure" cannot yet exhibit the correct shape of the empirical term structure, because it lacks the observed coherence. Nevertheless, the MMAR seems to be a great univariate contender as it delivers quite accurate interest rate values for each maturity at the end of simulated paths. This characteristic of the MMAR should be very crucial in a volatility modeling field. In addition, better modeling of the nonlinear interaction between the nodes is still required.
Figure 4.25: The Resulting Term Structure of Simulated MMAR. These graphs compare the empirical term structure with a simulated MMAR term structure.
Chapter 5

Discussions and Possible Extensions

The empirical results in the previous chapter provide answers to the hypothesis of this dissertation that the Treasury rate series for all eight nodal maturities and the Fed Funds rate series are nonlinear and time-frequency scaling. The multifractal model of Asset Return (MMAR) can synthesize the empirical Treasury rate series with all desired properties such as time-frequency scaling, thick-tails, and long-term dependence found in the empirical Treasury rate series. When compared with the traditional GBM and GARCH(1,1) models, the MMAR shows superiority over the two competing models in terms of the time-frequency scaling preservation. Interestingly, the MMAR does not perform well when applying to the Fed Funds rate series and cannot provide a dynamic system for the entire term structure. The findings themselves raise more challenging research questions whether such a complete dynamic system really exists.

According to the results of the time-scale analyses across the eight maturities, there is strong evidence supporting the existence of nonlinearity scaling properties in all studied Treasury rate series. The scaling function plots with curvature in its slope suggest the deviation of the interest rate process from the assumed Gaussian distribution. (A straight slope indicates Gaussianness.) Moreover, the partition function plots of the first five moments, which seem to be straight for all series, suggest that the higher moments, at least up to fifth moment, of the true distribution exist. This is not true for the case of the Fed Funds
series, where the partition functions quickly collapse, indicating that there exists no stable distribution. In all, these plots imply that the GBM is inappropriate for modeling the underlying process of the Treasury and Fed Funds series, and that interest rate models that rely heavily on the Gaussian assumption are falsified.

The detection of Hölder-Hurst exponents that are greater than 0.5 for all Treasury rates across maturities suggests that there exist persistent long memory effects in all Treasury rate series, while the Fed Funds rate is slightly anti-persistent. Thus, the US Treasury market can be addressed as (marginally) persistent markets. Nevertheless, the long memory effect is not very strong in the interest rates themselves because their Hölder-Hurst exponents are still fairly close to 0.5, particular the 10-year maturity series that has almost exact 0.5. The fact that the 10-year maturity bond is global and that many countries have such 10-year maturity markets implies that there are many institutional investors who maintain this investment horizon. Moreover, globalization and state-of-art technology now allows the investors to trade anywhere in the world. The 10-year Treasury market has a very large number of participants and is becoming very liquid and price-neutral.

Another interesting fact is that the shorter the Treasury maturities, the larger the Hölder-Hurst exponents are. The 3-month Treasury market exhibits a higher degree of persistence than all longer maturity markets, except the 6-month maturity. That result is attributable to the attention traders in these markets pay to short term news events. The Fed Funds market, which relies on overnight repurchase agreement with large numbers of commercial banks, is super liquid. Commercial banks have to enter Fed Funds transaction every two weeks to comply with their reserve requirement. This creates the very fast
automatic trading activities that lead to the mild degree of anti-persistence for the Fed Funds market.

Nevertheless, the results should be interpreted carefully because the Hölder-Hurst identification is not exactly unique. There are many available identification techniques of which the accuracy is sometimes doubtful. This dissertation uses a technique suggested by Mandelbrot et al. (1997). Although the resulting Hölder-Hurst exponents are similar to those reported in the prior literature, this identification method itself may have some drawbacks. As seen in the partition function plot, there is an increasing variation of the function over the larger scales. Since the Hurst exponent is obtained from the projection of the partition function, the accuracy of the projection technique can become a challenging issue. This dissertation also provides the Hölder-Hurst exponents for all maturities using a particular form of wavelet technology. All these Hölder-Hurst exponents identified by wavelets are very close to 0.5 and very often less than 0.5 indicating mild anti-persistence that is opposite to the results of Mandelbrot’s method used. This casts more doubt on the particular wavelet multiresolution method used than on the obtained persistence results.

This dissertation provides a proven technique to quantify the detection of the Hölder-Hurst exponent. The noise-data ratio allows one to find an optimal range of scales used to identify the Hölder-Hurst exponent. The technique is very powerful revealing insightful information that probably fills in the gap in the Long Memory literature, where Hölder-Hurst exponent identifications of the similar financial asset are not unique. In the 3-month rate analysis, the noise-data ratio analysis suggests that there are two regimes over the investment horizon where the Hölder-Hurst exponent can be either $H \approx 0.5$ or $H \approx 0.65$, while both regimes do not contain too much error in the projection process. The difference
might be explained by the fact that there are two groups of participants within the 3-month Treasury market who might have different investment horizons: short and long while having different trading patterns: frequent and less frequent. This finding raises an empirical research topic for the further study of the microstructure for the short-rate Treasury market. The readers should keep in mind that the MMAR can be further tuned so that the model can perform optimally for each financial process. Also, using the $2^6$ scales for all Treasury rate series assures that the results are not manipulated when we compare the results of the competing models.

The reported monofractal Hölder-Hurst exponents do not necessarily suggest that the Treasury rate series follow the monofractal fractional Brownian Motion (fBM). As suggested by Rogers (1997), the monofractal fBM may not be a proper model for financial asset price or return paths because the model allows for arbitrage opportunities. The multifractal spectrum plots exhibiting the concave exponent function strongly indicates that the studied series are multifractal. This multifractal process as modeled by the MMAR is deemed to be theoretically more suitable for financial series because the MMAR is a martingale. Apparently, the multifractal spectra of all studied interest rates are concave upward implying that all series follow the multifractal fractional Brownian Motion, as proposed by Mandelbrot et al. (1997) and by Elliott and Van der Hoek (2003).

The MMAR models a multifractal process requiring only a few parameters. The Hölder-Hurst exponent is one of them. Since the MMAR is a compounded process between the GBM and the persistent trading time, to use the model, one has to estimate additional three parameters that can be identified quite easily.
To investigate whether or not the MMAR can be used to synthesize the interest rate series of each maturity and reproduce the scaling, thick-tail, and long-term dependence properties, this dissertation executes simple Monte Carlo simulations. There are several techniques to simulate the fBM with corresponding Hölder-Hurst exponents. This dissertation uses the Cholesky decomposition, which in fact is the same as the Levinson/Hosking simulation method. There were earlier attempts to improve the simulation algorithm particularly dealing with the pseudo-randomness. Anti-thetic variance reduction was applied. Unfortunately, the MMAR is not a monotone function. Thus, the anti-thetic method is not applicable. Quasi-Monte Carlo simulation, which assumes that there exists a deterministic series that makes the simulation provide accurate results, was attempted. However, by default it can only generate one vector of deterministic variables. In this dissertation at least 1,000 independent vectors of variables are needed. Thus, that method was also discarded. Although the simulating results in this dissertation are not expected to deliver misleading findings, all mentioned caveats about the pseudo-randomness issue should be noted here.

The averaged Hölder-Hurst exponents calculated from the 1,000 MMAR simulations from each maturity are almost exactly the same as those found in the empirical series. This strong result at least assures that the simulation procedure is a precise replication. Moreover, the average mean scaling exponents for the first five moments of the entire distribution calculated from those simulated MMAR series for each maturity are extremely close to those of the empirical series relative, to those of the GBM and GARCH(1,1). This is strong evidential support for the fact that the MMAR can really preserve the scaling property and time distribution of the empirical process. Thus, the MMAR should be used
whenever empirical financial data exhibits multifractality. The MMAR should not only be limited to the Treasury rate series, because it can be a model for many other financial time series, including stock prices and foreign exchange rates.

Within the interest rate modeling arena, the interest rate process is often assumed to be a Geometric Brownian Motion. Most popular interest rate models such as (factor) affine and GARCH-based models rely on this assumption. Since the time-scale analysis of the Treasury series across eight maturities and the Fed Funds rate suggests that the series are multifractional Brownian Motion, the affine and GARCH-based models are also falsified. The fact that the shorter maturity rate series has a higher degree of persistence should discourage investors from using the short rate as a factor in affine models. However, academicians and practitioners still use these models. Two possible reasons are that, first, they are so familiar with Gaussian properties and that, second, there are not many elegant and simple alternatives.

To check the time-scale preservation of the MMAR, a wavelet scalogram analysis is conducted for all studied processes. The scalograms provide the information about the localized variance of the interested series. As expected, the MMAR provides scalograms that show a time-frequency pattern similar to that of the empirical series while the GBM’s scalogram produces white noise (as it should!). The scalogram of simulated GARCH(1,1) does not correctly scale the variance over time and scales. Moreover, the singularities of simulated MMAR are located at the proper time moments and scales relative to the observed series. This dissertation shows clearly that there are some forces that have permanent shocks in the volatility for at least 64 days.
Unfortunately, from the simulated term structure plot, the series of independent MMARs by themselves cannot deliver the appropriate system shapes of the empirical term structure. This should not be so surprising because by design the MMAR is a univariate model at this current stage. Moreover, the investment horizon of 64 days is used for all MMAR identification to avoid subjective manipulation of outcomes. Thus, the simulated MMARs were not tuned optimally to represent the Cash Flow theory proposed by Los (2003). To test that Cash Flow theory more rigorously, a multivariate MMAR needs to be developed. At this stage, the MMAR can be adapted to the volatility model for a single asset, since this dissertation has already shown that the model can essentially trace the second moment of the empirical distribution. Moreover, the current literature seems to commonly agree that there exist strong Long Memory effects in the volatility of financial asset series.
Chapter 6

Conclusion

Treasury rates are the basis for default risk-free rates in finance. A change in the level and
variation of these interest rates has a substantial impact on rational financial decision-
making. Many theories have been proposed to explain the changes of level and the variation
of the term structure. Asset pricing modelers attempt to identify price diffusion processes
from empirical financial market data. In particular, the Geometric Brownian Motion and
the GARCH models are currently popular in these efforts.

For the first time, this dissertation identifies Multifractal Models of Asset Return
(MMARs) - recently introduced in the finance literature by Mandelbrot et al. (1997) -
from the eight nodal maturity term structure series of US Treasury rates and Fed Funds
rate. The dissertation shows that the Treasury rate and the Fed Funds series are multi-
fractal Brownian Motion. This dissertation also improves the current MMARs with better
numerical methodologies for the identification of Hölder-Hurst exponents using wavelets
and noise-data ratio techniques.

After analytical, numerical, and graphical analyses of the Treasury rate series, the corre-
sponding MMARs are then properly synthesized and simulated with the exact Monte Carlo
method, following Los (2003) who advises that every identified model should be simulated
to ensure that it can replicate the original empirical data. The model performance results
of these Monte Carlo simulations are then compared with not only the original empirical
time series, but also with the simulated results from the corresponding Brownian Motion and GARCH(1,1) processes. Moreover, this dissertation visualizes, analyzes and compares the simulated results using advanced tools such as Wavelet-based distribution identification and Wavelet scalograms, which allow the analysis to be conducted over time and scales simultaneously.

All eight maturity Treasury rates and Fed Funds rate series are multifractal processes. The measure of degree of persistence suggests that all eight Treasury rate series are persistent with respect to their time to maturity while the Fed Funds rate series is anti-persistent. It was surprising to find that, on average, the higher the maturity, the less persistent the series is. For the model comparisons, the simulated MMARs most of the time outperforms the corresponding simulated GBM and GARCH(1,1). The MMARs are clearly superior in preserving the distributional scaling of the empirical data. In addition, the simulated MMARs can closely trace the volatility of the empirical Treasury series for all eight maturities. As expected, the MMAR, by default, cannot currently represent itself as a complete model for the term structure. It fails to provide coherence to the dynamic system relationship among the nine maturities. Since there is evidence suggesting that the term structure forms a coherent system, a multivariate MMAR is needed for testing Los’ (2003) cash flow theory of the term structure. As of now, the univariate MMAR can only be an advanced model of the Market Segmentation theory. These findings should be particularly interesting for typical investors, asset price modelers, fund managers, and policy makers.
Bibliography


APPENDIX A

Treasury 3-Month MMAR

This Appendix demonstrates the application of the MMAR. Using the techniques discussed above, the identified parameters of the 3-Month constant maturity Treasury rate are as follows:

<table>
<thead>
<tr>
<th>$H$</th>
<th>$\alpha_0$</th>
<th>$\lambda$</th>
<th>$\sigma^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.572</td>
<td>0.66</td>
<td>1.153</td>
<td>0.306</td>
</tr>
</tbody>
</table>

$H$ = Hölder-Hurst exponent of the rate of return series

$\alpha_0$ = most probable Hölder-Hurst exponent or modal Lipschitz alpha of the multifractal spectrum of the time deformation (warping) or trading time.

$\lambda = \frac{\alpha_0}{\lambda}$ or mean of the assumed Log-normal multifractal spectrum.

$\sigma^2$ = variance of the assumed Log-normal multifractal spectrum.
A.1 The Time Series, First Difference, Scaling Function, Multifractal Spectrum Plots and Model Simulation Results With Respect to the Estimated Parameters.

Figure A.1: The top panel of the above figure shows the time series plot of the 3-month US Treasury Bill rates, while the bottom panel plots its first log differences. The number of daily observations is 4096.
Figure A.2: After converting the interest rate series into log price series, the partition functions are calculated for moment $q = 0$ to $8$ with a unit increment over a dyadicity of two time scales. The partition functions are rescaled at their origin for comparison purpose. The lowest partition function curve indicates the zero$^{th}$ moment. The approximate straight lines suggest the existence of scaling properties in this time series.
Figure A.3: The scaling function can be calculated using a simple projection. The concavity of this \( \tau(q) \) curve suggests the existence of multifractal scaling. If this curve would have been a straight line, monofractal scaling would exist.
Figure A.4: With the Legendre Transformation, the multifractal spectrum is derived and shown above. The maximum of the spectrum is always 1 due to the definition of the transformation. As suggested, the approximate concave curve seems to fall into the lognormal multiplicative measure. Notice that the modal Lipschitz-$\alpha$ $\alpha_0 = 0.66$. 
Figure A.5: This figure shows the empirical multifractal spectrum and its best fit quadratic curve with some certain parameters. The approximately good fitting suggests that the trading time has similar multifractal spectrum except some discrepancies close to the mode, and at the other extreme of the small Lipschitz-alphas. This fact allows us to identify the required parameters of the MMAR.
Figure A.6: This figure shows the synthesized lognormal measure based on the estimated mean $\lambda = \frac{20}{T}$ and variance $\sigma^2$ of the fitted Log-normal measure. The simulated noise shows readily the periods of clustering of very large movements.
Figure A.7: The simulated trading time is simply a cumulative sum of the simulated multiplicative measure of the previous step. The horizontal axis measures the conventional time and the vertical axis measures the trading time.
Figure A.8: This figure presents the time series and its first log difference plots according to the identified monofractal Hölder-Hurst exponent using the monofractal fBM method with Hölder-Hurst exponent $H = 0.572$. Notice that this does not provide the needed intermittency or clustering of pricing events observed in the empirical series in Figure A.1, although it is definitely an improvement over the conventional GBM with Hölder-Hurst exponent $H = 0.5$. 
Figure A.9: Once compounding the trading time with the GBM according to the MMAR, a simple simulation generates a pricing process that is superior to the monofractal fBM itself and appears to be very comparable with the observed series in the first figure.
APPENDIX B

CASCADEING PROBABILITY MEASURES

By definition, the cascading equation

\[ \mu_{k,b_k} = M_{b_k}^k \cdot M_{b_k - 1}^{k-1} \cdot \ldots \cdot M_0^1 \cdot M_0^0 \]  

(B.1)

represents a measure function where \( \mu \) is called a measure, and \( M \) is called a multiplier or generator. \( M_0^0 \) is the initiator or original mass. The use of this measure function in the MMAR emerges from the fact that a series of \( \mu \)'s over a period \( T \) has similar properties found in financial data series. These properties include clustering volatility and intermittency with jumps (bursts).

There are several types of multifractal probability measures. Each type depends on its assigned distribution of the multipliers, \( M \)'s. The most basic measure is generated with the multipliers that represent the binomial distribution, and the generated measures are called “binomial measures.”

Practically, the measures represent the mass (or density) allocation of a function within a particular time interval. Let’s consider the uniform function on the \([0,1]\) interval. The original mass of this distribution simply is equal to 1. Thus, to synthesize (or generate) the series of the binomial measures, the iterative method (called “multiplicative cascade”) can be performed by simply sub-dividing the original interval, \([0,1]\) into two sub-intervals with
equal length in each iteration. The fraction of the original mass will, then, be distributed into each new sub-interval. The mass is reallocated according to how the multipliers, $M$, is preset under a condition that the original mass is preserved during the allocation process. In other words, the summation of the two new measures must be unity. This simple iteration procedure (only one step) has already generated two new measures from the original measure. Figure B.1 shows the resulted measures after some iterations.

B.1 Multiplicative Measure

The above binomial measure can be extended simply by randomly generating the log normal (non-negative) multipliers, $M$’s, instead of fixing their values as did in the binomial example. The generated measures are then called ‘multiplicative measures.’ Figure B.2 exhibits the plot of synthesized multiplicative measures, while figure B.3 suggested its theoretical scaling function and multifractal spectrum.

The multiplicative measures have the desired properties that

1) They are non-negative, $\mu \geq 0$

2) They have scaling properties, and

3) The multi-fractal spectrum of the cumulative sum of the generated measures – called Trading Time – is hump and symmetric around its mean while the closed form formula is available.

Fortunately, the spectrum and its shape are very similar to those we have found in the empirical financial series, particularly the spectrum of the trading time generated by the lognormal measures. Thus, the process is to compute the multifractal spectrum of the interest rate series and determine whether or not it is quadratic, similar to that of
Figure B.1: The Plot of Binomial Measure with $M_0 = 0.6$ and $M_1 = 0.4$. After a few iterations, the measure begins to show jumps that are similar to financial time series.
Figure B.2: The Plot of the Multiplicative Probability Measure. The synthesis of the measure is very similar to the binomial measure. Unlike the binomial measures, the multiplicative draw the multipliers randomly from a particular distribution and use them to generate the measure at each step of iterations.
Figure B.3: The Theoretical Multifractal Spectrum of the Binomial Measures. The plot shows that the multifractal spectrum of the binomial measures has a shape of a symmetric hump.
lognormal measure. This allows us to identify the first two moments of the lognormal measure generating the trading time of the studied interest rate series.
APPENDIX C

SIMULATION OF THE MONOFRACIAL FRACTIONAL BROWNIAN MOTION

Since few theoretical results are known in financial asset pricing, simulation is crucial in understanding an interested process or function. The more complex the function becomes, the more important simulation is, because one needs to verify if the simulated behavior can consistently replicate the original (empirical) inputs. This appendix aims to describe some simulation methods for fractional Brownian motion (fBM) used in this dissertation. Both of them provide exact simulation and theoretically should be the same. The difference only is their costs of computation.

Because simulation is only possible in a discrete time interval, the notation $P_0, P_1, \ldots$ is used for the value of the fBM at the discrete time moment $0, 1, \ldots$.

C.1 CHOLESKY METHOD

This Cholesky decomposition method can be done without setting the time horizon in advance. However, the method is very slow and requires large storage to contain a matrix $L(n)$ that grows in every stage of the recursion. Typically, a Personal Computer cannot really simulate a large dimension of fBM using this decomposition method. Fortunately, there is another exact method called Levinson/Hosking that is equivalent to the Cholesky method, but require less storage to keep a matrix $L(n)$ and is much faster.
C.2 Levinson/Hosking Method

The Levinson method - sometimes known as Hosking method originally introduced by Hosking (1981) - is an algorithm to simulate a fractional Brownian Motion. This method begins with simulations of fractional Gaussian noise (fGN), \( x_0, x_1, \ldots \), which is defined as \( \B(t) - \B(t-1) \) for \( t = 1 \) to \( n \). The simulation is performed with the corresponding Hölder-Hurst exponent. Once the fGN is generated, the fBM can be obtained from the fGN’s cumulative sum. The Levinson/Hosking method recursively generates \( x_{n+1} \) given \( x_n, \ldots, x_0 \). The method does not need to use specific properties of neither the fGN nor fBM. Thus, given the past information, the distribution of \( x_{n+1} \) can be calculated explicitly.

As defined above, let \( \gamma(\cdot) \) be the covariance function, \( \Gamma(n) = (\gamma(i-j))_{i,j=0,\ldots,n} \) be the covariance matrix, and \( c(n) \) be the \((n+1)\)-column vector with elements \( c(n)_k = \gamma(k+1), k = 0, \ldots, n \). Also, let the \((n+1) \times (n+1)\) matrix \( F(n) = (1(i = n-j))_{i,j=0,\ldots,n} \), where \( 1 \) denotes the indicator function.

The matrix \( \Gamma(n+1) \) can be decomposed to

\[
\Gamma(n+1) = \begin{pmatrix}
1 & c(n)' \\
(c(n) & \Gamma(n) \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\Gamma(n) & F(n)c(n) \\
(c(n)'F(n) & 1
\end{pmatrix}
\]

and the conditional distribution of \( x_{n+1} \) given \( x_n, \ldots, x_0 \) can be computed. This distribution has expectation and variance given by
\[
\mu_n := c(n)' \Gamma(n)' \begin{pmatrix} x_n \\ \vdots \\ x_1 \\ x_0 \end{pmatrix}, \text{ and} \tag{C.3}
\]

\[
\sigma_n^2 := 1 - c(n)' \Gamma(n)^{-1} c(n), \text{ respectively.} \tag{C.4}
\]

Since the distribution is known, the simulation can be done by generating a standard normal random variable \( x_0 \) and simulate \( x_{n+1} \) recursively for \( n = 0, 1, \ldots \).
APPENDIX D

PARAMETRIZATION AND SIMULATION OF GARCH($p, q$)

GARCH($p, q$) process is in the form

$$x_t = c + \varepsilon_t$$  \hspace{1cm} (D.1)

$$\sigma_t^2 = k + \sum_{i=1}^{p} a_i \sigma_{t-i}^2 + \sum_{i=1}^{q} b_i \varepsilon_{t-i}^2,$$  \hspace{1cm} (D.2)

To parametrize the GARCH($p, q$), Matlab function, garchfit, is used. The algorithm utilizes the maximum likelihood method to obtain the parameters. Simulation of GARCH($p, q$) needs a function called garchsim, a filter that can be used to generate a (possibly) correlated return series, $x_t$, from a white noise input series, $\varepsilon_t$.

The function garchsim generates stable output processes in (approximately) steady-state by attempting to eliminate transients in the data it simulates. The function garchsim first estimates the number of observations needed for the transients to decay to some arbitrarily small value, and then generates a number of observations equal to the sum of this estimated value and the number requested. The function garchsim then ignores the estimated number of initial samples needed for the transients to decay sufficiently and returns only the requested number of later observations. To do this, garchsim interprets the simulated GARCH($p,q$) conditional variance process as an ARMA($\max(p,q),p$) model for the squared innovations. It then interprets this ARMA($\max(p,q),p$) model as the correlated output of

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a linear filter and estimates its impulse response by finding the magnitude of the largest eigenvalue of its auto-regressive polynomial.

Based on this eigenvalue, garchsim estimates the number of observations needed for the magnitude of the impulse response (which begins at 1) to decay below 0.01 (i.e., 1 percent). Depending on the values of the parameters in the simulated conditional mean and variance models, one may need long pre-sample periods for the transients to die out. Although the simulation outputs may be relatively small matrices, the initial computation of these transients can result in a large memory burden and seriously impact performance.
APPENDIX E

WAVELET-BASED DENSITY IDENTIFICATION

Matlab provides a routine to identify a density function of a sample using wavelets. The key idea is to use the detail and approximation wavelet resonance coefficients to approximate the unknown distribution.

Let \( X_1, X_2, ..., X_n \) be a sequence of independent and identically distributed random variables, with a common density function

\[
p = p(x)
\]  

(E.1)

This density \( p \) of this sequence is unknown. Our goal is to identify it. Let also suppose that \( \int p(x)^2 \, dx \) is finite. This condition allows \( p \) to be expressed in the wavelet basis. Thus, \( p \) can be written in a usual basis of functions \( \Phi \) and \( \Psi \) and with \( J \) being an integer as (Los, 2001, 2003):

\[
p = \sum_k a_{j,k} \Phi_{j,k} + \sum_{j=-\infty}^{J} \sum_k d_{j,k} \Psi_{j,k}
\]  

(E.2)

\[
= A_J + \sum_{j=-\infty}^{J} D
\]  

(E.3)
Thus, to get the common distribution $\hat{p} = \hat{p}(x)$ we only need to estimate the coefficients $\hat{a}_{j,k}$ and $\hat{d}_{j,k}$ of $p(x)$ where the definitions of the coefficients are as follows:

\begin{align*}
    a_{j,k} &= \int \Phi_{j,k}(x)p(x)dx, \text{ and} \\
    d_{j,k} &= \int \Psi_{j,k}(x)p(x)dx
\end{align*}

(E.4) \hspace{1cm} (E.5)
APPENDIX F

MATLAB CODE

The following Matlab codes provide a complete analysis of multifractality and its simulation.

These codes are written in Matlab version 6.1 release 12.

F.1 MULTIFRACTAL ANALYSIS CODE

function [H, alphaZ, Lambda, Logvar, qplot, Tq, Spec, Speci]=log_pricenewfit1(PT, Inv);

  qmin=0;
  qmax=5;
  qstep=1;

  qn = (qmax-qmin)/qstep+1;
  alpmin = 0;
  alpmax = 5;
  alpstep = 0.01;
  alpn = (alpmax-alpmin)/alpstep+1;

  %Compute Log_Price of P(t)
  P=PT;
  n = length(P);
  Xt = zeros(n,1);
for i = 1:n

    Xt(i,1)=log(P(i,1))-log(P(1,1));

end

%Compute Partition Sum

%Prepare new figure

figure;

for j = 1:qn %Compute no. of column (5-1)/1+1

    q = qmin+(j-1)*qstep);

for dt=1:Inv

    N = floor((length(Xt)-1)/dt);

    SX = zeros(N,1);

    SX(1,1) = abs(Xt(dt+1,1)-Xt(1,1))\^q;

    for k=1:N-1

        SX(k+1,1) = abs(Xt(k*dt+dt+1,1)-Xt(k*dt+1,1))\^q;

    end

    Sq(dt,1) = sum(SX);

    SX = [];

    dtplot(dt,1) = dt;

end

logSq = log2(Sq);

logdtp = log2(dtplot);

%Rescale Partition Sum and Plot them

RlogSq = logSq-logSq(1,1);
hold on;

%Label all pictures
plot(logdtp,RlogSq),title('Partition function'),xlabel('Log_2(\delta)') ...
ylabel('S_q(T,\delta)');

%Collect Partition Sum in a Matrix
Tsq(:,j) = logSq;

%Collect Rescaled Partition Sum in a Matrix
RTsq(:,j) = Tsq(:,j)-Tsq(1,j);
logSq = [];
end
C = [ones(size(logdtp)) logdtp];
qplot = [qmin:qstep:qmax];

%Collect beta coefficient for Scaling function. This can be improved with
%better regression due to changing variance
for j = 1:qn
    b = regress(Tsq(:,j),C);
    Tq(j,1) = b(2,1);
    b=[];
end

%Plot Scaling function
hold off;
figure;
plot(qplot,Tq),title('Scaling function'),xlabel('q'),ylabel('\tau(q)');
Scf = [qplot Tq];

%Search for Tq=0 (using interpolation) and Compute H

y=interp1(Tq,qplot,0);

H = 1/y;

%Note that H might not be accurate. In the footnote of Calvet and Fisher page 395 discusses
the sensivity of H.

%Compute Spectrum by Legendre Transformation

for w = 1:alpn %Sometimes Plus 1 because of a bug in MATLAB

alph = alpmin+((w-1)*alpstep);

Spec(w,1) = min(alph.*qplot - Tq);

end

figure;

alphplot = [alpmin:alpstep:alpmax];

plot(alphplot,Spec),title('Multifractal Spectrum'),xlabel('\alpha'),ylabel('f(\alpha)');

SpecMat = [alphplot Spec];

%Show that the Spectrum can be fitted by Quaratic Function.

le=length(Spec);

%Probable exponent should mean the alpha-zero indicating

%Maximum of the Spectrum = 1 always

g1=max(Spec);

ty=ones(le,1);

%dsearch is used as a lookup table function
\begin{verbatim}
k = dsearchn(SpecMat(:,2),ty);
k = k(1,1);
alphaZ = SpecMat(k,1);

% Fitting Spectrum to get the most probable exponent alphazero
Tafx=[];
amin = .30;
amax = 1.5;
aastep = .01;

aan = (amax-amin)/aastep+1;
aaplot = [amin:aastep:amax]';
alphplot = alphplot(1:length(Spec),1);
Speci = find(Spec>0 & Spec<1);
Spec = Spec(Speci);
alphplot = alphplot(Speci);

% Set length of alphaplot to match the length of Spec
alphplot = alphplot(1:length(Spec),1);
figure;

for op = 1:aan
   aalphhz = aamin+((op-1)*aastep);
   for a = 1:alpn
      afx(:,1) = 1-(alphplot(:,1)-aalphhz).^2/(4*H*(aalphhz-H));
   end
   Tafx(:,op)=afx;
\end{verbatim}
%Compute the Residue from the fitting

Res = Spec - afx;

Fnorm(op,1) = norm(Res);

%Compute Alphazero of the Spectrum

%Get Alphazero that produce the least NORM of Residual Vector

hold on

plot(alphplot,afx,'g','LineWidth',3),title('Fitting Process ... of the Empirical Multifractal Spectrum'),xlabel('\alpha'),ylabel('f(\alpha)');

end

plot(alphplot,Spec,'r')

hold off

minorm=min(Fnorm);

%Create Vector of the Min for comparison

minormvec=ones(Fnle,1)*minorm;

%Creat a 2x2 matrix for alphazero’s and their norm

aalphzplot = [amin:aastep:aamax]';

FnormMat = [aalphzplot Fnorm];

%dsearch is used as a lookup table function

F = dsearchn(FnormMat(:,2),minormvec);

f=F(1,1);

%The alphaZero from fitting process is in the "alphaZ" variable

alphaZfit=FnormMat(f,1);

%Plot the best fit result with Estimated Spectrum
figure;

plot(alphaplot,Tafx(:,f),'g','LineWidth',3);

hold on;

plot(alphaplot,Spec,'r'),title('Empirical and Fitted Multifractal ... Spectrums'),xlabel('\alpha'),ylabel('f(\alpha)');

Hold off;

%Compute Estimated Parameters

Lambda = alphaZ/H;

LambdaFit = alphaZfit/H;

Logvar = 2.*(Lambda-1)/log2(2);

LogvarFit = 2.*(LambdaFit-1)/log2(2);

ScaleCoef = Lambda-1;

disp(' Hurst AlphaZ AlphaZfit Lambda LambdaFit Logvar LogvarFit')

disp([H alphaZ alphaZfit Lambda LambdaFit Logvar LogvarFit])

Tq

F.2 MMAR Simulation Code

function [vall, xall, tauall, Tqall, Hall, x, v, tau, PT] = MMAR0905(Lambda, C1, N, H, Loop, series, Inv);

    orig = price2ret(series);

    int = series(1,1);

    mu = mean(orig);

    sigma = std(orig);
for u=1:Loop;

%Generating lognormal measure and trading time
epsilon=lognormal_multifractal(Lambda,D,C1);

tau = cumsum(epsilon);

tau = tau/max(tau)*N;

tauall(:,u) = tau';

%Generating fBM
[x, y, r]=fbmhosking(N, H, sigma);
x = transformhk(x', mu, sigma, int);
xall(:,u)=x;

v = Interpolate(N, x', tau);
vall(:,u)=v;
end;
xf = mean(xall')';
vf = mean(vall')';

for t = 1:Loop

F.3 lognormal probabilty measure code

function epsilon = lognormal_multifractal(Lambda,D,C1);

Lambda_D = 1+(Lambda-1)*(D-1);
gamma=randn(Lambda_D, Lambda);
Ktmp(1:Lambda/2+1)=(1:Lambda/2+1);
Ktmp(Lambda/2+2:Lambda)=Ktmp(Lambda/2:-1:2);
K=repmat(Ktmp,Lambda_D,1).^2;
Kprime=K;
for a=1:D-1
Kprime=Kprime’;
end;
K=sqrt(Kprime+K);
ft_f=K.^(-D/2);
Gamma=real(ifft2(ft_f.*fft2(gamma)));
epsilon=exp(Gamma*sqrt(2*C1*log(Lambda)/mean(std(Gamma).^2));

F.4 Fractional Brownian Motion Code

function [x,y,r] = fbmosking(N,H,sigma) ;

tmax = N-1 ;

seed = rand(1) * 1e6 ;

shift = 1 ;

% linspace is to Generate N points between 0 and tmax

t = linspace(0,tmax,N) ;

s = 0 ;

alpha = 2*H ;

% r is autocovariance function of fGN (stationary increment of fBM)

r = sigma^2*(abs(t+shift-s).^alpha + abs(t-shift-s).^alpha - 2*abs(t-s).^alpha)/2 ;

randn(‘seed’,seed) ;

y = randn(N,1) ;
x = zeros(1,N) ;

inter1 = r ;

inter2 = [0 r(2:N) 0] ;

Y = y(1)*r ;

k = -inter2(2) ;

aa = sqrt(r(1)) ;

for j = 2:N

aa = aa*sqrt(1-k^2) ;

inter = k*inter2(j:N) + inter1(j-1:N-1) ;

inter2(j:N) = inter2(j:N) + k*inter1(j-1:N-1) ;

inter1(j:N) = inter ;

bb = y(j)/aa ;

x(j:N) = x(j:N) + bb*inter1(j:N) ;

k = -inter2(j+1)/(aa^2) ;

end

%Need to cumulative sum because only fGN is created above, x = cumsum(x(:)) ;